Machine Learning for Computer Graphics

Last Class

BRDF – Bidirectional Reflectance Distribution Function

Today’s Topic

The Monte Carlo Method

Intuitively (and informally)

- Estimate unknown quantities through multiple observations
- The observations are averaged to obtain the final estimate

Why will we be talking about MC?

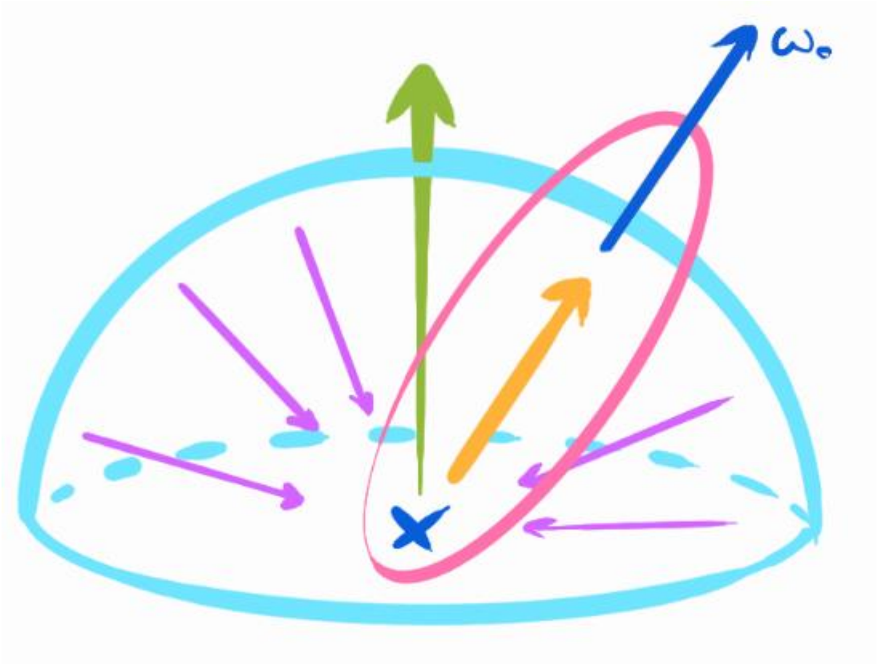
Preamble

From the BRDF to the Illumination Integral

Illumination Integral (or Reflection Equation)

Objective:

- Compute how much radiance arrives to each pixel of our image
- Determine the reflected radiance at a given point x in direction ω_o



$$L_r(\omega_o) = ?$$

Illumination Integral (or Reflection Equation)

Recall the BRDF equation:

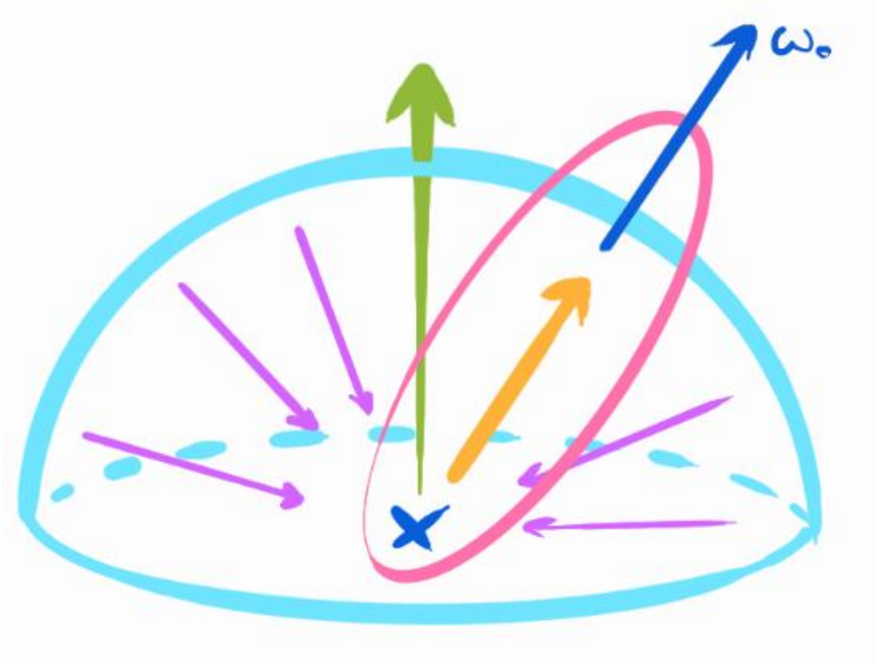
$$f_r(\omega_i, \omega_o) = \frac{\mathrm{d}L_r(\omega_o)}{L_i(\omega_i)\cos(\theta_i)\mathrm{d}\omega_i}$$

From where:

$$\mathrm{d}L_r(\omega_o) = f_r(\omega_i, \omega_o) L_i(\omega_i)\cos(\theta_i)\mathrm{d}\omega_i$$

And then:

$$L_r(\omega_o) = \int_{\Omega(x)} L_i(\omega_i) f_r(\omega_i, \omega_o) \cos\theta_i \, \mathrm{d}\omega_i$$



$L_r(\omega_o) = ?$

Characteristics of the Illumination Integral

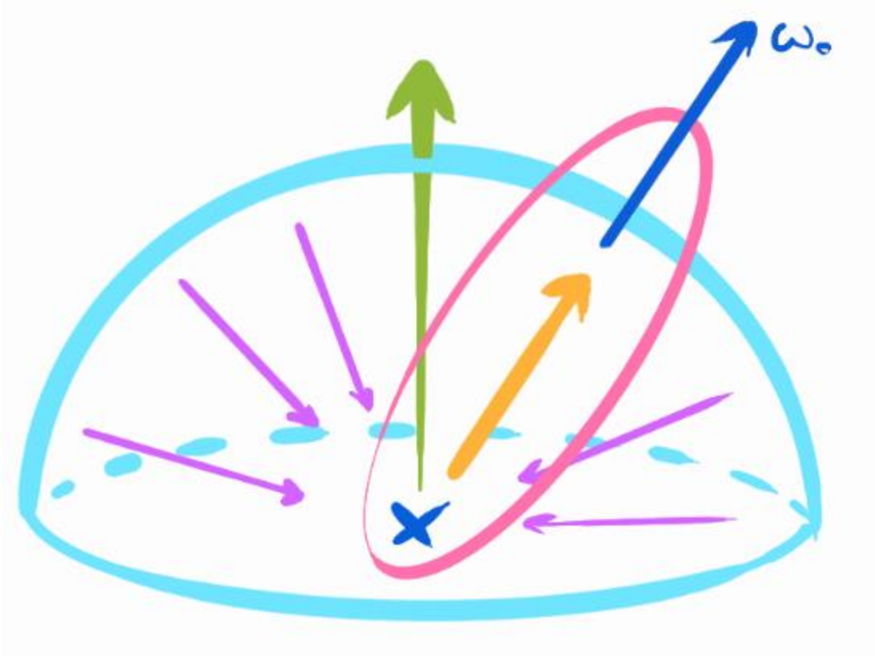
$$L_r(\omega_o) = \int_{\Omega(x)} L_i(\omega_i) f_r(\omega_i, \omega_o) \cos\theta_i \, \mathrm{d}\omega_i$$

