1 2 to 2 phase space

process:

$$\gamma^*(q) + g(k_1) \to Q(p_1) + \bar{Q}(p_2) \tag{1}$$

use c.m.s. of incoming particles:

$$q = \left(\frac{s+q^2}{2\sqrt{s}}, 0, 0, \ldots, -\frac{s-q^2}{2\sqrt{s}}\right) \tag{2}$$

$$k_1 = \frac{s-q^2}{2\sqrt{s}}\,(1, 0, 0, \ldots, 1) \tag{3}$$

such that

$$q + k_1 = (\sqrt{s}, \vec{0}) \qquad k_1^2 = 0 \tag{4}$$

for the outgoing particles it follows

$$p_1 = \frac{\sqrt{s}}{2}\,(1, 0, \beta\sin\theta, \ldots, \beta\cos\theta) \tag{5}$$

$$p_2 = \frac{\sqrt{s}}{2}\,(1, 0, -\beta\sin\theta, \ldots, -\beta\cos\theta) \tag{6}$$

with $\beta = \sqrt{1 - 4m^2/s}$ such that

$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \qquad p_1^2 = p_2^2 = m^2 \tag{7}$$

use n-sphere:

$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \tag{8}$$

compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_1}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q + k_1 - p_1 - p_2)\delta(p_1^2 - m^2)\delta(p_2^2 - m^2) \tag{9}$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1\, \delta((q + k_1 - p_2)^2 - m^2)\delta(p_1^2 - m^2) \tag{10}$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,||} d^2 p_{1,\perp} d^{n-4} \hat{p}_1\, \delta(s - 2p_{1,0}\sqrt{s})\delta(p_{1,0}^2 - p_{1,||}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{11}$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,||} dp_{1,\perp}^2 d^{n-4} \hat{p}_1\, \delta(s/4 - p_{1,||}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{12}$$

$$= \frac{\pi}{(2\pi)^{n-2} 2\sqrt{s}} \int dp_{1,||} d\hat{p}_1^2 \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \tag{13}$$

$$= \frac{1}{2\sqrt{s}\Gamma((n-4)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} d\hat{p}_1^2\, (\hat{p}_1^2)^{(n-6)/2} \tag{14}$$

Integration borders are

$$p_{1,||} \in \frac{\sqrt{s}}{2}\beta \cdot [-1, 1] \qquad \hat{p}_1^2 \in \left(\frac{s\beta^2}{4} - p_{1,||}^2\right) \cdot [0, 1] \tag{15}$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2\,(\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4}\left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{16}$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}}\int dp_{1,||}\left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{17}$$

rewrite $p_{1,||}$ to $\cos\theta$:

$$p_{1,||} = \frac{\sqrt{s}}{2}\beta\cos\theta \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{2}\beta\, d\cos\theta, \quad \cos\theta \in [-1,1],\, \hat{p}_1^2 \in \frac{s\beta^2}{4}\left(1-\cos^2\theta\right)\cdot[0,1] \tag{18}$$

rewrite $\cos\theta$ to $t_1 = (k_1-p_2)^2 - m^2$:

$$\cos\theta = t_1 \Rightarrow \quad d\cos\theta = dt_1, \quad t_1 \in [,],\, \hat{p}_1^2 \in \cdot[0,1] \tag{19}$$

2 2 to 3 phase space

process:

$$\gamma^*(q) + q(k_1) \to Q(p_1) + \bar{Q}(p_2) + q(k_2) \tag{20}$$

2.1 kinematic constraints

definitions of kinematic variables:

$$\begin{aligned}
s &= (q+k_1)^2 & &\Rightarrow & 2qk_1 &= s - q^2 & (21)\\
s_3 &= (k_2+p_2)^2 - m^2 & &\Rightarrow & 2k_2p_2 &= s_3 & (22)\\
s_4 &= (k_2+p_1)^2 - m^2 & &\Rightarrow & 2k_2p_1 &= s_4 & (23)\\
s_5 &= (p_1+p_2)^2 = -u_5 & &\Rightarrow & 2p_1p_2 &= s_5 - 2m^2 & (24)\\
t_1 &= (k_1-p_2)^2 - m^2 = t - m^2 & &\Rightarrow & 2k_1p_2 &= -t_1 & (25)\\
t' &= (k_1-k_2)^2 & &\Rightarrow & 2k_1k_2 &= -t' & (26)\\
u_1 &= (q-p_2)^2 - m^2 = u - m^2 & &\Rightarrow & 2qp_2 &= -u_1 + q^2 & (27)\\
u_6 &= (k_1-p_1)^2 - m^2 & &\Rightarrow & 2k_1p_1 &= -u_6 & (28)\\
u_7 &= (q-p_1)^2 - m^2 & &\Rightarrow & 2qp_1 &= -u_7 + q^2 & (29)\\
u' &= (q-k_2)^2 & &\Rightarrow & 2qk_2 &= -u' + q^2 & (30)
\end{aligned}$$

impose momentum conservation:

$$q + k_1 = p_1 + p_2 + k_2 \tag{31}$$

contract with 2 times momentum:

$$\begin{array}{rrlll}
2q^2 & +s - q^2 = -u_7 + q^2 & -u_1 + q^2 & -u' + q^2 & \Leftrightarrow \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad (32)\\
s - q^2 & +0 = -u_6 & -t_1 & -t' & \Leftrightarrow \quad 0 = s + t_1 + t' + u_6 - q^2 \quad (33)\\
-u_7 + q^2 & -u_6 = 2m^2 & +s_5 - 2m^2 & +s_4 & \Leftrightarrow \quad 0 = s_4 + s_5 + u_6 + u_7 - q^2 \quad (34)\\
-u_1 + q^2 & -t_1 = s_5 - 2m^2 & +2m^2 & +s_3 & \Leftrightarrow \quad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad (35)\\
-u' + q^2 & -t' = s_4 & +s_3 & +0 & \Leftrightarrow \quad 0 = s_3 + s_4 + t' + u' - q^2 \quad (36)
\end{array}$$

$$\frac{1}{2}\left((32)+(33)+(35)-(34)-(36)\right) = 0 = s - q^2 + t_1 + u_1 - s_4 \tag{37}$$

2.2 choose framework

use c.m.s. of recoiling heavy and light quark ($Q(p_1)$ and $q(k_2)$):

$$\begin{aligned}
k_2 &= (\omega_2, 0, \ldots, 0, \omega_2 \sin\theta \sin\phi, \omega_2 \sin\theta\cos\phi, \omega_2\cos\theta) && (38)\\
p_1 &= (E_1, 0, \ldots, 0, -\omega_2 \sin\theta \sin\phi, -\omega_2 \sin\theta\cos\phi, -\omega_2\cos\theta) && (39)\\
k_1 &= (\omega_1, 0, \ldots, 0, 0, \omega_1) && (40)\\
q &= (q_0, 0, \ldots, 0, |\vec{p}_2|\sin\psi, |\vec{p}_2|\cos\psi - \omega_1) && (41)\\
p_2 &= (E_2, 0, \ldots, 0, |\vec{p}_2|\sin\psi, |\vec{p}_2|\cos\psi) && (42)
\end{aligned}$$

light quark masses are already fixed: $k_1^2 = 0 = k_2^2$

constraints:

$$\begin{aligned}
q_0 + \omega_1 & &&= E_1 + E_2 + \omega_2 && (43)\\
m^2 &= p_1^2 &&= E_1^2 - \omega_2^2 && (44)\\
m^2 &= p_2^2 &&= E_2^2 - |\vec{p}_2|^2 && (45)\\
q^2 & &&= q_0^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\,\omega_1\cos\psi - \omega_1^2 && (46)\\
s &= (q+k_1)^2 &&= (q_0-\omega_1)^2 - |\vec{p}_2|^2 && (47)\\
t &= (k_1-p_2)^2 &&= (\omega_1 - E_2)^2 - |\vec{p}_2|^2 + 2\,|\vec{p}_2|\,\omega_1\cos\psi - \omega_1^2 && (48)\\
u &= (q-p_2)^2 &&= (q_0 - E_2)^2 - \omega_1^2 && (49)
\end{aligned}$$

solve:

$$
\begin{aligned}
(47)-(46)+(48)-(45)+(49) &= s-q^2+t-m^2+u && (50)\\
&= s_4+m^2=(E_1+\omega_2)^2 && (51)\\
(48)+(49)-(46) &= t+u-q^2=-2(E_1+\omega_2)E_2 && (52)\\
\Rightarrow E_2 &= -\frac{t+u-q^2}{2\sqrt{s_4+m^2}}=\frac{s-s_4-2m^2}{2\sqrt{s_4+m^2}} && (53)\\
(51)\wedge(44)\Rightarrow \omega_2 &= \frac{s_4}{2\sqrt{s_4+m^2}} && (54)\\
(51)\Rightarrow E_1 &= \frac{s_4+2m^2}{2\sqrt{s_4+m^2}} && (55)\\
(47)+(49)-(45) &= s+u-m^2=2q_0(E_1+\omega_2) && (56)\\
\Rightarrow q_0 &= \frac{s+u-m^2}{2\sqrt{s_4+m^2}} && (57)\\
(48)-(46) &= t-q^2=(\omega_1-E_2)^2-q_0^2 && (58)\\
\Rightarrow \omega_1 &= \frac{s_4+m^2-u}{2\sqrt{s_4+m^2}} && (59)\\
(45)\Rightarrow |\vec{p}_2| &= \sqrt{E_2^2-m^2}=\frac{\sqrt{(s-s_4)^2-4sm^2}}{2\sqrt{s_4+m^2}} && (60)\\
(46)\Rightarrow \cos\psi &= \frac{q^2-q_0^2+|\vec{p}_2|^2+\omega_1^2}{2\,|\vec{p}_2|\,\omega_1} && (61)\\
&= \frac{2u(q^2-s-m^2+t)-(2m^2-q^2-t)(s_4-u+m^2)}{(s_4+m^2-u)\sqrt{(s-s_4)^2-4sm^2}} && (62)
\end{aligned}
$$