1 2 to 2 phase space

following [1]:

process:
$$\gamma^*(q) + g(k_1) \to Q(p_1) + \bar{Q}(p_2) \tag{1}$$

kinematics:
$$s = (q + k_1)^2 \qquad\qquad s' = s - q^2 \tag{2}$$
$$t = (k_1 - p_1)^2 \qquad\qquad t_1 = t - m^2 \tag{3}$$
$$u = (k_1 - p_2)^2 \qquad\qquad u_1 = u - m^2 \tag{4}$$

use c.m.s. of incoming particles:
$$q = \left(\frac{s + q^2}{2\sqrt{s}}, 0, 0, \dots, -\frac{s - q^2}{2\sqrt{s}} \right) \tag{5}$$
$$k_1 = \frac{s - q^2}{2\sqrt{s}} \left(1, 0, 0, \dots, 1\right) \tag{6}$$

such that
$$q + k_1 = (\sqrt{s}, \vec{0}) \qquad k_1^2 = 0 \tag{7}$$

for the outgoing particles it follows
$$p_1 = \frac{\sqrt{s}}{2} \left(1, 0, \beta\sin\theta, \dots, \beta\cos\theta\right) \tag{8}$$
$$p_2 = \frac{\sqrt{s}}{2} \left(1, 0, -\beta\sin\theta, \dots, -\beta\cos\theta\right) \tag{9}$$

with $\beta = \sqrt{1 - 4m^2/s}$ such that
$$p_1 + p_2 = (\sqrt{s}, \vec{0}) \qquad p_1^2 = p_2^2 = m^2 \tag{10}$$

use n-sphere:
$$d^D x = \Omega_D x^{D-1} dx = \frac{2\pi^{D/2}}{\Gamma(D/2)} x^{D-1} dx = \frac{\pi^{D/2}}{\Gamma(D/2)} (x^2)^{(D-2)/2} dx^2 \tag{11}$$

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compute phase space:

$$PS_2 = \int \frac{d^n p_1}{(2\pi)^{n-1}} \frac{d^n p_1}{(2\pi)^{n-1}} (2\pi)^n \delta^{(n)}(q + k_1 - p_1 - p_2)\delta(p_1^2 - m^2)\delta(p_2^2 - m^2) \tag{12}$$

$$= \frac{1}{(2\pi)^{n-2}} \int d^n p_1 \, \delta((q + k_1 - p_2)^2 - m^2)\delta(p_1^2 - m^2) \tag{13}$$

$$= \frac{1}{(2\pi)^{n-2}} \int dp_{1,0} dp_{1,||} d^2 p_{1,\perp} d^{n-4}\hat{p}_1 \, \delta(s - 2p_{1,0}\sqrt{s})\delta(p_{1,0}^2 - p_{1,||}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{14}$$

$$= \frac{\pi}{(2\pi)^{n-2}2\sqrt{s}} \int dp_{1,||} dp_{1,\perp}^2 d^{n-4}\hat{p}_1 \, \delta(s/4 - p_{1,||}^2 - p_{1,\perp}^2 - \hat{p}_1^2 - m^2) \tag{15}$$

$$= \frac{\pi}{(2\pi)^{n-2}2\sqrt{s}} \int dp_{1,||} d\hat{p}_1^2 \frac{\pi^{(n-4)/2}}{\Gamma((n-4)/2)} (\hat{p}_1^2)^{(n-6)/2} \tag{16}$$

$$= \frac{1}{2\sqrt{s}\Gamma((n-4)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} d\hat{p}_1^2 (\hat{p}_1^2)^{(n-6)/2} \tag{17}$$

Integration borders are

$$p_{1,||} \in \frac{\sqrt{s}}{2}\beta \cdot [-1, 1] \qquad \hat{p}_1^2 \in \left(\frac{s\beta^2}{4} - p_{1,||}^2\right) \cdot [0, 1] \tag{18}$$

if cross section does not depend on hat-space:

$$\int d\hat{p}_1^2 (\hat{p}_1^2)^{(n-6)/2} = \frac{2}{n-4} \left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{19}$$

$$\Rightarrow PS_2 = \frac{1}{2\sqrt{s}\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dp_{1,||} \left(\frac{s\beta^2}{4} - p_{1,||}^2\right)^{(n-4)/2} \tag{20}$$

rewrite $p_{1,||}$ to $\cos\theta$:

