1 2 to 2 phase space

following [1]:

process:

$$\gamma^*(q) + g(k_1) \to Q(p_1) + \bar{Q}(p_2) \tag{1}$$

kinematics:

$$s = (q + k_1)^2 \qquad s' = s - q^2 \tag{2}$$
$$t = (k_1 - p_1)^2 \qquad t_1 = t - m^2 \tag{3}$$
$$u = (k_1 - p_2)^2 \qquad u_1 = u - m^2 \tag{4}$$

use c.m.s. of incoming particles:

$$q = \left(\frac{s + q^2}{2\sqrt{s}}, 0, 0, \ldots, -\frac{s - q^2}{2\sqrt{s}}\right) \tag{5}$$
$$k_1 = \frac{s - q^2}{2\sqrt{s}} (1, 0, 0, \ldots, 1) \tag{6}$$

such that

$$q + k_1 = (\sqrt{s}, \vec{0}) \qquad k_1^2 = 0 \tag{7}$$

for the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2} (1, 0, \beta \sin\theta, \ldots, \beta \cos\theta) \tag{8}$$
$$p_2 = \frac{\sqrt{s}}{2} (1, 0, -\beta \sin\theta, \ldots, -\beta \cos\theta) \tag{9}$$

with $\beta = \sqrt{1 - 4m^2/s}$ such that

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \qquad p_1^2 = p_2^2 = m^2 \tag{10}$$

use n-sphere:

$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \tag{11}$$

compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_1}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q+k_1-p_1-p_2)\delta(p_1^2-m^2)\delta(p_2^2-m^2) \tag{12}$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1\, \delta((q+k_1-p_2)^2-m^2)\delta(p_1^2-m^2) \tag{13}$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,||} d^2 p_{1,\perp} d^{n-4}\hat{p}_1\, \delta(s-2p_{1,0}\sqrt{s})\delta(p_{1,0}^2-p_{1,||}^2-p_{1,\perp}^2-\hat{p}_1^2-m^2) \tag{14}$$

$$= \frac{\pi}{(2\pi)^{n-2}2\sqrt{s}} \int dp_{1,||} dp_{1,\perp}^2 d^{n-4}\hat{p}_1\, \delta(s/4-p_{1,||}^2-p_{1,\perp}^2-\hat{p}_1^2-m^2) \tag{15}$$

$$= \frac{\pi}{(2\pi)^{n-2}2\sqrt{s}} \int dp_{1,||} d\hat{p}_1^2\, \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \tag{16}$$

$$= \frac{1}{2\sqrt{s}\Gamma((n-4)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} d\hat{p}_1^2\, (\hat{p}_1^2)^{(n-6)/2} \tag{17}$$

Integration borders are

$$p_{1,||} \in \frac{\sqrt{s}}{2}\beta \cdot [-1,1] \qquad \hat{p}_1^2 \in \left(\frac{s\beta^2}{4} - p_{1,||}^2\right) \cdot [0,1] \tag{18}$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2\, (\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4} \left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{19}$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} \left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{20}$$

rewrite $p_{1,||}$ to $\cos\theta$:

$$p_{1,||} = \frac{\sqrt{s}}{2}\beta\cos\theta \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{2}\beta\, d\cos\theta, \quad \cos\theta \in [-1,1],\ \hat{p}_1^2 \in \frac{s\beta^2}{4}\left(1-\cos^2\theta\right) \cdot [0,1] \tag{21}$$

rewrite $\cos\theta$ to $t_1 = (k_1-p_2)^2 - m^2$:

$$\cos\theta = \frac{2t_1/s'+1}{\beta} \Rightarrow \quad d\cos\theta = \frac{2}{\beta s'}dt_1, \quad t_1 \in \frac{s'}{2}[-\beta-1,\beta-1],\ \hat{p}_1^2 \in (-m^2 - \frac{st_1}{s'^2}(s'+t_1))\cdot[0,1] \tag{22}$$

$$p_{1,||} = \sqrt{s}\left(\frac{t_1}{s'} + \frac{1}{2}\right) \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{s'}dt_1 \tag{23}$$

$$\Rightarrow PS_2 = \frac{1}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dt_1 \left(\frac{(t_1(u_1-q^2)-s'm^2)s' - q^2t_1^2}{s'^2}\right)^{(n-4)/2} \tag{24}$$

