

1 2 to 2 phase space

process:

$$\gamma^*(q) + g(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) \quad (1)$$

kinematics:

$$s = (q + k_1)^2 \quad s' = s - q^2 \quad (2)$$

$$t = (k_1 - p_1)^2 \quad t_1 = t - m^2 \quad (3)$$

$$u = (k_1 - p_2)^2 \quad u_1 = u - m^2 \quad (4)$$

use c.m.s. of incoming particles:

$$q = \left(\frac{s + q^2}{2\sqrt{s}}, 0, 0, \dots, -\frac{s - q^2}{2\sqrt{s}} \right) \quad (5)$$

$$k_1 = \frac{s - q^2}{2\sqrt{s}} (1, 0, 0, \dots, 1) \quad (6)$$

such that

$$q + k_1 = (\sqrt{s}, \vec{0}) \quad k_1^2 = 0 \quad (7)$$

for the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2} (1, 0, \beta \sin \theta, \dots, \beta \cos \theta) \quad (8)$$

$$p_2 = \frac{\sqrt{s}}{2} (1, 0, -\beta \sin \theta, \dots, -\beta \cos \theta) \quad (9)$$

with $\beta = \sqrt{1 - 4m^2/s}$ such that

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \quad p_1^2 = p_2^2 = m^2 \quad (10)$$

use n-sphere:

$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \quad (11)$$

compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q + k_1 - p_1 - p_2) \delta(p_1^2 - m^2) \delta(p_2^2 - m^2) \quad (12)$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1 \delta((q + k_1 - p_2)^2 - m^2) \delta(p_1^2 - m^2) \quad (13)$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,\parallel} d^2 p_{1,\perp} d^{n-4} \hat{p}_1 \delta(s - 2p_{1,0}\sqrt{s}) \delta(p_{1,0}^2 - p_{1,\parallel}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \quad (14)$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,\parallel} d^2 p_{1,\perp} d^{n-4} \hat{p}_1 \delta(s/4 - p_{1,\parallel}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \quad (15)$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,\parallel} d\hat{p}_1^2 \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \quad (16)$$

$$= \frac{1}{2\sqrt{s} \Gamma((n-4)/2) (4\pi)^{(n-2)/2}} \int dp_{1,\parallel} d\hat{p}_1^2 (\hat{p}_1^2)^{(n-6)/2} \quad (17)$$

Integration borders are

$$p_{1,\parallel} \in \frac{\sqrt{s}}{2}\beta \cdot [-1, 1] \quad \hat{p}_1^2 \in \left(\frac{s\beta^2}{4} - p_{1,\parallel}^2 \right) \cdot [0, 1] \quad (18)$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2 (\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4} \left(\frac{s\beta^2}{4} - p_{1,\parallel}^2 \right)^{(n-4)/2} \quad (19)$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dp_{1,\parallel} \left(\frac{s\beta^2}{4} - p_{1,\parallel}^2 \right)^{(n-4)/2} \quad (20)$$

rewrite $p_{1,\parallel}$ to $\cos \theta$:

$$p_{1,\parallel} = \frac{\sqrt{s}}{2}\beta \cos \theta \Rightarrow dp_{1,\parallel} = \frac{\sqrt{s}}{2}\beta d\cos \theta, \quad \cos \theta \in [-1, 1], \quad \hat{p}_1^2 \in \frac{s\beta^2}{4} (1 - \cos^2 \theta) \cdot [0, 1] \quad (21)$$

rewrite $\cos \theta$ to $t_1 = (k_1 - p_2)^2 - m^2$:

$$\cos \theta = \frac{2t_1/s' + 1}{\beta} \Rightarrow d\cos \theta = \frac{2}{\beta s'} dt_1, \quad t_1 \in \frac{s'}{2}[-\beta - 1, \beta - 1], \quad \hat{p}_1^2 \in (-m^2 - \frac{st_1}{s'^2}(s' + t_1)) \cdot [0, 1] \quad (22)$$

$$p_{1,\parallel} = \sqrt{s} \left(\frac{t_1}{s'} + \frac{1}{2} \right) \Rightarrow dp_{1,\parallel} = \frac{\sqrt{s}}{s'} dt_1 \quad (23)$$

$$\Rightarrow PS_2 = \frac{1}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dt_1 \left(\frac{(t_1(u_1 - q^2) - s'm^2)s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} \quad (24)$$

