Question 1

$$A = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{bmatrix}$$

(**a**) What is the rank of A.

$$\begin{bmatrix} 2 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{bmatrix} \xrightarrow{2r_2-3r_1} \begin{bmatrix} 2 & 3 & 5 & 7 \\ 0 & 1 & -1 & -3 \\ 5 & 7 & 9 & 11 \end{bmatrix} \xrightarrow{2r_3-5r_1} \begin{bmatrix} 2 & 3 & 5 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & -7 & -13 \end{bmatrix} \xrightarrow{r_3+r_2} \begin{bmatrix} 2 & 3 & 5 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -8 & -16 \end{bmatrix}$$

It is clear that in row echelon form, there are 3 pivots, the dimension of the row space is 3 and therefore the rank is 3.

(**b**) Find a basis for the solution space of A.

$$\begin{bmatrix} 2 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As we have seen before, this matrix A is reduced to:

$$\left[\begin{array}{cccc|c} 2 & 3 & 5 & 7 & 0 \\ 0 & 1 & -1 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \quad \therefore \quad \begin{array}{ccccc} 2x_1 & +3x_2 & +5x_3 & & = 0 \\ & x_2 & -x_3 & -3x_4 = & 0 \\ & & x_3 & +2x_4 = & 0 \end{array} \quad \therefore x_3 = -2x_4$$

By substituting $x_3 = -2x_4$ back into the other equations we find we only have two remaining variables:

$$\begin{aligned} 2x_1 + 3x_2 - 10x_4 &= 0 \\ x_2 &= 5x_4 \\ x_3 &= -2x_4 \end{aligned} \quad \therefore \quad \begin{aligned} 2x_1 &= -13x_4 \\ x_2 &= 5x_4 \\ x_3 &= -2x_4 \end{aligned}$$

If we let $x_4 = a$ then:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{-13}{2} \\ 5 \\ -2 \\ 1 \end{bmatrix} \cdot a$$

Therefore a basis B for the solution space of the matrix A is:

$$B = \left\{ \left(\frac{-13}{2}, 5, -2, 1 \right) \right\}$$

(**c**) Find a basis for the row space of A.

By row reducing A as we have previously seen:

$$\begin{bmatrix} 2 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -8 & -16 \end{bmatrix} \quad \therefore B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -8 \\ -16 \end{bmatrix} \right\}$$

(**d**) State the theorem relating the dimensions of the solution space to the rank of the matrix. Verify that this theorem holds for A.

The rank-nullity theorem states that: $Rank(A) + Nullity(A) = dim\,V$. In the case of A:

$$Rank(A) = 3,\ Nullity(A) = 1,\ 3 + 1 = 4 = dim\,V$$

Question 2

Let $V = (-1, 1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$, and define the operations

$$x * y = \frac{x + y}{1 + xy} \qquad \text{(vector addition)}$$

$$a \odot x = \frac{(1 + x)^\alpha - (1 - x)^\alpha}{(1 + x)^\alpha + (1 - x)^\alpha} \qquad \text{(scalar multiplication)}$$

Where $x, y \in V$ and $a \in \mathbb{R}$. Determine whether V, with the operations listed is a vector space over $\mathbb{R}$.

Axiom 1 ($x + y$ is in V)

$$x * y = \frac{x + y}{(1 - x)(1 - y) + x + y}$$

As $x + y \to 2$, $x * y \to 1$.
As $x + y \to -2$, $x * y \to -1$.

