Euclidean Geometry via Complex Numbers

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- This makes things very general, but purely analytic proofs quickly become unrealistic.
- (Possible) Alternative: Define Euclidean geometry axiomatically, e.g., Tarski or Hilbert axioms.
- $\Rightarrow$ Analytic proofs still not feasible (or even less so).

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  - Analytic proofs become realistic.
- Cons:
  - Only 2-dimensional.
  - (basically) not in mathlib / from scratch $\Rightarrow$ much work!

Implementation

- Points are just complex numbers.
- Distance between points is the usual Euclidean metric.
- Three points are colinear iff a certain determinant is zero.
- A line is (basically) a set of colinear points.

Difficulties

- **Many** "trivial" statements to prove.
- Edge cases: For example, line through two points or angles.
- Proving nonlinear systems of equations is very tedious.
- Compilation time is sometimes very slow, and max heartbeats are sometimes too few.

Overview of Formalisation

- Foundation of Euclidean geometry (mostly) finished.
- "Advanced" statements include:
  - Thales' theorem (both directions).
  - Existence of orthocenter (via power of a point).
  - Existence of circumcircle.
  - Same angles imply triangles are similar.
  - Ceva's theorem (part of Freek's 100 problems list).

Outlook

- Improve code quality and remove redundancies.
- Finish missing "foundational" parts, e.g., Pasch's axiom or incircle.
- Develop a framework for more involved analytical proofs via the unit circle.