# 1 Introduction

This work is mainly based on the paper "Complete $O(\alpha_S)$ corrections to heavy-flavour structure functions in electroproduction" by Laenen et. al.[1] - that is, it recalculates all properties and formulas. It extends then the application to the equivalent *polarized* processes. The treating of the polarized processes can for example be found in [2] and we will use many ideas and technices from there. **FiXme Error: more** FiXme Error!

## 1.1 Motivation

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**FiXme Error: why do we do this**

## 1.2 Notation

We calculate in $n = 4 + \epsilon$ dimension to regularize all soft, collinear and ultraviolet poles. Unfortunaly this extension for *polarized* processes is nontrivial, because the occuring Levi-Civita tensors $\varepsilon_{\mu\nu\rho\sigma}$ and $\gamma_5$. A common choice to deal with these objects is the HVBM prescription[3] that keeps those two objects four dimensional at the price of splitting the full $n$-dimensional space into a $(n-4)$-dimensional space, called "hat-space", and a four-dimensional space (that is actually never used).

In leading order (LO) we have to consider the following processes

$$\gamma^*(q) + \mathrm{g}(k_1) \to Q(p_1) + \overline{Q}(p_2) \tag{1}$$

The corresponding parton structure tensor $W^{(0)}_{\mu\mu'}$ can then be written as **FiXme Error: avoid all order expr?** FiXme Error!

$$\begin{aligned} &W^{(0)}_{\mu\mu'}(k_1, q; s, t_1, u_1, q^2; \sigma_q) \\ &= \frac{1}{2} E_\epsilon K_{\mathrm{g}\gamma} \frac{1}{2s'} \int \frac{d^{n-1}p_1}{2E_1(2\pi)^{n-1}} \int \frac{d^{n-1}p_2}{2E_2(2\pi)^{n-1}} \delta(p_1^2 - m^2)\delta(p_2^2 - m^2) \\ &\quad (2\pi)^n \delta^{(n)}(k_1 + q - p_1 - p_2)\, \mathcal{M}^{(0)}_\mu \, \mathcal{M}^{(0)}_{\mu'} \end{aligned} \tag{2}$$

where the initial 1/2 is the initial state spin average, $K_{\mathrm{g}\gamma}$ is the color average,

$$E_\epsilon := \begin{cases} 1/(1+\epsilon/2) & \text{unpolarized} \\ 1 & \text{polarized} \end{cases} \tag{3}$$

accounts for additional degrees of freedom in $n$ dimensions for initial bosons. The Lorentz indices $\mu$ and $\mu'$ refer to the virtual photon that is exchanged with the scattering lepton. We have chosen to detect the heavy *antiquark* $\overline{Q}(p_2)$ and so we define the following Mandelstam variables:

$$s = (q + k_1)^2, \quad t_1 = t - m^2 = (k_1 - p_2)^2 - m^2, \quad u_1 = u - m^2 = (q - p_2)^2 - m^2 \tag{4}$$

For convenience we also define $s' = s - q^2$ and $u_1' = u_1 - q^2$. If the heavy *quark* $\overline{Q}(p_1)$ is detected, $p_2$ in eq. (4) has to be replaced by $p_1$ which effectively interchanges $t_1 \leftrightarrow u_1$.**FiXme Error: move to LO?** FiXme Error!

By using Lorentz covariance, hermiticity, parity invariance and current conservation the parton structure tensor can be decomposed into several parts:

$$\begin{aligned}W_{\mu\mu'}(k_1,q;s,t_1,u_1,q^2;\sigma_q) &= \left(-g_{\mu\mu'} + \frac{q_\mu q_{\mu'}}{q^2}\right)\frac{d^2\sigma_T(s,t_1,u_1,q^2)}{dt_1du_1}\\ &+ \left(k_{1,\mu} - \frac{k_1\cdot q}{q^2}q_\mu\right)\left(k_{1,\mu'} - \frac{k_1\cdot q}{q^2}q_{\mu'}\right)\left(\frac{-4q^2}{s'^2}\right)\\ &\cdot\left(\frac{d^2\sigma_T(s,t_1,u_1,q^2)}{dt_1du_1} + \frac{d^2\sigma_L(s,t_1,u_1,q^2)}{dt_1du_1}\right)\end{aligned} \tag{5}$$

**FiXme Error: extend** We can then define appropiate projection operators[1, 4] FiXme Error!

$$\hat{\mathcal{P}}^\gamma_{G,\mu\mu'} = -g_{\mu\mu'} \qquad b_G(\epsilon) = \frac{1}{2(1+\epsilon/2)} \tag{6}$$

$$\hat{\mathcal{P}}^\gamma_{L,\mu\mu'} = -\frac{4q^2}{s'^2}k_{1,\mu}k_{1,\mu'} \qquad b_L(\epsilon) = 1 \tag{7}$$

$$\hat{\mathcal{P}}^\gamma_{P,\mu\mu'} = i\varepsilon_{\mu\mu'\rho\rho'}\frac{q^\rho k_1^{\rho'}}{s'} \qquad b_P(\epsilon) = 1 \tag{8}$$

**FiXme Error: justify avoidance of $\boldsymbol{\Delta}$?** and we then find FiXme Error!

$$\frac{d^2\sigma_k(s,t_1,u_1,q^2)}{dt_1tu_1} = b_k(\epsilon)\hat{\mathcal{P}}^\gamma_{k,\mu\mu'}W^{\mu\mu'} \tag{9}$$

with $k \in \{G,L,P\}$ denoting (here and mostly ever after) the projection type. The transverse partonic cross section $d\sigma_T$ can be reconstructed from the above definitions by using

$$d\sigma_T = d\sigma_G + b_G(\epsilon)d\sigma_L \tag{10}$$

We also define accordingly

$$E_G(\epsilon) = E_L(\epsilon) = \frac{1}{1+\epsilon/2} \qquad E_P(\epsilon) = 1 \tag{11}$$

The final state spins are always summed over, but the initial spins have to be treated seperately: for unpolarized projections $k \in \{G,L\}$ they are also summed over, but for the polarized projection $k = P$ they are projected on their asymmetric part:

$$\hat{\mathcal{P}}^{\mathrm{g}}_G\varepsilon_\nu^{(\sigma_k)}(k_1)\varepsilon_{\nu'}^{*(\sigma_k)}(k_1) = \hat{\mathcal{P}}^{\mathrm{g}}_L\varepsilon_\nu^{(\sigma_k)}(k_1)\varepsilon_{\nu'}^{*(\sigma_k)}(k_1) = -g_{\nu\nu'} \tag{12}$$

$$\hat{\mathcal{P}}^{\mathrm{g}}_P\varepsilon_\nu^{(\sigma_k)}(k_1)\varepsilon_{\nu'}^{*(\sigma_k)}(k_1) = 2i\epsilon_{\nu\nu'\rho\rho'}\frac{k_1^\rho q^\rho}{s'} \tag{13}$$

By writing eq. (12) we decided to introduce Fadeev-Popov ghosts[2] as we got a single diagram in next-to-leading order with a triple-gluon vertex**FiXme Error: explain ghosts?**. As we can consider all quarks in the initial state as massless, we further find FiXme Error!

$$\hat{\mathcal{P}}_G^q u(k_1)\bar{u}(k_1) = \hat{\mathcal{P}}_L^q u(k_1)\bar{u}(k_1) = \not{k}_1 \qquad \hat{\mathcal{P}}_P^q u(k_1)\bar{u}(k_1) = -\gamma_5 \not{k}_1 \tag{14}$$

$$\hat{\mathcal{P}}_G^{\bar{q}} v(k_1)\bar{v}(k_1) = \hat{\mathcal{P}}_L^{\bar{q}} v(k_1)\bar{v}(k_1) = \not{k}_1 \qquad \hat{\mathcal{P}}_P^{\bar{q}} v(k_1)\bar{v}(k_1) = \gamma_5 \not{k}_1 \tag{15}$$

We further define a set of partonic variables:

$$0 \leq \rho = \frac{4m^2}{s} \leq 1 \qquad 0 \leq \beta = \sqrt{1-\rho} \leq 1 \qquad 0 \leq \chi = \frac{1-\beta}{1+\beta} \leq 1 \tag{16}$$

$$\rho_q = \frac{4m^2}{q^2} \leq 0 \qquad 1 \leq \beta_q = \sqrt{1-\rho_q} \qquad 0 \leq \chi_q = \frac{\beta_q - 1}{\beta_q + 1} \leq 1 \tag{17}$$

When computing Feynman diagrams a computer algebra system (CAS) is almost obligatory: common choices are `FORM`[5] or `Mathematica`[6] - for the later the most common choice is `TRACER`[7], but we have chosen `HEPMath`[8]. We used the Feynman rules given by [9].

## 2 Leading Order Calculations

In leading order we have to consider photon-gluon-fusion (PGF), that is

$$\gamma^*(q) + \mathrm{g}(k_1) \to Q(p_1) + \overline{Q}(p_2) \tag{18}$$

with two contributing diagrams depicted in figure **FiXme Error: todo**. The result can then be written as FiXme Error!

$$\hat{\mathcal{P}}_k^{\gamma,\mu\mu'} \hat{\mathcal{P}}_k^{\mathrm{g}} \sum_{j=1}^{2} \mathcal{M}_{j,\mu}^{(0)} \mathcal{M}_{j,\mu'}^{(0)*} = 8g^2 \mu_D^{-\epsilon} e^2 e_H^2 N_C C_F B_{k,QED} \tag{19}$$

where $g$ and $e$ are the strong and electromagnetic coupling constants respectively, $\mu_D$ is an arbitray mass parameter introduced to keep the couplings dimensionless and $e_H$ is the magnitude of the heavy quark in units of $e$. Further $N_C$ corresponds to the gauge group $SU(N_C)$ and the color factor $C_F = (N_C^2 - 1)/(2N_C)$ refers to the second Casimir constant of the fundamental representation for the quarks. We then find:

$$\begin{aligned}
B_{G,QED} &= \frac{t_1}{u_1} + \frac{u_1}{t_1} + \frac{4m^2 s'}{t_1 u_1}\left(1 - \frac{m^2 s'}{t_1 u_1}\right) + \frac{2s' q^2}{t_1 u_1} + \frac{2q^4}{t_1 u_1} + \frac{2m^2 q^2}{t_1 u_1}\left(2 - \frac{s'^2}{t_1 u_1}\right) \\
&\quad + \epsilon\left\{-1 + \frac{s'^2}{t_1 u_1} + \frac{s' q^2}{t_1 u_1} - \frac{q^4}{t_1 u_1} - \frac{m^2 q^2 s'^2}{t_1^2 u_1^2}\right\} + \epsilon^2 \frac{s'^2}{4 t_1 u_1} \qquad (20) \\
B_{L,QED} &= -\frac{4q^2}{s'}\left(\frac{s}{s'} - \frac{m^2 s'}{t_1 u_1}\right) \qquad (21) \\
B_{P,QED} &= \frac{1}{2}\left(\frac{t_1}{u_1} + \frac{u_1}{t_1}\right)\left(\frac{2m^2 s'}{t_1 u_1} - 1 - \frac{2q^2}{s'}\right) \qquad (22) \\
B_{k,QED} &= B^{(0)}_{k,QED} + \epsilon B^{(1)}_{k,QED} + \epsilon^2 B^{(2)}_{k,QED} \qquad (23)
\end{aligned}$$

By using eq. (2) we can derive the $n$-dimensional $2 \to 2$ phase space

$$dPS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} \Theta(p_{1,0})\delta(p_1^2 - m^2)\Theta(p_{2,0})\delta(p_2^2 - m^2)(2\pi)^n \delta^{(n)}(k_1 + q - p_1 - p_2) \qquad (24)$$

that can be solved by using the center-of-mass system (CMS) of the incoming particles[2]

$$q = \left(\frac{s + q^2}{2\sqrt{s}}, 0, 0, -\frac{s - q^2}{2\sqrt{s}}, \hat{0}\right) \qquad k_1 = \frac{s - q^2}{2\sqrt{s}}\left(1, 0, 0, 1, \hat{0}\right) \qquad (25)$$

such that $q + k_1 = (\sqrt{s}, \vec{0})$ and $k_1^2 = 0$. For the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2}\left(1, 0, \beta\sin\theta, \beta\cos\theta, \hat{0}\right) \qquad p_2 = \frac{\sqrt{s}}{2}\left(1, 0, -\beta\sin\theta, -\beta\cos\theta, \hat{0}\right) \qquad (26)$$

such that $p_1 + p_2 = (\sqrt{s}, \vec{0})$ and $p_1^2 = p_2^2 = m^2$. Finally we have to use the $n$-sphere

$$d^n x = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} dx = \frac{\pi^{n/2}}{\Gamma(n/2)} (x^2)^{(n-2)/2} dx^2 \qquad (27)$$

and arrive at the well known result[1]

$$\begin{aligned}
dPS_2 &= \frac{\delta(s' + t_1 + u_1)}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}}\left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{s'^2}\right)^{(n-4)/2} dt_1 du_1 \qquad (28) \\
&= \delta(s' + t_1 + u_1) h_2(n)\, dt_1 du_1 \qquad (29) \\
h_2(4+\epsilon) &= \frac{2\pi S_\epsilon}{s'\Gamma(1 + \epsilon/2)}\left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{s'^2}\right)^{\epsilon/2} \qquad (30)
\end{aligned}$$

with $S_\epsilon = (4\pi)^{(-2-\epsilon/2)}$.