2 2 to 3 phase space

following [2, 3, 1]:

process:

$$\gamma^*(q) + q(k_1) \to Q(p_1) + \bar{Q}(p_2) + q(k_2) \tag{25}$$

2.1 kinematic constraints

definitions of kinematic variables:

$$\begin{aligned}
s &= (q+k_1)^2 & &\Rightarrow & 2qk_1 &= s - q^2 = s' & (26)\\
s_3 &= (k_2+p_2)^2 - m^2 & &\Rightarrow & 2k_2p_2 &= s_3 & (27)\\
s_4 &= (k_2+p_1)^2 - m^2 & &\Rightarrow & 2k_2p_1 &= s_4 & (28)\\
s_5 &= (p_1+p_2)^2 = -u_5 & &\Rightarrow & 2p_1p_2 &= s_5 - 2m^2 & (29)\\
t_1 &= (k_1-p_2)^2 - m^2 = t - m^2 & &\Rightarrow & 2k_1p_2 &= -t_1 & (30)\\
t' &= (k_1-k_2)^2 & &\Rightarrow & 2k_1k_2 &= -t' & (31)\\
u_1 &= (q-p_2)^2 - m^2 = u - m^2 & &\Rightarrow & 2qp_2 &= -u_1 + q^2 & (32)\\
u_6 &= (k_1-p_1)^2 - m^2 & &\Rightarrow & 2k_1p_1 &= -u_6 & (33)\\
u_7 &= (q-p_1)^2 - m^2 & &\Rightarrow & 2qp_1 &= -u_7 + q^2 & (34)\\
u' &= (q-k_2)^2 & &\Rightarrow & 2qk_2 &= -u' + q^2 & (35)
\end{aligned}$$

impose momentum conservation:

$$q + k_1 = p_1 + p_2 + k_2 \tag{36}$$

contract with 2 times momentum:

$$\begin{aligned}
2q^2 & \quad +s - q^2 = -u_7 + q^2 & -u_1 + q^2 & \quad -u' + q^2 & \Leftrightarrow \quad 0 &= s + u_1 + u_7 + u' - 2q^2 & (37)\\
s - q^2 & \quad +0 = -u_6 & -t_1 & \quad -t' & \Leftrightarrow \quad 0 &= s + t_1 + t' + u_6 - q^2 & (38)\\
-u_7 + q^2 & \quad -u_6 = 2m^2 & +s_5 - 2m^2 & \quad +s_4 & \Leftrightarrow \quad 0 &= s_4 + s_5 + u_6 + u_7 - q^2 & (39)\\
-u_1 + q^2 & \quad -t_1 = s_5 - 2m^2 & +2m^2 & \quad +s_3 & \Leftrightarrow \quad 0 &= s_3 + s_5 + t_1 + u_1 - q^2 & (40)\\
-u' + q^2 & \quad -t' = s_4 & +s_3 & \quad +0 & \Leftrightarrow \quad 0 &= s_3 + s_4 + t' + u' - q^2 & (41)
\end{aligned}$$

$$\frac{1}{2}\left((37)+(38)+(40)-(39)-(41)\right)=0=s'+t_1+u_1-s_4 \tag{42}$$
$$\frac{1}{2}\left((37)+(38)+(41)-(39)-(40)\right)=0=s'+t'+u'-s_5 \tag{43}$$

2.2 choose framework

use c.m.s. of recoiling heavy and light quark ($Q(p_1)$ and $q(k_2)$):

$$k_2 = (\omega_2, k_{2,x}, \omega_2 \sin\theta_1 \cos\theta_2, \omega_2 \cos\theta_1, \hat{k}_2) \tag{44}$$
$$p_1 = (E_1, -k_{2,x}, -\omega_2 \sin\theta_1 \cos\theta_2, -\omega_2 \cos\theta_1, -\hat{k}_2) \tag{45}$$
$$k_1 = (\omega_1, 0, 0, \omega_1, \hat{0}) \tag{46}$$
$$q = (q_0, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi - \omega_1, \hat{0}) \tag{47}$$
$$p_2 = (E_2, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi, \hat{0}) \tag{48}$$

light quark masses are already fixed: $k_1^2 = 0 = k_2^2$

constraints:

$$q_0 + \omega_1 = E_1 + E_2 + \omega_2 \tag{49}$$
$$m^2 = p_1^2 = E_1^2 - \omega_2^2 \tag{50}$$
$$m^2 = p_2^2 = E_2^2 - |\vec{p}_2|^2 \tag{51}$$
$$q^2 = q_0^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\,\omega_1 \cos\psi - \omega_1^2 \tag{52}$$
$$s = (q+k_1)^2 = (q_0+\omega_1)^2 - |\vec{p}_2|^2 \tag{53}$$
$$t = (k_1-p_2)^2 = (\omega_1-E_2)^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\,\omega_1 \cos\psi - \omega_1^2 \tag{54}$$
$$u = (q-p_2)^2 = (q_0-E_2)^2 - \omega_1^2 \tag{55}$$

solve:

$$
\begin{aligned}
(53)-(52)+(54)-(51)+(55) &= s-q^2+t-m^2+u && (56)\\
&= s_4+m^2=(E_1+\omega_2)^2 && (57)\\
(54)+(55)-(52) &= t+u-q^2=-2(E_1+\omega_2)E_2 && (58)\\
\Rightarrow E_2 &= -\frac{t+u-q^2}{2\sqrt{s_4+m^2}}=\frac{s-s_4-2m^2}{2\sqrt{s_4+m^2}} && (59)\\
(57)\wedge(50)\Rightarrow \omega_2 &= \frac{s_4}{2\sqrt{s_4+m^2}} && (60)\\
(57)\Rightarrow E_1 &= \frac{s_4+2m^2}{2\sqrt{s_4+m^2}} && (61)\\
(53)+(55)-(51) &= s+u-m^2=2q_0(E_1+\omega_2) && (62)\\
\Rightarrow q_0 &= \frac{s+u_1}{2\sqrt{s_4+m^2}} && (63)\\
(54)-(52) &= t-q^2=(\omega_1-E_2)^2-q_0^2 && (64)\\
\Rightarrow \omega_1 &= \frac{s'+t_1}{2\sqrt{s_4+m^2}} && (65)\\
(51)\Rightarrow |\vec{p}_2| &= \sqrt{E_2^2-m^2}=\frac{\sqrt{(s-s_4)^2-4sm^2}}{2\sqrt{s_4+m^2}} && (66)\\
(52)\Rightarrow \cos\psi &= \frac{q^2-q_0^2+|\vec{p}_2|^2+\omega_1^2}{2\,|\vec{p}_2|\,\omega_1} && (67)\\
&= \frac{(u_1+m^2)(t_1-s')-(m^2-q^2-t_1)(s'+t_1)}{(s'+t_1)\sqrt{(s-s_4)^2-4sm^2}} && (68)\\
\Rightarrow \sin\psi &= 2\frac{\sqrt{s_4+m^2}\sqrt{m^2s'^2+q^2t_1(s'+t_1)-s't_1u_1}}{(s'+t_1)\sqrt{(s-s_4)^2-4sm}} && (69)
\end{aligned}
$$

$$
\begin{aligned}
t' &= -2k_1k_2=-2\omega_1\omega_2(1-\cos\theta_1) && (70)\\
u_6 &= -2k_1p_1=-2\omega_1(E_1+\omega_2\cos\theta_1) && (71)\\
(38):\quad 0 &= s+t_1+t'+u_6-q^2 \quad \checkmark && (72)
\end{aligned}
$$

t' is the only variable that can get collinear (for $-q^2>0$).

$$
\begin{aligned}
s_3 &= 2k_2p_2=2\omega_2(E_2-|\vec{p}_2|\,(\cos\psi\cos\theta_1+\sin\psi\sin\theta_1\cos\theta_2)) && (73)\\
s_5 &= (p_1+p_2)^2=2m^2+2p_1p_2 && (74)\\
&= 2(m^2+E_1E_2+\omega_2\,|\vec{p}_2|\,(\cos\psi\cos\theta_1+\sin\psi\sin\theta_1\cos\theta_2)) && (75)\\
(40):\quad 0 &= s_3+s_5+t_1+u_1-q^2 \quad \checkmark && (76)
\end{aligned}
$$

$$u' = (q-k_2)^2 = q^2 - 2qk_2 \tag{77}$$
$$= q^2 - 2(q_0\omega_2 - \omega_2(|\vec{p}_2|\cos\psi - \omega_1)\cos\theta_1 - \omega_2\,|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{78}$$
$$u_7 = q^2 - 2qp_1 \tag{79}$$
$$= q^2 - 2(q_0E_1 + \omega_2(|\vec{p}_2|\cos\psi - \omega_1)\cos\theta_1 + \omega_2\,|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{80}$$
$$(37): \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad \checkmark \tag{81}$$

