1 2 to 2 phase space

following [1]:

process:

$$\gamma^*(q) + \mathrm{g}(k_1) \to Q(p_1) + \overline{Q}(p_2) \tag{1}$$

kinematics:

$$s = (q+k_1)^2 \qquad s' = s - q^2 \tag{2}$$
$$t = (k_1 - p_2)^2 \qquad t_1 = t - m^2 \tag{3}$$
$$u = (q - p_2)^2 \qquad u_1 = u - m^2 \tag{4}$$

use c.m.s. of incoming particles:

$$q = \left(\frac{s+q^2}{2\sqrt{s}}, 0, 0, \ldots, -\frac{s-q^2}{2\sqrt{s}}\right) \tag{5}$$
$$k_1 = \frac{s-q^2}{2\sqrt{s}}\,(1, 0, 0, \ldots, 1) \tag{6}$$

such that

$$q + k_1 = (\sqrt{s}, \vec{0}) \qquad k_1^2 = 0 \tag{7}$$

for the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2}\,(1, 0, \beta\sin\theta, \ldots, \beta\cos\theta) \tag{8}$$
$$p_2 = \frac{\sqrt{s}}{2}\,(1, 0, -\beta\sin\theta, \ldots, -\beta\cos\theta) \tag{9}$$

with $\beta = \sqrt{1 - 4m^2/s}$ such that

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \qquad p_1^2 = p_2^2 = m^2 \tag{10}$$

use n-sphere:

$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \tag{11}$$

compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_1}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q + k_1 - p_1 - p_2)\delta(p_1^2 - m^2)\delta(p_2^2 - m^2) \tag{12}$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1 \, \delta((q + k_1 - p_1)^2 - m^2)\delta(p_1^2 - m^2) \tag{13}$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,||} d^2 p_{1,\perp} d^{n-4} \hat{p}_1 \, \delta(s - 2p_{1,0}\sqrt{s})\delta(p_{1,0}^2 - p_{1,||}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{14}$$

$$= \frac{1}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,||} dp_{1,\perp}^2 d^{n-4} \hat{p}_1 \, \delta(s/4 - p_{1,||}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{15}$$

$$= \frac{1}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,||} d\hat{p}_1^2 \, \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \tag{16}$$

$$= \frac{1}{2\sqrt{s}\Gamma((n-4)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} d\hat{p}_1^2 \, (\hat{p}_1^2)^{(n-6)/2} \tag{17}$$

Integration borders are

$$p_{1,||} \in \frac{\sqrt{s}}{2}\beta \cdot [-1, 1] \qquad \hat{p}_1^2 \in \left(\frac{s\beta^2}{4} - p_{1,||}^2\right) \cdot [0, 1] \tag{18}$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2 \, (\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4} \left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{19}$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} \left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{20}$$

rewrite $p_{1,||}$ to $\cos\theta$:

$$p_{1,||} = \frac{\sqrt{s}}{2}\beta\cos\theta \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{2}\beta \, d\cos\theta, \quad \cos\theta \in [-1, 1], \, \hat{p}_1^2 \in \frac{s\beta^2}{4}\left(1 - \cos^2\theta\right) \cdot [0, 1] \tag{21}$$

rewrite $\cos\theta$ to $t_1 = (k_1 - p_2)^2 - m^2$:

$$\cos\theta = \frac{2t_1/s' + 1}{\beta} \Rightarrow \quad d\cos\theta = \frac{2}{\beta s'} dt_1, \quad t_1 \in \frac{s'}{2}[-\beta - 1, \beta - 1], \, \hat{p}_1^2 \in (-m^2 - \frac{st_1}{s'^2}(s' + t_1)) \cdot [0, 1] \tag{22}$$

$$p_{1,||} = \sqrt{s}\left(\frac{t_1}{s'} + \frac{1}{2}\right) \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{s'} dt_1 \tag{23}$$

$$\Rightarrow PS_2 = \frac{1}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dt_1 \left(\frac{(t_1(u_1 - q^2) - s'm^2)s' - q^2 t_1^2}{s'^2}\right)^{(n-4)/2} \tag{24}$$

2 2 to 3 phase space

following [2, 3, 1]:

process:

$$\gamma^*(q) + q(k_1) \to Q(p_1) + \overline{Q}(p_2) + q(k_2) \tag{25}$$

$$\gamma^*(q) + \mathrm{g}(k_1) \to Q(p_1) + \overline{Q}(p_2) + \mathrm{g}(k_2) \tag{26}$$

2.1 kinematic constraints

definitions of kinematic variables:

$$\begin{aligned}
s &= (q+k_1)^2 & \Rightarrow \quad & 2qk_1 = s - q^2 = s' & (27)\\
s_3 &= (k_2+p_2)^2 - m^2 & \Rightarrow \quad & 2k_2p_2 = s_3 & (28)\\
s_4 &= (k_2+p_1)^2 - m^2 & \Rightarrow \quad & 2k_2p_1 = s_4 & (29)\\
s_5 &= (p_1+p_2)^2 = -u_5 & \Rightarrow \quad & 2p_1p_2 = s_5 - 2m^2 & (30)\\
t_1 &= (k_1-p_2)^2 - m^2 = t - m^2 & \Rightarrow \quad & 2k_1p_2 = -t_1 & (31)\\
t' &= (k_1-k_2)^2 & \Rightarrow \quad & 2k_1k_2 = -t' & (32)\\
u_1 &= (q-p_2)^2 - m^2 = u - m^2 & \Rightarrow \quad & 2qp_2 = -u_1 + q^2 & (33)\\
u_6 &= (k_1-p_1)^2 - m^2 & \Rightarrow \quad & 2k_1p_1 = -u_6 & (34)\\
u_7 &= (q-p_1)^2 - m^2 & \Rightarrow \quad & 2qp_1 = -u_7 + q^2 & (35)\\
u' &= (q-k_2)^2 & \Rightarrow \quad & 2qk_2 = -u' + q^2 & (36)
\end{aligned}$$

