Vector Calculus Mast 20009 | Felix McCuaig, 1159424
Assignment 3 | Tutorial Wed 5:15PM

Question 1

Let the path C traverse half of the ellipse $4x^2 + 9y^2 = 36$ from $(3, 0)$ to $(-3, 0)$ in a clockwise direction.

(a) Write down a parametrisation for C in terms of an increasing parameter t.

Since we are going in the clockwise direction, we can parametrise C by using:

$$x = a\cos(-t) \text{ and } y = b\sin(-t)$$

$$\begin{aligned}\frac{x^2}{9} + \frac{y^2}{4} &= 1\\ \frac{(a\cos(-t))^2}{9} + \frac{(b\sin(-t))^2}{4} &= 1\\ \frac{a^2\cos^2(t)}{9} + \frac{b^2\sin^2(t)}{4} &= 1\end{aligned}$$

Therefore, $a = 3$ and $b = 2$, then: $x = 3\cos t, y = -2\sin t$ so:

$$x = 3\cos t, y = -2\sin t \qquad t \in [0, \pi]$$

$$\boldsymbol{c}(t) = (3\cos t, -2\sin t) \qquad t \in [0, \pi]$$

(b) Using part **(a)**, determine the work done by the force:

$$\boldsymbol{F}(x, y) = 2y\boldsymbol{i} + 5x\boldsymbol{j}$$

to move a particle along C.

The work can be calculated as:

$$\text{Work} = \int_C \boldsymbol{F} \cdot \boldsymbol{ds}$$

$$
\begin{aligned}
&= \int_C \boldsymbol{F} \cdot \frac{\boldsymbol{ds}}{dt} dt \\
&= \int_0^\pi \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t)\ dt \\
&= \int_0^\pi \boldsymbol{F}(3\cos t, -2\sin t) \cdot (-3\sin t, -2\cos t)\ dt \\
&= \int_0^\pi (-4\sin t, 15\cos t) \cdot (-3\sin t, -2\cos t)\ dt \\
&= \int_0^\pi 12\sin^2 t - 30\cos^2 t\ dt \\
&= \int_0^\pi 12\sin^2 t + 12\cos^2 t - 42\cos^2 t\ dt \\
&= \int_0^\pi 12 - 42\cos^2 t\ dt \\
&= -\int_0^\pi 42\cos^2 t - 12\ dt \\
&= -\int_0^\pi 42\cos^2 t - 21 + 9\ dt \\
&= -\int_0^\pi 21\cos(2t) + 9\ dt \\
&= -\left[\frac{21\sin 2t}{2} + 9t\right]_0^\pi \\
&= -\left[\frac{21\sin 2\pi}{2} + 9\pi - \left(\frac{21\sin 0}{2} + 9\times 0\right)\right]_0^\pi \\
&= -9\pi \text{ Joules}
\end{aligned}
$$

Question 2

Let a and h be positive real numbers. Let Σ be the surface in $\mathbb{R}^3$ given by

$$4ax = y^2 + z^2$$

(a) Find the volume of the region enclosed between Σ and the plane $x = h$.

The above volume can be calculated by summing the area of circles created by the paraboloid from $x = 0$ to $x = h$ therefore, taking the following (where $y = 2a\sqrt{x}$):

$$\int_0^h \pi y^2\ dx$$

$$= \int_0^h \pi(2a\sqrt{x})^2\ dx$$

$$= \int_0^h 4\pi a^2 x\ dx$$

$$= 4\pi a^2 \int_0^h x\, dx$$

$$= 4\pi a^2 h^2$$

(b) Show that the surface area of the part of Σ with $x \leq h$ is

$$\frac{8\pi\sqrt{a}}{3}\left((a+h)^{\frac{3}{2}} - a^{\frac{3}{2}}\right).$$

We know the surface area of a parametrised surface:

$$\iint_S dS = \iint_D ||T_u \times T_v||\, dudv$$

Where T_u and T_v are the tangent vectors to u and v. We can parametrise our surface $4ax = y^2 + z^2$ using:

$$x = \frac{u^2+v^2}{4a},\ y = u,\ z = v$$

Now, we can compute T_u and T_v:

$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \left(\frac{u}{2a}, 1, 0\right)$$

$$T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(\frac{v}{2a}, 0, 1\right)$$

Now to compute $||T_u \times T_v||$:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{u}{2a} & 1 & 0 \\ \frac{v}{2a} & 0 & 1 \end{vmatrix} = \left(1, \frac{-u}{2a}, \frac{-v}{2a}\right)$$

The norm is:

$$\sqrt{1 + \frac{u^2+v^2}{4a^2}}$$

Now, our integral becomes:

$$\iint_D \sqrt{1 + \frac{u^2+v^2}{4a^2}}\, dudv$$

If we changed to polar coordinates, it would make our life easier since we could integrate from $r \in [0, h]$ and $\theta \in [0, 2\pi]$.
Let's let:

$$u = r\cos\theta,\ v = r\sin\theta$$

To make the change of variables:

$$dudv = \left|\frac{\partial(u,v)}{\partial(r,\theta)}\right| drd\theta$$

To compute the Jacobian determinant:

$$\begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\int_0^{2\pi}\int_0^h \sqrt{1+\frac{r^2}{4a^2}}\ r\ drd\theta$$

$$\int_0^{2\pi}\int_0^h \left(1+\frac{r^2}{4a^2}\right)^{\frac{1}{2}} r\ drd\theta$$

If we let $\beta = 1+\frac{r^2}{4a^2}$:

$$\frac{d\beta}{dr} = \frac{r}{2a^2} \Rightarrow d\beta = dr\frac{r}{2a^2}$$

$$2a^2\int_0^{2\pi}\int_0^{1+\frac{h^2}{4a^2}} (\beta)^{\frac{1}{2}}\ d\beta d\theta$$

$$2a^2\int_0^{2\pi}\left[\frac{2\beta^{\frac{3}{2}}}{3}\right]_0^{1+\frac{h^2}{4a^2}} d\theta$$

$$2a^2\int_0^{2\pi}\frac{2}{3}\left(1+\frac{h^2}{4a^2}\right)^{\frac{3}{2}} d\theta$$

$$2a^2\times 2\pi\times\frac{2}{3}\left(1+\frac{h^2}{4a^2}\right)^{\frac{3}{2}}$$

$$8\pi\times\frac{a^2}{3}\left(1+\frac{h^2}{4a^2}\right)^{\frac{3}{2}}$$

$$\frac{8\pi\sqrt{a}}{3}\left(a\left(1+\frac{h^2}{4a^2}\right)\right)^{\frac{3}{2}}$$

$$\frac{8\pi\sqrt{a}}{3}\left(a+\frac{h^2}{4a}\right)^{\frac{3}{2}}$$

Question 3

Let n be a positive integer. Let $f:[0,1]\to\mathbb{R}$ be a continuous function. Show that

$$\int_0^1\int_0^1 f(xy)x^{n+1}y(1-y)^{n-1}dx\ dy = \frac{1}{n}\int_0^1 t(1-t)^n f(t)\ dt$$

[Hint, perform the change of variables $s = x(1-y), t = xy$].

$$\begin{aligned}
&= \int_0^1\int_0^1 f(xy)x^{n+1}y(1-y)^{n-1}\ dxdy\\
&= \int_0^1\int_0^1 f(t)x^2y\times x^{n-1}(1-y)^{n-1}\ dxdy\\
&= \int_0^1\int_0^1 f(t)x^2y\times s^{n-1}\ dxdy
\end{aligned}$$

$$dsdt = \left|\frac{\partial(x,y)}{\partial(s,t)}\right| dxdy$$

$$dsdt = \begin{vmatrix} s_x & s_y \\ t_s & t_y \end{vmatrix} dxdy$$

$$dsdt = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} dxdy$$

$$dsdt = (x(1-y) + xy)dxdy$$

$$dsdt = (x)dxdy$$

$$\begin{aligned} &= \int_0^1 \int_0^{1-t} f(t)t \times s^{n-1} \ dsdt \\ &= \int_0^1 \frac{1}{n} t(1-t)^n f(t) \ dt \end{aligned}$$