The final double differential LO partonic cross section can then be written as

$$s'^2 \frac{d^2\sigma^{(0)}_{k,\mathrm{g}}(s',t_1,u_1,q^2)}{dt_1 du_1} = 2^6 \alpha\alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \delta(s'+t_1+u_1) \frac{\pi^3 S_\epsilon}{\Gamma(1+\epsilon/2)}$$
$$\left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{m^2 s'^2}\right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2} B_{k,QED} \qquad (31)$$

where we used $e^2 = 4\pi\alpha$ and $g^2 = 4\pi\alpha_s$. The color average is given by $K_{\mathrm{g}\gamma} = 1/(N_C^2 - 1)$.

From the results above we can easily obtain the *total* LO partonic cross sections

$$\sigma_G^{(0)}(s,q^2,m^2) = -4\pi\alpha\alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F \frac{1}{s'^3}\Big((s^2+q^4+4m^2 s)\beta + (s^2+q^4-4m^2(2m^2-s'))\ln(\chi)\Big) \qquad (32)$$

$$\sigma_L^{(0)}(s,q^2,m^2) = 16\pi\alpha\alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F \left(\frac{-q^2 s}{s'^3}\right)\left(\beta + \frac{2m^2}{s}\ln(\chi)\right) \qquad (33)$$

$$\sigma_P^{(0)}(s,q^2,m^2) = 4\pi\alpha\alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F \frac{1}{s'^2}\Big((3s+q^2)\beta + (s+q^2)\ln(\chi)\Big) \qquad (34)$$

from which we also find

$$\lim_{s\to 4m^2} \sigma_T^{(0)}(s',q^2) = 4\pi\alpha\alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F \frac{\beta}{4m^2-q^2} + O(\beta^3) = \lim_{s\to 4m^2} \sigma_P^{(0)}(s',q^2) \qquad (35)$$

$$\lim_{s\to 4m^2} \sigma_L^{(0)}(s',q^2) = -\frac{128}{3}\pi\alpha\alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F \frac{m^2 q^2 \beta^3}{(4m^2-q^2)^3} + O(\beta^5) \qquad (36)$$

(Note the missing factor of 2 in [1, eq. (5.9)].) **FiXme Error: shift to partonic?** FiXme Error!

## 3 Next-To-Leading Order Calculations

Next-to-leading order contributions can be split into three parts: one loop virtual contributions, one gluon radiation and the light quark processes. **FiXme Error: more?** FiXme Error!

### 3.1 One Loop Virtual Contributions

Virtual contributions have the same initial and final state as the Born process, but have a looping particle. All contributing Feynman diagrams are depicted in figure **FiXme Error: do**. The result can then be written as FiXme Error!

$$\hat{\mathcal{P}}_k^{\gamma,\mu\mu'} \hat{\mathcal{P}}_k^{\mathrm{g}} \sum_j \left[\mathcal{M}_{j,\mu}^{(1),V}\left(\mathcal{M}_{1,\mu'}^{(0)} + \mathcal{M}_{2,\mu'}^{(0)}\right)^* + c.c.\right]$$
$$= 8 g^4 \mu_D^{-\epsilon} e^2 e_H^2 N_C C_F C_\epsilon \left(C_A V_{k,OK} + 2 C_F V_{k,QED}\right) \qquad (37)$$

where $C_\epsilon = \exp(\epsilon/2(\gamma_E - \ln(4\pi)))/(16\pi^2)$ and $C_A$ is the second Casimir constant of the adjoint representation for the gluon (that introduces a non-abelian part).

For the computation of the loops the Passarino-Veltman-decomposition[10] in $n = 4 + \epsilon$ dimension is used as far as possible. The decomposition is based on Lorentz invariance and a good explanation is for example given in [2]. The needed scalar integrals are given in [11] and [1], but there is also one wrong integral: we find with [12, Box 16]:

$$\begin{aligned}
&D_0(m^2, 0, q^2, m^2, t, s, 0, m^2, m^2, m^2) \\
&= \frac{iC_\epsilon}{\beta s t_1} \left[ -\frac{2\ln(\chi)}{\epsilon} - 2\ln(\chi)\ln(-t_1/m^2) + \mathrm{Li}_2(1-\chi^2) - 4\zeta(2) + \ln^2(\chi_q) + 2\,\mathrm{Li}_2(-\chi\chi_q) \right. \\
&\qquad \left. + 2\,\mathrm{Li}_2(-\chi/\chi_q) + 2\ln(\chi\chi_q)\ln(1+\chi\chi_q) + 2\ln(\chi/\chi_q)\ln(1+\chi/\chi_q) \right]
\end{aligned} \tag{38}$$

where we used the argument ordering of `LoopTools`[13, 14] (and also checked it against `LoopTools`).

As the short example above shows, the full expressions for the $V_{k,OK}, V_{k,QED}$ are quite complicated and too long to be presented here, nevertheless the arising poles are quite compact:

$$V_{k,OK} = -2B_{k,QED}\left(\frac{4}{\epsilon^2} + \left(\ln(-t_1/m^2) + \ln(-u_1/m^2) - \frac{2m^2 - s}{s}\ln(\chi)\right)\frac{2}{\epsilon}\right) + O(\epsilon^0) \tag{39}$$

$$V_{k,QED} = -2B_{k,QED}\left(1 - \frac{2m^2 - s}{s}\ln(\chi)\right)\frac{2}{\epsilon} + O(\epsilon^0) \tag{40}$$

The above results already include the mass renormalization that we have performed *on-shell*, so all ultra-violet poles have been removed. For the renormalization of the strong coupling we use the $\overline{\mathrm{MS}}_m$ scheme defined in [2] and so the full (remaining) renormalization can be achieved by

$$\begin{aligned}
\frac{d^2\sigma_k^{(1),V,ren.}}{dt_1 du_1} &= \frac{d^2\sigma_k^{(1),V}}{dt_1 du_1} + \frac{\alpha_s(\mu_R^2)}{4\pi}\left[\left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(\mu_R^2/m^2) - \ln(\mu_D^2/m^2)\right)\beta_0^f \right. \\
&\qquad \left. + \frac{2}{3}\ln(\mu_R^2/m^2)\right]\frac{d^2\sigma_k^{(0)}}{dt_1 du_1} \qquad (41) \\
&= \frac{d^2\sigma_k^{(1),V}}{dt_1 du_1} + 4\pi\alpha_s(\mu_R^2)C_\epsilon\left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2}\left[\left(\frac{2}{\epsilon} + \ln(\mu_R^2/m^2)\right)\beta_0^f \right. \\
&\qquad \left. + \frac{2}{3}\ln(\mu_R^2/m^2)\right]\frac{d^2\sigma_k^{(0)}}{dt_1 du_1} \qquad (42)
\end{aligned}$$

with $\mu_R$ the renormalization scale introduced by the RGE, $\beta_0^f = (11C_A - 2n_f)/3$ the first coefficient of the beta function and $n_f$ the number of *total* flavours (i.e. $n_{lf} = n_f - 1$ active (light) flavours and one heavy flavour). The double poles occuring in $V_{k,OK}$ are introduced by the diagrams **FiXme Error: do** when the soft and collinear singularities FiXme Error!

coincide.

The double differential partonic cross section is given by

$$
\begin{aligned}
s'^2 \frac{d^2\sigma^{(1),V}_{k,\mathrm{g}}(s',t_1,u_1,q^2)}{dt_1 du_1} = & 2^8 \alpha\alpha_s^2 e_H^2 K_{\mathrm{g}\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon)\delta(s'+t_1+u_1)\frac{\pi^4 S_\epsilon}{\Gamma(1+\epsilon/2)} \\
& \left(\frac{(t_1u_1' - s'm^2)s' - q^2t_1^2}{m^2 s'^2}\right)^{\epsilon/2} C_\epsilon \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2} \\
& \left(C_A V_{k,OK} + 2C_F V_{k,QED}\right)
\end{aligned} \tag{43}
$$

The results agree in the photo-production limit ($q^2 \to 0$) with [15] **FiXme Error: Matrix elements available upon request**. FiXme Error!

### 3.2 Single Gluon Radiation

In next-to-leading order we have to consider the following process:

$$
\gamma^*(q) + \mathrm{g}(k_1) \to Q(p_1) + \overline{Q}(p_2) + \mathrm{g}(k_2) \tag{44}
$$

All contributing diagrams are depicted in figure **FiXme Error: do** and the result can be written as FiXme Error!

$$
\hat{\mathcal{P}}_k^{\gamma,\mu\mu'} \hat{\mathcal{P}}_k^{\mathrm{g}} \sum_{j,j'} \mathcal{M}^{(1),g}_{j,\mu} \mathcal{M}^{(1),g^*}_{j',\mu'} = 8g^4 \mu_D^{-2\epsilon} e^2 e_H^2 N_C C_F \left(C_A R_{k,OK} + 2C_F R_{k,QED}\right) \tag{45}
$$

and it will depend on ten kinematical invariants:

$$
\begin{aligned}
s &= (q+k_1)^2 & t_1 &= (k_1-p_2)^2 - m^2 & u_1 &= (q-p_2)^2 - m^2 && (46)\\
s_3 &= (k_2+p_2)^2 - m^2 & s_4 &= (k_2+p_1)^2 - m^2 & s_5 &= (p_1+p_2)^2 = -u_5 && (47)\\
t' &= (k_1-k_2)^2 & & & & && (48)\\
u' &= (q-k_2)^2 & u_6 &= (k_1-p_1)^2 - m^2 & u_7 &= (q-p_1)^2 - m^2 && (49)
\end{aligned}
$$

from which only five are independent as can be seen from momentum conservation $k_1+q = p_1+p_2+k_2$ and $s, t_1, u_1$ match to their leading order definition.

The $2 \to 3$ $n$-dimensional phase space is given by

$$
\begin{aligned}
dPS_3 = \int & \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} \frac{d^n k_2}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(k_1+q-p_1-p_2-k_2) \\
& \Theta(p_{1,0})\delta(p_1^2-m^2)\Theta(p_{2,0})\delta(p_2^2-m^2)\Theta(k_{2,0})\delta(k_2^2)
\end{aligned} \tag{50}
$$

This can be solved by writing eq. (50) as product of a $2 \to 2$ decay and a subsequent $1 \to 2$ decay[11]. We find

$$dPS_3 = \frac{1}{(4\pi)^n \Gamma(n-3) s'} \frac{s_4^{n-3}}{(s_4+m^2)^{n/2-1}} \left( \frac{(t_1 u_1' - s' m^2) s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} dt_1 du_1 d\Omega_n d\hat{\mathcal{I}} \tag{51}$$

$$= h_3(n)\, dt_1 du_1 d\Omega_n d\hat{\mathcal{I}} \tag{52}$$

with $d\Omega_n = \sin^{n-3}(\theta_1) d\theta_1 \sin^{n-4}(\theta_2) d\theta_2$ and $d\hat{\mathcal{I}}$ taking care of all occuring hat momenta:

$$d\hat{\mathcal{I}} = \frac{1}{B(1/2,(n-4)/2)} \frac{x^{(n-6)/2}}{\sqrt{1-x}} dx \qquad \text{with } x = \hat{p}_1^2/\hat{p}_{1,max} \tag{53}$$

$$\hat{p}_{1,max} = \frac{s_4^2}{4(s_4+m^2)} \sin^2(\theta_1)\sin^2(\theta_2) \tag{54}$$

$$\Rightarrow \int d\hat{\mathcal{I}} = 1 \qquad \int d\hat{\mathcal{I}}\, \hat{p}_1^2 = \epsilon \hat{p}_{1,max} + O(\epsilon^2) \tag{55}$$

The needed phase space integrals for $\theta_1$ and $\theta_2$ can be found in [11] and [2]. We find for the difference to the $2 \to 2$ phase space

$$\frac{h_3(4+\epsilon)}{h_2(4+\epsilon)} = \frac{S_\epsilon}{2\pi} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \frac{s_4^{1+\epsilon}}{(s_4+m^2)^{1+\epsilon/2}} \tag{56}$$

$$= \frac{C_\epsilon}{2\pi} \left(1 - \frac{3}{8}\zeta(2)\epsilon^2\right) \frac{s_4^{1+\epsilon}}{(s_4+m^2)^{1+\epsilon/2}} + O(\epsilon^3) \tag{57}$$

where $\zeta$ is Riemanns zeta function. **FiXme Error: introduce psLogs? in appendix?** FiXme Error!

