

## 1 2 to 2 phase space

following [1]:

process:

$$\gamma^*(q) + g(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) \quad (1)$$

kinematics:

$$s = (q + k_1)^2 \quad s' = s - q^2 \quad (2)$$

$$t = (k_1 - p_1)^2 \quad t_1 = t - m^2 \quad (3)$$

$$u = (k_1 - p_2)^2 \quad u_1 = u - m^2 \quad (4)$$

use c.m.s. of incoming particles:

$$q = \left( \frac{s + q^2}{2\sqrt{s}}, 0, 0, \dots, -\frac{s - q^2}{2\sqrt{s}} \right) \quad (5)$$

$$k_1 = \frac{s - q^2}{2\sqrt{s}} (1, 0, 0, \dots, 1) \quad (6)$$

such that

$$q + k_1 = (\sqrt{s}, \vec{0}) \quad k_1^2 = 0 \quad (7)$$

for the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2} (1, 0, \beta \sin \theta, \dots, \beta \cos \theta) \quad (8)$$

$$p_2 = \frac{\sqrt{s}}{2} (1, 0, -\beta \sin \theta, \dots, -\beta \cos \theta) \quad (9)$$

with  $\beta = \sqrt{1 - 4m^2/s}$  such that

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \quad p_1^2 = p_2^2 = m^2 \quad (10)$$

use n-sphere:

$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \quad (11)$$

compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_1}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q + k_1 - p_1 - p_2) \delta(p_1^2 - m^2) \delta(p_2^2 - m^2) \quad (12)$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1 \delta((q + k_1 - p_2)^2 - m^2) \delta(p_1^2 - m^2) \quad (13)$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,\parallel} d^2 p_{1,\perp} d^{n-4} \hat{p}_1 \delta(s - 2p_{1,0}\sqrt{s}) \delta(p_{1,0}^2 - p_{1,\parallel}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \quad (14)$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,\parallel} dp_{1,\perp}^2 d^{n-4} \hat{p}_1 \delta(s/4 - p_{1,\parallel}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \quad (15)$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,\parallel} d\hat{p}_1^2 \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \quad (16)$$

$$= \frac{1}{2\sqrt{s}\Gamma((n-4)/2)(4\pi)^{(n-2)/2}} \int dp_{1,\parallel} d\hat{p}_1^2 (\hat{p}_1^2)^{(n-6)/2} \quad (17)$$

Integration borders are

$$p_{1,\parallel} \in \frac{\sqrt{s}}{2} \beta \cdot [-1, 1] \quad \hat{p}_1^2 \in \left( \frac{s\beta^2}{4} - p_{1,\parallel}^2 \right) \cdot [0, 1] \quad (18)$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2 (\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4} \left( \frac{s\beta^2}{4} - p_{1,\parallel}^2 \right)^{(n-4)/2} \quad (19)$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dp_{1,\parallel} \left( \frac{s\beta^2}{4} - p_{1,\parallel}^2 \right)^{(n-4)/2} \quad (20)$$

rewrite  $p_{1,\parallel}$  to  $\cos \theta$ :

$$p_{1,\parallel} = \frac{\sqrt{s}}{2} \beta \cos \theta \Rightarrow dp_{1,\parallel} = \frac{\sqrt{s}}{2} \beta d\cos \theta, \quad \cos \theta \in [-1, 1], \quad \hat{p}_1^2 \in \frac{s\beta^2}{4} (1 - \cos^2 \theta) \cdot [0, 1] \quad (21)$$

rewrite  $\cos \theta$  to  $t_1 = (k_1 - p_2)^2 - m^2$ :

$$\cos \theta = \frac{2t_1/s' + 1}{\beta} \Rightarrow d\cos \theta = \frac{2}{\beta s'} dt_1, \quad t_1 \in \frac{s'}{2} [-\beta-1, \beta-1], \quad \hat{p}_1^2 \in (-m^2 - \frac{st_1}{s'^2} (s' + t_1)) \cdot [0, 1] \quad (22)$$

$$p_{1,\parallel} = \sqrt{s} \left( \frac{t_1}{s'} + \frac{1}{2} \right) \Rightarrow dp_{1,\parallel} = \frac{\sqrt{s}}{s'} dt_1 \quad (23)$$

$$\Rightarrow PS_2 = \frac{1}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dt_1 \left( \frac{(t_1(u_1 - q^2) - s'm^2)s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} \quad (24)$$

## 2 2 to 3 phase space

following [2, 3, 1]:

process:

$$\gamma^*(q) + q(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) + q(k_2) \quad (25)$$

