

1 Introduction

This work is mainly based on the paper “Complete $O(\alpha_S)$ corrections to heavy-flavour structure functions in electroproduction” by Laenen et. al.[1] - that is, it recalculates all properties and formulas. It extends then the application to the equivalent *polarized* processes. The treating of the polarized processes can for example be found in [2] and we will use many ideas and technics from there. **FiXme Error: more**

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1.1 Motivation

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1.2 Notation

To collect all soft and collinear poles we have to calculate in $n = 4 + \epsilon$ dimension. Unfortunaly the extension for *polarized* processes is nontrivial, because the occuring Levi-Civita tensors $\varepsilon_{\mu\nu\rho\sigma}$ and γ_5 . A common choice to deal with these objects is the HVBM prescription[3] that keeps those two objects four dimensional at the price for splitting the full n -dimensional space into a $(n - 4)$ -dimensional space, called “hat-space”, and a four-dimensional space (that is actually never used).

In leading order (LO) we have to consider the following processes

$$\gamma^*(q; \sigma_q) + g(k_1; \sigma_{k_1}) \rightarrow Q(p_1) + \bar{Q}(p_2) \quad (1)$$

The corresponding parton structure tensor $W_{\mu\mu'}^{(0)}$, can then be written as **FiXme Error: avoid all order expr?**

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$$\begin{aligned} & W_{\mu\mu'}^{(0)}(k_1, q; s, t_1, u_1, q^2; \sigma_{k_1} \sigma_q) \\ &= \frac{1}{2} E_k(\epsilon) K_{g\gamma} \int \frac{d^{n-1} p_1}{2E_1 (2\pi)^{n-1}} \int \frac{d^{n-1} p_2}{2E_2 (2\pi)^{n-1}} \delta(p_1^2 - m^2) \delta(p_2^2 - m^2) \\ & \quad (2\pi)^n \delta^{(n)}(k_1 + q - p_1 - p_2) \mathcal{M}_{\mu}^{(0)}(\sigma_{k_1}, \sigma_q) \mathcal{M}_{\mu'}^{(0)}(\sigma_{k_1}, \sigma_q) \end{aligned} \quad (2)$$

where the initial $1/2$ is the initial state spin average, $K_{g\gamma}$ is the color average,

$$E_\epsilon := \begin{cases} 1/(1 + \epsilon/2) & \text{unpolarized} \\ 1 & \text{polarized} \end{cases} \quad (3)$$

accounts for initial freedom in n dimensions for bosons and we defined the following Mandelstam variables:

$$s = (q + k_1)^2, \quad t_1 = t - m^2 = (k_1 - p_2)^2 - m^2, \quad u_1 = u - m^2 = (q - p_2)^2 - m^2 \quad (4)$$

$$s' = s - q^2, \quad u'_1 = u_1 - q^2 \quad (5)$$

FiXme Error: move to LO? The Lorentz indices μ and μ' refer to the virtual photon that is exchanged with the scattering lepton.

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By using Lorentz covariance, hermiticity, parity invariance and current conservation the parton structure tensor can be decomposed into several parts:

$$\begin{aligned} W_{\mu\mu'}(k_1, q; s, t_1, u_1, q^2; \sigma_{k_1}, \sigma_q) = & \left(-g_{\mu\mu'} + \frac{q_\mu q_{\mu'}}{q^2} \right) \frac{d^2 \sigma_T(s, t_1, u_1, q^2)}{dt_1 du_1} \\ & + \left(k_{1,\mu} - \frac{k_1 \cdot q}{q^2} q_\mu \right) \left(k_{1,\mu'} - \frac{k_1 \cdot q}{q^2} q_{\mu'} \right) \left(\frac{-4q^2}{s'^2} \right) \\ & \cdot \left(\frac{d^2 \sigma_T(s, t_1, u_1, q^2)}{dt_1 du_1} + \frac{d^2 \sigma_L(s, t_1, u_1, q^2)}{dt_1 du_1} \right) \end{aligned} \quad (6)$$

FiXme Error: extend We can then define appropriate projection operators[1, 4]:

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$$\mathcal{P}_{G,\mu\mu'} = -g_{\mu\mu'} \quad b_G(\epsilon) = \frac{1}{2(1 + \epsilon/2)} \quad (7)$$

$$\mathcal{P}_{L,\mu\mu'} = -\frac{4q^2}{s'^2} k_{1,\mu} k_{1,\mu'} \quad b_L(\epsilon) = 1 \quad (8)$$

$$\mathcal{P}_{P,\mu\mu'} = i\varepsilon_{\mu\mu'\rho\rho'} \frac{q^\rho k_1^{\rho'}}{s'} \quad b_P(\epsilon) = 1 \quad (9)$$

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$$\frac{d^2 \sigma_k(s, t_1, u_1, q^2)}{dt_1 tu_1} = b_k(\epsilon) \mathcal{P}_{k,\mu\mu'} W^{\mu\mu'} \quad (10)$$

with $k \in \{G, L, P\}$ denoting (here and mostly ever after) the projection type. The transverse partonic cross section $d\sigma_T$ can be reconstructed from the above definitions by using

$$d\sigma_T = d\sigma_G + b_G(\epsilon) d\sigma_L \quad (11)$$

We also define accordingly

$$E_G(\epsilon) = E_L(\epsilon) = \frac{1}{1 + \epsilon/2} \quad E_P(\epsilon) = 1 \quad (12)$$