Again when integrating the phase space angles the expressions become quite lengthy, but the (collinear) pole parts are compact:

$$\frac{s_4}{4\pi(s_4+m^2)} \int d\Omega_n d\hat{\mathcal{I}}\, C_A R_{k,OK} = -\frac{1}{u_1} B_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} P^H_{k,\mathrm{gg}}(x_1) \frac{2}{\epsilon} + O(\epsilon^0) \tag{58}$$

with $x_1 = -u_1/(s'+t_1)$ and the hard part of the Altarelli-Parisi splitting functions $P^H_{k,\mathrm{gg}}$[16, 17]:

$$P^H_{G,\mathrm{gg}}(x) = P^H_{L,\mathrm{gg}}(x) = C_A \left( \frac{2}{1-x} + \frac{2}{x} - 4 + 2x - 2x^2 \right) \tag{59}$$

$$P^H_{P,\mathrm{gg}}(x) = C_A \left( \frac{2}{1-x} - 4x + 2 \right) \tag{60}$$

The $R_{k,QED}$ do not contain poles. **FiXme Error: shift to factorization?** FiXme Error!

The double differential partonic cross section is given by

$$
\begin{aligned}
s'^2 \frac{d^2\sigma^{(1),R}_{k,\mathrm{g}}(s',t_1,u_1,q^2)}{dt_1 du_1} = {} & 2^7 \alpha \alpha_s^2 e_H^2 K_{\mathrm{g}\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \frac{\pi^3 S_\epsilon^2}{\Gamma(1+\epsilon)} \frac{s_4}{s_4+m^2} \\
& \left(\frac{(t_1 u_1' - s' m^2)s' - q^2 t_1^2}{m^2 s'^2}\right)^{\epsilon/2} \left(\frac{s_4^2}{m^2(s_4+m^2)}\right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon} \\
& \int d\Omega_n d\hat{\mathcal{I}} \, \big(C_A R_{k,OK} + 2C_F R_{k,QED}\big)
\end{aligned} \tag{61}
$$

From the above expression we can obtain the soft limit $k_2 \to 0$ and seperate their calculations:

$$
\lim_{k_2\to 0} \big(C_A R_{k,OK} + 2C_F R_{k,QED}\big) = \big(C_A S_{k,OK} + 2C_F S_{k,QED}\big) + O(1/s_4, 1/s_3, 1/t') \tag{62}
$$

$$
S_{k,OK} = 2\left(\frac{t_1}{t' s_3} + \frac{u_1}{t' s_4} - \frac{s-2m^2}{s_3 s_4}\right) B_{k,QED} \tag{63}
$$

$$
S_{k,QED} = 2\left(\frac{s-2m^2}{s_3 s_4} - \frac{m^2}{s_3^2} - \frac{m^2}{s_4^2}\right) B_{k,QED} \tag{64}
$$

Note that the einkonal factors multiplying the Born functions $B_{k,QED}$ neither depend on $q^2$ nor on the projection $k$. We can then split the phase space by introducing an infrared cut-off $\Delta$ and distinguish then between soft $s_4 \leq \Delta$ and hard $s_4 > \Delta$ contributions. Let $\mathcal{R}(s_4)$ be a function with a soft pole $s_4^{-1+\epsilon}\mathcal{S}(s_4)$ and a finite part $\mathcal{F}(s_4)$, we then find [2]:

$$
\int_0^{s_{4,max}} \mathcal{R}(s_4) = \int_0^{s_{4,max}} \left(s_4^{-1+\epsilon}\mathcal{S}(s_4) + \mathcal{F}(s_4)\right) \tag{65}
$$

$$
\simeq \frac{\Delta^\epsilon}{\epsilon}\mathcal{S}(0) + \int_\Delta^{s_{4,max}} \mathcal{R}(s_4) \tag{66}
$$

This expansion is valid for $\Delta$ being small, i.e. smaller then any leading order scale or $m^2$;

a typical choice is $\Delta/m^2 \sim 10^{-6}$. We then find

$$\frac{s_4^2}{4\pi(s_4+m^2)}\left(1-\frac{3}{8}\zeta(2)\epsilon^2\right)\int d\Omega_n d\hat{\mathcal{I}}\, S_{k,QED}$$
$$= B_{k,QED}\left[-\frac{2}{\epsilon}\left(1+\frac{s-2m^2}{s\beta}\ln(\chi)\right)+1-\frac{s-m^2}{s\beta}\left(\ln(\chi)\left(1+\ln(\chi)\right)+\mathrm{Li}_2(1-\chi^2)\right)\right] \tag{67}$$

$$\frac{s_4^2}{4\pi(s_4+m^2)}\left(1-\frac{3}{8}\zeta(2)\epsilon^2\right)\int d\Omega_n d\hat{\mathcal{I}}\, S_{k,OK}$$
$$= B_{k,QED}\left[\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\left(\ln(t_1/u_1)+\frac{s-2m^2}{s\beta}\ln(\chi)\right)-\ln^2(\chi)-\frac{3}{2}\zeta(2)+\frac{1}{2}\ln^2(t_1/(u_1\chi))\right.$$
$$\left.+\mathrm{Li}_2(1-t_1/(u_1\chi))-\mathrm{Li}_2(1-u_1/(t_1\chi))+\frac{s-2m^2}{s\beta}\left(\mathrm{Li}_2(1-\chi^2)+\ln^2(\chi)\right)\right] \tag{68}$$

(Note the mistyped sign of $\ln(\chi)^2$ in [1, eq. (3.25)]) The additional factors on the left hand sides originate from the difference between the $2\to 3$ phasespace of $R_k$ and the $2\to 2$ phasespace needed for $S_k$.

The double differential partonic cross section is given by

$$s'^2\frac{d^2\sigma_{k,\mathrm{g}}^{(1),S}(s',t_1,u_1,q^2)}{dt_1du_1}$$
$$= 2^8\alpha\alpha_s^2e_H^2K_{\mathrm{g}\gamma}N_CC_FE_k(\epsilon)b_k(\epsilon)\delta(s'+t_1+u_1)\frac{\pi^4S_\epsilon}{\Gamma(1+\epsilon/2)}$$
$$\left(\frac{(t_1u_1'-s'm^2)s'-q^2t_1^2}{m^2s'^2}\right)^{\epsilon/2}C_\epsilon\left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon}\left(\frac{\Delta}{m^2}\right)^{\epsilon}$$
$$\frac{s_4^2}{4\pi(s_4+m^2)}\left(1-\frac{3}{8}\zeta(2)\epsilon^2\right)\int d\Omega_n d\hat{\mathcal{I}}\,\left(C_AS_{k,OK}+2C_FS_{k,QED}\right) \tag{69}$$

The results agree in the photo-production limit ($q^2\to 0$) with [15].

### 3.3 Light Quark Processes

In next-to-leading order a new production mechanism enters that is induced by a light quark, so we have to consider the process

$$\gamma^*(q)+q(k_1)\to Q(p_1)+\overline{Q}(p_2)+q(k_2) \tag{70}$$

All contributing diagrams are depicted in figure **FiXme Error: do** and the result can be FiXme Error!

written as

$$\hat{\mathcal{P}}_k^{\gamma,\mu\mu'}\hat{\mathcal{P}}_k^q\sum_{j,j'}\mathcal{M}_{j,\mu}^{(1),q}\mathcal{M}_{j',\mu'}^{(1),q*} = 8g^4\mu_D^{-2\epsilon}e^2N_CC_F\left(e_H^2A_{k,1}+e_L^2A_{k,2}+e_Le_HA_{k,3}\right) \quad (71)$$

where $e_L$ denotes the charge of the light quark $q$ in units of $e$.

The needed $2\to 3$ phase space has already been calculated in section 3.2, so we can immediately quote the (collinear) poles:

$$\frac{s_4}{2\pi(s_4+m^2)}\int d\Omega_n d\hat{\mathcal{I}}\,C_FA_{k,1} = -\frac{1}{u_1}B_{k,QED}\begin{pmatrix} s'\to x_1s' \\ t_1\to x_1t_1\end{pmatrix}P_{k,\mathrm{g}q}(x_1)\frac{2}{\epsilon}+O(\epsilon^0) \quad (72)$$

with $x_1=-u_1/(s'+t_1)$ and the Altarelli-Parisi splitting functions $P_{k,\mathrm{g}q}$[16, 17]:

$$P_{G,\mathrm{g}q}(x) = P_{L,\mathrm{g}q}(x) = C_F\left(\frac{1}{x}+\frac{(1-x)^2}{x}\right) \quad (73)$$

$$P_{P,\mathrm{g}q}(x) = C_F\,(2-x) \quad (74)$$

$A_{k,2}$ does not contain poles and we find $\int dt_1du_1\int d\Omega_nd\hat{\mathcal{I}}\,A_{k,3}=0$. Note that in the limit $q^2\to 0$ $A_{k,2}$ will also get collinear poles.

The double differential partonic cross section is given by

$$\begin{aligned} s'^2\frac{d^2\sigma_{k,q}^{(1)}(s',t_1,u_1,q^2)}{dt_1du_1} &= 2^7\alpha\alpha_s^2K_{q\gamma}N_CC_Fb_k(\epsilon)\frac{\pi^3S_\epsilon^2}{\Gamma(1+\epsilon)}\frac{s_4}{s_4+m^2} \\ &\left(\frac{(t_1u_1'-s'm^2)s'-q^2t_1^2}{m^2s'^2}\right)^{\epsilon/2}\left(\frac{s_4^2}{m^2(s_4+m^2)}\right)^{\epsilon/2}\left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon} \\ &\int d\Omega_nd\hat{\mathcal{I}}\,\left(e_H^2A_{k,1}+e_L^2A_{k,2}+e_He_LA_{k,3}\right) \end{aligned} \quad (75)$$

with the color average $K_{q\gamma}=1/N_C$.

The results agree in the photo-production limit ($q^2\to 0$) with [15].

## 4 Mass Factorization

All collinear poles in the gluonic subprocess can be removed by mass factorization in the following way:

$$s'^2 \frac{d^2\sigma_{k,\mathrm{g}}^{(1),fin}(s',t_1,u_1,q^2,\mu_F)}{dt_1 du_1} = \lim_{\epsilon\to 0}\left[ s'^2 \frac{d^2\sigma_{k,\mathrm{g}}^{(1)}(s',t_1,u_1,q^2,\epsilon)}{dt_1 du_1} \right.$$

$$- \int_0^1 \frac{dx_1}{x_1} \Gamma_{k,\mathrm{gg}}^{(1)}(x_1,\mu_F^2,\mu_D,\epsilon) \tag{76}$$

$$\left. (x_1 s')^2 \frac{d^2\sigma_{k,\mathrm{g}}^{(0)}(x_1 s', x_1 t_1, u_1, q^2, \epsilon)}{d(x_1 t_1) du_1} \right] \tag{77}$$

$$\Gamma_{k,ij}^{(1)}(x,\mu_F^2,\mu_D,\epsilon) = \frac{\alpha_s}{2\pi}\left( P_{k,ij}(x)\frac{2}{\epsilon} + f_{k,ij}(x,\mu_F^2,\mu_D^2)\right) \tag{78}$$

where $\Gamma_{k,ij}^{(1)}$ is the first order correction to the transition functions $\Gamma_{k,ij}$ for *incoming* particle $j$ and *outgoing* particle $i$ in projection $k$. In the $\overline{\mathrm{MS}}$-scheme the $f_{k,ij}$ take their usual form and we find

$$\Gamma_{k,ij}^{(1),\overline{\mathrm{MS}}}(x,\mu_F^2,\mu_D,\epsilon) = \frac{\alpha_s}{2\pi} P_{k,ij}(x)\left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(\mu_F^2/m^2) - \ln(\mu_D^2/m^2)\right) \tag{79}$$

$$= 8\pi\alpha_s P_{k,ij}(x) C_\epsilon \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2} \left(\frac{2}{\epsilon} + \ln(\mu_F^2/m^2)\right) \tag{80}$$

The $P_{k,ij}(x)$ are the Altarelli-Parisi splitting functions for which we find[16, 17]

$$P_{k,\mathrm{gg}}(x) = \Theta(1-\delta-x) P_{k,\mathrm{gg}}^H(x) + \delta(1-x)\left(2C_A \ln(\delta) + \frac{\beta_0}{2}\right) \tag{81}$$

where we introduced another infrared cut-off $\delta$ to seperate soft $(x \geq 1-\delta)$ and hard $(x < 1-\delta)$ gluons that is connected to $\Delta$ via $\delta = \Delta/(s'+t_1)$. The structure here explains why we were able to write the equation (58).