$$p_{1,||} = \frac{\sqrt{s}}{2}\beta\cos\theta \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{2}\beta \, d\cos\theta, \quad \cos\theta \in [-1, 1], \hat{p}_1^2 \in \frac{s\beta^2}{4}\left(1 - \cos^2\theta\right) \cdot [0, 1] \tag{21}$$

rewrite $\cos\theta$ to $t_1 = (k_1 - p_2)^2 - m^2$:

$$\cos\theta = \frac{2t_1/s' + 1}{\beta} \Rightarrow \quad d\cos\theta = \frac{2}{\beta s'}dt_1, \quad t_1 \in \frac{s'}{2}[-\beta - 1, \beta - 1], \hat{p}_1^2 \in (-m^2 - \frac{st_1}{s'^2}(s' + t_1)) \cdot [0, 1] \tag{22}$$

$$p_{1,||} = \sqrt{s}\left(\frac{t_1}{s'} + \frac{1}{2}\right) \Rightarrow \quad dp_{1,||} = \frac{\sqrt{s}}{s'}dt_1 \tag{23}$$

$$\Rightarrow PS_2 = \frac{1}{2s'\Gamma((n-2)/2)(4\pi)^{(n-2)/2}} \int dt_1 \left(\frac{(t_1(u_1 - q^2) - s'm^2)s' - q^2 t_1^2}{s'^2}\right)^{(n-4)/2} \tag{24}$$

2 2 to 3 phase space

following [2, 3, 1]:

process:
$$\gamma^*(q) + q(k_1) \to Q(p_1) + \bar{Q}(p_2) + q(k_2) \tag{25}$$

2.1 kinematic constraints

definitions of kinematic variables:

$$
\begin{aligned}
s &= (q + k_1)^2 & \Rightarrow && 2qk_1 &= s - q^2 & (26) \\
s_3 &= (k_2 + p_2)^2 - m^2 & \Rightarrow && 2k_2p_2 &= s_3 & (27) \\
s_4 &= (k_2 + p_1)^2 - m^2 & \Rightarrow && 2k_2p_1 &= s_4 & (28) \\
s_5 &= (p_1 + p_2)^2 = -u_5 & \Rightarrow && 2p_1p_2 &= s_5 - 2m^2 & (29) \\
t_1 &= (k_1 - p_2)^2 - m^2 = t - m^2 & \Rightarrow && 2k_1p_2 &= -t_1 & (30) \\
t' &= (k_1 - k_2)^2 & \Rightarrow && 2k_1k_2 &= -t' & (31) \\
u_1 &= (q - p_2)^2 - m^2 = u - m^2 & \Rightarrow && 2qp_2 &= -u_1 + q^2 & (32) \\
u_6 &= (k_1 - p_1)^2 - m^2 & \Rightarrow && 2k_1p_1 &= -u_6 & (33) \\
u_7 &= (q - p_1)^2 - m^2 & \Rightarrow && 2qp_1 &= -u_7 + q^2 & (34) \\
u' &= (q - k_2)^2 & \Rightarrow && 2qk_2 &= -u' + q^2 & (35)
\end{aligned}
$$

impose momentum conservation:
$$q + k_1 = p_1 + p_2 + k_2 \tag{36}$$

contract with 2 times momentum:

$$
2q^2 \quad +s - q^2 = -u_7 + q^2 \quad -u_1 + q^2 \quad -u' + q^2 \quad \Leftrightarrow \quad 0 = s + u_1 + u_7 + u' - 2q^2 \tag{37}
$$

$$
s - q^2 \qquad +0 = -u_6 \qquad -t_1 \qquad -t' \qquad \Leftrightarrow \quad 0 = s + t_1 + t' + u_6 - q^2 \tag{38}
$$

$$
-u_7 + q^2 \qquad -u_6 = 2m^2 \quad +s_5 - 2m^2 \ +s_4 \qquad \Leftrightarrow \quad 0 = s_4 + s_5 + u_6 + u_7 - q^2 \tag{39}
$$

$$
-u_1 + q^2 \qquad -t_1 = s_5 - 2m^2 \ +2m^2 \qquad +s_3 \qquad \Leftrightarrow \quad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \tag{40}
$$

$$
-u' + q^2 \qquad -t' = s_4 \qquad +s_3 \qquad +0 \qquad \Leftrightarrow \quad 0 = s_3 + s_4 + t' + u' - q^2 \tag{41}
$$

$$
\frac{1}{2}\left((37) + (38) + (40) - (39) - (41)\right) = 0 = s - q^2 + t_1 + u_1 - s_4 \tag{42}
$$

