1 2 to 2 phase space

following [1]:

process:

$$\gamma^*(q) + g(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) \tag{1}$$

kinematics:

$$s = (q + k_1)^2 \qquad\qquad s' = s - q^2 \tag{2}$$
$$t = (k_1 - p_1)^2 \qquad\qquad t_1 = t - m^2 \tag{3}$$
$$u = (k_1 - p_2)^2 \qquad\qquad u_1 = u - m^2 \tag{4}$$

use c.m.s. of incoming particles:

$$q = \left(\frac{s + q^2}{2\sqrt{s}}, 0, 0, \ldots, -\frac{s - q^2}{2\sqrt{s}} \right) \tag{5}$$

$$k_1 = \frac{s - q^2}{2\sqrt{s}} (1, 0, 0, \ldots, 1) \tag{6}$$

such that

$$q + k_1 = (\sqrt{s}, \vec{0}) \qquad k_1^2 = 0 \tag{7}$$

for the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2} (1, 0, \beta \sin\theta, \ldots, \beta \cos\theta) \tag{8}$$

$$p_2 = \frac{\sqrt{s}}{2} (1, 0, -\beta \sin\theta, \ldots, -\beta \cos\theta) \tag{9}$$

with $\beta = \sqrt{1 - 4m^2/s}$ such that

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \qquad p_1^2 = p_2^2 = m^2 \tag{10}$$

use n-sphere:

$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \tag{11}$$

compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_1}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q + k_1 - p_1 - p_2) \delta(p_1^2 - m^2) \delta(p_2^2 - m^2) \tag{12}$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1 \, \delta((q + k_1 - p_2)^2 - m^2) \delta(p_1^2 - m^2) \tag{13}$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,\parallel} d^2 p_{1,\perp} d^{n-4}\hat{p}_1 \, \delta(s - 2p_{1,0}\sqrt{s}) \delta(p_{1,0}^2 - p_{1,\parallel}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{14}$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,\parallel} dp_{1,\perp}^2 d^{n-4}\hat{p}_1 \, \delta(s/4 - p_{1,\parallel}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{15}$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,\parallel} d\hat{p}_1^2 \, \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \tag{16}$$

$$= \frac{1}{2\sqrt{s}\Gamma((n-4)/2)(4\pi)^{(n-2)/2}} \int dp_{1,\parallel} d\hat{p}_1^2 \, (\hat{p}_1^2)^{(n-6)/2} \tag{17}$$

Integration borders are

$$p_{1,\parallel} \in \frac{\sqrt{s}}{2} \beta \cdot [-1, 1] \qquad \hat{p}_1^2 \in \left(\frac{s\beta^2}{4} - p_{1,\parallel}^2 \right) \cdot [0, 1] \tag{18}$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2 \, (\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4} \left(\frac{s\beta^2}{4} - p_{1,\parallel}^2 \right)^{(n-4)/2} \tag{19}$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dp_{1,\parallel} \left(\frac{s\beta^2}{4} - p_{1,\parallel}^2 \right)^{(n-4)/2} \tag{20}$$

rewrite $p_{1,\parallel}$ to $\cos\theta$:

$$p_{1,\parallel} = \frac{\sqrt{s}}{2} \beta \cos\theta \Rightarrow \quad dp_{1,\parallel} = \frac{\sqrt{s}}{2} \beta \, d\cos\theta, \quad \cos\theta \in [-1, 1], \hat{p}_1^2 \in \frac{s\beta^2}{4} \left(1 - \cos^2\theta \right) \cdot [0, 1] \tag{21}$$

rewrite $\cos\theta$ to $t_1 = (k_1 - p_2)^2 - m^2$:

$$\cos\theta = \frac{2t_1/s' + 1}{\beta} \Rightarrow \quad d\cos\theta = \frac{2}{\beta s'} dt_1, \quad t_1 \in \frac{s'}{2}[-\beta - 1, \beta - 1], \hat{p}_1^2 \in (-m^2 - \frac{st_1}{s'^2}(s' + t_1)) \cdot [0, 1] \tag{22}$$

$$p_{1,\parallel} = \sqrt{s} \left(\frac{t_1}{s'} + \frac{1}{2} \right) \Rightarrow \quad dp_{1,\parallel} = \frac{\sqrt{s}}{s'} dt_1 \tag{23}$$

$$\Rightarrow PS_2 = \frac{1}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dt_1 \left(\frac{(t_1(u_1 - q^2) - s'm^2)s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} \tag{24}$$