The light quark process can be regularized in a complete analogous way:

$$s'^2 \frac{d^2\sigma_{k,q}^{(1),fin}(s',t_1,u_1,q^2,\mu_F)}{dt_1 du_1} = \lim_{\epsilon\to 0}\left[ s'^2 \frac{d^2\sigma_{k,q}^{(1)}(s',t_1,u_1,q^2,\epsilon)}{dt_1 du_1} \right.$$

$$- \int_0^1 \frac{dx_1}{x_1} \Gamma_{k,\mathrm{g}q}^{(1)}(x_1,\mu_F^2,\mu_D,\epsilon)$$

$$\left. (x_1 s')^2 \frac{d^2\sigma_{k,\mathrm{g}}^{(0)}(x_1 s', x_1 t_1, u_1, q^2, \epsilon)}{d(x_1 t_1) du_1} \right] \tag{82}$$

The needed splitting functions $P_{k,\mathrm{g}q}$ have been already quoted in equations (73) and (74). Note that $K_{q\gamma} = 1/(N_C) = 2C_F K_{\mathrm{g}\gamma}$.

The final finite cross sections are then

$$
\begin{aligned}
s'^2 \frac{d^2\sigma^{(1),H,fin}_{k,\mathrm{g}}}{dt_1 du_1} = & \frac{1}{2\pi} K_{\mathrm{g}\gamma}\alpha\alpha_S e_H^2 N_C C_F b_k(0) \Bigg[ -\frac{1}{u_1} P^H_{k,\mathrm{gg}}(x_1) \\
& \left\{ 4\pi B^{(0)}_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \left( \ln\left( \frac{s_4^2}{m^2(s_4+m^2)} \right) - \ln(\mu_F^2/m^2) \right) \right. \\
& \left. -8\pi B^{(1)}_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \right\} \\
& + C_A \frac{s_4}{s_4+m^2} \left( \int d\Omega_n d\hat{\mathcal{I}}\, R_{k,OK} \right)^{finite} \\
& + 2C_F \frac{s_4}{s_4+m^2} \int d\Omega_4 d\hat{\mathcal{I}}\, R_{k,QED} \Bigg]
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
s'^2 \frac{d^2\sigma^{(1),S+V,fin}_{k,\mathrm{g}}}{dt_1 du_1} = & 4K_{\mathrm{g}\gamma}\alpha\alpha_S e_H^2 N_C C_F b_k(0) B^{(0)}_{k,QED} \delta(s'+t_1+u_1) \Big[ C_A \ln^2(\Delta/m^2) \\
& + \ln(\Delta/m^2) \Big( \Big( \ln(-t_1/m^2) - \ln(-u_1/m^2) - \ln(\mu_F^2/m^2) \Big) C_A \\
& - \frac{2m^2 - s}{s\beta} \ln(\chi)(C_A - 2C_F) - 2C_F \Big) \\
& + \frac{\beta_0^{lf}}{4} \Big( \ln(\mu_R^2/m^2) - \ln(\mu_F^2/m^2) \Big) + f_k(s', u_1, t_1, q^2) \Big]
\end{aligned} \tag{84}
$$

where $f_k$ contains lots of logarithms and dilogarithms, but does not depend on $\Delta, \mu_F^2, \mu_R^2$ nor $n_f$ and $\beta_0^{lf} = (11C_A - 2n_{lf})/3$.

$$
\begin{aligned}
s'^2 \frac{d^2\sigma^{(1),fin}_{k,q}}{dt_1 du_1} = & \frac{1}{2\pi} K_{q\gamma}\alpha\alpha_S N_C b_k(0) \Bigg[ -\frac{1}{u_1} e_H^2 P_{k,\mathrm{g}q}(x_1) \\
& \left\{ 2\pi B^{(0)}_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \left( \ln\left( \frac{s_4^2}{m^2(s_4+m^2)} \right) - \ln(\mu_F^2/m^2) + 1 - \delta_{k,P} \right) \right. \\
& \left. -4\pi B^{(1)}_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \right\} \\
& + C_F \frac{s_4}{s_4+m^2} \left( \int d\Omega_n d\hat{\mathcal{I}}\, e_H^2 A_{k,1} \right)^{finite} \\
& + C_F \frac{s_4}{s_4+m^2} \int d\Omega_4 d\hat{\mathcal{I}}\, e_L^2 A_{k,2} + C_F \frac{s_4}{s_4+m^2} \int d\Omega_4 d\hat{\mathcal{I}}\, e_H e_L A_{k,3} \Bigg]
\end{aligned} \tag{85}
$$

where $1 - \delta_{k,P}$ may also be written as $-2\partial_\epsilon E_k(\epsilon = 0)$ as it originates from the additional factor of $E_k(\epsilon)$ in the subtraction part of equation (82).

## 5 Partonic Results

The *total* partonic cross sections can be computed by

$$\sigma_{k,j}^{(n)}(s,q^2,m^2) = \int\limits_{-s'(1+\beta)/2}^{-s'(1-\beta)/2} dt_1 \int\limits_{0}^{s_{4,max}} ds_4 \frac{d^2\sigma_{k,j}^{(n),fin}(s',t_1,u_1,q^2)}{dt_1 ds_4} \tag{86}$$

$$s_{4,max} = \frac{s}{s' t_1}\left(t_1 + \frac{s'(1-\beta)}{2}\right)\left(t_1 + \frac{s'(1+\beta)}{2}\right) \tag{87}$$

where $k$ denotes as usual projection, $j \in \{\mathrm{g}, q, \bar{q}\}$ is a parton index and we used $s_4 = s' + t_1 + u_1$. The result can then be parametrised as

$$\begin{aligned}
&\sigma_{k,j}(s,q^2,m^2) \\
&= \sigma_{k,j}^{(0)}(s,q^2,m^2) + \sigma_{k,j}^{(1)}(s,q^2,m^2) && (88)\\
&= \frac{\alpha\alpha_S}{m^2}\left(f_{k,j}^{(0)}(\eta,\xi) + 4\pi\left(f_{k,j}^{(1)}(\eta,\xi) + \ln(\mu_F^2/m^2)\bar{f}_{k,j}^{F,(1)}(\eta,\xi) + \ln(\mu_R^2/m^2)\bar{f}_{k,j}^{R,(1)}(\eta,\xi)\right)\right) && (89)
\end{aligned}$$

where each function $f_{k,j}$ depends on the scaling variables $\eta = 1/\rho - 1$ and $\xi = -q^2/m^2$ and can be further decomposed by the electric charges

$$f_{k,\mathrm{g}}(\eta,\xi) = e_H^2 c_{k,\mathrm{g}}(\eta,\xi) \tag{90}$$

$$f_{k,q}(\eta,\xi) = e_H^2 c_{k,q}(\eta,\xi) + e_L^2 d_{k,q}(\eta,\xi) \tag{91}$$

As already mentioned the interference term proportional to $e_H e_L$ vanishes for total cross sections. FiXme Error!

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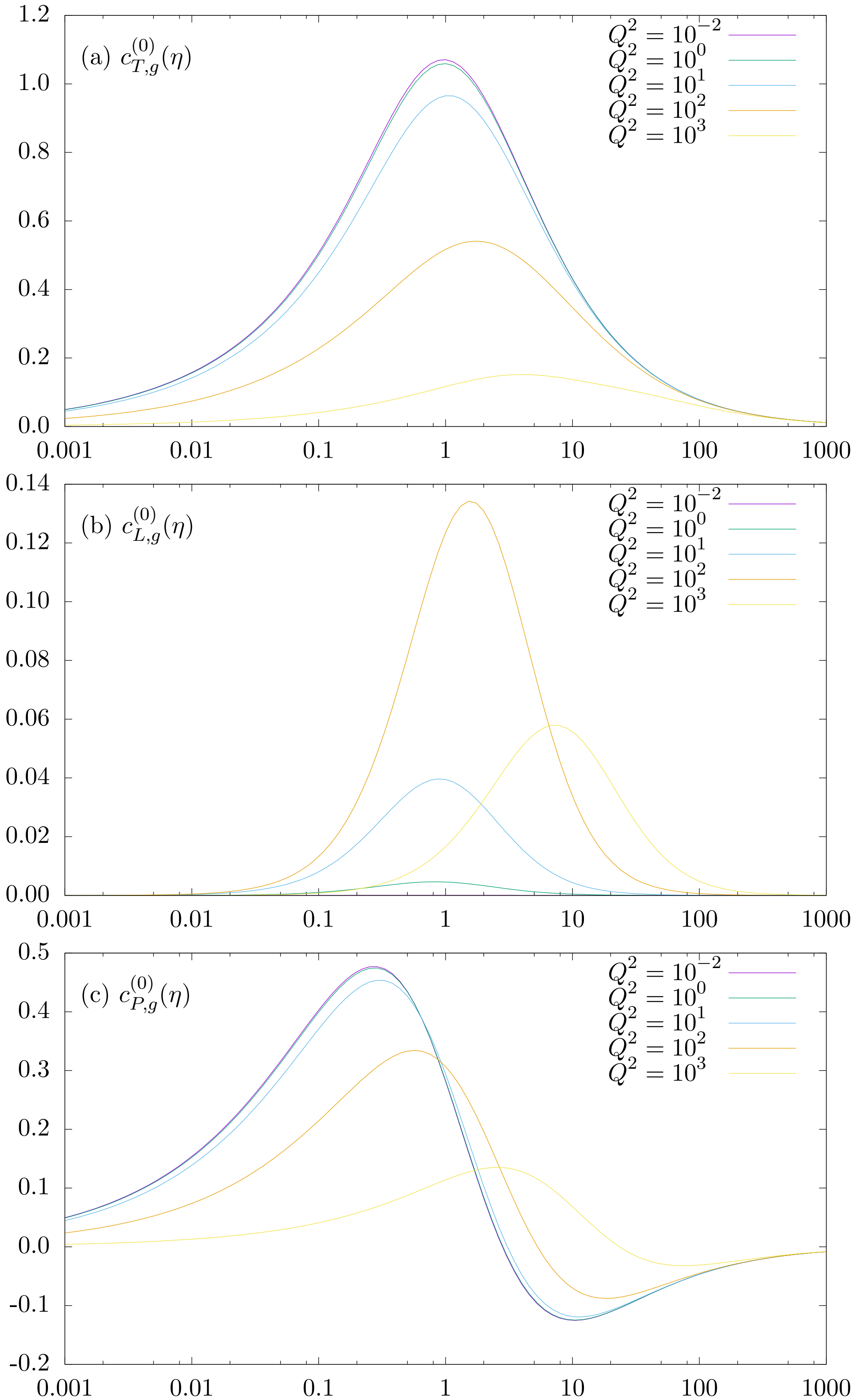


Figure 1: leading order scaling functions $c^{(0)}_{k,g}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of GeV$^2$ at $m = 4.75\,$GeV (i.e. different values of $\xi = Q^2/m^2$)