impose momentum conservation:

$$q + k_1 = p_1 + p_2 + k_2 \tag{37}$$

contract with 2 times momentum:

$$\begin{aligned}
2q^2 \quad +s' &= -u_7 + q^2 \quad -u_1 + q^2 \quad -u' + q^2 & \Leftrightarrow \quad & 0 = s' + u_1 + u_7 + u' - q^2 & (38)\\
s - q^2 \quad +0 &= -u_6 \quad -t_1 \quad -t' & \Leftrightarrow \quad & 0 = s' + t_1 + t' + u_6 & (39)\\
-u_7 + q^2 \quad -u_6 &= 2m^2 \quad +s_5 - 2m^2 \quad +s_4 & \Leftrightarrow \quad & 0 = s_4 + s_5 + u_6 + u_7 - q^2 & (40)\\
-u_1 + q^2 \quad -t_1 &= s_5 - 2m^2 \quad +2m^2 \quad +s_3 & \Leftrightarrow \quad & 0 = s_3 + s_5 + t_1 + u_1 - q^2 & (41)\\
-u' + q^2 \quad -t' &= s_4 \quad +s_3 \quad +0 & \Leftrightarrow \quad & 0 = s_3 + s_4 + t' + u' - q^2 & (42)
\end{aligned}$$

$$\frac{1}{2}((38)+(39)+(41)-(40)-(42)) = 0 = s' + t_1 + u_1 - s_4 \quad (43)$$
$$\frac{1}{2}((38)+(39)+(42)-(40)-(41)) = 0 = s' + t' + u' - s_5 \quad (44)$$
$$\frac{1}{2}((41)+(42)-(38)-(39)-(40)) = 0 = s_3 - s' - u_6 - u_7 \quad (45)$$
$$\frac{1}{2}((40)+(41)+(42)-(38)-(39)) = 0 = s_3 + s_4 + s_5 - s \quad (46)$$
$$\frac{1}{2}((38)+(40)-(39)-(41)-(42)) = 0 = -s_3 - t_1 - t' + u_7 \quad (47)$$
$$\frac{1}{2}((39)+(40)+(42)-(38)-(41)) = 0 = s_4 + t' - u_1 + u_6 \quad (48)$$

2.2 choose framework

use c.m.s. of recoiling heavy and light quark ($Q(p_1)$ and $q(k_2)$):

$$k_2 = (\omega_2, k_{2,x}, \omega_2 \sin\theta_1 \cos\theta_2, \omega_2 \cos\theta_1, \hat{k}_2) \quad (49)$$
$$p_1 = (E_1, -k_{2,x}, -\omega_2 \sin\theta_1 \cos\theta_2, -\omega_2 \cos\theta_1, -\hat{k}_2) \quad (50)$$

where $k_{2,x} = k_{2,x}(\omega_2, \theta_1, \theta_2, \hat{k}_2)$ is such, that $k_2^2 = 0$.

The only remainding choice is then the alignment of the z-axis: either along k_1, called "Set I"(2.2.1), along q, called "Set II"(2.2.2) or along p_1, called "Set III"(2.2.3).

2.2.1 Set I

align k_1 to z-axis:

$$k_1 = (\omega_1, 0, 0, \omega_1, \hat{0}) \quad (51)$$
$$q = (q_0, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi - \omega_1, \hat{0}) \quad (52)$$
$$p_2 = (E_2, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi, \hat{0}) \quad (53)$$

constraints, from energy conservation, masses (light quark masses are already fixed: $k_1^2 =$

$0 = k_2^2$) and external Mandelstam variables:

$$
\begin{aligned}
q_0 + \omega_1 \quad & = E_1 + E_2 + \omega_2 && (54)\\
m^2 = p_1^2 \quad & = E_1^2 - \omega_2^2 && (55)\\
m^2 = p_2^2 \quad & = E_2^2 - |\vec{p}_2|^2 && (56)\\
q^2 \quad & = q_0^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\,\omega_1 \cos\psi - \omega_1^2 && (57)\\
s = (q+k_1)^2 \quad & = (q_0 + \omega_1)^2 - |\vec{p}_2|^2 && (58)\\
t = (k_1 - p_2)^2 \quad & = (\omega_1 - E_2)^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\,\omega_1 \cos\psi - \omega_1^2 && (59)\\
u = (q - p_2)^2 \quad & = (q_0 - E_2)^2 - \omega_1^2 && (60)
\end{aligned}
$$

solve:

$$
\begin{aligned}
(58) - (57) + (59) - (56) + (60) &= s - q^2 + t - m^2 + u && (61)\\
&= s_4 + m^2 = (E_1 + \omega_2)^2 && (62)\\
(59) + (60) - (57) &= t + u - q^2 = -2(E_1 + \omega_2)E_2 && (63)\\
\Rightarrow E_2 &= -\frac{t + u - q^2}{2\sqrt{s_4 + m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4 + m^2}} && (64)\\
(62) \wedge (55) \Rightarrow \omega_2 &= \frac{s_4}{2\sqrt{s_4 + m^2}} && (65)\\
(62) \Rightarrow E_1 &= \frac{s_4 + 2m^2}{2\sqrt{s_4 + m^2}} && (66)\\
(58) + (60) - (56) &= s + u - m^2 = 2q_0(E_1 + \omega_2) && (67)\\
\Rightarrow q_0 &= \frac{s + u_1}{2\sqrt{s_4 + m^2}} && (68)\\
(59) - (57) &= t - q^2 = (\omega_1 - E_2)^2 - q_0^2 && (69)\\
\Rightarrow \omega_1 &= \frac{s' + t_1}{2\sqrt{s_4 + m^2}} = \frac{s_4 - u_1}{2\sqrt{s_4 + m^2}} && (70)\\
(56) \Rightarrow |\vec{p}_2| &= \sqrt{E_2^2 - m^2} = \frac{\sqrt{(s - s_4)^2 - 4sm^2}}{2\sqrt{s_4 + m^2}} && (71)\\
(57) \Rightarrow \cos\psi &= \frac{q^2 - q_0^2 + |\vec{p}_2|^2 + \omega_1^2}{2\,|\vec{p}_2|\,\omega_1} && (72)\\
&= \frac{(u_1 + m^2)(t_1 - s') - (m^2 - q^2 - t_1)(s' + t_1)}{(s' + t_1)\sqrt{(s - s_4)^2 - 4sm^2}} && (73)\\
\Rightarrow \sin\psi &= 2\frac{\sqrt{s_4 + m^2}\sqrt{m^2 s'^2 + q^2 t_1 (s' + t_1) - s' t_1 u_1}}{(s' + t_1)\sqrt{(s - s_4)^2 - 4sm}} && (74)
\end{aligned}
$$

$$t' = -2k_1k_2 = -2\omega_1\omega_2(1-\cos\theta_1) \tag{75}$$
$$u_6 = -2k_1p_1 = -2\omega_1(E_1+\omega_2\cos\theta_1) \tag{76}$$
$$(39): \qquad 0 = s + t_1 + t' + u_6 - q^2 \quad \checkmark \tag{77}$$

t' is the only variable that can get collinear (for $-q^2 > 0$).

$$s_3 = 2k_2p_2 = 2\omega_2(E_2 - |\vec{p}_2|(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{78}$$
$$s_5 = (p_1+p_2)^2 = 2m^2 + 2p_1p_2 \tag{79}$$
$$= 2(m^2 + E_1E_2 + \omega_2|\vec{p}_2|(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{80}$$
$$(41): \qquad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad \checkmark \tag{81}$$

$$u' = (q-k_2)^2 = q^2 - 2qk_2 \tag{82}$$
$$= q^2 - 2(q_0\omega_2 - \omega_2(|\vec{p}_2|\cos\psi - \omega_1)\cos\theta_1 - \omega_2|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{83}$$
$$u_7 = q^2 - 2qp_1 \tag{84}$$
$$= q^2 - 2(q_0E_1 + \omega_2(|\vec{p}_2|\cos\psi - \omega_1)\cos\theta_1 + \omega_2|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{85}$$
$$(38): \qquad 0 = s + u_1 + u_7 + u' - 2q^2 \quad \checkmark \tag{86}$$

2.2.2 Set II

align q to z-axis:

$$q = (q_0, 0, 0, q_z, \hat{0}) \tag{87}$$
$$k_1 = (\omega_1, 0, |\vec{p}_2|\sin\psi, |\vec{p}_2|\cos\psi - q_z, \hat{0}) \tag{88}$$
$$p_2 = (E_2, 0, |\vec{p}_2|\sin\psi, |\vec{p}_2|\cos\psi, \hat{0}) \tag{89}$$

constraints, from energy conservation, masses ($k_2^2 = 0$ is already fixed) and external

Mandelstam variables:

$$
\begin{aligned}
q_0 + \omega_1 \qquad\qquad &= E_1 + E_2 + \omega_2 && (90)\\
m^2 = p_1^2 \qquad\qquad &= E_1^2 - \omega_2^2 && (91)\\
m^2 = p_2^2 \qquad\qquad &= E_2^2 - |\vec{p}_2|^2 && (92)\\
q^2 \qquad\qquad &= q_0^2 - q_z^2 && (93)\\
0 \qquad\qquad &= \omega_1^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\, q_z \cos\psi - q_z^2 && (94)\\
s = (q + k_1)^2 \qquad\qquad &= (q_0 + \omega_1)^2 - |\vec{p}_2|^2 && (95)\\
t = (k_1 - p_2)^2 \qquad\qquad &= (\omega_1 - E_2)^2 - q_z^2 && (96)\\
u = (q - p_2)^2 \qquad\qquad &= (q_0 - E_2)^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\, q_z \cos\psi - q_z^2 && (97)
\end{aligned}
$$

solve:

$$(97) - (94) = u = (q_0 - E_2)^2 - \omega_1^2 \tag{98}$$

$$(95) - (93) + (96) - (92) + (98) = s - q^2 + t - m^2 + u \tag{99}$$

$$= s_4 + m^2 = (\omega_1 + q_0 - E_2)^2 \tag{100}$$

$$(96) + (97) - (93) = t + u - q^2 = 2(E_2 - \omega_1 - q_0)E_2 \tag{101}$$

$$\Rightarrow E_2 = -\frac{t + u - q^2}{2\sqrt{s_4 + m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4 + m^2}} = (64) \tag{102}$$

$$(95) - (93) + (96) - (92) = s - q^2 + t - m^2 = -2(E_2 - \omega_1 - q_0)\omega_1 \tag{103}$$

