# 1 Passarino-Veltman decomposition

## 1.1 Definitions

[1]:

$$A(m) = \frac{1}{i\pi^2} \int d^n q \frac{1}{q^2 + m^2} \tag{1}$$

$$B_0(p, m_1, m_2) = \frac{1}{i\pi^2} \int d^n q \frac{1}{(q^2 + m_1^2)((q+p)^2 + m_2^2)} \tag{2}$$

and apart from their pole term (called $\Delta$ - see [1, eq. D.1]), they keep $n = 4$.

[2, 3]:

$$A(m) = \mu^{-\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 - m^2} \tag{3}$$

$$B(q_1, m_1, m_2) = \mu^{-\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_1^2)((q+q_1)^2 - m_2^2)} \tag{4}$$

and $n = 4 + \epsilon$. ([2] writes “The notations for the one-, two-, three-, and four-point functions have been taken over from Ref. [1].” - obviously they do not.)

`HEPMath`[4] and `FeynCalc`[5, 6] refer to `LoopTools`[7, 8]. [8, eq. (1.1)] and [9, eq. (2.6)],`QCDLoop`[10]:

$$T^N_{\mu_1 \ldots \mu_P} = \frac{\mu^{4-D}}{i\pi^{D/2} r_\Gamma} \int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_P}}{\left[q^2 - m_1^2\right] \left[(q+k_1)^2 - m_2^2\right] \cdots \left[(q + k_{N-1})^2 - m_N^2\right]} \tag{5}$$

$$r_\Gamma = \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}, \quad D = 4 - 2\varepsilon$$

later in the code they use a different signature (to avoid any vector structure):

$$\begin{aligned} &A(m^2), B_0(p^2, m_1^2, m_2^2), C_0(p_1^2, p_2^2, (p_1+p_2)^2, m_1^2, m_2^2, m_3^2) \\ &D_0(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2) \end{aligned} \tag{6}$$

[11]:

$$T^N_{\mu_1 \ldots \mu_P}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_P}}{L_0 L_1 \cdots L_{N-1}} \tag{7}$$

$$L_0 = q^2 - m_0^2 + i\varepsilon \tag{8}$$

$$L_i = (q + p_i)^2 - m_i^2 + i\varepsilon \, i = 1, \ldots, N-1 \tag{9}$$

I will stick to the integrals of [3] as it is the most natural form, I think, and to the non-vector signature, if possible.

The transformation of the *analytic* results from the notation in [10] is given by

$$[10]\colon \frac{\mu^{4-n}}{i\pi^{n/2} r_\Gamma}\mathcal{I} \leftrightarrow [3]\colon \frac{\mu^{4-n}}{(2\pi)^n}\mathcal{I} \tag{10}$$

with $\mathcal{I}$ denonting the *raw* integral. We then need to solve (B=Bojak[3],E=Ellis[10]):

$$\begin{aligned} &\Rightarrow iC_\epsilon \left(\frac{a_2^B}{(n-4)^2} + \frac{a_1^B}{n-4} + a_0^B + O(n-4)\right) \\ &\stackrel{!}{=} \frac{\mu^{4-n}}{(2\pi)^n}\frac{i\pi^{n/2} r_\Gamma}{\mu^{4-n}}\left(\frac{a_2^E}{(n-4)^2} + \frac{a_1^E}{n-4} + a_0^E + O(n-4)\right) \end{aligned} \tag{11}$$

$$\Rightarrow a_2^B = a_2^E \tag{12}$$

$$a_1^B = a_1^E - \frac{1}{2}a_2^E \ln(m^2/\mu^2) \tag{13}$$

$$a_0^B = a_0^E - \frac{a_2^E}{8}\zeta(2) + \frac{a_2^E}{8}\ln^2(m^2/\mu^2) - \frac{a_1^E}{2}\ln(m^2/\mu^2) \tag{14}$$

