

# 1 Passarino-Veltman decomposition

## 1.1 Definitions

[1]:

$$A(m) = \frac{1}{i\pi^2} \int d^n q \frac{1}{q^2 + m^2} \quad (1)$$

$$B_0(p, m_1, m_2) = \frac{1}{i\pi^2} \int d^n q \frac{1}{(q^2 + m_1^2)((q + p)^2 + m_2^2)} \quad (2)$$

and apart from their pole term (called  $\Delta$  - see [1, eq. D.1]), they keep  $n = 4$ .

[2, 3]:

$$A(m) = \mu^{-\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 - m^2} \quad (3)$$

$$B(q_1, m_1, m_2) = \mu^{-\epsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_1^2)((q + q_1)^2 - m_2^2)} \quad (4)$$

and  $n = 4 + \epsilon$ . ([2] writes “The notations for the one-, two-, three-, and four-point functions have been taken over from Ref. [1].” - obviously they do not.)

HEPMath[4] and FeynCalc[5, 6] refer to LoopTools[7, 8]. [8, eq. (1.1)] and [9, eq. (2.6)]:

$$T_{\mu_1 \dots \mu_P}^N = \frac{\mu^{4-D}}{i\pi^{D/2} r_\Gamma} \int d^D q \frac{q_{\mu_1} \dots q_{\mu_P}}{[q^2 - m_1^2] [(q + k_1)^2 - m_2^2] \dots [(q + k_{N-1})^2 - m_N^2]} \quad (5)$$

$$r_\Gamma = \frac{\Gamma^2(1 - \epsilon)\Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)}, \quad D = 4 - 2\epsilon$$

later in the code they use a different signature (to avoid any vector structure):

$$\begin{aligned} A(m^2), B_0(p^2, m_1^2, m_2^2), C_0(p_1^2, p_2^2, (p_1 + p_2)^2, m_1^2, m_2^2, m_3^2) \\ D_0(p_1^2, p_2^2, p_3^2, p_4^2, (p_1 + p_2)^2, (p_2 + p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2) \end{aligned} \quad (6)$$

[10]:

$$T_{\mu_1 \dots \mu_P}^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \dots q_{\mu_P}}{L_0 L_1 \dots L_{N-1}} \quad (7)$$

$$L_0 = q^2 - m_0^2 + i\epsilon \quad (8)$$

$$L_i = (q + p_i)^2 - m_i^2 + i\epsilon \quad i = 1, \dots, N-1 \quad (9)$$

I will stick to the integrals of [3] as it is the most natural form, I think, and to the non-vector signature, if possible.

## 1.2 Decomposition Labeling

[1, 3]:

$$B_\mu(p, m_1, m_2) = p_\mu B_1(p, m_1, m_2) \quad (10)$$

$$B_{\mu\nu} = p_\mu p_\nu B_{21} + g_{\mu\nu} B_{22} \quad (11)$$

$$C_\mu(p_1, p_2, m_1, m_2, m_3) = p_{1,\mu} C_{11} + p_{2,\mu} C_{12} \quad (12)$$

$$C_{\mu\nu} = p_{1,\mu} p_{1,\nu} C_{21} + p_{2,\mu} p_{2,\nu} C_{22} + (p_{1,\mu} p_{2,\nu} + p_{1,\nu} p_{2,\mu}) C_{23} + g_{\mu\nu} C_{24} \quad (13)$$

The arguments of the functions are always inherited.

HEPMath, FeynCalc, LoopTools, [9]:

$$B_\mu(p, m_1, m_2) = p_\mu B_1(p, m_1, m_2) \quad (14)$$

$$B_{\mu\nu} = g_{\mu\nu} B_{00} + p_\mu p_\nu B_{11} \quad (15)$$

$$C_\mu(p_1, p_2, m_1, m_2, m_3) = p_{1,\mu} C_1 + p_{2,\mu} C_2 = \sum_{j=1}^2 p_{j,\mu} C_j \quad (16)$$

$$C_{\mu\nu} = p_{1,\mu} p_{1,\nu} C_{11} + p_{2,\mu} p_{2,\nu} C_{22} + (p_{1,\mu} p_{2,\nu} + p_{1,\nu} p_{2,\mu}) C_{12} + g_{\mu\nu} C_{00} \quad (17)$$

$$= g_{\mu\nu} C_{00} + \sum_{j,k=1}^2 p_{j,\mu} p_{k,\nu} C_{jk} \quad (18)$$

The arguments of the functions are always inherited.