Axiom 2 ($x + y = y + x$)

$$x * y = \frac{x + y}{1 + x \cdot y} = \frac{y + x}{1 + y \cdot x} = y * x \qquad \text{(fulfills axiom 2)}$$

Axiom 3 $(u + v) + w = u + (v + w)$

$$(u * v) * w = \left(\frac{u + v}{1 + u \cdot v}\right) * w = \frac{\frac{u+v}{1+uv} + w}{1 + w\frac{u+v}{1+uv}} = \frac{\frac{u+v}{1+uv} + w\frac{1+uv}{1+uv}}{\frac{1+uv}{1+uv} + w\frac{u+v}{1+uv}} = \frac{u + v + w + uvw}{1 + uv + wu + wv} = \frac{v + w + u(1 + wv)}{1 + wv + u(w + v)}$$

$$\therefore \frac{\frac{v+w}{1+wv} + v}{1 + u\frac{w+v}{1+vw}} = u * \frac{w + v}{1 + wv} = u * (w * v) \qquad \text{(fulfills axiom 3)}$$

Axiom 4 There is a zero vector in V where $u + 0 = u$

$$0 * u = \frac{u + 0}{1 + u \cdot 0} = \frac{u}{1} = u \qquad \text{(fulfills axiom 4)}$$

Axiom 5 For each vector u in V there is a vector $-u$ satisfying $u + (-u) = 0$.

$$u * -u = \frac{u - u}{1 + u^2} = \frac{0}{1 = u^2} = 0 \qquad \text{(fulfills axiom 5)}$$

Axiom 6 $\alpha \odot x$ is in V

$$\text{As } x \to 1, \qquad (1 + x)^\alpha \to 2^\alpha \qquad (1 - x)^\alpha \to 0 \qquad \therefore \frac{2^\alpha - 0}{2^\alpha + 0} \to 1$$

$$\text{As } x \to -1, \qquad (1 + x)^\alpha \to 0 \qquad (1 - x)^\alpha \to 2^\alpha \qquad \therefore \frac{0 - 2^\alpha}{0 + 2^\alpha} \to -1$$

$$\text{When } x = 0, \qquad \frac{(1)^\alpha - (1)^\alpha}{(1)^\alpha + (1)^\alpha} = 0. \qquad \text{(fulfills axiom 6)}$$

Axiom 7 $\alpha \odot (u+v) = \alpha \odot u + \alpha \odot v$
Let $u = \frac{1}{4}$ and $v = \frac{1}{2}$:

$$u * v = \frac{\frac{1}{2}+\frac{1}{4}}{1+\frac{1}{2}\cdot\frac{1}{4}} = \frac{\frac{3}{4}}{\frac{9}{8}} = \frac{2}{3}$$

Now, let $\alpha = 4$:

$$4 \odot \left(\frac{1}{2} * \frac{1}{4}\right) = \frac{\left(1+\frac{2}{3}\right)^4 - \left(1-\frac{2}{3}\right)^4}{\left(1+\frac{2}{3}\right)^4 + \left(1-\frac{2}{3}\right)^4} = \frac{312}{313}$$

Now, if we take $\alpha \odot \frac{1}{2} + \alpha \odot \frac{1}{4}$:

$$4 \odot \frac{1}{2} + 4 \odot \frac{1}{4} = \frac{\left(1+\frac{1}{2}\right)^4 - \left(1-\frac{1}{2}\right)^4}{\left(1+\frac{1}{2}\right)^4 + \left(1-\frac{1}{2}\right)^4} + \frac{\left(1+\frac{1}{4}\right)^4 - \left(1-\frac{1}{4}\right)^4}{\left(1+\frac{1}{4}\right)^4 + \left(1-\frac{1}{4}\right)^4} = \frac{25272}{14473}$$

Clearly, they are not equal and therefore V doesn't fulfil the requirements to be a vector space.

Question 3

(**a**) Find a subset of the set

$$\mathcal{S} = \{(0,-1,-3,3),(-1,-1,-3,2),(3,1,3,0),(0,-1,-2,1)\}$$

of vectors in $\mathbb{R}^4$, that form a basis for $Span(\mathcal{S})$.

