EE16A - Lecture 28 Notes

Name: Felix Su SID: 25794773

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Decoupling State Space Equation

> 1. Summary

- State Space equation (SSE): $\vec{s}[n+1] = A\vec{s}[n]$
- $\vec{s}[n] = V\vec{q}[n] \implies \vec{s}[n+1] = V\vec{q}[n+1]$
  - SSE becomes $V\vec{q}[n+1] = AV\vec{q}[n]$
  - $A = V\Lambda V^{-1} \implies V^{-1}AV = \Lambda$
  - Premultiply by $V^{-1} \implies \vec{q}[n+1] = V^{-1}AV\vec{q}[n]$
- So, you have a new equation: $\vec{q}[n+1] = \Lambda\vec{q}[n]$
- Good because:
  - Original SSE: $\vec{s}_k[n+1] = \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kN} \end{bmatrix} \begin{bmatrix} \vec{s}_1[n] & \vec{s}_2[n] & \cdots & \vec{s}_N[n] \end{bmatrix}^T = \sum_{l=1}^{N} \vec{a}_{kl}\vec{s}_k[n]$
  - Can't solve this independently of other state variables, because they are coupled and appear in the right hand side of the equation
  - Shift from $\vec{s}$ to $\vec{q}$ = decoupled state variables
  - Can solve for the state $\vec{q}$ independently of the others

Change of Basis:

$$\begin{bmatrix} \vec{q}_1[n+1] \\ \vdots \\ \vec{q}_k[n+1] \\ \vdots \\ \vec{q}_N[n+1] \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & \ddots & \\ 0 & & & & \lambda_N \end{bmatrix} \begin{bmatrix} \vec{q}_1[n] \\ \vdots \\ \vec{q}_k[n] \\ \vdots \\ \vec{q}_N[n] \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{q}_1[n] \\ \vdots \\ \lambda_k\vec{q}_k[n] \\ \vdots \\ \lambda_N\vec{q}_n[n] \end{bmatrix} \tag{1}$$

Decoupled State Evolution Equation

$$\vec{q}_k[n+1] = \lambda_k\vec{q}_k[n], \text{ for } k = 1, \ldots, N \tag{2}$$

Solving independent Decoupled States
Given $\vec{s}[0]$, $\vec{q}[0] = V\vec{s}[0]$, so:

$$\vec{q}_k[n] = \lambda_k^n\vec{q}_k[0] \text{ for } k = 1, \ldots, N \tag{3}$$

Solving $\vec{q}_k$

- $\vec{s}[n] = V\vec{q}[n] \to \vec{q}[n] = V^{-1}\vec{s}[n]$
- We know initial state $\vec{s}[0]$ so we also know initial state $\vec{q}[0] \to \vec{q}[0] = V^{-1}\vec{s}[0]$
- So, we have $\vec{q}_k[0]$ (known) and the new state evolution equation $\vec{q}_k[n+1] = \lambda_k\vec{q}_k[n]$
- This means, $\vec{q}_k[n] = \lambda_k^n\vec{q}_k[0]$ for $k = 1, \ldots, N$

Basis Transformation

- Canonical basis: set of vectors e_i that form the identity matrix
- Want to use another set of lin. indep. vectors $\vec{z_1}, \ldots, \vec{z}_n$ as our principal axes
  - $\vec{s} = Z\vec{x}^{[Z]}$ where $Z = span\{\vec{z_1}, \ldots, \vec{z}_n\}$
  - Want to express vecotr x in terms of vectors in z
  - Z is a square matrix of linearly independent cols, so it is an invertible matrix
  - $\vec{x}^{[Z]}$ represents $\vec{x}$ in the corrdinate sys. given by the z's
- $x = Z\vec{x}^{[Z]} \implies \vec{x}^{[Z]} = Z^{-1}\vec{x}$, one to one mapping between x and $x^{[Z]}$ because Z is invertible.

Representation of a Linear Transformation in a new Coordinate System

- $\vec{y} = A\vec{x}$ where $\vec{x} = Z\vec{x}^{[Z]}$ and $\vec{y} = Z\vec{y}^{[Z]}$
- So, $Z\vec{y}^{[Z]} = AZ\vec{x}^{[Z]}$ and Z is invertible, so,
- $\vec{y}^{[Z]} = Z^{-1}AZ\vec{x}^{[Z]} \to y = Ax$
  - A takes the vector x and maps it to the vector y in the orginal coordinate system
  - In the new coordinate system, the same linear transformation maps the coordinate vector for x into the coordinate vector for y by $A^{[Z]} = Z^{-1}AZ$, the transfomation matrix in the new coordinate system
- If $Z = V$ (eigenvector matrix for A) where $A = V\Lambda V^{-1}$
  - Then the coordinate transformation diagonlizes A
  - $A^{[Z]} = V^{-1}AV = V^{-1}V\Lambda V^{-1}V = \Lambda$
  - Switching to this new coordinate system and diagonalizing matrix A decouples the state variables in the new coordinate system

From Canonical Coordinate System to Coordinate System of Z

$$\vec{y}^{[Z]} = A^{[Z]}\vec{x}^{[Z]} \text{ where } A^{[Z]} = Z^{-1}AZ \tag{4}$$

Transformations Across Coordinate Systems

$$y \to y\vec{y}^{[Z]} = Z^{-1}y \tag{5}$$

$$x \to x\vec{x}^{[Z]} = Z^{-1}x \tag{6}$$

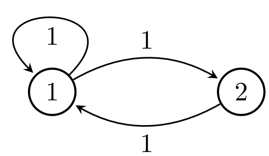
$$A \to A^{[Z]} = Z^{-1}AZ \tag{7}$$

If $Z = V$ (eigenvector matrix for A) where $A = V\Lambda V^{-1}$

$$A^{[Z]} = \Lambda \tag{8}$$

Example:
Fibonacci Sequence

$$\vec{s}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$\vec{s}[n+1] = A\vec{s}[n]$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$\vec{s}[1] = A\vec{s}[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{s}[2] = A\vec{s}[1] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{s}[3] = A\vec{s}[2] = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \vec{s}[4] = A\vec{s}[3] = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \vec{s}[5] = A\vec{s}[4] = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

Eigendecomposition of A

Find $(\lambda_1, \vec{v}_1)$ $(\lambda_2, \vec{v}_2)$

$\vec{s}[n] = \alpha_1 \lambda_1^n \vec{v}_1 + \alpha_2 \lambda_2^n \vec{v}_2$ where α_1, α_2 are determined by initial state: $\vec{s}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ s.t. $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **Solve**

$A - \lambda I$ $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$ where $\det(A - \lambda I) = -\lambda(1-\lambda) - 1 = la^2 - \lambda - 1 = 0$

Roots: $\frac{1 \pm \sqrt{5}}{2}$, $\lambda_1 = \frac{1-\sqrt{5}}{2}$, $\lambda_2 = \frac{1+\sqrt{5}}{2}$ (**Golden Ratio**)