# 1 Introduction

This work is mainly based on the paper "Complete $O(\alpha_S)$ corrections to heavy-flavour structure functions in electroproduction" by Laenen et. al.[1] - that is, it recalculates all properties and formulas. It extends then the application to the equivalent *polarized* processes. The treating of the polarized processes can for example be found in [2] and we will use many ideas and technices from there. **FiXme Error: more** FiXme Error!

## 1.1 Motivation

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**FiXme Error: why do we do this**

## 1.2 Notation

To collect all soft and collinear poles we have to calculate in $n = 4 + \epsilon$ dimension. Unfortunaly the extension for *polarized* processes is nontrivial, because the occuring Levi-Civita tensors $\varepsilon_{\mu\nu\rho\sigma}$ and $\gamma_5$. A common choice to deal with these objects is the HVBM prescription[3] that keeps those two objects four dimensional at the price for splitting the full $n$-dimensional space into a $(n-4)$-dimensional space, called "hat-space", and a four-dimensional space (that is actually never used).

In leading order (LO) we have to consider the following processes

$$\gamma^*(q;\sigma_q) + \mathrm{g}(k_1;\sigma_{k_1}) \to Q(p_1) + \overline{Q}(p_2) \tag{1}$$

The corresponding parton structure tensor $W^{(0)}_{\mu\mu'}$ can then be written as **FiXme Error: avoid all order expr?** FiXme Error!

$$\begin{aligned}
&W^{(0)}_{\mu\mu'}(k_1,q;s,t_1,u_1,q^2;\sigma_{k_1}\sigma_q)\\
&= \frac{1}{2}E_k(\epsilon)K_{\mathrm{g}\gamma}\int\frac{d^{n-1}p_1}{2E_1(2\pi)^{n-1}}\int\frac{d^{n-1}p_2}{2E_2(2\pi)^{n-1}}\delta(p_1^2-m^2)\delta(p_2^2-m^2)\\
&\quad (2\pi)^n\delta^{(n)}(k_1+q-p_1-p_2)\,\mathcal{M}^{(0)}_{\mu}(\sigma_{k_1},\sigma_q)\,\mathcal{M}^{(0)}_{\mu'}(\sigma_{k_1},\sigma_q)
\end{aligned} \tag{2}$$

where the initial 1/2 is the initial state spin average, $K_{\mathrm{g}\gamma}$ is the color average,

$$E_\epsilon := \begin{cases} 1/(1+\epsilon/2) & \text{unpolarized}\\ 1 & \text{polarized}\end{cases} \tag{3}$$

accounts for initial freedom in $n$ dimensions for bosons and we defined the following Mandelstam variables:

$$s = (q+k_1)^2,\quad t_1 = t - m^2 = (k_1-p_2)^2 - m^2,\quad u_1 = u - m^2 = (q-p_2)^2 - m^2 \tag{4}$$

$$s' = s - q^2, \quad u_1' = u_1 - q^2 \tag{5}$$

**FiXme Error: move to LO?** The Lorentz indices $\mu$ and $\mu'$ refer to the virtual photon that is exchanged with the scattering lepton. FiXme Error!

By using Lorentz covariance, hermiticity, parity invariance and current conservation the parton structure tensor can be decomposed into several parts:

$$\begin{aligned} W_{\mu\mu'}(k_1, q; s, t_1, u_1, q^2; \sigma_{k_1}, \sigma_q) = &\left(-g_{\mu\mu'} + \frac{q_\mu q_{\mu'}}{q^2}\right) \frac{d^2\sigma_T(s, t_1, u_1, q^2)}{dt_1 du_1} \\ &+ \left(k_{1,\mu} - \frac{k_1 \cdot q}{q^2} q_\mu\right) \left(k_{1,\mu'} - \frac{k_1 \cdot q}{q^2} q_{\mu'}\right) \left(\frac{-4q^2}{s'^2}\right) \\ &\cdot \left(\frac{d^2\sigma_T(s, t_1, u_1, q^2)}{dt_1 du_1} + \frac{d^2\sigma_L(s, t_1, u_1, q^2)}{dt_1 du_1}\right) \end{aligned} \tag{6}$$

**FiXme Error: extend** We can then define appropiate projection operators[1, 4]: FiXme Error!

$$\mathcal{P}_{G,\mu\mu'} = -g_{\mu\mu'} \qquad b_G(\epsilon) = \frac{1}{2(1+\epsilon/2)} \tag{7}$$

$$\mathcal{P}_{L,\mu\mu'} = -\frac{4q^2}{s'^2} k_{1,\mu} k_{1,\mu'} \qquad b_L(\epsilon) = 1 \tag{8}$$

$$\mathcal{P}_{P,\mu\mu'} = i\varepsilon_{\mu\mu'\rho\rho'} \frac{q^\rho k_1^{\rho'}}{s'} \qquad b_P(\epsilon) = 1 \tag{9}$$

**FiXme Error: justify avoidance of Δ?** FiXme Error!

$$\frac{d^2\sigma_k(s, t_1, u_1, q^2)}{dt_1 t u_1} = b_k(\epsilon) \mathcal{P}_{k,\mu\mu'} W^{\mu\mu'} \tag{10}$$

with $k \in \{G, L, P\}$ denoting (here and mostly ever after) the projection type. The transverse partonic cross section $d\sigma_T$ can be reconstructed from the above definitions by using

$$d\sigma_T = d\sigma_G + b_G(\epsilon) d\sigma_L \tag{11}$$

We also define accordingly

$$E_G(\epsilon) = E_L(\epsilon) = \frac{1}{1+\epsilon/2} \qquad E_P(\epsilon) = 1 \tag{12}$$

The final state spins are always summed over, but the initial spins have to be treated seperately: for unpolarized projections $k \in \{G, L\}$ they are also summed over, but for polarized $k = P$ they are combined as follows

$$\hat{\sum}_{G,\sigma} f(\sigma_{k_1}, \sigma_q) = \hat{\sum}_{L,\sigma} f(\sigma_{k_1}, \sigma_q) = f(+,+) + f(-,-) + f(+,-) + f(-,+) \tag{13}$$

$$\hat{\sum}_{P,\sigma} f(\sigma_{k_1}, \sigma_q) = f(+,+) + f(-,-) - f(+,-) - f(-,+) \tag{14}$$

which keeps spin asymmetries well behaving.

When computing total partonic cross sections we define a set of partonic variables:

$$0 \le \rho = \frac{4m^2}{s} \le 1 \qquad 0 \le \beta = \sqrt{1-\rho} \le 1 \qquad 0 \le \chi = \frac{1-\beta}{1+\beta} \le 1 \tag{15}$$

$$\rho_q = \frac{4m^2}{q^2} \le 0 \qquad 1 \le \beta_q = \sqrt{1-\rho_q} \qquad 0 \le \chi_q = -\frac{1-\beta_q}{1+\beta_q} \le 1 \tag{16}$$

When computing Feynman diagrams a computer algebra system (CAS) is almost obligatory: common choices are `FORM`[5] or `Mathematica`[6] - for the later the most common choice is `TRACER`[7], but we have chosen `HEPMath`[8]. We used the Feynman rules given by [9].**FiXme Error: explain ghosts?** FiXme Error!

## 2 Leading Order Calculations

In leading order we have to consider photon-gluon-fusion (PGF), that is

$$\gamma^*(q;\sigma_q) + \mathrm{g}(k_1;\sigma_{k_1}) \to Q(p_1) + \overline{Q}(p_2) \tag{17}$$

with two contributing diagrams depicted in figure **FiXme Error: todo**. The result can then be written as FiXme Error!

$$\hat{\sum_{k,\sigma}} \mathcal{P}_k^{\mu\mu'} \sum_{j=1}^{2} \mathcal{M}_{j,\mu}^{(0)}(\sigma_{k_1},\sigma_q)\, \mathcal{M}_{j,\mu'}^{(0)*}(\sigma_{k_1},\sigma_q) = 8g^2\mu_D^{-\epsilon}e^2e_H^2 N_C C_F B_{k,QED} \tag{18}$$

where $g$ and $e$ are the strong and electromagnetic coupling constants respectively, $\mu_D$ is an arbitray mass parameter introduced to keep the couplings dimensionless and $e_H$ is the magnitude of the heavy quark in units $e$. Further $N_C$ corresponds to the gauge group $SU(N_C)$ and the color factor $C_F = (N_C^2-1)/(2N_C)$ refers to the second Casimir constant of the fundamental representation for the quarks. We then find:

$$\begin{aligned} B_{G,QED} = &\frac{t_1}{u_1} + \frac{u_1}{t_1} + \frac{4m^2s'}{t_1u_1}\left(1 - \frac{m^2s'}{t_1u_1}\right) + \frac{2s'q^2}{t_1u_1} + \frac{2q^4}{t_1u_1} + \frac{2m^2q^2}{t_1u_1}\left(2 - \frac{s'^2}{t_1u_1}\right) \\ &+ \epsilon\left\{-1 + \frac{s'^2}{t_1u_1} + \frac{s'q^2}{t_1u_1} - \frac{q^4}{t_1u_1} - \frac{m^2q^2s'^2}{t_1^2u_1^2}\right\} + \epsilon^2\frac{s'^2}{4t_1u_1} \end{aligned} \tag{19}$$

$$B_{L,QED} = -\frac{4q^2}{s'}\left(\frac{s}{s'} - \frac{m^2s'}{t_1u_1}\right) \tag{20}$$

$$B_{P,QED} = \frac{1}{2}\left(\frac{t_1}{u_1} + \frac{u_1}{t_1}\right)\left(\frac{2m^2s'}{t_1u_1} - 1 - \frac{2q^2}{s'}\right) \tag{21}$$

$$B_{k,QED} = B_{k,QED}^{(0)} + \epsilon B_{k,QED}^{(1)} + \epsilon^2 B_{k,QED}^{(2)} \tag{22}$$

By using eq. (2) we can derive the $n$-dimensional $2 \to 2$ phase space

$$dPS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_2}{(2\pi)^{n-1}} \Theta(p_{1,0}) \delta(p_1^2 - m^2) \Theta(p_{2,0}) \delta(p_2^2 - m^2) (2\pi)^n \delta^{(n)}(k_1 + q - p_1 - p_2) \tag{23}$$

that can be solved by using the center-of-mass system (CMS) of the incoming particles[2]

$$q = \left( \frac{s + q^2}{2\sqrt{s}}, 0, 0, -\frac{s - q^2}{2\sqrt{s}}, \hat{0} \right) \qquad\qquad k_1 = \frac{s - q^2}{2\sqrt{s}} \left(1, 0, 0, 1, \hat{0}\right) \tag{24}$$

such that $q + k_1 = (\sqrt{s}, \vec{0})$ and $k_1^2 = 0$. For the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2} \left(1, 0, \beta \sin\theta, \beta \cos\theta, \hat{0}\right) \qquad p_2 = \frac{\sqrt{s}}{2} \left(1, 0, -\beta \sin\theta, -\beta \cos\theta, \hat{0}\right) \tag{25}$$

such that $p_1 + p_2 = (\sqrt{s}, \vec{0})$ and $p_1^2 = p_2^2 = m^2$. Finally we have to use the $n$-sphere

$$d^n x = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} dx = \frac{\pi^{n/2}}{\Gamma(n/2)} (x^2)^{(n-2)/2} dx^2 \tag{26}$$

and arrive at the well known result[1]

$$\begin{aligned} dPS_2 &= \frac{\delta(s' + t_1 + u_1)}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \left( \frac{(t_1 u_1' - s' m^2) s' - q^2 t_1^2}{s'^2} \right)^{(n-4)/2} dt_1 du_1 && (27) \\ &= \delta(s' + t_1 + u_1) h_2(n)\, dt_1 du_1 && (28) \\ h_2(4+\epsilon) &= \frac{2\pi S_\epsilon}{s'\Gamma(1+\epsilon/2)} \left( \frac{(t_1 u_1' - s' m^2) s' - q^2 t_1^2}{s'^2} \right)^{\epsilon/2} && (29) \end{aligned}$$

with $S_\epsilon = (4\pi)^{(-2-\epsilon/2)}$.