2.3 phase space integrals

at phase space integration there occur integrations over propagators[4, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1\ \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2\ \sin^{n-4}(\theta_2)(a+b\cos(\theta_1))^{-k}(A+B\cos(\theta_1)+C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{82}$$
$$= \int d\Omega_n\,(a+b\cos(\theta_1))^{-k}(A+B\cos(\theta_1)+C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{83}$$

the integrals can be further destinguished by the range of k, l and the type of collinearity (following the notation in [4]):

- "non collinear": $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \to I_{0,n}^{(k,l)}$
- "single collinear a": $a = -b \wedge A^2 \neq B^2 + C^2 \to I_{a,n}^{(k,l)}$
- "single collinear A": $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \to I_{A,n}^{(k,l)}$
- "double collinear": $a = -b \wedge A = -\sqrt{B^2+C^2} \to I_{aA,n}^{(k,l)}$

Use $n = 4 + \epsilon$.

2.3.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int_0^\pi dt\ \sin^{\nu-3}(t)\cos^q(t) \tag{84}$$

It is[5, eq. 5.12.6]:

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha\, B\left((\alpha+\beta+1)/2, (\alpha-\beta+1)/2\right)} \qquad \text{if } \Re(\alpha) > 0 \tag{85}$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{B((n-1)/2,(n-1)/2)} \tag{86}$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{B((n-2)/2,(n-2)/2)} = B((n-3)/2, 1/2) \tag{87}$$

If q is odd: $\hat{I}^{(q)} = 0$, due to symetry of kernel; if q is even: $q = 2p$ with $p \in \mathbb{N}$:

$$\hat{I}^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t)\, dt \tag{88}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2+(k-p),(\nu-1)/2-(k-p))} \tag{89}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^{p} \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2+l,(\nu-1)/2-l)} \tag{90}$$

$$\begin{aligned} &= \frac{\pi\Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right. \\ &\qquad \left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \end{aligned} \tag{91}$$

$$\begin{aligned} &= \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right. \\ &\qquad \left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \end{aligned} \tag{92}$$

TODO: prove

FiXme Error: prove

$$p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{93}$$

$$= \frac{1}{p!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l p!}{(p+l)!(p-l)!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \tag{94}$$

$$= 1 \tag{95}$$

$$\Rightarrow \hat{I}^{(2p)}(\nu) = \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p!)}{2^p p!} \tag{96}$$

$$= \frac{\sqrt{\pi}(2p)!}{2^{2p}p!} \frac{\Gamma((\nu-2)/2)}{\Gamma(\frac{\nu-1}{2}+p)} \tag{97}$$

2.3.2 any collinearity and $-k, -l \in \mathbb{N}_0$

If $-k, -l \in \mathbb{N}_0$ $I_n^{(k,l)}$ can always be reduced in a straight forward manner to combinations of $\hat{I}^{(q)}(n)$ and this way one finds[4, Ch. 5][2, App. C]:

$$I_n^{(0,0)} = \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n) = \frac{2\pi}{n-3} \tag{98}$$

$$I_4^{(0,0)} = 2\pi \tag{99}$$

$$I_n^{(-1,0)} = \hat{I}^{(0)}(n-1) \cdot (a\hat{I}^{(0)}(n) + b\hat{I}^{(1)}(n)) = \frac{2\pi a}{n-3} \tag{100}$$

$$I_4^{(-1,0)} = 2\pi a \tag{101}$$

$$I_n^{(0,-1)} = \hat{I}^{(0)}(n-1) \cdot (A\hat{I}^{(0)}(n) + B\hat{I}^{(1)}(n)) + C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \tag{102}$$

$$= \frac{2\pi A}{n-3} \tag{103}$$

$$I_4^{(0,-1)} = 2\pi A \tag{104}$$

$$I_n^{(-2,0)} = \hat{I}^{(0)}(n-1) \cdot (a^2\hat{I}^{(0)}(n) + 2ab\hat{I}^{(1)}(n) + b^2\hat{I}^{(2)}(n)) \tag{105}$$