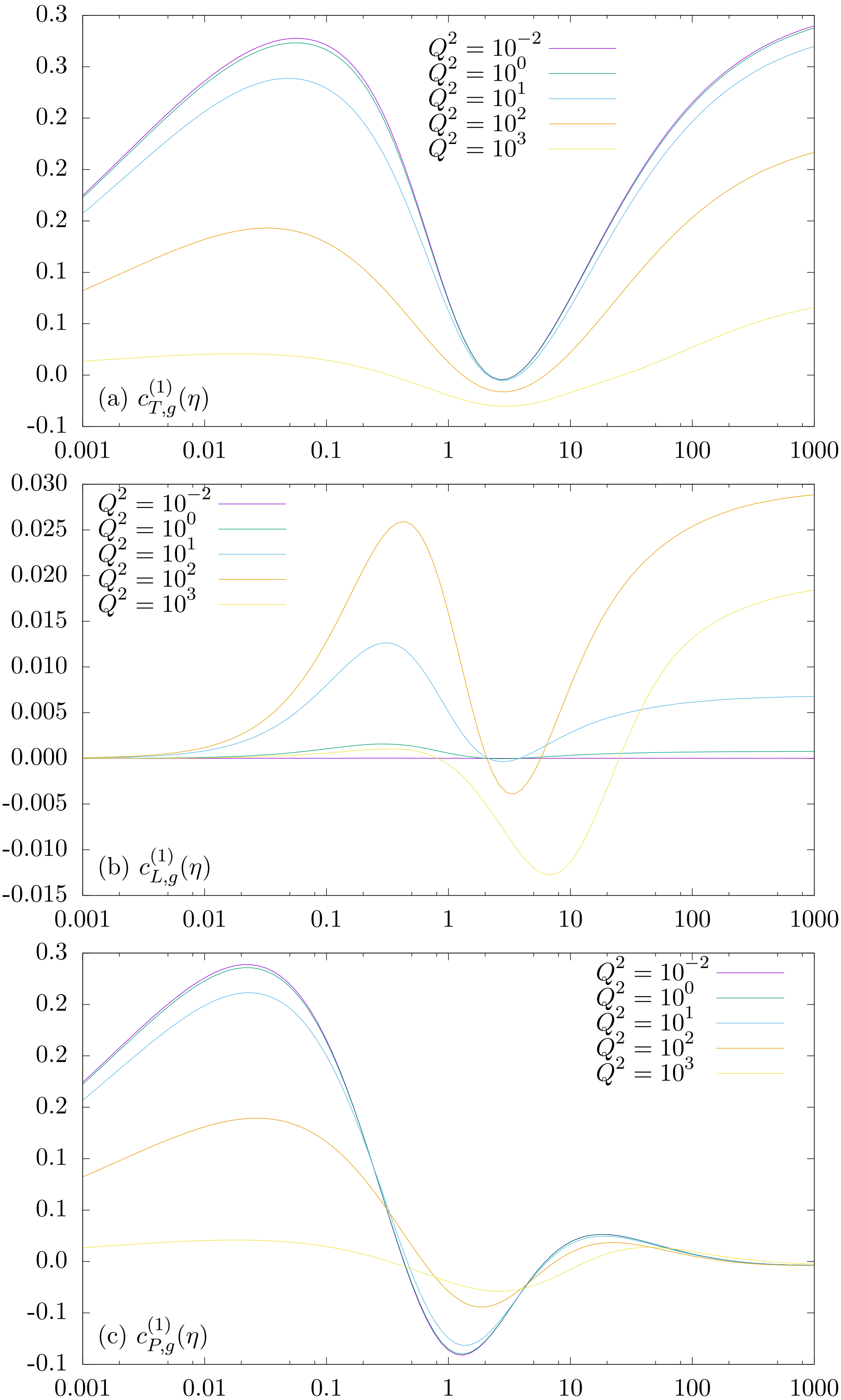


Figure 2: next-to-leading order scaling functions $c^{(1)}_{k,g}(\eta, \xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\text{GeV}^2$ at $m = 4.75\,\text{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

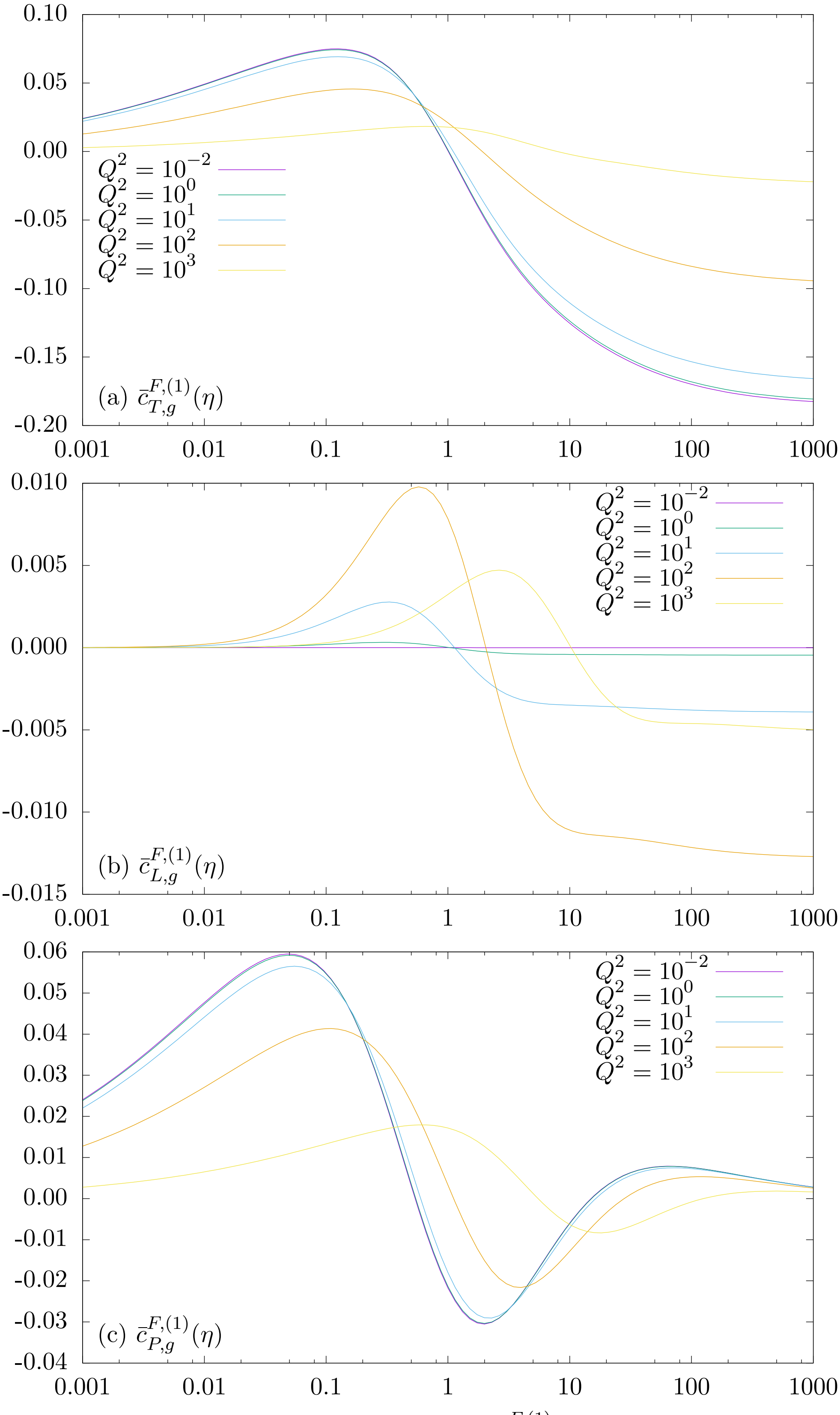


Figure 3: next-to-leading order scaling functions $\bar{c}^{F,(1)}_{k,g}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of GeV$^2$ at $m = 4.75\,$GeV (i.e. different values of $\xi = Q^2/m^2$) and $n_{lf} = 4$

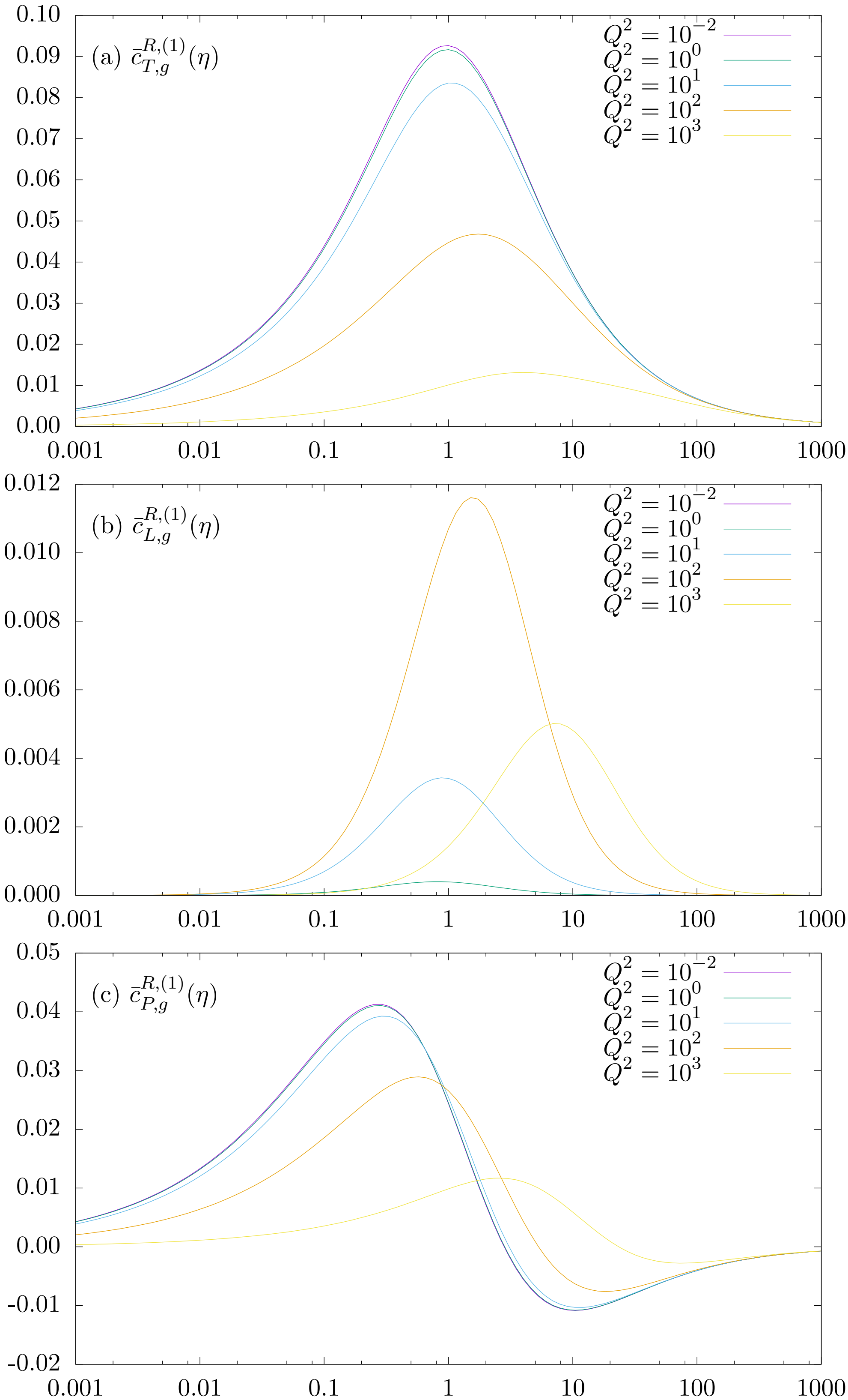


Figure 4: next-to-leading order scaling functions $\bar{c}_{k,g}^{R,(1)}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of GeV$^2$ at $m = 4.75\,$GeV (i.e. different values of $\xi = Q^2/m^2$)