2.2 choose framework

use c.m.s. of recoiling heavy and light quark $(Q(p_1)$ and $q(k_2))$:

$$k_2 = (\omega_2, k_{2,x}, \omega_2 \sin\theta_1 \cos\theta_2, \omega_2 \cos\theta_1, \hat{k}_2) \tag{43}$$

$$p_1 = (E_1, -k_{2,x}, -\omega_2 \sin\theta_1 \cos\theta_2, -\omega_2 \cos\theta_1, -\hat{k}_2) \tag{44}$$

$$k_1 = (\omega_1, 0, 0, \omega_1, \hat{0}) \tag{45}$$

$$q = (q_0, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi - \omega_1, \hat{0}) \tag{46}$$

$$p_2 = (E_2, 0, |\vec{p}_2| \sin\psi, |\vec{p}_2| \cos\psi, \hat{0}) \tag{47}$$

light quark masses are already fixed: $k_1^2 = 0 = k_2^2$

constraints:

$$q_0 + \omega_1 = E_1 + E_2 + \omega_2 \tag{48}$$

$$m^2 = p_1^2 = E_1^2 - \omega_2^2 \tag{49}$$

$$m^2 = p_2^2 = E_2^2 - |\vec{p}_2|^2 \tag{50}$$

$$q^2 = q_0^2 - |\vec{p}_2|^2 + 2 |\vec{p}_2| \omega_1 \cos\psi - \omega_1^2 \tag{51}$$

$$s = (q + k_1)^2 = (q_0 + \omega_1)^2 - |\vec{p}_2|^2 \tag{52}$$

$$t = (k_1 - p_2)^2 = (\omega_1 - E_2)^2 - |\vec{p}_2|^2 + 2 |\vec{p}_2| \omega_1 \cos\psi - \omega_1^2 \tag{53}$$

$$u = (q - p_2)^2 = (q_0 - E_2)^2 - \omega_1^2 \tag{54}$$

solve:

$$(52) - (51) + (53) - (50) + (54) = s - q^2 + t - m^2 + u \tag{55}$$

$$= s_4 + m^2 = (E_1 + \omega_2)^2 \tag{56}$$

$$(53) + (54) - (51) = t + u - q^2 = -2(E_1 + \omega_2)E_2 \tag{57}$$

$$\Rightarrow E_2 = -\frac{t + u - q^2}{2\sqrt{s_4 + m^2}} = \frac{s - s_4 - 2m^2}{2\sqrt{s_4 + m^2}} \tag{58}$$

$$(56) \wedge (49) \Rightarrow \omega_2 = \frac{s_4}{2\sqrt{s_4 + m^2}} \tag{59}$$

$$(56) \Rightarrow E_1 = \frac{s_4 + 2m^2}{2\sqrt{s_4 + m^2}} \tag{60}$$

$$(52) + (54) - (50) = s + u - m^2 = 2q_0(E_1 + \omega_2) \tag{61}$$

$$\Rightarrow q_0 = \frac{s + u_1}{2\sqrt{s_4 + m^2}} \tag{62}$$

$$(53) - (51) = t - q^2 = (\omega_1 - E_2)^2 - q_0^2 \tag{63}$$

$$\Rightarrow \omega_1 = \frac{s_4 - u_1}{2\sqrt{s_4 + m^2}} \tag{64}$$

$$(50) \Rightarrow |\vec{p}_2| = \sqrt{E_2^2 - m^2} = \frac{\sqrt{(s - s_4)^2 - 4sm^2}}{2\sqrt{s_4 + m^2}} \tag{65}$$

$$(51) \Rightarrow \cos\psi = \frac{q^2 - q_0^2 + |\vec{p}_2|^2 + \omega_1^2}{2\,|\vec{p}_2|\,\omega_1} \tag{66}$$

$$= \frac{(u_1 + m^2)(q^2 - s + t_1) - (m^2 - q^2 - t_1)(s_4 - u_1)}{(s_4 - u_1)\sqrt{(s - s_4)^2 - 4sm^2}} \tag{67}$$

$$\Rightarrow \sin\psi = 2\frac{\sqrt{s_4 + m^2}\sqrt{(s - q^2)t_1 u_1 - q^2 t_1 (s - q^2 + t_1) - m^2(s - q^2)^2}}{(s_4 - u_1)\sqrt{(s - s_4)^2 - 4sm}} \tag{68}$$

$$t' = -2k_1 k_2 = -2\omega_1\omega_2(1 - \cos\theta_1) \tag{69}$$

$$u_6 = -2k_1 p_1 = -2\omega_1(E_1 + \omega_2\cos\theta_1) \tag{70}$$

$$(38): \qquad 0 = s + t_1 + t' + u_6 - q^2 \quad \checkmark \tag{71}$$

t' is the only variable that can get collinear (for $-q^2 > 0$).