### 2.1 kinematic constraints

definitions of kinematic variables:

$$s = (q + k_1)^2 \Rightarrow 2qk_1 = s - q^2 \quad (26)$$

$$s_3 = (k_2 + p_2)^2 - m^2 \Rightarrow 2k_2p_2 = s_3 \quad (27)$$

$$s_4 = (k_2 + p_1)^2 - m^2 \Rightarrow 2k_2p_1 = s_4 \quad (28)$$

$$s_5 = (p_1 + p_2)^2 = -u_5 \Rightarrow 2p_1p_2 = s_5 - 2m^2 \quad (29)$$

$$t_1 = (k_1 - p_2)^2 - m^2 = t - m^2 \Rightarrow 2k_1p_2 = -t_1 \quad (30)$$

$$t' = (k_1 - k_2)^2 \Rightarrow 2k_1k_2 = -t' \quad (31)$$

$$u_1 = (q - p_2)^2 - m^2 = u - m^2 \Rightarrow 2qp_2 = -u_1 + q^2 \quad (32)$$

$$u_6 = (k_1 - p_1)^2 - m^2 \Rightarrow 2k_1p_1 = -u_6 \quad (33)$$

$$u_7 = (q - p_1)^2 - m^2 \Rightarrow 2qp_1 = -u_7 + q^2 \quad (34)$$

$$u' = (q - k_2)^2 \Rightarrow 2qk_2 = -u' + q^2 \quad (35)$$

impose momentum conservation:

$$q + k_1 = p_1 + p_2 + k_2 \quad (36)$$

contract with 2 times momentum:

$$2q^2 + s - q^2 = -u_7 + q^2 \quad -u_1 + q^2 \quad -u' + q^2 \Leftrightarrow 0 = s + u_1 + u_7 + u' - 2q^2 \quad (37)$$

$$s - q^2 + 0 = -u_6 \quad -t_1 \quad -t' \Leftrightarrow 0 = s + t_1 + t' + u_6 - q^2 \quad (38)$$

$$-u_7 + q^2 \quad -u_6 = 2m^2 \quad +s_5 - 2m^2 \quad +s_4 \Leftrightarrow 0 = s_4 + s_5 + u_6 + u_7 - q^2 \quad (39)$$

$$-u_1 + q^2 \quad -t_1 = s_5 - 2m^2 \quad +2m^2 \quad +s_3 \Leftrightarrow 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad (40)$$

$$-u' + q^2 \quad -t' = s_4 \quad +s_3 \quad +0 \Leftrightarrow 0 = s_3 + s_4 + t' + u' - q^2 \quad (41)$$

$$\frac{1}{2} ((37) + (38) + (40) - (39) - (41)) = 0 = s - q^2 + t_1 + u_1 - s_4 \quad (42)$$

$$\frac{1}{2} ((37) + (38) + (41) - (39) - (40)) = 0 = s - q^2 + t' + u' - s_5 \quad (43)$$

## 2.2 choose framework

use c.m.s. of recoiling heavy and light quark ( $Q(p_1)$  and  $q(k_2)$ ):

$$k_2 = (\omega_2, k_{2,x}, \omega_2 \sin \theta_1 \cos \theta_2, \omega_2 \cos \theta_1, \hat{k}_2) \quad (44)$$

$$p_1 = (E_1, -k_{2,x}, -\omega_2 \sin \theta_1 \cos \theta_2, -\omega_2 \cos \theta_1, -\hat{k}_2) \quad (45)$$

$$k_1 = (\omega_1, 0, 0, \omega_1, \hat{0}) \quad (46)$$

$$q = (q_0, 0, |\vec{p}_2| \sin \psi, |\vec{p}_2| \cos \psi - \omega_1, \hat{0}) \quad (47)$$

$$p_2 = (E_2, 0, |\vec{p}_2| \sin \psi, |\vec{p}_2| \cos \psi, \hat{0}) \quad (48)$$

light quark masses are already fixed:  $k_1^2 = 0 = k_2^2$

constraints:

$$q_0 + \omega_1 = E_1 + E_2 + \omega_2 \quad (49)$$

$$m^2 = p_1^2 = E_1^2 - \omega_2^2 \quad (50)$$

$$m^2 = p_2^2 = E_2^2 - |\vec{p}_2|^2 \quad (51)$$

$$q^2 = q_0^2 - |\vec{p}_2|^2 + 2 |\vec{p}_2| \omega_1 \cos \psi - \omega_1^2 \quad (52)$$

$$s = (q + k_1)^2 = (q_0 + \omega_1)^2 - |\vec{p}_2|^2 \quad (53)$$

$$t = (k_1 - p_2)^2 = (\omega_1 - E_2)^2 - |\vec{p}_2|^2 + 2 |\vec{p}_2| \omega_1 \cos \psi - \omega_1^2 \quad (54)$$

$$u = (q - p_2)^2 = (q_0 - E_2)^2 - \omega_1^2 \quad (55)$$

solve:

$$(53) - (52) + (54) - (51) + (55) = s - q^2 + t - m^2 + u \quad (56)$$

$$= s_4 + m^2 = (E_1 + \omega_2)^2 \quad (57)$$

$$(54) + (55) - (52) = t + u - q^2 = -2(E_1 + \omega_2)E_2 \quad (58)$$

$$\Rightarrow E_2 = -\frac{t + u - q^2}{2\sqrt{s_4 + m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4 + m^2}} \quad (59)$$

$$(57) \wedge (50) \Rightarrow \omega_2 = \frac{s_4}{2\sqrt{s_4 + m^2}} \quad (60)$$

$$(57) \Rightarrow E_1 = \frac{s_4 + 2m^2}{2\sqrt{s_4 + m^2}} \quad (61)$$

$$(53) + (55) - (51) = s + u - m^2 = 2q_0(E_1 + \omega_2) \quad (62)$$

$$\Rightarrow q_0 = \frac{s + u_1}{2\sqrt{s_4 + m^2}} \quad (63)$$

$$(54) - (52) = t - q^2 = (\omega_1 - E_2)^2 - q_0^2 \quad (64)$$

$$\Rightarrow \omega_1 = \frac{s_4 - u_1}{2\sqrt{s_4 + m^2}} \quad (65)$$

$$(51) \Rightarrow |\vec{p}_2| = \sqrt{E_2^2 - m^2} = \frac{\sqrt{(s - s_4)^2 - 4sm^2}}{2\sqrt{s_4 + m^2}} \quad (66)$$

$$(52) \Rightarrow \cos \psi = \frac{q^2 - q_0^2 + |\vec{p}_2|^2 + \omega_1^2}{2|\vec{p}_2|\omega_1} \quad (67)$$

$$= \frac{(u_1 + m^2)(q^2 - s + t_1) - (m^2 - q^2 - t_1)(s_4 - u_1)}{(s_4 - u_1)\sqrt{(s - s_4)^2 - 4sm^2}} \quad (68)$$

$$\Rightarrow \sin \psi = 2 \frac{\sqrt{s_4 + m^2} \sqrt{m^2(s - q^2)^2 + q^2 t_1(s - q^2 + t_1) - (s - q^2)t_1 u_1}}{(s_4 - u_1)\sqrt{(s - s_4)^2 - 4sm^2}} \quad (69)$$

$$t' = -2k_1 k_2 = -2\omega_1 \omega_2 (1 - \cos \theta_1) \quad (70)$$

$$u_6 = -2k_1 p_1 = -2\omega_1 (E_1 + \omega_2 \cos \theta_1) \quad (71)$$

$$(38) : \quad 0 = s + t_1 + t' + u_6 - q^2 \quad \checkmark \quad (72)$$

$t'$  is the only variable that can get collinear (for  $-q^2 > 0$ ).

$$s_3 = 2k_2 p_2 = 2\omega_2 (E_2 - |\vec{p}_2| (\cos \psi \cos \theta_1 + \sin \psi \sin \theta_1 \cos \theta_2)) \quad (73)$$

$$s_5 = (p_1 + p_2)^2 = 2m^2 + 2p_1 p_2 \quad (74)$$

$$= 2(m^2 + E_1 E_2 + \omega_2 |\vec{p}_2| (\cos \psi \cos \theta_1 + \sin \psi \sin \theta_1 \cos \theta_2)) \quad (75)$$

$$(40) : \quad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad \checkmark \quad (76)$$

$$u' = (q - k_2)^2 = q^2 - 2qk_2 \quad (77)$$

$$= q^2 - 2(q_0\omega_2 - \omega_2(|\vec{p}_2| \cos \psi - \omega_1) \cos \theta_1 - \omega_2 |\vec{p}_2| \sin \psi \sin \theta_1 \cos \theta_2) \quad (78)$$

$$u_7 = q^2 - 2qp_1 \quad (79)$$

$$= q^2 - 2(q_0E_1 + \omega_2(|\vec{p}_2| \cos \psi - \omega_1) \cos \theta_1 + \omega_2 |\vec{p}_2| \sin \psi \sin \theta_1 \cos \theta_2) \quad (80)$$

$$(37) : \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad \checkmark \quad (81)$$

## 2.3 phase space integrals

at phase space integration there occur integrations over propagators[4, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1 \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2 \sin^{n-4}(\theta_2) (a + b \cos(\theta_1))^{-k} (A + B \cos(\theta_1) + C \sin(\theta_1) \cos(\theta_2))^{-l} \quad (82)$$

$$= \int d\Omega_n (a + b \cos(\theta_1))^{-k} (A + B \cos(\theta_1) + C \sin(\theta_1) \cos(\theta_2))^{-l} \quad (83)$$

the integrals can be further destinguished by the range of  $k, l$  and the type of collinearity (following the notation in [4]):

- "non collinear":  $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \rightarrow I_{0,n}^{(k,l)}$
- "single collinear a":  $a = -b \wedge A^2 \neq B^2 + C^2 \rightarrow I_{a,n}^{(k,l)}$
- "single collinear A":  $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \rightarrow I_{A,n}^{(k,l)}$
- "double collinear":  $a = -b \wedge A = -\sqrt{B^2 + C^2} \rightarrow I_{aA,n}^{(k,l)}$

Use  $n = 4 + \epsilon$ .