$$= 2\pi \left(\frac{a^2(n-1) + b^2}{(n-1)(n-3)} \right) \tag{106}$$

$$I_4^{(-2,0)} = 2\pi(a^2 + b^2/3) \tag{107}$$

$$I_n^{(0,-2)} = \hat{I}^{(0)}(n-1) \cdot (A^2\hat{I}^{(0)}(n) + B^2\hat{I}^{(2)}(n)) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \tag{108}$$

$$= 2\pi \left(\frac{A^2(n-1) + B^2 + C^2}{(n-1)(n-3)} \right) \tag{109}$$

$$I_4^{(0,-2)} = 2\pi(A^2 + (B^2 + C^2)/3) \tag{110}$$

$$I_n^{(-1,-1)} = \hat{I}^{(0)}(n-1) \cdot (aA\hat{I}^{(0)}(n) + bB\hat{I}^{(2)}(n)) = 2\pi \left(\frac{aA(n-1) + bB}{(n-1)(n-3)} \right) \tag{111}$$

$$I_4^{(-1,-1)} = 2\pi(aA + bB/3) \tag{112}$$

2.3.3 single collinear a

If $-l \in \mathcal{N}$ one finds:

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1-\cos(t))^k} \cos^q(t)\, dt \tag{113}$$

$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1-\cos^2(t))^k} \cos^q(t)(1+\cos(t))^k\, dt \tag{114}$$

$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t)(1+\cos(t))^k\, dt \tag{115}$$

$$= \sum_{l=0}^{k} \binom{k}{l} \hat{I}^{(q+l)}(\nu-2k) \tag{116}$$

this way one finds[4, Ch. 5][2, App. C]:

$$I_{a,n}^{(1,0)} = \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot \hat{I}^{(0)}(n-2) \tag{117}$$

$$= \frac{2\pi}{a(n-4)} \tag{118}$$

$$I_{a,n}^{(1,-1)} = \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot \Big(A\hat{I}^{(0)}(n-2) + B\hat{I}^{(2)}(n-2)\Big) \tag{119}$$

$$= \frac{2\pi}{a}\frac{(A(n-3)+B)}{(n-3)(n-4)} \approx \frac{2\pi}{a}\left(\frac{A+B}{\epsilon} - 2B + O(\epsilon)\right) \tag{120}$$

$$I_{a,n}^{(1,-2)} = \frac{1}{a}\Big(\hat{I}^{(0)}(n-1)\cdot \Big(A^2\hat{I}^{(0)}(n-2) + (B^2+2AB)\hat{I}^{(2)}(n-2)\Big) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n)\Big) \tag{121}$$

$$= \frac{2\pi}{a}\left(\frac{A^2}{n-4} + \frac{2AB+B^2}{(n-4)(n-3)} + \frac{C^2}{(n-3)(n-2)}\right) \tag{122}$$

$$\approx \frac{2\pi}{a}\left(\frac{(A+B)^2}{\epsilon} + \frac{C^2}{2} - 2AB - B^2 + O(\epsilon)\right) \tag{123}$$

$$
\begin{aligned}
I_{a,n}^{(2,0)} &= \frac{1}{a^2}\hat{I}^{(0)}(n-1)\cdot\left(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4)\right) && (124)\\
&= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2}+O(\epsilon) && (125)\\
I_{a,n}^{(2,-1)} &= \frac{1}{a^2}\hat{I}^{(0)}(n-1)\cdot\left(A\left(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4)\right)+2B\hat{I}^{(2)}(n-4)\right) && (126)\\
&= \frac{2\pi}{a^2}\left(\frac{A}{n-6}+\frac{2B}{(n-6)(n-4)}\right) \approx -\frac{2\pi}{a^2}\left(\frac{B}{\epsilon}+\frac{A+B}{2}\right)+O(\epsilon) && (127)\\
I_{a,n}^{(2,-2)} &= \frac{1}{a^2}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4))+4AB\hat{I}^{(2)}(n-4)\right.\right.\\
&\qquad \left.\left.+B^2(\hat{I}^{(2)}(n-4)+\hat{I}^{(4)}(n-4))\right)+C^2\hat{I}^{(2)}(n-1)(\hat{I}^{(0)}(n-2)+\hat{I}^{(2)}(n-2))\right) && (128)\\
&= \frac{2\pi}{a^2}\left(\frac{A^2}{n-6}+\frac{4AB}{(n-6)(n-4)}+\frac{B^2n}{(n-6)(n-4)(n-3)}+\frac{C^2}{(n-4)(n-3)}\right) && (129)\\
&\approx \frac{2\pi}{a^2}\left(\frac{-2AB-2B^2+C^2}{\epsilon}+\frac{B^2-A^2}{2}-AB-C^2+O(\epsilon)\right) && (130)\\
& && (131)
\end{aligned}
$$