$$s_3 = 2k_2 p_2 = 2\omega_2(E_2 - |\vec{p}_2|\,(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{72}$$

$$s_5 = (p_1 + p_2)^2 = 2m^2 + 2p_1 p_2 \tag{73}$$

$$= 2(m^2 + E_1 E_2 + \omega_2 |\vec{p}_2|\,(\cos\psi\cos\theta_1 + \sin\psi\sin\theta_1\cos\theta_2)) \tag{74}$$

$$(40): \qquad 0 = s_3 + s_5 + t_1 + u_1 - q^2 \quad \checkmark \tag{75}$$

$$u' = (q - k_2)^2 = q^2 - 2qk_2 \tag{76}$$

$$= q^2 - 2(q_0\omega_2 - \omega_2(|\vec{p}_2|\cos\psi - \omega_1)\cos\theta_1 - \omega_2|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{77}$$

$$u_7 = q^2 - 2qp_1 \tag{78}$$

$$= q^2 - 2(q_0E_1 + \omega_2(|\vec{p}_2|\cos\psi - \omega_1)\cos\theta_1 + \omega_2|\vec{p}_2|\sin\psi\sin\theta_1\cos\theta_2) \tag{79}$$

$$(37): \quad 0 = s + u_1 + u_7 + u' - 2q^2 \quad \checkmark \tag{80}$$

2.3 phase space integrals

at phase space integration there occur integrations over propagators[4, 2, 3]; the propagators can be decomposed in 2 types: [ab] and [ABC]; the needed integrals then reduce to the master formula:

$$I_n^{(k,l)} = \int_0^\pi d\theta_1 \, \sin^{n-3}(\theta_1) \int_0^\pi d\theta_2 \, \sin^{n-4}(\theta_2)(a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{81}$$

$$= \int d\Omega_n \, (a + b\cos(\theta_1))^{-k}(A + B\cos(\theta_1) + C\sin(\theta_1)\cos(\theta_2))^{-l} \tag{82}$$

the integrals can be further destinguished by the range of k, l and the type of collinearity (following the notation in [4]):

- "non collinear": $a^2 \neq b^2 \wedge A^2 \neq B^2 + C^2 \to I_{0,n}^{(k,l)}$

- "single collinear a": $a = -b \wedge A^2 \neq B^2 + C^2 \to I_{a,n}^{(k,l)}$

- "single collinear A": $a^2 \neq b^2 \wedge A^2 = B^2 + C^2 \to I_{A,n}^{(k,l)}$

- "double collinear": $a = -b \wedge A = -\sqrt{B^2 + C^2} \to I_{aA,n}^{(k,l)}$

Use $n = 4 + \epsilon$.

2.3.1 integral helper

define helper integral

$$\hat{I}^{(q)}(\nu) := \int_0^\pi dt \, \sin^{\nu-3}(t) \cos^q(t) \tag{83}$$

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It is[5, eq. 5.12.6]:

$$\int_0^\pi (\sin t)^{\alpha-1} e^{i\beta t} dt = \frac{\pi}{2^{\alpha-1}} \frac{e^{i\pi\beta/2}}{\alpha\, B\left((\alpha+\beta+1)/2, (\alpha-\beta+1)/2\right)} \qquad \text{if } \Re(\alpha) > 0 \qquad (84)$$

$$\Rightarrow \hat{I}^{(0)}(n) = \frac{\pi}{2^{n-3}(n-2)} \frac{1}{B((n-1)/2, (n-1)/2)} \tag{85}$$

$$\Rightarrow \hat{I}^{(0)}(n-1) = \frac{\pi}{2^{n-4}(n-3)} \frac{1}{B((n-2)/2, (n-2)/2)} = B((n-3)/2, 1/2) \tag{86}$$