2

2 2 to 3 phase space

following [2, 3, 1]:

process:
$$\gamma^*(q) + q(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) + q(k_2) \tag{25}$$

2.1 kinematic constraints

definitions of kinematic variables:

$$s = (q + k_1)^2 \qquad \Rightarrow \qquad 2qk_1 = s - q^2 \tag{26}$$
$$s_3 = (k_2 + p_2)^2 - m^2 \qquad \Rightarrow \qquad 2k_2p_2 = s_3 \tag{27}$$
$$s_4 = (k_2 + p_1)^2 - m^2 \qquad \Rightarrow \qquad 2k_2p_1 = s_4 \tag{28}$$
$$s_5 = (p_1 + p_2)^2 = -u_5 \qquad \Rightarrow \qquad 2p_1p_2 = s_5 - 2m^2 \tag{29}$$
$$t_1 = (k_1 - p_2)^2 - m^2 = t - m^2 \qquad \Rightarrow \qquad 2k_1p_2 = -t_1 \tag{30}$$
$$t' = (k_1 - k_2)^2 \qquad \Rightarrow \qquad 2k_1k_2 = -t' \tag{31}$$
$$u_1 = (q - p_2)^2 - m^2 = u - m^2 \qquad \Rightarrow \qquad 2qp_2 = -u_1 + q^2 \tag{32}$$
$$u_6 = (k_1 - p_1)^2 - m^2 \qquad \Rightarrow \qquad 2k_1p_1 = -u_6 \tag{33}$$
$$u_7 = (q - p_1)^2 - m^2 \qquad \Rightarrow \qquad 2qp_1 = -u_7 + q^2 \tag{34}$$
$$u' = (q - k_2)^2 \qquad \Rightarrow \qquad 2qk_2 = -u' + q^2 \tag{35}$$

impose momentum conservation:

$$q + k_1 = p_1 + p_2 + k_2 \tag{36}$$

contract with 2 times momentum:

$$2q^2 \quad +s - q^2 = -u_7 + q^2 \quad -u_1 + q^2 \quad -u' + q^2 \quad \Leftrightarrow \quad 0 = s + u_1 + u_7 + u' - 2q^2 \tag{37}$$

$$s - q^2 \quad +0 = -u_6 \quad -t_1 \quad -t' \quad \Leftrightarrow \quad 0 = s + t_1 + t' + u_6 - q^2 \tag{38}$$

$$-u_7 + q^2 \quad -u_6 = 2m^2 \quad +s_5 - 2m^2 \quad +s_4 \quad \Leftrightarrow \quad 0 = s_4 + s_5 + u_6 + u_7 - q^2 \tag{39}$$

$$-u_1 + q^2 \quad -t_1 = s_5 - 2m^2 \quad +2m^2 \quad +s_3 \quad \Leftrightarrow \quad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \tag{40}$$

$$-u' + q^2 \quad -t' = s_4 \quad +s_3 \quad +0 \quad \Leftrightarrow \quad 0 = s_3 + s_4 + t' + u' - q^2 \tag{41}$$

$$\frac{1}{2}\left((37)+(38)+(40)-(39)-(41)\right)=0=s-q^2+t_1+u_1-s_4 \tag{42}$$

$$\frac{1}{2}\left((37)+(38)+(41)-(39)-(40)\right)=0=s-q^2+t'+u'-s_5 \tag{43}$$

2.2 choose framework

use c.m.s. of recoiling heavy and light quark ($Q(p_1)$ and $q(k_2)$):