$$\Rightarrow \omega_1 = \frac{s' + t_1}{2\sqrt{s_4 + m^2}} = \frac{s_4 - u_1}{2\sqrt{s_4 + m^2}} = (70) \tag{104}$$

$$(100) \wedge (102) \wedge (104) \Rightarrow q_0 = \frac{s + u_1}{2\sqrt{s_4 + m^2}} = (68) \tag{105}$$

$$(95) \wedge (105) \wedge (104) \Rightarrow |\vec{p}_2| = \frac{\sqrt{(s - s_4)^2 - 4m^2 s}}{2\sqrt{s_4 + m^2}} = (71) \tag{106}$$

$$(93) \wedge (105) \Rightarrow q_z = \frac{\sqrt{(s' + u_1')^2 - 4q^2 t}}{2\sqrt{s_4 + m^2}} \tag{107}$$

$$(91) \Rightarrow \omega_2 = \frac{s_4}{2\sqrt{s_4 + m^2}} = (65) \tag{108}$$

$$(90) \Rightarrow E_1 = \frac{s_4 + 2m^2}{2\sqrt{s_4 + m^2}} = (66) \tag{109}$$

$$(93) \Rightarrow \cos\psi = \frac{s(q^2 - t - m^2) - 2q^2(s_4 + m^2) + s_4 u_1}{\sqrt{((s - s_4)^2 - 4m^2 s)((s' + u_1')^2 - 4q^2 t)}} \neq (73) \tag{110}$$

$$u' = (q-k_2)^2 = q^2 - 2qk_2 \qquad = q^2 - 2(q_0\omega_2 - q_z\omega_2\cos\theta_1 \tag{111}$$

$$u_7 = q^2 - 2qp_1 \qquad = q^2 - 2(q_0E_1 + q_z\omega_2\cos\theta_1) \tag{112}$$

$$(38): \qquad 0 = s' + u_1' + u_7 + u' \quad \checkmark \tag{113}$$

$$s_3 = 2k_2p_2 = 2\omega_2(E_2 - |\vec{p}_2|(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) = (78) \tag{114}$$

$$s_5 = (p_1+p_2)^2 = 2m^2 + 2p_1p_2 \tag{115}$$

$$= 2(m^2 + E_1E_2 + \omega_2|\vec{p}_2|(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) = (80) \tag{116}$$

$$(41): \qquad 0 = s_3 + s_5 + t_1 + u_1' \quad \checkmark \tag{117}$$

$$t' = -2k_1k_2 \tag{118}$$

$$= -2\omega_2(\omega_1 - (|\vec{p}_2|\cos\psi - q_z)\cos\theta_1 - |\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{119}$$

$$u_6 = -2k_1p_1 \tag{120}$$

$$= -2(\omega_1E_1 + \omega_2(|\vec{p}_2|\cos\psi - q_z)\cos\theta_1 + \omega_2|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{121}$$

$$(39): \qquad 0 = s' + t_1 + t' + u_6 \quad \checkmark \tag{122}$$

same as before: t' is the only variable that can get collinear (for $-q^2 > 0$) - now collinear as [ABC] variable.

2.2.3 Set III

align p_2 to z-axis:

$$q = (q_0, 0, |\vec{q}|\sin\psi, |\vec{q}|\cos\psi, \hat{0}) \tag{123}$$

$$k_1 = (\omega_1, 0, -|\vec{q}|\sin\psi, -|\vec{q}|\cos\psi + p_{2,z}, \hat{0}) \tag{124}$$

$$p_2 = (E_2, 0, 0, p_{2,z}, \hat{0}) \tag{125}$$

constraints, from energy conservation, masses ($k_2^2 = 0$ is already fixed) and external

Mandelstam variables:

$$
\begin{align}
q_0 + \omega_1 & = E_1 + E_2 + \omega_2 \tag{126} \\
m^2 = p_1^2 & = E_1^2 - \omega_2^2 \tag{127} \\
m^2 = p_2^2 & = E_2^2 - p_{2,z}^2 \tag{128} \\
q^2 & = q_0^2 - |\vec{q}\,|^2 \tag{129} \\
0 & = \omega_1^2 - |\vec{q}\,|^2 + 2\,|\vec{q}\,|\,p_{2,z}\cos\psi - p_{2,z}^2 \tag{130} \\
s = (q + k_1)^2 & = (q_0 + \omega_1)^2 - p_{2,z}^2 \tag{131} \\
t = (k_1 - p_2)^2 & = (\omega_1 - E_2)^2 - |\vec{q}\,|^2 \tag{132} \\
u = (q - p_2)^2 & = (q_0 - E_2)^2 - |\vec{q}\,|^2 + 2\,|\vec{q}\,|\,p_{2,z}\cos\psi - p_{2,z}^2 \tag{133}
\end{align}
$$

solve:

$$(133)-(130) = u = (q_0 - E_2)^2 - \omega_1^2 \tag{134}$$

$$(131)-(129)+(132)-(128)+(134) = s - q^2 + t - m^2 + u \tag{135}$$

$$= s_4 + m^2 = (\omega_1 + q_0 - E_2)^2 = (100) \tag{136}$$

$$(132)+(133)-(129) = t + u - q^2 = 2(E_2 - \omega_1 - q_0)E_2 \tag{137}$$

$$\Rightarrow E_2 = -\frac{t+u-q^2}{2\sqrt{s_4+m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4+m^2}} = (64) = (102) \tag{138}$$

$$(131)-(129)+(132)-(128) = s - q^2 + t - m^2 = -2(E_2 - \omega_1 - q_0)\omega_1 \tag{139}$$