The basis vectors can be described as a matrix:

$$\mathcal{S} = \begin{bmatrix} 0 & -1 & -3 & 3 \\ -1 & -1 & -3 & 2 \\ 3 & 1 & 3 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

When row-reduced this becomes:

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & 1 & 0 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore a subset of $\mathcal{S}$, designated as basis $\mathcal{B}$ which forms a subspace of $\mathbb{R}^3$ in $\mathbb{R}^4$ and is a basis for $Span(\mathcal{S})$ is:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(**b**) Show that the set

$$\mathcal{B} = \{(-1,-3,-8,6),(-1,0,-1,1),(-1,-2,-6,5)\}$$

is another basis for $Span(\mathcal{S})$.

$\mathcal{B}$ can be described in matrix form and let this new basis be $\mathcal{M}$:

$$\mathcal{M} = \begin{bmatrix} -1 & -3 & -8 & 6 \\ -1 & -0 & -1 & 1 \\ -1 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\mathcal{M} = \left\{ \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-2\end{bmatrix} \right\}$$

It is clear this is similar to the result we found in (**a**), this set is indeed another basis for $Span(\mathcal{S})$. (**c**) Let $\mathbf{v} = (3, 7, 20, -16) \in \mathbb{R}^4$. Write the coordinate vector of $\mathbf{v}$ with respect to

(i) the basis found in part (**a**);

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&3&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}3\\7\\20\\-16\end{bmatrix} \sim \left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&7\\0&0&1&20\\1&3&-2&-16\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&7\\0&0&1&20\\0&0&0&0\end{array}\right] \therefore a = 3, b = 7, c = 20$$

$$\therefore \mathbf{v} = 3\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + 7\begin{bmatrix}0\\1\\0\\3\end{bmatrix} + 20\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}$$

(ii) the basis $\mathcal{B}$.

$$\mathbf{v} = a\begin{bmatrix}-1&-1&-1\\-3&0&-2\\-8&-1&-6\\0&1&5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}3\\7\\20\\-16\end{bmatrix} \sim \left[\begin{array}{ccc|c}-1&-1&-1&3\\-3&0&-2&7\\-8&-1&-6&20\\6&1&5&-16\end{array}\right] \xrightarrow{r_4+2r_2} \left[\begin{array}{ccc|c}-1&-1&-1&3\\-3&0&-2&7\\-8&-1&-6&20\\0&1&1&-2\end{array}\right] \xrightarrow{r_2-3r_1}$$

$$\left[\begin{array}{ccc|c}-1&-1&-1&3\\0&3&1&-5\\-8&-1&-6&20\\0&1&1&-2\end{array}\right] \xrightarrow{r_3-8r_1} \left[\begin{array}{ccc|c}-1&-1&-1&3\\0&3&1&-5\\0&7&2&-4\\0&1&1&-2\end{array}\right] \xrightarrow{r_3-2r_4} \left[\begin{array}{ccc|c}-1&-1&-1&3\\0&3&1&-5\\0&5&0&0\\0&1&1&-2\end{array}\right] \xrightarrow{r_2-r_4} \left[\begin{array}{ccc|c}-1&-1&-1&3\\0&2&0&-3\\0&5&0&0\\0&1&1&-2\end{array}\right]$$

$$\xrightarrow{2r_4-r_2} \left[\begin{array}{ccc|c}-1&-1&-1&3\\0&1&0&0\\0&0&1&-2\\0&2&0&-3\end{array}\right] \xrightarrow{r_1+r_2} \left[\begin{array}{ccc|c}-1&0&-1&3\\0&1&0&0\\0&0&1&-2\\0&2&0&-3\end{array}\right] \xrightarrow{r_1+r_3} \left[\begin{array}{ccc|c}-1&0&0&1\\0&1&0&0\\0&0&1&-2\\0&2&0&-3\end{array}\right] \therefore a = -1, b = 0, c = -2$$

To test these values for a, b, c we can back-substitute them into the original equation.

