AST3310 Project 1: Modelling the energy production of a star

(Dated: March 9, 2025)

I. INTRODUCTION

For the first project in the course AST3310 I have modelled the energy production in the core of a star given its temperature and density using a custom `python`-script. The model seeks to calculate the energy produced from the fusion of ${}^1_1\mathrm{H}$ into ${}^4_2\mathrm{He}$ through the PPI, PPII and PPIII branches and the CNO cycle, as well as how the relative energy production between these branches evolve with the temperature.

We restrict ourselves to considering a homogeneous and isotropic stellar core with uniform temperature and density such that the mass fraction of each atomic species is independent of location in the star. The model only considers the energy production of the star at a particular moment in time and does not take into account the time evolution of the elemental abundances. We assume the mass fractions at this particular time are those given in table I. In addition, we also assume that all the elements are fully ionized.

Mass fractions in the core		
Atomic species		Mass fraction
Hydrogen	${}^1_1\mathrm{H}$	$X = 0.7$
Helium 3	${}^3_2\mathrm{He}$	$Y_{{}^3_2\mathrm{He}} = 10^{-10}$
Helium 4	${}^4_2\mathrm{He}$	$Y = 0.29$
Lithium 7	${}^7_3\mathrm{Li}$	$Z_{{}^7_3\mathrm{Li}} = 10^{-7}$
Beryllium 7	${}^7_4\mathrm{Be}$	$Z_{{}^7_4\mathrm{Be}} = 10^{-7}$
Nitrogen 14	${}^{14}_7\mathrm{N}$	$Z_{{}^{14}_7\mathrm{N}} = 10^{-11}$

Table I. The mass fractions of the different atomic species present in the stellar core. These are assumed to be independent of location in the star.

II. THEORY

Inside the core of main sequence stars there is a massive production of energy that maintains the hydrostatic equilibrium of the star and keeps it from cooling and contracting. This energy, sustaining the star and ultimately released as electromagnetic radiation from its surface, is generated through nuclear fusion. The most important of these fusion processes, and the one we will be considering in the development of our model, is the fusion of ${}^1_1\mathrm{H}$ nuclei into ${}^4_2\mathrm{He}$. We know two main sets of fusion reactions that achieve this; the proton-proton chain (PP chain) and the Carbon-Nitrogen-Oxygen cycle (CNO cycle). The specifics of these will be discussed in their proper subsections.

For now, we turn our attention to some general theory that let's us calculate the power generated per unit volume for each individual nuclear reaction. The power generated per unit mass from the nuclear reaction between two atomic species i and k is given by

$$\varepsilon_{ik} = r_{ik} Q'_{ik}$$

where ε_{ik} has SI units [W kg^{-1}]. The energy output Q'_{ik} from the reaction, with SI units [J/reaction] but often given in units of [MeV/reaction], comes from the conversion of mass into energy which can be calculated using Einstein's mass-energy equivalency principle; $Q'_{ik} = \delta m\, c^2 - Q_\nu$, where δm is the mass difference between the reactants and the products of the given reaction, c is the speed of light and Q_ν is the energy lost to neutrino emission (which only happens in some of the reactions).

The reaction rate r_{ik} has units [reactions kg^{-1} s^{-1}] and is a measure of how many reactions occur per unit mass per unit time. It depends on the probability of the two reactants overcoming the Coulomb barrier, which arises due to the repulsive force between their like charges. From equation (3.24) in the lecture notes, we know that the reaction rate is given by

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik}$$

where n_i and n_k are the number densities [m^{-3}] of the two atomic species, ρ is the density [kg m^{-3}] of the core, δ_{ik} is the Kronecker-delta, and λ_{ik} is the so called *proportionality function* [reactions m^3 s^{-1}]. If we assume that the energies of the particles follow a Maxwellian distribution, we can write the proportionality function as

$$\lambda_{ik} = \sqrt{\frac{8}{m\pi(k_B T)^3}} \int_0^\infty \exp\Big(-\frac{E}{k_B T}\Big) E\sigma(E) dE$$

where $m \equiv m_i m_k/(m_i + m_k)$ is the reduced mass, k_B is Boltzmann's constant, T is the temperature and $E = mv^2/2$ is the kinetic energy in the center of mass system.