If q is odd: $\hat{I}^{(q)} = 0$, due to symetry of kernel; if q is even: $q = 2p$ with $p \in \mathbb{N}$:

$$\hat{I}^{(2p)}(\nu) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} \int_0^\pi \sin^{\nu-3}(t) \exp(2i(k-p)t)\, dt \tag{87}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{k=0}^{2p} \binom{2p}{k} \frac{\exp(i\pi(k-p))}{B((\nu-1)/2 + (k-p), (\nu-1)/2 - (k-p))} \tag{88}$$

$$= \frac{\pi}{2^{2p+\nu-3}(\nu-2)} \sum_{l=-p}^{p} \binom{2p}{p+l} \frac{(-1)^l}{B((\nu-1)/2 + l, (\nu-1)/2 - l)} \tag{89}$$

$$= \frac{\pi\Gamma(\nu-1)(2p)!}{2^{2p+\nu-3}(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2})} \right.$$

$$\left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2}-p)}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{90}$$

$$= \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}+p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \cdot p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} \right.$$

$$\left. + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{91}$$

TODO: prove

$$p! \left(\frac{1}{(p!)^2} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l}{(p+l)!(p-l)!} \frac{\Gamma(\frac{\nu-1}{2}+p)}{\Gamma(\frac{\nu-1}{2}+l)} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu-1}{2}-l)} \right) \tag{92}$$

$$= \frac{1}{p!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2})} + 2\sum_{l=1}^{p} \frac{(-1)^l p!}{(p+l)!(p-l)!} \frac{\Gamma(-\frac{1}{2}+p)}{\Gamma(-\frac{1}{2}+l)} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}-l)} \tag{93}$$

$$= 1 \tag{94}$$

$$\Rightarrow \hat{I}^{(2p)}(\nu) = \frac{2^{3-\nu}\pi\Gamma(\nu-1)}{(\nu-2)\Gamma(\frac{n-1}{2}+p)\Gamma(\frac{n-1}{2}-p)} \cdot \frac{\Gamma(\frac{\nu-1}{2}-p)}{2^p\Gamma(\frac{\nu-1}{2})} \cdot \frac{(2p)!}{2^p p!} \tag{95}$$

$$= \frac{\sqrt{\pi}(2p)!}{2^{2p}p!} \frac{\Gamma((\nu-2)/2)}{\Gamma(\frac{\nu-1}{2}+p)} \tag{96}$$

FiXme
Error:
prove

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2.3.2 any collinearity and $-k, -l \in \mathbb{N}_0$

If $-k, -l \in \mathbb{N}_0$ $I_n^{(k,l)}$ can always be reduced in a straight forward manner to combinations of $\hat{I}^{(q)}(n)$ and this way one finds[4, Ch. 5][2, App. C]:

$$I_n^{(0,0)} = \hat{I}^{(0)}(n-1) \cdot \hat{I}^{(0)}(n) = \frac{2\pi}{n-3} \tag{97}$$

$$I_4^{(0,0)} = 2\pi \tag{98}$$

$$I_n^{(-1,0)} = \hat{I}^{(0)}(n-1) \cdot (a\hat{I}^{(0)}(n) + b\hat{I}^{(1)}(n)) = \frac{2\pi a}{n-3} \tag{99}$$

$$I_4^{(-1,0)} = 2\pi a \tag{100}$$

$$I_n^{(0,-1)} = \hat{I}^{(0)}(n-1) \cdot (A\hat{I}^{(0)}(n) + B\hat{I}^{(1)}(n)) + C\hat{I}^{(1)}(n-1)\hat{I}^{(0)}(n) \tag{101}$$

$$= \frac{2\pi A}{n-3} \tag{102}$$

$$I_4^{(0,-1)} = 2\pi A \tag{103}$$

$$I_n^{(-2,0)} = \hat{I}^{(0)}(n-1) \cdot (a^2\hat{I}^{(0)}(n) + 2ab\hat{I}^{(1)}(n) + b^2\hat{I}^{(2)}(n)) \tag{104}$$

$$= 2\pi \left(\frac{a^2(n-1) + b^2}{(n-1)(n-3)} \right) \tag{105}$$

$$I_4^{(-2,0)} = 2\pi(a^2 + b^2/3) \tag{106}$$

$$I_n^{(0,-2)} = \hat{I}^{(0)}(n-1) \cdot (A^2\hat{I}^{(0)}(n) + B^2\hat{I}^{(2)}(n)) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n+2) \tag{107}$$