### 2.3.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int_0^\pi dt \sin^{\nu-3}(t) \cos^q(t) \quad (84)$$

It is [5, eq. 5.12.6]:

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha B((\alpha+\beta+1)/2, (\alpha-\beta+1)/2)} \quad \text{if } \Re(\alpha) > 0 \quad (85)$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{B((n-1)/2, (n-1)/2)} \quad (86)$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{B((n-2)/2, (n-2)/2)} = B((n-3)/2, 1/2) \quad (87)$$

If  $q$  is odd:  $\hat{I}^{(q)} = 0$ , due to symmetry of kernel; if  $q$  is even:  $q = 2p$  with  $p \in \mathbb{N}$ :

$$\hat{I}^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t) dt \quad (88)$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2 + (k-p), (\nu-1)/2 - (k-p))} \quad (89)$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^p \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2 + l, (\nu-1)/2 - l)} \quad (90)$$

$$= \frac{\pi \Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2) \Gamma(\frac{n-1}{2} + p) \Gamma(\frac{n-1}{2} + p)} \left( \frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2} + p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2} - p)}{\Gamma(\frac{\nu-1}{2})} \right. \\ \left. + 2 \sum_{l=1}^p \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2} + p)}{\Gamma(\frac{\nu-1}{2} + l)} \frac{\Gamma(\frac{\nu-1}{2} - p)}{\Gamma(\frac{\nu-1}{2} - l)} \right) \quad (91)$$

$$= \frac{2^{3-\nu} \pi \Gamma(\nu-1)}{(\nu-2) \Gamma(\frac{n-1}{2} + p) \Gamma(\frac{n-1}{2} + p)} \cdot \frac{\Gamma(\frac{\nu-1}{2} - p)}{2^p \Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left( \frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2} + p)}{\Gamma(\frac{\nu-1}{2})} \right. \\ \left. + 2 \sum_{l=1}^p \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2} + p)}{\Gamma(\frac{\nu-1}{2} + l)} \frac{\Gamma(\frac{\nu-1}{2} - p)}{\Gamma(\frac{\nu-1}{2} - l)} \right) \quad (92)$$

TODO: prove

$$p! \left( \frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2} + p)}{\Gamma(\frac{\nu-1}{2})} + 2 \sum_{l=1}^p \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2} + p)}{\Gamma(\frac{\nu-1}{2} + l)} \frac{\Gamma(\frac{\nu-1}{2} - p)}{\Gamma(\frac{\nu-1}{2} - l)} \right) \quad (93)$$

$$= \frac{1}{p!} \frac{\Gamma(-\frac{1}{2} + p)}{\Gamma(-\frac{1}{2})} + 2 \sum_{l=1}^p \frac{(-1)^l p!}{(p+l)!(p-l)!} \frac{\Gamma(-\frac{1}{2} + p)}{\Gamma(-\frac{1}{2} + l)} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2} - l)} \quad (94)$$

$$= 1 \quad (95)$$

$$\Rightarrow \hat{I}^{(2p)}(\nu) = \frac{2^{3-\nu} \pi \Gamma(\nu-1)}{(\nu-2) \Gamma(\frac{n-1}{2} + p) \Gamma(\frac{n-1}{2} - p)} \cdot \frac{\Gamma(\frac{\nu-1}{2} - p)}{2^p \Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \quad (96)$$

$$= \frac{\sqrt{\pi} (2p)! \Gamma((\nu-2)/2)}{2^{2p} p! \Gamma(\frac{\nu-1}{2} + p)} \quad (97)$$