In main sequence stars, such as our Sun, the Coloumb barrier is overcome due to quantum tunneling. In fact, due to Heisenberg's uncertainty principle, the particles' wavefunctions can "bleed" through the Coloumb potential and thus have a non-zero probabilty of appearing on the other side of the barrier – which is what we call tunneling. For these reactions, the cross section factor $\sigma(E)$ takes the form

$$\sigma(E) = E^{-1} S(E) \exp\Big(-\sqrt{\frac{m}{2E}}\frac{Z_i Z_k e^2 \pi}{\epsilon_0 h}\Big)$$

where Z_i and Z_k are the atomic numbers of each of the nuclei, e is the elemental charge, ϵ_0 is the vacuum permittivity and h is Planck's constant. The function $S(E)$

is a slowly varying function and we assume it to be a constant with units [J m^2] as stated in the lecture notes. Plugging this into the expression for λ_{ik}, we get

$$\lambda_{ik} \sim \int_0^\infty S(E) \exp\Big(- \frac{E}{k_B T} - \sqrt{\frac{m}{2E}} \frac{Z_i Z_k e^2 \pi}{\epsilon_0 h} \Big) dE$$

The exponential function in this last expression for λ_{ik} is a result of the product of two exponential functions; the Maxwellian distribution of the energy of the particles and the Coloumb barrier between the charges of the particles. This function will be sharply peaked and is called the *Gamow peak*. The peak appears since the probability of of overcoming the Coloumb barrier increases rapidly as the energy of the particles increase, while the probability of the particles such high energies in the Maxwellian distribution decreases rapidly. The exponentially decreasing tails of these two distributions kills off everything except a narrow overlapping window, thus producing a peak (Wikipedia 1., 2025).

The lecture notes tells us that if we approximate this peak by a Gaussian and compute the integral for λ_{ik}, we get

$$\lambda_{ik} = 4S(E_{\max}) \frac{(2/3m)^{1/2}(b/2)^{1/3}}{(k_B T)^{2/3}} \exp\left[- 3\Big(\frac{b^2}{4 k_b T} \Big)^{1/3} \right]$$

where $E_{\max}$ is the energy for which the Gamow peak is assumed to have its maximum and

$$b \equiv \frac{m}{2} \frac{e^2 \pi}{\epsilon_0 h} Z_i Z_k$$

Using this approximation of the λ's, we see that they are dependent on the temperature T. In our model we will use the expressions for the different λ's given in table 3.1 in the lecture notes. These expressions involve the unitless quantity $T_9 \equiv T[\mathrm{K}]/10^9\mathrm{K}$.

Finally, when we have calculated the different ε's, we can get the power Π_{ik}[W m^{-3}] generated by the given nuclear reaction per unit volume by simply multiplying by the density, i.e.

$$\Pi_{ik} = \varepsilon_{ik} \rho = r_{ik} Q'_{ik} \rho$$

The Π's are what we are going to calculate in our model and use for subsequent discussions.

A. The PP chain

The PP chain fuses ${}^1_1\mathrm{H}$ into ${}^4_2\mathrm{He}$ through three different branches: PPI, PPII and PPIII. All three of these produce the same amount of energy per ${}^4_2\mathrm{He}$ nucleus due to the mass difference between four free protons and ${}^4_2\mathrm{He}$:

$$\begin{aligned} Q_{4\,{}^1_1\mathrm{H}\to{}^4_2\mathrm{He}} &= (4m_p - m_{{}^4_2\mathrm{He}})c^2 \\ &= 4.7654 \times 10^{-29}\mathrm{kg} \cdot (2.9979 \times 10^8 \mathrm{m/s})^2 \\ &= 4.2829 \times 10^{-12}\mathrm{J} \approx 26.732\ \mathrm{MeV} \end{aligned}$$

The three branch share the same first two steps, namely

$$\begin{aligned} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} &\to {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_e \\ {}^2_1\mathrm{D} + {}^1_1\mathrm{H} &\to {}^3_2\mathrm{He} + \gamma \end{aligned}$$

where the positron e^+ quickly annihilates with an electron e^- and releases two photons γ, both of energy 511 keV. Since the reaction rate of proton-proton fusion is much slower than the proton-deuterium fusion, and the deuterium only acts as an intermediary product in the synthesis of ${}^3_2\mathrm{He}$, we can combine these two steps into just one. We will assume that the proton-deuterium fusion happens almost instantaneously compared to the first step such that the reaction rate of the single step will be r_{pp} associated with the proton-proton fusion.

After this, the branches deviate and produce ${}^4_2\mathrm{He}$ in different ways. In the PPI branch, the produced ${}^3_2\mathrm{He}$ simply fuses to make one ${}^4_2\mathrm{He}$ and two ${}^1_1\mathrm{H}$:

$${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \to {}^4_2\mathrm{He} + 2\,{}^1_1\mathrm{H}$$

On the other hand, the PPII and PPIII branches have more intermediary steps involving heavier elements like lithium, beryllium and boron. The complete sets of nuclear reactions in each branch are summarized in table II along with the generated energy Q' of each step, as well as the energy Q_ν lost to neutrino emission.