The final state spins are always summed over, but the initial spins have to be treated seperately: for unpolarized projections $k \in \{G, L\}$ they are also summed over, but for polarized $k = P$ they are combined as follows

$$\sum_{G,\sigma} f(\sigma_{k_1}, \sigma_q) = \sum_{L,\sigma} f(\sigma_{k_1}, \sigma_q) = f(+, +) + f(-, -) + f(+, -) + f(-, +) \quad (13)$$

$$\sum_{P,\sigma} f(\sigma_{k_1}, \sigma_q) = f(+, +) + f(-, -) - f(+, -) - f(-, +) \quad (14)$$

which keeps spin asymmetries well behaving.

When computing total partonic cross sections we define a set of partonic variables:

$$0 \leq \rho = \frac{4m^2}{s} \leq 1 \quad 0 \leq \beta = \sqrt{1 - \rho} \leq 1 \quad 0 \leq \chi = \frac{1 - \beta}{1 + \beta} \leq 1 \quad (15)$$

$$\rho_q = \frac{4m^2}{q^2} \leq 0 \quad 1 \leq \beta_q = \sqrt{1 - \rho_q} \quad 0 \leq \chi_q = -\frac{1 - \beta_q}{1 + \beta_q} \leq 1 \quad (16)$$

When computing Feynman diagrams a computer algebra system (CAS) is almost obligatory: common choices are **FORM**[5] or **Mathematica**[6] - for the later the most common choice is **TRACER**[7], but we have chosen **HEPMath**[8]. We used the Feynman rules given by [9]. **FiXme Error: explain ghosts?**

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2 Leading Order Calculations

In leading order we have to consider photon-gluon-fusion (PGF), that is

$$\gamma^*(q; \sigma_q) + g(k_1; \sigma_{k_1}) \rightarrow Q(p_1) + \bar{Q}(p_2) \quad (17)$$

with two contributing diagrams depicted in figure **FiXme Error: todo**. The result can then be written as

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$$\hat{\sum}_{k, \sigma} \mathcal{P}_k^{\mu\mu'} \sum_{j=1}^2 \mathcal{M}_{j, \mu}^{(0)}(\sigma_{k_1}, \sigma_q) \mathcal{M}_{j, \mu'}^{(0)*}(\sigma_{k_1}, \sigma_q) = 8g^2 \mu_D^{-\epsilon} e^2 e_H^2 N_C C_F B_{k, QED} \quad (18)$$

where g and e are the strong and electromagnetic coupling constants respectively, μ_D is an arbitray mass parameter introduced to keep the couplings dimensionless and e_H is the magnitude of the heavy quark in units e . Further N_C corresponds to the gauge group $SU(N_C)$ and the color factor $C_F = (N_C^2 - 1)/(2N_C)$ refers to the second Casimir constant of the fundamental representation for the quarks. We then find:

$$B_{G, QED} = \frac{t_1}{u_1} + \frac{u_1}{t_1} + \frac{4m^2 s'}{t_1 u_1} \left(1 - \frac{m^2 s'}{t_1 u_1} \right) + \frac{2s' q^2}{t_1 u_1} + \frac{2q^4}{t_1 u_1} + \frac{2m^2 q^2}{t_1 u_1} \left(2 - \frac{s'^2}{t_1 u_1} \right) + \epsilon \left\{ -1 + \frac{s'^2}{t_1 u_1} + \frac{s' q^2}{t_1 u_1} - \frac{q^4}{t_1 u_1} - \frac{m^2 q^2 s'^2}{t_1^2 u_1^2} \right\} + \epsilon^2 \frac{s'^2}{4t_1 u_1} \quad (19)$$

$$B_{L, QED} = -\frac{4q^2}{s'} \left(\frac{s}{s'} - \frac{m^2 s'}{t_1 u_1} \right) \quad (20)$$

$$B_{P, QED} = \frac{1}{2} \left(\frac{t_1}{u_1} + \frac{u_1}{t_1} \right) \left(\frac{2m^2 s'}{t_1 u_1} - 1 - \frac{2q^2}{s'} \right) \quad (21)$$

$$B_{k, QED} = B_{k, QED}^{(0)} + \epsilon B_{k, QED}^{(1)} + \epsilon^2 B_{k, QED}^{(2)} \quad (22)$$

