Vector Calculus Mast 20009 | Felix McCuaig, 1159424
Assignment 3 | Tutorial Wed 5:15PM

Question 1

Let S be the surface given by

$$x^2 + 2y^4 + 3z^6 = 1 \text{ with } z \geqslant 0.$$

Let Γ be the curve

$$\gamma(t) = \left(\cos(\pi t), \sqrt{t+1}, \frac{2t}{t^2+1}\right), \qquad \text{for} \qquad 0 \leqslant t \leqslant 1$$

and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (a - 4z^2 x)\boldsymbol{i} + 2by\boldsymbol{j} + cx^2 z\boldsymbol{k}$$

where $a, b, c \in \mathbb{R}$

(a) Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{S}.$$

According to Stokes' theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \mathbf{F} \cdot d\boldsymbol{S}.$$

Where ∂S is the boundary of S, which occurs when $z = 0$. Hence, ∂S is given by:

$$x^2 + 2y^4 = 1$$

Now, ∂S should parametrised:

$$x = \pm\cos t, \qquad y = \pm\frac{\sqrt{\sin t}}{\sqrt{\sqrt{2}}}, \qquad z = 0$$

To verify:

$$\begin{aligned} &= \cos^2 t + 2\left(\frac{\sin t}{\sqrt{2}}\right)^2 \\ &= \cos^2 t + 2\sin^2 t \\ &= 1 \end{aligned}$$

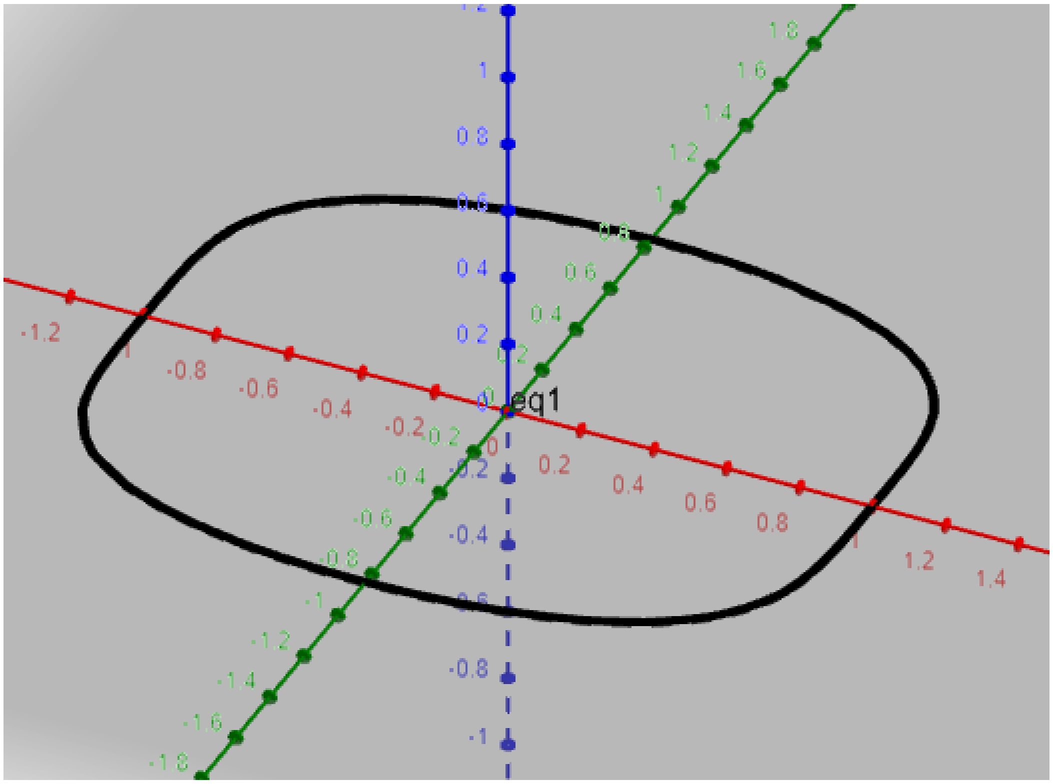
Nice, now we need to traverse our squary ellipsey thing in an anticlockwise direction:
That is, for (x, y, z) :