The final double differential LO partonic cross section can then be written as

$$\begin{aligned} s'^2 \frac{d^2\sigma^{(0)}_{k,\mathrm{g}}(s', t_1, u_1, q^2)}{dt_1 du_1} &= 2^6 \alpha \alpha_s e_H^2 K_{\mathrm{g}\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \delta(s' + t_1 + u_1) \frac{\pi^3 S_\epsilon}{\Gamma(1+\epsilon/2)} \\ &\quad \left( \frac{(t_1 u_1' - s' m^2) s' - q^2 t_1^2}{m^2 s'^2} \right)^{\epsilon/2} \left( \frac{\mu_D^2}{m^2} \right)^{-\epsilon/2} B_{k,QED} \end{aligned} \tag{30}$$

where we used $e^2 = 4\pi\alpha$ and $g^2 = 4\pi\alpha_s$ and introduced the arbitrary mass parameter $\mu_D$ to keep the strong coupling dimensionless. The color average is given by $K_{\mathrm{g}\gamma} = 1/(N_C^2 - 1)$.

From the results above we can easily obtain the *total* LO partonic cross sections

$$\sigma_G^{(0)}(s,q^2,m^2) = -4\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{1}{s'^3}\left((s^2+q^4+4m^2s)\beta + (s^2+q^4-4m^2(2m^2-s'))\ln(\chi)\right) \tag{31}$$

$$\sigma_L^{(0)}(s,q^2,m^2) = 16\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \left(\frac{-q^2 s}{s'^3}\right)\left(\beta + \frac{2m^2}{s}\ln(\chi)\right) \tag{32}$$

$$\sigma_P^{(0)}(s,q^2,m^2) = 4\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{1}{s'^2}\left((3s+q^2)\beta + (s+q^2)\ln(\chi)\right) \tag{33}$$

from which we also find

$$\lim_{s\to 4m^2} \sigma_T^{(0)}(s',q^2) = 4\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{\beta}{4m^2-q^2} + O(\beta^3) = \lim_{s\to 4m^2} \sigma_P^{(0)}(s',q^2) \tag{34}$$

$$\lim_{s\to 4m^2} \sigma_L^{(0)}(s',q^2) = -\frac{128}{3}\pi\alpha\alpha_s e_H^2 K_{g\gamma} N_C C_F \frac{m^2 q^2 \beta^3}{(4m^2-q^2)^3} + O(\beta^5) \tag{35}$$

(Note the missing factor of 2 in [1, eq. (5.9)].) **FiXme Error: shift to partonic?** FiXme Error!

## 3 Next-To-Leading Order Calculations

Next-to-leading order contributions can be split into three parts: one loop virtual contributions, one gluon radiation and light quark processes. **FiXme Error: more?** FiXme Error!

### 3.1 One Loop Virtual Contributions

Virtual contributions have the same initial and final state as the Born process, but have a looping particle. All contributing Feynman diagrams are depicted in figure **FiXme Error: do**. The result can then be written as FiXme Error!

$$\begin{aligned}\hat{\sum}_{k,\sigma} \mathcal{P}_k^{\mu\mu'} \sum_j \left[\mathcal{M}_{j,\mu}^{(1),V}\left(\mathcal{M}_{1,\mu'}^{(0)} + \mathcal{M}_{2,\mu'}^{(0)}\right)^* + c.c.\right] \\ = 8g^4\mu_D^{-\epsilon} e^2 e_H^2 N_C C_F C_\epsilon \left(C_A V_{k,OK} + 2C_F V_{k,QED}\right)\end{aligned} \tag{36}$$

where $C_\epsilon = \exp(\epsilon/2(\gamma_E - \ln(4\pi)))/(16\pi^2)$ and $C_A$ is the second Casimir constant of the adjoint representation for the gluon (that introduces a non-abelian part).

For the computation of the loops the Passarino-Veltman-decomposition[10] in $n = 4+\epsilon$ dimension is used as far as possible. The decomposition is based on Lorentz invariance and a good explanation is for example given in [2]. The needed scalar integrals are given in [11] and [1], but there is also one wrong integral: we find with [12, Box 16]:

$$
\begin{aligned}
&D_0(m^2,0,q^2,m^2,t,s,0,m^2,m^2,m^2)\\
&= \frac{iC_\epsilon}{\beta s t_1}\left[-\frac{2\ln(\chi)}{\epsilon} - 2\ln(\chi)\ln(-t_1/m^2) + \mathrm{Li}_2(1-\chi^2) - 4\zeta(2) + \ln^2(\chi_q) + 2\,\mathrm{Li}_2(-\chi\chi_q)\right.\\
&\qquad\left. +2\,\mathrm{Li}_2(-\chi/\chi_q) + 2\ln(\chi\chi_q)\ln(1+\chi\chi_q) + 2\ln(\chi/\chi_q)\ln(1+\chi/\chi_q)\right]
\end{aligned}
\tag{37}
$$

where we used the argument ordering of `LoopTools`[13, 14] (and also checked it against `LoopTools`).

As the short example above shows are the full expressions for the $V_{k,OK}, V_{k,QED}$ quite complicated and too long to be presented here, nevertheless the arising poles are quite compact:

$$
V_{k,OK} = -2B_{k,QED}\left(\frac{4}{\epsilon^2} + \left(\ln(-t_1/m^2) + \ln(-u_1/m^2) - \frac{2m^2-s}{s}\ln(\chi)\right)\frac{2}{\epsilon}\right) + O(\epsilon^0)
\tag{38}
$$

$$
V_{k,QED} = -2B_{k,QED}\left(1 - \frac{2m^2-s}{s}\ln(\chi)\right)\frac{2}{\epsilon} + O(\epsilon^0)
\tag{39}
$$

The above results already include the mass renormalization that we have performed *on-shell*, so all ultra-violet poles have been removed. For the renormalization of the strong coupling we use the $\overline{\mathrm{MS}}_m$ scheme defined in [2] and so the full renormalization can be achieved by

$$
\begin{aligned}
\frac{d^2\sigma_k^{(1),V,ren.}}{dt_1du_1} &= \frac{d^2\sigma_k^{(1),V}}{dt_1du_1} + \frac{\alpha_s(\mu_R^2)}{4\pi}\left[\left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(\mu_R^2/m^2) - \ln(\mu_D^2/m^2)\right)\beta_0^f\right.\\
&\qquad\left. +\frac{2}{3}\ln(\mu_R^2/m^2)\right]\frac{d^2\sigma_k^{(0)}}{dt_1du_1}
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&= \frac{d^2\sigma_k^{(1),V}}{dt_1du_1} + 4\pi\alpha_s(\mu_R^2)C_\epsilon\left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2}\left[\left(\frac{2}{\epsilon} + \ln(\mu_R^2/m^2)\right)\beta_0^f\right.\\
&\qquad\left. +\frac{2}{3}\ln(\mu_R^2/m^2)\right]\frac{d^2\sigma_k^{(0)}}{dt_1du_1}
\end{aligned}
\tag{41}
$$

with $\mu_R$ the renormalization scale introduced by the RGE, $\beta_0^f = (11C_A - 2n_f)/3$ the first coefficient of the beta function and $n_f$ the number of *total* flavours (i.e. $n_{lf} = n_f - 1$ active (light) flavours and one heavy flavour). The double poles occuring in $V_{k,OK}$ are introduced by the diagrams **FiXme Error: do** when the soft and collinear singularities coincide. FiXme Error!

The double differential partonic cross section is given by

$$s'^2\frac{d^2\sigma^{(1),V}_{k,\mathrm{g}}(s',t_1,u_1,q^2)}{dt_1du_1} = 2^8\alpha\alpha_s^2e_H^2K_{\mathrm{g}\gamma}N_CC_FE_k(\epsilon)b_k(\epsilon)\delta(s'+t_1+u_1)\frac{\pi^4S_\epsilon}{\Gamma(1+\epsilon/2)}$$
$$\left(\frac{(t_1u_1'-s'm^2)s'-q^2t_1^2}{m^2s'^2}\right)^{\epsilon/2}C_\epsilon\left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2}$$
$$\left(C_AV_{k,OK}+2C_FV_{k,QED}\right) \tag{42}$$

### 3.2 Single Gluon Radiation

In next-to-leading order we have to consider the following process:

$$\gamma^*(q;\sigma_q)+\mathrm{g}(k_1;\sigma_{k_1})\to Q(p_1)+\overline{Q}(p_2)+\mathrm{g}(k_2) \tag{43}$$

All contributing diagrams are depicted in figure **FiXme Error: do** and the result can be written as FiXme Error!

$$\hat{\sum}_{k,\sigma}\mathcal{P}_k^{\mu\mu'}\sum_{j,j'}\mathcal{M}^{(1),g}_{j,\mu}\mathcal{M}^{(1),g*}_{j',\mu'} = 8g^4\mu_D^{-2\epsilon}e^2e_H^2N_CC_F\left(C_AR_{k,OK}+2C_FR_{k,QED}\right) \tag{44}$$

and it will depend on ten kinematical invariants:

$$s=(q+k_1)^2 \qquad t_1=(k_1-p_2)^2-m^2 \qquad u_1=(q-p_2)^2-m^2 \tag{45}$$
$$s_3=(k_2+p_2)^2-m^2 \qquad s_4=(k_2+p_1)^2-m^2 \qquad s_5=(p_1+p_2)^2=-u_5 \tag{46}$$
$$t'=(k_1-k_2)^2 \tag{47}$$
$$u'=(q-k_2)^2 \qquad u_6=(k_1-p_1)^2-m^2 \qquad u_7=(q-p_1)^2-m^2 \tag{48}$$

from which only five are independent as can be seen from momentum conservation $k_1+q=p_1+p_2+k_2$ and $s,t_1,u_1$ match to their leading order definition.

The $2\to 3$ $n$-dimensional phase space is given by

$$dPS_3=\int\frac{d^np_1}{(2\pi)^{n-1}}\frac{d^np_2}{(2\pi)^{n-1}}\frac{d^nk_2}{(2\pi)^{n-1}}(2\pi)^n\delta^{(n)}(k_1+q-p_1-p_2-k_2)$$
$$\Theta(p_{1,0})\delta(p_1^2-m^2)\Theta(p_{2,0})\delta(p_2^2-m^2)\Theta(k_{2,0})\delta(k_2^2) \tag{49}$$

This can be solved by writing eq. (49) as product of a $2\to 2$ decay and a subsequent $1\to 2$ decay [11]. We find

$$dPS_3=\frac{1}{(4\pi)^n\Gamma(n-3)s'}\frac{s_4^{n-3}}{(s_4+m^2)^{n/2-1}}\left(\frac{(t_1u_1'-s'm^2)s'-q^2t_1^2}{s'^2}\right)^{(n-4)/2}dt_1du_1d\Omega_nd\hat{\mathcal{I}} \tag{50}$$
$$=h_3(n)\,dt_1du_1d\Omega_nd\hat{\mathcal{I}} \tag{51}$$

with $d\Omega_n = \sin^{n-3}(\theta_1)d\theta_1 \sin^{n-4}(\theta_2)d\theta_2$ and $d\hat{\mathcal{I}}$ taking care of all occuring hat momenta:

$$d\hat{\mathcal{I}} = \frac{1}{B(1/2,(n-4)/2)} \frac{x^{(n-6)/2}}{\sqrt{1-x}} dx \qquad \text{with } x = \hat{p}_1^2/\hat{p}_{1,max} \tag{52}$$

$$\hat{p}_{1,max} = \frac{s_4^2}{4(s_4+m^2)} \sin^2(\theta_1)\sin^2(\theta_2) \tag{53}$$

$$\Rightarrow \int d\hat{\mathcal{I}} = 1 \qquad \int d\hat{\mathcal{I}}\, \hat{p}_1^2 = \epsilon \hat{p}_{1,max} + O(\epsilon^2) \tag{54}$$

The needed phase space integrals for $\theta_1$ and $\theta_2$ can be found in [11] and [2]. We find for the difference to the $2 \to 2$ phase space

$$\frac{h_3(4+\epsilon)}{h_2(4+\epsilon)} = \frac{S_\epsilon}{2\pi} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \frac{s_4^{1+\epsilon}}{(s_4+m^2)^{1+\epsilon/2}} \tag{55}$$

$$= \frac{C_\epsilon}{2\pi}\left(1 - \frac{3}{8}\zeta(2)\epsilon^2\right) \frac{s_4^{1+\epsilon}}{(s_4+m^2)^{1+\epsilon/2}} + O(\epsilon^3) \tag{56}$$

where $\zeta$ is Riemanns zeta function. **FiXme Error: introduce psLogs? in appendix?** FiXme Error!