To compare *numeric* results form `LoopTools` or `QCDLoop` one need to solve

$$\begin{aligned} &\Rightarrow \left(\frac{b_2^B}{(n-4)^2} + \frac{b_1^B}{n-4} + b_0^B + O(n-4)\right) \\ &\stackrel{!}{=} \frac{\mu^{4-n}}{(2\pi)^n}\frac{i\pi^{n/2} r_\Gamma}{\mu^{4-n}}\left(\frac{b_2^E}{(n-4)^2} + \frac{b_1^E}{n-4} + b_0^E + O(n-4)\right) \end{aligned} \tag{15}$$

$$\Rightarrow b_2^B = \frac{i}{16\pi^2} b_2^E \tag{16}$$

$$b_1^B = \frac{i}{16\pi^2}\left(b_1^E + \frac{b_2^E}{2}(\gamma_E - \ln(4\pi))\right) \tag{17}$$

$$b_0^B = \frac{i}{16\pi^2}\left(b_0^E + \frac{b_1^E}{2}(\gamma_E - \ln(4\pi)) + \frac{b_2^E}{8}\left((\gamma_E - \ln(4\pi))^2 - \zeta(2)\right)\right) \tag{18}$$

$$\tag{19}$$

## 1.2 Decomposition Labeling

[1, 3]:

$$B_\mu(p, m_1, m_2) = p_\mu B_1(p, m_1, m_2) \tag{20}$$

$$B_{\mu\nu} = p_\mu p_\nu B_{21} + g_{\mu\nu} B_{22} \tag{21}$$

$$C_\mu(p_1, p_2, m_1, m_2, m_3) = p_{1,\mu} C_{11} + p_{2,\mu} C_{12} \tag{22}$$

$$C_{\mu\nu} = p_{1,\mu}p_{1,\nu}C_{21} + p_{2,\mu}p_{2,\nu}C_{22} + (p_{1,\mu}p_{2,\nu} + p_{1,\nu}p_{2,\mu})C_{23} + g_{\mu\nu}C_{24} \tag{23}$$

The arguments of the functions are always inherited.

`HEPMath`, `FeynCalc`, `LoopTools`, [9]:

$$B_\mu(p, m_1, m_2) = p_\mu B_1(p, m_1, m_2) \tag{24}$$

$$B_{\mu\nu} = g_{\mu\nu} B_{00} + p_\mu p_\nu B_{11} \tag{25}$$

$$C_\mu(p_1, p_2, m_1, m_2, m_3) = p_{1,\mu} C_1 + p_{2,\mu} C_2 = \sum_{j=1}^{2} p_{j,\mu} C_j \tag{26}$$

$$C_{\mu\nu} = p_{1,\mu} p_{1,\nu} C_{11} + p_{2,\mu} p_{2,\nu} C_{22} + (p_{1,\mu} p_{2,\nu} + p_{1,\nu} p_{2,\mu}) C_{12} + g_{\mu\nu} C_{00} \tag{27}$$

$$= g_{\mu\nu} C_{00} + \sum_{j,k=1}^{2} p_{j,\mu} p_{k,\nu} C_{jk} \tag{28}$$

The arguments of the functions are always inherited.

I will stick to `HEPMath` as it is the more generic and extensible form, I think.

### 1.3 Decomposition momenta

The tensor coefficients have to be expanded by momenta and there are also different conventions: `HEPMath` and `LoopTools`(LT) use the *internal* momenta $k_i$ and [1, 3, 9](E) use the *external* momenta $p_i$. They are related by

$$k_1 = p_1, \quad k_2 = k_1 + p_2 = p_1 + p_2, \quad k_i = k_{i-1} + p_i, \forall i > 1 \tag{29}$$