I will stick to HEPMath as it is the more generic and extensible form, I think.

## 1.3 B Decomposition

define

$$f_1 = m_1^2 - m_0^2 - p^2 \quad (19)$$

then one finds easily

$$B_1(p^2, m_0^2, m_1^2) = \frac{1}{2p^2} \left( f_1 B_0(p^2, m_0^2, m_1^2) + A_0(m_0^2) - A_0(m_1^2) \right) \quad (20)$$

$$B_{00}(p^2, m_0^2, m_1^2) = \frac{1}{2(n-1)} \left( 2m_0^2 B_0(p^2, m_0^2, m_1^2) + A_0(m_1^2) - f_1 B_1(p^2, m_0^2, m_1^2) \right) \quad (21)$$

$$B_{11}(p^2, m_0^2, m_1^2) = \frac{1}{2p^2} \left( f_1 B_0(p^2, m_0^2, m_1^2) + A_0(m_1^2) - 2B_{00}(p^2, m_0^2, m_1^2) \right) \quad (22)$$

in accordance with [3, 9].

Concerning  $B_1$  [1] and **LoopTools** use the following identity

$$A_0(m_0^2) - A_0(m_1^2) = (m_0^2 - m_1^2)B_0(0, m_0^2, m_1^2) \quad (23)$$

that might help away with

In case  $m_1$  and/or  $m_2$  are very large the expression on the right-hand side of eq. (20) suffers very strong cancellations: the total is very much smaller than the individual terms. For this reason we have not used these algebraic relations, except to rewrite self-energy diagrams as much as possible in a form most suitable for numerical evaluation. ([1, below eq. D.6])

To compare the other results to [1] and **LoopTools** one has to use the *strict*  $n \rightarrow 4$  limit and the following identities[10]:

$$(n-4)B_{00}(p^2, m_0^2, m_1^2) = \frac{1}{6}(p^2 - 3m_0^2 - 3m_1^2) \quad (24)$$

$$(n-4)B_{11}(p^2, m_0^2, m_1^2) = -\frac{2}{3} \quad (25)$$

## 2 Scalar Integrals

We focus on:

$$\gamma^*(q) + g(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2) \quad (26)$$

$$k_1^2 = 0 \quad p_1^2 = p_2^2 = m^2 \quad (p_1 + p_2)^2 = s \quad (p_2 - q)^2 = t \quad (p_1 - q)^2 = u \quad (27)$$

define some shortcuts

$$0 \leq \rho = \frac{4m^2}{s} \leq 1 \quad 0 \leq \beta = \sqrt{1 - \rho} \leq 1 \quad 0 \leq \chi = \frac{1 - \beta}{1 + \beta} \leq 1 \quad (28)$$

$$\rho_q = \frac{4m^2}{q^2} \leq 0 \quad 1 \leq \beta_q = \sqrt{1 - \rho_q} \quad 0 \leq \chi_q = -\frac{1 - \beta_q}{1 + \beta_q} \leq 1 \quad (29)$$

## 2.1 One-Point Function $A_0$

[10]:

$$A_0(m^2) = -\frac{i}{16\pi^2} m^2 \left( \frac{m^2}{4\pi\mu^2} \right)^{(n-4)/2} \Gamma(1 - n/2) \quad (30)$$

$$= \frac{im^2}{16\pi^2} \left( \Delta - \log(m^2/\mu^2) + 1 \right) + O(n-4) \quad (31)$$

$$= iC_\epsilon m^2 \left( -\frac{2}{\epsilon} + 1 \right) + O(n-4) \quad (32)$$

$$\Delta = \frac{2}{4-n} - \gamma_E + \log(4\pi) \quad (33)$$

$$C_\epsilon = \frac{1}{16\pi^2} \exp \left( \left( \gamma_E - \log(4\pi) + \log(m^2/\mu^2) \right) \frac{\epsilon}{2} \right) \quad (34)$$

this is *up to order*  $O(n-4)$  in accordance with [3][11], but NOT beyond - see also [3, eq. (A.12)]. So we can treat  $C_\epsilon$  and  $\Delta$  as equal.