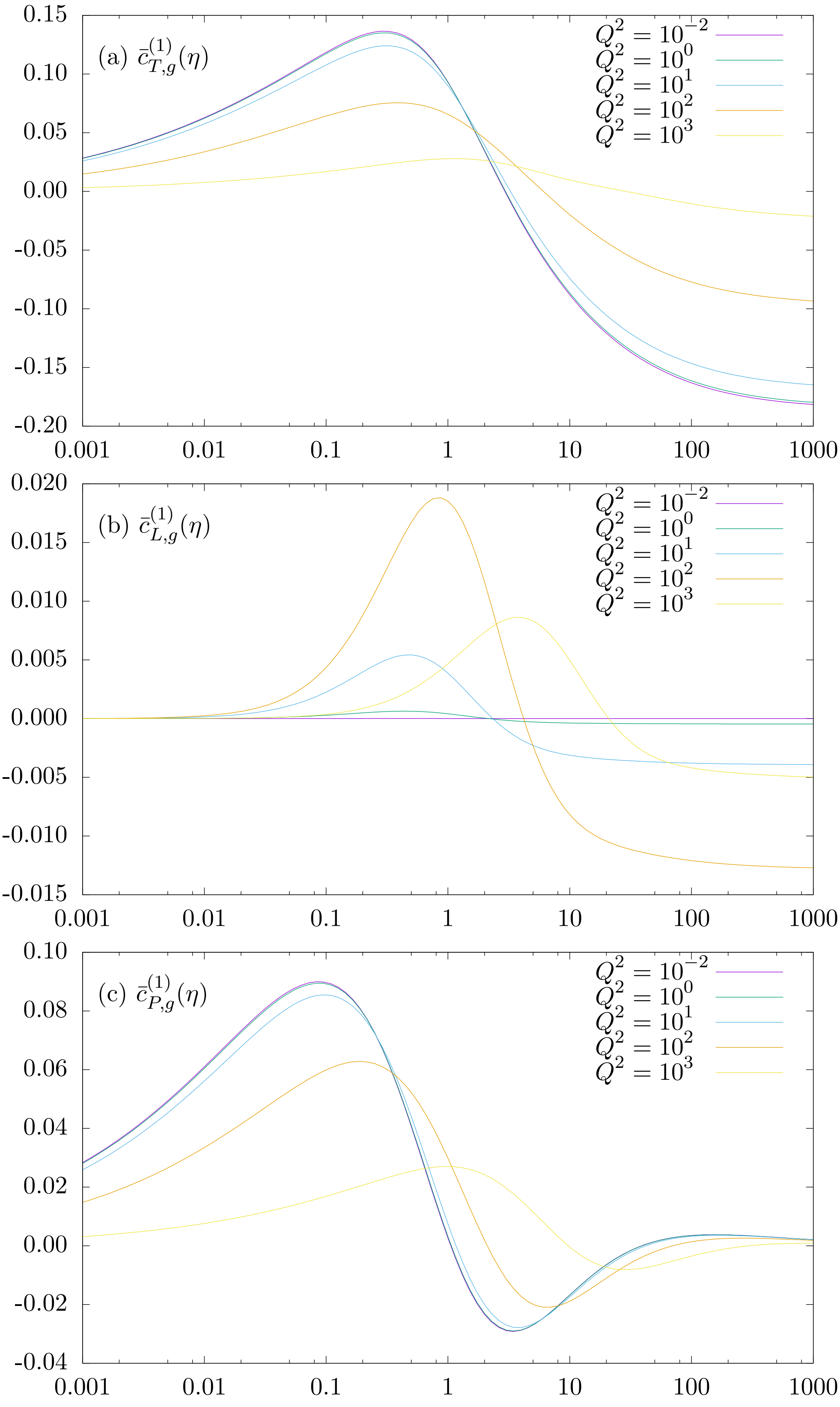


Figure 5: next-to-leading order scaling functions $\bar{c}^{(1)}_{k,g}(\eta,\xi) = \bar{c}^{R,(1)}_{k,g}(\eta,\xi) + \bar{c}^{F,(1)}_{k,g}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$) and $n_{lf} = 4$

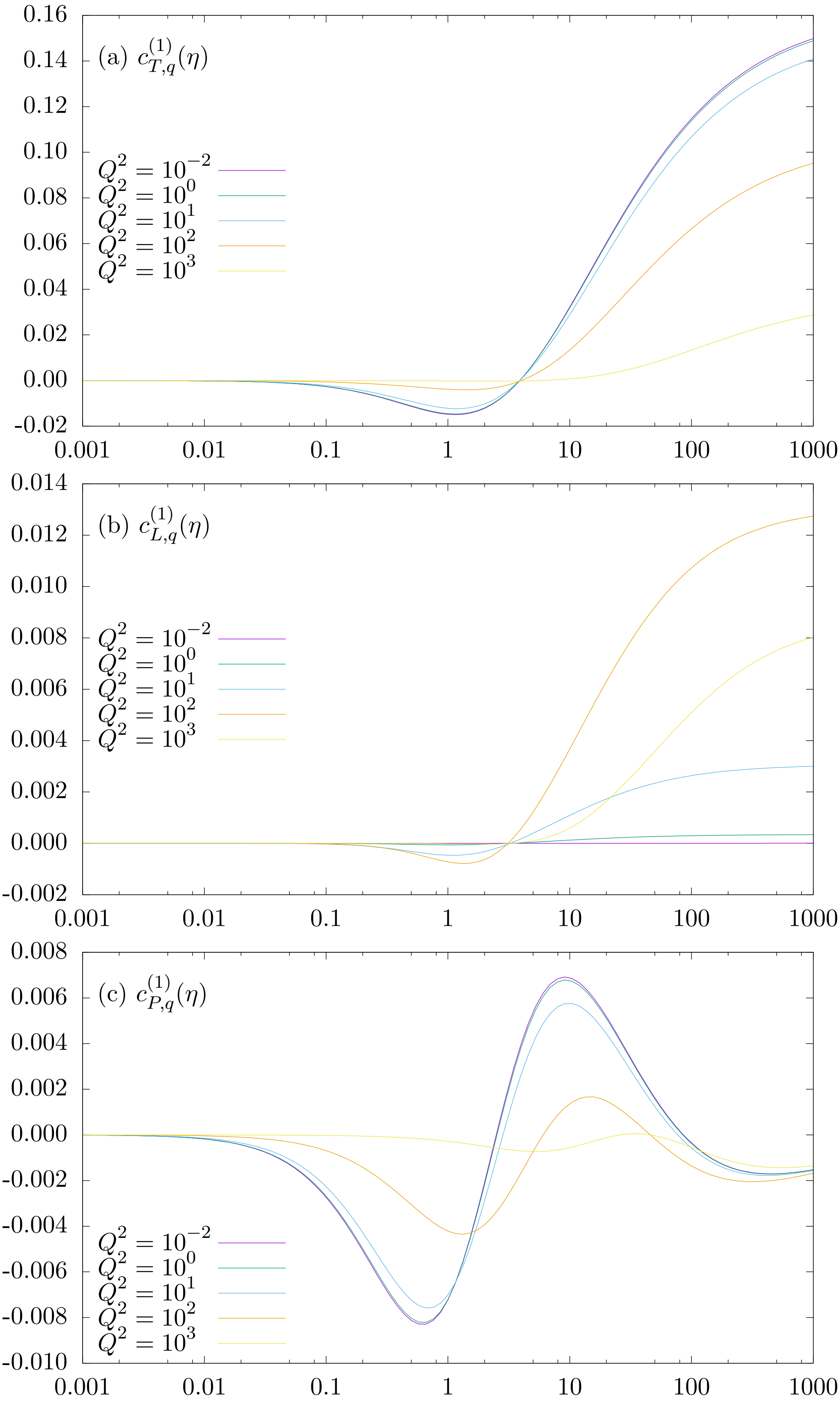


Figure 6: next-to-leading order scaling functions $c^{(1)}_{k,q}(\eta, \xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$). Note that [1, Fig. 9 (b)] is wrong.

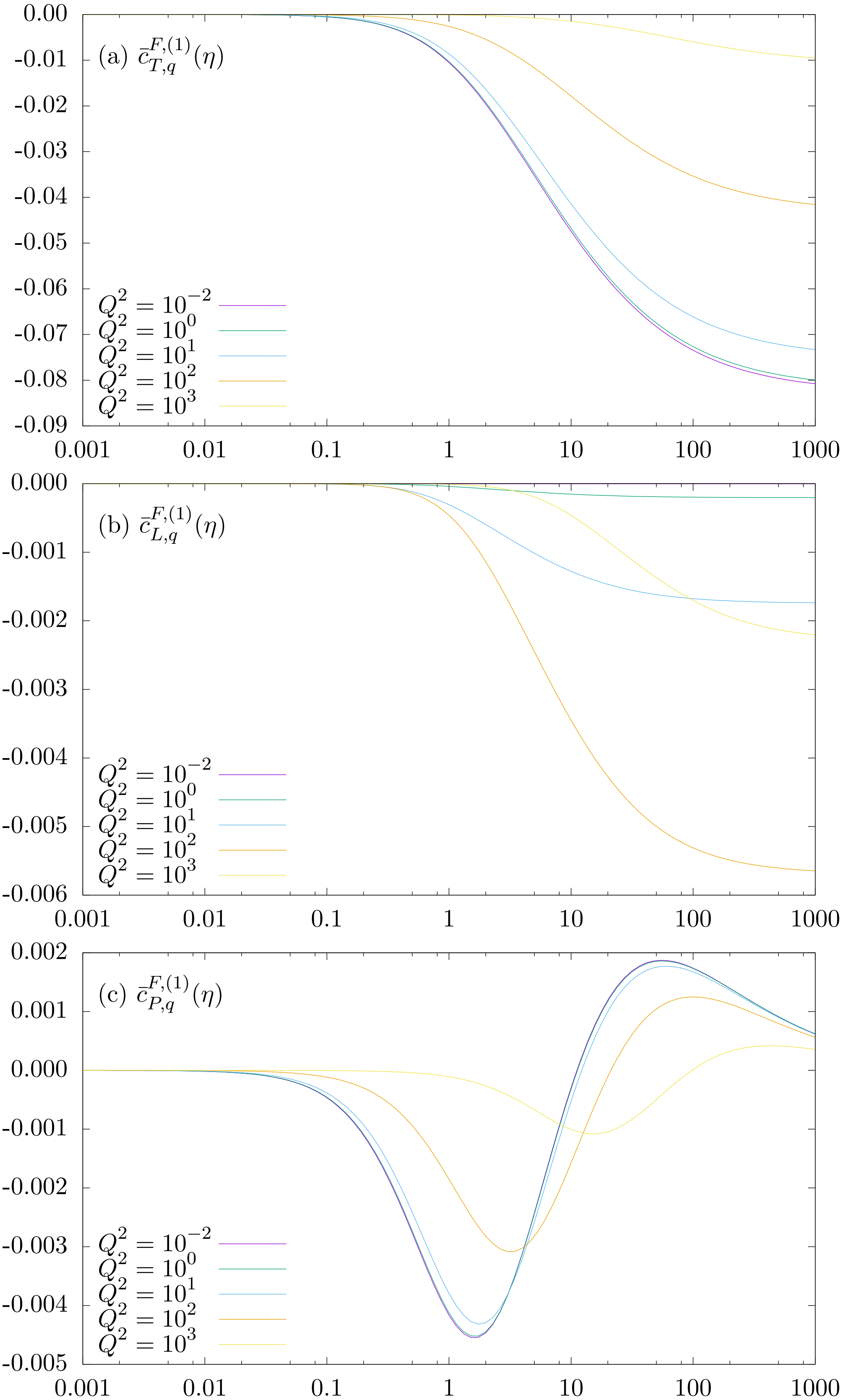


Figure 7: next-to-leading order scaling functions $\bar{c}_{k,q}^{F,(1)}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\text{GeV}^2$ at $m = 4.75\,\text{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

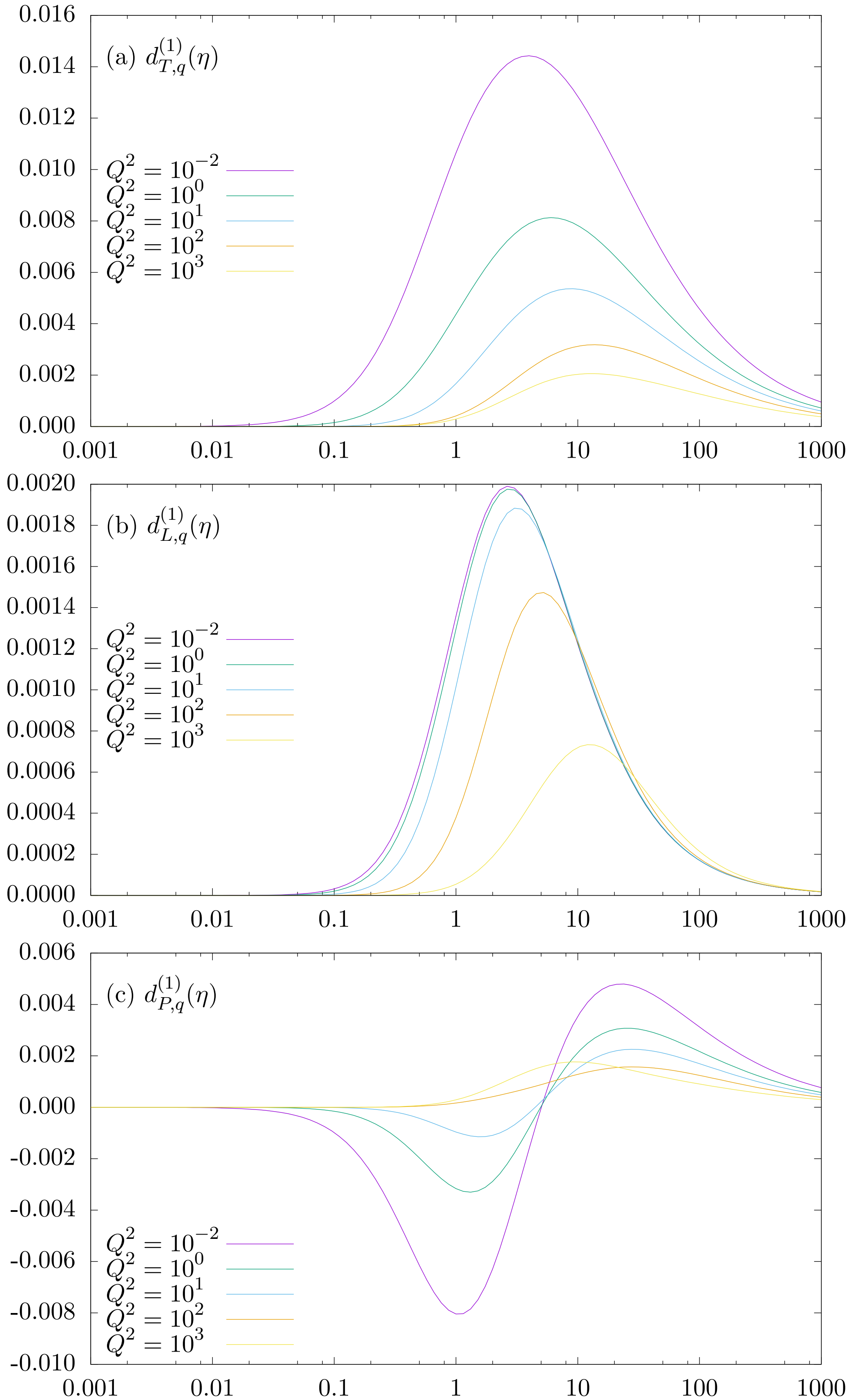


Figure 8: next-to-leading order scaling functions $d^{(1)}_{k,q}(\eta, \xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