$$= 2\pi \left(\frac{A^2(n-1) + B^2 + C^2}{(n-1)(n-3)} \right) \tag{108}$$

$$I_4^{(0,-2)} = 2\pi(A^2 + (B^2 + C^2)/3) \tag{109}$$

$$I_n^{(-1,-1)} = \hat{I}^{(0)}(n-1) \cdot (aA\hat{I}^{(0)}(n) + bB\hat{I}^{(2)}(n)) = 2\pi \left(\frac{aA(n-1) + bB}{(n-1)(n-3)} \right) \tag{110}$$

$$I_4^{(-1,-1)} = 2\pi(aA + bB/3) \tag{111}$$

2.3.3 single collinear a and $k, -l \in \mathbb{N}_0$

It is

$$\hat{I}_a^{(k,q)}(\nu) = \int_0^\pi \frac{\sin^{\nu-3} t}{(1-\cos(t))^k} \cos^q(t)\, dt \tag{112}$$

$$= \int_0^\pi \frac{\sin^{\nu-3}(t)}{(1-\cos^2(t))^k} \cos^q(t)(1+\cos(t))^k\, dt \tag{113}$$

$$= \int_0^\pi \sin^{\nu-3-2k}(t) \cos^q(t)(1+\cos(t))^k\, dt \tag{114}$$

$$= \sum_{l=0}^k \binom{k}{l} \hat{I}^{(q+l)}(\nu - 2k) \tag{115}$$

this way one finds[4, Ch. 5][2, App. C]:

$$I_{a,n}^{(1,0)} = \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot\hat{I}^{(0)}(n-2) \tag{116}$$

$$= \frac{2\pi}{a(n-4)} \tag{117}$$

$$I_{a,n}^{(2,0)} = \frac{1}{a^2}\hat{I}^{(0)}(n-1)\cdot\left(\hat{I}^{(0)}(n-4) + \hat{I}^{(2)}(n-4)\right) \tag{118}$$

$$= \frac{2\pi}{a^2(n-6)} \approx -\frac{\pi}{a^2} + O(\epsilon) \tag{119}$$

$$I_{a,n}^{(1,-1)} = \frac{1}{a}\hat{I}^{(0)}(n-1)\cdot\left(A\hat{I}^{(0)}(n-2) + B\hat{I}^{(2)}(n-2)\right) \tag{120}$$

$$= \frac{2\pi}{a}\frac{(A(n-3)+B)}{(n-3)(n-4)} \approx \frac{2\pi}{a}\left(\frac{A+B}{\epsilon} - 2B + O(\epsilon)\right) \tag{121}$$

$$I_{a,n}^{(1,-2)} = \frac{1}{a}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2\hat{I}^{(0)}(n-2) + (B^2+2AB)\hat{I}^{(2)}(n-2)\right) + C^2\hat{I}^{(2)}(n-1)\hat{I}^{(0)}(n)\right) \tag{122}$$

$$= \frac{2\pi}{a}\left(\frac{A^2}{n-4} + \frac{2AB+B^2}{(n-4)(n-3)} + \frac{C^2}{(n-3)(n-2)}\right) \tag{123}$$

$$\approx \frac{2\pi}{a}\left(\frac{(A+B)^2}{\epsilon} + \frac{C^2}{2} - 2AB - B^2 + O(\epsilon)\right) \tag{124}$$

$$I_{a,n}^{(2,-2)} = \frac{1}{a^2}\left(\hat{I}^{(0)}(n-1)\cdot\left(A^2(\hat{I}^{(0)}(n-4)+\hat{I}^{(2)}(n-4)) + 4AB\hat{I}^{(2)}(n-4)\right.\right.$$
$$\left.\left. +B^2(\hat{I}^{(2)}(n-4)+\hat{I}^{(4)}(n-4))\right) + C^2\hat{I}^{(2)}(n-1)(\hat{I}^{(0)}(n-2)+\hat{I}^{(2)}(n-2))\right) \tag{125}$$

$$= \frac{2\pi}{a^2}\left(\frac{A^2}{n-6} + \frac{4AB}{(n-6)(n-4)} + \frac{B^2n}{(n-6)(n-4)(n-3)} + \frac{C^2}{(n-4)(n-3)}\right) \tag{126}$$

$$\approx \frac{2\pi}{a^2}\left(\frac{-2AB-2B^2+C^2}{\epsilon} + \frac{B^2-A^2}{2} - AB - C^2 + O(\epsilon)\right) \tag{127}$$