By using eq. (2) we can derive the n -dimensional $2 \rightarrow 2$ phase space

$$dPS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} \Theta(p_{1,0}) \delta(p_1^2 - m^2) \Theta(p_{2,0}) \delta(p_2^2 - m^2) (2\pi)^n \delta^{(n)}(k_1 + q - p_1 - p_2) \quad (23)$$

that can be solved by using the center-of-mass system (CMS) of the incoming particles[2]

$$q = \left(\frac{s + q^2}{2\sqrt{s}}, 0, 0, -\frac{s - q^2}{2\sqrt{s}}, \hat{0} \right) \quad k_1 = \frac{s - q^2}{2\sqrt{s}} (1, 0, 0, 1, \hat{0}) \quad (24)$$

such that $q + k_1 = (\sqrt{s}, \vec{0})$ and $k_1^2 = 0$. For the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2} (1, 0, \beta \sin \theta, \beta \cos \theta, \hat{0}) \quad p_2 = \frac{\sqrt{s}}{2} (1, 0, -\beta \sin \theta, -\beta \cos \theta, \hat{0}) \quad (25)$$

such that $p_1 + p_2 = (\sqrt{s}, \vec{0})$ and $p_1^2 = p_2^2 = m^2$. Finally we have to use the n -sphere

$$d^n x = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} dx = \frac{\pi^{n/2}}{\Gamma(n/2)} (x^2)^{(n-2)/2} dx^2 \quad (26)$$

and arrive at the well known result[1]

$$dPS_2 = \frac{\delta(s' + t_1 + u_1)}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} dt_1 du_1 \quad (27)$$

$$= \delta(s' + t_1 + u_1) h_2(n) dt_1 du_1 \quad (28)$$

$$h_2(4 + \epsilon) = \frac{2\pi S_\epsilon}{s'\Gamma(1 + \epsilon/2)} \left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{s'^2} \right)^{\epsilon/2} \quad (29)$$

with $S_\epsilon = (4\pi)^{(-2-\epsilon/2)}$.

The final double differential LO partonic cross section can then be written as

$$s'^2 \frac{d^2 \sigma_{k,g}^{(0)}(s', t_1, u_1, q^2)}{dt_1 du_1} = 2^6 \alpha \alpha_s e_H^2 K_{g\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \delta(s' + t_1 + u_1) \frac{\pi^3 S_\epsilon}{\Gamma(1 + \epsilon/2)} \left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{m^2 s'^2} \right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon/2} B_{k,QED} \quad (30)$$

where we used $e^2 = 4\pi\alpha$ and $g^2 = 4\pi\alpha_s$ and introduced the arbitrary mass parameter μ_D to keep the strong coupling dimensionless. The color average is given by $K_{g\gamma} = 1/(N_C^2 - 1)$.

From the results above we can easily obtain the total LO partonic cross sections

$$\sigma_G^{(0)}(s', q^2) = -4\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{1}{s'^3} \left((s^2 + q^4 + 4m^2 s)\beta + (s^2 + q^4 - 4m^2(2m^2 - s')) \ln(\chi) \right) \quad (31)$$

$$\sigma_L^{(0)}(s', q^2) = 16\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \left(\frac{-q^2 s}{s'^3} \right) \left(\beta + \frac{2m^2}{s} \ln(\chi) \right) \quad (32)$$

$$\sigma_P^{(0)}(s', q^2) = 4\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{1}{s'^2} \left((3s + q^2)\beta + (s + q^2) \ln(\chi) \right) \quad (33)$$

from which we also see

$$\lim_{s \rightarrow 4m^2} \sigma_T^{(0)}(s', q^2) = 4\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{\beta}{4m^2 - q^2} + O(\beta^3) = \lim_{s \rightarrow 4m^2} \sigma_P^{(0)}(s', q^2) \quad (34)$$

3 Next-To-Leading Order Calculations

Next-to-leading order contributions can be split into three parts: one loop virtual contributions, one gluon radiation and light quark processes. **FiXme Error: more?**

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3.1 One Loop Virtual Contributions

Virtual contributions have the same initial and final state as the Born process, but have a looping particle. All contributing Feynman diagrams are depicted in figure **FiXme Error: do.** The result can then be written as

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$$\begin{aligned} & \sum_{k,\sigma} \hat{\mathcal{P}}_k^{\mu\mu'} \sum_j \left[\mathcal{M}_{j,\mu}^{(1),V} \left(\mathcal{M}_{1,\mu'}^{(0)} + \mathcal{M}_{2,\mu'}^{(0)} \right)^* + c.c. \right] \\ & = 8g^4 \mu_D^{-\epsilon} e^2 e_H^2 N_C C_F C_\epsilon (C_A V_{k,OK} + 2C_F V_{k,QED}) \end{aligned} \quad (35)$$

where $C_\epsilon = \exp(\epsilon/2(\gamma_E - \ln(4\pi)))/(16\pi^2)$ and C_A is the second Casimir constant of the adjoint representation for the gluon (that introduces a non-abelian part).

For the computation of the loops the Passarino-Veltman-decomposition[10] in $n = 4 + \epsilon$ dimension is used as far as possible. The decomposition is based on Lorentz invariance and a good explanation is for example given in [2]. The needed scalar integrals are given in [11] and [1], but there is also one wrong integral: we find with [12, Box 16]:

$$\begin{aligned} & D_0(m^2, 0, q^2, m^2, t, s, 0, m^2, m^2, m^2) \\ & = \frac{iC_\epsilon}{\beta s t_1} \left[-\frac{2\ln(\chi)}{\epsilon} - 2\ln(\chi) \ln(-t_1/m^2) + \text{Li}_2(1 - \chi^2) - 4\zeta(2) + \ln^2(\chi_q) + 2\text{Li}_2(-\chi\chi_q) \right. \\ & \quad \left. + 2\text{Li}_2(-\chi/\chi_q) + 2\ln(\chi\chi_q) \ln(1 + \chi\chi_q) + 2\ln(\chi/\chi_q) \ln(1 + \chi/\chi_q) \right] \end{aligned} \quad (36)$$

where we used the argument ordering of `LoopTools`[13, 14] (and also checked it against `LoopTools`).

