

## 1 2 to 3 phase space

at phase space integration there occur integrations over propagators[1, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1 \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2 \sin^{n-4}(\theta_2) (a + b \cos(\theta_1))^{-k} (A + B \cos(\theta_1) + C \sin(\theta_1) \cos(\theta_2))^{-l} \quad (1)$$

(2)

the integrals can be further destinguished by the range of  $k, l$  and the type of collinearity (following the notation in [1]):

- "non collinear":  $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \rightarrow I_{0,n}^{(k,l)}$
- "single collinear a":  $a = -b \wedge A^2 \neq B^2 + C^2 \rightarrow I_{aA,n}^{(k,l)}$
- "single collinear A":  $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \rightarrow I_{aA,n}^{(k,l)}$
- "double collinear":  $a = -b \wedge A^2 = B^2 + C^2 \rightarrow I_{aA,n}^{(k,l)}$

### 1.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int_0^\pi dt \sin^{\nu-3}(t) \cos^q(t) \quad (3)$$

It is[4, eq. 5.12.6]:

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha \text{B}((\alpha + \beta + 1)/2, (\alpha - \beta + 1)/2)} \quad \text{if } \Re(\alpha) > 0 \quad (4)$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{\text{B}((n-1)/2, (n-1)/2)} \quad (5)$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{\text{B}((n-2)/2, (n-2)/2)} = \text{B}((n-3)/2, 1/2) \quad (6)$$

If  $q$  is odd:  $\hat{I}^{(q)} = 0$ , due to symmetry of kernel; if  $q$  is even:  $q = 2p$  with  $p \in \mathbb{N}$ :

$$\hat{I}^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t) dt \quad (7)$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2 + (k-p), (\nu-1)/2 - (k-p))} \quad (8)$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^p \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2 + l, (\nu-1)/2 - l)} \quad (9)$$

$$= \frac{\pi \Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{\nu-1}{2}+p)\Gamma(\frac{\nu-1}{2}-p)} \left( \frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right. \\ \left. + 2 \sum_{l=1}^p \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \quad (10)$$

$$= \frac{2^{3-\nu} \pi \Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{\nu-1}{2}+p)\Gamma(\frac{\nu-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p \Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left( \frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right. \\ \left. + 2 \sum_{l=1}^p \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \quad (11)$$

TODO: prove

$$p! \left( \frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} + 2 \sum_{l=1}^p \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \quad (12)$$

$$= \frac{1}{p!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})} + 2 \sum_{l=1}^p \frac{(-1)^l p!}{(p+l)!(p-l)!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \quad (13)$$

$$= 1 \quad (14)$$

$$\Rightarrow \hat{I}^{(2p)}(\nu) = \frac{2^{3-\nu} \pi \Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{\nu-1}{2}+p)\Gamma(\frac{\nu-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p \Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \quad (15)$$

$$= \frac{\sqrt{\pi}(2p)! \Gamma((\nu-2)/2)}{2^{2p} p! \Gamma(\frac{\nu-1}{2}+p)} \quad (16)$$

FiXme  
Error:  
prove

## 1.2 $-k, -l \in \mathbb{N}_0$

If  $-k, -l \in \mathbb{N}_0$   $I_n^{(k,l)}$  can always be reduced in a straight forward manner to combination of  $\hat{I}^{(q)}(n)$  and this way one finds[1, Ch. 5][2, App. C]:

$$I_n^{(0,0)} = \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n) = \frac{2\pi}{n-3} \quad (17)$$

$$I_4^{(0,0)} = 2\pi \quad (18)$$

$$I_n^{(-1,0)} = \hat{I}^{(0)}(n-1) \cdot (a\hat{I}^{(0)}(n) + b\hat{I}^{(1)}(n)) = \frac{2\pi a}{n-3} \quad (19)$$

$$I_4^{(-1,0)} = 2\pi a \quad (20)$$

$$I_n^{(0,-1)} = \hat{I}^{(0)}(n-1) \cdot (A\hat{I}^{(0)}(n) + B\hat{I}^{(1)}(n)) + C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \quad (21)$$

$$= \frac{2\pi A}{n-3} \quad (22)$$

$$I_4^{(0,-1)} = 2\pi A \quad (23)$$

$$I_n^{(-2,0)} = \hat{I}^{(0)}(n-1) \cdot (a^2\hat{I}^{(0)}(n) + 2ab\hat{I}^{(1)}(n) + b^2\hat{I}^{(2)}(n)) \quad (24)$$