$$k_2 = (\omega_2, k_{2,x}, \omega_2 \sin\theta_1 \cos\theta_2, \omega_2 \cos\theta_1, \hat{k}_2) \tag{44}$$

$$p_1 = (E_1, -k_{2,x}, -\omega_2 \sin\theta_1 \cos\theta_2, -\omega_2 \cos\theta_1, -\hat{k}_2) \tag{45}$$

$$k_1 = (\omega_1, 0, 0, \omega_1, \hat{0}) \tag{46}$$

$$q = (q_0, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi - \omega_1, \hat{0}) \tag{47}$$

$$p_2 = (E_2, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi, \hat{0}) \tag{48}$$

light quark masses are already fixed: $k_1^2 = 0 = k_2^2$

constraints:

$$q_0 + \omega_1 \qquad = E_1 + E_2 + \omega_2 \tag{49}$$

$$m^2 = p_1^2 \qquad = E_1^2 - \omega_2^2 \tag{50}$$

$$m^2 = p_2^2 \qquad = E_2^2 - |\vec{p}_2|^2 \tag{51}$$

$$q^2 \qquad = q_0^2 - |\vec{p}_2|^2 + 2|\vec{p}_2|\,\omega_1 \cos\psi - \omega_1^2 \tag{52}$$

$$s = (q+k_1)^2 \qquad = (q_0 + \omega_1)^2 - |\vec{p}_2|^2 \tag{53}$$

$$t = (k_1 - p_2)^2 \qquad = (\omega_1 - E_2)^2 - |\vec{p}_2|^2 + 2|\vec{p}_2|\,\omega_1 \cos\psi - \omega_1^2 \tag{54}$$

$$u = (q - p_2)^2 \qquad = (q_0 - E_2)^2 - \omega_1^2 \tag{55}$$

solve:

$$(53) - (52) + (54) - (51) + (55) = s - q^2 + t - m^2 + u \tag{56}$$

$$= s_4 + m^2 = (E_1 + \omega_2)^2 \tag{57}$$

$$(54) + (55) - (52) = t + u - q^2 = -2(E_1 + \omega_2)E_2 \tag{58}$$

$$\Rightarrow E_2 = -\frac{t + u - q^2}{2\sqrt{s_4 + m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4 + m^2}} \tag{59}$$

$$(57) \wedge (50) \Rightarrow \omega_2 = \frac{s_4}{2\sqrt{s_4 + m^2}} \tag{60}$$

$$(57) \Rightarrow E_1 = \frac{s_4 + 2m^2}{2\sqrt{s_4 + m^2}} \tag{61}$$

$$(53) + (55) - (51) = s + u - m^2 = 2q_0(E_1 + \omega_2) \tag{62}$$

$$\Rightarrow q_0 = \frac{s + u_1}{2\sqrt{s_4 + m^2}} \tag{63}$$

$$(54) - (52) = t - q^2 = (\omega_1 - E_2)^2 - q_0^2 \tag{64}$$

$$\Rightarrow \omega_1 = \frac{s_4 - u_1}{2\sqrt{s_4 + m^2}} \tag{65}$$

$$(51) \Rightarrow |\vec{p}_2| = \sqrt{E_2^2 - m^2} = \frac{\sqrt{(s - s_4)^2 - 4sm^2}}{2\sqrt{s_4 + m^2}} \tag{66}$$

$$(52) \Rightarrow \cos\psi = \frac{q^2 - q_0^2 + |\vec{p}_2|^2 + \omega_1^2}{2|\vec{p}_2|\omega_1} \tag{67}$$

$$= \frac{(u_1 + m^2)(q^2 - s + t_1) - (m^2 - q^2 - t_1)(s_4 - u_1)}{(s_4 - u_1)\sqrt{(s - s_4)^2 - 4sm^2}} \tag{68}$$

$$\Rightarrow \sin\psi = 2\frac{\sqrt{s_4 + m^2}\sqrt{m^2(s - q^2)^2 + q^2 t_1(s - q^2 + t_1) - (s - q^2)t_1 u_1}}{(s_4 - u_1)\sqrt{(s - s_4)^2 - 4sm}} \tag{69}$$

$$t' = -2k_1 k_2 = -2\omega_1\omega_2(1 - \cos\theta_1) \tag{70}$$

$$u_6 = -2k_1 p_1 = -2\omega_1(E_1 + \omega_2\cos\theta_1) \tag{71}$$

$$(38): \qquad 0 = s + t_1 + t' + u_6 - q^2 \quad \checkmark \tag{72}$$

t' is the only variable that can get collinear (for $-q^2 > 0$).

