THE SNAKE LEMMA AND LONG EXACT HOMOLOGY SEQUENCE

Consider a commutative diagram of R-modules of the form

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

If the rows are exact, there is an exact sequence

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c$$

constituting the picture of a snake in the diagram

$$\begin{tikzcd}
 & \ker a \arrow[r, dashed] \arrow[d, "\iota"] & \ker b \arrow[r, dashed] \arrow[d] & \ker c \arrow[d] \arrow[dddll, dashed, rounded corners, "d"] & \\
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d, "\pi"] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} a \arrow[r, dashed] & \operatorname{coker} b \arrow[r, dashed] & \operatorname{coker} c &
\end{tikzcd}$$

where the ι are the natural inclusion morphisms of the kernel and the π are the natural projection morphisms.

Proof.

Morphisms between the kernel

First we define the morphisms between the kernel. For example we define the map
$$\bar{f} : \ker a \to \ker b$$
by
$$x \mapsto f(x).$$
This is well defined, since
$$b(f(x)) = f'(a(x)) = f'(0) = 0,$$

meaning that indeed $f(x) \in \ker b$. Of course, the upper left square then commutes and $\bar{f}$ is a homomorphism of R-modules.

Of course, $\bar{g} : \ker b \to \ker c$ is defined analogously.

Morphisms between the cokernel

We proceed to define the morphisms between the cokernel. Here we, for example, define the map

$$\bar{f}' : \operatorname{coker} a \to \operatorname{coker} b$$

by

$$[y] \mapsto [f'(y)]$$

for y in A'. Again we check that this gives us a well defined map:

Let y_1, y_2 be in A' with $[y_1] = [y_2]$. This means that $y_1 - y_2 \in \operatorname{im} a$, therefore we have z in A, such that $a(z) = y_1 - y_2$. Then

$$\begin{aligned} \bar{f}'([y_1]) - \bar{f}'([y_2]) &= \bar{f}'([y_1 - y_2]) = [f'(y_1 - y_2)] \\ &= [f'(a(z))] = [b(f(z))] = 0 \end{aligned}$$

thus $\bar{f}'[y_1] = \bar{f}'[y_2]$. Here we have already used the homomorphism property of $\bar{f}'$ that needs a quick proof:

$$\begin{aligned} \bar{f}'([x] + [y]) &= \bar{f}'([x + y]) = [f'(x) + f'(y)] \\ &= [f'(x)] + [f'(y)] = \bar{f}'([x]) + \bar{f}'([x]) \end{aligned}$$

$$\bar{f}'(r[x]) = \bar{f}'([rx]) = [f'(rx)] = r[f'(x)] = r\bar{f}'([x])$$

for $r \in R$. Of course, the lower left square commutes, in fact, the way we defined $\bar{f}'$ was the only chance to have it commute.

Again we use an analogous construction to obtain the morphism

$$\bar{g}' : \operatorname{coker} b \to \operatorname{coker} c.$$

The morphism $d : \ker c \to \operatorname{coker} a$

Now we define the morphism $d : \ker c \to \operatorname{coker} a$. Starting with an element $x \in \ker c$, we chase x through the diagram until we arrive in $\operatorname{coker} a$:

First we obviously want to map x to $\iota(x) \in C$ and then use the surjectivity of $g : B \to C$ to get an (not necessarily unique) element y in B, such that $g(b) = \iota(x)$. We then take $b(y) \in B'$ and from

$$g'(b(y)) = c(g(y)) = c(\iota(x)) = 0,$$

we use the exactness of the lower rower to obtain a unique(!) element z in A', with $f'(z) = b(y)$. With $\pi(z)$ we finally arrive in $\operatorname{coker} a$.

We now immediately want to define d as

$$d : \ker c \to \operatorname{coker} a, \;\; x \mapsto \pi(z),$$

however, we first need to show that this way, this gives a well-defined map.

For this take y_1 and y_2 in B, with

$$g(y_1) = g(y_2) = \iota(x).$$

We want to show that this gives the same element $\pi(z)$ in $\operatorname{coker} a$. Since $y_2 - y_1 \in \ker g$, we have a $w \in \ker g$ such that $y_2 = y_1 + w$. Using the exactness of the upper row we can rewrite this as

$$y_2 = y_1 + f(u)$$

with $u \in A$.

We then have

$$z_1 = f'^{-1}(b(y_1)) \text{ and } z_2 = f'^{-1}(b(y_2))$$

as the elements of A' the result from chasing y_1 and y_2 to A' and compute

$$\begin{aligned} z_2 &= f'^{-1}(b(y_2)) = f'^{-1}(b(y_1 + f(u)) \\ &= f'^{-1}(b(y_1)) + f'^{-1}(b(f(u))) \\ &= z_1 + f'^{-1}(f'(a(u))) = z_1 + a(u) \end{aligned}$$

(mind the injectivity of f'), giving us

$$\pi(z_2) = \pi(z_1) + \pi((a(u)) = \pi(z_1)$$

as desired.

Again, d was the only choice to make the diagram commute, we now to need to ascertain that d is morphism of R-modules.

Taking $x_1 + x_2 \in \ker c$, we have $\iota(x_1) + \iota(x_2) \in C$ and since d is well defined, we may take $y_1 + y_2 \in B$, where $b(y_i) = \iota(x_i)$,$i = 1, 2$, as the pre-image. Then the pre-image of $b(y_1)+b(y_2)$ clearly is z_1+z_2, where $f'(z_i) = b(y_i)$,$i = 1, 2$ and finally obtain $\pi(z_1) + \pi(z_2)$ for $d(x_1 + x_2)$, thus

$$d(x_1 + x_2) = d(x_1) + d(x_2).$$

For $rx \in \ker c$, $r \in R$, the argumentation is completely analoguous:

$\square$

We now want to use the snake lemma to show the existence of the long exact homology sequence:

Given a short exact sequence of chain complexes

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

there is a long exact sequence

$$\dots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \dots$$