So the decomposition is different, e.g. $C_\mu$ :

$$C_\mu^{LT} = k_{1\mu} C_1^{LT} + k_{2\nu} C_2^{LT} \tag{30}$$

$$= p_{1\mu}(C_1^{LT} - C_2^{LT}) + p_{2\mu} C_2^{LT} \tag{31}$$

$$C_\mu^E = p_{1\mu} C_1^E + p_{2\mu} C_2^E \tag{32}$$

Scalar PaVe-Coefficients do not need any transformation, because they do not depend on any momenta. B-Coefficients also do not need any transformation, because $k_1 = p_1$. The transformation for all other coefficients is given by:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^{LT} \qquad \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}^E = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}^{LT} \tag{33}$$

$$\begin{pmatrix} C_{00} \\ C_{11} \\ C_{12} \\ C_{22} \end{pmatrix}^E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_{00} \\ C_{11} \\ C_{12} \\ C_{22} \end{pmatrix}^{LT} \tag{34}$$

$$
\begin{pmatrix} D_{00} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{22} \\ D_{23} \\ D_{33} \end{pmatrix}^{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_{00} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{22} \\ D_{23} \\ D_{33} \end{pmatrix}^{LT} \tag{35}
$$

$$
\begin{pmatrix} D_{001} \\ D_{002} \\ D_{003} \\ D_{111} \\ D_{112} \\ D_{113} \\ D_{122} \\ D_{123} \\ D_{133} \\ D_{222} \\ D_{223} \\ D_{333} \end{pmatrix}^{E} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 & 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_{001} \\ D_{002} \\ D_{003} \\ D_{111} \\ D_{112} \\ D_{113} \\ D_{122} \\ D_{123} \\ D_{133} \\ D_{222} \\ D_{223} \\ D_{333} \end{pmatrix}^{LT} \tag{36}
$$

### 1.4 B Decomposition

define

$$
f_1 = m_1^2 - m_0^2 - p_1^2 \tag{37}
$$

then one finds easily

$$
B_1(p_1^2, m_0^2, m_1^2) = \frac{1}{2p_1^2} \Big( f_1 B_0(p_1^2, m_0^2, m_1^2) + A_0(m_0^2) - A_0(m_1^2) \Big) \tag{38}
$$

$$
B_{00}(p_1^2, m_0^2, m_1^2) = \frac{1}{2(n-1)} \Big( 2m_0^2 B_0(p_1^2, m_0^2, m_1^2) + A_0(m_1^2) - f_1 B_1(p_1^2, m_0^2, m_1^2) \Big) \tag{39}
$$

$$
B_{11}(p_1^2, m_0^2, m_1^2) = \frac{1}{2p_1^2} \Big( f_1 B_0(p_1^2, m_0^2, m_1^2) + A_0(m_1^2) - 2B_{00}(p_1^2, m_0^2, m_1^2) \Big) \tag{40}
$$

in accordance with [3, 9].

Concering $B_1$ [1] and `LoopTools` use the following identity

$$
A_0(m_0^2) - A_0(m_1^2) = (m_0^2 - m_1^2) B_0(0, m_0^2, m_1^2) \tag{41}
$$

that might help away with

In case $m_1$ and/or $m_2$ are very large the expression on the right-hand side of eq. (38) suffers very strong cancellations: the total is very much smaller than the individual terms. For this reason we have not used these algebraic relations, except to rewrite self-energy diagrams as much as possible in a form most suitable for numerical evaluation. ([1, below eq. D.6])

To compare the other results to [1] and `LoopTools` one has to use the *strict* $n \to 4$ limit and the following identities[12]:

$$(n-4)B_{00}(p_1^2, m_0^2, m_1^2) = \frac{1}{6}(p_1^2 - 3m_0^2 - 3m_1^2) \tag{42}$$

$$(n-4)B_{11}(p_1^2, m_0^2, m_1^2) = -\frac{2}{3} \tag{43}$$

## 2 Scalar Integrals

We focus on:

$$\gamma^*(q) + \mathrm{g}(k_1) \to Q(p_1) + \overline{Q}(p_2) \tag{44}$$

$$k_1^2 = 0 \quad p_1^2 = p_2^2 = m^2 \quad (p_1+p_2)^2 = s \quad (p_2-q)^2 = t \quad (p_1-q)^2 = u \tag{45}$$

define some shortcuts

$$0 \leq \rho = \frac{4m^2}{s} \leq 1 \qquad 0 \leq \beta = \sqrt{1-\rho} \leq 1 \qquad 0 \leq \chi = \frac{1-\beta}{1+\beta} \leq 1 \tag{46}$$

$$\rho_q = \frac{4m^2}{q^2} \leq 0 \qquad 1 \leq \beta_q = \sqrt{1-\rho_q} \qquad 0 \leq \chi_q = -\frac{1-\beta_q}{1+\beta_q} \leq 1 \tag{47}$$

