1 2 to 3 phase space

at phase space integration there occur integrations over propagators[1, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1 \, \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2 \, \sin^{n-4}(\theta_2)(a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{1}$$

the integrals can be further destinguished by the range of k, l and the type of collinearity (following the notation in [1]):

- "non collinear": $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \to I_{0,n}^{(k,l)}$
- "single collinear a": $a = -b \wedge A^2 \neq B^2 + C^2 \to I_{a,n}^{(k,l)}$
- "single collinear A": $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \to I_{A,n}^{(k,l)}$
- "double collinear": $a = -b \wedge A = -\sqrt{B^2 + C^2} \to I_{aA,n}^{(k,l)}$

Use $n = 4 + \epsilon$.

1.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int_0^\pi dt \, \sin^{\nu-3}(t)\cos^q(t) \tag{2}$$

It is[4, eq. 5.12.6]:

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha \, B\left((\alpha+\beta+1)/2, (\alpha-\beta+1)/2\right)} \qquad \text{if } \Re(\alpha) > 0 \tag{3}$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{B((n-1)/2, (n-1)/2)} \tag{4}$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{B((n-2)/2, (n-2)/2)} = B((n-3)/2, 1/2) \tag{5}$$

If q is odd: $\hat{I}^{(q)} = 0$, due to symetry of kernel; if q is even: $q = 2p$ with $p \in \mathbb{N}$:

$$\hat{I}^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int\limits_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t)\, dt \tag{6}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2+(k-p),(\nu-1)/2-(k-p))} \tag{7}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^{p} \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2+l,(\nu-1)/2-l)} \tag{8}$$

$$= \frac{\pi\Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right.$$
$$\left. +2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{9}$$

$$= \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right.$$
$$\left. +2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{10}$$

TODO: prove

FiXme Error: prove

$$p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{11}$$

$$= \frac{1}{p!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l p!}{(p+l)!(p-l)!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \tag{12}$$

$$= 1 \tag{13}$$

$$\Rightarrow \hat{I}^{(2p)}(\nu) = \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p!)}{2^p p!} \tag{14}$$

$$= \frac{\sqrt{\pi}(2p)!}{2^{2p}p!} \frac{\Gamma((\nu-2)/2)}{\Gamma(\frac{\nu-1}{2}+p)} \tag{15}$$

1.2 any collinearity and $-k, -l \in \mathbb{N}_0$

If $-k, -l \in \mathbb{N}_0$ $I_n^{(k,l)}$ can always be reduced in a straight forward manner to combinations of $\hat{I}^{(q)}(n)$ and this way one finds[1, Ch. 5][2, App. C]:

$$I_n^{(0,0)} = \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n) = \frac{2\pi}{n-3} \tag{16}$$

$$I_4^{(0,0)} = 2\pi \tag{17}$$

$$I_n^{(-1,0)} = \hat{I}^{(0)}(n-1) \cdot (a\hat{I}^{(0)}(n) + b\hat{I}^{(1)}(n)) = \frac{2\pi a}{n-3} \tag{18}$$

$$I_4^{(-1,0)} = 2\pi a \tag{19}$$

$$I_n^{(0,-1)} = \hat{I}^{(0)}(n-1) \cdot (A\hat{I}^{(0)}(n) + B\hat{I}^{(1)}(n)) + C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \tag{20}$$

$$= \frac{2\pi A}{n-3} \tag{21}$$

$$I_4^{(0,-1)} = 2\pi A \tag{22}$$

$$I_n^{(-2,0)} = \hat{I}^{(0)}(n-1) \cdot (a^2\hat{I}^{(0)}(n) + 2ab\hat{I}^{(1)}(n) + b^2\hat{I}^{(2)}(n)) \tag{23}$$

$$= 2\pi \left(\frac{a^2(n-1) + b^2}{(n-1)(n-3)} \right) \tag{24}$$

$$I_4^{(-2,0)} = 2\pi(a^2 + b^2/3) \tag{25}$$

$$I_n^{(0,-2)} = \hat{I}^{(0)}(n-1) \cdot (A^2\hat{I}^{(0)}(n) + B^2\hat{I}^{(2)}(n)) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \tag{26}$$

$$= 2\pi \left(\frac{A^2(n-1) + B^2 + C^2}{(n-1)(n-3)} \right) \tag{27}$$

$$I_4^{(0,-2)} = 2\pi(A^2 + (B^2 + C^2)/3) \tag{28}$$

$$I_n^{(-1,-1)} = \hat{I}^{(0)}(n-1) \cdot (aA\hat{I}^{(0)}(n) + bB\hat{I}^{(2)}(n)) = 2\pi \left(\frac{aA(n-1) + bB}{(n-1)(n-3)} \right) \tag{29}$$

$$I_4^{(-1,-1)} = 2\pi(aA + bB/3) \tag{30}$$

1.3 single collinear a and $k, -l \in \mathbb{N}_0$

It is

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1-\cos(t))^k} \cos^q(t)\, dt \tag{31}$$

$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1-\cos^2(t))^k} \cos^q(t)(1+\cos(t))^k\, dt \tag{32}$$