No analytic solution in the general case

- $L_i(\omega_i)$ does not have an analytic expression
- $L_i(\omega_i)$ can only be known through sampling

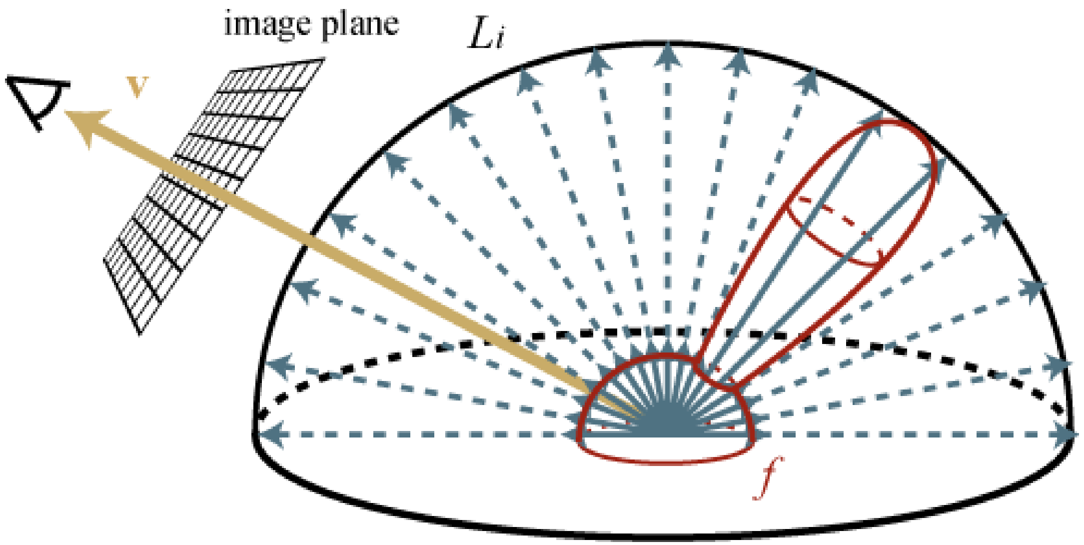
Need to resort to numerical methods

- Most common: *Monte Carlo Integration*



Solving the Illumination Integral

$$L_r(\omega_o) \approx \hat{L}_r(\omega_o) = \frac{1}{N}\sum_{n=1}^{N} \frac{L_i(\omega_n) f_r(\omega_n, \omega_o) \cos\theta_n}{p(\omega_n)}$$

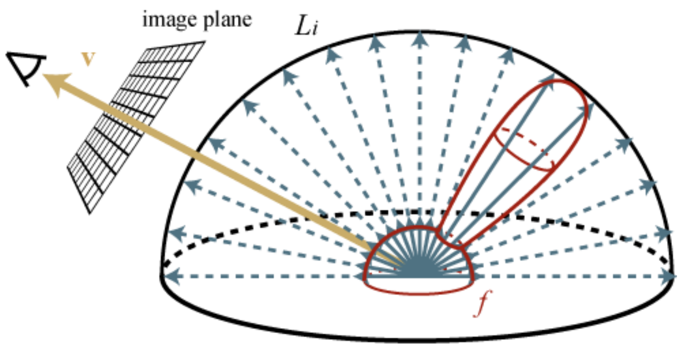


Solving the Illumination Integral

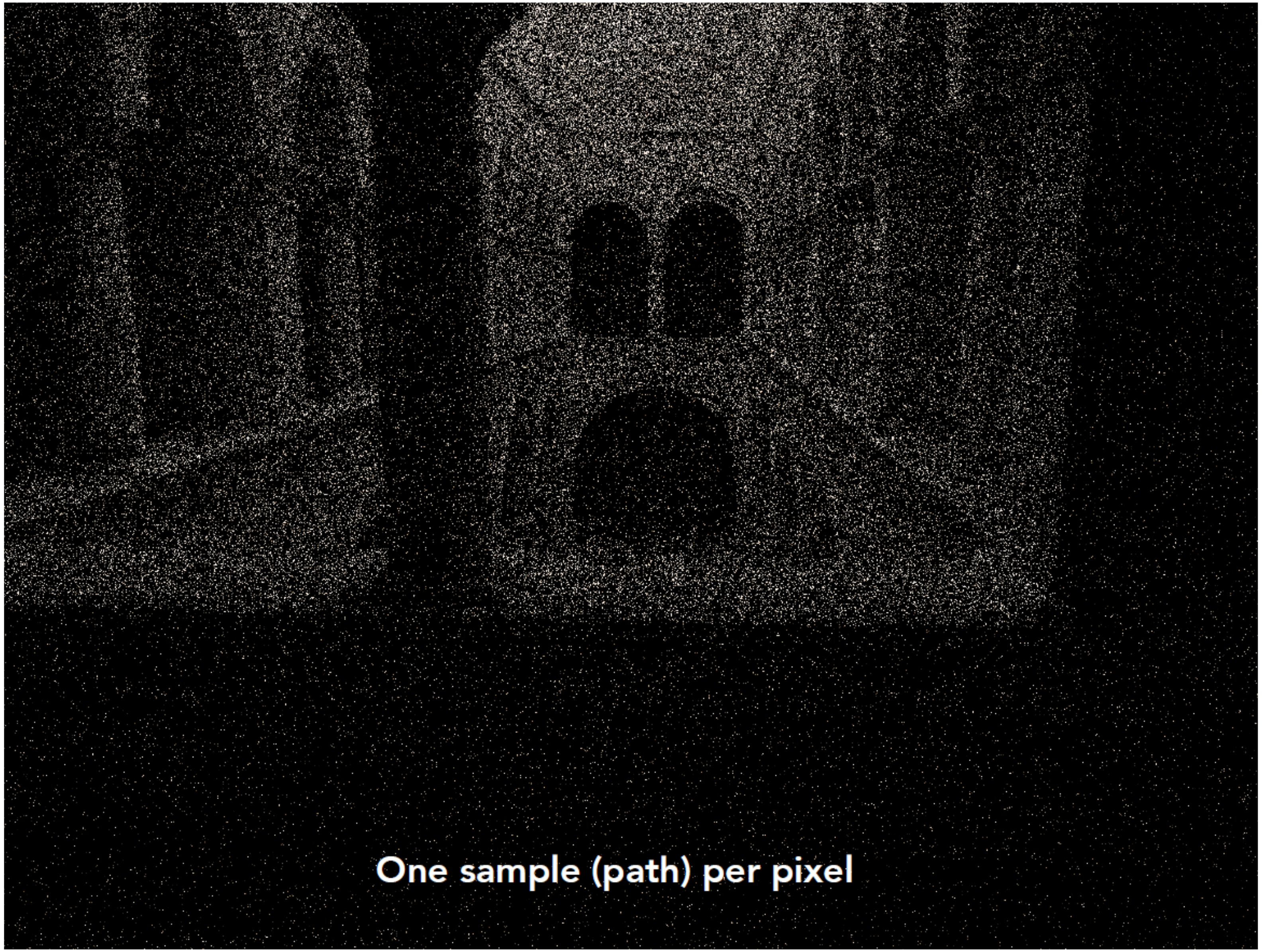
Monte Carlo Estimator:

$$L_r(\omega_o) \approx \hat{L}_r(\omega_o) = \frac{1}{N}\sum_{n=1}^{N}\frac{L_i(\omega_n) f_r(\omega_n,\omega_o)\cos\theta_n}{p(\omega_n)}$$

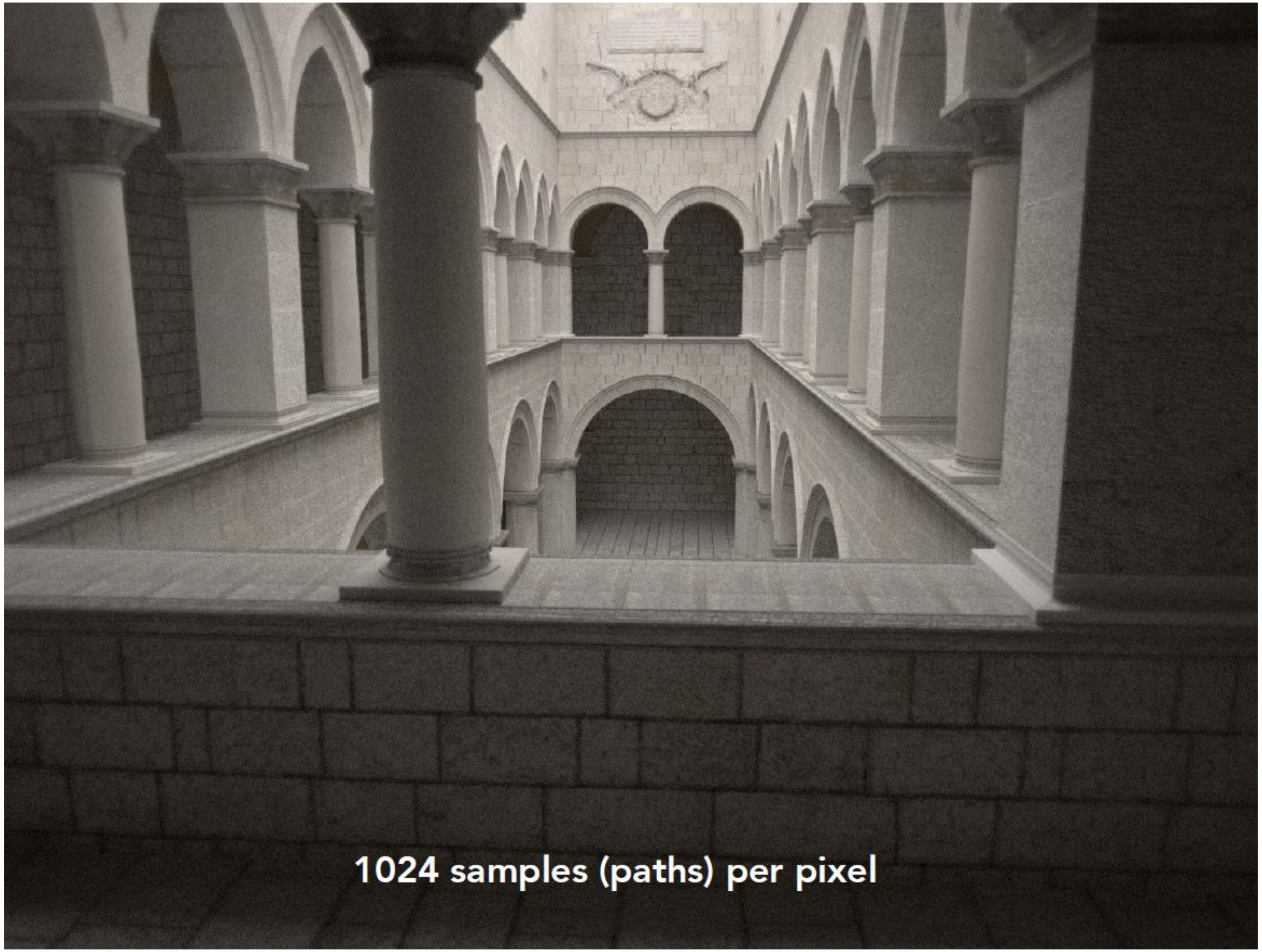
- Weighted average of integrand samples
- $p(\omega_n)$ is the probability of sampling direction ω_n
- Different choices for $p(\omega_n)$ are possible…
  - ... but some work much better than others
  - Can we learn the best pdf? More details later.



