UMEP Linear Algebra
Assignment 3

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Q1.

Let V be an inner product space with inner product $\langle -, - \rangle : V \times V \to \mathbb{R}$.
(a) For any arbitrary $\mathbf{v} \in V$, show that the function $\varphi_{\mathbf{v}} : V \to \mathbb{R}$ given by

$$\varphi_{\mathbf{v}}(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$$

is a linear transformation.

For linearity we must prove:

$$\varphi_{\mathbf{v}}(\mathbf{w} + \mathbf{a}) = \langle \mathbf{v}, \mathbf{w} + \mathbf{a} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{a} \rangle = \varphi_{\mathbf{v}}(\mathbf{w}) + \varphi_{\mathbf{v}}(\mathbf{a})$$

And

$$\varphi_{\mathbf{v}}(\alpha \cdot \mathbf{w}) = \langle \mathbf{v}, \alpha \cdot \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{w} \rangle = \alpha \varphi_{\mathbf{v}}(\mathbf{w})$$

Therefore the inner product is a linear transformation.
(b) When $V = \mathbb{R}^3$, where the inner product is the usual dot product, and $\mathbf{v} = (1, 2, 3)$, find the matrix for the linear transformation $\varphi_{\mathbf{v}}$ with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}$.

A matrix of dimension 1×3 could be constructed as a transformation matrix for $\varphi_{\mathbf{v}}$ and vectors in $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix}$$

(c) What can you notice about the matrix for $\varphi_{\mathbf{v}}$? For arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, how can we write down the dot product of $\mathbf{x} \cdot \mathbf{y}$ in terms of matrix multiplication?

The matrix for $\varphi_{\mathbf{v}}$ is a $1 \times n$ matrix where $\mathbf{v}$ is a vector in $\mathbb{R}^n$.

The dot product can be written as:

$$\mathbf{x}^T \cdot \mathbf{y}$$

$$\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_n \end{bmatrix} \cdot \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

Where x is a row vector of dimension $1 \times n$ and y is a column vector of $n \times 1$.

Q2.

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Find the eigenvalues, and associated eigenspaces of the matrix A. State the dimensions of the eigenspaces.

To find the eigenvalues we must evaluate $\text{Det}(\lambda \cdot I_4 - A) = 0$

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \lambda & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \lambda & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \lambda \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \lambda & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \lambda & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \lambda \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \lambda & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \lambda & 0 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} \lambda & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \lambda & 0 \end{vmatrix} - \lambda\begin{vmatrix} \lambda & 0 & \frac{1}{2} \\ 0 & \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \lambda \end{vmatrix}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} - \lambda\begin{vmatrix} 0 & -\frac{1}{2} \\ \lambda & -\frac{1}{2} \end{vmatrix}\right) + \frac{1}{2}\left(\frac{1}{2}\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} - \lambda\begin{vmatrix} \lambda & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{vmatrix}\right) - \lambda\left(\frac{1}{2}\begin{vmatrix} 0 & \frac{1}{2} \\ \lambda & \frac{1}{2} \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \lambda & \frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} + \lambda\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{2}(0) - \lambda\left(\frac{\lambda}{2}\right)\right) + \frac{1}{2}\left(\frac{1}{2}(0) - \lambda\left(-\frac{\lambda}{2}\right)\right) - \lambda\left(\frac{1}{2}\left(\frac{-\lambda}{2}\right) - \frac{1}{2}\left(\frac{\lambda}{2}\right) + \lambda\left(\lambda^2\right)\right)$$

$$= -\frac{1}{2}\left(\frac{\lambda^2}{2}\right) + \frac{1}{2}\left(\frac{-\lambda^2}{2}\right) - \lambda\left(-\frac{\lambda}{4} - \frac{\lambda}{4} + \lambda^3\right)$$

$$= -\frac{\lambda^2}{4} - \frac{\lambda^2}{4} - \frac{\lambda^2}{2} - \lambda^4$$

Then

$$0 = -\frac{\lambda^2}{2} - \frac{\lambda^2}{2} - \lambda^4$$

$$0 = \lambda^4 - \lambda^2$$

$$\therefore \lambda = 1, -1, 0$$

Now, to find the eigenvectors we must find the nullspace of lambda matrix thing.