B. The CNO cycle

In addition to the PP chain, the CNO cycle also fuses ${}^1_1\mathrm{H}$ into ${}^4_2\mathrm{He}$ generating the same 26.732 MeV of energy per produced ${}^4_2\mathrm{He}$ nucleus. The nuclear reactions in CNO cycle along with their generated energies and the energies lost to neutrino emission are summarized in table III.

The first step in the cycle requires a presence of ${}^{12}_6\mathrm{C}$ nuclei in the stellar core. These are synthesized through the *triple-alpha process* (Wikipedia 2., 2025). In the triple-alpha process, two ${}^4_2\mathrm{He}$ nuclei initially fuse to create ${}^8_4\mathrm{Be}$ which has a half-life of about 8.19×10^{-17}s. If however another ${}^4_2\mathrm{He}$ nucleus manages to fuse with the unstable ${}^8_4\mathrm{Be}$ before it decays into smaller nuclei, it will form a ${}^{12}_6\mathrm{C}$ nucleus as well as two photons γ. In other words, the ${}^{12}_6\mathrm{C}$ nuclei required for the CNO cycle to commence are produced through the two nuclear reactions

$$\begin{aligned} {}^4_2\mathrm{He} + {}^4_2\mathrm{He} &\to {}^8_4\mathrm{Be} \\ {}^8_4\mathrm{Be} + {}^4_2\mathrm{He} &\to {}^{12}_6\mathrm{C} + 2\gamma \end{aligned}$$

Even though the half-life of ${}^8_4\mathrm{Be}$ is very small, it is still four orders of magnitude larger than the time it takes for two ${}^4_2\mathrm{He}$ nuclei to scatter inside a star (Wikipedia 2., 2025), and since our stellar core is 29% ${}^4_2\mathrm{He}$ by mass, we can assume with high degree of certainty that there has been a significant accumulation of ${}^{12}_6\mathrm{C}$ in our core. Especially if the temperature of the core is above $T \sim 10^8$ K (Wikipedia 2., 2025).

As the name suggests, the CNO cycle is cyclic, so we regain each nucleus of ${}^{12}_{6}\mathrm{C}$ we start with in the last step of the cycle. Thus, the amount of ${}^{12}_{6}\mathrm{C}$ in the stellar will not be reduced. Since this is the case for the other catalysts that appear in the cycle (the different isotopes of carbon, nitrogen and oxygen), their number densities will remain unchanged and the entire CNO cycle can be modelled as a single step converting $4\,{}^{1}_{1}\mathrm{H}$ into ${}^{4}_{2}\mathrm{He}$ and some biproducts:

$$4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + 2\mathrm{e}^{+} + 2\nu_e + 3\gamma$$

This process will be limited by the slowest reaction in the cycle, which is the the proton capture by ${}^{14}_{7}\mathrm{N}$ (Wikipedia 3., 2025). Thus, we can model the entire CNO cycle as one nuclear reaction with the reaction rate r_{p14} associated with the proton capture by ${}^{14}_{7}\mathrm{N}$.

III. METHOD

The `python`-script I wrote has this general structure;

1. it calculates r_{ik}, Q'_{ik} and Π_{ik} for each *relevant* reaction in the PP branches and the CNO cycle;
2. then it performs a "sanity check" to see of the calculated values for the Π_{ik}'s match the reference values given in the assignment;
3. if it passes the "sanity check", it uses the r_{ik}'s and Q'_{ik}'s to find the energy production ε per unit mass from each of the PP branches and the CNO cycle as a function of temperature;
4. lastly, it plots the relative energy production of the PP branches and the CNO cycle in the temperature range $T \in [10^4, 10^9]$ K. It also plots the Gamow peaks of the *relevant* fusion reactions in the energy range $E \in [10^{-17}, 10^{-13}]$ J.

First of all, we have already seen that we can combine the first two steps of the PP branches into a single step, and that the CNO cycle can be regarded as a single step limited by the proton capture on ${}^{14}_{7}\mathrm{N}$. Similarly, we can combine the last three steps of the PPIII branch into one step limited by the reaction rate r_{17} associated with the proton capture by ${}^{7}_{4}\mathrm{Be}$, since the following two steps depend on the production of the intermediary product ${}^{8}_{5}\mathrm{B}$. These three simplifications allow us to reduce the number of reaction rates we need to calculate down to seven: r_{pp}, r_{33}, r_{34}, r_{e7}, r'_{17}, r_{17} and r_{p14}. These seven are what I refer to as the *relevant* reactions.