- Requires a very large number of samples to converge to the correct solution


One sample (path) per pixel


32 samples (paths) per pixel


1024 samples (paths) per pixel

End of Preamble

From the BRDF to the Illumination Integral

Introduction to Monte Carlo

Brief History of Monte Carlo

Term “Monte Carlo” coined in the 1940s

- Possible with the advent of electronic computing

Class of techniques which rely on repeated random sampling

Originally devised for atomic bomb development

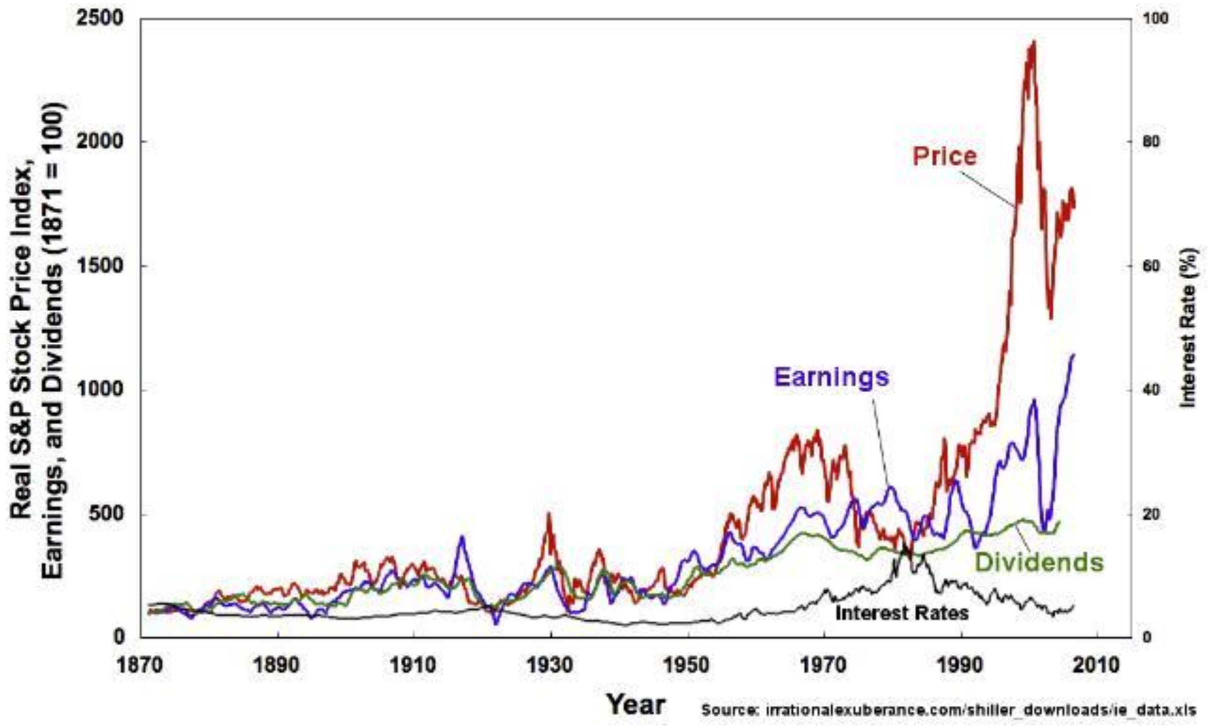
- Los Alamos (John von Neumann, Nicholas Metropolis, ...)

Intuition:

- Randomly simulate possible realizations the same type of event

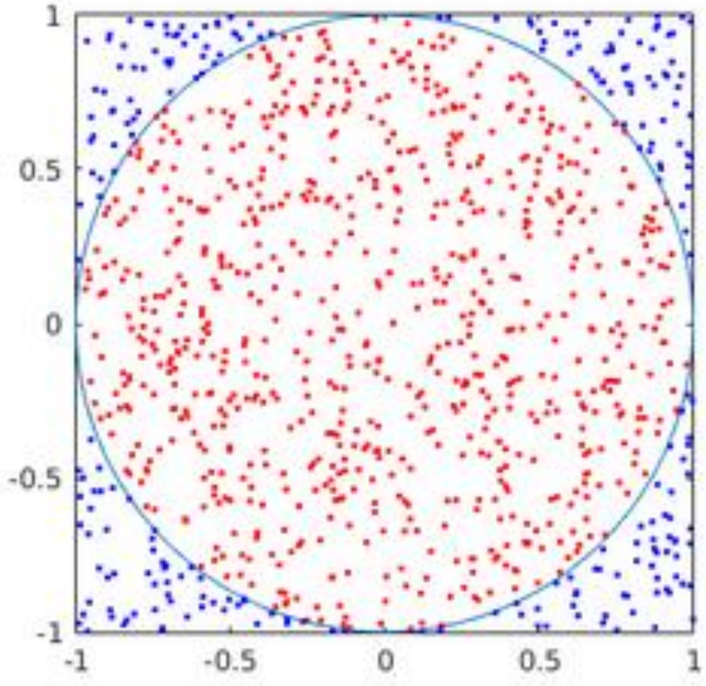
Applications

- Financial market simulations
- Astrophysics
- Computational Biology
- Fluid Dynamics
- Autonomous Robotics
- Computer Graphics
  - Light Transport Simulation
- ...



Monte Carlo – A Simple Example

Estimate the value of π

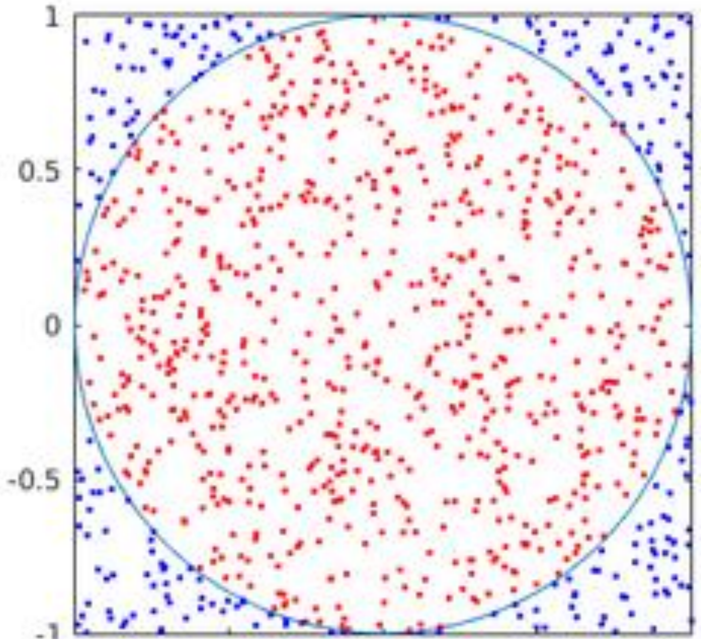


We know:

- Area of the square $A_s = 2 \times 2 = 4$
- Area of the circle $A_c = \pi$ (radius 1)
- Ratio between A_c and A_s