$$\Rightarrow \omega_1 = \frac{s' + t_1}{2\sqrt{s_4+m^2}} = \frac{s_4 - u_1}{2\sqrt{s_4+m^2}} = (70) = (104) \tag{140}$$

$$(136) \wedge (138) \wedge (140) \Rightarrow q_0 = \frac{s + u_1}{2\sqrt{s_4+m^2}} = (68) = (105) \tag{141}$$

$$(129) \wedge (141) \Rightarrow |\vec{q}\,| = \frac{\sqrt{(s' + u_1')^2 - 4q^2 t}}{2\sqrt{s_4+m^2}} = (107) \tag{142}$$

$$(127) \wedge (126) \Rightarrow E_1 = \frac{s_4 + 2m^2}{2\sqrt{s_4+m^2}} = (66) = (109) \tag{143}$$

$$(126) \Rightarrow \omega_2 = \frac{s_4}{2\sqrt{s_4+m^2}} = (65) = (108) \tag{144}$$

$$(128) \wedge (138) \Rightarrow p_{2,z} = \frac{\sqrt{(s - s_4)^2 - 4m^2 s}}{2\sqrt{s_4+m^2}} = (71) = (106) \tag{145}$$

$$(129) \Rightarrow \cos\psi = \frac{s(q^2 - t - m^2) - 2q^2(s_4 + m^2) + s_4 u_1}{\sqrt{((s-s_4)^2 - 4m^2 s)((s' + u_1')^2 - 4q^2 t)}} \tag{146}$$

$$= (110) \neq (73)$$

$$s_3 = 2k_2 p_2 = 2\omega_2(E_2 - p_{2,z}\cos\theta_1) \tag{147}$$

$$s_5 = (p_1 + p_2)^2 = 2m^2 + 2p_1 p_2 \tag{148}$$

$$= 2(m^2 + E_1 E_2 + \omega_2 p_{2,z}\cos\theta_1) \tag{149}$$

$$(41): \qquad 0 = s_3 + s_5 + t_1 + u_1' \quad \checkmark \tag{150}$$

$$u' = (q - k_2)^2 = q^2 - 2qk_2 \tag{151}$$
$$= q^2 - 2\omega_2(q_0 - |\vec{q}|\,(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{152}$$
$$u_7 = q^2 - 2qp_1 \tag{153}$$
$$= q^2 - 2(q_0E_1 + |\vec{q}|\,\omega_2(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2))) \tag{154}$$
$$(38): \qquad 0 = s' + u_1' + u_7 + u' \quad \checkmark \tag{155}$$

$$t' = -2k_1k_2 \tag{156}$$
$$= -2\omega_2(\omega_1 + (|\vec{q}|\cos\psi - p_{2,z})\cos\theta_1 + |\vec{q}|\sin\psi\sin\theta_1\cos\theta_2) \tag{157}$$
$$u_6 = -2k_1p_1 \tag{158}$$
$$= -2(\omega_1E_1 - \omega_2(|\vec{q}|\cos\psi - p_{2,z})\cos\theta_1 - \omega_2\,|\vec{q}|\sin\psi\sin\theta_1\cos\theta_2) \tag{159}$$
$$(39): \qquad 0 = s' + t_1 + t' + u_6 \quad \checkmark \tag{160}$$

same as before: t' is the only variable that can get collinear (for $-q^2 > 0$) - now collinear as [ABC] variable.

2.3 phase space integrals

at phase space integration there occur integrations over propagators[4, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1\,\sin^{n-3}(\theta_1)\int_0^\pi d\theta_2\,\sin^{n-4}(\theta_2)(a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{161}$$
$$= \int d\Omega_n\,(a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{162}$$

the integrals can be further destinguished by the range of k, l and the type of collinearity (following the notation in [4]):

- "non collinear": $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \to I_{0,n}^{(k,l)}$
- "single collinear a": $a = -b \wedge A^2 \neq B^2 + C^2 \to I_{a,n}^{(k,l)}$
- "single collinear A": $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \to I_{A,n}^{(k,l)}$

- "double collinear": $a = -b \wedge A = -\sqrt{B^2 + C^2} \to I_{aA,n}^{(k,l)}$

Use $n = 4 + \epsilon$.

2.3.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int\limits_0^\pi dt\, \sin^{\nu-3}(t) \cos^q(t) \tag{163}$$

It is[5, eq. 5.12.6]:

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha\, B\left((\alpha+\beta+1)/2, (\alpha-\beta+1)/2\right)} \qquad \text{if} \Re(\alpha) > 0 \tag{164}$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{B((n-1)/2, (n-1)/2)} \tag{165}$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{B((n-2)/2, (n-2)/2)} = B((n-3)/2, 1/2) \tag{166}$$

If q is odd: $\hat{I}^{(q)} = 0$, due to symetry of kernel; if q is even: $q = 2p$ with $p \in \mathbb{N}$:

$$\begin{aligned}
\hat{I}^{(2p)}(\nu) &= \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int\limits_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t)\, dt && (167)\\
&= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2 + (k-p), (\nu-1)/2 - (k-p))} && (168)\\
&= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^{p} \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2 + l, (\nu-1)/2 - l)} && (169)\\
&= \frac{\pi\Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right.\\
&\qquad \left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) && (170)\\
&= \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right.\\
&\qquad \left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) && (171)
\end{aligned}$$