After giving the temperature T and density ρ as parameters, the program starts calculating the power Π_{ik} per unit volume for each individual relevant reaction. For this calculation we require r_{ik} and Q'_{ik}. The Q'_{ik}'s we get from using Einstein's mass-energy equivalency as explained in the theory section. When I calculated the Q'_{ik}'s, I assumed the neutrinos to be massless. Also, in the reactions producing a positron, we have to add 1.022 MeV due to it annihiliating with a free electron and subtract the energy of the emitted neutrino.

To calculate the reaction rates, we need the number densities n_i and n_k, as well as the proportionality function λ_{ik}.

The number densities are defined as $n_i = \frac{M_i/m_i}{V}$ where M_i is the total amount of mass of the atomic species i we have in the core, m_i is the mass of one nucleus of i and V is the volume of the stellar core. We can rewrite this using the uniform density $\rho = M/V$ (where M is the total mass of the core) and the mass fractions $\chi_i = M_i/M$ (which are given in table I):

$$n_i = \frac{M_i/m_i}{V} = \frac{M_i}{M}\frac{M}{V}\frac{1}{m_i} = \frac{\rho\chi_i}{m_i}$$

For these calculations I assumed the mass of the nucleus to be $m_i = A_i m_u$, where A_i is the number of nucleons in the nucleus of species i and $m_u = 1.6605 \times 10^{-27}$ kg is the atomic mass constant. We then end up with the final expression for the number densities:

$$n_i = \frac{\rho\chi_i}{A_i m_u}$$

When we calculate the electron density we assume that all the elements in the core are fully ionized. The expression for the electron density can then be written as $n_{\mathrm{e}} = \sum_i Z_i n_i$. However, since the mass fraction of every element except hydrogen and helium 4 are less than or equal to 10^{-7}, we can neglect the contribution of these elements to the electron density. Then we get

$$\begin{aligned} n_{\mathrm{e}} &= n_{{}^{1}_{1}\mathrm{H}} + 2n_{{}^{4}_{2}\mathrm{He}} = \left(X + \frac{Y}{2}\right)\frac{\rho}{m_u} \\ &= (1+X)\frac{\rho}{2m_u} \end{aligned}$$

where we use $X + Y \approx 1$.

Finally, we get the λ's by plugging in the temperature into the expressions given in table 3.1 in the lecture notes. This is pretty straight forward. One technicality we must must take into account is that the electron capture by ${}^{7}_{4}\mathrm{Be}$ has an upper limit of $\lambda_{\mathrm{e7}} \leq 1.51 \times 10^{-7}/n_{\mathrm{e}}$ for temperatures below 10^6 K, according to the lecture notes. I implemented this in the model by choosing the smallest value between the calculated λ_{e7} and the upper limit if the temperature is below 10^6 K.

After we have found the number densities and the proportionality functions, we are ready to calculate the reaction rates.

A. Making sure there is no overconsumption of elements

Some of the steps in the PP branches share reactants we must thus allocate the elements available such that no step consumes more of an element than the previous

steps are able to produce. The way I chose to do this was to naively calculate the reaction rates following the steps described previously and then adjust them afterwards to avoid overconsumption.

The first problem we encounter is that the ${}^3_2$He produced in the first two steps with a rate of r_{pp} needs to be shared between the helium 3-helium 3 (with rate r_{33}) fusion of the PPI branch and the helium 3-helium 4 fusion (with rate r_{34}) of the PPII and PPIII branches. Since the helium 3-helium 3 fusion consumes two ${}^3_2$He nuclei for each reaction completed, we must make sure that

$$2r_{33} + r_{34} \leq r_{\text{pp}}$$

In the case where $2r_{33} + r_{34} > r_{\text{pp}}$ we must adjust the reaction rates in a way that respects the following conditions

$$\begin{cases} 2\tilde{r}_{33} + \tilde{r}_{34} = r_{\text{pp}} \\ \dfrac{2\tilde{r}_{33}}{\tilde{r}_{34}} = \dfrac{2r_{33}}{r_{34}} \end{cases}$$

where the $\tilde{r}$'s are the adjusted rates. The first condition basically says that the amount of ${}^3_2$He consumed by the two reactions should equal the amount available from the previous step. The second conditions states that the ratio between the adjusted rates should be the same as for the original rates, i.e. if one of the reactions is X times faster than the other, the adjusted rates should also reflect this fact.

Solving this set of equations for the adjusted rates $\tilde{r}_{33}$ and $\tilde{r}_{34}$, we get

$$\tilde{r}_{33} = \frac{a r_{\text{pp}}}{2a+1}, \qquad \tilde{r}_{34} = \frac{r_{\text{pp}}}{2a+1}$$

where we define $a \equiv r_{33}/r_{34}$.