2 2 to 3 phase space

process:

$$\gamma^*(q) + q(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) + q(k_2) \quad (25)$$

2.1 kinematic constraints

definitions of kinematic variables:

$$s = (q + k_1)^2 \Rightarrow 2qk_1 = s - q^2 \quad (26)$$

$$s_3 = (k_2 + p_2)^2 - m^2 \Rightarrow 2k_2 p_2 = s_3 \quad (27)$$

$$s_4 = (k_2 + p_1)^2 - m^2 \Rightarrow 2k_2 p_1 = s_4 \quad (28)$$

$$s_5 = (p_1 + p_2)^2 = -u_5 \Rightarrow 2p_1 p_2 = s_5 - 2m^2 \quad (29)$$

$$t_1 = (k_1 - p_2)^2 - m^2 = t - m^2 \Rightarrow 2k_1 p_2 = -t_1 \quad (30)$$

$$t' = (k_1 - k_2)^2 \Rightarrow 2k_1 k_2 = -t' \quad (31)$$

$$u_1 = (q - p_2)^2 - m^2 = u - m^2 \Rightarrow 2qp_2 = -u_1 + q^2 \quad (32)$$

$$u_6 = (k_1 - p_1)^2 - m^2 \Rightarrow 2k_1 p_1 = -u_6 \quad (33)$$

$$u_7 = (q - p_1)^2 - m^2 \Rightarrow 2qp_1 = -u_7 + q^2 \quad (34)$$

$$u' = (q - k_2)^2 \Rightarrow 2qk_2 = -u' + q^2 \quad (35)$$

impose momentum conservation:

$$q + k_1 = p_1 + p_2 + k_2 \quad (36)$$

contract with 2 times momentum:

$$2q^2 \quad +s - q^2 = -u_7 + q^2 \quad -u_1 + q^2 \quad -u' + q^2 \quad \Leftrightarrow \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad (37)$$

$$s - q^2 \quad +0 = -u_6 \quad -t_1 \quad -t' \quad \Leftrightarrow \quad 0 = s + t_1 + t' + u_6 - q^2 \quad (38)$$

$$-u_7 + q^2 \quad -u_6 = 2m^2 \quad +s_5 - 2m^2 \quad +s_4 \quad \Leftrightarrow \quad 0 = s_4 + s_5 + u_6 + u_7 - q^2 \quad (39)$$

$$-u_1 + q^2 \quad -t_1 = s_5 - 2m^2 \quad +2m^2 \quad +s_3 \quad \Leftrightarrow \quad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad (40)$$

$$-u' + q^2 \quad -t' = s_4 \quad +s_3 \quad +0 \quad \Leftrightarrow \quad 0 = s_3 + s_4 + t' + u' - q^2 \quad (41)$$

$$\frac{1}{2} ((37) + (38) + (40) - (39) - (41)) = 0 = s - q^2 + t_1 + u_1 - s_4 \quad (42)$$

2.2 choose framework

use c.m.s. of recoiling heavy and light quark ($Q(p_1)$ and $q(k_2)$):

$$k_2 = (\omega_2, k_{2,x}, \omega_2 \sin \theta_1 \cos \theta_2, \omega_2 \cos \theta_1, \hat{k}_2) \quad (43)$$

$$p_1 = (E_1, -k_{2,x}, -\omega_2 \sin \theta_1 \cos \theta_2, -\omega_2 \cos \theta_1, -\hat{k}_2) \quad (44)$$

$$k_1 = (\omega_1, 0, 0, \omega_1, \hat{0}) \quad (45)$$

$$q = (q_0, 0, |\vec{p}_2| \sin \psi, |\vec{p}_2| \cos \psi - \omega_1, \hat{0}) \quad (46)$$

$$p_2 = (E_2, 0, |\vec{p}_2| \sin \psi, |\vec{p}_2| \cos \psi, \hat{0}) \quad (47)$$

light quark masses are already fixed: $k_1^2 = 0 = k_2^2$

constraints:

$$q_0 + \omega_1 = E_1 + E_2 + \omega_2 \quad (48)$$

$$m^2 = p_1^2 = E_1^2 - \omega_1^2 \quad (49)$$

$$m^2 = p_2^2 = E_2^2 - |\vec{p}_2|^2 \quad (50)$$

$$q^2 = q_0^2 - |\vec{p}_2|^2 + 2|\vec{p}_2|\omega_1 \cos \psi - \omega_1^2 \quad (51)$$

$$s = (q + k_1)^2 = (q_0 - \omega_1)^2 - |\vec{p}_2|^2 \quad (52)$$

$$t = (k_1 - p_2)^2 = (\omega_1 - E_2)^2 - |\vec{p}_2|^2 + 2|\vec{p}_2|\omega_1 \cos \psi - \omega_1^2 \quad (53)$$

$$u = (q - p_2)^2 = (q_0 - E_2)^2 - \omega_1^2 \quad (54)$$

solve:

$$(52) - (51) + (53) - (50) + (54) = s - q^2 + t - m^2 + u \quad (55)$$

$$= s_4 + m^2 = (E_1 + \omega_2)^2 \quad (56)$$

$$(53) + (54) - (51) = t + u - q^2 = -2(E_1 + \omega_2)E_2 \quad (57)$$

$$\Rightarrow E_2 = -\frac{t + u - q^2}{2\sqrt{s_4 + m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4 + m^2}} \quad (58)$$

$$(56) \wedge (49) \Rightarrow \omega_2 = \frac{s_4}{2\sqrt{s_4 + m^2}} \quad (59)$$

$$(56) \Rightarrow E_1 = \frac{s_4 + 2m^2}{2\sqrt{s_4 + m^2}} \quad (60)$$

$$(52) + (54) - (50) = s + u - m^2 = 2q_0(E_1 + \omega_2) \quad (61)$$

$$\Rightarrow q_0 = \frac{s + u - m^2}{2\sqrt{s_4 + m^2}} \quad (62)$$

$$(53) - (51) = t - q^2 = (\omega_1 - E_2)^2 - q_0^2 \quad (63)$$

$$\Rightarrow \omega_1 = \frac{s_4 + m^2 - u}{2\sqrt{s_4 + m^2}} \quad (64)$$

$$(50) \Rightarrow |\vec{p}_2| = \sqrt{E_2^2 - m^2} = \frac{\sqrt{(s - s_4)^2 - 4sm^2}}{2\sqrt{s_4 + m^2}} \quad (65)$$

$$(51) \Rightarrow \cos \psi = \frac{q^2 - q_0^2 + |\vec{p}_2|^2 + \omega_1^2}{2|\vec{p}_2|\omega_1} \quad (66)$$

$$= \frac{2u(q^2 - s - m^2 + t) - (2m^2 - q^2 - t)(s_4 + m^2 - u)}{(s_4 + m^2 - u)\sqrt{(s - s_4)^2 - 4sm^2}} \quad (67)$$

$$t' = -2k_1k_2 = -2\omega_1\omega_2(1 - \cos \theta_1) \quad (68)$$

$$u_6 = -2k_1p_1 = -2\omega_1(E_1 + \omega_2 \cos \theta_1) \quad (69)$$

$$(38) : \quad 0 = s + t_1 + t' + u_6 - q^2 \quad \checkmark \quad (70)$$

$$s_3 = 2k_2p_2 = 2\omega_2(E_2 - |\vec{p}_2|(\sin \psi \sin \theta_1 \cos \theta_2 + \cos \psi \cos \theta_1)) \quad (71)$$

$$s_5 = (p_1 + p_2)^2 = 2m^2 + 2p_1p_2 \quad (72)$$

$$= 2(m^2 + E_1E_2 + \omega_2|\vec{p}_2|(\sin \psi \sin \theta_1 \cos \theta_2 + \cos \psi \cos \theta_1)) \quad (73)$$

$$(40) : \quad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad (74)$$

$$u' = (q - k_2)^2 = q^2 - 2qk_2 \quad (75)$$

$$= q^2 - 2(q_0\omega_2 - \omega_2|\vec{p}_2|(\sin \psi \sin \theta_1 \cos \theta_2 + \cos \psi \cos \theta_1) - \omega_1\omega_2 \cos \theta_1) \quad (76)$$

$$(37) : \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad (77)$$