$$s_3 = 2k_2 p_2 = 2\omega_2(E_2 - |\vec{p}_2|(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{73}$$

$$s_5 = (p_1 + p_2)^2 = 2m^2 + 2p_1 p_2 \tag{74}$$

$$= 2(m^2 + E_1 E_2 + \omega_2|\vec{p}_2|(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{75}$$

$$(40): \qquad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad \checkmark \tag{76}$$

$$u' = (q - k_2)^2 = q^2 - 2qk_2 \tag{77}$$

$$= q^2 - 2(q_0\omega_2 - \omega_2(|\vec{p_2}|\cos\psi - \omega_1)\cos\theta_1 - \omega_2|\vec{p_2}|\sin\psi\sin\theta_1\cos\theta_2) \tag{78}$$

$$u_7 = q^2 - 2qp_1 \tag{79}$$

$$= q^2 - 2(q_0 E_1 + \omega_2(|\vec{p_2}|\cos\psi - \omega_1)\cos\theta_1 + \omega_2|\vec{p_2}|\sin\psi\sin\theta_1\cos\theta_2) \tag{80}$$

$$(37): \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad \checkmark \tag{81}$$

2.3 phase space integrals

at phase space integration there occur integrations over propagators[4, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1 \, \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2 \, \sin^{n-4}(\theta_2)(a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l}$$

$$\tag{82}$$

$$= \int d\Omega_n \, (a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{83}$$

the integrals can be further destinguished by the range of k, l and the type of collinearity (following the notation in [4]):

- "non collinear": $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \rightarrow I_{0,n}^{(k,l)}$

- "single collinear a": $a = -b \wedge A^2 \neq B^2 + C^2 \rightarrow I_{a,n}^{(k,l)}$

- "single collinear A": $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \rightarrow I_{A,n}^{(k,l)}$

- "double collinear": $a = -b \wedge A = -\sqrt{B^2 + C^2} \rightarrow I_{aA,n}^{(k,l)}$

Use $n = 4 + \epsilon$.

2.3.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int_0^\pi dt \, \sin^{\nu-3}(t)\cos^q(t) \tag{84}$$

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It is[5, eq. 5.12.6]:

$$\int_0^{\pi} (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha\, B\left((\alpha+\beta+1)/2, (\alpha-\beta+1)/2\right)} \qquad \text{if } \Re(\alpha) > 0 \qquad (85)$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{B((n-1)/2, (n-1)/2)} \qquad (86)$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{B((n-2)/2, (n-2)/2)} = B((n-3)/2, 1/2) \qquad (87)$$

If q is odd: $\hat{I}^{(q)} = 0$, due to symetry of kernel; if q is even: $q = 2p$ with $p \in \mathbb{N}$:

$$\hat{I}^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int_0^{\pi} \sin^{\nu-3}(t) \exp(2i(k-p)t)\, dt \qquad (88)$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2 + (k-p), (\nu-1)/2 - (k-p))} \qquad (89)$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^{p} \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2 + l, (\nu-1)/2 - l)} \qquad (90)$$

$$= \frac{\pi\Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right.$$
$$\left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \qquad (91)$$

$$= \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p \Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right.$$
$$\left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \qquad (92)$$

TODO: prove

$$p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \qquad (93)$$

$$= \frac{1}{p!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l p!}{(p+l)!(p-l)!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \qquad (94)$$

$$= 1 \qquad (95)$$

$$\Rightarrow \hat{I}^{(2p)}(\nu) = \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \qquad (96)$$

$$= \frac{\sqrt{\pi}(2p)!}{2^{2p}p!} \frac{\Gamma((\nu-2)/2)}{\Gamma(\frac{\nu-1}{2}+p)} \qquad (97)$$