When $\lambda = 1$:

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 \end{array}\right] \xrightarrow{2R_3, 2R_4} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 2 & 0 \\ -1 & -1 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_4+(R_1+R_2)} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1-(\frac{1}{2}R_4)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2-(\frac{1}{2}R_4)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

So:

$$x_1 = x_4, \qquad x_2 = x_4, \qquad x_3 = -x_4$$

Let $x_4 = \alpha$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \quad \text{Forming a one dimensional eigenspace.}$$

$$\text{When } \lambda = 1\text{, the eigenvector is: } \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 0$:

$$\left[\begin{array}{cccc|c} 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_1, 2R_2, 2R_3, 2R_4} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{array}\right]$$

So:

$$x_1 = -x_2, \qquad x_3 = x_4$$

Let $x_1 = \alpha$ and $x_2 = \beta$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \alpha, \beta \in \mathbb{R}, \quad \text{Forming a two dimensional eigenspace.}$$

$$\text{When } \lambda = 0\text{, the eigenvectors are: } \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

When $\lambda = -1$:

$$\left[\begin{array}{cccc|c} -1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 \end{array}\right] \xrightarrow{2R_1, 2R_2, 2R_3, 2R_4} \left[\begin{array}{cccc|c} -2 & 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ -1 & -1 & 0 & -2 & 0 \end{array}\right] \xrightarrow{R_3 + R_4, 2R_4} \left[\begin{array}{cccc|c} -2 & 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ -2 & -2 & 0 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_4, 2R_4} \left[\begin{array}{cccc|c} -2 & 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right]$$

Then let $x_4 = \alpha$:

$$-2x_1 + x_3 - x_4 = 0, \qquad -2x_2 + x_3 - x_4 = 0, \qquad x_3 = x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \quad \text{Forming a one dimensional eigenspace.}$$

$$\text{When } \lambda = -1\text{, the eigenvector is: } \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

The eigenvectors for A are:

$$\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Q3.

Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree less than or equal to three with real coefficients, and $M^{2,2}$ is the vector space of 2×2 matrices with real entries. Consider the function $T : \mathcal{P}_3 \to M^{2,2}$ given by

$$T(p) = p(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + p(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$

(a) Prove that T is a linear transformation.

Let there be two polynomials in $\mathcal{P}_3 : p, q$.

$$T(p+q) = (p+q)(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + (p+q)(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$

$$\begin{aligned}T(p+q) &= p(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + q(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + p(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + q(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\\ &= p(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + p(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + q(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + q(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\\ &= T(p) + T(q)\end{aligned}$$

And, let $\alpha \in \mathbb{R}$.

$$\begin{aligned}T(\alpha \cdot p) &= (\alpha \cdot p)(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + (\alpha \cdot p)(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\\ &= \alpha \cdot p(1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + \alpha \cdot p(-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\\ &= \alpha \cdot T(p)\end{aligned}$$

Therefore, T is a linear transformation.

(b) Find the matrix for the linear transformation T with respect to the basis

$$\mathcal{B} = \left\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\right\}$$

For $\mathcal{P}_\ni$ and the standard basis

$$\mathcal{S} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

for $M^{2,2}$.

The transformation matrix for T can be found by squashing together the basis vectors from $\mathcal{B}$ to $\mathcal{S}$.

$$[T]_{B,S} = \left[[T(1)]_{\mathcal{S}} \quad [T(1+x)]_{\mathcal{S}} \quad [T(1+x+x^2)]_{\mathcal{S}} \quad [T(1+x+x^2+x^3)]_{\mathcal{S}}\right]$$

Then:

$$T(1) = 1\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + 1\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}$$

$$T(1+x) = (1+1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + (1-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & -2\end{bmatrix}$$

$$T(1+x+x^2) = (1+1+1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + (1-1+1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}3 & 1\\-1 & -3\end{bmatrix}$$

$$T(1+x+x^2+x^3) = (1+1+1+1)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + (1-1+1-1)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & -4\end{bmatrix}$$

Therefore, when columnvectorify these matrices and squash them together:

$$[T]_{\mathcal{B},\mathcal{S}} = \begin{bmatrix}1 & 2 & 3 & 4\\1 & 0 & 1 & 0\\-1 & 0 & -1 & 0\\-1 & -2 & -3 & -4\end{bmatrix}$$

(c) What is the kernel for the linear transformation T? You only need to provide a brief explaination for this.