FiXme  
Error:  
prove

### 2.3.2 any collinearity and $-k, -l \in \mathbb{N}_0$

If  $-k, -l \in \mathbb{N}_0$   $I_n^{(k,l)}$  can always be reduced in a straight forward manner to combinations of  $\hat{I}^{(q)}(n)$  and this way one finds[4, Ch. 5][2, App. C]:

$$I_n^{(0,0)} = \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n) = \frac{2\pi}{n-3} \quad (98)$$

$$I_4^{(0,0)} = 2\pi \quad (99)$$

$$I_n^{(-1,0)} = \hat{I}^{(0)}(n-1) \cdot (a\hat{I}^{(0)}(n) + b\hat{I}^{(1)}(n)) = \frac{2\pi a}{n-3} \quad (100)$$

$$I_4^{(-1,0)} = 2\pi a \quad (101)$$

$$I_n^{(0,-1)} = \hat{I}^{(0)}(n-1) \cdot (A\hat{I}^{(0)}(n) + B\hat{I}^{(1)}(n)) + C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \quad (102)$$

$$= \frac{2\pi A}{n-3} \quad (103)$$

$$I_4^{(0,-1)} = 2\pi A \quad (104)$$

$$I_n^{(-2,0)} = \hat{I}^{(0)}(n-1) \cdot (a^2\hat{I}^{(0)}(n) + 2ab\hat{I}^{(1)}(n) + b^2\hat{I}^{(2)}(n)) \quad (105)$$

$$= 2\pi \left( \frac{a^2(n-1) + b^2}{(n-1)(n-3)} \right) \quad (106)$$

$$I_4^{(-2,0)} = 2\pi(a^2 + b^2/3) \quad (107)$$

$$I_n^{(0,-2)} = \hat{I}^{(0)}(n-1) \cdot (A^2\hat{I}^{(0)}(n) + B^2\hat{I}^{(2)}(n)) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \quad (108)$$

$$= 2\pi \left( \frac{A^2(n-1) + B^2 + C^2}{(n-1)(n-3)} \right) \quad (109)$$

$$I_4^{(0,-2)} = 2\pi(A^2 + (B^2 + C^2)/3) \quad (110)$$

$$I_n^{(-1,-1)} = \hat{I}^{(0)}(n-1) \cdot (aA\hat{I}^{(0)}(n) + bB\hat{I}^{(2)}(n)) = 2\pi \left( \frac{aA(n-1) + bB}{(n-1)(n-3)} \right) \quad (111)$$

$$I_4^{(-1,-1)} = 2\pi(aA + bB/3) \quad (112)$$



### 2.3.3 single collinear a

If  $-l \in \mathcal{N}$  one finds:

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1 - \cos(t))^k} \cos^q(t) dt \quad (113)$$

$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1 - \cos^2(t))^k} \cos^q(t) (1 + \cos(t))^k dt \quad (114)$$

$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t) (1 + \cos(t))^k dt \quad (115)$$

$$= \sum_{l=0}^k \binom{k}{l} \hat{I}^{(q+l)}(\nu - 2k) \quad (116)$$

this way one finds[4, Ch. 5][2, App. C]:

$$I_{a,n}^{(1,0)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n-2) \quad (117)$$

$$= \frac{2\pi}{a(n-4)} \quad (118)$$

$$I_{a,n}^{(1,-1)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \left( A \hat{I}^{(0)}(n-2) + B \hat{I}^{(2)}(n-2) \right) \quad (119)$$

$$= \frac{2\pi}{a} \frac{(A(n-3) + B)}{(n-3)(n-4)} \approx \frac{2\pi}{a} \left( \frac{A+B}{\epsilon} - 2B + O(\epsilon) \right) \quad (120)$$

$$I_{a,n}^{(1,-2)} = \frac{1}{a} \left( \hat{I}^{(0)}(n-1) \cdot \left( A^2 \hat{I}^{(0)}(n-2) + (B^2 + 2AB) \hat{I}^{(2)}(n-2) \right) + C^2 \hat{I}^{(2)}(n-1) \hat{I}^{(0)}(n) \right) \quad (121)$$

$$= \frac{2\pi}{a} \left( \frac{A^2}{n-4} + \frac{2AB + B^2}{(n-4)(n-3)} + \frac{C^2}{(n-3)(n-2)} \right) \quad (122)$$

$$\approx \frac{2\pi}{a} \left( \frac{(A+B)^2}{\epsilon} + \frac{C^2}{2} - 2AB - B^2 + O(\epsilon) \right) \quad (123)$$

$$I_{a,n}^{(2,0)} = \frac{1}{a^2} \hat{I}^{(0)}(n-1) \cdot \left( \hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4) \right) \quad (124)$$

$$= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2} + O(\epsilon) \quad (125)$$

$$I_{a,n}^{(2,-1)} = \frac{1}{a^2} \hat{I}^{(0)}(n-1) \cdot \left( A \left( \hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4) \right) + 2B\hat{I}^{(2)}(n-4) \right) \quad (126)$$

$$= \frac{2\pi}{a^2} \left( \frac{A}{n-6} + \frac{2B}{(n-6)(n-4)} \right) \approx -\frac{2\pi}{a^2} \left( \frac{B}{\epsilon} + \frac{A+B}{2} \right) + O(\epsilon) \quad (127)$$