### 2.1 One-Point Function $A_0$

[11]:

$$A_0(m^2) = -\frac{i}{16\pi^2} m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{(n-4)/2} \Gamma(1-n/2) \tag{48}$$

$$= \frac{im^2}{16\pi^2}\left(\Delta - \log(m^2/\mu^2) + 1\right) + O(n-4) \tag{49}$$

$$= iC_\epsilon m^2 \left(-\frac{2}{\epsilon} + 1\right) + O(n-4) \tag{50}$$

$$\Delta = \frac{2}{4-n} - \gamma_E + \log(4\pi) \tag{51}$$

$$C_\epsilon = \frac{1}{16\pi^2} \exp\left((\gamma_E - \log(4\pi))\frac{\epsilon}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{\epsilon/2} \tag{52}$$

this is *up to order* $O(n-4)$ in accordance with [3][13], but NOT beyond - see also [3, eq. (A.12)]. So we can treat $C_\epsilon$ and $\Delta$ as equal. We also find with [3]

$$A_0(0) = 0 \tag{53}$$

### 2.2 Two-Point Function $B_0$

In [11, eq. (4.23)] is a generic function given and we end up with

$$B_0(s, m^2, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 + \beta \log(\chi) \right) \tag{54}$$

$$B_0(q^2, m^2, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 + \beta_q \log(\chi_q) \right) \tag{55}$$

$$B_0(0, m^2, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} \right) \tag{56}$$

$$B_0(m^2, 0, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 \right) \tag{57}$$

$$B_0(t, 0, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 - \frac{t-m^2}{t} \ln \left( -\frac{t-m^2}{m^2} \right) \right) \tag{58}$$

focussing on imaginary part *only*; this in accordance with [3][13]. We also find with [3]

$$B_0(0,0,0) = 0 \tag{59}$$

### 2.3 Three-Point Function $C_0$

Again, in [11, eq. (4.26)] is a generic function given.

First, we compute $C_0(s, q^2, 0, m^2, m^2, m^2)$ and by taking the limit $k_1^2 \to 0$ (or equivalenty $s_4 \to 0$) we end up with:

$$\begin{aligned} C_0(s, q^2, 0, m^2, m^2, m^2) = \frac{i}{16\pi^2} \cdot \frac{1}{s-q^2} \Bigg( &\mathrm{Li}_2\left(\frac{2}{1+\beta_q}\right) + \mathrm{Li}_2\left(\frac{2}{1-\beta_q}\right) \\ &- \mathrm{Li}_2\left(\frac{2}{1+\beta}\right) - \mathrm{Li}_2\left(\frac{2}{1-\beta}\right) \Bigg) \end{aligned} \tag{60}$$

with [14] we find:

$$\mathrm{Li}_2\left(\frac{2}{1+b}\right) + \mathrm{Li}_2\left(\frac{2}{1-b}\right) = 3\zeta(2) + \frac{1}{2}\ln^2\left(\frac{1-b}{1+b}\right) - \ln\left(\frac{1-b}{1+b}\right)\ln\left(-\frac{1-b}{1+b}\right) \tag{61}$$

and if we focus on real part *only*, we find:

$$\mathrm{Li}_2\left(\frac{2}{1+\beta}\right) + \mathrm{Li}_2\left(\frac{2}{1-\beta}\right) = 3\zeta(2) - \frac{1}{2}\ln^2(\chi) \tag{62}$$

$$\mathrm{Li}_2\left(\frac{2}{1+\beta_q}\right) + \mathrm{Li}_2\left(\frac{2}{1-\beta_q}\right) = -\frac{1}{2}\ln^2(\chi_q) \tag{63}$$