$$\frac{A_c}{A_s} = \frac{\pi}{4} \Leftrightarrow \pi = 4\frac{A_c}{A_s}$$

Monte Carlo – A Simple Example

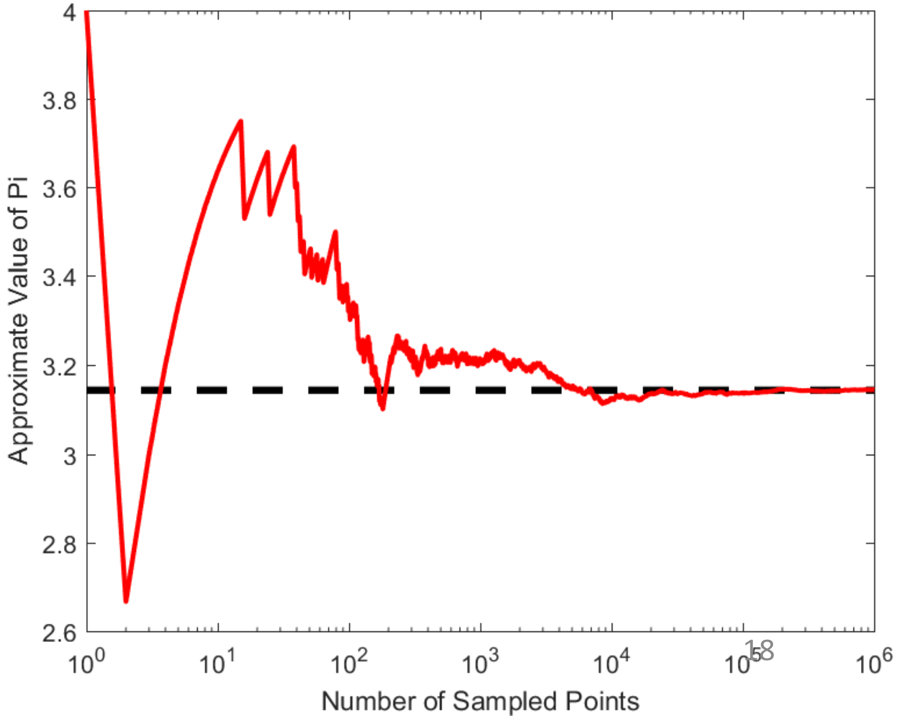


Estimate the value of π

$$\pi = 4\frac{A_c}{A_s}$$

Generate N random points inside the square

$$\frac{A_c}{A_s} \approx \frac{N_c}{N_s} \Rightarrow \pi \approx 4\frac{N_c}{N_s}$$



Brief Review of Probabilities

Random Variables

X (random variable)

- Variable whose values depend on a random phenomenon

D (probability distribution)

- Characterizes the possible values that a random variable might take, as well as its likelihoods

$X \sim D$

- Reads: X follows a probability distribution D
- If D is uniform, X assumes different values with the same probability

Discrete Random Variables

Random variables which can take a finite set of values $\mathrm{X} = \{x_i\}_{i=1}^{N}$

Each value is associated with a given probability p_i

$p_i \stackrel{\text{def}}{=} \Pr(X = x_i)$

$\sum_{i=1}^{N} p_i = 1$

Example of a fair die:

$X_{die} = \{1, 2, 3, 4, 5, 6\}$

$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6$

Expected Value ($\mathbb{E}$)

Characterizes the “average” value attained by the random variable

Weighted average of all possible outcomes

$$\mathrm{E}[X] \stackrel{\text{def}}{=} \sum_{i=1}^{N} x_i \, p_i$$

Example of a fair die:

$$\mathrm{E}[X_{die}] = \sum_{i=1}^{N} x_i \, \frac{1}{6} = \frac{(1+2+3+4+5+6)}{6} = 3.5$$

Variance (V)

Captures the dispersion of the realizations of a random variable X from its average value (i.e., from $\mathrm{E}[X]$)

$$
\begin{aligned}
\mathrm{V}[X] &= \mathrm{E}[(X - \mathrm{E}[X])^2] \\
&= \mathrm{E}[X^2 - 2X\mathrm{E}[X] + \mathrm{E}[X]^2] \\
&= \mathrm{E}[X^2] - 2\mathrm{E}[X]\mathrm{E}[X] + \mathrm{E}[X]^2 \\
&= E[X^2] - \mathrm{E}[X]^2
\end{aligned}
$$

Variance (V)

Properties of variance:

$V\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} V[X_i]$ (if X_i are independent random variables)

$V[aX] = a^2 V[X]$

Example of a fair die:

$$V[X_{die}] = \frac{1}{6}\,[\,(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2\,] = 2.91$$

Continuous Random Variables

Random variables which can take an ***infinite*** set of values

Probability mass is always zero

$$\Pr(X = x_i) = 0$$

Resort to probability ***density*** function $p(x)$, such that:

$$\Pr(X \in [a, b]) = \int_a^b p(\boldsymbol{x})\,\mathrm{d}\mathbf{x}$$

Continuous Random Variables

Conditions on the PDF:

$$\begin{cases} p(\boldsymbol{x}) \geq 0 \\ \int p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1 \end{cases}$$

Expected value:

$$\mathrm{E}[X] = \int \boldsymbol{x}\, p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

Function of a Random Variable

A function Y of a random variable X is also a random variable

$$X \sim p(\boldsymbol{x})$$

$$Y = f(X)$$

Its expected value is given by:

$$\mathrm{E}[Y] = \int f(\boldsymbol{x})\, p(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

Monte Carlo Integration

Using Monte Carlo to estimate the value of a given integral

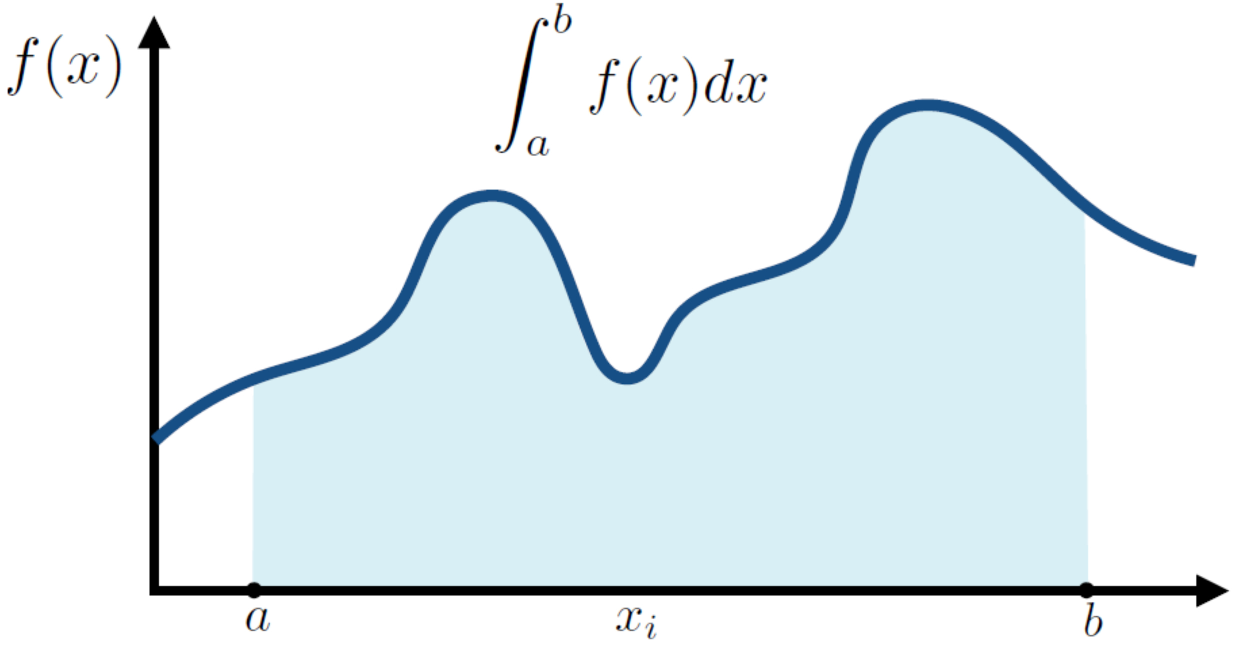
Monte Carlo Integration – Problem Setting