2.3.2 any collinearity and $-k, -l \in \mathbb{N}_0$

If $-k, -l \in \mathbb{N}_0$ $I_n^{(k,l)}$ can always be reduced in a straight forward manner to combinations of $\hat{I}^{(q)}(n)$ and this way one finds[4, Ch. 5][2, App. C]:

$$I_n^{(0,0)} = \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n) = \frac{2\pi}{n-3} \tag{98}$$

$$I_4^{(0,0)} = 2\pi \tag{99}$$

$$I_n^{(-1,0)} = \hat{I}^{(0)}(n-1) \cdot (a\hat{I}^{(0)}(n) + b\hat{I}^{(1)}(n)) = \frac{2\pi a}{n-3} \tag{100}$$

$$I_4^{(-1,0)} = 2\pi a \tag{101}$$

$$I_n^{(0,-1)} = \hat{I}^{(0)}(n-1) \cdot (A\hat{I}^{(0)}(n) + B\hat{I}^{(1)}(n)) + C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \tag{102}$$

$$= \frac{2\pi A}{n-3} \tag{103}$$

$$I_4^{(0,-1)} = 2\pi A \tag{104}$$

$$I_n^{(-2,0)} = \hat{I}^{(0)}(n-1) \cdot (a^2\hat{I}^{(0)}(n) + 2ab\hat{I}^{(1)}(n) + b^2\hat{I}^{(2)}(n)) \tag{105}$$

$$= 2\pi \left(\frac{a^2(n-1) + b^2}{(n-1)(n-3)} \right) \tag{106}$$

$$I_4^{(-2,0)} = 2\pi(a^2 + b^2/3) \tag{107}$$

$$I_n^{(0,-2)} = \hat{I}^{(0)}(n-1) \cdot (A^2\hat{I}^{(0)}(n) + B^2\hat{I}^{(2)}(n)) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \tag{108}$$

$$= 2\pi \left(\frac{A^2(n-1) + B^2 + C^2}{(n-1)(n-3)} \right) \tag{109}$$

$$I_4^{(0,-2)} = 2\pi(A^2 + (B^2 + C^2)/3) \tag{110}$$

$$I_n^{(-1,-1)} = \hat{I}^{(0)}(n-1) \cdot (aA\hat{I}^{(0)}(n) + bB\hat{I}^{(2)}(n)) = 2\pi \left(\frac{aA(n-1) + bB}{(n-1)(n-3)} \right) \tag{111}$$

$$I_4^{(-1,-1)} = 2\pi(aA + bB/3) \tag{112}$$

2.3.3 single collinear a

If $-l \in \mathcal{N}$ one finds:

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1-\cos(t))^k} \cos^q(t)\, dt \tag{113}$$

$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1-\cos^2(t))^k} \cos^q(t)(1+\cos(t))^k\, dt \tag{114}$$

$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t)(1+\cos(t))^k\, dt \tag{115}$$

$$= \sum_{l=0}^k \binom{k}{l} \hat{I}^{(q+l)}(\nu-2k) \tag{116}$$

this way one finds[4, Ch. 5][2, App. C]:

$$I_{a,n}^{(1,0)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n-2) \tag{117}$$

$$= \frac{2\pi}{a(n-4)} \tag{118}$$

$$I_{a,n}^{(1,-1)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \left(A\hat{I}^{(0)}(n-2) + B\hat{I}^{(2)}(n-2)\right) \tag{119}$$

$$= \frac{2\pi}{a} \frac{(A(n-3)+B)}{(n-3)(n-4)} \approx \frac{2\pi}{a}\left(\frac{A+B}{\epsilon} - 2B + O(\epsilon)\right) \tag{120}$$