Additionally, we find

$$\lim_{q^2\to 0}\left[\mathrm{Li}_2\left(\frac{2}{1+\beta_q}\right)+\mathrm{Li}_2\left(\frac{2}{1-\beta_q}\right)\right]=0 \tag{64}$$

So we get:

$$C_0(s,q^2,0,m^2,m^2,m^2)=iC_\epsilon\frac{1}{s-q^2}\left(\frac{1}{2}\ln^2(\chi)-\frac{1}{2}\ln^2(\chi_q)-3\zeta(2)\right) \tag{65}$$

$$C_0(s,0,0,m^2,m^2,m^2)=iC_\epsilon\frac{1}{s}\left(\frac{1}{2}\ln^2(\chi)-3\zeta(2)\right) \tag{66}$$

in accordance with [3][13][15]. These results can also be obtained by the methods described in [3, chap. 3].

Next, we compute $C_0(m^2,0,t,0,m^2,m^2)$ again by taking the limit $k_1^2\to 0$ we end up with:

$$\begin{aligned}C_0(m^2,0,t,0,m^2,m^2)=\frac{i}{16\pi^2}\cdot\frac{1}{t-m^2}\Bigg(&2\,\mathrm{Li}_2(2)+\mathrm{Li}_2(m^2/t)-\frac{\pi^2}{6}\\&-\mathrm{Li}_2((t+m^2)/m^2)-\mathrm{Li}_2((m^2+t)/t)\Bigg)\end{aligned} \tag{67}$$

Using [14] and focussing on real part, we find

$$\mathrm{Li}_2(2)=\frac{\pi^2}{4}-i\pi\ln(2) \tag{68}$$

$$2\,\mathrm{Li}_2(2)+\mathrm{Li}_2(1/z)-\frac{\pi^2}{6}-\mathrm{Li}_2(1+z)-\mathrm{Li}_2(1+1/z)=\frac{\pi^2}{6}-\mathrm{Li}_2(z) \tag{69}$$

So we get:

$$C_0(m^2,0,t,0,m^2,m^2)=iC_\epsilon\frac{1}{t-m^2}\left(\zeta(2)-\mathrm{Li}_2(t/m^2)\right) \tag{70}$$

in accordance with [3][13].

To compute $C_0(m^2,s,m^2,0,m^2,m^2)$ we use [3] and find

$$\begin{aligned}C_0(m^2,s,m^2,0,m^2,m^2)=\frac{iC_\epsilon}{s\beta}\Bigg(&-\frac{2}{\epsilon}\ln(\chi)-\frac{\pi^2}{2}+\frac{1}{2}\ln^2(\chi)-\ln(\chi)\ln(1-\chi)\\&-\mathrm{Li}_2(1/(1-\chi))+\mathrm{Li}_2(\chi/(\chi-1))\Bigg)\end{aligned} \tag{71}$$

Using [14] and focussing on real part, we find

$$-\mathrm{Li}_2(1/(1-\chi))+\mathrm{Li}_2(\chi/(\chi-1))=-2\,\mathrm{Li}_2(\chi)-\ln(\chi)\ln(1-\chi)-\frac{\pi^2}{6} \tag{72}$$

So we get:

$$C_0(m^2,s,m^2,0,m^2,m^2) = iC_\epsilon \frac{1}{s\beta}\left(-\frac{2}{\epsilon}\ln(\chi) - 2\ln(\chi)\ln(1-\chi) - 2\,\mathrm{Li}_2(\chi) + \frac{1}{2}\ln^2(\chi) - 4\zeta(2)\right) \tag{73}$$

in accordance with [3][13].