## 6 Hadronic Results

The hadronic reaction to study is deep-inelastic lepton-proton scattering:

$$\ell^-(l_1) + \mathrm{p}(p) \to \ell^-(l_2) + Q(p_1)(\overline{Q}(p_2)) + X \tag{92}$$

where one either detects the heavy quark $Q$ *or* the heavy antiquark $\overline{Q}$ and $X$ stands for any final hadronic state allowed by quantum-number conservation. We define then the hadronic Bjorken variables

$$q = l_1 - l_2, \quad Q^2 = -q^2, \quad x = \frac{Q^2}{2p\cdot q}, \quad z = \frac{p\cdot q}{p\cdot l_1} \tag{93}$$

We can then define the measurable deep-inelastic hadron structure functions

$$F_k(x,Q^2,m^2) = \sum_{n=0}^{\infty} F_k^{(n)}(x,Q^2,m^2) \tag{94}$$

$$F_k^{(n)}(x,Q^2,m^2) = \frac{Q^2}{4\pi^2\alpha} \sum_{j\in\{\mathrm{g},q,\bar{q}\}} \int\limits_x^{z_{max}} \frac{dz}{z} f_j(x/z,\mu_F^2)\sigma_{k,j}^{(n)}(s,q^2,m^2) \tag{95}$$

where $k \in \{G,L,P\}$ denotes as usual projection, $z = Q^2/s'$, $z_{max} = Q^2/(4m^2+Q^2)$ and $f_j(x/z,\mu_F^2)$ denotes parton momentum density functions[18, 19]. We can then split the contributions whether there is a gluon in the initial state $F_{k,g}$ or a (anti)quark $F_{k,q}$. In leading order we find:

$$F_{k,g}^{(0)}(x,Q^2,m^2) = \frac{\alpha_s Q^2}{4\pi^2 m^2} e_H^2 \int\limits_x^{z_{max}} \frac{dz}{z} f_g(x/z,\mu_F^2) c_{k,g}^{(0)}(\eta,\xi) \tag{96}$$

We find for the gluonic part in next-to-leading order:

$$\begin{aligned}
&F_{k,g}^{(1)}(x,Q^2,m^2)\\
&= \frac{\alpha_s^2 Q^2}{\pi m^2} e_H^2 \int\limits_x^{z_{max}} \frac{dz}{z} f_g(x/z,\mu_F^2) \left(c_{k,g}^{(1)}(\eta,\xi) + \ln(\mu_F^2/m^2)\bar{c}_{k,g}^{F,(1)}(\eta,\xi) + \ln(\mu_R^2/m^2)\bar{c}_{k,g}^{R,(1)}(\eta,\xi)\right)
\end{aligned} \tag{97}$$

We find for the quark part in next-to-leading order:

$$\begin{aligned}
&F_{k,q}^{(1)}(x,Q^2,m^2)\\
&= \frac{\alpha_s^2 Q^2}{\pi m^2} e_H^2 \int\limits_x^{z_{max}} \frac{dz}{z} \left(\sum_{j=1}^{n_{lf}} f_{q(j)}(x/z,\mu_F^2) + f_{q(-j)}(x/z,\mu_F^2)\right)\\
&\qquad \cdot \left(c_{k,q}^{(1)}(\eta,\xi) + \ln(\mu_F^2/m^2)\bar{c}_{k,q}^{F,(1)}(\eta,\xi)\right)\\
&\quad + \frac{\alpha_s^2 Q^2}{\pi m^2} \int\limits_x^{z_{max}} \frac{dz}{z} \left(\sum_{j=1}^{n_{lf}} e_{q(j)}^2 \left(f_{q(j)}(x/z,\mu_F^2) + f_{q(-j)}(x/z,\mu_F^2)\right)\right) d_{k,q}(\eta,\xi)
\end{aligned} \tag{98}$$

where we used the PDG particle labeling[20]: $q(1) = u, q(-1) = \bar{u}, q(2) = d, q(-2) = \bar{d}$, $\ldots$ and $e_u = e_c = e_t = 2/3, e_d = e_s = e_b = -1/3$.

We can then also define some more practical functions:

$$
\begin{aligned}
F_2(x, Q^2, m^2) &= F_T(x, Q^2, m^2) + F_L(x, Q^2, m^2) && (99)\\
&= F_G(x, Q^2, m^2) + \frac{3}{2} F_L(x, Q^2, m^2) && (100)\\
F_1(x, Q^2, m^2) &= (F_2(x, Q^2, m^2) - F_L(x, Q^2, m^2))/(2x) && (101)\\
&= \left( F_G(x, Q^2, m^2) + \frac{1}{2} F_L(x, Q^2, m^2) \right) /(2x) && (102)\\
g_1(x, Q^2, m^2) &= F_P(x, Q^2, m^2)/(2x) && (103)
\end{aligned}
$$

and we define

$$
R_{k'}(x, Q^2, m^2) = \frac{F_{k'}^{(0)}(x, Q^2, m^2) + F_{k'}^{(1)}(x, Q^2, m^2)}{F_{k'}^{(0)}(x, Q^2, m^2)} \tag{104}
$$

with $k' \in \{2, L, P\}$ to better observe next-to-leading order effects.

We define the spin asymmetry by

$$
A_1(x, Q^2, m^2) = \frac{g_1(x, Q^2, m^2)}{F_1(x, Q^2, m^2)} = \frac{F_P(x, Q^2, m^2)}{F_2(x, Q^2, m^2) - F_L(x, Q^2, m^2)} \tag{105}
$$

For the plots we focused on charm production ($n_{lf} = 3$) with $m_c = 1.5\,\mathrm{GeV}$ and we used the two-loop running coupling of [21]:

$$
\alpha_s(\mu_R^2) = \frac{1}{\beta_0^4 \ln(\mu_R^2/\Lambda_4)} \left( 1 - \frac{\beta_1^4}{(\beta_0^4)^2} \ln(\ln(\mu_R^2/\Lambda_4)) \right) \tag{106}
$$

with the first two coefficients of the QCD beta function $\beta_0^f = (33 - 2f)/(12\pi)$ and $\beta_1^f = (306 - 38f)/(48\pi^2)$**FiXme Error: find CA?** and $\Lambda_4 = 0.194\,\mathrm{GeV}^2$. We set $\mu_F^2 = \mu_R^2 = 4m^2 - q^2$ in analogy to [1]. We used the PDF set MSTW2008nlo90cl[18, 22, 23] provided by LHAPDF[24] for the unpolarized structure functions $(F_2, F_1, F_G, F_L)$ and DSSV2014[19] for the polarized structure function $(g_1, F_G)$.

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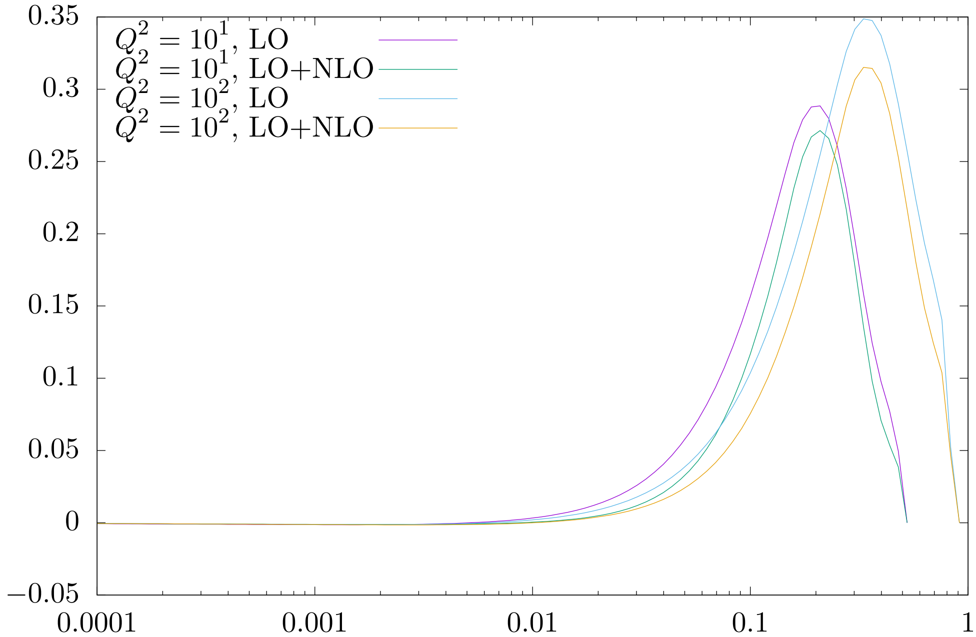


Figure 9: spin asymmetry $A_1(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

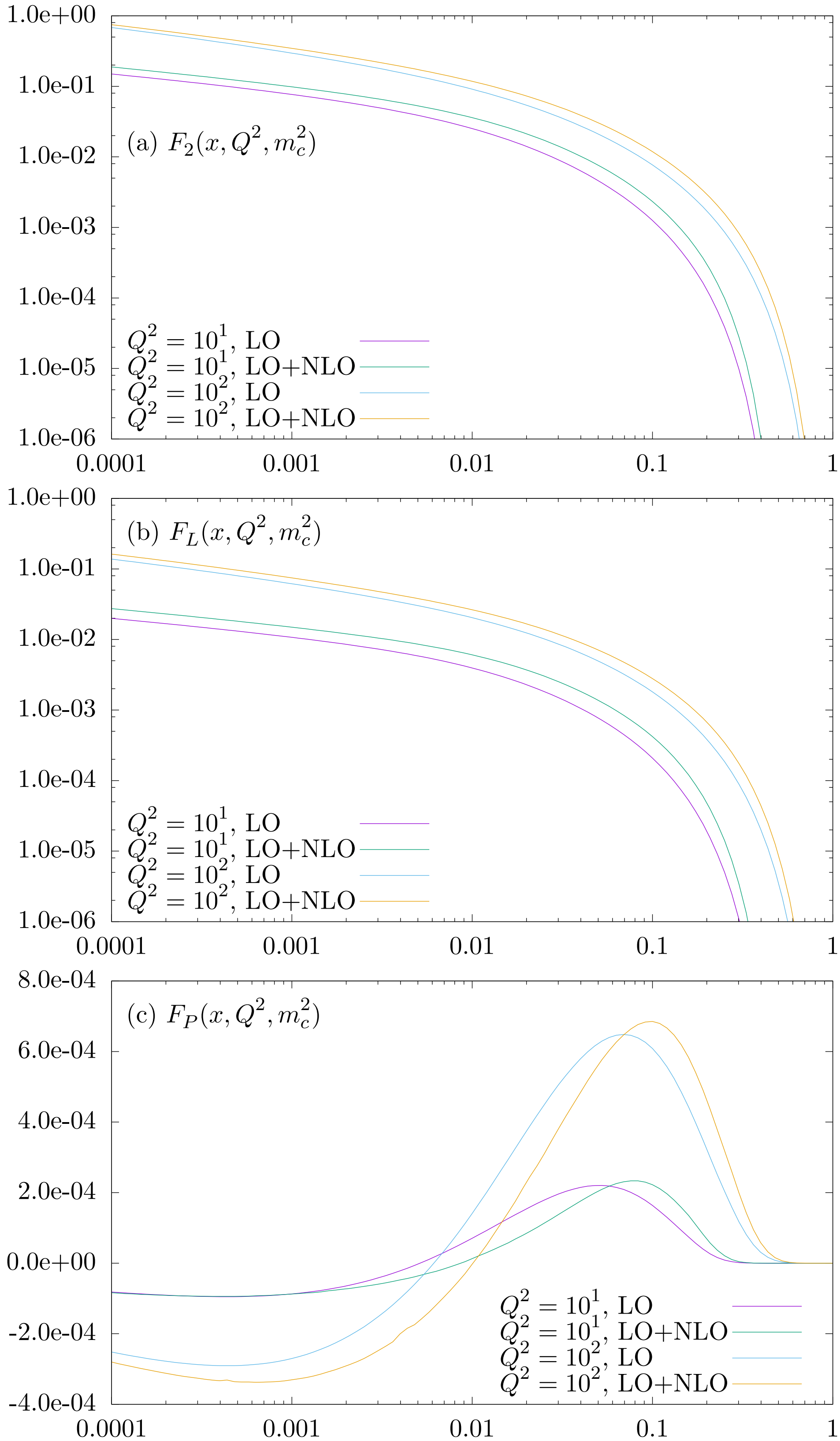


Figure 10: hadronic structure functions $F_k(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

