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Question 1

Consider the function:

$$f(x) = \begin{cases} \frac{\log(1-2x)}{x} & x < 0 \\ a & x = 0 \\ x^2 \cos\left(\frac{1}{x}\right) & x > 0 \end{cases}$$

For which values of $a, b \in \mathbb{R}$ is f continuous at $x = 0$?

At the point $x = 0$, the following should be true for f to be continuous:

$$\lim_{x \to 0^-} \frac{\log(1-2x)}{x} = \lim_{x \to 0} a = \lim_{x \to 0^+} x^2 \cos\left(\frac{1}{x}\right)$$

We can't just take the limits of f when $x > 0$ and $x < 0$ directly, we need to use some tricks first.

For $f(x)$ when $x < 0$, we can use l'hopital's rule, take the derivative of both the top and the bottom and happy days:

$$\lim_{x \to 0^-} \frac{\log(1-2x)}{x} = \lim_{x \to 0^-} \frac{\frac{-2}{1-2x}}{1} = \lim_{x \to 0^-} \frac{-2}{1-2x} = -2$$

When $x = 0$, evaluating the limit yields:

$$\lim_{x \to 0} a = a$$

Finally to evaluate when $x > 0$, we can use the sandwich theorem:

$$\lim_{x \to 0^+} x^2 \cos\left(\frac{1}{x}\right) + b$$

$$x^2 + b \geq x^2 \cos\left(\frac{1}{x}\right) + b \geq -x^2 + b$$

Then:

$$\lim_{x \to 0^+} x^2 + b = \lim_{x \to 0^+} x^2 \cos\left(\frac{1}{x}\right) + b = \lim_{x \to 0^+} -x^2 + b = b$$

Therefore, combining our results, we find that:

$$-2 = a = b$$

So

$$a = -2 \qquad b = -2$$

Question 2

Evaluate the following integrals:

(a)

$$\int (\log x)^2 dx$$

Using integration by parts we find:

$$\int u\ dv = uv - \int v\ du$$

$$u = \log^2 x \qquad du = \frac{2\log x}{x} \text{ and } dv = 1\ dx \qquad v = x$$

$$\therefore \int \log^2 x\ dx = x\log^2 x - \int 2\log x\ dx$$

Next, we must take the integral:

$$\int \log x\ dx$$

So:

$$u = \log x \qquad du = \frac{1}{x} \text{ and } dv = \ dx \qquad v = x$$

$$\therefore \int \log x = x\log x - x$$

Therefore, if we put all the parts together we find that:

$$\begin{aligned}\int \log^2 x\ dx =& x\log^2 x - 2(x\log x - x) + C \\ =& x\log^2 x - 2x\log x + 2x + C \\ =& x(\log^2 x - 2\log x + 2) + C\end{aligned}$$

Where $C \in \mathbb{R}$.

(b)

$$\int_0^1 \frac{\cosh x}{\sinh^2 x + 2\sinh x + 2} dx$$

Since:

$$\frac{dy}{dx}\sinh x = \cosh x$$

We can apply the substitution let $u = \sinh x$, $du = \cosh x\ dx$.
Don't forget the terminals: $\sinh 1$ and $\sinh 0 = 0$

$$\begin{aligned}&= \int_0^{\sinh 1} \frac{1}{u^2 + 2u + 2} du \\ &= \int_0^{\sinh 1} \frac{1}{(u+1)^2 + 1} du \\ &= [\arctan(u+1)]_0^{\sinh 1} \\ &= \arctan(\sinh 1 + 1) - \frac{\pi}{4}\end{aligned}$$

Question 3

(a) Show that

$$\int \sec^n x\ dx = \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$$

We can use integration by parts to determine the reduction of $\int \sec^n x\ dx$:

$$\int \sec^n x\ dx = \int \sec^2 x \sec^{n-2} x\ dx$$

If we let:

$$du = \sec^2 x\ dx \qquad u = \tan x \text{ and } v = \sec^{n-2} x \qquad dv = (n-2)\sin x \cos x \sec^n x\ dx$$

$$\therefore \int \sec^n x\ dx = \tan x \sec^{n-2} x - \int \tan x (n-2) \sin x \cos x \sec^n x\ dx$$

$$\therefore \int \sec^n x\ dx = \tan x \sec^{n-2} x - (n-2) \int \frac{\sin^2 x}{\cos^n x}\ dx$$

Recall that $\sin^2 x + \cos^2 x = 1$ then:

$$\therefore \int \sec^n x\ dx = \tan x \sec^{n-2} x - (n-2) \int \frac{1 - \cos^2 x}{\cos^n x}\ dx$$

$$\therefore \int \sec^n x\ dx = \tan x \sec^{n-2} x - (n-2) \left[\int \frac{1}{\cos^n x}\ dx - \int \frac{\cos^2 x}{\cos^n x}\ dx \right]$$

$$\therefore \int \sec^n x\ dx = \tan x \sec^{n-2} x - (n-2) \left[\int \sec^n x\ dx - \int \sec^{n-2} x\ dx \right]$$

$$\therefore \int \sec^n x\ dx = \tan x \sec^{n-2} x - (n-2) \int \sec^n x\ dx + (n-2) \int \sec^{n-2} x\ dx$$

Rearrange to get all our $\sec^n x$ to one side:

$$\therefore \int \sec^n x\ dx + (n-2) \int \sec^n x\ dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x\ dx$$

And factorize to yield:

$$\therefore \int \sec^n x\ dx (n-2+1) = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x\ dx$$

Which yields our final formula:

$$\therefore \int \sec^n x\ dx = \frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} \int \sec^{n-2} x\ dx$$

(b) Evaluate

$$\int \sec^5 x\ dx$$

Use the formula from **(a)**:

$$\int \sec^5 x\ dx = \frac{1}{4} \tan x \sec^3 x + \frac{3}{4} \int \sec^3 x\ dx$$

Apply again to $\sec^3 x$:

$$\int \sec^3 x\ dx = \frac{1}{2} \tan x \sec x + \frac{3}{4} \int \sec x\ dx$$

We know:

$$\int \sec x\ dx = \log |\sec x + \tan x| + C$$

Therefore:

$$\int \sec^5 x\ dx = \frac{1}{4} \tan x \sec^3 x + \frac{3}{8} \tan x \sec x + \frac{3}{4} \log |\sec x + \tan x| + C$$

Where $C \in \mathbb{R}$.

Question 4

Use an appropriate substitution to evaluate the following integral

$$\int \sqrt{x^2 + 4x + 3}\; dx$$

First let's express as a perfect square:

$$\int \sqrt{(x+2)^2 - 1}\; dx$$

Now, let $x + 2 = \cosh u$

$$\frac{dx}{du} = \sinh u$$

$$\int \sqrt{\cosh^2 u - 1}\, \sinh u\; du$$

Remembering that $\cosh^2 u - \sinh^2 u = 1$:

$$\int \sqrt{\sinh^2 u}\, \sinh u\; du$$

$$\int \sinh^2 u\; du$$

Rearranging the formula $\cosh 2u = 1 + 2\sinh^2 u$:

$$\int \frac{\cosh 2u - 1}{2}\; du = \frac{1}{2} \int \cosh 2u - 1\; du$$

$$\frac{1}{2}\left(\frac{1}{2}\sinh 2u - u\right) + C$$

Recall that $u = \operatorname{arccosh}(x+2)$

$$\frac{1}{4}\sinh(2\operatorname{arccosh}(x+2)) - \frac{\operatorname{arccosh}(x+2)}{2} + C$$

Where $C \in \mathbb{R}$.