As the short example above shows are the full expressions for the $V_{k,OK}, V_{k,QED}$ quite complicated and too long to be presented here, nevertheless the arising poles are quite compact:

$$V_{k,OK} = -2B_{k,QED} \left(\frac{4}{\epsilon^2} + \left(\ln(-t_1/m^2) + \ln(-u_1/m^2) - \frac{2m^2 - s}{s} \ln(\chi) \right) \frac{2}{\epsilon} \right) + O(\epsilon^0) \quad (37)$$

$$V_{k,QED} = -2B_{k,QED} \left(1 - \frac{2m^2 - s}{s} \ln(\chi) \right) \frac{2}{\epsilon} + O(\epsilon^0) \quad (38)$$

The above results already include the mass renormalization that we have performed *on-shell*, so all ultra-violet poles have been removed. For the renormalization of the strong coupling we use the $\overline{\text{MS}}_m$ scheme defined in [2] and so the full renormalization can be achieved by

$$\begin{aligned} \frac{d^2 \sigma_k^{(1),V,ren.}}{dt_1 du_1} &= \frac{d^2 \sigma_k^{(1),V}}{dt_1 du_1} + \frac{\alpha_s(\mu_R^2)}{4\pi} \left[\left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(\mu_R^2/m^2) - \ln(\mu_D^2/m^2) \right) \beta_0 \right. \\ &\quad \left. + \frac{2}{3} \ln(\mu_R^2/m^2) \right] \frac{d^2 \sigma_k^{(0)}}{dt_1 du_1} \end{aligned} \quad (39)$$

$$\begin{aligned} &= \frac{d^2 \sigma_k^{(1),V}}{dt_1 du_1} + 4\pi \alpha_s(\mu_R^2) C_\epsilon \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon/2} \left[\left(\frac{2}{\epsilon} + \ln(\mu_R^2/m^2) \right) \beta_0 \right. \\ &\quad \left. + \frac{2}{3} \ln(\mu_R^2/m^2) \right] \frac{d^2 \sigma_k^{(0)}}{dt_1 du_1} \end{aligned} \quad (40)$$

with μ_R the renormalization scale introduced by the RGE, $\beta_0 = (11C_A - 2n_f)/3$ the first coefficient of the beta function and n_f the number of *total* flavours (i.e. $n_{lf} = n_f - 1$ active (light) flavours and one heavy flavour). The double poles occurring in $V_{k,OK}$ are introduced by the diagrams **FiXme Error: do** when the soft and collinear singularities coincide. FiXme Error!

The double differential partonic cross section is given by

$$\begin{aligned} s'^2 \frac{d^2 \sigma_{k,g}^{(1),V}(s', t_1, u_1, q^2)}{dt_1 du_1} &= 2^8 \alpha_s^2 e_H^2 K_{g\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \delta(s' + t_1 + u_1) \frac{\pi^4 S_\epsilon}{\Gamma(1 + \epsilon/2)} \\ &\quad \left(\frac{(t_1 u'_1 - s' m^2) s' - q^2 t_1^2}{m^2 s'^2} \right)^{\epsilon/2} C_\epsilon \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon/2} \\ &\quad (C_A V_{k,OK} + 2C_F V_{k,QED}) \end{aligned} \quad (41)$$

3.2 Single Gluon Radiation

In next-to-leading order we have to consider the following process:

$$\gamma^*(q; \sigma_q) + g(k_1; \sigma_{k_1}) \rightarrow Q(p_1) + \bar{Q}(p_2) + g(k_2) \quad (42)$$

All contributing diagrams are depicted in figure **FiXme Error: do** and the result can be written as

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$$\sum_{k,\sigma} \hat{\mathcal{P}}_k^{\mu\mu'} \sum_{j,j'} \mathcal{M}_{j,\mu}^{(1),g} \mathcal{M}_{j',\mu'}^{(1),g*} = 8g^4 \mu_D^{-2\epsilon} e^2 e_H^2 N_C C_F (C_A R_{k,OK} + 2C_F R_{k,QED}) \quad (43)$$

and it will depend on ten kinematical invariants:

$$s = (q + k_1)^2 \quad t_1 = (k_1 - p_2)^2 - m^2 \quad u_1 = (q - p_2)^2 - m^2 \quad (44)$$

$$s_3 = (k_2 + p_2)^2 - m^2 \quad s_4 = (k_2 + p_1)^2 - m^2 \quad s_5 = (p_1 + p_2)^2 = -u_5 \quad (45)$$

$$t' = (k_1 - k_2)^2 \quad (46)$$

$$u' = (q - k_2)^2 \quad u_6 = (k_1 - p_1)^2 - m^2 \quad u_7 = (q - p_1)^2 - m^2 \quad (47)$$

from which only five are independent as can be seen from momentum conservation $k_1 + q = p_1 + p_2 + k_2$ and s, t_1, u_1 match to their leading order definition.

