1. **Gaussian Elimination Matching**

   Consider, a system of equations $A\vec{x} = \vec{b}$ where A is $m \times n$. Suppose you row-reduce the augmented matrix $\left[A \mid \vec{b} \right]$ to the form $\left[G \mid \vec{c} \right]$.

   Match each scenario on the first table with one of the properties of systems of equations in the second table.

Scenarios	
(1)	A is square. G is the identity
(2)	A is tall $(m > n)$. The bottom $m - n + 1$ rows of both G and $\vec{c}$ are 0's.
(3)	A is fat $(n > m)$. The last row of G is 0's, but the last element of $\vec{c}$ is 1.

Properties of System	
(a)	The system of equations has many solutions.
(b)	The system of equations has a unique solution.
(c)	The system of equations has no solutions.

   Solutions:

2. **Diagon Alley**

   Let A be an $n \times n$ diagonalizable matrix. **Show that the determinant of A is the product of its eigenvalues.**

   Hint: Recall these properties that you have already seen established and that follow from the meaning of the determinant as an oriented volume.

   - For square matrices X, Y: $\det(XY) = \det(X)\det(Y)$.
   - For invertible matrices X: $\det(X^{-1}) = 1/\det(X)$.
   - The determinant of a diagonal matrix is the product of its diagonal entries.

   Solutions: $X = V\Lambda V^{-1}$ so

$$
\begin{aligned}
\det(X) &= \det(V\Lambda V^{-1}) \\
&= \det(V)\det(\Lambda)\det(V^{-1}) \\
&= \det(V)(\prod_i \lambda_i)\det(V^{-1}) \\
&= (\prod_i \lambda_i)\det(V)\det(V^{-1}) \\
&= (\prod_i \lambda_i)\det(VV^{-1}) \\
&= (\prod_i \lambda_i)\det(I) \\
&= (\prod_i \lambda_i)
\end{aligned}
$$

3. Diagonalization

Diagonalize $M = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}$, that is, write $M = PDP^{-1}$ where D is a diagonal matrix.

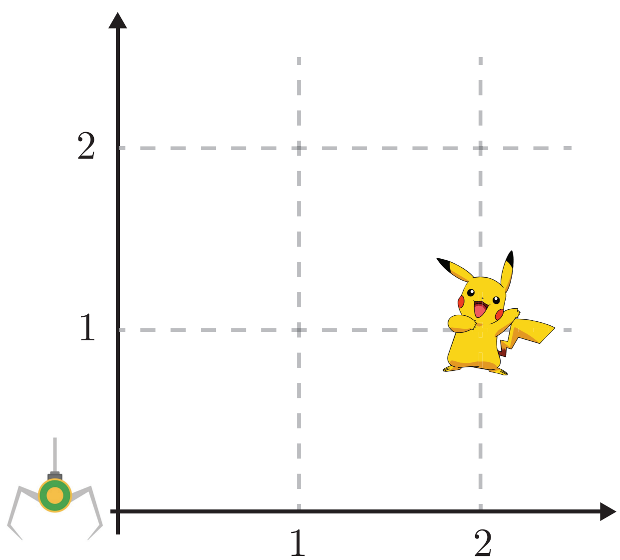
Solutions:

$$\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

4. Collect or Broke

Finals are finally over, and your friends decide to go to the nearest arcade to play some games and hang out. You're wondering what to play when you spot the Claw Crane Toy Pickup, and you think to yourself "I was so busy studying I didn't have time to get presents for my friends and family! If I can pick up some toys, that would be the perfect present!" You set your sights on a Pikachu plushie.

The crane starts at the lower left corner of the machine $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and you can navigate it with the controller. Then, you press a button for the claw to shoot down and attempt to pick up the plushie.



We will say that the commands you give the controller can be represented in a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. So for example, if the Pikachu was at location (2, 1) (as seen in the picture) you would navigate to it with the control $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) The Claw Crane is broken! Someone reversed the wiring during setup, and every time you steer it right it goes left, if you steer it down it goes up. If the pikachu is again at (2, 1), what command would you give the controller to reach it?

