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Tutorial Wed 5:15PM

Question 1

Consider the function

$$f(x,y) = \begin{cases} \frac{-2x^4y}{x^4+2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $y = kx$, where $k \neq 0$.

If we take the path $y = kx$, the limit becomes:

$$\begin{aligned} &= \lim_{x\to 0} \frac{-2x^4kx}{x^4 + 2(kx)^2} \\ &= \lim_{x\to 0} \frac{-2kx^5}{x^4 + 2k^2x^2} \end{aligned}$$

Direct substitution of $x = 0$ would yield an indeterminate form of $\frac{0}{0}$. Therefore, we should use L'hopital's rule, and keep applying it until a determinate form is found.

$$\begin{aligned} &= \lim_{x\to 0} \frac{-10kx^4}{4x^3 + 4k^2x} \\ &= \lim_{x\to 0} \frac{-40kx^3}{12x^2 + 4k^2} \\ &= \lim_{x\to 0} \frac{-120kx^2}{24x} \\ &= \lim_{x\to 0} -5kx \\ &= 0 \end{aligned}$$

(b) Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists or justify why the limit doesn't exist.

We could use polar coordinates to evaluate the limit and let $x = r\cos\theta$, $y = r\sin\theta$:

$$\begin{aligned} &= \lim_{(x,y)\to(0,0)} \frac{-2x^4y}{x^4+2y^2} \\ &= \lim_{r\to 0} \frac{-2(r\cos\theta)^4 r\sin\theta}{(r\cos\theta)^4 + 2(r\sin\theta)^2} \\ &= \lim_{r\to 0} \frac{-2r^5\cos^4\theta\sin\theta}{r^4\cos^4\theta + 2r^2\sin^2\theta} \\ &= \lim_{r\to 0} \frac{-2r^3\cos^4\theta\sin\theta}{r^2\cos^4\theta + 2\sin^2\theta} \\ &= \lim_{r\to 0} \frac{-2r^3\cos^4\theta\sin\theta}{r^2\cos^4\theta + 2\sin^2\theta} \\ &= 0 \end{aligned}$$

Therefore the $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(c) Determine where f is C^1. Justify your answer, referring to any theorems you use.

Theorem:
For a function f to be C^n its nth partial derivatives must exist and be continuous over the domain of f.

Let $f_1 = \frac{-2x^4y}{x^4+2y^2}$ and $f_2 = 0$.

$$\begin{aligned}
\frac{\partial f_1}{\partial x} =& \frac{(x^4+2y^2)(-8x^3y)+(2x^4y)(4x^3)}{(x^4+2y^2)^2} = \frac{-16x^3y^3}{(x^4+2y^2)^2} \\
\frac{\partial f_1}{\partial y} =& \frac{(x^4+2y^2)(-2x^4)+(2x^4y)(4y)}{(x^4+2y^2)^2} = \frac{2(2y^2-x^4)x^4}{(x^4+2y^2)^2} \\
\frac{\partial f_2}{\partial x} =& 0 \\
\frac{\partial f_1}{\partial y} =& 0
\end{aligned}$$

To check that our partial derivatives are continuous over the domain of f, we expect:

$$\lim_{(x,y)\to(0,0)} \frac{\partial f_1}{\partial x} = 0 \qquad \lim_{(x,y)\to(0,0)} \frac{\partial f_1}{\partial y} = 0$$

For our case $\frac{\partial f_1}{\partial x}$:

$$\lim_{(x,y)\to(0,0)} \frac{\partial f_1}{\partial x} = \lim_{(x,y)\to(0,0)} \frac{-16x^3y^3}{(x^4+2y^2)^2} \tag{1}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
=& \lim_{r\to 0} \frac{-16(r\cos\theta)^3(r\sin\theta)^3}{((r\cos\theta)^4+2(r\sin\theta)^2)^2} \\
=& \lim_{r\to 0} \frac{-16r^6\cos^3\theta\sin^3\theta}{((r^4\cos^4\theta)+2r^2\sin^2\theta)^2} \\
=& \lim_{r\to 0} \frac{-16r^6\cos^3\theta\sin^3\theta}{(r^4\cos^4\theta)^2+2(r^4\cos^4\theta)(2r^2\sin^2\theta)+(2r^2\sin^2\theta)^2} \\
=& \lim_{r\to 0} \frac{-16r^6\cos^3\theta\sin^3\theta}{r^8\cos^8\theta+4r^6\cos^4\theta\sin^2\theta+4r^4\sin^4\theta} \\
=& \lim_{r\to 0} \frac{-16r^2\cos^3\theta\sin^3\theta}{r^4\cos^8\theta+4r^2\cos^4\theta\sin^2\theta+4\sin^4\theta} \\
=& 0
\end{aligned}$$

For our case $\frac{\partial f_1}{\partial y}$:

$$\lim_{(x,y)\to(0,0)} \frac{\partial f_1}{\partial y} = \lim_{(x,y)\to(0,0)} \frac{2(2y^2-x^4)x^4}{(x^4+2y^2)^2} \tag{2}$$