The $2 \rightarrow 3$ n -dimensional phase space is given by

$$dPS_3 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} \frac{d^n k_2}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(k_1 + q - p_1 - p_2 - k_2) \Theta(p_{1,0}) \delta(p_1^2 - m^2) \Theta(p_{2,0}) \delta(p_2^2 - m^2) \Theta(k_{2,0}) \delta(k_2^2) \quad (48)$$

This can be solved by writing eq. (48) as product of a $2 \rightarrow 2$ decay and a subsequent $1 \rightarrow 2$ decay [11]. We find

$$dPS_3 = \frac{1}{(4\pi)^n \Gamma(n-3) s'} \frac{s_4^{n-3}}{(s_4 + m^2)^{n/2-1}} \left(\frac{(t_1 u'_1 - s' m^2) s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} dt_1 du_1 d\Omega_n d\hat{\mathcal{I}} \quad (49)$$

$$= h_3(n) dt_1 du_1 d\Omega_n d\hat{\mathcal{I}} \quad (50)$$

with $d\Omega_n = \sin^{n-3}(\theta_1) d\theta_1 \sin^{n-4}(\theta_2) d\theta_2$ and $d\hat{\mathcal{I}}$ taking care of all occuring hat momenta:

$$d\hat{\mathcal{I}} = \frac{1}{B(1/2, (n-4)/2)} \frac{x^{(n-6)/2}}{\sqrt{1-x}} dx \quad \text{with } x = \hat{p}_1^2 / \hat{p}_{1,max} \quad (51)$$

$$\hat{p}_{1,max} = \frac{s_4^2}{4(s_4 + m^2)} \sin^2(\theta_1) \sin^2(\theta_2) \quad (52)$$

$$\Rightarrow \int d\hat{\mathcal{I}} = 1 \quad \int d\hat{\mathcal{I}} \hat{p}_1^2 = \epsilon \hat{p}_{1,max} + O(\epsilon^2) \quad (53)$$

The needed phase space integrals for θ_1 and θ_2 can be found in [11] and [2]. We find for the difference to the $2 \rightarrow 2$ phase space

$$\frac{h_3(4+\epsilon)}{h_2(4+\epsilon)} = \frac{S_\epsilon}{2\pi} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \frac{s_4^{1+\epsilon}}{(s_4+m^2)^{1+\epsilon/2}} \quad (54)$$

$$= \frac{C_\epsilon}{2\pi} \left(1 - \frac{3}{8}\zeta(2)\epsilon^2\right) \frac{s_4^{1+\epsilon}}{(s_4+m^2)^{1+\epsilon/2}} + O(\epsilon^3) \quad (55)$$

where ζ is Riemanns zeta function. **FiXme Error: introduce psLogs? in appendix?**

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Again when integrating the phase space angles the expressions become quite lengthy, but the (collinear) pole parts are compact:

$$\frac{s_4}{4\pi(s_4+m^2)} \int d\Omega_n d\hat{\mathcal{I}} C_A R_{k,OK} = -\frac{1}{u_1} B_{k,QED} \left(\begin{matrix} s' \rightarrow x_1 s' \\ t_1 \rightarrow x_1 t_1 \end{matrix} \right) P_{k,gg}^H(x_1) \frac{2}{\epsilon} + O(\epsilon^0) \quad (56)$$

with $x_1 = -u_1/(s' + t_1)$ and the hard part of the Altarelli-Parisi splitting functions $P_{k,gg}^H$ [15, 16]:

$$P_{G,gg}^H(x) = P_{L,gg}^H(x) = C_A \left(\frac{2}{1-x} + \frac{2}{x} - 4 + 2x - 2x^2 \right) \quad (57)$$

$$P_{P,gg}^H(x) = C_A \left(\frac{2}{1-x} - 4x + 2 \right) \quad (58)$$

The $R_{k,QED}$ do not contain poles. **FiXme Error: shift to factorization?**

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The double differential partonic cross section is given by

$$\begin{aligned} s'^2 \frac{d^2 \sigma_{k,g}^{(1),R}(s', t_1, u_1, q^2)}{dt_1 du_1} &= 2^7 \alpha_s^2 e_H^2 K_{g\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \frac{\pi^3 S_\epsilon^2}{\Gamma(1+\epsilon)} \frac{s_4}{s_4+m^2} \\ &\quad \left(\frac{(t_1 u'_1 - s' m^2) s' - q^2 t_1^2}{m^2 s'^2} \right)^{\epsilon/2} \left(\frac{s_4^2}{m^2 (s_4+m^2)} \right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon} \\ &\quad \int d\Omega_n d\hat{\mathcal{I}} (C_A R_{k,OK} + 2C_F R_{k,QED}) \end{aligned} \quad (59)$$

From the above expression we can obtain the soft limit $k_2 \rightarrow 0$ and separate their calculations:

$$\lim_{k_2 \rightarrow 0} (C_A R_{k,OK} + 2C_F R_{k,QED}) = (C_A S_{k,OK} + 2C_F S_{k,QED}) + O(1/s_4, 1/s_3, 1/t') \quad (60)$$

$$S_{k,OK} = 2 \left(\frac{t_1}{t' s_3} + \frac{u_1}{t' s_4} - \frac{s-2m^2}{s_3 s_4} \right) B_{k,QED} \quad (61)$$

$$S_{k,QED} = 2 \left(\frac{s-2m^2}{s_3 s_4} - \frac{m^2}{s_3^2} - \frac{m^2}{s_4^2} \right) B_{k,QED} \quad (62)$$