Again when integrating the phase space angles the expressions become quite lengthy, but the (collinear) pole parts are compact:

$$\frac{s_4}{4\pi(s_4+m^2)} \int d\Omega_n d\hat{\mathcal{I}}\, C_A R_{k,OK} = -\frac{1}{u_1} B_{k,QED}\begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} P^H_{k,\mathrm{gg}}(x_1)\frac{2}{\epsilon} + O(\epsilon^0) \tag{57}$$

with $x_1 = -u_1/(s'+t_1)$ and the hard part of the Altarelli-Parisi splitting functions $P^H_{k,\mathrm{gg}}$[15, 16]:

$$P^H_{G,\mathrm{gg}}(x) = P^H_{L,\mathrm{gg}}(x) = C_A\left(\frac{2}{1-x} + \frac{2}{x} - 4 + 2x - 2x^2\right) \tag{58}$$

$$P^H_{P,\mathrm{gg}}(x) = C_A\left(\frac{2}{1-x} - 4x + 2\right) \tag{59}$$

The $R_{k,QED}$ do not contain poles. **FiXme Error: shift to factorization?** FiXme Error!

The double differential partonic cross section is given by

$$s'^2 \frac{d^2\sigma^{(1),R}_{k,\mathrm{g}}(s',t_1,u_1,q^2)}{dt_1 du_1} = 2^7 \alpha\alpha_s^2 e_H^2 K_{\mathrm{g}\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \frac{\pi^3 S_\epsilon^2}{\Gamma(1+\epsilon)} \frac{s_4}{s_4+m^2}$$
$$\left(\frac{(t_1 u_1' - s'm^2)s' - q^2 t_1^2}{m^2 s'^2}\right)^{\epsilon/2} \left(\frac{s_4^2}{m^2(s_4+m^2)}\right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon}$$
$$\int d\Omega_n d\hat{\mathcal{I}}\, \left(C_A R_{k,OK} + 2C_F R_{k,QED}\right) \tag{60}$$

From the above expression we can obtain the soft limit $k_2 \to 0$ and seperate their calculations:

$$\lim_{k_2\to 0}\left(C_A R_{k,OK} + 2C_F R_{k,QED}\right) = \left(C_A S_{k,OK} + 2C_F S_{k,QED}\right) + O(1/s_4, 1/s_3, 1/t') \quad (61)$$

$$S_{k,OK} = 2\left(\frac{t_1}{t' s_3} + \frac{u_1}{t' s_4} - \frac{s-2m^2}{s_3 s_4}\right) B_{k,QED} \quad (62)$$

$$S_{k,QED} = 2\left(\frac{s-2m^2}{s_3 s_4} - \frac{m^2}{s_3^2} - \frac{m^2}{s_4^2}\right) B_{k,QED} \quad (63)$$

Note that the einkonal factors multiplying the Born functions $B_{k,QED}$ neither depend on $q^2$ nor on the projection $k$. We can then split the phase space by introducing an infrared cut-off $\Delta$ and distinguish then between soft $s_4 \le \Delta$ and hard $s_4 > \Delta$ contributions. Let $\mathcal{R}(s_4)$ be a function with a soft pole $s_4^{-1+\epsilon}\mathcal{S}(s_4)$ and a finite part $\mathcal{F}(s_4)$, we then find [2]:

$$\int_0^{s_{4,max}} \mathcal{R}(s_4) = \int_0^{s_{4,max}} \left(s_4^{-1+\epsilon}\mathcal{S}(s_4) + \mathcal{F}(s_4)\right) \quad (64)$$

$$\simeq \frac{\Delta^\epsilon}{\epsilon}\mathcal{S}(0) + \int_\Delta^{s_{4,max}} \mathcal{R}(s_4) \quad (65)$$

This expansion is valid for $\Delta$ being small, i.e. smaller then any leading order scale or $m^2$; a typical choice is $\Delta/m^2 \sim 10^{-6}$. We then find

$$\begin{aligned}&\frac{s_4^2}{4\pi(s_4+m^2)}\left(1-\frac{3}{8}\zeta(2)\epsilon^2\right)\int d\Omega_n d\hat{\mathcal{I}}\, S_{k,QED}\\ &= B_{k,QED}\left[-\frac{2}{\epsilon}\left(1+\frac{s-2m^2}{s\beta}\ln(\chi)\right)+1-\frac{s-m^2}{s\beta}\left(\ln(\chi)\left(1+\ln(\chi)\right)+\mathrm{Li}_2(1-\chi^2)\right)\right]\end{aligned} \quad (66)$$

$$\begin{aligned}&\frac{s_4^2}{4\pi(s_4+m^2)}\left(1-\frac{3}{8}\zeta(2)\epsilon^2\right)\int d\Omega_n d\hat{\mathcal{I}}\, S_{k,OK}\\ &= B_{k,QED}\left[\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\left(\ln(t_1/u_1)+\frac{s-2m^2}{s\beta}\ln(\chi)\right)-\ln^2(\chi)-\frac{3}{2}\zeta(2)+\frac{1}{2}\ln^2(t_1/(u_1\chi))\right.\\ &\left.\quad +\mathrm{Li}_2(1-t_1/(u_1\chi))-\mathrm{Li}_2(1-u_1/(t_1\chi))+\frac{s-2m^2}{s\beta}\left(\mathrm{Li}_2(1-\chi^2)+\ln^2(\chi)\right)\right]\end{aligned} \quad (67)$$

(Note the mistyped sign of $\ln(\chi)^2$ in [1, eq. (3.25)]) The additional factors originate from the difference between the $2 \to 3$ phasespace of $R_k$ and the $2 \to 2$ phasespace needed for $S_k$.

The double differential partonic cross section is given by

$$
\begin{aligned}
&s'^2 \frac{d^2\sigma^{(1),S}_{k,\mathrm{g}}(s',t_1,u_1,q^2)}{dt_1du_1} \\
&= 2^8 \alpha\alpha_s^2 e_H^2 K_{\mathrm{g}\gamma} N_C C_F E_k(\epsilon) b_k(\epsilon) \delta(s'+t_1+u_1) \frac{\pi^4 S_\epsilon}{\Gamma(1+\epsilon/2)} \\
&\quad \left(\frac{(t_1u_1'-s'm^2)s'-q^2t_1^2}{m^2s'^2}\right)^{\epsilon/2} C_\epsilon \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon} \left(\frac{\Delta}{m^2}\right)^{\epsilon} \\
&\quad \frac{s_4^2}{4\pi(s_4+m^2)} \left(1-\frac{3}{8}\zeta(2)\epsilon^2\right) \int d\Omega_n d\hat{\mathcal{I}} \, \left(C_A S_{k,OK} + 2C_F S_{k,QED}\right)
\end{aligned} \tag{68}
$$

### 3.3 Light Quark Processes

In next-to-leading order a new production mechanism enters, that is induced by a light quark, so we have to consider the process

$$\gamma^*(q;\sigma_q) + q(k_1;\sigma_{k_1}) \to Q(p_1) + \overline{Q}(p_2) + q(k_2) \tag{69}$$

All contributing diagrams are depicted in figure **FiXme Error: do** and the result can be written as FiXme Error!

$$\hat{\sum_{k,\sigma}} \mathcal{P}_k^{\mu\mu'} \sum_{j,j'} \mathcal{M}^{(1),q}_{j,\mu} \mathcal{M}^{(1),q^*}_{j',\mu'} = 8g^4\mu_D^{-2\epsilon}e^2 N_C C_F \left(e_H^2 A_{k,1} + e_L^2 A_{k,2} + e_L e_H A_{k,3}\right) \tag{70}$$

where $e_L$ denotes the charge of the light quark $q$ in units of $e$.

The needed $2 \to 3$ phase space has already been calculated in section 3.2, so we can immediately quote the (collinear) poles:

$$\frac{s_4}{2\pi(s_4+m^2)} \int d\Omega_n d\hat{\mathcal{I}} \, C_F A_{k,1} = -\frac{1}{u_1} B_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} P_{k,\mathrm{g}q}(x_1) \frac{2}{\epsilon} + O(\epsilon^0) \tag{71}$$

with $x_1 = -u_1/(s'+t_1)$ and the Altarelli-Parisi splitting functions $P_{k,\mathrm{g}q}$[15, 16]:

$$P_{G,\mathrm{g}q}(x) = P_{L,\mathrm{g}q}(x) = C_F\left(\frac{1}{x} + \frac{(1-x)^2}{x}\right) \tag{72}$$

$$P_{P,\mathrm{g}q}(x) = C_F\,(2-x) \tag{73}$$

$A_{k,2}$ does not contain poles and we find $\int dt_1 du_1 \int d\Omega_n d\hat{\mathcal{I}} \, A_{k,3} = 0$. Note that in the limit $q^2 \to 0$ $A_{k,2}$ will also get collinear poles.

The double differential partonic cross section is given by

$$s'^2 \frac{d^2\sigma^{(1)}_{k,q}(s',t_1,u_1,q^2)}{dt_1 du_1} = 2^7 \alpha\alpha_s^2 K_{q\gamma} N_C C_F b_k(\epsilon) \frac{\pi^3 S_\epsilon^2}{\Gamma(1+\epsilon)} \frac{s_4}{s_4+m^2}$$

$$\left(\frac{(t_1u_1' - s'm^2)s' - q^2t_1^2}{m^2 s'^2}\right)^{\epsilon/2} \left(\frac{s_4^2}{m^2(s_4+m^2)}\right)^{\epsilon/2} \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon}$$

$$\int d\Omega_n d\hat{\mathcal{I}} \left(e_H^2 A_{k,1} + e_L^2 A_{k,2} + e_H e_L A_{k,3}\right) \tag{74}$$

with the color average $K_{q\gamma} = 1/N_C$.

## 4 Mass Factorization

All collinear poles can be removed by mass factorization in the following way:

$$s'^2 \frac{d^2\sigma^{(1),fin}_{k,\mathrm{g}}(s',t_1,u_1,q^2,\mu_F)}{dt_1 du_1} = \lim_{\epsilon\to 0} \left[ s'^2 \frac{d^2\sigma^{(1)}_{k,\mathrm{g}}(s',t_1,u_1,q^2,\epsilon)}{dt_1 du_1} \right.$$

$$- \int_0^1 \frac{dx_1}{x_1} \Gamma^{(1)}_{k,\mathrm{gg}}(x_1,\mu_F^2,\mu_D,\epsilon) \tag{75}$$

$$\left. (x_1 s')^2 \frac{d^2\sigma^{(0)}_{k,\mathrm{g}}(x_1 s', x_1 t_1, u_1, q^2, \epsilon)}{d(x_1 t_1) du_1} \right] \tag{76}$$

$$\Gamma^{(1)}_{k,ij}(x,\mu_F^2,\mu_D,\epsilon) = \frac{\alpha_s}{2\pi} \left(P_{k,ij}(x)\frac{2}{\epsilon} + f_{k,ij}(x,\mu_F^2,\mu_D^2)\right) \tag{77}$$

where $\Gamma^{(1)}_{k,ij}$ is the first order correction to the transition functions $\Gamma_{k,ij}$ for *incoming* particle $j$ and *outgoing* particle $i$ in projection $k$. In the $\overline{\mathrm{MS}}$-scheme the $f_{k,ij}$ take their usual form and we find

$$\Gamma^{(1),\overline{\mathrm{MS}}}_{k,ij}(x,\mu_F^2,\mu_D,\epsilon) = \frac{\alpha_s}{2\pi} P_{k,ij}(x) \left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(\mu_F^2/m^2) - \ln(\mu_D^2/m^2)\right) \tag{78}$$

$$= 8\pi\alpha_s P_{k,ij}(x) C_\epsilon \left(\frac{\mu_D^2}{m^2}\right)^{-\epsilon/2} \left(\frac{2}{\epsilon} + \ln(\mu_F^2/m^2)\right) \tag{79}$$

The $P_{k,ij}(x)$ are the Altarelli-Parisi splitting functions for which we find[15, 16]

$$P_{k,\mathrm{gg}}(x) = \Theta(1-\delta-x) P^H_{k,\mathrm{gg}}(x) + \delta(1-x)\left(2C_A \ln(\delta) + \frac{\beta_0}{2}\right) \tag{80}$$

where we introduced another infrared cut-off $\delta$ to seperate soft ($x \geq 1-\delta$) and hard ($x < 1-\delta$) gluons that is connected to $\Delta$ via $\delta = \Delta/(s'+t_1)$. The structure here explains why we were able to write the equation (57).