Take the limit along the path $x = 0$.

$$\begin{aligned}
=& \lim_{(x,y)\to(0,0)} \frac{2(2y^2-x^4)x^4}{(x^4+2y^2)^2} \\
=& \lim_{y\to 0} \frac{0}{4y^4} \\
=& 0
\end{aligned}$$

Take the limit along the path $y = 0$.

$$
\begin{aligned}
&= \lim_{(x,y)\to(0,0)} \frac{2(2y^2 - x^4)x^4}{(x^4 + 2y^2)^2} \\
&= \lim_{x\to 0} \frac{-2x^8}{x^8} \\
&= -2
\end{aligned}
$$

Since $0 \neq -2$, the limit doesn't exist.

Since not all partial derivatives of $f(x, y)$ exist, the function is not C^1.

Question 2

Consider the function

$$q(x, y) = xe^{y+2x}$$

(a) Determine the second order Taylor polynomial for q about the point $(1, 1)$.

We know the 2nd order Taylor approximation of a function of two variables (x, y) near the point (a, b) follows the formula:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

To create a Taylor polynomial for q around $(1, 1)$, we should find the partial derivatives of q.

$$
\begin{aligned}
q_x &= 2xe^{y+2x} + e^{y+2x} = (2x + 1)e^{y+2x} \\
q_y &= xe^{y+2x} \\
q_{xy} &= xe^{y+2x} \\
q_{xx} &= 2e^{y+2x} + 2e^{y+2x}(2x + 1) = 2(2x + 2)e^{y+2x} \\
q_{yy} &= xe^{y+2x}
\end{aligned}
$$

Therefore, the Taylor formula for $q(x, y)$ around the point $(1, 1)$ is:

$$
\begin{aligned}
q(x,y) \approx\, & q(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) + \frac{f_{xx}}{2}(1,1)(x-1)^2 \\
& + \frac{f_{yy}}{2}(1,1)(y-1)^2 + f_{xy}(1,1)(x-1)(y-1) \\
q(x,y) \approx\, & q(1,1) + (3)e^3(x-1) + (1)e^3(y-1) + \frac{2(4)e^3}{2}(x-1)^2 + \frac{1e^3}{2}(y-1)^2 + (1e^3)(x-1)(y-1) \\
q(x,y) \approx\, & e^3 + 3e^3(x-1) + e^3(y-1) + 4e^3(x-1)^2 + \frac{e^3}{2}(y-1)^2 + (e^3)(x-1)(y-1)
\end{aligned}
$$

(b) Using your Taylor polynomial in part (a), approximate $q(1.1, 1.1)$.

To find $q(1.1, 1.1)$, we should let $x = y = 1.1$.

$$\begin{aligned}
q(1.1, 1.1) \approx& e^3 + 3e^3(1.1-1) + e^3(1.1-1) + 4e^3(1.1-1)^2 + \frac{e^3}{2}(1.1-1)^2 + (e^3)(1.1-1)(1.1-1) \\
q(1.1, 1.1) \approx& e^3 + 3e^3(0.1) + e^3(0.1) + 4e^3(0.01) + \frac{e^3}{2}(0.01) + (e^3)(0.01) \\
q(1.1, 1.1) \approx& e^3 + 4e^3(0.1) + \frac{7}{2}(e^3)(0.01) \\
q(1.1, 1.1) \approx& 28.823
\end{aligned}$$

(c) Using Taylor's remainder formula, determine an upper bound for the error in your approximation of $q(1.1, 1.1)$.

Remember that the formula to calculate the Taylor remainder term where $f = q$ is:

$$R_n(\boldsymbol{x}) = \sum_{|\alpha|=n+1} \frac{(\boldsymbol{x}-\boldsymbol{a})^\alpha}{\alpha!}(\partial^\alpha f)(\boldsymbol{a}+\xi(\boldsymbol{x}-\boldsymbol{a}))$$

For some $\xi \in (0, 1)$.

$$\begin{aligned}
R_2(\boldsymbol{x}) =& \sum_{|\alpha|=3} \frac{(\boldsymbol{x}-\boldsymbol{a})^\alpha}{\alpha!}(\partial^\alpha f)(\boldsymbol{a}+\xi(\boldsymbol{x}-\boldsymbol{a})) \\
R_2(x, y) =& \frac{1}{3!}f_{xxx}(\boldsymbol{p})x^3 + \frac{1}{2!}f_{xxy}(\boldsymbol{p})x^2y + \frac{1}{2!}f_{xyy}(\boldsymbol{p})xy^2 + \frac{1}{3!}f_{yyy}(\boldsymbol{p})y^3
\end{aligned}$$