$$(1, 0, 0) \to (-1, 0, 0) \to (1, 0, 0)$$

When $\cos t = 1, t = 0$, when $\cos t = -1, t = \pi$.
Since y has a $\pm$ thingy, we should traverse the curve in two segments:

$$\mathbf{p}_1(t) = \left(\cos t, \frac{\sqrt{\sin t}}{\sqrt{\sqrt{2}}}, 0\right) t \in [0, \pi]$$

$$\mathbf{p}_2(t) = \left(-\cos t, -\frac{\sqrt{\sin t}}{\sqrt{\sqrt{2}}}, 0\right) t \in [0, \pi]$$



Note: $d\mathbf{S}_1$ corresponds to $\mathbf{p}_1$ and $d\mathbf{S}_2$ to $\mathbf{p}_2$

$$\int_0^{\pi} \mathbf{F}\left(\cos t, \frac{\sqrt{\sin t}}{\sqrt{\sqrt{2}}}, 0\right) \cdot d\mathbf{S}_1 - \int_0^{\pi} \mathbf{F}\left(-\cos t, -\frac{\sqrt{\sin t}}{\sqrt{\sqrt{2}}}, 0\right) \cdot d\mathbf{S}_2$$

Note, the integrals must be subtracted or we get 0.

If we take out the negatives from our second integral we have twice our first integral. So let's just do the integral once and double it later.
Now, we know that $d\mathbf{S}$ is:

$$d\mathbf{S} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) dt$$

$$d\mathbf{S}_1 = \left(-\sin t, \frac{1}{2} \times \frac{\cos t}{\sqrt{2}} \times \left(\frac{\sin t}{\sqrt{2}}\right)^{-\frac{1}{2}}, 0\right) dt$$

$$\mathbf{F}(\mathbf{c}(t)) = \left(a - 4(0)^2 \cos t, 2b \times \left(\frac{\sin t}{\sqrt{2}}\right)^{\frac{1}{2}}, 0\right) = \left(a, 2b\left(\frac{\sin t}{\sqrt{2}}\right)^{\frac{1}{2}}, 0\right)$$

$$\int_0^{\pi} \left(a, 2b\left(\frac{\sin t}{\sqrt{2}}\right)^{\frac{1}{2}}, 0\right) \cdot \left(-\sin t, \frac{\cos t}{2\sqrt{2}}\left(\frac{\sin t}{\sqrt{2}}\right)^{-\frac{1}{2}}, 0\right) dt$$

$$\int_0^{\pi} -a \sin t + \frac{b}{\sqrt{2}} \cos t \ dt$$

$$\left[a \cos t + \frac{b}{\sqrt{2}} \sin t\right]_0^{\pi} = -2a$$

Now, let's double our result (yielding our final answer):

$$\therefore \iint_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{S} = -4a$$

(b) Find all values of a, b and c such that the vector field $\mathbf{F}$ is conservative.

Since

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \mathbf{F} \cdot d\boldsymbol{S}.$$

Our vector field $\mathbf{F}$ is conservative when:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{S} = 0.$$

Therefore $a = 0,\ \ b \in \mathbb{R}\ c \in \mathbb{R}$.
Note, this visually makes sense as when $a = 0$, the vector field is symmetric upon all axes, but when $a \neq 0$, it's wonky (technical term for asymmetric).

(c) If a, b, c are chosen such that $\mathbf{F}$ is conservative, compute the work done by $\mathbf{F}$ to move a particle along Γ in the direction of increasing t.

To choose a, b, c such that $\mathbf{F}$ is conservative, let $a = 0,\ \ b \in \mathbb{R}\ c \in \mathbb{R}$. Then, $\mathbf{F}$ is:

$$\mathbf{F}(x, y, z) = (-4z^2x)\boldsymbol{i} + 2by\boldsymbol{j} + cx^2z\boldsymbol{k}$$

We want to calculate the work done as $t \in [0, 1]$.