1 2 to 3 phase space

FiXme Error: Quote Phys.Rev.I

$$I_n^{(k,l)} =: I_{abABC}^{(k,l)}(n) \tag{1}$$

$$= \int_0^\pi d\theta_1\ \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2\ \sin^{n-4}(\theta_2)(a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{2}$$

$$I_b^{(q)}(\nu) := \int_0^\pi dt\ \sin^{\nu-3}(t)\cos^q(t) \tag{3}$$

It is:

FiXme Error: Quote DLMF 5.12.6

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha\, \mathrm{B}((\alpha+\beta+1)/2, (\alpha-\beta+1)/2)} \qquad \text{if } \Re(\alpha) > 0 \tag{4}$$

$$\Rightarrow I_b^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{\mathrm{B}((n-1)/2, (n-1)/2)} \tag{5}$$

$$\Rightarrow I_b^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{\mathrm{B}((n-2)/2, (n-2)/2)} = \mathrm{B}((n-3)/2, 1/2) \tag{6}$$

If q is odd: $I_b^{(q)} = 0$, due to symetry of kernel; if q is even: $q = 2p$ with $p \in \mathbb{N}$:

$$I_b^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t)\, dt \tag{7}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{\mathrm{B}((\nu-1)/2 + (k-p), (\nu-1)/2 - (k-p))} \tag{8}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^{p} \binom{2p}{p+l} \frac{(-1)^l}{\mathrm{B}((\nu-1)/2 + l, (\nu-1)/2 - l)} \tag{9}$$

$$= \frac{\pi\Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right.$$
$$\left. +2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{10}$$

$$= \frac{2^{3-\nu}\pi\Gamma(\nu-1)(2p)!}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right.$$
$$\left. +2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{11}$$

TODO: prove the brackets resolve to $\mathcal{N}(p)$ ...

FiXme Error: prove

$$I_b^{(2p)}(\nu) = \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \mathcal{N}(p) \tag{12}$$

$$= \frac{\mathcal{N}(p)\sqrt{\pi}}{2^{p-1}} \frac{\Gamma(\nu/2)}{(\nu-2)\Gamma(\frac{\nu-1}{2}+p)} \tag{13}$$

with

$$\mathcal{N}(p) = \frac{(2p)!}{2^p(p!)^2}\frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})} + 2\sum_{l=1}^{p}\frac{(-1)^l(2p)!}{2^p(p+l)!(p-l)!}\frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)}\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \tag{14}$$

If $-k, -l \in \mathbb{N}_0$ $I^{(k,l)}_{abABC}(n)$ can always be reduced to combination of $I^{(q)}_b(n)$ and this way one for example finds:

FiXme Error: quote Phys.Rev.I

$$I^{(0,0)}_{abABC}(4) = I^{(0)}_b(3) \cdot I^{(0)}_b(4) = 2\pi \tag{15}$$

$$I^{(-1,0)}_{abABC}(4) = I^{(0)}_b(3) \cdot (aI^{(0)}_b(4) + bI^{(1)}_b(4)) = 2\pi a \tag{16}$$

$$I^{(0,-1)}_{abABC}(4) = I^{(0)}_b(3) \cdot (AI^{(0)}_b(4) + BI^{(1)}_b(4)) + CI^{(1)}_b(3)I^{(0)}_b(4) = 2\pi A \tag{17}$$

$$I^{(-2,0)}_{abABC}(4) = I^{(0)}_b(3) \cdot (a^2I^{(0)}_b(4) + 2abI^{(1)}_b(4) + b^2I^{(2)}_b(4)) = 2\pi(a^2 + b^2/3) \tag{18}$$

$$I^{(0,-2)}_{abABC}(4) = I^{(0)}_b(3) \cdot (A^2I^{(0)}_b(4) + B^2I^{(2)}_b(4)) + C^2I^{(2)}_b(3)I^{(0)}_b(6) \tag{19}$$

$$= 2\pi(A^2 + (B^2 + C^2)/3) \tag{20}$$

$$I^{(-1,-1)}_{abABC}(4) = I^{(0)}_b(3) \cdot (aAI^{(0)}_b(4) + bBI^{(2)}_b(4)) = 2\pi(aA + bB/3) \tag{21}$$