Goal:

- Compute $I = \int_a^b f(x)\,\mathrm{d}x$

Typical constraints:

- $f(x)$ is analytically unknown
- $f(x)$ can only be known through sampling

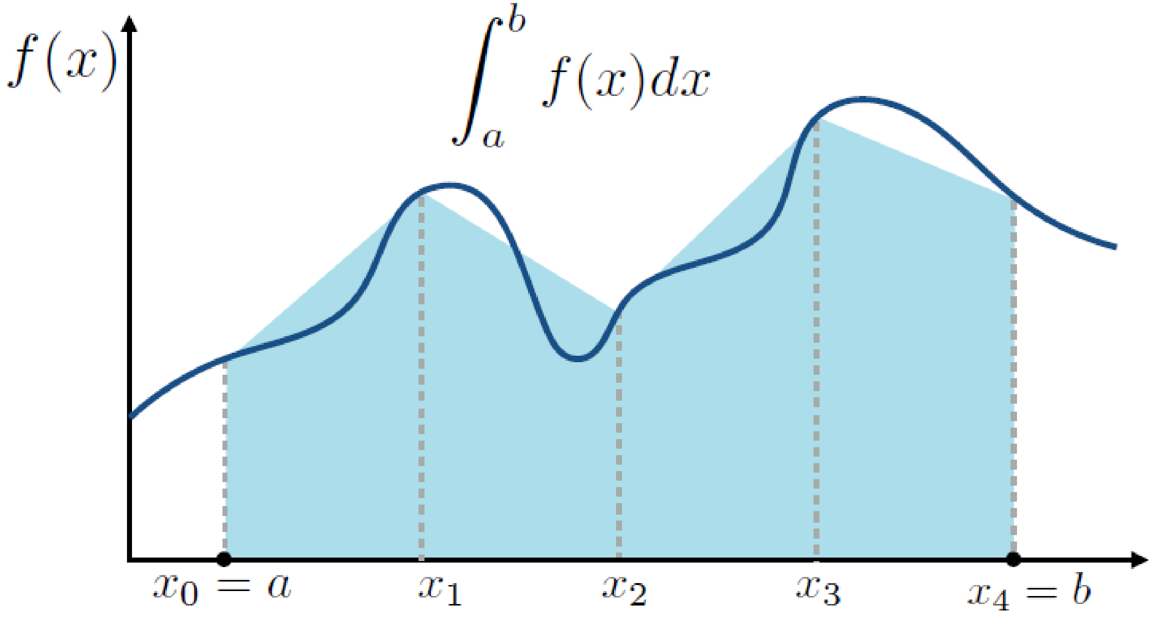


Quadrature Rules for Numerical Integration

E.g. trapezoidal rule: assume function is piece-wise linear

General formula

$$\hat{I} = \sum_{j=1}^{N} w_j \, f(x_j)$$



w_j are the quadrature weights

x_j are the sample positions

Samples position chosen ***deterministically***

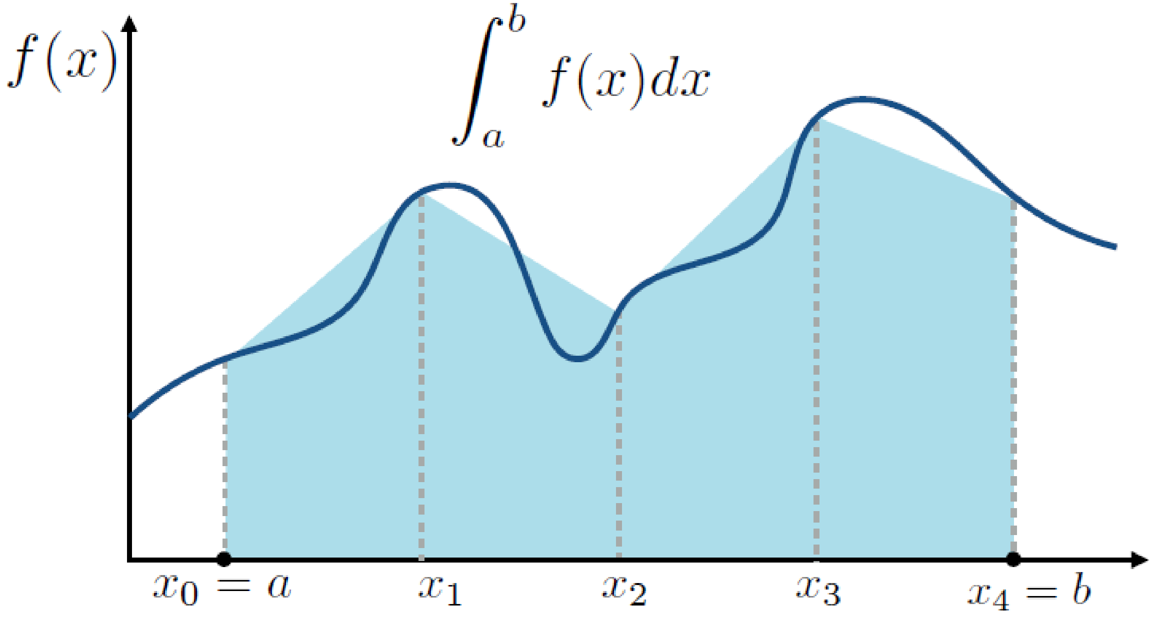
Quadrature Rules for Numerical Integration

Advantages:

- Choice of sample positions and weights
- Can lead to better estimates

Disadvantages:

- Computational overhead
- Scales poorly to higher dimensions

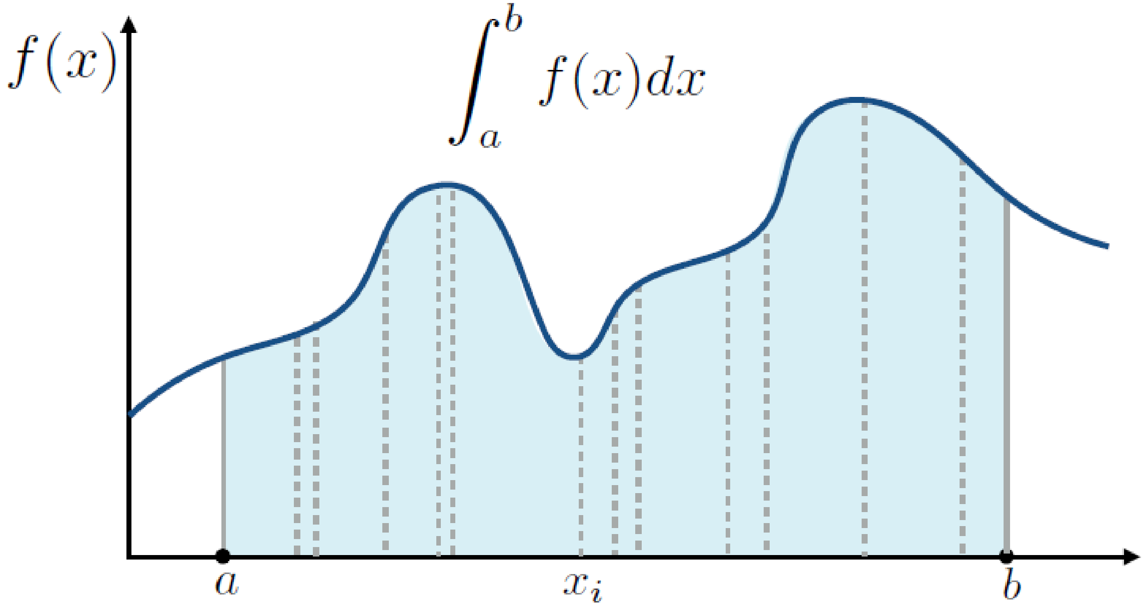


Monte Carlo Integration - Overview

Estimate the integral by ***randomly*** sampling the integrand

General formula

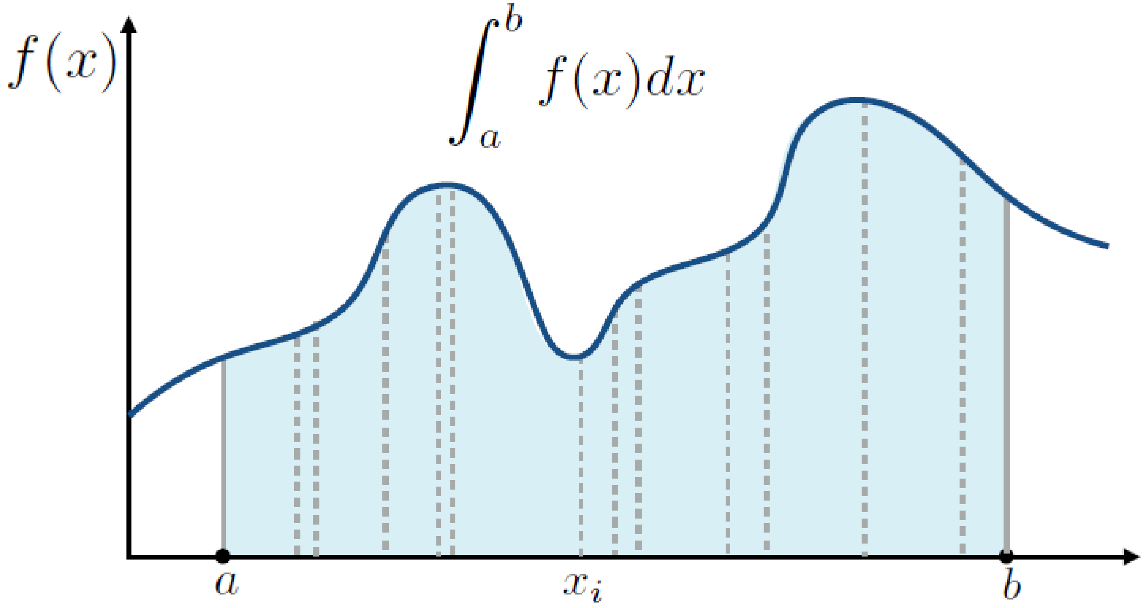
$$\hat{I} = \frac{1}{N} \sum_{j=1}^{N} \frac{f(x_j)}{p(x_j)}$$



N is the number of used samples
x_j is the sample location
$f(x_j)$ is the sample value (or observation)
$p(x_j)$ is the probability of choosing x_j

Monte Carlo Integration - Overview

$$\hat{I} = \frac{1}{N}\sum_{j=1}^{N}\frac{f(x_j)}{p(x_j)}$$



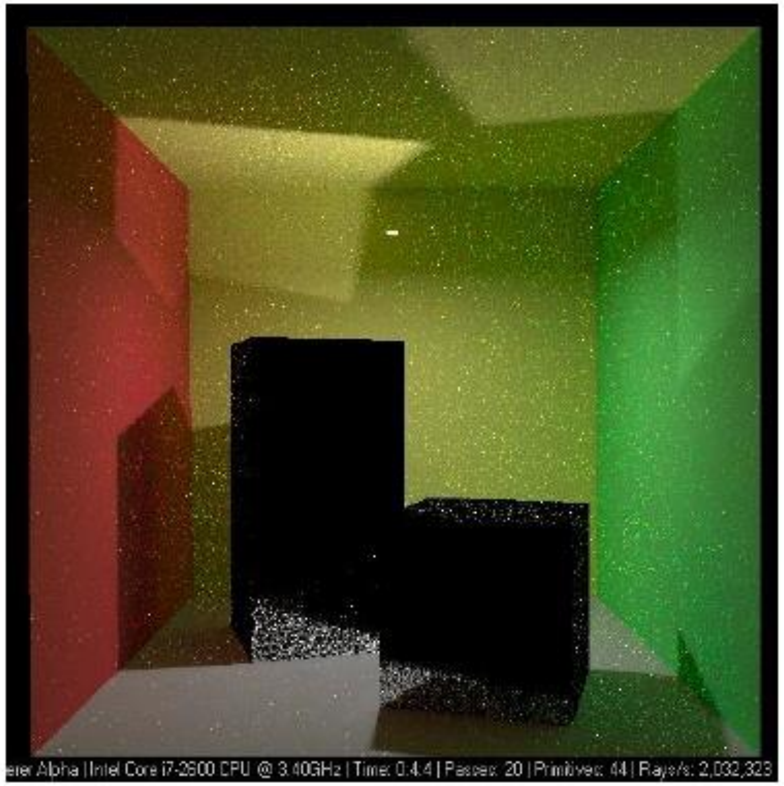
Advantages:

- Simple (both conceptually and in practice)
- Dimension independent
- Works for complex functions

Disadvantages:

- Slow convergence (needs many samples)
- Variance (leads to noisy images)

Monte Carlo Integration Noise



Monte Carlo Estimator

$$I = \int_a^b f(x)\,\mathrm{d}x$$

$$\hat{I} = \frac{1}{N}\sum_{j=1}^{N}\frac{f(x)}{p(x)}$$

How can we characterize $\hat{I}$ as an estimator of I ?