$$I_{a,n}^{(2,-2)} = \frac{1}{a^2} \left( \hat{I}^{(0)}(n-1) \cdot \left( A^2(\hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4)) + 4AB\hat{I}^{(2)}(n-4) \right. \right. \\ \left. \left. + B^2(\hat{I}^{(2)}(n-4) + \hat{I}^{(4)}(n-4)) \right) + C^2\hat{I}^{(2)}(n-1)(\hat{I}^{(0)}(n-2) + \hat{I}^{(2)}(n-2)) \right) \quad (128)$$

$$= \frac{2\pi}{a^2} \left( \frac{A^2}{n-6} + \frac{4AB}{(n-6)(n-4)} + \frac{B^2n}{(n-6)(n-4)(n-3)} + \frac{C^2}{(n-4)(n-3)} \right) \quad (129)$$

$$\approx \frac{2\pi}{a^2} \left( \frac{-2AB - 2B^2 + C^2}{\epsilon} + \frac{B^2 - A^2}{2} - AB - C^2 + O(\epsilon) \right) \quad (130)$$

$$(131)$$

It is [4, Ch. 5]:

$$I_{a,n}^{(1,1)} = \frac{\pi}{a(A+B)} \left( \frac{2}{\epsilon} + \ln \left( \frac{(A+B)^2}{A^2 - B^2 - C^2} \right) \right) + O(\epsilon) \quad (132)$$

$$I_{a,n}^{(2,1)} = \frac{\pi}{a^2(A+B)} \left( \frac{B^2 + AB + C^2}{(A+B)^2} \left( \frac{2}{\epsilon} + \ln \left( \frac{(A+B)^2}{A^2 - B^2 - C^2} \right) \right) \right. \\ \left. - \frac{2C^2}{(A+B)^2} - 1 \right) + O(\epsilon) \quad (133)$$

From this are the following integrals derived [4, Ch. 5]:

$$I_{a,n}^{(1,2)} = -\frac{\partial}{\partial A} I_{a,n}^{(1,1)} \quad (134)$$

$$= \frac{\pi}{a(A+B)^2} \left( \frac{2}{\epsilon} + \ln \left( \frac{(A+B)^2}{A^2 - B^2 - C^2} \right) + \frac{2A(A+B)}{A^2 - B^2 - C^2} - 2 \right) + O(\epsilon) \quad (135)$$

$$I_{a,n}^{(2,2)} = -\frac{\partial}{\partial A} I_{a,n}^{(2,1)} \quad (136)$$

$$= \frac{\pi}{a^2(A+B)^2} \left( \frac{2B^2 + 2AB + 3C^2}{(A+B)^2} \left( \frac{2}{\epsilon} + \ln \left( \frac{(A+B)^2}{A^2 - B^2 - C^2} \right) \right) \right. \\ \left. + \frac{2A^2}{A^2 - B^2 - C^2} - \frac{8C^2}{(A+B)^2} - 3 \right) + O(\epsilon) \quad (137)$$

If  $-k \in \mathcal{N}$  use  $I_0$  with  $b = -a$ .

### 2.3.4 double collinear

as said in [4, Ch. 5]: if  $0 \leq -\frac{C}{A}, \frac{B}{A} \leq 1$  use [3, eq. A11] with  $\cos \kappa = -\frac{B}{A}$ :

$$I_{aA,n}^{(k,l)} = \frac{2\pi 2^{-(k+l)}}{a^k A^l} \frac{\Gamma(1+\epsilon)}{\Gamma^2(1+\epsilon/2)} B(1+\frac{\epsilon}{2}-k, 1+\frac{\epsilon}{2}-l) {}_2F_1\left(k, l; 1+\frac{\epsilon}{2}; \frac{A-B}{2A}\right) \quad (138)$$

but we will not need it here.

### 2.3.5 non collinear

If  $-l \in \mathcal{N}$  the  $\theta_2$  integration can be performed using the integral helper and the problem reduces then to the following integral:

$$\begin{aligned} \hat{I}_0^{(k,q,p)}(\epsilon) &= \int_0^\pi d\theta_1 \frac{\sin^{1+\epsilon}(\theta_1) \sin^q(\theta_1) \cos^p(\theta_1)}{(a + b \cos(\theta_1))^k} \\ &= \frac{1}{2a^k} \left( (1 + (-1)^p) B\left(\frac{2+q+\epsilon}{2}, \frac{1+p}{2}\right) {}_3F_2\left(\frac{1+k}{2}, \frac{k}{2}, \frac{1+p}{2}; \frac{1}{2}, \frac{3+q+p+\epsilon}{2}; \frac{b^2}{a^2}\right) \right. \\ &\quad \left. + \frac{b}{a} k (-1 + (-1)^p) B\left(\frac{2+q+\epsilon}{2}, \frac{2+p}{2}\right) {}_3F_2\left(\frac{1+k}{2}, \frac{2+k}{2}, \frac{2+p}{2}; \frac{3}{2}, \frac{4+q+p+\epsilon}{2}; \frac{b^2}{a^2}\right) \right) \end{aligned} \quad (139)$$