We get a similiar problem for distributing the ${}^7_4$Be produced at a rate of r_{34} (or $\tilde{r}_{34}$ if we had to adjust it – the argument is the same) to the electron-beryllium 7 reaction of PPII (with rate $r_{\text{e}7}$) and the proton-beryliium 7 reaction of PPIII (with rate r_{17}). In the case where $r_{\text{e}7} + r_{17} > r_{34}$, the conditions for the adjusted rates are

$$\begin{cases} \tilde{r}_{\text{e}7} + \tilde{r}_{17} = r_{34} \\ \dfrac{\tilde{r}_{\text{e}7}}{\tilde{r}_{17}} = \dfrac{r_{\text{e}7}}{r_{17}} \end{cases}$$

giving the solutions

$$\tilde{r}_{\text{e}7} = \frac{b r_{34}}{b+1}, \qquad \tilde{r}_{17} = \frac{r_{34}}{b+1}$$

with $b \equiv r_{\text{e}7}/r_{17}$.

Finally, we must make sure that $r'_{17} \leq r_{\text{e}7}$ (or $r'_{17} \leq \tilde{r}_{\text{e}7}$ if we needed to adjust it – again, the argument stays the same) such the last step in the PPII branch does not consume more ${}^7_3$Li than what is produced in the previous step. This is simply achieved by setting $\tilde{r}'_{17} = r_{\text{e}7}$ if $r'_{17} > r_{\text{e}7}$.

B. Calculating the energy production of each PP branch and the CNO cycle

We only want to consider the energy generated by the completed branches of the PP chain and completed CNO cycles. The number of completed PP branches per unit time per unit mass is just the last reaction rate of that particular branch, while the number of completed CNO cycles is just $r_{\text{p}14}$. To get the power ε generated per unit mass, we then multiply these reaction rates with the sum of the energies emitted at each step, i.e.

$$\begin{aligned} \varepsilon_{\text{PPI}} &= r_{33}[Q'_{33} + 2(Q'_{\text{pd}} + Q'_{\text{pp}})] \\ \varepsilon_{\text{PPII}} &= r'_{17}(Q'_{17'} + Q'_{\text{e}7} + Q'_{34} + Q'_{\text{pd}} + Q'_{\text{pp}}) \\ \varepsilon_{\text{PPIII}} &= r_{17}(Q'_{8'} + Q'_{8} + Q'_{17} + Q'_{34} + Q'_{\text{pd}} + Q'_{\text{pp}}) \\ \varepsilon_{\text{CNO}} &= r_{\text{p}14}(Q'_{\text{p}12} + Q'_{13} + Q'_{\text{p}13} + Q'_{\text{p}14} + Q'_{15} + Q'_{\text{p}15}) \end{aligned}$$

To get the relative energy productions, we normalize the ε's to the total energy production

$$\varepsilon_{\text{TOT}} = \varepsilon_{\text{PPI}} + \varepsilon_{\text{PPII}} + \varepsilon_{\text{PPIII}} + \varepsilon_{\text{CNO}}$$

C. Plotting the Gamow peaks

We are interested in looking at the Gamow peaks for each of the relevant fusion reaction. This means that we do not include the electron capture by ${}^7_4$Be, since this isn't a fusion reaction. We thus have six different relevant fusion reactions we want to plot the Gamow peak for.

As a reminder from our discussion in the theory section, the Gamow peak is the probability distribution

$$P_{ik}(E) = \exp\Big(-\frac{E}{k_B T} - \sqrt{\frac{m}{2E}} \frac{Z_i Z_k e^2 \pi}{\epsilon_0 h} \Big) \tag{1}$$

For the assignment we had to plot each of these distributions normalized to their respective maxima in the energy range $E \in [10^{-17}, 10^{-13}]$ J. I chose to look at the distributions for our Sun's temperature: $T = T_\odot = 1.57 \times 10^7$ K.

IV. RESULTS

A. Calculated Q' values and percentage of energy lost to neutrino emission

The calculated values for the generated energies Q' resulting from the mass difference between reactants and products in each individual nuclear reaction, as well as the reference values given in the lecture notes are given in table IV.

The percentage of the energy lost to neutrino emission in branch B per ${}^4_2$He nucleus produced, is given by

$$\varrho^\nu_B = \frac{\sum_B Q_\nu}{N_{{}^4_2\text{He}} \sum_B (Q' + Q_\nu)} = \frac{\sum_B Q_\nu}{N_{{}^4_2\text{He}} Q_{4\,{}^1_1\text{H} \to {}^4_2\text{He}}}$$