$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t)(1+\cos(t))^k\, dt \tag{33}$$

$$= \sum_{l=0}^{k} \binom{k}{l} \hat{I}^{(q+l)}(\nu - 2k) \tag{34}$$

this way one finds[1, Ch. 5][2, App. C]:

$$
\begin{aligned}
I^{(1,0)}_{a,n} &= \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot\hat{I}^{(0)}(n-2) && (35)\\
&= \frac{2\pi}{a(n-4)} && (36)\\
I^{(2,0)}_{a,n} &= \frac{1}{a}^2\hat{I}^{(0)}(n-1)\cdot\left(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4)\right) && (37)\\
&= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2}+O(\epsilon) && (38)\\
I^{(1,-1)}_{a,n} &= \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot\left(A\hat{I}^{(0)}(n-2)+B\hat{I}^{(2)}(n-2)\right) && (39)\\
&= \frac{2\pi}{a}\frac{(A(n-3)+B)}{(n-3)(n-4)} \approx \frac{2\pi}{a}\left(\frac{A+B}{\epsilon}-2B+O(\epsilon)\right) && (40)\\
I^{(1,-2)}_{a,n} &= \frac{1}{a}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2\hat{I}^{(0)}(n-2)+(B^2+2AB)\hat{I}^{(2)}(n-2)\right)+C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n)\right) && (41)\\
&= \frac{2\pi}{a}\left(\frac{A^2}{n-4}+\frac{2AB+B^2}{(n-4)(n-3)}+\frac{C^2}{(n-3)(n-2)}\right) && (42)\\
&\approx \frac{2\pi}{a}\left(\frac{(A+B)^2}{\epsilon}+\frac{C^2}{2}-2AB-B^2+O(\epsilon)\right) && (43)\\
I^{(2,-2)}_{a,n} &= \frac{1}{a^2}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4))+4AB\hat{I}^{(2)}(n-4)\right.\right.\\
&\qquad \left.\left.+B^2(\hat{I}^{(2)}(n-4)+\hat{I}^{(4)}(n-4))\right)+C^2\hat{I}^{(2)}(n-1)(\hat{I}^{(0)}(n-2)+\hat{I}^{(2)}(n-2))\right) && (44)\\
&= \frac{2\pi}{a^2}\left(\frac{A^2}{n-6}+\frac{4AB}{(n-6)(n-4)}+\frac{B^2n}{(n-6)(n-4)(n-3)}+\frac{C^2}{(n-4)(n-3)}\right) && (45)\\
&\approx \frac{2\pi}{a^2}\left(\frac{-2AB-2B^2+C^2}{\epsilon}+\frac{B^2-A^2}{2}-AB-C^2+O(\epsilon)\right) && (46)\\
& && (47)
\end{aligned}
$$

1.4 double collinear and $k, l \in \mathbb{N}$

as said in [1, Ch. 5]: if $0 \le -\frac{C}{A}, \frac{B}{A} \le 1$ use [3, eq. A11] with $\cos\kappa = -\frac{B}{A}$:

$$
I^{(k,l)}_{aA,n} = \frac{2\pi 2^{-(k+l)}}{a^k A^l}\frac{\Gamma(1+\epsilon)}{\Gamma^2(1+\epsilon/2)}B(1+\frac{\epsilon}{2}-k, 1+\frac{\epsilon}{2}-l)_2F_1\left(k,l;1+\frac{\epsilon}{2};\frac{A-B}{2A}\right) \quad (48)
$$

1.5 non collinear and $k, -l \in \mathbb{N}$

Next we want to compute $I^{1,-3}_{0,n}$.

The θ_2 integration can be performed using the integral helper and the problem reduces then to the following integral:

$$
\begin{aligned}
&\hat{I}^{(k,q,p)}_{0,\cos}(\epsilon) \\
&= \int_0^\pi d\theta_1 \, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))^k} \qquad (49)\\
&= \frac{1}{2a^k}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{k}{2},\frac{1+p}{2};\frac{1}{2},\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.\\
&\quad \left.\frac{b}{a}k(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{2+k}{2},\frac{2+p}{2};\frac{3}{2},\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) \qquad (50)
\end{aligned}
$$

for $k = 1$ this simplifies to

$$
\begin{aligned}
&\hat{I}^{(1,q,p)}_{0,\cos}(\epsilon) \\
&= \int_0^\pi d\theta_1 \, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))} \qquad (51)\\
&= \frac{1}{2a}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_2F_1\left(1,\frac{1+p}{2};\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.\\
&\quad \left.\frac{b}{a}(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_2F_1\left(1,\frac{2+p}{2};\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) \qquad (52)
\end{aligned}
$$

A References

[1] I. Bojak, NLO QCD corrections to the polarized photoproduction and hadroproduction of heavy quarks. PhD thesis, Dortmund U., 2000. `arXiv:hep-ph/0005120 [hep-ph]`.

[2] W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith, "QCD Corrections to Heavy Quark Production in p anti-p Collisions," Phys. Rev. **D40** (1989) 54–82.

[3] W. L. Van Neerven, "Dimensional regularization of mass and infrared singularities in two-loop on-shell vertex functions," Nuclear Physics B **268** no. 2, (May, 1986) 453–488. `http://www.sciencedirect.com/science/article/pii/0550321386901653`.

[4] "NIST Digital Library of Mathematical Functions." Http://dlmf.nist.gov/, release 1.0.10 of 2015-08-07. `http://dlmf.nist.gov/`. Online companion to [5].

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List of Corrections