The light quark process can be regularized in a complete analogous way:

$$s'^2 \frac{d^2\sigma^{(1),fin}_{k,q}(s',t_1,u_1,q^2,\mu_F)}{dt_1 du_1} = \lim_{\epsilon\to 0}\left[ s'^2 \frac{d^2\sigma^{(1)}_{k,q}(s',t_1,u_1,q^2,\epsilon)}{dt_1 du_1} \right.$$
$$- \int_0^1 \frac{dx_1}{x_1} \Gamma^{(1)}_{k,\mathrm{g}q}(x_1,\mu_F^2,\mu_D,\epsilon)$$
$$\left. (x_1 s')^2 \frac{d^2\sigma^{(0)}_{k,\mathrm{g}}(x_1 s', x_1 t_1, u_1, q^2, \epsilon)}{d(x_1 t_1) du_1} \right] \qquad (81)$$

The needed splitting functions $P_{k,\mathrm{g}q}$ have been already quoted in equations (72) and (73). Note that $K_{q\gamma} = 1/(N_C) = 2C_F K_{\mathrm{g}\gamma}$.

The final finite cross sections are then

$$s'^2 \frac{d^2\sigma^{(1),H,fin}_{k,\mathrm{g}}}{dt_1 du_1} = \frac{1}{2\pi} K_{\mathrm{g}\gamma}\alpha\alpha_S e_H^2 N_C C_F b_k(0) \left[ -\frac{1}{u_1} P^H_{k,\mathrm{gg}}(x_1) \right.$$
$$\left\{ 4\pi B^{(0)}_{k,QED}\begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \left( \ln\left( \frac{s_4^2}{m^2(s_4+m^2)} \right) - \ln(\mu_F^2/m^2) \right) \right.$$
$$\left. -8\pi B^{(1)}_{k,QED}\begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \right\}$$
$$+ C_A \frac{s_4}{s_4+m^2} \left( \int d\Omega_n d\hat{\mathcal{I}}\, R_{k,OK} \right)^{finite}$$
$$\left. + 2C_F \frac{s_4}{s_4+m^2} \int d\Omega_4 d\hat{\mathcal{I}}\, R_{k,QED} \right] \qquad (82)$$

$$s'^2 \frac{d^2\sigma^{(1),S+V,fin}_{k,\mathrm{g}}}{dt_1 du_1} = 4K_{\mathrm{g}\gamma}\alpha\alpha_S e_H^2 N_C C_F b_k(0) B^{(0)}_{k,QED}\delta(s'+t_1+u_1) \left[ C_A \ln^2(\Delta/m^2) \right.$$
$$+ \ln(\Delta/m^2)\left( \left( \ln(-t_1/m^2) - \ln(-u_1/m^2) - \ln(\mu_F^2/m^2) \right) C_A \right.$$
$$\left. - \frac{2m^2 - s}{s\beta} \ln(\chi)(C_A - 2C_F) - 2C_F \right)$$
$$\left. + \frac{\beta_0^{lf}}{4}\left( \ln(\mu_R^2/m^2) - \ln(\mu_F^2/m^2) \right) + f_k(s',u_1,t_1,q^2) \right] \qquad (83)$$

where $f_k$ contains lots of logarithms and dilogarithms, but does not depend on $\Delta, \mu_F^2, \mu_R^2$

nor $n_f$ and $\beta_0^{lf} = (11C_A - 2n_{lf})/3$.

$$
\begin{aligned}
s'^2 \frac{d^2\sigma^{(1),fin}_{k,q}}{dt_1 du_1} = \frac{1}{2\pi} K_{q\gamma} \alpha\alpha_S N_C b_k(0) \Bigg[ -\frac{1}{u_1} e_H^2 P_{k,gq}(x_1) \\
&\Bigg\{ 2\pi B^{(0)}_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \left( \ln\left( \frac{s_4^2}{m^2(s_4+m^2)} \right) - \ln(\mu_F^2/m^2) + 1 - \delta_{k,P} \right) \\
&-4\pi B^{(1)}_{k,QED} \begin{pmatrix} s' \to x_1 s' \\ t_1 \to x_1 t_1 \end{pmatrix} \Bigg\} \\
&+ C_F \frac{s_4}{s_4+m^2} \left( \int d\Omega_n d\hat{\mathcal{I}}\, e_H^2 A_{k,1} \right)^{finite} \\
&+ C_F \frac{s_4}{s_4+m^2} \int d\Omega_4 d\hat{\mathcal{I}}\, e_L^2 A_{k,2} + C_F \frac{s_4}{s_4+m^2} \int d\Omega_4 d\hat{\mathcal{I}}\, e_H e_L A_{k,3} \Bigg]
\end{aligned} \tag{84}
$$

where $1 - \delta_{k,P}$ may also be written as $-2\partial_\epsilon E_k(\epsilon = 0)$ as it originates from the additional factor of $E_k(\epsilon)$ in the subtraction part of equation (81).

## 5 Partonic Results

The *total* partonic cross sections can be computed by

$$
\sigma^{(n)}_{k,j}(s, q^2, m^2) = \int\limits_{-s'(1+\beta)/2}^{-s'(1-\beta)/2} dt_1 \int\limits_0^{s_{4,max}} ds_4 \frac{d^2\sigma^{(n),fin}_{k,j}(s', t_1, u_1, q^2)}{dt_1 ds_4} \tag{85}
$$

$$
s_{4,max} = \frac{s}{s' t_1} \left( t_1 + \frac{s'(1-\beta)}{2} \right) \left( t_1 + \frac{s'(1+\beta)}{2} \right) \tag{86}
$$

where $k$ denotes as usual projection, $j \in \{\mathrm{g}, q, \bar{q}\}$ is a parton index and we used $s_4 = s' + t_1 + u_1$. The result can then be parametrised as

$$
\begin{aligned}
&\sigma_{k,j}(s, q^2, m^2) \\
&= \sigma^{(0)}_{k,j}(s, q^2, m^2) + \sigma^{(1)}_{k,j}(s, q^2, m^2) && (87) \\
&= \frac{\alpha\alpha_S}{m^2} \left( f^{(0)}_{k,j}(\eta, \xi) + 4\pi \left( f^{(1)}_{k,j}(\eta, \xi) + \ln(\mu_F^2/m^2) \bar{f}^{F,(1)}_{k,j}(\eta, \xi) + \ln(\mu_R^2/m^2) \bar{f}^{R,(1)}_{k,j}(\eta, \xi) \right) \right) && (88)
\end{aligned}
$$

where each function $f_{k,j}$ depends on the scaling variables $\eta = 1/\rho - 1$ and $\xi = -q^2/m^2$ and can be further decomposed by the electric charges

$$
f_{k,\mathrm{g}}(\eta, \xi) = e_H^2 c_{k,\mathrm{g}}(\eta, \xi) \tag{89}
$$

$$
f_{k,q}(\eta, \xi) = e_H^2 c_{k,q}(\eta, \xi) + e_L^2 d_{k,q}(\eta, \xi) \tag{90}
$$

As already mentioned the interference term proportional to $e_H e_L$ vanishes for total cross sections.

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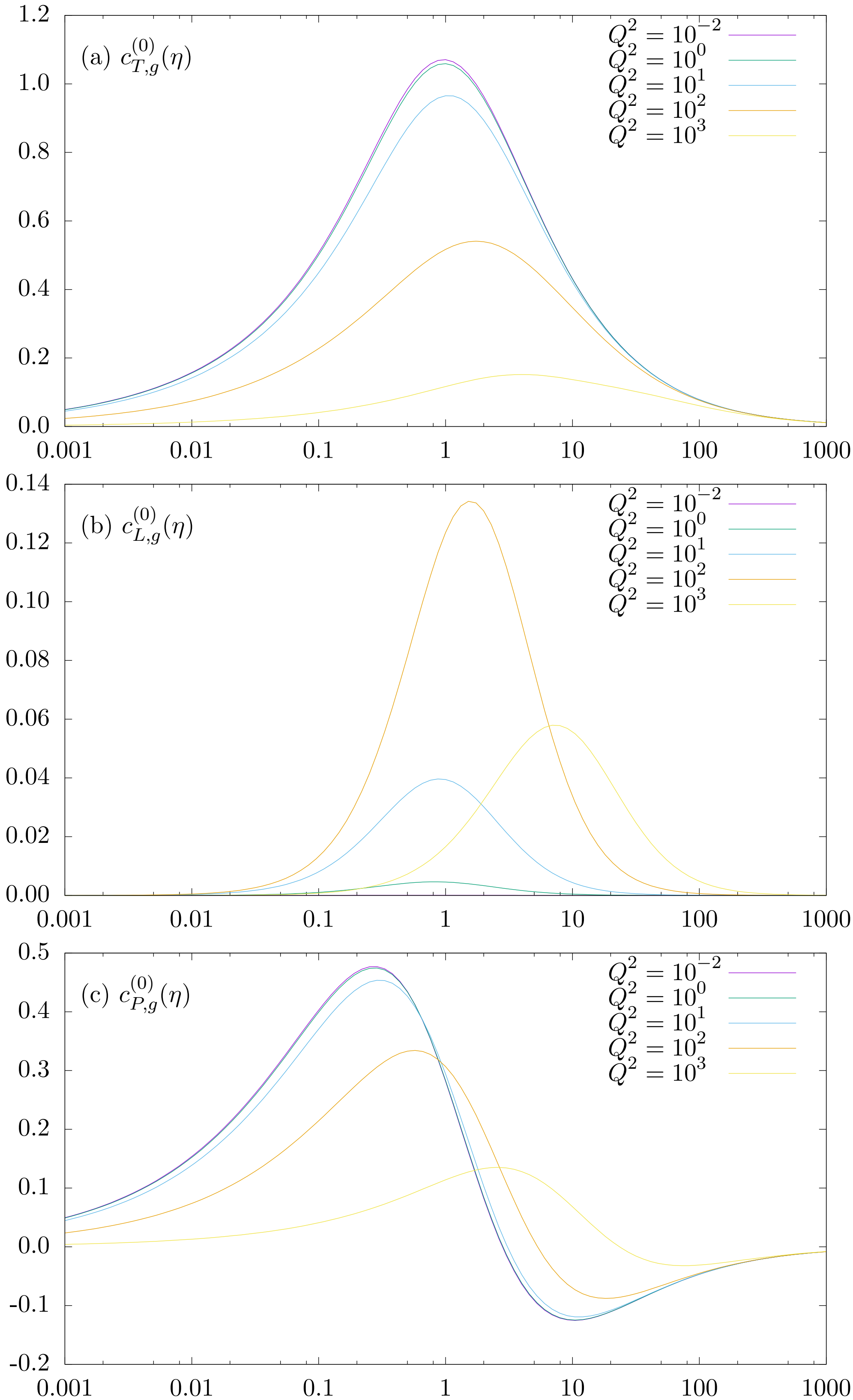


Figure 1: leading order scaling functions $c_{k,g}^{(0)}(\eta, \xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