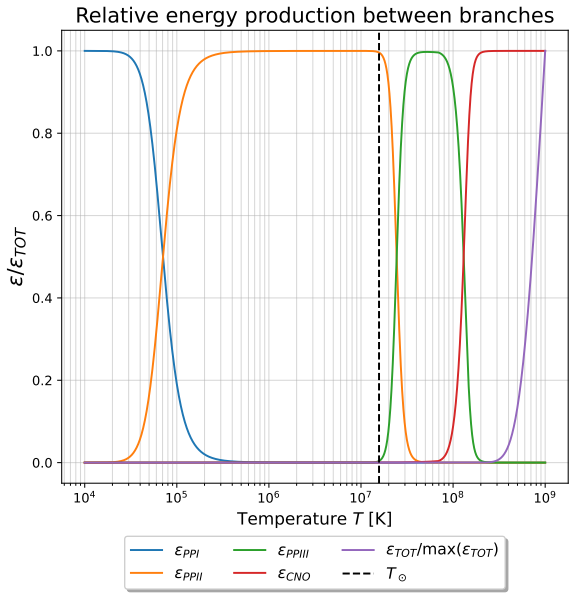

Figure 1. The relative energy productions between the different PP branches and the CNO cycle normalized to the total power output $\varepsilon_{\mathrm{TOT}}$ as functions of temperature. All of them are calculated for density $\rho = \rho_\odot = 1.62 \times 10^5$ kg/m^3. The total $\varepsilon_{\mathrm{TOT}}$ is shown in purple normalized to its maximum. The vertical dashed line shows where $T = T_\odot$.

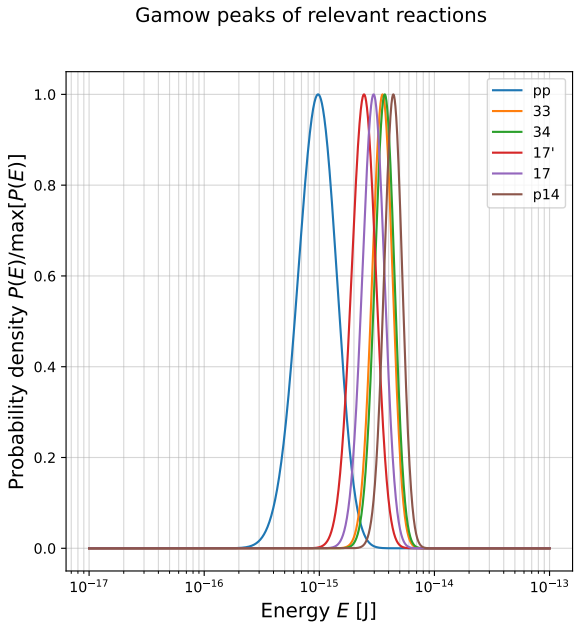

Figure 2. The normalized probability distributions $P_{ik}(E)$, or "Gamow peaks" of the relevant fusion reactions as given in eq 1. Plotted for energies in the range $E \in [10^{-17}, 10^{-13}]$ J. All of the Gamow peaks were calculated for $T = T_\odot$.

The numerator is just the total energy lost to neutrino emission in the branch, $N_{^4_2\mathrm{He}}$ is the number of $^4_2$He nuclei produced and $Q_{4\,^1_1\mathrm{H}\to{}^4_2\mathrm{He}} = 26.732$ MeV is the total energy generated per $^4_2$He nucleus. Using the reference values for the Q_ν's, we get

$$\varrho^\nu_{\mathrm{PPI}} = \frac{2Q_{\nu,\mathrm{pp}}}{Q_{4\,^1_1\mathrm{H}\to{}^4_2\mathrm{He}}} = 1.983\%$$

$$\varrho^\nu_{\mathrm{PPII}} = \frac{Q_{\nu,\mathrm{pp}} + Q_{\nu,\mathrm{e7}}}{2Q_{4\,^1_1\mathrm{H}\to{}^4_2\mathrm{He}}} = 2.020\%$$

$$\varrho^\nu_{\mathrm{PPIII}} = \frac{Q_{\nu,\mathrm{pp}} + Q_{\nu,8}}{2Q_{4\,^1_1\mathrm{H}\to{}^4_2\mathrm{He}}} = 13.048\%$$

$$\varrho^\nu_{\mathrm{CNO}} = \frac{Q_{\nu,13} + Q_{\nu,15}}{Q_{4\,^1_1\mathrm{H}\to{}^4_2\mathrm{He}}} = 6.374\%$$

B. Relative energy production from each of the PP branches and the CNO cycle

Figure 1 shows the relative energy production between the PP branches and the CNO cycle, all normalized to the total $\varepsilon_{\mathrm{TOT}}$. The density was set to the Sun's density $\rho_\odot = 1.62 \times 10^5$ k/m^3 for all temperatures. Also included in the plot is the total energy production $\varepsilon_{\mathrm{TOT}}$ normalized to its maximum.