$$I_{a,n}^{(1,-2)} = \frac{1}{a}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2\hat{I}^{(0)}(n-2) + (B^2+2AB)\hat{I}^{(2)}(n-2)\right) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n)\right) \tag{121}$$

$$= \frac{2\pi}{a}\left(\frac{A^2}{n-4} + \frac{2AB+B^2}{(n-4)(n-3)} + \frac{C^2}{(n-3)(n-2)}\right) \tag{122}$$

$$\approx \frac{2\pi}{a}\left(\frac{(A+B)^2}{\epsilon} + \frac{C^2}{2} - 2AB - B^2 + O(\epsilon)\right) \tag{123}$$

$$I_{a,n}^{(2,0)} = \frac{1}{a^2}\hat{I}^{(0)}(n-1) \cdot \left(\hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4)\right) \tag{124}$$

$$= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2} + O(\epsilon) \tag{125}$$

$$I_{a,n}^{(2,-1)} = \frac{1}{a^2}\hat{I}^{(0)}(n-1) \cdot \left(A\left(\hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4)\right) + 2B\hat{I}^{(2)}(n-4)\right) \tag{126}$$

$$= \frac{2\pi}{a^2}\left(\frac{A}{n-6} + \frac{2B}{(n-6)(n-4)}\right) \approx -\frac{2\pi}{a^2}\left(\frac{B}{\epsilon} + \frac{A+B}{2}\right) + O(\epsilon) \tag{127}$$

$$I_{a,n}^{(2,-2)} = \frac{1}{a^2}\left(\hat{I}^{(0)}(n-1) \cdot \left(A^2(\hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4)) + 4AB\hat{I}^{(2)}(n-4)\right.\right.$$
$$\left.\left. + B^2(\hat{I}^{(2)}(n-4) + \hat{I}^{(4)}(n-4))\right) + C^2\hat{I}^{(2)}(n-1)(\hat{I}^{(0)}(n-2) + \hat{I}^{(2)}(n-2))\right) \tag{128}$$

$$= \frac{2\pi}{a^2}\left(\frac{A^2}{n-6} + \frac{4AB}{(n-6)(n-4)} + \frac{B^2 n}{(n-6)(n-4)(n-3)} + \frac{C^2}{(n-4)(n-3)}\right) \tag{129}$$

$$\approx \frac{2\pi}{a^2}\left(\frac{-2AB - 2B^2 + C^2}{\epsilon} + \frac{B^2 - A^2}{2} - AB - C^2 + O(\epsilon)\right) \tag{130}$$

$$\tag{131}$$

It is[4, Ch. 5]:

$$I_{a,n}^{(1,1)} = \frac{\pi}{a(A+B)}\left(\frac{2}{\epsilon} + \ln\left(\frac{(A+B)^2}{A^2 - B^2 - C^2}\right)\right) + O(\epsilon) \tag{132}$$

$$I_{a,n}^{(2,1)} = \frac{\pi}{a^2(A+B)}\left(\frac{B^2 + AB + C^2}{(A+B)^2}\left(\frac{2}{\epsilon} + \ln\left(\frac{(A+B)^2}{A^2 - B^2 - C^2}\right)\right)\right.$$
$$\left. - \frac{2C^2}{(A+B)^2} - 1\right) + O(\epsilon) \tag{133}$$

From this are the following integrals derived:

$$I_{a,n}^{(1,2)} = -\frac{\partial}{\partial A}I_{a,n}^{(1,1)} \tag{134}$$

$$= \frac{\pi}{a(A+B)^2}\left(\frac{2}{\epsilon} + \ln\left(\frac{(A+B)^2}{A^2 - B^2 - C^2}\right) + \frac{2A(A+B)}{A^2 - B^2 - C^2} - 2\right) + O(\epsilon) \tag{135}$$

$$I_{a,n}^{(2,2)} = -\frac{\partial}{\partial A}I_{a,n}^{(2,1)} \tag{136}$$

$$= \frac{\pi}{a^2(A+B)^2}\left(\frac{2B^2 + 2AB + 3C^2}{(A+B)^2}\left(\frac{2}{\epsilon} + \ln\left(\frac{(A+B)^2}{A^2 - B^2 - C^2}\right)\right)\right.$$
$$\left. + \frac{2A^2}{A^2 - B^2 - C^2} - \frac{8C^2}{(A+B)^2} - 3\right) + O(\epsilon) \tag{137}$$