To compute $C_0(t,m^2,q^2,0,m^2,m^2)$ we use [16] and find immediatelty:

$$\begin{aligned} C_0(t,m^2,q^2,0,m^2,m^2) = \frac{iC_\epsilon}{\alpha}\Bigg[ & -\zeta(2) + 2\,\mathrm{Li}_2\left(\frac{t_1+\alpha}{t_1}\right) + \mathrm{Li}_2\left(\frac{q^2-t-m^2+\alpha}{q^2-t-m^2-\alpha}\right) \\ & + \mathrm{Li}_2\left(\frac{t_1-q^2\beta_q^2+\alpha}{t_1-q^2\beta_q^2-\beta_q\alpha}\right) - \mathrm{Li}_2\left(\frac{t_1-q^2\beta_q^2-\alpha}{t_1-q^2\beta_q^2+\beta_q\alpha}\right) \\ & + \mathrm{Li}_2\left(\frac{t_1-q^2\beta_q^2+\alpha}{t_1+q^2\beta_q^2-\beta_q\alpha}\right) - \mathrm{Li}_2\left(\frac{t_1-q^2\beta_q^2-\alpha}{t_1+q^2\beta_q^2+\beta_q\alpha}\right) \\ & - \mathrm{Li}_2\left(\frac{t_1(q^2-t-m^2-\alpha)-2m^2\alpha}{t_1(q^2-t-m^2+\alpha)}\right) \\ & - \mathrm{Li}_2\left(\frac{t_1(q^2-t-m^2-\alpha)-2m^2\alpha}{t_1(q^2-t-m^2-\alpha)}\right)\Bigg] \end{aligned} \tag{74}$$

with $\alpha = \kappa(t,q^2,m^2)$ and the Källén function (as defined in [16, eq. (4.27)])

$$\kappa(x,y,z) = \sqrt{x^2+y^2+z^2-2(xy+xz+yx)} \tag{75}$$

This is in accordance with [15, eq. (A.8)](Note the typo there!).

Additionally, we find

$$\lim_{q^2\to 0} C_0(t,m^2,q^2,0,m^2,m^2) = C_0(t,m^2,0,0,m^2,m^2) = C_0(m^2,0,t,0,m^2,m^2) \tag{76}$$

To compute $C_0(0,m^2,t,0,0,m^2)$ we use [3] and find

$$C_0(0,m^2,t,0,0,m^2) = \frac{iC_\epsilon}{t_1}\left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln(-t_1/m^2) + \ln^2(-t_1/m^2) + \mathrm{Li}_2(t/m^2) + \frac{\zeta(2)}{4}\right) \tag{77}$$

### 2.4 Four-Point Function $D_0$

To compute $D_0(m^2, 0, q^2, m^2, t, s, 0, m^2, m^2, m^2)$ we follow Ingos way[3] of computing his $D_0(p_1, -k_1, -k_2, 0, m, m, m) = D_0(m^2, 0, 0, m^2, t, s, 0, m^2, m^2, m^2)$ and find

$$\tilde{t} = -\frac{t_1}{m^2} \tag{78}$$

$$K = \frac{x}{\rho\rho_q}\left[4x(-1+y)yz\rho + yz\rho\rho_q\tilde{t} + x(-4(-1+y)y(-1+z) + \rho - \tilde{t}yz\rho)\rho_q\right] \tag{79}$$

$$I_{xy} = \frac{2x^{\epsilon/2}\rho\rho_q^{2-\epsilon/2}\left[\tilde{t}y\rho_q + x(\rho_q + y(4(y-1) - \tilde{t}\rho_q))\right]^{-1+\epsilon/2}}{(-2+\epsilon)\left[4x(-1+y)\rho + \tilde{t}\rho\rho_q - x(4(y-1)+\tilde{t}\rho)\rho_q\right]} \tag{80}$$

$$II_{xy} = -\frac{2x^{-1+\epsilon}\rho^{2-\epsilon/2}\rho_q\left[4(-1+y)y+\rho\right]^{-1+\epsilon/2}}{(-2+\epsilon)\left[4x(-1+y)\rho + \tilde{t}\rho\rho_q - x(4(y-1)+\tilde{t}\rho)\rho_q\right]} \tag{81}$$