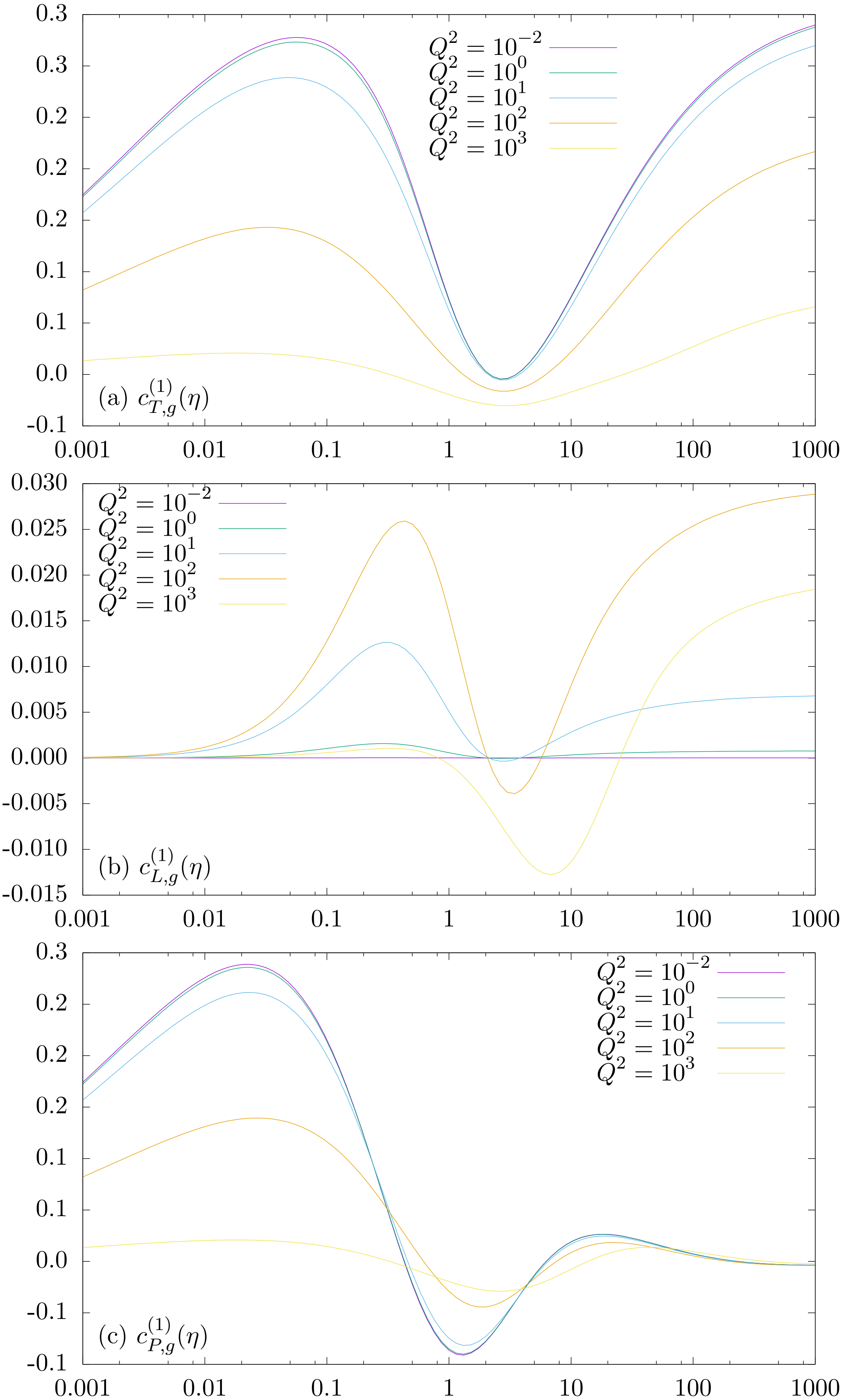


Figure 2: next-to-leading order scaling functions $c^{(1)}_{k,g}(\eta, \xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\text{GeV}^2$ at $m = 4.75\,\text{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

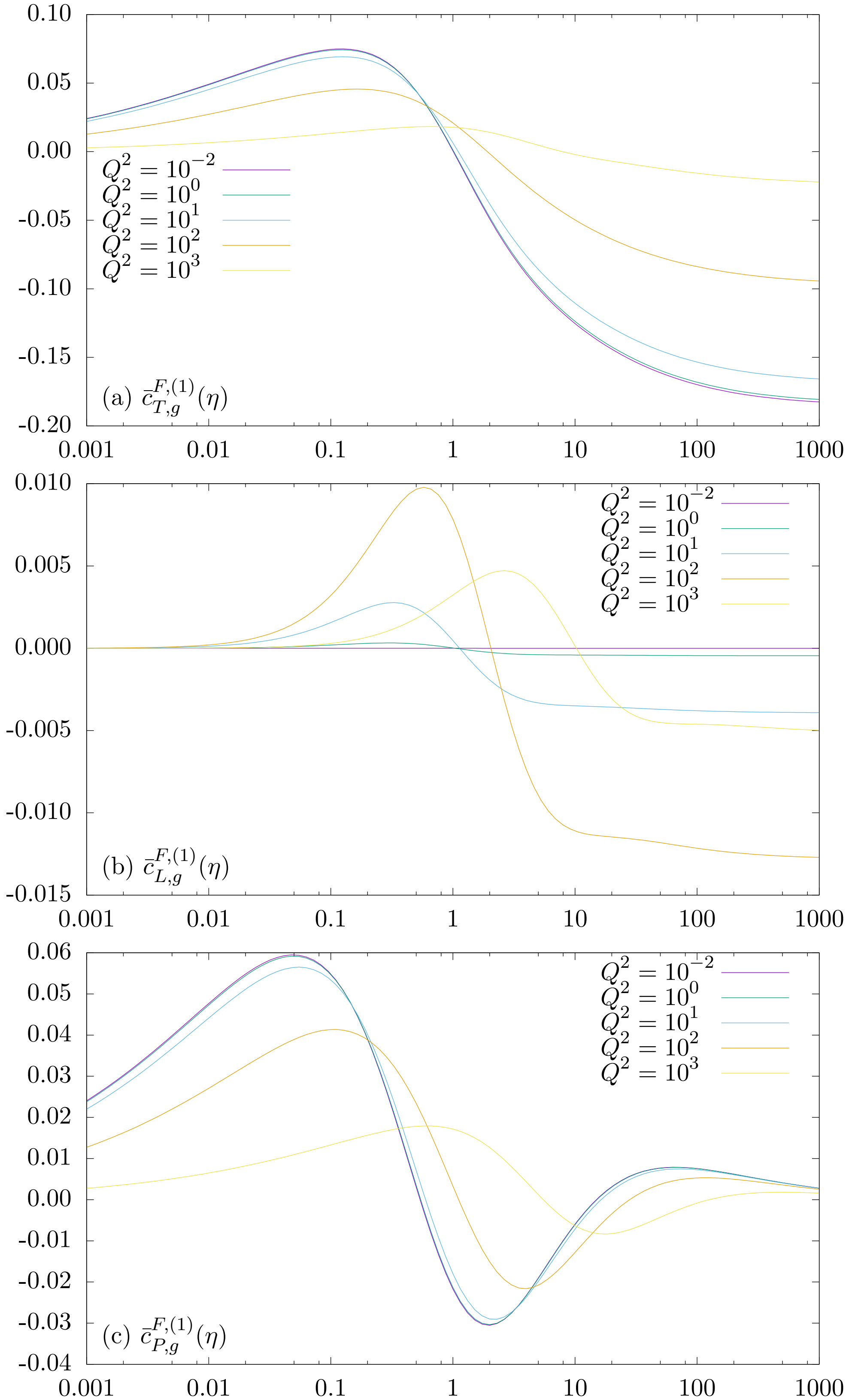


Figure 3: next-to-leading order scaling functions $\bar{c}^{F,(1)}_{k,g}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$) and $n_{lf} = 4$

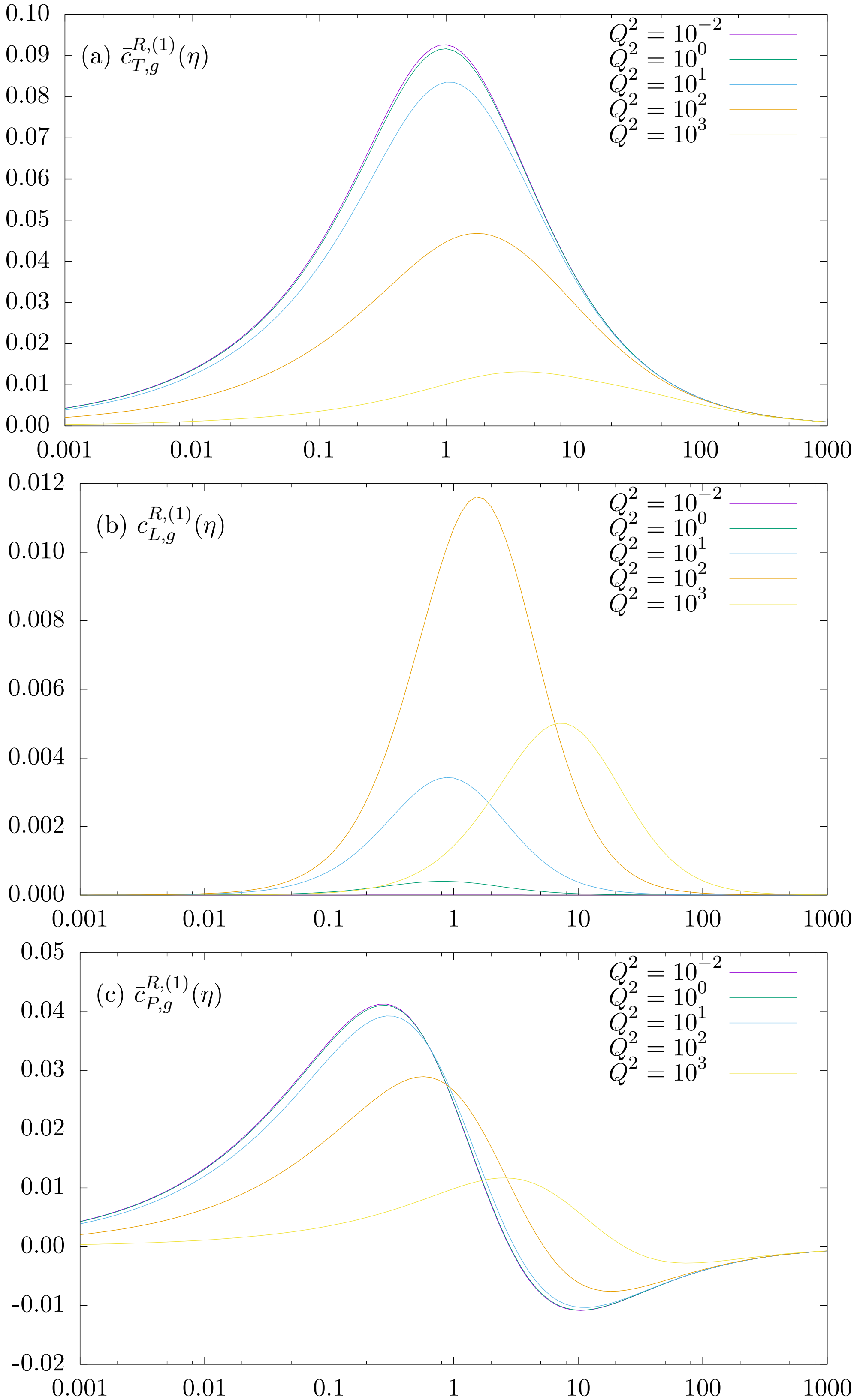


Figure 4: next-to-leading order scaling functions $\bar{c}^{R,(1)}_{k,g}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

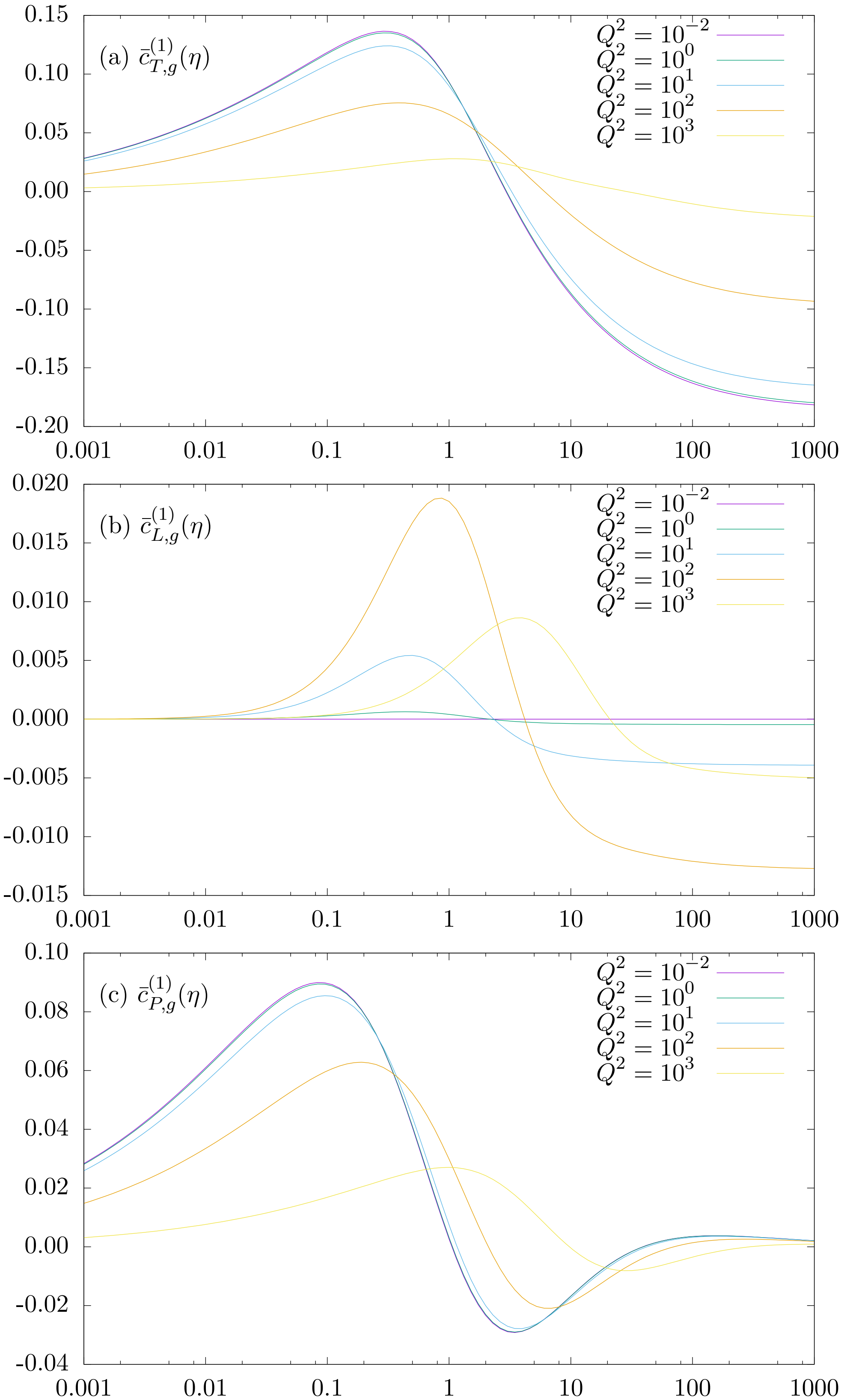


Figure 5: next-to-leading order scaling functions $\bar{c}^{(1)}_{k,g}(\eta,\xi) = \bar{c}^{R,(1)}_{k,g}(\eta,\xi) + \bar{c}^{F,(1)}_{k,g}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$) and $n_{lf} = 4$