Note that the einkonal factors multiplying the Born functions $B_{k,QED}$ neither depend on q^2 nor on the projection k . We can then split the phase space by introducing an infrared cut-off Δ and distinguish then between soft $s_4 \leq \Delta$ and hard $s_4 > \Delta$ contributions. Let $\mathcal{R}(s_4)$ be a function with a soft pole $s_4^{-1+\epsilon}\mathcal{S}(s_4)$ and a finite part $\mathcal{F}(s_4)$, we then find [2]:

$$\int_0^{s_{4,max}} \mathcal{R}(s_4) = \int_0^{s_{4,max}} \left(s_4^{-1+\epsilon} \mathcal{S}(s_4) + \mathcal{F}(s_4) \right) \quad (63)$$

$$\simeq \frac{\Delta^\epsilon}{\epsilon} \mathcal{S}(0) + \int_\Delta^{s_{4,max}} \mathcal{R}(s_4) \quad (64)$$

This expansion is valid for Δ being small, i.e. smaller than any leading order scale or m^2 ; a typical choice is $\Delta/m^2 \sim 10^{-6}$. We then find

$$\begin{aligned} & \frac{s_4^2}{4\pi(s_4 + m^2)} \left(1 - \frac{3}{8}\zeta(2)\epsilon^2 \right) \int d\Omega_n d\hat{\mathcal{L}} S_{k,QED} \\ &= B_{k,QED} \left[-\frac{2}{\epsilon} \left(1 + \frac{s - 2m^2}{s\beta} \ln(\chi) \right) + 1 - \frac{s - m^2}{s\beta} \left(\ln(\chi) (1 + \ln(\chi)) + \text{Li}_2(1 - \chi^2) \right) \right] \end{aligned} \quad (65)$$

$$\begin{aligned} & \frac{s_4^2}{4\pi(s_4 + m^2)} \left(1 - \frac{3}{8}\zeta(2)\epsilon^2 \right) \int d\Omega_n d\hat{\mathcal{L}} S_{k,OK} \\ &= B_{k,QED} \left[\frac{4}{\epsilon^2} + \frac{2}{\epsilon} \left(\ln(t_1/u_1) + \frac{s - 2m^2}{s\beta} \ln(\chi) \right) - \ln^2(\chi) - \frac{3}{2}\zeta(2) + \frac{1}{2} \ln^2(t_1/(u_1\chi)) \right. \\ & \quad \left. + \text{Li}_2(1 - t_1/(u_1\chi)) - \text{Li}_2(1 - u_1/(t_1\chi)) + \frac{s - 2m^2}{s\beta} \left(\text{Li}_2(1 - \chi^2) + \ln^2(\chi) \right) \right] \end{aligned} \quad (66)$$

(Note the mistyped sign of $\ln(\chi)^2$ in [1, eq. (3.25)]) The additional factors originate from the difference between the $2 \rightarrow 3$ phasespace of R_k and the $2 \rightarrow 2$ phasespace needed for S_k .

The double differential partonic cross section is given by

$$\begin{aligned} & s'^2 \frac{d^2 \sigma_{k,g}^{(1),S}(s', t_1, u_1, q^2)}{dt_1 du_1} \\ &= 2^8 \alpha_s^2 e_H^2 K_{g\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \delta(s' + t_1 + u_1) \frac{\pi^4 S_\epsilon}{\Gamma(1 + \epsilon/2)} \\ & \quad \left(\frac{(t_1 u'_1 - s' m^2) s' - q^2 t_1^2}{m^2 s'^2} \right)^{\epsilon/2} C_\epsilon \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon} \left(\frac{\Delta}{m^2} \right)^\epsilon \\ & \quad \frac{s_4^2}{4\pi(s_4 + m^2)} \left(1 - \frac{3}{8}\zeta(2)\epsilon^2 \right) \int d\Omega_n d\hat{\mathcal{L}} (C_A S_{k,OK} + 2C_F S_{k,QED}) \end{aligned} \quad (67)$$

3.3 Light Quark Processes

In next-to-leading order a new production mechanism enters, that is induced by a light quark, so we have to consider the process

$$\gamma^*(q; \sigma_q) + q(k_1; \sigma_{k_1}) \rightarrow Q(p_1) + \bar{Q}(p_2) + q(k_2) \quad (68)$$

All contributing diagrams are depicted in figure **FiXme Error: do** and the result can be written as

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$$\sum_{k,\sigma} \hat{\mathcal{P}}_k^{\mu\mu'} \sum_{j,j'} \mathcal{M}_{j,\mu}^{(1),q} \mathcal{M}_{j',\mu'}^{(1),q*} = 8g^4 \mu_D^{-2\epsilon} e^2 N_C C_F \left(e_H^2 A_{k,1} + e_L^2 A_{k,2} + e_L e_H A_{k,3} \right) \quad (69)$$

where e_L denotes the charge of the light quark q in units of e .