2.3.4 double collinear

as said in [4, Ch. 5]: if $0 \leq -\frac{C}{A}, \frac{B}{A} \leq 1$ use [3, eq. A11] with $\cos\kappa = -\frac{B}{A}$:

$$I_{aA,n}^{(k,l)} = \frac{2\pi 2^{-(k+l)}}{a^k A^l} \frac{\Gamma(1+\epsilon)}{\Gamma^2(1+\epsilon/2)} B(1+\frac{\epsilon}{2}-k, 1+\frac{\epsilon}{2}-l)_2F_1\left(k,l;1+\frac{\epsilon}{2};\frac{A-B}{2A}\right) \quad (138)$$

2.3.5 non collinear

If $-l \in \mathcal{N}$ the θ_2 integration can be performed using the integral helper and the problem reduces then to the following integral:

$$\hat{I}_{0,\cos}^{(k,q,p)}(\epsilon)$$

$$= \int_0^\pi d\theta_1 \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))^k} \quad (139)$$

$$= \frac{1}{2a^k}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right) {}_3F_2\left(\frac{1+k}{2},\frac{k}{2},\frac{1+p}{2};\frac{1}{2},\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.$$
$$\left.\frac{b}{a}k(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right) {}_3F_2\left(\frac{1+k}{2},\frac{2+k}{2},\frac{2+p}{2};\frac{3}{2},\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right)$$

$$(140)$$

for $k=1$ this simplifies to

$$\hat{I}_{0,\cos}^{(1,q,p)}(\epsilon)$$

$$= \int_0^\pi d\theta_1 \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))} \quad (141)$$

$$= \frac{1}{2a}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right) {}_2F_1\left(1,\frac{1+p}{2};\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.$$
$$\left.\frac{b}{a}(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right) {}_2F_1\left(1,\frac{2+p}{2};\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) \quad (142)$$

TODO: structure does NOT match [4, Ch. 5] - neither A,B,C nor ϵ

FiXme
Error: fix
to Bojak

For $I_{0,n}^{1,1}$ one finds[4, Ch. 5]:

$$I_{0,n}^{1,1} = \frac{\pi}{\sqrt{X}} \ln\left(\frac{aA-bB+\sqrt{X}}{aA-bC-\sqrt{X}}\right) \quad (143)$$

$$X = (aB-bA)^2 - (a^2-b^2)C^2 \quad (144)$$

2.4 needed set in matrix element

define a shortcut:

$$(\mathcal{V}(x,y))_{(k,l)} = x^{-2+k}y^{-2+l} \qquad k,l = \{0,1,2,3\} \tag{145}$$

It is

$$A_1 = \sum_{k,l=0}^{3} (\mathcal{C}_{A_1})_{(k,l)} \, t'^{-2+k} u_7^{-2+l} = \text{tr}(\mathcal{C}_{A_1} \mathcal{V}(t', u_7)^t) \tag{146}$$

and we will thus need the integrals

$$(\mathcal{I}_{A_1})_{(k,l)} = \int d\Omega_n \, \left(\mathcal{V}(t', u_7)\right)_{(k,l)} \tag{147}$$

with

$$a(t') = -b(t') = -2\omega_1\omega_2 \qquad\qquad = -\frac{s_4(s_4 - u_1)}{2(s_4 + m^2)} \tag{148}$$

$$A(u_7) = q^2 - 2q_0 E_1 \qquad\qquad = q^2 - \frac{(s_4 + 2m^2)(s + u_1)}{2(s_4 + m^2)} \tag{149}$$

$$B(u_7) = -2\omega_2(|\vec{p}_2| \cos\psi - \omega_1) \quad = \frac{s_4}{2}\left(1 - \frac{s + u_1}{s_4 + m^2} + \frac{s - q^2 - t_1}{s_4 - u_1}\right) \tag{150}$$

$$C(u_7) = -2\omega_2 \, |\vec{p}_2| \sin\psi \tag{151}$$

that is $I_{a,n}^{(-1\ldots2,-1\ldots2)}$. With this we get

$$\int d\Omega_n \, A_1 = \text{tr}(\mathcal{C}_{A_1} \, (\mathcal{I}_{A_1})^t) \tag{152}$$