for  $k = 1$  this simplifies to

$$\begin{aligned} \hat{I}_0^{(1,q,p)}(\epsilon) &= \int_0^\pi d\theta_1 \frac{\sin^{1+\epsilon}(\theta_1) \sin^q(\theta_1) \cos^p(\theta_1)}{(a + b \cos(\theta_1))} \\ &= \frac{1}{2a} \left( (1 + (-1)^p) B\left(\frac{2+q+\epsilon}{2}, \frac{1+p}{2}\right) {}_2F_1\left(1, \frac{1+p}{2}; \frac{3+q+p+\epsilon}{2}; \frac{b^2}{a^2}\right) \right. \\ &\quad \left. + \frac{b}{a} (-1 + (-1)^p) B\left(\frac{2+q+\epsilon}{2}, \frac{2+p}{2}\right) {}_2F_1\left(1, \frac{2+p}{2}; \frac{4+q+p+\epsilon}{2}; \frac{b^2}{a^2}\right) \right) \end{aligned} \quad (141)$$

*TODO: structure does NOT match [4, Ch. 5] - neither A,B,C nor  $\epsilon$*

For  $I_{0,n}^{(1,1)}$  one finds [4, Ch. 5]:

$$I_{0,n}^{(1,1)} = \frac{\pi}{\sqrt{X}} \ln \left( \frac{aA - bB + \sqrt{X}}{aA - bC - \sqrt{X}} \right) \quad (143)$$

$$\text{with } X = (aB - bA)^2 - (a^2 - b^2)C^2 \quad (144)$$

FiXme  
Error: fix  
to Bojak

$I_{0,n}^{(1,-3)}$  is given by [4, Ch. 5] and from those two all other integrals can be deduced using the techniques described in [4, Ch. 5]: increase  $k$  or  $l$  by differentiation or interchange  $k$  and  $l$  by a rotation:

$$I_n^{(k,l)} = I_n^{(l,k)} \left( a \leftrightarrow A, b \rightarrow -\sqrt{B^2 + C^2}, B \rightarrow \frac{-bB}{\sqrt{B^2 + C^2}}, C \rightarrow \frac{-bC}{\sqrt{B^2 + C^2}} \right) \quad (145)$$

## 2.4 needed set in matrix element

define a shortcut:

$$\mathcal{V}_{a,b}(x, y) = \left( x^k y^l \right)_{k=0..a, l=0..b} \quad (146)$$

It is

$$A_{G,1} = \sum_{k,l=0}^3 (\mathcal{C}_{A_{G,1}})_{(k,l)} t'^{-2+k} u_7^{-2+l} = \text{tr} \left( \mathcal{C}_{A_{G,1}} \frac{\mathcal{V}_{3,3}(t', u_7)^t}{t'^2 u_7^2} \right) \quad (147)$$

$$A_{L,1} = \sum_{k=0}^4 \sum_{l=0}^2 (\mathcal{C}_{A_{L,1}})_{(k,l)} t'^{-2+k} u_7^{-2+l} = \text{tr} \left( \mathcal{C}_{A_{L,1}} \frac{\mathcal{V}_{4,2}(t', u_7)^t}{t'^2 u_7^2} \right) \quad (148)$$

and we will thus need the integrals

$$(\mathcal{I}_{A_{G,1}})_{(k,l)} = \int d\Omega_n \frac{1}{t'^2 u_7^2} (\mathcal{V}_{3,3}(t', u_7))_{(k,l)} \quad (149)$$

$$(\mathcal{I}_{A_{L,1}})_{(k,l)} = \int d\Omega_n \frac{1}{t'^2 u_7^2} (\mathcal{V}_{4,2}(t', u_7))_{(k,l)} \quad (150)$$

with

$$a(t') = -b(t') = -2\omega_1 \omega_2 = -\frac{s_4(s_4 - u_1)}{2(s_4 + m^2)} \quad (151)$$

$$A(u_7) = q^2 - 2q_0 E_1 = q^2 - \frac{(s_4 + 2m^2)(s + u_1)}{2(s_4 + m^2)} \quad (152)$$

$$B(u_7) = -2\omega_2(|\vec{p}_2| \cos \psi - \omega_1) = \frac{s_4}{2} \left( 1 - \frac{s + u_1}{s_4 + m^2} + \frac{s - q^2 - t_1}{s_4 - u_1} \right) \quad (153)$$

$$C(u_7) = -2\omega_2 |\vec{p}_2| \sin \psi \quad (154)$$

that is  $I_{a,n}^{(-2...2, -1...2)}$ . With this we get

$$\int d\Omega_n A_{j,1} = \text{tr}(\mathcal{C}_{A_{j,1}} (\mathcal{I}_{A_{j,1}})^t) \quad j = G, L \quad (155)$$