The needed $2 \rightarrow 3$ phase space has already been calculated in section 3.2, so we can immediately quote the (collinear) poles:

$$\frac{s_4}{2\pi(s_4 + m^2)} \int d\Omega_n d\hat{\mathcal{I}} C_F A_{k,1} = -\frac{1}{u_1} B_{k,QED} \left(\begin{matrix} s' \rightarrow x_1 s' \\ t_1 \rightarrow x_1 t_1 \end{matrix} \right) P_{k,gq}(x_1) \frac{2}{\epsilon} + O(\epsilon^0) \quad (70)$$

with $x_1 = -u_1/(s' + t_1)$ and the Altarelli-Parisi splitting functions $P_{k,gq}$ [15, 16]:

$$P_{G,gq}(x) = P_{L,gq}(x) = C_F \left(\frac{1}{x} + \frac{(1-x)^2}{x} \right) \quad (71)$$

$$P_{P,gq}(x) = C_F (2 - x) \quad (72)$$

$A_{k,2}$ does not contain poles and we find $\int d\Omega_n d\hat{\mathcal{I}} A_{k,3} = 0$. Note that in the limit $q^2 \rightarrow 0$ $A_{k,2}$ will also get collinear poles.

The double differential partonic cross section is given by

$$\begin{aligned} s'^2 \frac{d^2 \sigma_{k,q}^{(1)}(s', t_1, u_1, q^2)}{dt_1 du_1} &= 2^7 \alpha \alpha_s^2 K_{q\gamma} N_C C_F b_k(\epsilon) \frac{\pi^3 S_\epsilon^2}{\Gamma(1+\epsilon)} \frac{s_4}{s_4 + m^2} \\ &\quad \left(\frac{(t_1 u'_1 - s' m^2) s' - q^2 t_1^2}{m^2 s'^2} \right)^{\epsilon/2} \left(\frac{s_4^2}{m^2 (s_4 + m^2)} \right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon} \\ &\quad \int d\Omega_n d\hat{\mathcal{I}} \left(e_H^2 A_{k,1} + e_L^2 A_{k,2} + e_H e_L A_{k,3} \right) \end{aligned} \quad (73)$$

with the color average $K_{q\gamma} = 1/N_C$.

4 Mass Factorization

All collinear poles can be removed by mass factorization in the following way:

$$s'^2 \frac{d^2 \sigma_{k,g}^{(1),fin}(s', t_1, u_1, q^2, \mu_F)}{dt_1 du_1} = \lim_{\epsilon \rightarrow 0} \left[s'^2 \frac{d^2 \sigma_{k,g}^{(1)}(s', t_1, u_1, q^2, \epsilon)}{dt_1 du_1} - \int_0^1 \frac{dx_1}{x_1} \Gamma_{k,gg}^{(1)}(x_1, \mu_F^2, \mu_D, \epsilon) \right. \quad (74)$$

$$\left. (x_1 s')^2 \frac{d^2 \sigma_{k,g}^{(0)}(x_1 s', x_1 t_1, u_1, q^2, \epsilon)}{d(x_1 t_1) du_1} \right] \quad (75)$$

$$\Gamma_{k,ij}^{(1)}(x, \mu_F^2, \mu_D, \epsilon) = \frac{\alpha_s}{2\pi} \left(P_{k,ij}(x) \frac{2}{\epsilon} + f_{k,ij}(x, \mu_F^2, \mu_D^2) \right) \quad (76)$$

where $\Gamma_{k,ij}^{(1)}$ is the first order correction to the transition functions $\Gamma_{k,ij}$ for *incoming* particle j and *outgoing* particle i in projection k . In the $\overline{\text{MS}}$ -scheme the $f_{k,ij}$ take their usual form and we find

$$\Gamma_{k,ij}^{(1),\overline{\text{MS}}}(x, \mu_F^2, \mu_D, \epsilon) = \frac{\alpha_s}{2\pi} P_{k,ij}(x) \left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(\mu_F^2/m^2) - \ln(\mu_D^2/m^2) \right) \quad (77)$$

$$= 8\pi\alpha_s P_{k,ij}(x) C_\epsilon \left(\frac{\mu_D^2}{m^2} \right)^{-\epsilon/2} \left(\frac{2}{\epsilon} + \ln(\mu_F^2/m^2) \right) \quad (78)$$

The $P_{k,ij}(x)$ are the Altarelli-Parisi splitting functions for which we find [15, 16]

$$P_{k,gg}(x) = \Theta(1 - \delta - x) P_{k,gg}^H(x) + \delta(1 - x) \left(2C_A \ln(\delta) + \frac{\beta_0}{2} \right) \quad (79)$$

where we introduced another infrared cut-off δ to separate soft ($x \geq 1 - \delta$) and hard ($x < 1 - \delta$) gluons that is connected to Δ via $\delta = \Delta/(s' + t_1)$. The structure here explains why we were able to write the equation (56).