It is[4, Ch. 5]:

$$
\begin{aligned}
I_{a,n}^{(1,1)} &= \frac{\pi}{a(A+B)}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)\right)+O(\epsilon) && (132)\\
I_{a,n}^{(2,1)} &= \frac{\pi}{a^2(A+B)}\left(\frac{B^2+AB+C^2}{(A+B)^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)\right)\right.\\
&\qquad \left.-\frac{2C^2}{(A+B)^2}-1\right)+O(\epsilon) && (133)
\end{aligned}
$$

From this are the following integrals derived[4, Ch. 5]:

$$
\begin{aligned}
I_{a,n}^{(1,2)} &= -\frac{\partial}{\partial A}I_{a,n}^{(1,1)} && (134)\\
&= \frac{\pi}{a(A+B)^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)+\frac{2A(A+B)}{A^2-B^2-C^2}-2\right)+O(\epsilon) && (135)\\
I_{a,n}^{(2,2)} &= -\frac{\partial}{\partial A}I_{a,n}^{(2,1)} && (136)\\
&= \frac{\pi}{a^2(A+B)^2}\left(\frac{2B^2+2AB+3C^2}{(A+B)^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{(A+B)^2}{A^2-B^2-C^2}\right)\right)\right.\\
&\qquad \left.+\frac{2A^2}{A^2-B^2-C^2}-\frac{8C^2}{(A+B)^2}-3\right)+O(\epsilon) && (137)
\end{aligned}
$$

If $-k\in\mathcal{N}$ use I_0 with $b=-a$.

2.3.4 double collinear

as said in [4, Ch. 5]: if $0 \leq -\frac{C}{A}, \frac{B}{A} \leq 1$ use [3, eq. A11] with $\cos\kappa = -\frac{B}{A}$:

$$I_{aA,n}^{(k,l)} = \frac{2\pi 2^{-(k+l)}}{a^k A^l}\frac{\Gamma(1+\epsilon)}{\Gamma^2(1+\epsilon/2)}B(1+\frac{\epsilon}{2}-k,1+\frac{\epsilon}{2}-l)_2F_1\left(k,l;1+\frac{\epsilon}{2};\frac{A-B}{2A}\right) \tag{138}$$

but we will not need it here.

2.3.5 non collinear

If $-l \in \mathcal{N}$ the θ_2 integration can be performed using the integral helper and the problem reduces then to the following integral:

$$\begin{aligned}
&\hat{I}_0^{(k,q,p)}(\epsilon)\\
&= \int_0^\pi d\theta_1\, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))^k} && (139)\\
&= \frac{1}{2a^k}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{k}{2},\frac{1+p}{2};\frac{1}{2},\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.\\
&\quad\left.\frac{b}{a}k(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{2+k}{2},\frac{2+p}{2};\frac{3}{2},\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) && (140)
\end{aligned}$$

for $k=1$ this simplifies to

$$\begin{aligned}
&\hat{I}_0^{(1,q,p)}(\epsilon)\\
&= \int_0^\pi d\theta_1\, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))} && (141)\\
&= \frac{1}{2a}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_2F_1\left(1,\frac{1+p}{2};\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.\\
&\quad\left.\frac{b}{a}(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_2F_1\left(1,\frac{2+p}{2};\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) && (142)
\end{aligned}$$