C. The Gamow peaks

Figure 2 shows the Gamow peaks for the six relevant fusion reactions in the PP chain and the CNO cycle in the energy range $E \in [10^{-17}, 10^{-13}]$ J. All of the Gamow peaks were calculated for $T = T_\odot$.

V. DISCUSSION

First of all, the model ended up passing the sanity check with none of the relative differences between the calculated Π_{ik}'s and the reference values given in the assignment surpassing 1%. This gives some confidence as to the legitimacy of the models results. However, there were some problems, as I will highlight in the next subsections.

A. Discrepancies between the calculated Q' and the reference values

Observing the values in table IV, we see that 6 out of 15 of the Q''s the model calculated had relative difference larger than 1% compared to the reference values given in the lecture notes. Particularly Q'_{e7} stands out as being highly inaccurate, having a relative difference of 10.388. The rest of the discrepancies can probably be explained by differing masses used to calculate the mass differences between reactants and products. I used the masses listed

on the isotopes' respective Wikipedia pages, with all the available significant figures. However, the Q'_{e7} must be due to a bigger error that I sadly couldn't identify.

These discrepancies led me to use the reference values for the Q''s in the rest of the calculation in the model.

B. The relative energy productions

First of all, in figure 1, we see that $\varepsilon_{\text{TOT}}/\max(\varepsilon_{\text{TOT}})$ increases exponentially after around 3×10^8 K. This makes sense since the total energy production should increase rapidly for higher temperatures. We also see that the PPI branch dominates the energy production in the core for temperature up to $\sim 7 \times 10^4$ K, while PPII dominates between $\sim 7 \times 10^4$ K and $\sim 3 \times 10^7$ K, and PPIII between $\sim 3 \times 10^7$ K and $\sim 10^8$ K. The CNO cycle dominates for temperature above $\sim 10^8$ K. The order of these periods of domination make sense, since each of them requires more thermal energy to fuse heavier and heavier nuclei.

However, the temperature limits of each are quite off compared to what's mentioned in the literature: e.g. according to the Wikipedia page on the proton-proton chain (Wikipedia 4., 2025), the PPI branch should dominate for temperatures up to 1.8×10^6 K, not 7×10^4 K!

Also, looking at where the temperature is $T_\odot$, for which the conditions in the stellar core should equal that of our Sun since the density is $\rho_\odot$, the PPII branch is clearly dominating. This disagrees with the fact that around 83.3% of the produced $^4_2$He in our Sun comes from the PPI branch (Wikipedia 4., 2025). So, $T = T_\odot$ the PPI branch should dominate, which it clearly does not!

I fear that I have used the wrong expressions for calculating the energies produced in each branch of the PP chain.

C. The Gamow peaks

Looking at figure 2, we notice that the centers of the Gamow peaks move towards higher and higher energies as the reactants of the fusion reactions become heavier and the repelling charges become larger: e.g. the leftmost peak is the one associated with proton-proton fusion, which involve the lightest reactants, while the rightmost peak is associated with the proton capture by $^{14}_7$N, which involves the heaviest reactants out of the six reactions. The peaks tell us for which particle energies the reactants are most likely to overcome the Coloumb barrier between them and fuse. Thus it makes intuitive sense that the value of this energy should increase as the repelling charges and masses increase.

I unfortunately didn't have the time to compare the specific values of the central energies of the peaks with the literature, but qualitatively it seems to make sense.

VI. CONCLUSION

In this first assignment of AST3310, I have constructed a model for the energy production inside a homogeneous and isotropic stellar core given its temperature and density. We have looked at some nucleosynthesis theory that allow us to calculate the energies produced in the individual nuclear reactions in the PP chain and the CNO cycle: two of the known sets of reactions that fuse $^1_1$H into $^4_2$He, which is the most important fusion process inside main sequence stars.

My model ended up passing the "sanity check" imposed by the assignment and succeeded in calculating the power produced per unit volume per unit time for the individual nuclear reactions. Sadly, it didn't give adequate results for the relative energy production between the PP branches and the CNO cycle as functions of temperature, even though the qualitative trends matched our intuition.

VII. REFLECTION

For me, one of the main takeaways from this first project is that I have to better manage my time! I think I used too much time on the programming. The goal was to make it easy to implement in the next project, but I should probably have moved on to the report sooner than I did. It also didn't help that I got sick in the middle of the project period!

However, I still learned a lot about how the fusion reaction in stellar cores happen. I must admit that I hadn't really considered how the nuclei overcome the Coloumb barrier between them and how unlikely this really is. I didn't know that quantum tunneling was the main reason why the fusion reactions happen in main sequence stars. It is really fascinating to see that quantum mechanical effects play such an important role in stellar nucleosynthesis.

Another thing that frustrated me slightly was the discrepancies between the Q' values I calculated and those given in the lecture notes. It would have been nice to have the masses of the different isotopes listed somewhere in the lecture notes for example so we could use the same masses in our calculations.