The light quark process can be regularized in a complete analogous way:

$$s'^2 \frac{d^2 \sigma_{k,q}^{(1),fin}(s', t_1, u_1, q^2, \mu_F)}{dt_1 du_1} = \lim_{\epsilon \rightarrow 0} \left[s'^2 \frac{d^2 \sigma_{k,q}^{(1)}(s', t_1, u_1, q^2, \epsilon)}{dt_1 du_1} - \int_0^1 \frac{dx_1}{x_1} \Gamma_{k,gq}^{(1)}(x_1, \mu_F^2, \mu_D, \epsilon) \right. \quad (80)$$

$$\left. (x_1 s')^2 \frac{d^2 \sigma_{k,g}^{(0)}(x_1 s', x_1 t_1, u_1, q^2, \epsilon)}{d(x_1 t_1) du_1} \right]$$

The needed splitting functions $P_{k,gq}$ have been already quoted in equations (71) and (72). Note that $K_{q\gamma} = 1/(N_C) = 2C_F K_{g\gamma}$.

The final finite cross sections are then

$$\begin{aligned}
s'^2 \frac{d^2 \sigma_{k,g}^{(1),H,fin}}{dt_1 du_1} &= \frac{1}{2\pi} K_{g\gamma} \alpha \alpha_S e_H^2 N_C C_F b_k(0) \left[-\frac{1}{u_1} P_{k,gg}^H(x_1) \right. \\
&\quad \left\{ 4\pi B_{k,QED}^{(0)} \left(\begin{matrix} s' \rightarrow x_1 s' \\ t_1 \rightarrow x_1 t_1 \end{matrix} \right) \left(\ln \left(\frac{s_4^2}{m^2(s_4 + m^2)} \right) - \ln(\mu_F^2/m^2) \right) \right. \\
&\quad \left. \left. - 8\pi B_{k,QED}^{(1)} \left(\begin{matrix} s' \rightarrow x_1 s' \\ t_1 \rightarrow x_1 t_1 \end{matrix} \right) \right\} \right. \\
&\quad + C_A \frac{s_4}{s_4 + m^2} \left(\int d\Omega_n d\hat{\mathcal{I}} R_{k,OK} \right)^{finite} \\
&\quad \left. + 2C_F \frac{s_4}{s_4 + m^2} \int d\Omega_4 d\hat{\mathcal{I}} R_{k,QED} \right] \quad (81)
\end{aligned}$$

$$\begin{aligned}
s'^2 \frac{d^2 \sigma_{k,g}^{(1),S+V,fin}}{dt_1 du_1} &= 4K_{g\gamma} \alpha \alpha_S e_H^2 N_C C_F b_k(0) B_{k,QED}^{(0)} \delta(s' + t_1 + u_1) \left[C_A \ln^2(\Delta/m^2) \right. \\
&\quad + \ln(\Delta/m^2) \left(\left(\ln(-t_1/m^2) - \ln(-u_1/m^2) - \ln(\mu_F^2/m^2) \right) C_A \right. \\
&\quad \left. \left. - \frac{2m^2 - s}{s\beta} \ln(\chi)(C_A - 2C_F) - 2C_F \right) \right. \\
&\quad \left. + \frac{\beta_0}{4} \left(\ln(\mu_R^2/m^2) - \ln(\mu_F^2/m^2) \right) + f_k(s', u_1, t_1, q^2) \right] \quad (82)
\end{aligned}$$

where f_k contains lots of logarithms and dilogarithms, but does not depend on Δ, μ_F^2, μ_R^2 nor n_f .

$$\begin{aligned}
s'^2 \frac{d^2 \sigma_{k,q}^{(1),fin}}{dt_1 du_1} &= \frac{1}{2\pi} K_{q\gamma} \alpha \alpha_S N_C b_k(0) \left[-\frac{1}{u_1} e_H^2 P_{k,gq}(x_1) \right. \\
&\quad \left\{ 2\pi B_{k,QED}^{(0)} \left(\begin{matrix} s' \rightarrow x_1 s' \\ t_1 \rightarrow x_1 t_1 \end{matrix} \right) \left(\ln \left(\frac{s_4^2}{m^2(s_4 + m^2)} \right) - \ln(\mu_F^2/m^2) + 1 - \delta_{k,P} \right) \right. \\
&\quad \left. \left. - 4\pi B_{k,QED}^{(1)} \left(\begin{matrix} s' \rightarrow x_1 s' \\ t_1 \rightarrow x_1 t_1 \end{matrix} \right) \right\} \right. \\
&\quad + C_F \frac{s_4}{s_4 + m^2} \left(\int d\Omega_n d\hat{\mathcal{I}} e_H^2 A_{k,1} \right)^{finite} \\
&\quad \left. + C_F \frac{s_4}{s_4 + m^2} \int d\Omega_4 d\hat{\mathcal{I}} e_L^2 A_{k,2} + C_F \frac{s_4}{s_4 + m^2} \int d\Omega_4 d\hat{\mathcal{I}} e_H e_L A_{k,3} \right] \quad (83)
\end{aligned}$$

where $1 - \delta_{k,P}$ may also be written as $-2\partial_\epsilon E_k(\epsilon = 0)$ as it originates from the additional factor of $E_k(\epsilon)$ in the subtraction part of equation (80).

5 Partonic Results

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6 Hadronic Results

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7 Summary

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