It is

$$A_2 = \sum_{k,l=0}^{3} (\mathcal{C}_{A_2})_{(k,l)} \, s_5^{-2+k} u'^{-2+l} = \text{tr}(\mathcal{C}_{A_2} \mathcal{V}(s_5, u')^t) \tag{153}$$

and we will thus need the integrals

$$(\mathcal{I}_{A_2})_{(k,l)} = \int d\Omega_n \, \left(\mathcal{V}(s_5, u')\right)_{(k,l)} \tag{154}$$

but as both s_5 and u' are of $[ABC]$ type we have to apply partial fractioning, following the ideas of [4, Ch. 5]. It is

$$(43): \qquad s_5 = s - q^2 + t' + u' \tag{155}$$

so end up with a form of $\mathcal{V}(p+q, q)$ where p is $[ab]$ and q and $p+q$ are $[ABC]$. The aim is then to get to a form with fractions of $\frac{p}{q}$ and/or $\frac{p+q}{p}$ and indeed this can be achieved. Define

$$\mathcal{T} = \begin{pmatrix} \{-2,1,2,1\} & \{1, \ 0,-1,-1\} & \{0,0,0,1\} & \{0,0,1,-1\} \\ \{-1,1,1,0\} & \{1,-1, \ 0, \ 0\} & \{0,0,1,0\} & \{0,0,1,-1\} \\ \{ \ 0,1,0,0\} & \{1, \ 0, \ 0, \ 0\} & \{1,0,0,0\} & \{0,0,1,-1\} \\ \{ \ 1,1,0,0\} & \{1, \ 1, \ 0, \ 0\} & \{1,1,0,0\} & \{0,0,1,-1\} \end{pmatrix} \tag{156}$$

$$(\mathcal{W}(x,y))_{(k,l)} = x^{-2+p_x(k+l)}y^{-2+p_y(k+l)} \tag{157}$$

$$p_x(s) = \begin{cases} k+l & \text{if } k+l \leq 3 \\ 3 & \text{else} \end{cases} \qquad p_y(s) = \begin{cases} 0 & \text{if } k+l \leq 3 \\ k+l-3 & \text{else} \end{cases} \tag{158}$$

$$\Rightarrow \mathcal{W}(x,y) = \begin{pmatrix} x^{-2}y^{-2} & x^{-1}y^{-2} & y^{-2} & xy^{-2} \\ x^{-1}y^{-2} & y^{-2} & xy^{-2} & xy^{-1} \\ y^{-2} & xy^{-2} & xy^{-1} & x \\ xy^{-2} & xy^{-1} & x & xy \end{pmatrix} \tag{159}$$

$$(\mathcal{W}^*(p,q))_{(k,l)} = \left\{ \frac{q}{p}\mathcal{W}(p,q), \mathcal{W}(p,q), \frac{p+q}{p}\mathcal{W}(p,p+q), \mathcal{W}(p,p+q) \right\}_{(k,l)} \tag{160}$$

we then find:

$$(\mathcal{V}(p+q,q))_{(k,l)} = \mathcal{T}_{(k,l)} \cdot \mathcal{W}^*(p,q)_{(k,l)} \tag{161}$$

where the equation has to be read separate for each (k,l), i.e. "$\cdot$" applies to the scalar $4x4$-product of the $\{\}$s. As $\mathcal{W}^t = \mathcal{W}$, $\mathcal{W}$ has 10 unique entries and thus for $\mathcal{W}^*$ are 40 unique integrals needed: $I_0^{(-1...3,-2...2)}$.

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List of Corrections