TODO: prove

FiXme Error: prove

$$p!\left(\frac{1}{(p!)^2}\frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})}+2\sum_{l=1}^{p}\frac{(-1)^l}{(p+l)!(p-l)!}\frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)}\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)}\right) \tag{172}$$

$$=\frac{1}{p!}\frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})}+2\sum_{l=1}^{p}\frac{(-1)^l p!}{(p+l)!(p-l)!}\frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)}\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \tag{173}$$

$$=1 \tag{174}$$

$$\Rightarrow \hat{I}^{(2p)}(\nu)=\frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}-p)}\cdot\frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})}\cdot\frac{(2p!)}{2^p p!} \tag{175}$$

$$=\frac{\sqrt{\pi}(2p)!}{2^{2p}p!}\frac{\Gamma((\nu-2)/2)}{\Gamma(\frac{\nu-1}{2}+p)} \tag{176}$$

2.3.2 any collinearity and $-k, -l \in \mathbb{N}_0$

If $-k, -l \in \mathbb{N}_0$ $I_n^{(k,l)}$ can always be reduced in a straight forward manner to combinations of $\hat{I}^{(q)}(n)$ and this way one finds[4, Ch. 5][2, App. C]:

$$I_n^{(0,0)}=\hat{I}^{(0)}(n-1)\cdot\hat{I}^{(0)}(n)=\frac{2\pi}{n-3} \tag{177}$$

$$I_4^{(0,0)}=2\pi \tag{178}$$

$$I_n^{(-1,0)}=\hat{I}^{(0)}(n-1)\cdot(a\hat{I}^{(0)}(n)+b\hat{I}^{(1)}(n))=\frac{2\pi a}{n-3} \tag{179}$$

$$I_4^{(-1,0)}=2\pi a \tag{180}$$

$$I_n^{(0,-1)}=\hat{I}^{(0)}(n-1)\cdot(A\hat{I}^{(0)}(n)+B\hat{I}^{(1)}(n))+C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \tag{181}$$

$$=\frac{2\pi A}{n-3} \tag{182}$$

$$I_4^{(0,-1)}=2\pi A \tag{183}$$

$$I_n^{(-2,0)}=\hat{I}^{(0)}(n-1)\cdot(a^2\hat{I}^{(0)}(n)+2ab\hat{I}^{(1)}(n)+b^2\hat{I}^{(2)}(n)) \tag{184}$$

$$=2\pi\left(\frac{a^2(n-1)+b^2}{(n-1)(n-3)}\right) \tag{185}$$

$$I_4^{(-2,0)}=2\pi(a^2+b^2/3) \tag{186}$$

$$I_n^{(0,-2)}=\hat{I}^{(0)}(n-1)\cdot(A^2\hat{I}^{(0)}(n)+B^2\hat{I}^{(2)}(n))+C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \tag{187}$$

$$=2\pi\left(\frac{A^2(n-1)+B^2+C^2}{(n-1)(n-3)}\right) \tag{188}$$

$$I_4^{(0,-2)}=2\pi(A^2+(B^2+C^2)/3) \tag{189}$$

$$I_n^{(-1,-1)}=\hat{I}^{(0)}(n-1)\cdot(aA\hat{I}^{(0)}(n)+bB\hat{I}^{(2)}(n))=2\pi\left(\frac{aA(n-1)+bB}{(n-1)(n-3)}\right) \tag{190}$$

$$I_4^{(-1,-1)}=2\pi(aA+bB/3) \tag{191}$$

2.3.3 single collinear a

If $-l \in \mathcal{N}$ one finds:

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1-\cos(t))^k} \cos^q(t)\, dt \tag{192}$$
$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1-\cos^2(t))^k} \cos^q(t)(1+\cos(t))^k\, dt \tag{193}$$
$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t)(1+\cos(t))^k\, dt \tag{194}$$
$$= \sum_{l=0}^{k} \binom{k}{l} \hat{I}^{(q+l)}(\nu-2k) \tag{195}$$

this way one finds[4, Ch. 5][2, App. C]:

$$I_{a,n}^{(1,0)} = \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot\hat{I}^{(0)}(n-2) \tag{196}$$
$$= \frac{2\pi}{a(n-4)} \tag{197}$$
$$I_{a,n}^{(1,-1)} = \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot\Big(A\hat{I}^{(0)}(n-2)+B\hat{I}^{(2)}(n-2)\Big) \tag{198}$$
$$= \frac{2\pi}{a}\frac{(A(n-3)+B)}{(n-3)(n-4)} \approx \frac{2\pi}{a}\left(\frac{A+B}{\epsilon}-2B+O(\epsilon)\right) \tag{199}$$
$$I_{a,n}^{(1,-2)} = \frac{1}{a}\Big(\hat{I}^{(0)}(n-1)\cdot\Big(A^2\hat{I}^{(0)}(n-2)+(B^2+2AB)\hat{I}^{(2)}(n-2)\Big)+C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n)\Big) \tag{200}$$
$$= \frac{2\pi}{a}\left(\frac{A^2}{n-4}+\frac{2AB+B^2}{(n-4)(n-3)}+\frac{C^2}{(n-3)(n-2)}\right) \tag{201}$$
$$\approx \frac{2\pi}{a}\left(\frac{(A+B)^2}{\epsilon}+\frac{C^2}{2}-2AB-B^2+O(\epsilon)\right) \tag{202}$$