"The integration of $I_{xy}$ does not diverge and one easily gets upon setting $\epsilon \to 0$"

$$I = \frac{m^4}{st_1\beta}\Bigg[\ln^2(\chi) + 4\,\mathrm{Li}_2(-\chi) + \frac{\pi^2}{3} + 2\ln(\chi_q)\ln\left(\frac{\beta_q+\beta}{\beta_q-\beta}\right) - 2\ln(\chi)\ln(1-q^2/s)$$
$$+2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q-\beta}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q+1}{\beta_q-\beta}\right) + 2\,\mathrm{Li}_2\left(\frac{\beta_q+1}{\beta_q+\beta}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q+\beta}\right)\Bigg] \tag{82}$$

$$= \frac{m^4}{st_1\beta}\Bigg[\ln^2(\chi) + 4\,\mathrm{Li}_2(-\chi) + \frac{\pi^2}{3} + \ln\left(\frac{\beta_q^2-\beta^2}{(\beta_q-1)^2}\right)\ln\left(\frac{\beta_q-\beta}{\beta_q+\beta}\right) - 2\ln(\chi)\ln(1-q^2/s)$$
$$+2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q-\beta}\right) + 2\,\mathrm{Li}_2\left(\frac{\beta_q-\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q+\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q+\beta}\right)\Bigg] \tag{83}$$

with

$$\lim_{q^2\to 0}\Bigg[\ln\left(\frac{\beta_q^2-\beta^2}{(\beta_q-1)^2}\right)\ln\left(\frac{\beta_q-\beta}{\beta_q+\beta}\right) - 2\ln(\chi)\ln(1-q^2/s)$$
$$+2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q-\beta}\right) + 2\,\mathrm{Li}_2\left(\frac{\beta_q-\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q+\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q+\beta}\right)\Bigg] = 0 \tag{84}$$

"Integrating $II_{xy}$ over $x$ gives"

$$II_y = -\frac{2}{\tilde{t}(-2+\epsilon)\epsilon}\left(\frac{\rho-4y(1-y)}{\rho}\right)^{-1+\epsilon/2} {}_2F_1\left(1,\epsilon;1+\epsilon;1-\frac{4(1-y)(\rho_q-\rho)}{\tilde{t}\rho\rho_q}\right) \tag{85}$$

"The integration over $y$ does not give an additional pole, so we can expand to $O(1)$ using ([3, eq. B.5]) and then integrate to obtain"

$$II = -\frac{m^4}{\beta st_1}\left(\frac{2\ln(\chi)}{\epsilon} + \ln(\chi)\left(1 + 2\ln(\beta\tilde{t}) + \ln(\chi) - 2\ln\left(1-q^2/s\right)\right) + \mathrm{Li}_2(\chi^2) + \frac{5\pi^2}{6}\right) \tag{86}$$

with

$$\lim_{q^2\to 0} \ln(1-q^2/s) = 0 \tag{87}$$

"The final result is then"

$$\begin{aligned}
&D_0(m^2,0,q^2,m^2,t,s,0,m^2,m^2,m^2)\\
&= \frac{iC_\epsilon}{\beta s t_1}\Bigg[-\frac{2\ln(\chi)}{\epsilon} - 2\ln(\chi)\ln(\beta\tilde{t}) + 2\,\mathrm{Li}_2(-\chi) - 2\,\mathrm{Li}_2(\chi) - 3\zeta(2)\\
&\qquad + \ln\left(\frac{\beta_q^2-\beta^2}{(\beta_q-1)^2}\right)\ln\left(\frac{\beta_q-\beta}{\beta_q+\beta}\right) - 2\ln(\chi)\ln(1-q^2/s)\\
&\qquad +2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q-\beta}\right) + 2\,\mathrm{Li}_2\left(\frac{\beta_q-\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q+\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q+\beta}\right)\Bigg]
\end{aligned} \tag{88}$$

This is NOT in accordance with [15, eq. (A.3)] - but I suspect a bunch of typos there.