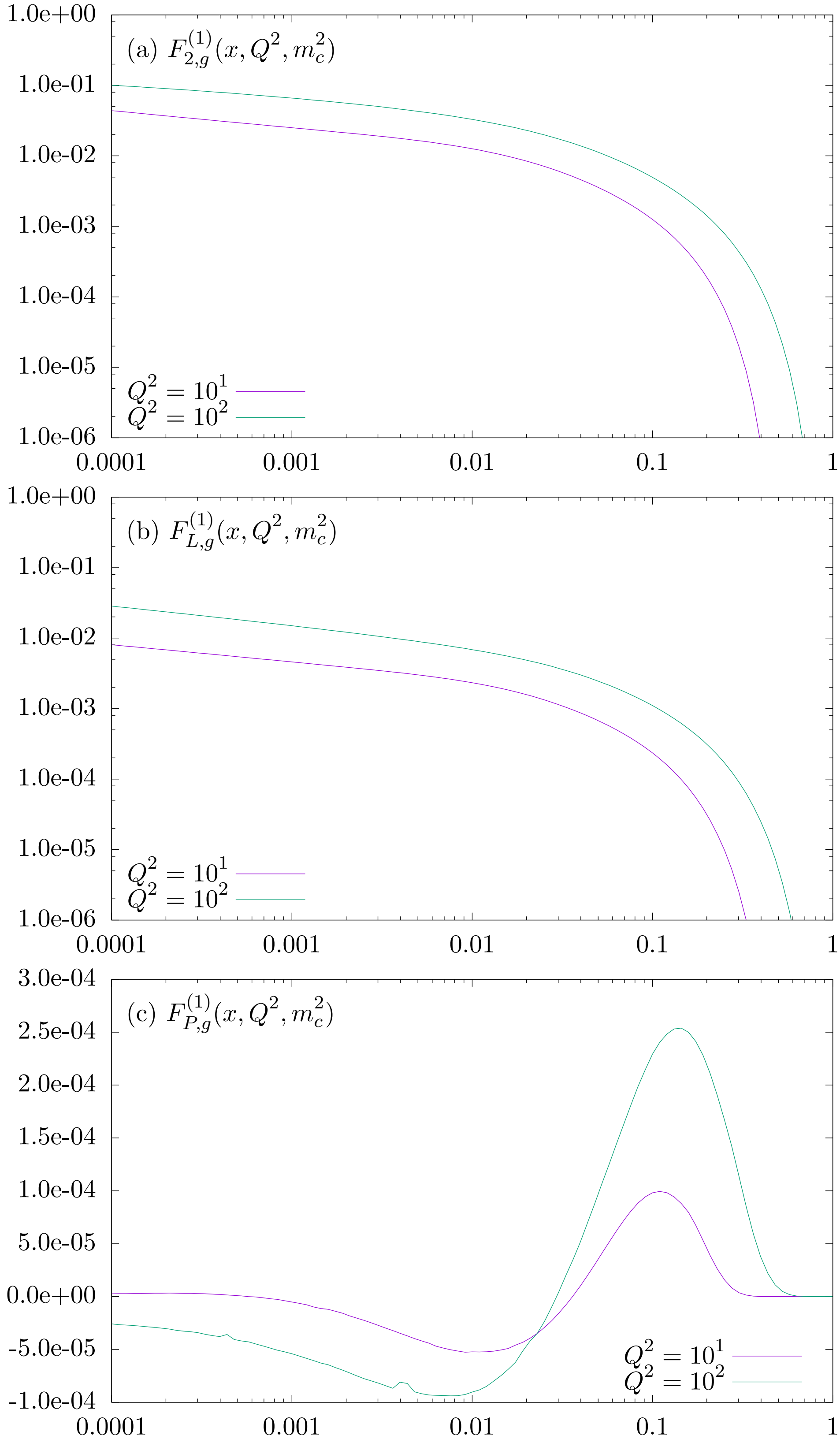


Figure 11: next-to-leading order hadronic structure functions $F^{(1)}_{k,g}(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

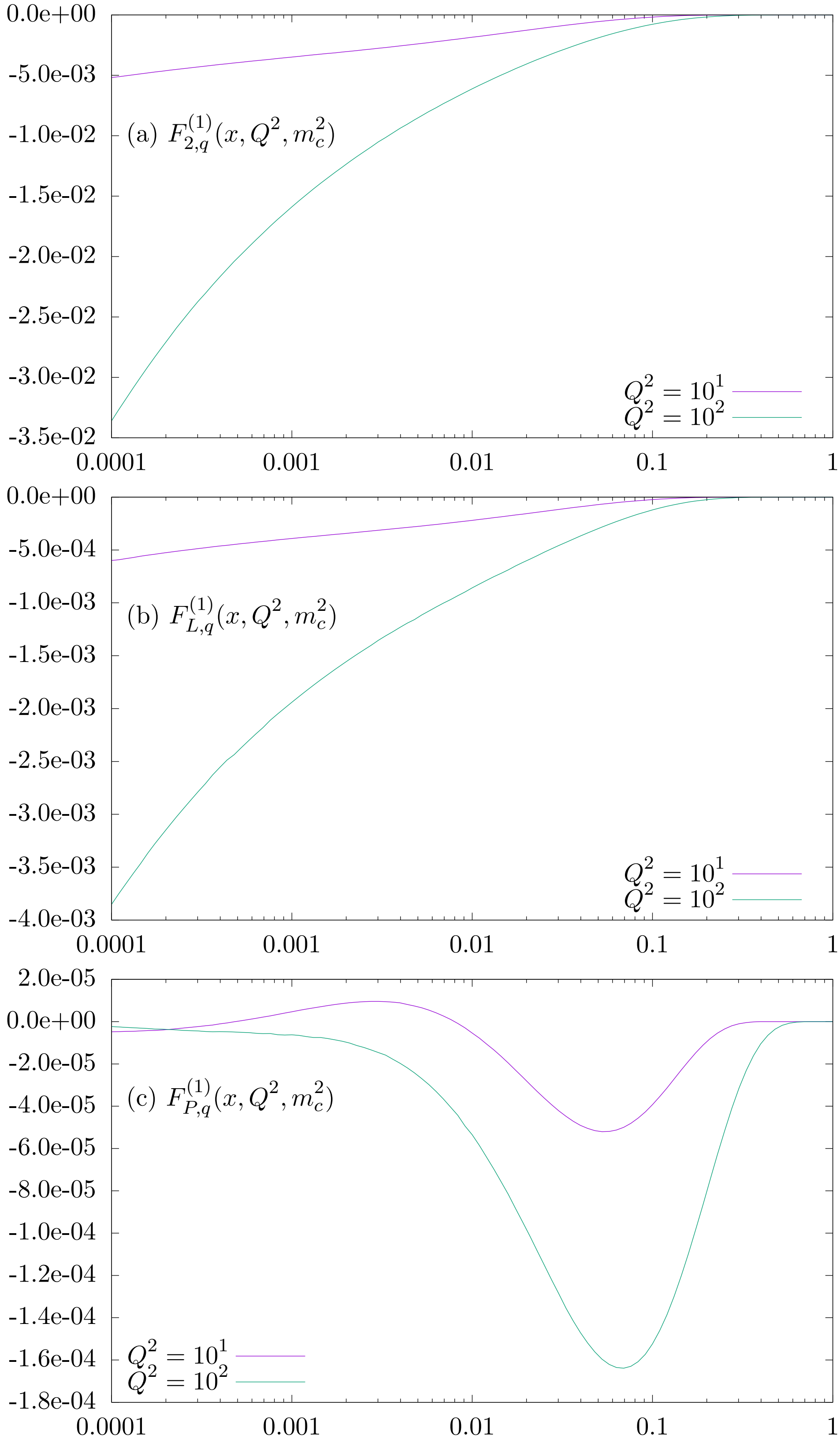


Figure 12: next-to-leading order hadronic structure functions $F^{(1)}_{k,q}(x,Q^2,m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of GeV$^2$

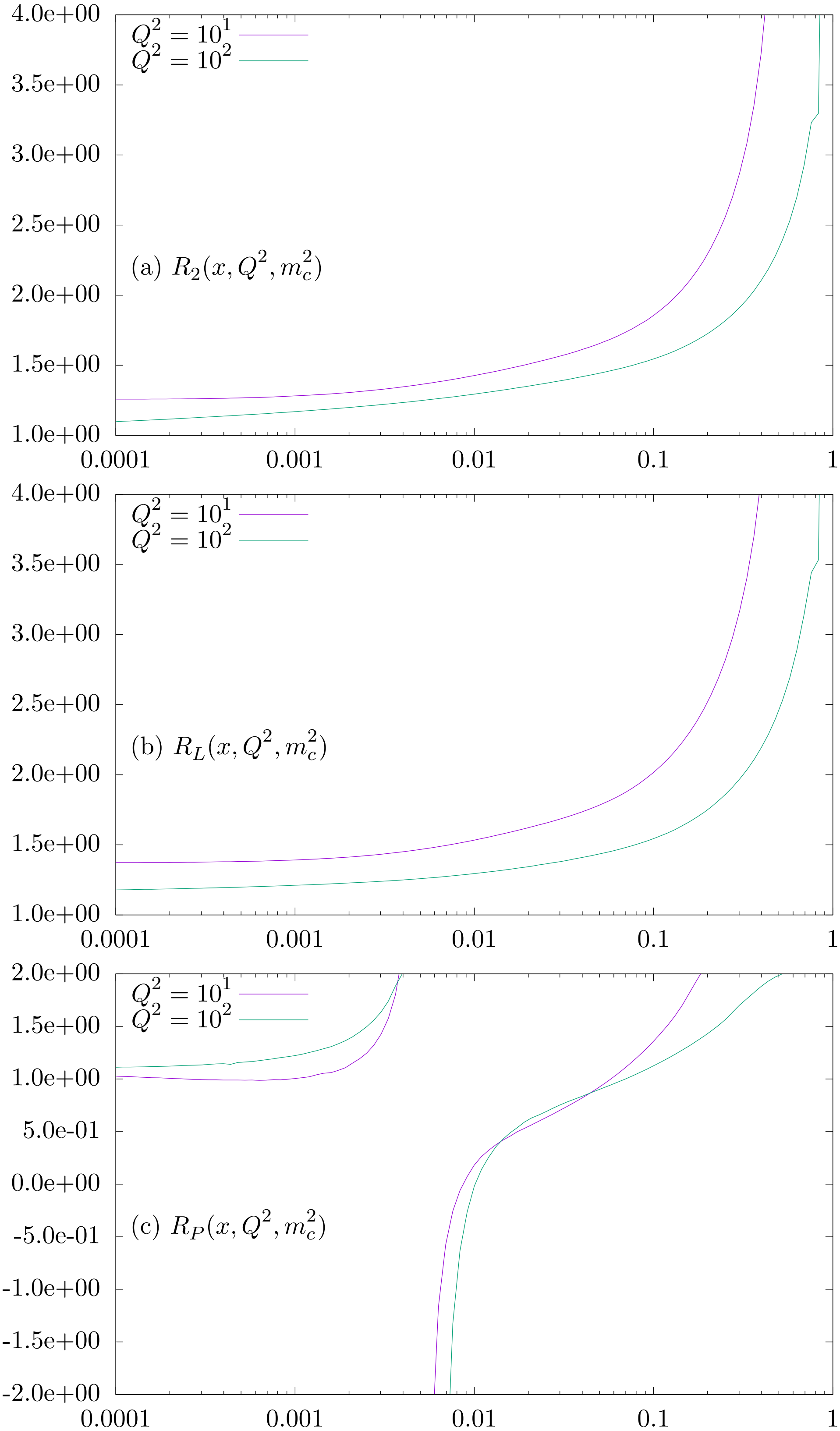


Figure 13: ratio of hadronic structure functions $R_{k'}^{(1)}(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of GeV$^2$

# 7 Summary

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## List of Corrections