$$
\begin{aligned}
I_{a,n}^{(2,0)} &= \frac{1}{a^2}\hat{I}^{(0)}(n-1)\cdot\left(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4)\right) && (203)\\
&= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2}+O(\epsilon) && (204)\\
I_{a,n}^{(2,-1)} &= \frac{1}{a^2}\hat{I}^{(0)}(n-1)\cdot\left(A\left(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4)\right)+2B\hat{I}^{(2)}(n-4)\right) && (205)\\
&= \frac{2\pi}{a^2}\left(\frac{A}{n-6}+\frac{2B}{(n-6)(n-4)}\right) \approx -\frac{2\pi}{a^2}\left(\frac{B}{\epsilon}+\frac{A+B}{2}\right)+O(\epsilon) && (206)\\
I_{a,n}^{(2,-2)} &= \frac{1}{a^2}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4))+4AB\hat{I}^{(2)}(n-4)\right.\right.\\
&\qquad \left.\left.+B^2(\hat{I}^{(2)}(n-4)+\hat{I}^{(4)}(n-4))\right)+C^2\hat{I}^{(2)}(n-1)(\hat{I}^{(0)}(n-2)+\hat{I}^{(2)}(n-2))\right) && (207)\\
&= \frac{2\pi}{a^2}\left(\frac{A^2}{n-6}+\frac{4AB}{(n-6)(n-4)}+\frac{B^2n}{(n-6)(n-4)(n-3)}+\frac{C^2}{(n-4)(n-3)}\right) && (208)\\
&\approx \frac{2\pi}{a^2}\left(\frac{-2AB-2B^2+C^2}{\epsilon}+\frac{B^2-A^2}{2}-AB-C^2+O(\epsilon)\right) && (209)\\
& && (210)
\end{aligned}
$$

It is[4, Ch. 5]:

$$
\begin{aligned}
I_{a,n}^{(1,1)} &= \frac{\pi}{a(A+B)}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)\right)+O(\epsilon) && (211)\\
I_{a,n}^{(2,1)} &= \frac{\pi}{a^2(A+B)}\left(\frac{B^2+AB+C^2}{(A+B)^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)\right)\right.\\
&\qquad \left.-\frac{2C^2}{(A+B)^2}-1\right)+O(\epsilon) && (212)
\end{aligned}
$$

From this are the following integrals derived[4, Ch. 5]:

$$
\begin{aligned}
I_{a,n}^{(1,2)} &= -\frac{\partial}{\partial A}I_{a,n}^{(1,1)} && (213)\\
&= \frac{\pi}{a(A+B)^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)+\frac{2A(A+B)}{A^2-B^2-C^2}-2\right)+O(\epsilon) && (214)\\
I_{a,n}^{(2,2)} &= -\frac{\partial}{\partial A}I_{a,n}^{(2,1)} && (215)\\
&= \frac{\pi}{a^2(A+B)^2}\left(\frac{2B^2+2AB+3C^2}{(A+B)^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)\right)\right.\\
&\qquad \left.+\frac{2A^2}{A^2-B^2-C^2}-\frac{8C^2}{(A+B)^2}-3\right)+O(\epsilon) && (216)
\end{aligned}
$$

If $-k \in \mathcal{N}$ use I_0 with $b = -a$.

2.3.4 double collinear

as said in [4, Ch. 5]: if $0 \leq -\frac{C}{A}, \frac{B}{A} \leq 1$ use [3, eq. A11] with $\cos\kappa = -\frac{B}{A}$:

$$I_{aA,n}^{(k,l)} = \frac{2\pi 2^{-(k+l)}}{a^k A^l} \frac{\Gamma(1+\epsilon)}{\Gamma^2(1+\epsilon/2)} B(1+\frac{\epsilon}{2}-k, 1+\frac{\epsilon}{2}-l)_2F_1\left(k,l;1+\frac{\epsilon}{2};\frac{A-B}{2A}\right) \tag{217}$$

but we will not need it here.

2.3.5 non collinear

If $-l \in \mathcal{N}$ the θ_2 integration can be performed using the integral helper and the problem reduces then to the following integral:

$$\begin{aligned}
&\hat{I}_0^{(k,q,p)}(\epsilon) \\
&= \int_0^\pi d\theta_1 \, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))^k} && (218)\\
&= \frac{1}{2a^k}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{k}{2},\frac{1+p}{2};\frac{1}{2},\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.\\
&\quad \left.\frac{b}{a}k(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{2+k}{2},\frac{2+p}{2};\frac{3}{2},\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) && (219)
\end{aligned}$$

for $k = 1$ this simplifies to

$$\begin{aligned}
&\hat{I}_0^{(1,q,p)}(\epsilon) \\
&= \int_0^\pi d\theta_1 \, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))} && (220)\\
&= \frac{1}{2a}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_2F_1\left(1,\frac{1+p}{2};\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.\\
&\quad \left.\frac{b}{a}(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_2F_1\left(1,\frac{2+p}{2};\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) && (221)
\end{aligned}$$

TODO: structure does NOT match [4, Ch. 5] - neither A,B,C nor ϵ

FiXme Error: fix to Bojak

For $I_{0,n}^{(1,1)}$ one finds[4, Ch. 5]:

$$I_{0,n}^{(1,1)} = \frac{\pi}{\sqrt{X}} \ln \left(\frac{aA - bB + \sqrt{X}}{aA - bC - \sqrt{X}} \right) \tag{222}$$

$$\text{with } X = (aB - bA)^2 - (a^2 - b^2)C^2 \tag{223}$$

$I_{0,n}^{(1,-3)}$ is given by [4, Ch. 5] and from those two all other integrals can be deduced using the techiques described in [4, Ch. 5]: increase k or l by differentiation or interchange k and l by a rotation:

$$I_n^{(k,l)} = I_n^{(l,k)} \left(a \leftrightarrow A, b \to -\sqrt{B^2 + C^2}, B \to \frac{-bB}{\sqrt{B^2 + C^2}}, C \to \frac{-bC}{\sqrt{B^2 + C^2}} \right) \tag{224}$$