We get the match to [3] and [13] by using

$$\begin{aligned}
&\lim_{q^2\to 0}\Bigg[\ln\left(\frac{\beta_q^2-\beta^2}{(\beta_q-1)^2}\right)\ln\left(\frac{\beta_q-\beta}{\beta_q+\beta}\right) - 2\ln(\chi)\ln(1-q^2/s)\\
&\quad 2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q-\beta}\right) + 2\,\mathrm{Li}_2\left(\frac{\beta_q-\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q+\beta}{\beta_q+1}\right) - 2\,\mathrm{Li}_2\left(\frac{\beta_q-1}{\beta_q+\beta}\right)\Bigg] \quad = 0
\end{aligned} \tag{89}$$

FiXme Error: fix

Nevertheless this is probably wrong - because we find with [10, Box 16]:

$$\begin{aligned}
&D_0(m^2,0,q^2,m^2,t,s,0,m^2,m^2,m^2)\\
&= \frac{iC_\epsilon}{\beta s t_1}\Bigg[-\frac{2\ln(\chi)}{\epsilon} - 2\ln(\chi)\ln(\tilde{t}) + \mathrm{Li}_2(1-\chi^2) - 4\zeta(2) + \ln^2(\chi_q) + 2\,\mathrm{Li}_2(-\chi\chi_q)\\
&\qquad +2\,\mathrm{Li}_2(-\chi/\chi_q) + 2\ln(\chi\chi_q)\ln(1+\chi\chi_q) + 2\ln(\chi/\chi_q)\ln(1+\chi/\chi_q)\Bigg]
\end{aligned} \tag{90}$$

We get the match to [3] and [13] by using

$$\begin{aligned}
&\lim_{q^2\to 0}\Big[\mathrm{Li}_2(1-\chi^2) - 4\zeta(2) + \ln^2(\chi_q) + 2\,\mathrm{Li}_2(-\chi\chi_q) + 2\,\mathrm{Li}_2(-\chi/\chi_q)\\
&\qquad +2\ln(\chi\chi_q)\ln(1+\chi\chi_q) + 2\ln(\chi/\chi_q)\ln(1+\chi/\chi_q)\Big]\\
&= -2\ln(\chi)\ln(\beta) - 3\zeta(2) + 2\,\mathrm{Li}_2(2,-\chi) - 2\,\mathrm{Li}_2(2,\chi)
\end{aligned} \tag{91}$$

To compute $D_0(0,m^2,q^2,m^2,t,u,0,0,m^2,m^2)$ I neither suceeded with [3] nor [16], but one can use [10, Box 11] (transformation see sec. 1.1) So we get

$$\begin{aligned}
D_0(0,m^2,q^2,m^2,t,u,0,0,m^2,m^2) &= \frac{iC_\epsilon}{t_1u_1}\left(\frac{4}{\epsilon^2} + \frac{2}{\epsilon}\left(\ln(-t_1/m^2) + \ln(-u_1/m^2)\right)\right.\\
&\qquad \left. +2\ln(-t_1/m^2)\ln(-u_1/m^2) - \frac{7}{2}\zeta(2) - \ln^2(\chi_q)\right)
\end{aligned} \tag{92}$$

This is in accordance with [15, eq. (A.4)] using [14] (as above):

$$2\,\mathrm{Li}_2\left(\frac{q^2(1+\beta_q)}{2m^2}\right)+2\,\mathrm{Li}_2\left(\frac{q^2(1-\beta_q)}{2m^2}\right)=2\,\mathrm{Li}_2\left(\frac{2}{1-\beta_q}\right)+2\,\mathrm{Li}_2\left(\frac{2}{1-\beta_q}\right) \tag{93}$$

$$= -\ln^2(\chi_q) \tag{94}$$

(The question, why they use a complicated dilogarithm remains open ...)

We get the match to [3] and [13] by using

$$\lim_{q^2\to 0}\ln^2(\chi_q)=0 \tag{95}$$

to find

$$D_0(0,m^2,0,m^2,t,u,0,0,m^2,m^2)=\frac{iC_\epsilon}{t_1u_1}\left(\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\left(\ln(-t_1/m^2)+\ln(-u_1/m^2)\right)\right.$$

$$\left.+2\ln(-t_1/m^2)\ln(-u_1/m^2)-\frac{7}{2}\zeta(2)\right) \tag{96}$$

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## List of Corrections