VERTICAL AND HORIZONTAL COMPOSITION

Abstract. There are two ways to connect natural transformations between functors. One is called vertical composition and the other is called horizontal composition. Both are explained in this article.

Given natural transformations $\eta : F \to G$, $\varepsilon : G \to H$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, "G"{description, name=M}] \arrow[r, bend right=50, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, phantom, "\Downarrow\eta"] \arrow[from=M, to=D, phantom, "\Downarrow\varepsilon"] \end{tikzcd}$$

we obtain a natural transformation $\varepsilon\eta : F \to H$ with components $(\varepsilon\eta)_x := \varepsilon_x \circ \eta_x$.

Then $\varepsilon\eta$ is called *vertical composition* of η and ε. To see that $\varepsilon\eta$ is indeed a natural transforamation one only needs to consult the following diagram:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"'] \\ G(X) \arrow[r, "G(f)"] \arrow[d, "\varepsilon_X"'] & G(Y) \arrow[d, "\varepsilon_Y"'] \\ H(X) \arrow[r, "H(f)"] & H(Y) \end{tikzcd}$$

$$\begin{aligned} (\varepsilon\eta)_Y \circ F(f) &= \varepsilon_Y \circ \eta_Y \circ F(f) \\ &= \varepsilon_Y \circ G(f) \circ \eta_X \\ &= H(f) \circ \varepsilon_X \circ \eta_X \\ &= H(f) \circ (\varepsilon\eta)_X \end{aligned}$$

Given natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=A}] \arrow[r, bend right, "G"'{name=B}] & \mathcal{D} \arrow[r, bend left, "J"{name=C}] \arrow[r, bend right, "K"'{name=E}] & \mathcal{E} \arrow[from=A, to=B, phantom, "\Downarrow\eta"] \arrow[from=C, to=E, phantom, "\Downarrow\varepsilon"] \end{tikzcd}$$

$\eta : F \to G$, $\varepsilon : J \to K$ we obtain a natural transformation $\varepsilon\eta : JF \to KG$ by *horizontal composition*:

The components of $\varepsilon\eta$ are given by

$$(\varepsilon\eta)_X = \varepsilon_{G(X)} \circ J(\eta_X)$$

$$
\begin{tikzcd}
JF(X) \arrow[r, "JF(f)"] \arrow[d, "J(\eta_X)"'] & JF(Y) \arrow[d, "J(\eta_Y)"] \\
JG(X) \arrow[r, "JG(f)"] \arrow[d, "\varepsilon_{G(X)}"'] & JG(Y) \arrow[d, "\varepsilon_{G(Y)}"] \\
KG(X) \arrow[r, "KG(f)"] & KG(Y)
\end{tikzcd}
$$

$$
\begin{aligned}
(\varepsilon\eta)_Y \circ JF(f) &= \varepsilon_{G(Y)} \circ J(\eta_Y) \circ JF(f) \\
&= \varepsilon_{G(Y)} \circ J(\eta_Y \circ F(f) \\
&= \varepsilon_{G(Y)} \circ JG(f) \circ J(\eta_X) \\
&= KG(f) \circ \varepsilon_{G(X)} J(\eta_X) \\
&= KG(f) \circ (\varepsilon\eta)_X
\end{aligned}
$$

References

[PlanMath] PlanetMath.org - compositions of natural transformations