TODO: structure does NOT match [4, Ch. 5] - neither A,B,C nor ϵ

FiXme Error: fix to Bojak

For $I_{0,n}^{(1,1)}$ one finds[4, Ch. 5]:

$$I_{0,n}^{(1,1)} = \frac{\pi}{\sqrt{X}}\ln\left(\frac{aA-bB+\sqrt{X}}{aA-bC-\sqrt{X}}\right) \tag{143}$$

$$\text{with } X = (aB-bA)^2-(a^2-b^2)C^2 \tag{144}$$

$I_{0,n}^{(1,-3)}$ is given by [4, Ch. 5] and from those two all other integrals can be deduced using the techiques described in [4, Ch. 5]: increase k or l by differentiation or interchange k and l by a rotation:

$$I_n^{(k,l)} = I_n^{(l,k)}\left(a \leftrightarrow A, b \to -\sqrt{B^2+C^2}, B \to \frac{-bB}{\sqrt{B^2+C^2}}, C \to \frac{-bC}{\sqrt{B^2+C^2}}\right) \tag{145}$$

2.4 needed set in matrix element

define a shortcut:

$$\mathcal{V}_{a,b}(x,y) = \left(x^k y^l\right)_{k=0..a,l=0..b} \tag{146}$$

It is

$$A_{G,1} = \sum_{k,l=0}^{3} \left(\mathcal{C}_{A_{G,1}}\right)_{(k,l)} t'^{-2+k} u_7{}^{-2+l} \qquad = \operatorname{tr}\left(\mathcal{C}_{A_{G,1}} \frac{\mathcal{V}_{3,3}(t',u_7)^t}{t'^2 u_7{}^2}\right) \tag{147}$$

$$A_{L,1} = \sum_{k=0}^{4}\sum_{l=0}^{2} \left(\mathcal{C}_{A_{L,1}}\right)_{(k,l)} t'^{-2+k} u_7{}^{-2+l} \qquad = \operatorname{tr}\left(\mathcal{C}_{A_{L,1}} \frac{\mathcal{V}_{4,2}(t',u_7)^t}{t'^2 u_7{}^2}\right) \tag{148}$$

and we will thus need the integrals

$$\left(\mathcal{I}_{A_{G,1}}\right)_{(k,l)} = \int d\Omega_n \frac{1}{t'^2 u_7{}^2} \left(\mathcal{V}_{3,3}(t',u_7)\right)_{(k,l)} \tag{149}$$

$$\left(\mathcal{I}_{A_{L,1}}\right)_{(k,l)} = \int d\Omega_n \frac{1}{t'^2 u_7{}^2} \left(\mathcal{V}_{4,2}(t',u_7)\right)_{(k,l)} \tag{150}$$

with

$$a(t') = -b(t') = -2\omega_1\omega_2 \qquad = -\frac{s_4(s'+t_1)}{2(s_4+m^2)} \tag{151}$$

$$A(u_7) = q^2 - 2q_0E_1 \qquad = q^2 - \frac{(s_4+2m^2)(s+u_1)}{2(s_4+m^2)} \tag{152}$$

$$B(u_7) = -2\omega_2(|\vec{p}_2|\cos\psi - \omega_1) \qquad = \frac{s_4}{2}\left(1 - \frac{s+u_1}{s_4+m^2} + \frac{s'-t_1}{s'+t_1}\right) \tag{153}$$

$$C(u_7) = -2\omega_2\,|\vec{p}_2|\sin\psi \tag{154}$$

that is $I_{a,n}^{(-2...2,-1...2)}$. With this we find

$$A + B = -\frac{t_1u_1}{s'+t_1} \tag{155}$$

$$\frac{(A+B)^2}{A^2-B^2-C^2} = \frac{(s_4+m^2)t_1^2u_1^2}{(s'+t_1)^2(s_4q^2t_1 + m^2(s'+u_1)^2)} \tag{156}$$

$$2B(A+B) + 3C^2 = -\frac{s_4(m^2s'(3s's_4+2t_1u_1) + t_1(q^2(s_4-u_1)(3s_4-u_1) - u_1(s's_4+t_1u_1)))}{(s_4+m^2)(s'+t_1)^2} \tag{157}$$

With this we get

$$\int d\Omega_n\, A_{j,1} = \mathrm{tr}(\mathcal{C}_{A_{j,1}}\, \left(\mathcal{I}_{A_{j,1}}\right)^t) \qquad j = G, L \tag{158}$$

