Vector Calculus Mast 20009
Assignment 2

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Tutorial Wed 5:15PM

Question 1

Let $D = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 \leq 1\}$ and consider the function

$$f(x, y) = 2x^2 + y^2 - y + 3$$

(a) Find and classify the local extrema of f on the interior of the domain D.

Let's use $\nabla f = 0$ to find extrema of f, then we can check if it's in D.

$$\nabla f = (4x, 2y - 1)$$

When $\nabla f = 0$:

$$4x = 0 \rightarrow x = 0$$
$$2y - 1 = 0 \rightarrow y = \frac{1}{2}$$

To check that $\left(0, \frac{1}{2}\right)$ lies in D, we can compute $x^2 + y^2$ and check that the result is ≤ 1.

$$0^2 + \frac{1}{2}^2 = \frac{1}{4}, \qquad \frac{1}{4} \leq 1$$

Now, let's classify the extrema using the Hessian and partial derivatives.

$$f_{xx} = 4$$
$$f_{yy} = 2$$
$$f_{xy} = 0$$

$$H_f = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8$$

Since $H_f > 0$ and $f_{xx} > 0$, the point $(0, \frac{1}{2})$ is a local min.

(b) Use Lagrange multipliers to find the extrema of f on the boundary of D.

If we let $g(x, y) = x^2 + y^2 - 1$, then we can find critical points on the boundary of D using:

$$\nabla f = \lambda \nabla g$$
$$\begin{bmatrix} 4x \\ 2y - 1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

This yields two equations:

$$4x = \lambda 2x \tag{1}$$
$$2y - 1 = \lambda 2y \tag{2}$$

From (1), we get:

$$x(4-2\lambda)=0, \qquad x=0 \text{ or } \lambda=2$$

For the case $x=0$:

$$0^2+y^2=1, \quad y=\pm 1$$

$$\Rightarrow (0,1)\,,(0,-1)$$

For the case $\lambda=2$:

$$2y-1=2\times 2y \rightarrow -1=2y \quad \therefore y=-\frac{1}{2}$$

Then, as before, we can use:

$$x^2+\left(\frac{-1}{2}\right)^2=1 \rightarrow x^2=\frac{3}{4} \qquad \therefore x=\pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2},\frac{-1}{2}\right),\left(\frac{-\sqrt{3}}{2},\frac{-1}{2}\right)$$

(c) Use the results of the previous parts to determine the extrema of f on the domain D.

$$f\left(0,\frac{1}{2}\right)=2(0)^2+\left(\frac{1}{2}\right)^2-\frac{1}{2}+3=\frac{1}{4}-\frac{2}{4}+\frac{12}{4}=\frac{11}{4}$$

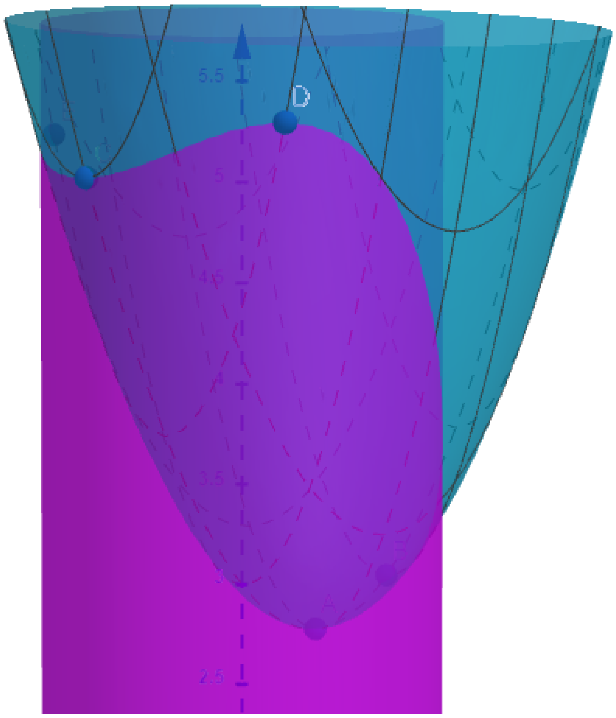
$$f(0,1)=2(0)^2+(1)^2-1+3=3$$

$$f(0,-1)=2(0)^2+(-1)^2+1+3=5$$

$$f\left(\frac{\sqrt{3}}{2},\frac{-1}{2}\right)=2\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2+\frac{1}{2}+3=2\cdot\frac{3}{4}+\frac{1}{4}+\frac{1}{2}+3=5+\frac{1}{4}$$

$$f\left(\frac{-\sqrt{3}}{2},\frac{-1}{2}\right)=2\left(-\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2+\frac{1}{2}+3=2\cdot\frac{3}{4}+\frac{1}{4}+\frac{1}{2}+3=5+\frac{1}{4}$$