## 2.2 Two-Point Function $B_0$

In [10, eq. (4.23)] is a generic function given and we end up with

$$B_0(s, m^2, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 + \beta \log(\chi) \right) \quad (35)$$

$$B_0(q^2, m^2, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 + \beta_q \log(\chi_q) \right) \quad (36)$$

$$B_0(0, m^2, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} \right) \quad (37)$$

$$B_0(m^2, 0, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 \right) \quad (38)$$

$$B_0(t, 0, m^2) = iC_\epsilon \left( -\frac{2}{\epsilon} + 2 - \frac{t-m^2}{t} \ln \left( -\frac{t-m^2}{m^2} \right) \right) \quad (39)$$

focussing on imaginary part *only*; this in accordance with [3][11].

## 2.3 Three-Point Function $C_0$

Again, in [10, eq. (4.26)] is a generic function given.

First, we compute  $C_0(s, q^2, 0, m^2, m^2, m^2)$  and by taking the limit  $k_1^2 \rightarrow 0$  (or equivalently

$s_4 \rightarrow 0$ ) we end up with:

$$C_0(s, q^2, 0, m^2, m^2, m^2) = \frac{i}{16\pi^2} \cdot \frac{1}{s - q^2} \left( \text{Li}_2\left(\frac{2}{1 + \beta_q}\right) + \text{Li}_2\left(\frac{2}{1 - \beta_q}\right) - \text{Li}_2\left(\frac{2}{1 + \beta}\right) - \text{Li}_2\left(\frac{2}{1 - \beta}\right) \right) \quad (40)$$

with [12] we find:

$$\text{Li}_2\left(\frac{2}{1 + b}\right) + \text{Li}_2\left(\frac{2}{1 - b}\right) = 3\zeta(2) + \frac{1}{2} \ln^2\left(\frac{1 - b}{1 + b}\right) - \ln\left(\frac{1 - b}{1 + b}\right) \ln\left(-\frac{1 - b}{1 + b}\right) \quad (41)$$

and if we focus on real part *only*, we find:

$$\text{Li}_2\left(\frac{2}{1 + \beta}\right) + \text{Li}_2\left(\frac{2}{1 - \beta}\right) = 3\zeta(2) - \frac{1}{2} \ln^2(\chi) \quad (42)$$

$$\text{Li}_2\left(\frac{2}{1 + \beta_q}\right) + \text{Li}_2\left(\frac{2}{1 - \beta_q}\right) = -\frac{1}{2} \ln^2(\chi_q) \quad (43)$$

Additionally, we find

$$\lim_{q^2 \rightarrow 0} \left[ \text{Li}_2\left(\frac{2}{1 + \beta_q}\right) + \text{Li}_2\left(\frac{2}{1 - \beta_q}\right) \right] = 0 \quad (44)$$

So we get:

$$C_0(s, q^2, 0, m^2, m^2, m^2) = iC_\epsilon \frac{1}{s - q^2} \left( \frac{1}{2} \ln^2(\chi) - \frac{1}{2} \ln^2(\chi_q) - 3\zeta(2) \right) \quad (45)$$

$$C_0(s, 0, 0, m^2, m^2, m^2) = iC_\epsilon \frac{1}{s} \left( \frac{1}{2} \ln^2(\chi) - 3\zeta(2) \right) \quad (46)$$

in accordance with [3][11][13]. These results can also be obtained by the methods described in [3, chap. 3].