$$\tag{128}$$

2.3.4 double collinear and $k,l \in \mathbb{N}$

as said in [4, Ch. 5]: if $0 \leq -\frac{C}{A}, \frac{B}{A} \leq 1$ use [3, eq. A11] with $\cos\kappa = -\frac{B}{A}$:

$$I_{aA,n}^{(k,l)} = \frac{2\pi 2^{-(k+l)}}{a^k A^l}\frac{\Gamma(1+\epsilon)}{\Gamma^2(1+\epsilon/2)}B(1+\frac{\epsilon}{2}-k, 1+\frac{\epsilon}{2}-l)_2F_1\left(k,l;1+\frac{\epsilon}{2};\frac{A-B}{2A}\right) \tag{129}$$

2.3.5 non collinear and $k,-l \in \mathbb{N}$

Next we want to compute $I_{0,n}^{1,-3}$.

The θ_2 integration can be performed using the integral helper and the problem reduces then to the following integral:

$$\hat{I}_{0,\cos}^{(k,q,p)}(\epsilon)$$

$$= \int_0^\pi d\theta_1 \, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))^k} \tag{130}$$

$$= \frac{1}{2a^k}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{k}{2},\frac{1+p}{2};\frac{1}{2},\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.$$

$$\left.\frac{b}{a}k(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_3F_2\left(\frac{1+k}{2},\frac{2+k}{2},\frac{2+p}{2};\frac{3}{2},\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) \tag{131}$$

for $k=1$ this simplifies to

$$\hat{I}_{0,\cos}^{(1,q,p)}(\epsilon)$$

$$= \int_0^\pi d\theta_1 \, \frac{\sin^{1+\epsilon}(\theta_1)\sin^q(\theta_1)\cos^p(\theta_1)}{(a+b\cos(\theta_1))} \tag{132}$$

$$= \frac{1}{2a}\left((1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{1+p}{2}\right){}_2F_1\left(1,\frac{1+p}{2};\frac{3+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right.$$

$$\left.\frac{b}{a}(-1+(-1)^p)B\left(\frac{2+q+\epsilon}{2},\frac{2+p}{2}\right){}_2F_1\left(1,\frac{2+p}{2};\frac{4+q+p+\epsilon}{2};\frac{b^2}{a^2}\right)\right) \tag{133}$$

2.3.6 needed set

we will need for A_1

$$\mathcal{I}_{A1}^{(j,k)} = \int d\Omega_n \, t'^{-2+j} u_7^{-2+k} \qquad j,k = \{0,1,2,3\} \tag{134}$$

with

$$a(t') = -b(t') = -2\omega_1\omega_2 \qquad\qquad = -\frac{s_4(s_4-u_1)}{2(s_4+m^2)} \tag{135}$$

$$A(u_7) = q^2 - 2q_0 E_1 \qquad\qquad = q^2 - \frac{(s_4+2m^2)(s+u_1)}{2(s_4+m^2)} \tag{136}$$

$$B(u_7) = -2\omega_2(|\vec{p}_2|\cos\psi - \omega_1) \qquad = \frac{s_4}{2}\left(1 - \frac{s+u_1}{s_4+m^2} + \frac{s-q^2-t_1}{s_4-u_1}\right) \tag{137}$$

$$C(u_7) = -2\omega_2\,|\vec{p}_2|\sin\psi \tag{138}$$

A References

[1] M. Stratmann. Private communications, 2015.

[2] W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith, "QCD Corrections to Heavy Quark Production in p anti-p Collisions," Phys. Rev. **D40** (1989) 54–82.

[3] W. L. Van Neerven, "Dimensional regularization of mass and infrared singularities in two-loop on-shell vertex functions," Nuclear Physics B **268** no. 2, (May, 1986) 453–488.
http://www.sciencedirect.com/science/article/pii/0550321386901653.

[4] I. Bojak, NLO QCD corrections to the polarized photoproduction and hadroproduction of heavy quarks. PhD thesis, Dortmund U., 2000. arXiv:hep-ph/0005120 [hep-ph].

[5] "NIST Digital Library of Mathematical Functions." Http://dlmf.nist.gov/, release 1.0.10 of 2015-08-07. http://dlmf.nist.gov/. Online companion to [6].

[6] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., NIST Handbook of Mathematical Functions. Cambridge University Press, New York, NY, 2010. Print companion to [5].

List of Corrections