REFERENCES

1. *Gamow factor*, Wikipedia. [Online]. Available: https://en.wikipedia.org/wiki/Gamow_factor. [Accessed: 09.03.2025].

2. *Triple-alpha process*, Wikipedia. [Online]. Available: https://en.wikipedia.org/wiki/Triple-alpha_process. [Accessed: 07.03.2025].

3. *CNO cycle*, Wikipedia. [Online]. Available:

https://en.wikipedia.org/wiki/CNO_cycle. [Accessed: 08.03.2025].

4. *Proton-proton chain*, Wikipedia. [Online]. Available: https://en.wikipedia.org/wiki/Proton%E2%80%93proton_chain. [Accessed: 09.03.2025].

Appendix A: The branches of the PP chain

The nuclear reactions in the PPI branch

Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_e$	1.177	0.265	λ_{pp}
${}^2_1\mathrm{D} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma$	5.494		λ_{pd}
${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + 2{}^1_1\mathrm{H}$	12.860		λ_{33}

The nuclear reactions in the PPII branch

Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_e$	1.177	0.265	λ_{pp}
${}^2_1\mathrm{D} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma$	5.494		λ_{pd}
${}^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma$	1.586		λ_{34}
${}^7_4\mathrm{Be} + \mathrm{e}^- \rightarrow {}^7_3\mathrm{Li} + \nu_e + \gamma$	0.049	0.815	$\lambda_{\mathrm{e}7}$
${}^7_3\mathrm{Li} + {}^1_1\mathrm{H} \rightarrow 2{}^4_2\mathrm{He}$	17.346		λ'_{17}

The nuclear reactions in the PPIII branch

Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_e$	1.177	0.265	λ_{pp}
${}^2_1\mathrm{D} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma$	5.494		λ_{pd}
${}^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma$	1.586		λ_{34}
${}^7_4\mathrm{Be} + {}^1_1\mathrm{H} \rightarrow {}^8_5\mathrm{B} + \gamma$	0.137		λ_{17}
${}^8_5\mathrm{B} \rightarrow {}^8_4\mathrm{Be} + \mathrm{e}^+ + \nu_e$	11.270	6.711	λ_8
${}^8_4\mathrm{Be} \rightarrow 2{}^4_2\mathrm{He}$	0.092		λ'_8

Table II. The nuclear reactions in the PP branches and their respective released energies and rates. Taken from the lecture notes.

Appendix B: The CNO cycle

The nuclear reactions in the CNO cycle

Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
${}^{12}_6\mathrm{C} + {}^1_1\mathrm{H} \rightarrow {}^{13}_7\mathrm{N} + \gamma$	1.944		$\lambda_{\mathrm{p}12}$
${}^{13}_7\mathrm{N} \rightarrow {}^{13}_6\mathrm{C} + \mathrm{e}^+ + \nu_e$	1.513	0.707	λ_{13}
${}^{13}_6\mathrm{C} + {}^1_1\mathrm{H} \rightarrow {}^{14}_7\mathrm{N} + \gamma$	7.551		$\lambda_{\mathrm{p}13}$
${}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} \rightarrow {}^{15}_8\mathrm{O} + \gamma$	7.297		$\lambda_{\mathrm{p}14}$
${}^{15}_8\mathrm{O} \rightarrow {}^{15}_7\mathrm{N} + \mathrm{e}^+ + \nu_e$	1.757	0.997	λ_{15}
${}^{15}_7\mathrm{N} + {}^1_1\mathrm{H} \rightarrow {}^{12}_6\mathrm{C} + {}^4_2\mathrm{He}$	4.966		$\lambda_{\mathrm{p}15}$

Table III. The nuclear reactions in the CNO cycle and their respective released energies and rates. Taken from the lecture notes.

Appendix C: The generated energies Q'

Values for Q'

Reaction	Calculated Q' [MeV]	Reference value [MeV]	Relative difference
pp	1.716	1.177	0.458
pd	5.508	5.494	0.003
33	12.833	12.860	-0.002
34	1.587	1.586	0.001
e7	0.558	0.049	10.388
$17'$	17.362	17.346	0.001
17	0.150	0.137	0.098
8	11.781	11.270	0.045
$8'$	0.092	0.092	-0.002
p12	1.958	1.944	0.007
13	2.024	1.513	0.338
p13	7.565	7.551	0.002
p14	7.311	7.297	0.002
15	2.268	1.757	0.291
p15	4.980	4.966	0.003

Table IV. The calculated values for the generated energies Q' resulting from the mass difference between reactants and products in each individual nuclear reaction compared with the reference values given in the lecture notes.