The kernel for the linear transformation of T, is the nullspace of the transformation matrix, all the things which T transforms to 0.
So:

$$\left[\begin{array}{cccc|c}1 & 2 & 3 & 4 & 0\\1 & 0 & 1 & 0 & 0\\-1 & 0 & -1 & 0 & 0\\-1 & -2 & -3 & -4 & 0\end{array}\right] \xrightarrow{R_4+R_1, R_3+R_2} \left[\begin{array}{cccc|c}1 & 2 & 3 & 4 & 0\\1 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

Let $x_4 = \alpha$ and $x_3 = \beta$.

$$x_1 = -x_3, \quad x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \alpha\begin{bmatrix}0\\-2\\0\\1\end{bmatrix} + \beta\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

$$Ker(T) = span\left(\left\{\begin{bmatrix}0\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}\right\}\right)$$

Q4.

(a) use the Gram Schmidt procedure to turn $\{1, x, x^2\}$ into an orthonormal basis for $\mathcal{P}_2$, with respect to the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$.

We want to transform our original basis, the standard basis $\mathcal{S}$ to a new basis $\mathcal{B}$. Let the original polynomial W.R.T standard basis be a.

$$\therefore b_1 = \frac{a_1}{\sqrt{\langle a_1, a_1\rangle}}$$

$$b_1 = \frac{1}{\sqrt{\int_0^1 1 \cdot 1 dx}} = 1$$

$$a_2^{'} = a_2 - \langle a_2, b_1 \rangle b_1$$

$$\therefore a_2^{'} = x - \langle x, 1 \rangle 1 = x - \int_0^1 x \cdot dx = x - \frac{1}{2}$$

$$b_2 = \frac{a_2^{'}}{\sqrt{\langle a_2^{'}, a_2^{'} \rangle}} = \frac{x - \frac{1}{2}}{\sqrt{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle}} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 x^2 - x + \frac{1}{4} dx}} = \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = \sqrt{3}(2x - 1)$$

$$a_3^{'} = a_3 - \langle a_2, b_1 \rangle b_1 - \langle a_3, b_2 \rangle b_2$$

$$\therefore a_3^{'} = x^2 - \langle x^2, 1 \rangle 1 - \langle x^2, \sqrt{3}(2x-1) \rangle \sqrt{3}(2x-1)$$

$$= x^2 - \int_0^1 x^2 dx \cdot 1 - \int_0^1 x^2 \cdot \sqrt{3}(2x-1) dx \cdot \sqrt{3}(2x-1)$$

$$= x^2 - x + \frac{1}{2} - \frac{1}{3}$$

$$= x^2 - x + \frac{1}{6}$$

$$b_3 = \frac{a_3^{'}}{\sqrt{\langle a_3^{'}, a_3^{'} \rangle}} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6} \rangle}} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 x^4 - 2x^3 + \frac{4x^2}{3} - \frac{x}{3} + \frac{1}{36} dx}}$$

$$= \sqrt{5}(6x^2 - 6x + 1)$$

Our new orthonormal basis $\mathcal{B}$ is now:

$$\left\{ 1, \sqrt{3}(2x-1), \sqrt{5}(6x^2 - 6x + 1) \right\}$$

(b) Prove that if $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, with respect to the usual dot product, then the matrix formed by putting the columns of $\mathcal{B}$ together is orthogonal. That is, show that the matrix

$$M = [\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_n]$$

satisfies

$$M^T M = I$$

where I is the $n \times n$ identity matrix.

Since the basis is orthonormal, the dot product of columns $\mathbf{v}_n \cdot \mathbf{v}_m = 1$ if $m = n$.
If $n \neq m$ then $\mathbf{v}_n \cdot \mathbf{v}_m = 0$. We know from **Q1** that $x^T \cdot y = x \cdot y$ so:

$$M \cdot M = I_n$$

$$M^T \cdot M = I_n$$