It is

$$A_{G,2} = \sum_{k,l=0}^{3} \left(\mathcal{C}_{A_{G,2}}\right)_{(k,l)} {s_5}^{-2+k} {u'}^{-2+l} \qquad\qquad = \mathrm{tr}\left(\mathcal{C}_{A_{G,2}} \frac{\mathcal{V}_{3,3}(s_5, u')^t}{{s_5}^2 {u'}^2}\right) \tag{159}$$

$$A_{L,2} = \sum_{k=0}^{4} \sum_{l=0}^{3} \left(\mathcal{C}_{A_{G,2}}\right)_{(k,l)} {s_5}^{-2+k} {u'}^{-2+l} \qquad\qquad = \mathrm{tr}\left(\mathcal{C}_{A_{L,2}} \frac{\mathcal{V}_{4,3}(s_5, u')^t}{{s_5}^2 {u'}^2}\right) \tag{160}$$

and we will thus need the integrals

$$\left(\mathcal{I}_{A_{G,2}}\right)_{(k,l)} = \int d\Omega_n \frac{1}{{s_5}^2 {u'}^2} \left(\mathcal{V}_{3,3}(s_5, u')\right)_{(k,l)} \tag{161}$$

$$\left(\mathcal{I}_{A_{L,2}}\right)_{(k,l)} = \int d\Omega_n \frac{1}{{s_5}^2 {u'}^2} \left(\mathcal{V}_{4,3}(s_5, u')\right)_{(k,l)} \tag{162}$$

but as both s_5 and u' are of $[ABC]$ type we have to apply partial fractioning, following the ideas of [4, Ch. 5]. It is

$$(43): \qquad s_5 = s - q^2 + t' + u' \tag{163}$$

so end up with a form of $\mathcal{V}(p+q, q)$ where p is $[ab]$ and q and $p+q$ are $[ABC]$. The aim is then to get to a form with fractions of $\frac{p}{q}$ and/or $\frac{p+q}{p}$ and indeed this can be achieved. Define

$$\mathcal{T} = \begin{pmatrix} \{-2,1,2,1\} & \{1,\ \ 0,-1,-1\} & \{0,0,0,1\} & \{0,0,1,-1\} \\ \{-1,1,1,0\} & \{1,-1,\ \ 0,\ \ 0\} & \{0,0,1,0\} & \{0,0,1,-1\} \\ \{\ \ 0,1,0,0\} & \{1,\ \ 0,\ \ 0,\ \ 0\} & \{1,0,0,0\} & \{0,0,1,-1\} \\ \{\ \ 1,1,0,0\} & \{1,\ \ 1,\ \ 0,\ \ 0\} & \{1,1,0,0\} & \{0,0,1,-1\} \end{pmatrix} \tag{164}$$

$$\left(\mathcal{W}(x,y)\right)_{(k,l)} = x^{-2+p_x(k+l)} y^{-2+p_y(k+l)} \tag{165}$$

$$p_x(s) = \begin{cases} s & \text{if } s \le 3 \\ 3 & \text{else} \end{cases} \qquad p_y(s) = 3 - p_x(6-s) \begin{cases} 0 & \text{if } s \le 3 \\ s-3 & \text{else} \end{cases} \tag{166}$$

$$\Rightarrow \mathcal{W}(x,y) = \begin{pmatrix} x^{-2}y^{-2} & x^{-1}y^{-2} & y^{-2} & xy^{-2} \\ x^{-1}y^{-2} & y^{-2} & xy^{-2} & xy^{-1} \\ y^{-2} & xy^{-2} & xy^{-1} & x \\ xy^{-2} & xy^{-1} & x & xy \end{pmatrix} \tag{167}$$

$$\left(\mathcal{W}^*(p,q)\right)_{(k,l)} = \left\{\frac{q}{p}\mathcal{W}(p,q), \mathcal{W}(p,q), \frac{p+q}{p}\mathcal{W}(p,p+q), \mathcal{W}(p,p+q)\right\}_{(k,l)} \tag{168}$$

we then find:

$$\left(\mathcal{V}(p+q,q)\right)_{(k,l)} = \mathcal{T}_{(k,l)} \cdot \mathcal{W}^*(p,q)_{(k,l)} \tag{169}$$

where the equation has to be read separate for each (k,l), i.e. "$\cdot$" applies to the scalar 4x4-product of the {}s. As $\mathcal{W}^t = \mathcal{W}$, $\mathcal{W}$ has 10 unique entries and thus for $\mathcal{W}^*$ are 40 unique integrals needed: $I_0^{(-1\ldots3,-2\ldots2)}$.

A References

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List of Corrections