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Given a category $\mathcal{C}$, we can form a category $\mathbf{Set}^{\mathcal{C}}$, consisting of all functors $\mathcal{C} \to \mathbf{Set}$ with natural transformations as morphisms. The composition of morphisms in $\mathbf{Set}^{\mathcal{C}}$ is then just the vertical composition of natural transformations.

One easily sees that this composition is asscociative and for all functors $F : \mathcal{C} \to \mathbf{Set}$ the natural transformation $F \to F$ with components id_C is an identity, thus we have indeed a category. We call $\mathbf{Set}^{\mathcal{C}}$ the *functor category* of functors from $\mathcal{C}$ to $\mathbf{Set}$.

For each object A in $\mathcal{C}$ we obtain a special functor $\mathcal{C} \to \mathbf{Set}$ called the *Hom-functor* of A. It sends each object X in $\mathcal{C}$ to the Hom-Set $\mathrm{Mor}_{\mathcal{C}}$.

$$\begin{aligned} h^A : \quad & \mathcal{C} \longrightarrow \mathbf{Set} \\ & X \longmapsto \mathrm{Mor}_{\mathcal{C}}(A,X) \\ & \left(X \xrightarrow{f} Y\right) \longmapsto \left(\begin{array}{c} \mathrm{Mor}_{\mathcal{C}}(A,X) \xrightarrow{h^A(f)} \mathrm{Mor}_{\mathcal{C}}(A,Y) \\ A \to X \longmapsto \begin{tikzcd} A \arrow[rr] \arrow[dr] & & Y \\ & X \arrow[ur, "f"] & \end{tikzcd} \end{array}\right) \end{aligned}$$

In order to secure that the Hom-sets are indeed sets, we demand that $\mathcal{C}$ is *locally small.*

Yoneda's lemma now states that there is a bijection between the set of natural transformations $h^A \to F$ and the set $F(A)$ for any functor $F : \mathcal{C} \to \mathbf{Set}$.

Proof.

Given a natural transformation $h^A \to F$ with components m_C we define the map

$$\begin{aligned} f : \left\{ {\text{natural transformations} \atop h^A \to F} \right\} & \longrightarrow F(A) \\ \mathrm{h}^A \to F \qquad & \longmapsto m_A(\mathrm{id}_A) \end{aligned}$$

Obviously it is well defined, we now check for surjectivity and injectivity.

For $c \in F(A)$ we just define a natural transformation by assigning $\mathrm{id}_A \mapsto c$ via the component m_A. This not only gives a natural transformation - it is actually completely determined by that assignment:

To see this consider any object X in $\mathcal{C}$. We need to give a map

$$h^A(X) = \mathrm{Mor}(A, X) \to F(X).$$

If $\mathrm{Mor}(A, X)$ is empty, there there is only one possible choice. Else let $g \in \mathrm{Mor}(A, X)$. Then we consider the diagram

$$\begin{tikzcd}
\mathrm{Mor}(A, A) \arrow[r, "g\circ -"] \arrow[d, "m_A"'] & \mathrm{Mor}(A, X) \arrow[d, "m_X"] \\
F(A) \arrow[r, "F(g)"] & F(X)
\end{tikzcd}$$

in which we don't know the vertical arrows, but we *do* know two things:

$$\mathrm{id}_A \mapsto m_A(\mathrm{id}_A) = c$$

and that the diagram has to constitute a naturality square.

Thus we get, by tracking id_A,

$$\boxed{m_X(g \circ \mathrm{id}_A) = F(g) \circ m_A(\mathrm{id}_A)}.$$

Thus $m_X(g)$ is completely determined by $m_A(\mathrm{id}_A)$. □

Lemma 0.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be fully faithful functor. Then, whenever $F(X) \xrightarrow{F(f)} F(X')$ is an isomorphism in $\mathcal{D}$, $X \xrightarrow{f} X'$ is an isomorphism in $\mathcal{C}$.*

Corollary 0.2. *Fully faithful functors are essentially injective.*

Proof.

Let $F(X) \xrightarrow{F(f)} F(X')$ be an isomorphism. Then there exists

$$\varphi \in \mathrm{Mor}_{\mathcal{D}}(F(X'), F(X))$$

such that $F(f) \circ \varphi = \mathrm{id}_{F(X')}$, $\varphi \circ F(f) = \mathrm{id}_{F(X)}$.

Since F is fully faithful there exists a morphism $g \in \mathrm{Mor}(X, X')$, such that $F(g) = \varphi$. (This is the injectivity of the map $\mathrm{Mor}(X, X') \to \mathrm{Mor}(F(X), F(X'))$.)

Then we have

$$F(g \circ f) = F(g) \circ F(f) = \mathrm{id}_{F(X)} = F(\mathrm{id}_X).$$

Injectivity then gives $g \circ f = \mathrm{id}_X$ amd similarily $f \circ g = \mathrm{id}_{X'}$. □

In summary one can say:

Fully faithful functors reflect isomorphisms.