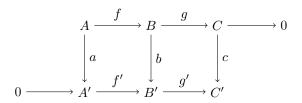
## THE SNAKE LEMMA AND LONG EXACT HOMOLOGY SEQUENCE

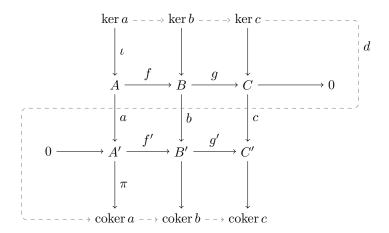
Consider a commutative diagram of R-modules of the form



If the rows are exact, there is an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c$$

constituting the picture of a snake in the diagram



where the  $\iota$  are the natural inclusion morphisms of the kernel and the  $\pi$  are the natural projection morphisms.

Proof.

## Morphisms between the kernel

First we define the morphisms between the kernel. For example we define the map

$$\bar{f}: \ker a \to \ker b$$

by

$$x \mapsto f(x)$$
.

This is well defined, since

$$b(f(x)) = f'(a(x)) = f'(0) = 0,$$

meaning that indeed  $f(x) \in \ker b$ . Of course, the upper left square then commutes and  $\bar{f}$  is a homomorphism of R-modules.

Of course,  $\bar{g}: \ker b \to \ker c$  is defined analogously.

## Morphisms between the cokernel

We proceed to define the morphisms between the cokernel. Here we, for example, define the map

$$\bar{f}': \operatorname{coker} a \to \operatorname{coker} b$$

by

$$[y] \mapsto [f'(y)]$$

for y in A'. Again we check that this gives us a well defined map:

Let  $y_1, y_2$  be in A' with  $[y_1] = [y_2]$ . This means that  $y_1 - y_2 \in \text{im } a$ , therefore we have z in A, such that  $a(z) = y_1 - y_2$ . Then

$$\bar{f}'([y_1]) - \bar{f}'([y_2]) = \bar{f}'([y_1 - y_2]) = [f'(y_1 - y_2)]$$
  
=  $[f'(a(z))] = [b(f(z))] = 0$ 

thus  $\bar{f}'[y_1] = \bar{f}'[y_2]$ . Here we have already used the homomorphism property of  $\bar{f}'$  that needs a quick proof:

$$\begin{split} \bar{f}'([x] + [y]) &= \bar{f}'([x + y]) = [f'(x) + f'(y)] \\ &= [f'(x)] + [f'(y)] = \bar{f}'([x]) + \bar{f}'([x]) \\ \bar{f}'(r[x]) &= \bar{f}'([rx]) = [f'(rx)] = r[f'(x)] = r\bar{f}'([x]) \end{split}$$

for  $r \in R$ . Of course, the lower left square commutes, in fact, the way we defined  $\bar{f}'$  was the only chance to have it commute.

Again we use an analogous construction to obtain the morphism

$$\bar{g'}$$
: coker  $b \to \operatorname{coker} c$ .

## The morphism $d: \ker c \to \operatorname{coker} a$

Now we define the morphism  $d: \ker c \to \operatorname{coker} a$ . Starting with an element  $x \in \ker c$ , we chase x through the diagram until we arrive in  $\operatorname{coker} a$ :

First we obviously want to map x to  $\iota(x) \in C$  and then use the surjectivity of  $g: B \to C$  to get an (not necessarily unique) element y in B, such that  $g(b) = \iota(x)$ . We then take  $b(y) \in B'$  and from

$$g'(b(y)) = c(g(y)) = c(\iota(x)) = 0,$$

we use the exactness of the lower rower to obtain a unique(!) element z in A', with f'(z) = b(y). With  $\pi(z)$  we finally arrive in coker a.

We now immediately want to define d as

$$d: \ker c \to \operatorname{coker} a, \ x \mapsto \pi(z),$$

however, we first need to show that this way, this gives a well-defined map.

For this take  $y_1$  and  $y_2$  in B, with

$$g(y_1) = g(y_2) = \iota(x).$$

We want to show that this gives the same element  $\pi(z)$  in coker a. Since  $y_2 - y_1 \in \ker g$ , we have a  $w \in \ker g$  such that  $y_2 = y_1 + w$ . Using the exactness of the upper row we can rewrite this as

$$y_2 = y_1 + f(u)$$

with  $u \in A$ .

We then have

$$z_1 = f'^{-1}(b(y_1))$$
 and  $z_2 = f'^{-1}(b(y_2))$ 

as the elements of A' the result from chasing  $y_1$  and  $y_2$  to A' and compute

$$z_2 = f'^{-1}(b(y_2)) = f'^{-1}(b(y_1 + f(u)))$$
  
=  $f'^{-1}(b(y_1)) + f'^{-1}(b(f(u)))$   
=  $z_1 + f'^{-1}(f'(a(u))) = z_1 + a(u)$ 

(mind the injectivity of f'), giving us

$$\pi(z_2) = \pi(z_1) + \pi((a(u)) = \pi(z_1)$$

as desired.

Again, d was the only choice to make the diagram commute, we now to need to ascertain that d is morphism of R-modules.

Taking  $x_1 + x_2 \in \ker c$ , we have  $\iota(x_1) + \iota(x_2) \in C$  and since d is well defined, we may take  $y_1 + y_2 \in B$ , where  $b(y_i) = \iota(x_i), i = 1, 2$ , as the pre-image. Then the pre-image of  $b(y_1) + b(y_2)$  clearly is  $z_1 + z_2$ , where  $f'(z_i) = b(y_i), i = 1, 2$  and finally obtain  $\pi(z_1) + \pi(z_2)$  for  $d(x_1 + x_2)$ , thus

$$d(x_1 + x_2) = d(x_1) + d(x_2).$$

For  $rx \in \ker c$ ,  $r \in R$ , the argumentation is completely analoguous:

We now want to use the snake lemma to show the existence of the long exact homology sequence:

Given a short exact sequence of chain complexes

$$0 \longrightarrow A_{\bullet} \longrightarrow B_{\bullet} \longrightarrow C_{\bullet} \longrightarrow 0$$

there is a long exact sequence

$$\dots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \dots$$