2.4 needed set in matrix element

define a shortcut:

$$\mathcal{V}_{a,b}(x,y) = \left(x^k y^l \right)_{k=0..a,l=0..b} \tag{225}$$

It is

$$A_{G,1} = \sum_{k,l=0}^{3} \left(\mathcal{C}_{A_{G,1}} \right)_{(k,l)} {t'}^{-2+k} {u_7}^{-2+l} \qquad = \operatorname{tr} \left(\mathcal{C}_{A_{G,1}} \frac{\mathcal{V}_{3,3}(t', u_7)^t}{{t'}^2 {u_7}^2} \right) \tag{226}$$

$$A_{L,1} = \sum_{k=0}^{4} \sum_{l=0}^{2} \left(\mathcal{C}_{A_{L,1}} \right)_{(k,l)} {t'}^{-2+k} {u_7}^{-2+l} \qquad = \operatorname{tr} \left(\mathcal{C}_{A_{L,1}} \frac{\mathcal{V}_{4,2}(t', u_7)^t}{{t'}^2 {u_7}^2} \right) \tag{227}$$

and we will thus need the integrals

$$\left(\mathcal{I}_{A_{G,1}} \right)_{(k,l)} = \int d\Omega_n \frac{1}{{t'}^2 {u_7}^2} \left(\mathcal{V}_{3,3}(t', u_7) \right)_{(k,l)} \tag{228}$$

$$\left(\mathcal{I}_{A_{L,1}} \right)_{(k,l)} = \int d\Omega_n \frac{1}{{t'}^2 {u_7}^2} \left(\mathcal{V}_{4,2}(t', u_7) \right)_{(k,l)} \tag{229}$$

with

$$a(t') = -b(t') = -2\omega_1\omega_2 \qquad = -\frac{s_4(s' + t_1)}{2(s_4 + m^2)} \tag{230}$$

$$A(u_7) = q^2 - 2q_0E_1 \qquad = q^2 - \frac{(s_4 + 2m^2)(s + u_1)}{2(s_4 + m^2)} \tag{231}$$

$$B(u_7) = -2\omega_2(|\vec{p}_2| \cos\psi - \omega_1) \qquad = \frac{s_4}{2} \left(1 - \frac{s + u_1}{s_4 + m^2} + \frac{s' - t_1}{s' + t_1} \right) \tag{232}$$

$$C(u_7) = -2\omega_2 \, |\vec{p}_2| \sin\psi \tag{233}$$

that is $I_{a,n}^{(-2\dots 2,-1\dots 2)}$. With this we find

$$A+B=-\frac{t_1u_1}{s'+t_1} \tag{234}$$

$$\frac{(A+B)^2}{A^2-B^2-C^2}=\frac{(s_4+m^2)t_1^2u_1^2}{(s'+t_1)^2(s_4q^2t_1+m^2(s'+u_1)^2)} \tag{235}$$

$$2B(A+B)+3C^2=-\frac{s_4(m^2s'(3s's_4+2t_1u_1)+t_1(q^2(s_4-u_1)(3s_4-u_1)-u_1(s's_4+t_1u_1)))}{(s_4+m^2)(s'+t_1)^2} \tag{236}$$

With this we get

$$\int d\Omega_n\, A_{j,1}=\mathrm{tr}(\mathcal{C}_{A_{j,1}}\left(\mathcal{I}_{A_{j,1}}\right)^t) \qquad j=G,L \tag{237}$$

It is

$$A_{G,2}=\sum_{k,l=0}^{3}\left(\mathcal{C}_{A_{G,2}}\right)_{(k,l)}{s_5}^{-2+k}{u'}^{-2+l} \qquad =\mathrm{tr}\left(\mathcal{C}_{A_{G,2}}\frac{\mathcal{V}_{3,3}(s_5,u')^t}{{s_5}^2{u'}^2}\right) \tag{238}$$

$$A_{L,2}=\sum_{k=0}^{4}\sum_{l=0}^{3}\left(\mathcal{C}_{A_{G,2}}\right)_{(k,l)}{s_5}^{-2+k}{u'}^{-2+l} \qquad =\mathrm{tr}\left(\mathcal{C}_{A_{L,2}}\frac{\mathcal{V}_{4,3}(s_5,u')^t}{{s_5}^2{u'}^2}\right) \tag{239}$$

and we will thus need the integrals

$$\left(\mathcal{I}_{A_{G,2}}\right)_{(k,l)}=\int d\Omega_n\,\frac{1}{{s_5}^2{u'}^2}\left(\mathcal{V}_{3,3}(s_5,u')\right)_{(k,l)} \tag{240}$$

$$\left(\mathcal{I}_{A_{L,2}}\right)_{(k,l)}=\int d\Omega_n\,\frac{1}{{s_5}^2{u'}^2}\left(\mathcal{V}_{4,3}(s_5,u')\right)_{(k,l)} \tag{241}$$

but as both s_5 and u' are of $[ABC]$ type we have to apply partial fractioning, following the ideas of [4, Ch. 5]. It is

$$(44): \qquad s_5=s-q^2+t'+u' \tag{242}$$

so end up with a form of $\mathcal{V}(p+q,q)$ where p is $[ab]$ and q and $p+q$ are $[ABC]$. The aim is then to get to a form with fractions of $\frac{p}{q}$ and/or $\frac{p+q}{p}$ and indeed this can be achieved. [TODO]

FiXme Error: rewrite

A References

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List of Corrections