Bayesian Synthetic Likelihood - Summary - Author: Felix Germaine

I. Concept and Theory

Goal:

The overarching goal of **Bayesian Synthetic Likelihood** (thereafter BSL) is to sample from posterior parameter distributions from **intractable models**:

$$p(\theta \mid y) \propto \overbrace{p(y \mid \theta)}^{\text{intractable}}\, p(\theta)$$

Approximation 1:

In order to circumvent the "curse-of-dimensionality" issue, BSL makes use of **summary statistics**. Therefore, BSL aims at sampling from:

$$p(\theta \mid \boldsymbol{s_y}) \propto \overbrace{p(\boldsymbol{s_y} \mid \theta)}^{\text{intractable}}\, p(\theta)$$

If the summary statistics are not **sufficient**, the above posterior is an approximation.

Approximation 2:

BSL approximates the likelihood of the summary statistic, $p(s_y \mid \theta)$ with a **normal pdf**:

$$p(s_y \mid \theta) \approx p_A(s_y \mid \theta) = \mathcal{N}\Big(s_y; \overbrace{\mu(\theta), \Sigma(\theta)}^{\text{Unknown}}\Big)$$

Approximation 3:

For a given θ^*, $\mu(\theta^*), \Sigma(\theta^*)$ can be estimated, yielding : $\hat{p}(s_y \mid \theta^*) = \mathcal{N}(s_y; \mu_{\boldsymbol{n}}(\theta^*), \Sigma_{\boldsymbol{n}}(\theta^*))$
BSL uses this estimation within a **Pseudo-Marginal Metropolis Hastings (PMMH) algorithm** [**1**] that targets:

$$p_{A,n}(\theta \mid s_y) \propto \underbrace{p_{A,n}(s_y \mid \theta)}\, p(\theta)$$

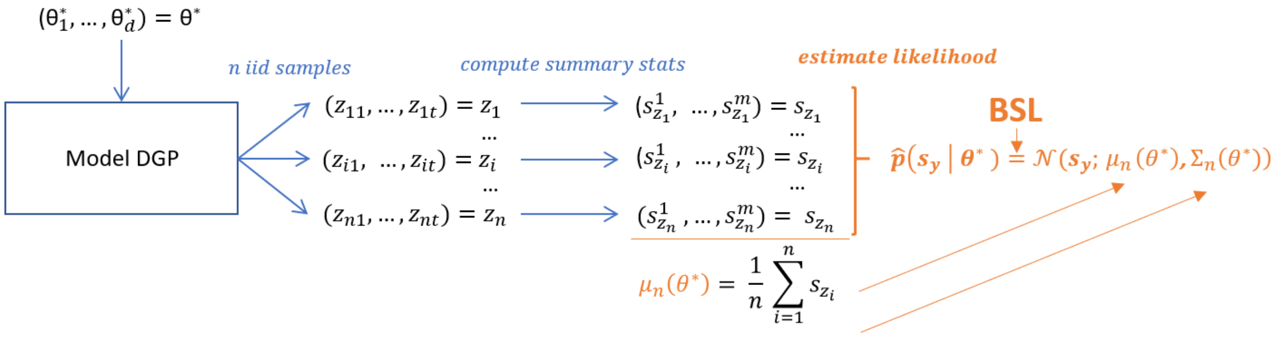
$$p_{A,n}(s_y \mid \theta) = E\Big[\hat{p}(s_y \mid \theta)\Big] = \int_{\mathrm{S}^n} \mathcal{N}(s_y; \mu_n(\theta), \Sigma_n(\theta)) \prod_{i=1}^{n} p(s_i \mid \theta)\, ds_{1:n} \neq p_A(s_y \mid \theta)$$

Note: the BSL target depends on the number of simulations per round within the PMMH algorithm, n. However, it can be shown that: $p_{A,n} \xrightarrow{n\to\infty} p_A$ [2]

II. Computational Method - BSL as a PMMH algorithm:

- **0. Initialize :** set θ^0, compute the prior $p(\theta^0)$ and *simulate and compute* $\hat{\mathbf{p}}(\mathbf{s_y} \mid \boldsymbol{\theta^0})$
- **1. For i in (1 to T) do:**

1.1. Draw $\theta^* \sim q(\theta^* \mid \theta^{i-1})$ → *the proposal density*
1.2. **Simulate** $\mathbf{s}_{\mathbf{z_i}}$ and **Estimate the Likelihood** $\mathbf{p}(\mathbf{s_y} \mid \boldsymbol{\theta^*})$:



$$\Sigma_n(\theta^*) = \frac{1}{n-1}\sum_{i=1}^{n}(s_{z_i} - \mu_n(\theta^*))(s_{z_i} - \mu_n(\theta^*))^T$$

1.3. Accept / Reject θ^* :

$$r = min\left(1, \frac{\hat{p}(s_y \mid \theta^*) \cdot p(\theta^*) \cdot q(\theta^{i-1} \mid \theta^*)}{\hat{p}(s_y \mid \theta^{i-1}) \cdot p(\theta_{i-1}) \cdot q(\theta^* \mid \theta^{i-1})}\right) \quad \boldsymbol{IF}\ r > U(0,1)\ \boldsymbol{THEN}\ \theta^i = \theta^*\ \boldsymbol{ELSE}\ \theta^i = \theta^{i-1}$$

III. ABC as a "state-of-the-art" benchmark:

Similary to BSL:

- ABC is a summary statistics based method that enables to sample from an approximate posterior distribution in models with an intractable likelihood.
- ABC-MCMC also uses the PMMH algorithm, estimating the likelihood with: $\hat{p}\left(s_y \mid \theta\right)=\frac{1}{n} \sum_{i=1}^n K_\epsilon\left(\rho\left(\boldsymbol{s}_y, \boldsymbol{s}_{z_i}\right)\right)$

IV. Strengths and Weaknesses:

Distributional assumptions of the summary statistics:

BSL:

- Requires summary statistics that are "close" to normally distributed.
- Empirical results suggest that BSL is rather robust to "smaller" deviations from normality [3].

ABC:

- No distributional assumption

Selection of Tuning Parameters:

BSL:

- Weak dependence of target to tuning parameter n (Suggested by empirical results in [3]).
- Could enable to select n based on computational efficiency [3] (e.g based on $ESS/n_{simulations}$).

ABC:

- Target distribution is sensitive to its bandwidth parameter ϵ [3])
- No standard way to select ϵ [3]

Computational Efficiency in Higher Dimensions:

- Possible efficiency edge of BSL over ABC-MCMC that could improve on the "curse of dimensionality" issue (in a limited way).
- Intuition: Parametric vs. Non-Parametric estimation of the auxiliary likelihood.
- First clues given by a toy example and empirical comparisons [3]

References

[1] C. Andrieu and G Roberts. "The Pseudo-Marginal Approach for Efficient Monte-Carlo Computations". In: *The Annals of Statistics* 37.2 (2009), pp. 697–725.

[2] C. C. Drovandi, A. N. Petitt, and A. Lee. "Bayesian Indirect Inference Using a Parametric Auxiliary Model". In: *Statistical Science* 37.2 (2015), pp. 72–95.

[3] C. C. Drovandi et al. "Bayesian Synthetic Likelihood". In: *Journal of Computational and Graphical Statistics* 27.1 (2018), pp. 1–11.