Hence, $\left(1,\frac{1}{2}\right)$ is the global min of f on D. $(0,1)$ is a local min on the boundary of D, $(0,-1)$ is a local min on the boundary of D and $\left(\frac{\sqrt{3}}{2},\frac{-1}{2}\right)$ and $\left(\frac{-\sqrt{3}}{2},\frac{-1}{2}\right)$ are local maximums on the boundary of D. See cool graphic:



Question 2

Compute the torsion of the curve

$$\boldsymbol{\gamma}(t) = (at, \sin(bt), \cos(bt)), \qquad a, b > 0$$

Let's first find the unit tangent vector for some $\boldsymbol{c}(t)$ which so happens to be $\boldsymbol{\gamma}(t)$:

$$\boldsymbol{T}(t) = \frac{\frac{d\boldsymbol{c}}{dt}}{\left|\frac{d\boldsymbol{c}}{dt}\right|} = \frac{\frac{d\boldsymbol{\gamma}}{dt}}{\left|\frac{d\boldsymbol{\gamma}}{dt}\right|}$$

$$\frac{d\gamma}{dt} = (a, b\cos(bt), -b\sin(bt))$$

$$\left|\frac{d\gamma}{dt}\right| = \sqrt{a^2 + b^2\cos(bt) + b^2\sin(bt)} = \sqrt{a^2 + b^2}$$

$$\boldsymbol{T}(t) = \frac{(a, b\cos(bt), -b\sin(bt))}{\sqrt{a^2 + b^2}}$$

Now, let's find a unit normal vector $\boldsymbol{N}(t)$:

$$\boldsymbol{N}(t) = \frac{\frac{d\boldsymbol{T}}{dt}}{\left|\frac{d\boldsymbol{T}}{dt}\right|}$$

$$\frac{d\boldsymbol{T}}{dt} = \frac{1}{\sqrt{a^2 + b^2}}\left(0, -b^2\sin(bt), b^2\cos(bt)\right)$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$\boldsymbol{N}(t) = \frac{1}{\sqrt{a^2+b^2}}\left(0, -\sin(bt), \cos(bt)\right)$$

$$\boldsymbol{B}(t) = \boldsymbol{T} \times \boldsymbol{N} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{a}{\sqrt{a^2+b^2}} & \frac{b\cos(bt)}{\sqrt{a^2+b^2}} & \frac{-b\sin(bt)}{\sqrt{a^2+b^2}} \\ 0 & \frac{-\sin(bt)}{\sqrt{a^2+b^2}} & \frac{\cos(bt)}{\sqrt{a^2+b^2}} \end{vmatrix} = \begin{bmatrix} \frac{b\cos^2(bt)}{a^2+b^2} - \frac{b\sin^2(bt)}{a^2+b^2} \\ -\frac{a\cos(bt)}{a^2+b^2} \\ -\frac{a\sin(bt)}{a^2+b^2} \end{bmatrix}$$

$$\boldsymbol{B}(t) = \left(\frac{b\cos^2(bt)}{a^2+b^2} - \frac{b\sin^2(bt)}{a^2+b^2}, -\frac{a\cos(bt)}{a^2+b^2}, -\frac{a\sin(bt)}{a^2+b^2}\right)$$

The torsion τ can be calculated as:

$$\tau = \frac{\frac{d\boldsymbol{B}}{dt}}{\left|\frac{d\boldsymbol{c}}{dt}\right|} \cdot \boldsymbol{N}$$

$$\frac{d\boldsymbol{B}}{dt} = \left(0, \frac{ab\sin(bt)}{a^2+b^2}, -\frac{ab\cos(bt)}{a^2+b^2}\right)$$

Remember that:

$$\left|\frac{d\gamma}{dt}\right| = \sqrt{a^2 + b^2\cos(bt) + b^2\sin(bt)} = \sqrt{a^2+b^2}$$

So:

$$\tau = \frac{\left(0, \frac{ab\sin(bt)}{a^2+b^2}, -\frac{ab\cos(bt)}{a^2+b^2}\right)}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}}\left(0, -\sin(bt), \cos(bt)\right)$$

$$\tau = \frac{\left(0, \frac{ab\sin(bt)}{a^2+b^2}, -\frac{ab\cos(bt)}{a^2+b^2}\right)}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}}\left(0, -\sin(bt), \cos(bt)\right)$$

$$\tau = \frac{-a \cdot b}{(a^2+b^2)^2}$$

Question 3

Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = ||\boldsymbol{r}||$. Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable scalar function. Prove that

$$\nabla \cdot (f(r)\boldsymbol{r}) = r\frac{df}{dr} + 3f(r)$$

Let's start with:

$$\begin{aligned} &= \nabla \cdot (f(r)\boldsymbol{r}) \\ &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} x \times f(r) \\ y \times f(r) \\ z \times f(r) \end{bmatrix} \\ &= \frac{\partial}{\partial x}(x \times f(r)) + \frac{\partial}{\partial y}(y \times f(r)) + \frac{\partial}{\partial z}(z \times f(r)) \end{aligned}$$