Next, we compute  $C_0(m^2, 0, t, 0, m^2, m^2)$  again by taking the limit  $k_1^2 \rightarrow 0$  we end up with:

$$C_0(m^2, 0, t, 0, m^2, m^2) = \frac{i}{16\pi^2} \cdot \frac{1}{t - m^2} \left( 2\text{Li}_2(2) + \text{Li}_2(m^2/t) - \frac{\pi^2}{6} - \text{Li}_2((t + m^2)/m^2) - \text{Li}_2((m^2 + t)/t) \right) \quad (47)$$

Using [12] and focussing on real part, we find

$$\text{Li}_2(2) = \frac{\pi^2}{4} - i\pi \ln(2) \quad (48)$$

$$2\text{Li}_2(2) + \text{Li}_2(1/z) - \frac{\pi^2}{6} - \text{Li}_2(1 + z) - \text{Li}_2(1 + 1/z) = \frac{\pi^2}{6} - \text{Li}_2(z) \quad (49)$$

So we get:

$$C_0(m^2, 0, t, 0, m^2, m^2) = iC_\epsilon \frac{1}{t - m^2} \left( \zeta(2) - \text{Li}_2(t/m^2) \right) \quad (50)$$

in accordance with [3][11].

To compute  $C_0(m^2, s, m^2, 0, m^2, m^2)$  we use [3] and find

$$C_0(m^2, s, m^2, 0, m^2, m^2) = \frac{iC_\epsilon}{s\beta} \left( -\frac{2}{\epsilon} \ln(\chi) - \frac{\pi^2}{2} + \frac{1}{2} \ln^2(\chi) - \ln(\chi) \ln(1 - \chi) \right. \\ \left. - \text{Li}_2(1/(1 - \chi)) + \text{Li}_2(\chi/(\chi - 1)) \right) \quad (51)$$

Using [12] and focussing on real part, we find

$$-\text{Li}_2(1/(1 - x)) + \text{Li}_2(x/(x - 1)) = -2\text{Li}_2(z) - \ln(x) \ln(1 - x) - \frac{\pi^2}{6} \quad (52)$$

So we get:

$$C_0(m^2, s, m^2, 0, m^2, m^2) = iC_\epsilon \frac{1}{s\beta} \left( -\frac{2}{\epsilon} \ln(\chi) - 2\ln(\chi) \ln(1 - \chi) - 2\text{Li}_2(\chi) \right. \\ \left. + \frac{1}{2} \ln^2(\chi) - 4\zeta(2) \right) \quad (53)$$

in accordance with [3][11].

To compute  $C_0(t, m^2, q^2, 0, m^2, m^2)$  we use [10] and find immediatly:

$$C_0(t, m^2, q^2, 0, m^2, m^2) = \frac{iC_\epsilon}{\alpha} \left[ -\zeta(2) + 2\text{Li}_2 \left( \frac{t_1 + \alpha}{t_1} \right) + \text{Li}_2 \left( \frac{q^2 - t - m^2 + \alpha}{q^2 - t - m^2 - \alpha} \right) \right. \\ \text{Li}_2 \left( \frac{t_1 - q^2 \beta_q^2 + \alpha}{t_1 - q^2 \beta_q^2 - \beta_q \alpha} \right) - \text{Li}_2 \left( \frac{t_1 - q^2 \beta_q^2 - \alpha}{t_1 - q^2 \beta_q^2 + \beta_q \alpha} \right) \\ \text{Li}_2 \left( \frac{t_1 - q^2 \beta_q^2 + \alpha}{t_1 + q^2 \beta_q^2 - \beta_q \alpha} \right) - \text{Li}_2 \left( \frac{t_1 - q^2 \beta_q^2 - \alpha}{t_1 + q^2 \beta_q^2 + \beta_q \alpha} \right) \\ \left. - \text{Li}_2 \left( \frac{t_1(q^2 - t - m^2 - \alpha) - 2m^2 \alpha}{t_1(q^2 - t - m^2 + \alpha)} \right) \right. \\ \left. - \text{Li}_2 \left( \frac{t_1(q^2 - t - m^2 - \alpha) - 2m^2 \alpha}{t_1(q^2 - t - m^2 - \alpha)} \right) \right] \quad (54)$$

with  $\alpha = \kappa(t, q^2, m^2)$  and the Källén function (as defined in [10, eq. (4.27)])

$$\kappa(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2(xy + xz + yz)} \quad (55)$$

This is in accordance with [13, eq. (A.8)](Note the typo there!).

Additionally, we find

$$\lim_{q^2 \rightarrow 0} C_0(t, m^2, q^2, 0, m^2, m^2) = C_0(t, m^2, 0, 0, m^2, m^2) = C_0(m^2, 0, t, 0, m^2, m^2) \quad (56)$$

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**List of Corrections**