Unbiased Estimator

An estimator is said unbiased if its expected value is the dessired value

$$\mathrm{E}\left[\hat{I}\right] = I$$

Intuition: on average, over multiple estimates, the estimator $\hat{I}$ will yield the correct value of the estimated quantity I

Expected Value of $\hat{I}$ - Proof

$$\mathrm{E}\left[\hat{I}\right] = \mathrm{E}\left[\frac{1}{N}\sum_{j=1}^{N}\frac{f\left(x_j\right)}{p\left(x_j\right)}\right]$$

$$= \frac{1}{N}\ \mathrm{E}\left[\sum_{j=1}^{N}\frac{f\left(x_j\right)}{p\left(x_j\right)}\right]$$

$$= \ \frac{1}{N} N\ \mathrm{E}\left[\frac{f(x)}{p(x)}\right]$$

$$\stackrel{\text{def}}{=} \int_a^b \frac{f(x)}{p(x)} p(x)\ \mathrm{d}x = I$$

Expected Value - Proof

$$\mathrm{E}[\hat{I}] = \mathrm{E}\left[\frac{1}{N}\sum_{j=1}^{N}\frac{f(x_j)}{p(x_j)}\right]$$

$$= \frac{1}{N}\,\mathrm{E}\left[\sum_{j=1}^{N}\frac{f(x_j)}{p(x_j)}\right]$$

$$= \frac{1}{N} N\,\mathrm{E}\left[\frac{f(x)}{p(x)}\right]$$

$$\stackrel{\text{def}}{=} \int_a^b \frac{f(x)}{p(x)} p(x)\,\mathrm{d}x = I$$

$\hat{I}$ is an **unbiased** estimator of I

- No systematic error
- But there is (random) error...

For biased estimators, we have:

$$E[\hat{I}] = I + B,$$

where B is the bias

Variance

$$V[\hat{I}] = V\left[\frac{1}{N}\sum_{j=1}^{N}\frac{f(x_j)}{p(x_j)}\right] = \frac{1}{N^2}\sum_{j=1}^{N} V\left[\frac{f(x_j)}{p(x_j)}\right]$$

$$= \frac{1}{N^2} N\, V\left[\frac{f(x)}{p(x)}\right] = \frac{1}{N}\left(\mathrm{E}\left[\left(\frac{f(x)}{p(x)}\right)^2\right] - \mathrm{E}\left[\frac{f(x)}{p(x)}\right]^2\right)$$

$$= \frac{1}{N}\left(\int_a^b \frac{f(x)^2}{p(x)}\,\mathrm{d}x - I^2\right) = \sigma^2$$

Variance

Unbiased MC estimator:

- Error is due to variance ($B = 0$)

The error $\sigma \propto \frac{1}{\sqrt{N}}$ (slow convergence)

- Need to multiply N by 4 to reduce σ by half

$$V[\hat{I}] = V\left[\frac{1}{N}\sum_{j=1}^{N}\frac{f(x_j)}{p(x_j)}\right] = \frac{1}{N^2}\sum_{j=1}^{N} V\left[\frac{f(x_j)}{p(x_j)}\right]$$

$$= \frac{1}{N^2} N\, V\left[\frac{f(x)}{p(x)}\right] = \frac{1}{N}\left(\mathrm{E}\left[\left(\frac{f(x)}{p(x)}\right)^2\right]\right.$$

$$= \frac{1}{N}\left(\int_a^b \frac{f(x)^2}{p(x)}\,\mathrm{d}x - I^2\right) = \sigma^2$$

Lecture's Summary

Motivated the need for MC methods in rendering

- Estimate the illumination integral

History of MC and Applications

Details on the MC integration method

- General case
- Next week: application to physically-based rendering

Course Roadmap

Week no.	Class Date	Theory	Labs	Notes
Week 1	22/02/2022	1.1 Intro	-	-
Week 2	02/03/2022	1.2 Ray-Tracing Phong	Lab 1	P1 Phong
Week 3	08/03/2022	1.3 Radiometry and BRDF	Lab 2	
Week 4	**15/03/2022**	**2.1 Monte Carlo (MC)**	**Lab 3**	**P2 MC**
Week 5	22/03/2022	2.2 MC Rendering	Lab 4	
Week 6	29/03/2022	2.3 Bayesian Monte Carlo (BMC)	Lab 5	P3 BMC
	05/04/2022	PARTIAL EXAMS (NO CLASS)		
	12/04/2022	EASTER HOLYDAYS (NO CLASS)		
Week 7	19/04/2022	2.4 BMC Rendering	Lab 6	
Week 8	26/04/2022	3.1 MC Importance Sampling	Lab 7	P4 AMC
Week 9	03/05/2022	3.2 BMC Importance Sampling	Lab 8	Papers Out
	10/05/2022	AI SEMINAR MAI (NO CLASS)		
Week 10	17/05/2022	Presentations I		
Week 11	24/05/2022	Presentations II		

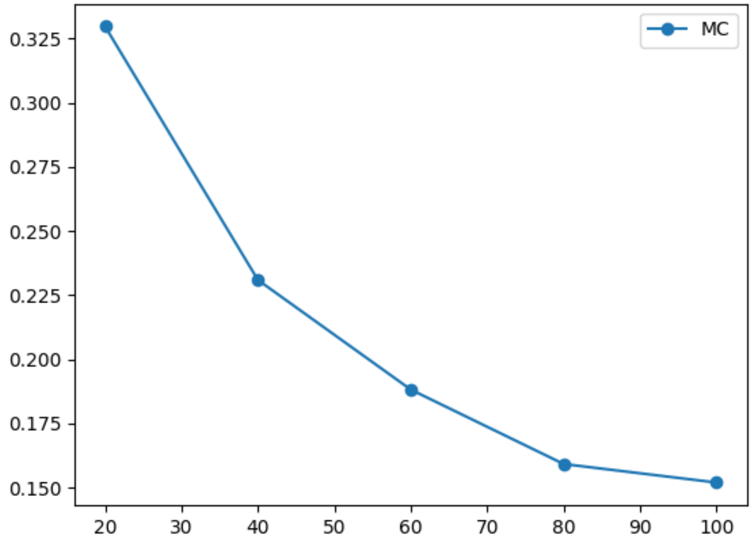
Hands on the Code!

Download the PDF with the Practice 2

- Read it carefully before starting

Download the base code

Apply the concepts related to MC estimates



The End

Questions?