Because of the chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \times \frac{\partial r}{\partial x}$$

$$\frac{\partial}{\partial x}(x)f(r) + \frac{\partial}{\partial x}(f(r))x = f(r) + \frac{x^2}{\sqrt{x^2+y^2+z^2}}\frac{\partial f}{\partial r}$$

If we repeat this for y, z and add them together, when we find that:

$$= 3f(r) + \frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} \frac{df}{dr}$$

$$= 3f(r) + \sqrt{x^2 + y^2 + z^2} \frac{df}{dr}$$

$$= r\frac{df}{dr} + 3f(r)$$

Question 4

Consider the vector field

$$\boldsymbol{F}(x, y, z) = (1 - xz^2, yz^2, x^2 - 1)$$

If it exists, find a vector potential for $\boldsymbol{F}$.

To check if there exists a vector potential, take $\nabla \cdot \boldsymbol{F}$ and hope its $= 0$:

$$\frac{\partial}{\partial x}(1 - xz^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(x^2 - 1)$$

$$-z^2 + z^2 + 0 = 0$$

Great, there exists a vector potential, let's find it:

$$\boldsymbol{F} = \nabla \times \boldsymbol{V}$$

We want to find the $\boldsymbol{V}$, let's let $\boldsymbol{V} = (V_1, V_2, V_3)$

$$\begin{bmatrix} 1 - xz^2 \\ yz^2 \\ x^2 - 1 \end{bmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\begin{bmatrix} 1 - xz^2 \\ yz^2 \\ x^2 - 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \\ \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \\ \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{bmatrix}$$

This yields:

$$1 - xz^2 = \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \tag{3}$$

$$yz^2 = \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \tag{4}$$

$$x^2 - 1 = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \tag{5}$$

Because we don't really care about the general solution to this differential equationy thing, we can let one of our V_1, V_2, V_3 equal zero, and hence find a particular solution.

Let's let $V_1 = 0$. Now (4) and (5) become:

$$\frac{\partial V_3}{\partial x} = -yz^2$$

$$\frac{\partial V_2}{\partial x} = x^2 - 1$$

If we integrate, we get:

$$V_3 = -xyz^2 + h(y,z)$$
$$V_2 = \frac{1}{3}x^3 - x + g(y,z)$$

Now, we want to put these equations back into (3), so we differentiate:

$$\frac{\partial V_3}{\partial y} = \frac{\partial h}{\partial y} - xz^2$$
$$\frac{\partial V_2}{\partial z} = \frac{\partial g}{\partial z}$$

Therefore (3) becomes:

$$1 - xz^2 = \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}$$
$$1 - xz^2 = -xz^2 + \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}$$
$$1 = \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}$$

Now, since we only want a particular solution, not a general one, we can let $g = 0$.

$$\frac{\partial h}{\partial y} = 1$$

If we integrate, we get:

$$h(y,z) = y + c(z)$$

And then we can let $c(z) = 0$, because we only want a single solution:

$$h(y,z) = y$$

Now:

$$V_1 = 0$$

$$V_2 = \frac{1}{3}x^3 - x$$

$$V_3 = y - xyz^2$$

Now we have our $\boldsymbol{V} = \left(0, \frac{1}{3}x^3 - x, z - xyz^2\right)$. To check that it is in fact a vector potential for $\boldsymbol{F}$, we can take:

$$\nabla \times \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{1}{3}x^3 - x & y - xyz^2 \end{vmatrix}$$
$$= \begin{bmatrix} 1 - xz^2 \\ yz^2 \\ x^2 - 1 \end{bmatrix}$$
$$= \boldsymbol{F}$$

Question 5

Find the condition that needs to be satisfied by the real numbers a, b and c so that the function

$$f(x, y, z) = e^{ax}\sin(by)\cos(cz)$$

satisfies $\nabla^2 f = 0$.

$$\begin{aligned} f_{xx} &= a^2 e^{ax}\sin(by)\cos(cz) \\ f_{yy} &= -b^2 e^{ax}\sin(by)\cos(cz) \\ f_{zz} &= -c^2 e^{ax}\sin(by)\cos(cz) \end{aligned}$$

So:

$$a^2 e^{ax}\sin(by)\cos(cz) - b^2 e^{ax}\sin(by)\cos(cz) - c^2 e^{ax}\sin(by)\cos(cz) = 0$$

$$e^{ax}\sin(by)\cos(cz)(a^2 - b^2 - c^2) = 0$$

Because of the null factor law, either:

$$a^2 - b^2 - c^2 = 0 \text{ or } b = 0$$