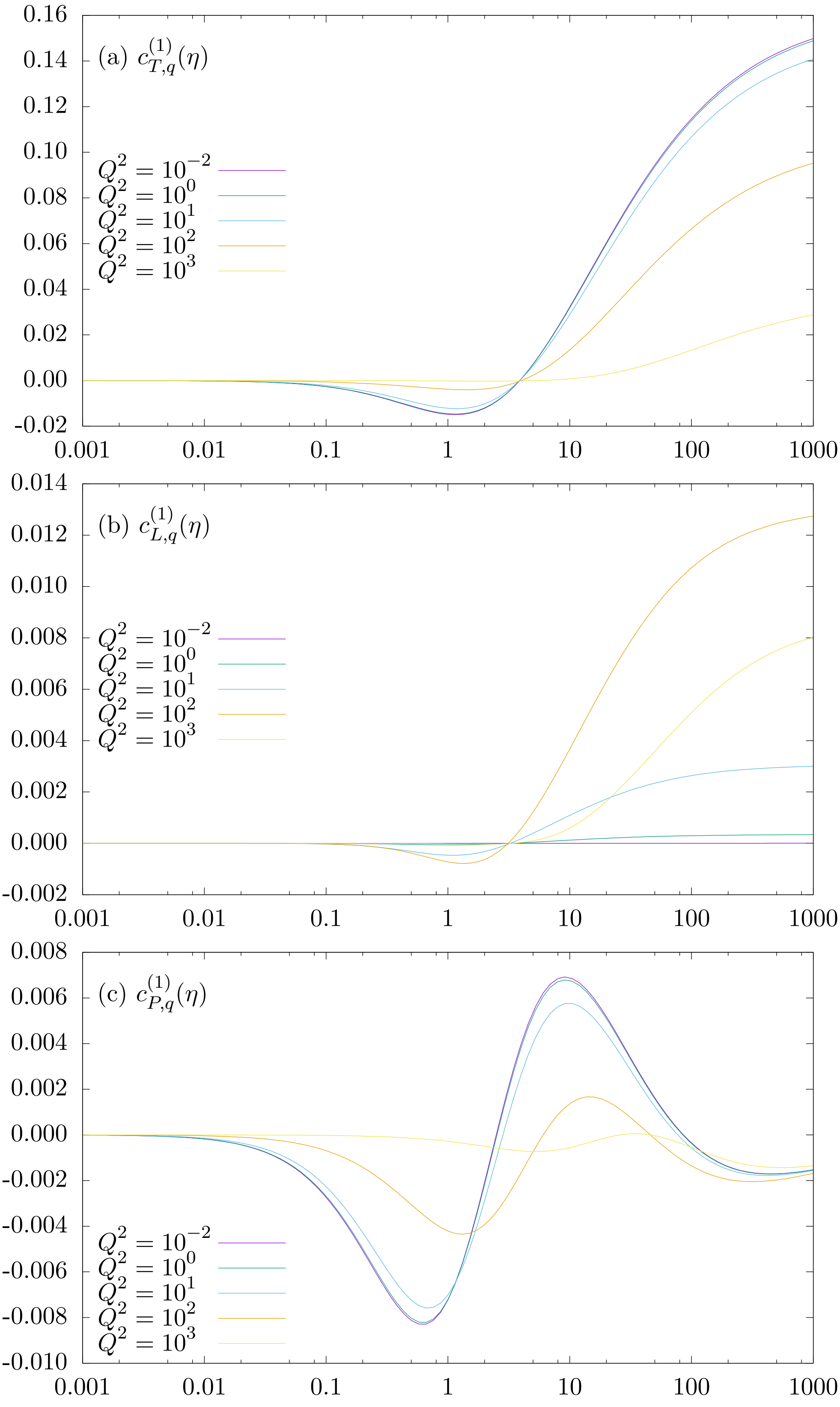


Figure 6: next-to-leading order scaling functions $c^{(1)}_{k,q}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$). Note that [1, Fig. 9 (b)] is wrong.

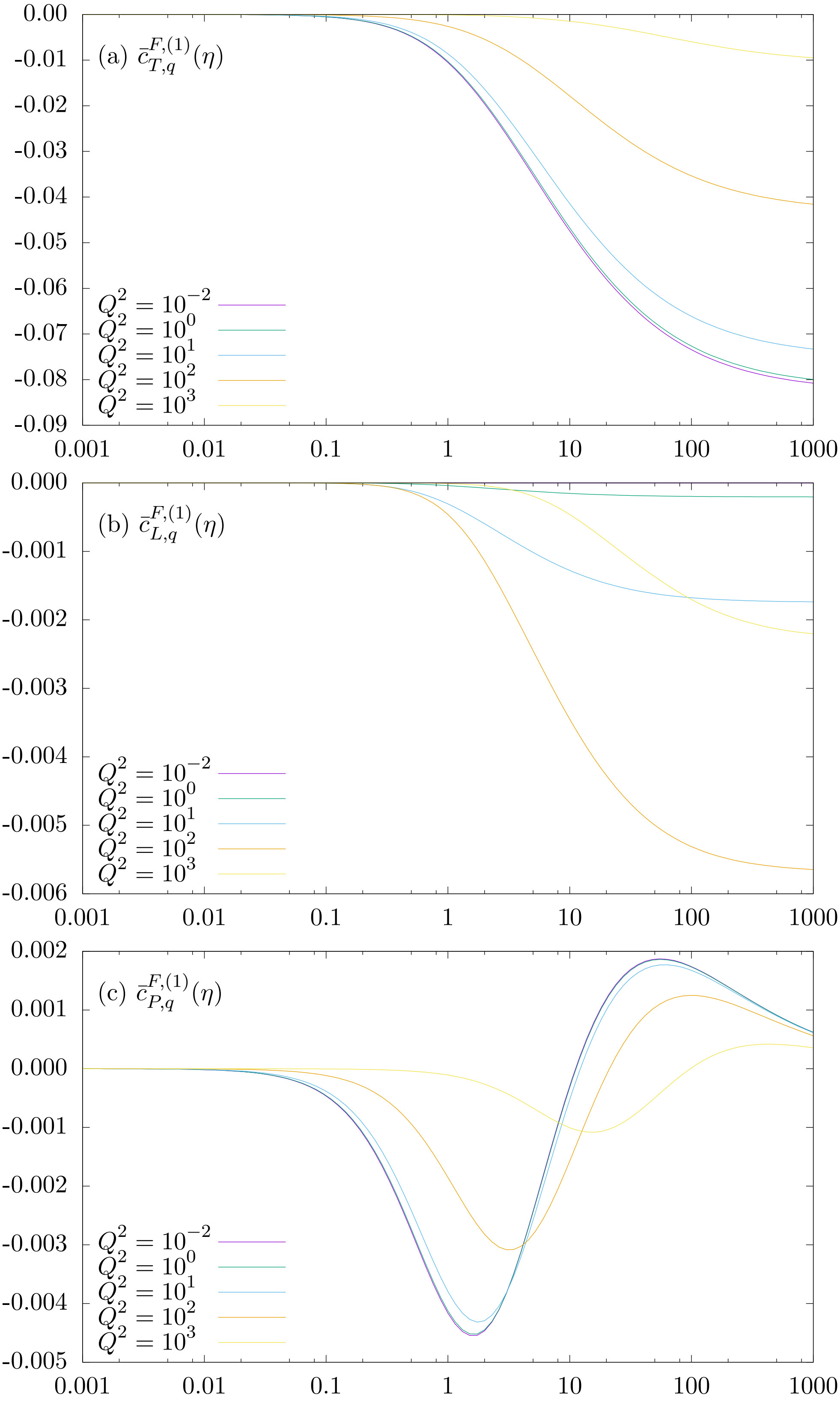


Figure 7: next-to-leading order scaling functions $\bar{c}_{k,q}^{F,(1)}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

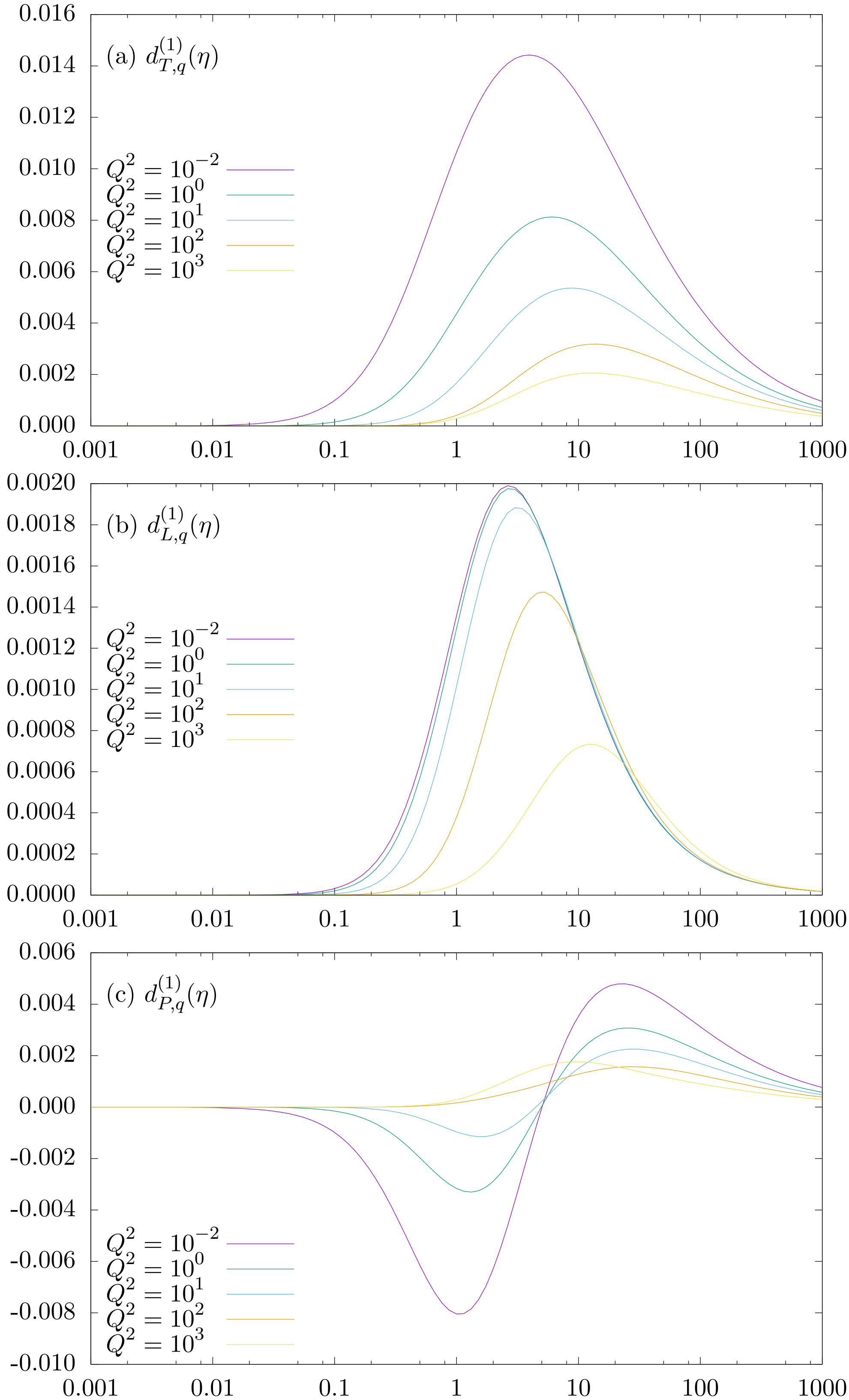


Figure 8: next-to-leading order scaling functions $d^{(1)}_{k,q}(\eta,\xi)$ plotted as function of $\eta = s/(4m^2) - 1$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$ at $m = 4.75\,\mathrm{GeV}$ (i.e. different values of $\xi = Q^2/m^2$)