Where $\boldsymbol{p} = \boldsymbol{a} + \xi(\boldsymbol{x} - \boldsymbol{a})$.
If we let $(x, y) = (1.1, 1.1)$ and $(a, b) = (1, 1)$, then our expression becomes:

$$\begin{aligned}
R_2(1.1, 1.1) =& \frac{1}{3!}f_{xxx}(1+0.1\xi, 1+0.1\xi)(1.1)^3 + \frac{1}{2!}f_{xxy}(1+0.1\xi, 1+0.1\xi)(1.1)^2(1.1) \\
&+ \frac{1}{2!}f_{xyy}(1+0.1\xi, 1+0.1\xi)(1.1)(1.1)^2 + \frac{1}{3!}f_{yyy}(1+0.1\xi, 1+0.1\xi)(1.1)^3 \\
\therefore R_2(1.1, 1.1) =& \frac{1}{3!}f_{xxx}(1+0.1\xi, 1+0.1\xi)(1.1)^3 + \frac{1}{2!}f_{xxy}(1+0.1\xi, 1+0.1\xi)(1.1)^3 \\
&+ \frac{1}{2!}f_{xyy}(1+0.1\xi, 1+0.1\xi)(1.1)^3 + \frac{1}{3!}f_{yyy}(1+0.1\xi, 1+0.1\xi)(1.1)^3
\end{aligned}$$

And:

$$\begin{aligned}
q_{xxx} =& 4(2x+3)e^{2x+y} \\
q_{xxy} =& 4(x+1)e^{2x+y} \\
q_{xyy} =& (2x+1)e^{y+2x} \\
q_{yyy} =& xe^{y+2x}
\end{aligned}$$

So:

$$\begin{aligned}
R_2(1.1,1.1) =& \frac{1}{6}f_{xxx}(1+0.1\xi,1+0.1\xi)(1.1)^3+\frac{1}{2}f_{xxy}(1+0.1\xi,1+0.1\xi)(1.1)^3\\
&+\frac{1}{2}f_{xyy}(1+0.1\xi,1+0.1\xi)(1.1)^3+\frac{1}{6}f_{yyy}(1+0.1\xi,1+0.1\xi)(1.1)^3\\
R_2(1.1,1.1) =& \frac{1}{6}\cdot\frac{4(\xi+25)e^{\frac{3\xi}{10}+3}}{5}(1.1)^3+\frac{1}{2}\cdot\frac{2(\xi+20)e^{\frac{3\xi}{10}+3}}{5}(1.1)^3\\
&+\frac{1}{2}\cdot\frac{(\xi+15)e^{\frac{3\xi}{10}+3}}{5}(1.1)^3+\frac{1}{6}\cdot\frac{(\xi+10)e^{\frac{3\xi}{10}+3}}{10}(1.1)^3\\
R_2(1.1,1.1) =& \frac{11979(\xi+20)e^{\frac{3\xi}{10}+3}}{20000}
\end{aligned}$$

Taking $\xi = 0.1$

$$R_2(1.1,1.1) = 249.172$$

Question 3

Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be the function

$$f(x,y) = (x\cos y + y\cos x, 2 + y - 2e^{-x})$$

(a) Find $\mathbf{D}f$, the matrix of partial derivatives of f.

$$\mathbf{D}f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$\mathbf{D}f = \begin{bmatrix} \cos y - y\sin x & \cos x - x\sin y \\ 2e^{-x} & 1 \end{bmatrix}$$

(b) Let $g(x,y)$ be a differentiable function such that $g(0,0)=(0,0)$ and $f(g(x,y))=(x,y)$ for all (x,y) sufficiently close to $(0,0)$. Find $\mathbf{D}g(0,0)$.

Since $f(g(x,y))=(x,y)$ sufficiently close to $(0,0)$, we know that $g=f^{-1}$, close to $(0,0)$. Therefore:

$$\mathbf{D}(f\circ g) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And since:

$$\mathbf{D}(f\circ g) = \mathbf{D}f\mathbf{D}g$$

Then:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos y - y\sin x & \cos x - x\sin y \\ 2e^{-x} & 1 \end{bmatrix}\mathbf{D}g$$

So that means $\mathbf{D}g$ is just the inverse Jacobian matrix of $\mathbf{D}f$:

$$\mathbf{D}g = \frac{1}{(\cos y - y\sin x)-(2e^{-x})(\cos x - x\sin y)}\begin{bmatrix} 1 & -(\cos x - x\sin y) \\ -2e^{-x} & \cos y - y\sin x \end{bmatrix}$$

When $(x, y) = (0, 0)$

$$\therefore \mathbf{D}g(\mathbf{0}) = \frac{1}{(\cos 0 - 0\sin 0) - (2e^{-0})(\cos 0 - 0\sin 0)} \begin{bmatrix} 1 & -(\cos 0 - 0\sin 0) \\ -2e^{-0} & \cos 0 - 0\sin 0 \end{bmatrix}$$

$$\therefore \mathbf{D}g(\mathbf{0}) = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$