EE16A - Lecture 21 Notes

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Determine Distance from Beacon

1. Sent signal (periodic) = $a[n] = a[n+N], \forall n \in \mathbb{Z}$
2. Received signal (Time delayed) = $y[n] = \alpha a[n - N_A]$
3. Find time delay N_A by finding the maximum inner product of each shifted version of the original signal and the received signal ($max(|\langle S^k a, y\rangle|), k = 1, \cdots, N$) where resulting $k = N_A$
4. Determine distance (d_A) using time delay N_A and given speed of wave signal
5. **Note:** When doing inner products to get correct shifts $S^k, k =$ shift, you want everything to be as close as possible to mutually orthogonal with shifted version, so only the inner product of the original signal with itself will be large (peak) and everything else will be noticably smaller

- Beacon A sends periodic signal with period N ($a[n] = a[n+N], \forall n \in \mathbb{Z}$)
- Receive attenuated/timed delayed version of the sent signal
  - Goal: Figure out time delay and derive distance using the given speed of the wave being emitted
- Need 3 beacons to pinpoint one point on a 2D plane
- Find d_A, d_B, d_C from signals A, B, C which are all $N - periodic$
- Receiver gets some linear combination the 3 signals ($y[n] = \alpha a[n - N_A] + \beta b[n - N_B] + \gamma c[n - N_C] = \alpha S^{N_A} a + \beta S^{N_B} b + \gamma S^{N_C} c$)
- Linear waves: Superposition principles applies (Electromagnetic and acoustic waves are approx. linear)
- Determine $N_A, N_B, N_C \rightarrow d_A, d_B, d_C$, by using $max(\langle S^k a, y\rangle), k = 1, \cdots, N$

Vector Model of a Signal Period:

$$y = \begin{bmatrix} y[n] \\ \vdots \\ y[N-1] \end{bmatrix} \tag{1}$$

Triangulation

1. When subtracting vectors, arrow point towards "positive" value
2. **Minimizing Error**: Find x s.t. $\|\varepsilon\|^2$ is minimized where $\varepsilon = b - Ax$ (Result data we receive $Ax = b - \varepsilon$)
3. $\varepsilon \perp a_1$ & $\varepsilon \perp a_2$: ε has to be orthogonal to each of the columns of the A matrix

- Retrieve position after distances from each beacon received

- $\|a_n - x\|^2 = \langle a_n - x, a_n - x\rangle = \langle a_n, a_n\rangle - 2\langle a_n, x\rangle + \langle x, x\rangle = \|a\|^2 + 2\langle a_n, x\rangle + \|x\|^2 = d_n^2$
- For 3-beacon system, subtract d_2 and d_3 from d_1:
    * $2\langle a_2, x\rangle - 2\langle a_1, x\rangle = 2\langle a_2 - a_1, x\rangle = d_1^2 - d_2^2 + \|a_2\|^2 - \|a_1\|^2$
    * $2\langle a_3, x\rangle - 2\langle a_1, x\rangle = 2\langle a_3 - a_1, x\rangle = d_1^2 - d_3^2 + \|a_3\|^2 - \|a_1\|^2$
- Matrix multiplication and solve for b_1, b_2
    - $Ax = b$ has a unique soln. if $a_2^T - a_1^T$ and $a_3^T - a_1^T$ are lin. independent
    - As long as the three beacons are not co-linear, the above will be lin. ind.
    - Use more beacons to do pairwise subtractions and account for error. This will just make A an $n \times 2$ matrix, and we will get $b \in \mathbb{R}^n$
- **Problem:** $b \notin$ span of cols of A because there is error in the signals.
    - $Ax = b$ has no exact soln.
    - b is some vector outside the subspace of $x_1A_1 + x_2A_2$ (outside of the plane formed by the two vectors of A)
    - Error $\varepsilon = b - Ax$
    - Find x s.t. $\|\varepsilon\|^2$ is minimized
    - $Ax = a_1x_1 + a_2x_2$, which is a line between the two vectors (a_1, a_2).
    - Choose point on Ax that creates a perpendicular line from that point to b in order to get the smallest ε
    - ε is perpendicular to the plane of the subspace between a_1, a_2

Matrix Multiplication Step for Triangulation

$$\begin{bmatrix} 2(a_2^T - a_1^T) \\ 2(a_3^T - a_1^T) \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{2}$$

Basic Projection Problem

> 1. $\langle \vec{a}, \varepsilon = b - \vec{a}x\rangle = 0$, Solve for x
> 2. Best estimate for $\vec{b}$ in the least square error sense is $\hat{b} = \vec{a}x - \frac{\vec{a}^T\vec{b}}{\vec{a}^Ta}\vec{a}$

- Have one dimensional vector $\vec{a}$ and $\vec{b}$ outside of the subspace.
- Want to get value x on $\vec{a}$ s.t. $\varepsilon = b - \vec{a}x$ is minimized.
    - Optimal x makes ε orthogonal to $\vec{a}$
    - $\varepsilon \perp a \implies \langle \vec{a}, \varepsilon\rangle = 0 \implies \langle \vec{a}, b - \vec{a}x\rangle = 0$
- Solve for x:
    - $\langle \vec{a}, \vec{b}\rangle - \langle \vec{a}, \vec{a}x\rangle = 0$
    - $\langle \vec{a}, \vec{b}\rangle - x\langle \vec{a}, \vec{a}\rangle = 0$
    - $x = \frac{\langle \vec{a}, \vec{b}\rangle}{\langle \vec{a}, \vec{a}\rangle} = \frac{\vec{a}^T\vec{b}}{\vec{a}^Ta}$
- Best estimate for $\vec{b}$ in the least square error sense is $\hat{b} = \vec{a}x - \frac{\vec{a}^T\vec{b}}{\vec{a}^Ta}\vec{a}$
- Can derive Cauchy-Schwarz from $\|\varepsilon\| = \|\vec{b} - \hat{b}\|^2 \geq 0$

Cauchy-Schwarz Inequality

$$|\langle \vec{a}, \vec{b}\rangle| \leq \|\vec{a}\|\|\vec{b}\| \tag{3}$$

Multi-Dimension Projection Problem

> 1. $x = (A^T A)^{-1} A^T b$

- $A_{n\times m} x_{m\times 1} = b_{n\times 1}$
- Let $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$
- Optimal x makes ε orthogonal to each a_n (column vec) of A
  - $\varepsilon \perp a_n \implies \langle \vec{a_n}, \varepsilon \rangle = \langle \vec{a_n}, b - Ax \rangle = \vec{a_n}^T (\vec{b} - Ax) = 0$
- Solve for x: $A^T(\vec{b} - Ax) = 0$
  - $A^T(\vec{b} - Ax) = 0 \implies A^T b - A^T A x = 0 \implies A^T A x = A^T b$
  - $A^T A$ is invertible iff A has full column rank (lin. indep. columns)
  - $x = (A^T A)^{-1} A^T b$

Solve for x:

$$\begin{bmatrix} \vec{a_1}^T \\ \vdots \\ \vec{a_n}^T \end{bmatrix} (\vec{b} - Ax) = 0 \tag{4}$$