## 6 Hadronic Results

The hadronic reaction to study is deep-inelastic lepton-proton scattering:

$$\ell^-(l_1) + \mathrm{p}(p) \to \ell^-(l_2) + Q(p_1)(\overline{Q}(p_2)) + X \tag{91}$$

where one either detects the heavy quark $Q$ *or* the heavy anti quark $\overline{Q}$ and $X$ stands for any final hadronic state allowed by quantum-number conservation. We define then the hadronic Bjorken variables

$$q = l_1 - l_2, \quad Q^2 = -q^2, \quad x = \frac{Q^2}{2p\cdot q}, \quad z = \frac{p\cdot q}{p\cdot l_1} \tag{92}$$

We can then define the measurable deep-inelastic hadron structure functions

$$F_k(x,Q^2,m^2) = \sum_{n=0}^{\infty} F_k^{(n)}(x,Q^2,m^2) \tag{93}$$

$$F_k^{(n)}(x,Q^2,m^2) = \frac{Q^2}{4\pi^2\alpha} \sum_{j\in\{\mathrm{g},q,\bar{q}\}} \int_x^{z_{max}} \frac{dz}{z} f_j(x/z,\mu_F^2)\sigma_{k,j}^{(n)}(s,q^2,m^2) \tag{94}$$

where $k \in \{G,L,P\}$ denotes as usual projection, $z = Q^2/s'$, $z_{max} = Q^2/(4m^2+Q^2)$ and $f_j(x/z,\mu_F^2)$ denotes parton momentum density functions[17, 18]. We can then split the contributions whether there is a gluon in the initial state $F_{k,g}$ or a (anti)quark $F_{k,q}$. In leading order we find:

$$F_{k,g}^{(0)}(x,Q^2,m^2) = \frac{\alpha_s Q^2}{4\pi^2 m^2} e_H^2 \int_x^{z_{max}} \frac{dz}{z} f_g(x/z,\mu_F^2) c_{k,g}^{(0)}(\eta,\xi) \tag{95}$$

We find for the gluonic part in next-to-leading order:

$$\begin{aligned} &F_{k,g}^{(1)}(x,Q^2,m^2) \\ &= \frac{\alpha_s^2 Q^2}{\pi m^2} e_H^2 \int_x^{z_{max}} \frac{dz}{z} f_g(x/z,\mu_F^2) \left( c_{k,g}^{(1)}(\eta,\xi) + \ln(\mu_F^2/m^2)\bar{c}_{k,g}^{F,(1)}(\eta,\xi) + \ln(\mu_R^2/m^2)\bar{c}_{k,g}^{R,(1)}(\eta,\xi)\right) \end{aligned} \tag{96}$$

We find for the quark part in next-to-leading order:

$$\begin{aligned} &F_{k,q}^{(1)}(x,Q^2,m^2) \\ &= \frac{\alpha_s^2 Q^2}{\pi m^2} e_H^2 \int_x^{z_{max}} \frac{dz}{z} \left( \sum_{j=1}^{n_{lf}} f_{q(j)}(x/z,\mu_F^2) + f_{q(-j)}(x/z,\mu_F^2) \right) \\ &\qquad \cdot \left( c_{k,q}^{(1)}(\eta,\xi) + \ln(\mu_F^2/m^2)\bar{c}_{k,q}^{F,(1)}(\eta,\xi)\right) \\ &\quad + \frac{\alpha_s^2 Q^2}{\pi m^2} \int_x^{z_{max}} \frac{dz}{z} \left( \sum_{j=1}^{n_{lf}} e_{q(j)}^2 \left( f_{q(j)}(x/z,\mu_F^2) + f_{q(-j)}(x/z,\mu_F^2)\right)\right) d_{k,q}(\eta,\xi) \end{aligned} \tag{97}$$

where we used the PDG particle labeling[19]: $q(1) = u, q(-1) = \bar{u}, q(2) = d, q(-2) = \bar{d}$, $\ldots$ and $e_u = e_c = e_t = 2/3, e_d = e_s = e_b = -1/3$.

We can then also define some more practical functions:

$$F_2(x, Q^2, m^2) = F_T(x, Q^2, m^2) + F_L(x, Q^2, m^2) \tag{98}$$

$$= F_G(x, Q^2, m^2) + \frac{3}{2} F_L(x, Q^2, m^2) \tag{99}$$

$$F_1(x, Q^2, m^2) = (F_2(x, Q^2, m^2) - F_L(x, Q^2, m^2))/(2x) \tag{100}$$

$$= \left( F_G(x, Q^2, m^2) + \frac{1}{2} F_L(x, Q^2, m^2) \right) /(2x) \tag{101}$$

$$g_1(x, Q^2, m^2) = F_P(x, Q^2, m^2)/(2x) \tag{102}$$

and we define

$$R_{k'}(x, Q^2, m^2) = \frac{F_{k'}^{(0)}(x, Q^2, m^2) + F_{k'}^{(1)}(x, Q^2, m^2)}{F_{k'}^{(0)}(x, Q^2, m^2)} \tag{103}$$

with $k' \in \{2, L, P\}$ to better observe next-to-leading order effects.

We define the spin asymmetry by

$$A_1(x, Q^2, m^2) = \frac{g_1(x, Q^2, m^2)}{F_1(x, Q^2, m^2)} = \frac{F_P(x, Q^2, m^2)}{F_2(x, Q^2, m^2) - F_L(x, Q^2, m^2)} \tag{104}$$

For the plots we focused on charm production ($n_{lf} = 3$) with $m_c = 1.5\,\mathrm{GeV}$ and we used the two-loop running coupling of [20]:

$$\alpha_s(\mu_R^2) = \frac{1}{\beta_0^4 \ln(\mu_R^2/\Lambda_4)} \left( 1 - \frac{\beta_1^4}{(\beta_0^4)^2} \ln(\ln(\mu_R^2/\Lambda_4)) \right) \tag{105}$$

with $\beta_0^f = (33 - 2n_f)/(12\pi)$, $\beta_1^f = (306 - 38n_f)/(48\pi^2)$**FiXme Error: find CA?** and $\Lambda_4 = 0.194\,\mathrm{GeV}^2$. We set $\mu_F^2 = \mu_R^2 = 4m^2 - q^2$ in analogy to [1].We used the PDF set MSTW2008nlo90cl[17, 21, 22] provided by LHAPDF[23] for the unpolarized structure functions ($F_2, F_1, F_G, F_L$) and DSSV2014[18] for the polarized structure function ($F_G$).

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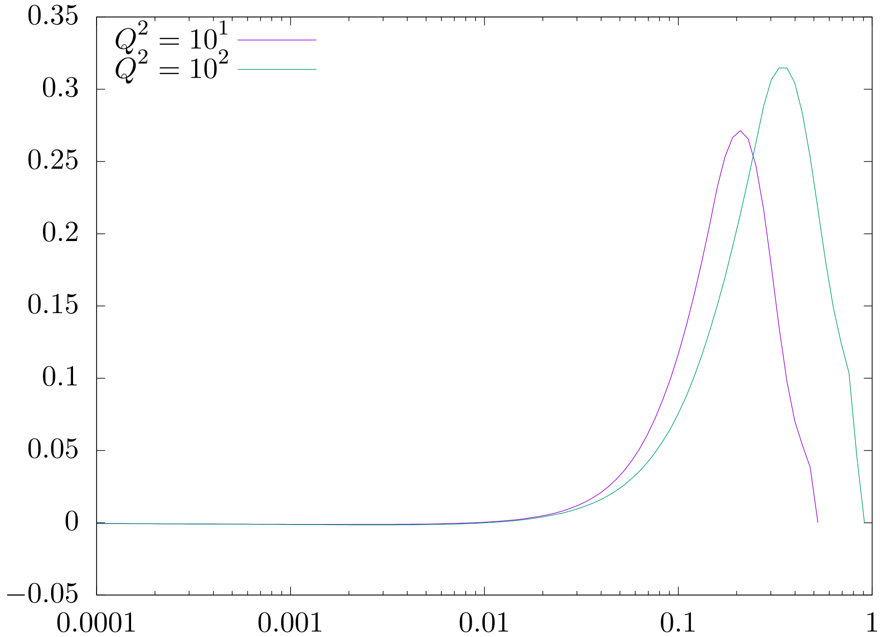


Figure 9: spin asymmetry $A_1(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

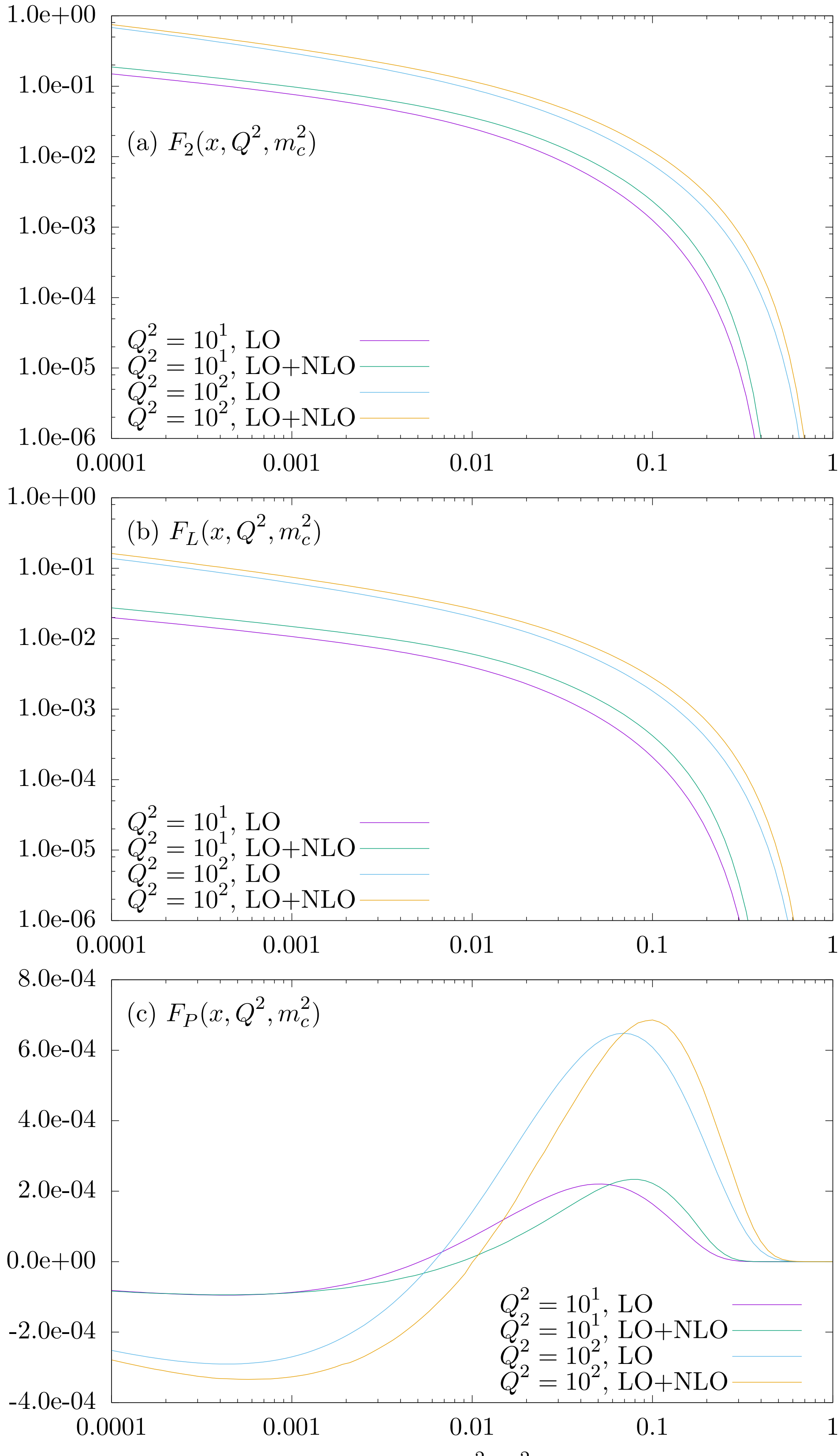


Figure 10: hadronic structure functions $F_k(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