$$\mathbf{v} = -1\begin{bmatrix}-1\\-3\\-8\\6\end{bmatrix} + 0\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix} - 2\begin{bmatrix}-1\\-2\\-6\\5\end{bmatrix} = \begin{bmatrix}1+2\\3+4\\8+12\\-6-10\end{bmatrix} = \begin{bmatrix}3\\7\\20\\-16\end{bmatrix}$$

(**d**) Consider the set of polynomials

$$\mathcal{P} = \{-1 - 3x - 8x^2 + 6x^3, \quad -1 - x^2 + x^3, \quad -1 - 2x - 6x^2 + 5x^3\}$$

Can the polynomial $p(x) = -3 - 5x - 15x^2 + 12x^3$ be written as a linear combination of the polynomials in $\mathcal{P}$ in more than one way?

$$\begin{bmatrix}-1&-1&-1\\-3&0&-2\\-8&-1&-6\\6&1&5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}-3\\-5\\-15\\12\end{bmatrix} \sim \left[\begin{array}{ccc|c}-1&-1&-1&-3\\-3&0&-2&-5\\-8&-1&-6&-15\\6&1&5&12\end{array}\right] \xrightarrow{r_4+2r_2} \left[\begin{array}{ccc|c}-1&-1&-1&-3\\-3&0&-2&-5\\-8&-1&-6&-15\\0&1&1&2\end{array}\right]$$

$$\xrightarrow{r_2-3r_1} \left[\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ -3 & 0 & -2 & -5 \\ -8 & -1 & -6 & -15 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_3-8r_1} \left[\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ -3 & 0 & -2 & -5 \\ 0 & 7 & 2 & 9 \\ 0 & 1 & -3 & 12 \end{array}\right] \xrightarrow{r_2-3r_1} \left[\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ 0 & 3 & 1 & 4 \\ 0 & 7 & 2 & 9 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{7r_2-3r_3} \left[\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 7 & 2 & 9 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_3-2r_2} \left[\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 7 & 0 & 7 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_4-(r_2+r_3)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_1-(r_2+r_3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \therefore a=1, b=1, c=1$$

To check that this satisfies $p(x)$, we can substitute the values into the original equation:

$$\begin{bmatrix} -1 \\ -3 \\ -8 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ -15 \\ 12 \end{bmatrix} = p(x)$$

Since the equation above has a single solution, there is only one way $p(x)$ can be written in terms of $\mathcal{P}$.

Question 4

Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle -, - \rangle : V \times V \to V$.
Let W be a subspace of V. Define the set

$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$$

(**a**) Show that $W^{\perp}$ is a subspace of V.

Let $\{v_1, v_2, w_1\} \in W^{\perp}$

(1) $\quad \langle v_1 + v_2, w_1 \rangle = \langle v_1, w_1 \rangle + \langle v_2, w_1 \rangle = 0 + 0 = 0 \qquad \therefore$ (Closed under addition)

(2) $\quad \alpha \cdot \langle v_1, w_1 \rangle = \alpha \cdot 0 = 0 \qquad \therefore$ (Closed under multiplication)

(3) $\quad \langle v_1, w_1 \rangle = 0 \qquad \therefore$ (Non empty)

(**b**) Prove that $dim\ W + dim\ W^{\perp} = dim\ V$

Recalling the rank nullity theorem from earlier:

$$rank\ A + Nullity\ A = dim\ V$$

Where V is some vector space over $\mathbb{R}^n$.
Suppose we create a matrix B which spans W.

$$B = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{bmatrix}$$

$$\therefore rank\ B^T + nullity\ B^T = dim\ V$$

But $rank\ B^T = rank\ B$, so:

$$\therefore rank\ B + nullity\ B^T = dim\ V$$

Rank of A is just the dimension of the column space of A and nullity the dimension of the null space so:

$$\therefore dim\ C(A) + dim\ N(A^T) = dim\ V$$

We can recall that $C(A)$ spans W and $N(A^T) = C(A)^\perp = W^\perp$.

$$\therefore dim\ W + dim\ W^\perp = dim\ V$$