$$= 2\pi \left( \frac{a^2(n-1) + b^2}{(n-1)(n-3)} \right) \quad (25)$$

$$I_4^{(-2,0)} = 2\pi(a^2 + b^2/3) \quad (26)$$

$$I_n^{(0,-2)} = \hat{I}^{(0)}(n-1) \cdot (A^2\hat{I}^{(0)}(n) + B^2\hat{I}^{(2)}(n)) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \quad (27)$$

$$= 2\pi \left( \frac{A^2(n-1) + B^2 + C^2}{(n-1)(n-3)} \right) \quad (28)$$

$$I_4^{(0,-2)} = 2\pi(A^2 + (B^2 + C^2)/3) \quad (29)$$

$$I_n^{(-1,-1)} = \hat{I}^{(0)}(n-1) \cdot (aA\hat{I}^{(0)}(n) + bB\hat{I}^{(2)}(n)) = 2\pi \left( \frac{aA(n-1) + bB}{(n-1)(n-3)} \right) \quad (30)$$

$$I_4^{(-1,-1)} = 2\pi(aA + bB/3) \quad (31)$$

### 1.3 single collinear a and $k, -l \in \mathbb{N}_0$

It is

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1 - \cos(t))^k} \cos^q(t) dt \quad (32)$$

$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1 - \cos^2(t))^k} \cos^q(t) (1 + \cos(t))^k dt \quad (33)$$

$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t) (1 + \cos(t))^k dt \quad (34)$$

$$= \sum_{l=0}^k \binom{k}{l} \hat{I}^{(q+l)}(\nu - 2k) \quad (35)$$

this way one finds[1, Ch. 5][2, App. C]:

$$I_{a,n}^{(1,0)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n-2) \quad (36)$$

$$= \frac{2\pi}{a(n-4)} \quad (37)$$

$$I_{a,n}^{(2,0)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \left( \hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4) \right) \quad (38)$$

$$= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2} + O(\epsilon) \quad (39)$$

$$I_{a,n}^{(1,-1)} = \frac{1}{a} \hat{I}^{(0)}(n-1) \cdot \left( A \hat{I}^{(0)}(n-2) + B \hat{I}^{(2)}(n-2) \right) \quad (40)$$

$$= \frac{2\pi}{a} \frac{(A(n-3) + B)\Gamma(n-4)}{\Gamma(n-2)} \approx \frac{2\pi}{a} \left( \frac{A+B}{\epsilon} - 2B + O(\epsilon) \right) \quad (41)$$

$$I_{a,n}^{(1,-2)} = \frac{1}{a} \left( \hat{I}^{(0)}(n-1) \cdot \left( A^2 \hat{I}^{(0)}(n-2) + (B^2 + 2AB) \hat{I}^{(2)}(n-2) \right) + C^2 \hat{I}^{(2)}(n-1) \hat{I}^{(0)}(n) \right) \quad (42)$$

$$= \frac{2\pi}{a} \left( \frac{A^2}{n-4} + \frac{2AB + B^2}{(n-4)(n-3)} + \frac{C^2}{(n-3)(n-2)} \right) \quad (43)$$

$$\approx \frac{2\pi}{a} \left( \frac{(A+B)^2}{\epsilon} + \frac{C^2}{2} - 2AB - B^2 + O(\epsilon) \right) \quad (44)$$

### 1.4 non collinear and $k, l \in \mathbb{N}$

see [3]

## A References

- [1] I. Bojak,  
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- [2] W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith, “QCD Corrections to Heavy Quark Production in p anti-p Collisions,” Phys. Rev. **D40** (1989) 54–82.
- [3] W. L. Van Neerven, “Dimensional regularization of mass and infrared singularities in two-loop on-shell vertex functions,” Nuclear Physics B **268** no. 2, (May, 1986) 453–488.  
<http://www.sciencedirect.com/science/article/pii/0550321386901653>.
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- [5] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds.,  
NIST Handbook of Mathematical Functions. Cambridge University Press, New York, NY, 2010. Print companion to [4].

## List of Corrections

Error: prove . . . . . 2