It is

$$A_{G,2} = \sum_{k,l=0}^3 (\mathcal{C}_{A_{G,2}})_{(k,l)} s_5^{-2+k} u'^{-2+l} = \text{tr} \left( \mathcal{C}_{A_{G,2}} \frac{\mathcal{V}_{3,3}(s_5, u')^t}{s_5^2 u'^2} \right) \quad (156)$$

$$A_{L,2} = \sum_{k=0}^4 \sum_{l=0}^3 (\mathcal{C}_{A_{L,2}})_{(k,l)} s_5^{-2+k} u'^{-2+l} = \text{tr} \left( \mathcal{C}_{A_{L,2}} \frac{\mathcal{V}_{4,3}(s_5, u')^t}{s_5^2 u'^2} \right) \quad (157)$$

and we will thus need the integrals

$$(\mathcal{I}_{A_{G,2}})_{(k,l)} = \int d\Omega_n \frac{1}{s_5^2 u'^2} (\mathcal{V}_{3,3}(s_5, u'))_{(k,l)} \quad (158)$$

$$(\mathcal{I}_{A_{L,2}})_{(k,l)} = \int d\Omega_n \frac{1}{s_5^2 u'^2} (\mathcal{V}_{4,3}(s_5, u'))_{(k,l)} \quad (159)$$

but as both  $s_5$  and  $u'$  are of  $[ABC]$  type we have to apply partial fractioning, following the ideas of [4, Ch. 5]. It is

$$(43) : \quad s_5 = s - q^2 + t' + u' \quad (160)$$

so end up with a form of  $\mathcal{V}(p+q, q)$  where  $p$  is  $[ab]$  and  $q$  and  $p+q$  are  $[ABC]$ . The aim is then to get to a form with fractions of  $\frac{p}{q}$  and/or  $\frac{p+q}{p}$  and indeed this can be achieved. Define

$$\mathcal{T} = \begin{pmatrix} \{-2, 1, 2, 1\} & \{1, 0, -1, -1\} & \{0, 0, 0, 1\} & \{0, 0, 1, -1\} \\ \{-1, 1, 1, 0\} & \{1, -1, 0, 0\} & \{0, 0, 1, 0\} & \{0, 0, 1, -1\} \\ \{0, 1, 0, 0\} & \{1, 0, 0, 0\} & \{1, 0, 0, 0\} & \{0, 0, 1, -1\} \\ \{1, 1, 0, 0\} & \{1, 1, 0, 0\} & \{1, 1, 0, 0\} & \{0, 0, 1, -1\} \end{pmatrix} \quad (161)$$

$$(\mathcal{W}(x, y))_{(k,l)} = x^{-2+p_x(k+l)} y^{-2+p_y(k+l)} \quad (162)$$

$$p_x(s) = \begin{cases} s & \text{if } s \leq 3 \\ 3 & \text{else} \end{cases} \quad p_y(s) = 3 - p_x(6-s) \begin{cases} 0 & \text{if } s \leq 3 \\ s-3 & \text{else} \end{cases} \quad (163)$$

$$\Rightarrow \mathcal{W}(x, y) = \begin{pmatrix} x^{-2}y^{-2} & x^{-1}y^{-2} & y^{-2} & xy^{-2} \\ x^{-1}y^{-2} & y^{-2} & xy^{-2} & xy^{-1} \\ y^{-2} & xy^{-2} & xy^{-1} & x \\ xy^{-2} & xy^{-1} & x & xy \end{pmatrix} \quad (164)$$

$$(\mathcal{W}^*(p, q))_{(k,l)} = \left\{ \frac{q}{p} \mathcal{W}(p, q), \mathcal{W}(p, q), \frac{p+q}{p} \mathcal{W}(p, p+q), \mathcal{W}(p, p+q) \right\}_{(k,l)} \quad (165)$$

we then find:

$$(\mathcal{V}(p+q, q))_{(k,l)} = \mathcal{T}_{(k,l)} \cdot \mathcal{W}^*(p, q)_{(k,l)} \quad (166)$$

where the equation has to be read separate for each  $(k, l)$ , i.e. “ $\cdot$ ” applies to the scalar  $4 \times 4$ -product of the  $\{\}$ s. As  $\mathcal{W}^t = \mathcal{W}$ ,  $\mathcal{W}$  has 10 unique entries and thus for  $\mathcal{W}^*$  are 40 unique integrals needed:  $I_0^{(-1\dots 3, -2\dots 2)}$ .

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