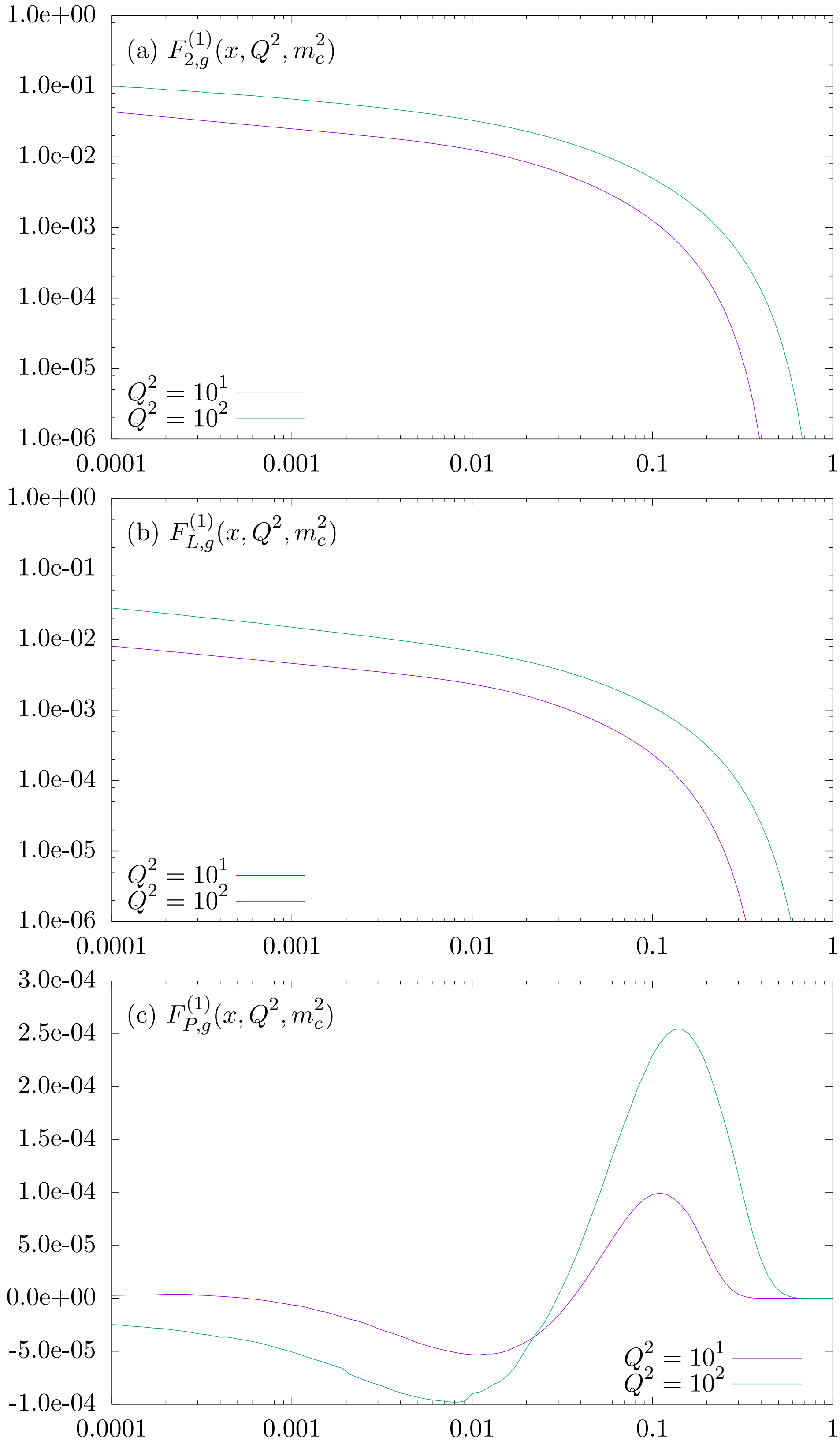


Figure 11: next-to-leading order hadronic structure functions $F_{k,g}^{(1)}(x,Q^2,m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

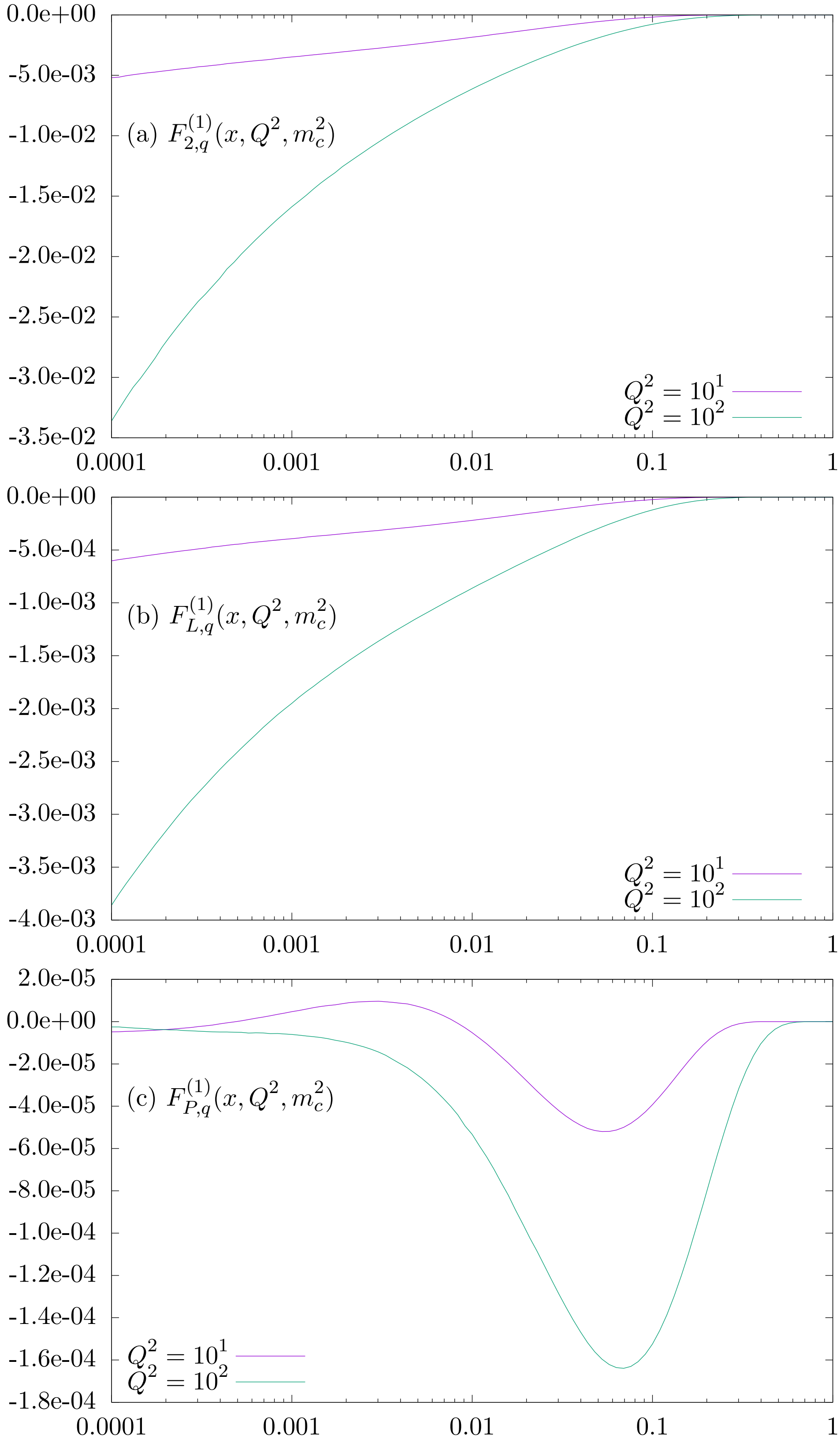


Figure 12: next-to-leading order hadronic structure functions $F_{k,q}^{(1)}(x,Q^2,m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

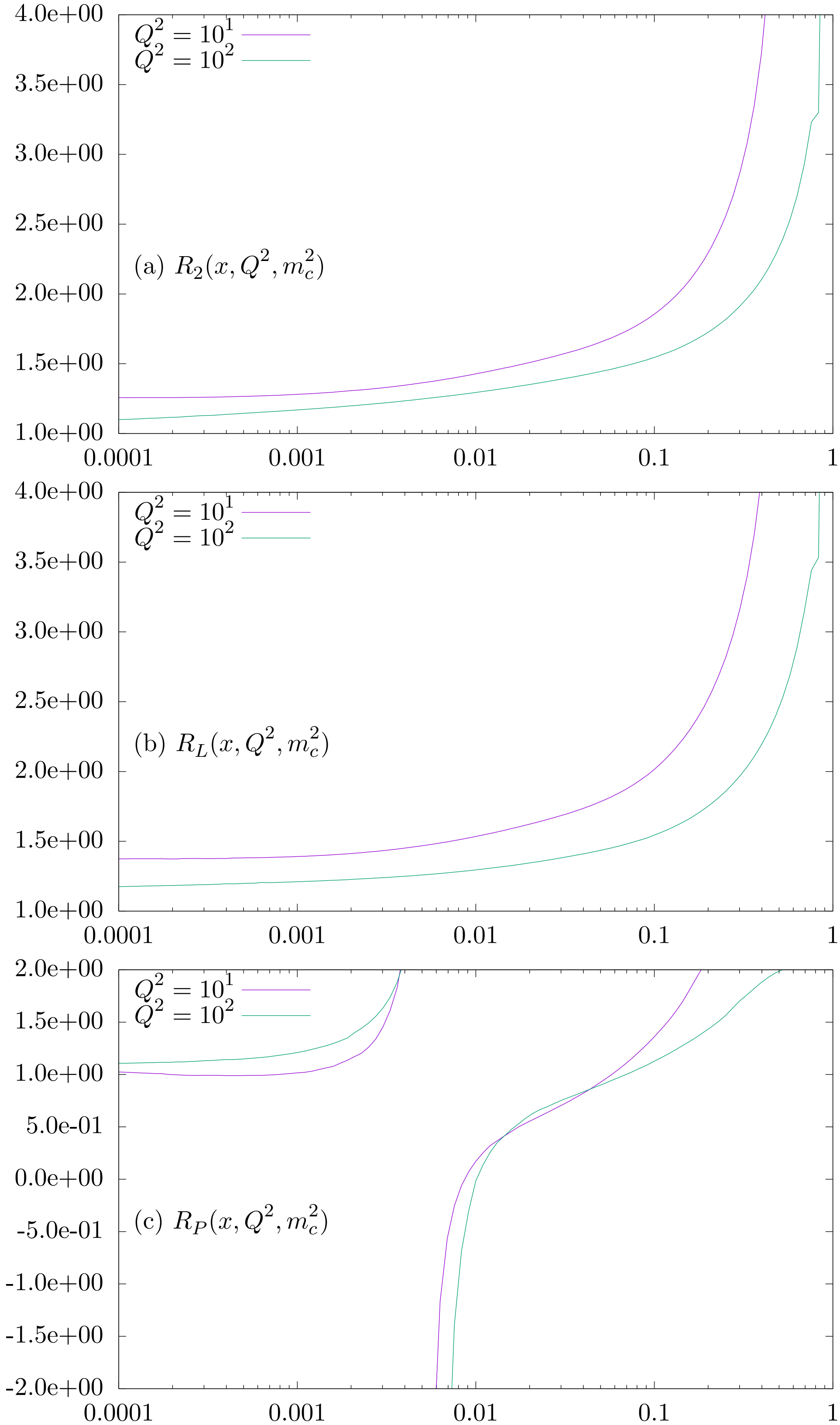


Figure 13: ratio of hadronic structure functions $R_{k'}^{(1)}(x, Q^2, m_c^2)$ plotted as function of $x$ for different values of $Q^2$ in units of $\mathrm{GeV}^2$

## 7 Summary

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## List of Corrections