Solutions: $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$

(b) Write the above setup as a change of basis. What is the basis where the wiring is reversed? What is the basis you want to change to? What does the controller do to your input command (what matrix does it multiply by) to output its action? Given a toy location, how would you find the command to give the controller?

Solutions: The wiring reversed basis is $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$.

We want to change to standard basis $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

The controller is in the wiring reversed basis, so it takes the input command and changes to standard basis.

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2\\ -1\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$$

Given a toy location in the standard basis, you find the input command by changing to the controller's basis.

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}^{-1}\begin{bmatrix}-2\\ -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-2\\ -1\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$$

(c) The Claw is worse off than you previously thought! When you input "right" it goes up, "left" it goes down, "up" it goes left, and "down" it goes right. Set it up as a change of basis and find the input command to navigate to the pikachu at (2, 1).

Solutions: Recall a rotation matrix is $\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$. The commands have been rotated 90°, so the basis is $\begin{bmatrix}\cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right)\\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right)\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

To find the input command, change the toy location basis to controller basis.

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}1\\ -2\end{bmatrix}$$

(d) The Claw is very subtly rigged, such that on control input $\begin{bmatrix}x\\ y\end{bmatrix}$, the claw moves to

$$\begin{bmatrix}\cos(3^\circ) & \cos(95^\circ)\\ \sin(3^\circ) & \sin(95^\circ)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

What command would you input to reach the location (2, 1)?
Use iPython to calculate the command, and answer to the second decimal place.

(There won't be IPython questions on the exam, but this is here to help you practice the underlying understanding.)

Solutions: Recall the original rotation matrix is: $\begin{bmatrix}\cos(\theta) & \cos(90^\circ+\theta)\\ \sin(\theta) & \sin(90^\circ+\theta)\end{bmatrix}$. Thus, the controller basis is $\begin{bmatrix}\cos(3^\circ) & \cos(90^\circ+5^\circ)\\ \sin(3^\circ) & \sin(90^\circ+5^\circ)\end{bmatrix}$.

$$\begin{bmatrix}\cos(3^\circ) & \cos(95^\circ)\\ \sin(3^\circ) & \sin(95^\circ)\end{bmatrix}^{-1}\begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}2.08\\ 0.89\end{bmatrix}$$

5. Orthonormal matrices

(a) You are given two n-dimensional complex-valued vectors $\vec{s}$ and $\vec{t}$ and an $n \times n$ orthonormal complex-valued matrix P.

Recall that an orthnormal matrix has orthonormal columns, which means that for two columns $\vec{p}_i$ and $\vec{p}_j$, $\langle \vec{p}_i, \vec{p}_j \rangle$ is 1 if $i = j$ and 0 otherwise. **Show that the inner product stays the same after transforming to the orthonormal basis of P:**

$$\langle \vec{s}, \vec{t} \rangle = \langle P^* \vec{s}, P^* \vec{t} \rangle$$

Recall that the inner-product is defined as

$$\langle \vec{u}, \vec{v} \rangle := \vec{u}^* \vec{v}$$

where * denotes conjugate-transpose. Recall further that * obeys similar algebraic properties as ^{T}, for example $(AB)^* = B^* A^*$ for any matrices A, B.

(b) Suppose P is the matrix of DFT basis vectors:

$$P = \begin{bmatrix} | & & | \\ \vec{p}_0 & \cdots & \vec{p}_{n-1} \\ | & & | \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{\frac{2\pi i}{n}} & e^{\frac{2\pi i(2)}{n}} & \cdots & e^{\frac{2\pi i(n-1)}{n}} \\ 1 & e^{\frac{2\pi i(2)}{n}} & e^{\frac{2\pi i(4)}{n}} & \cdots & e^{\frac{2\pi i 2(n-1)}{n}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{\frac{2\pi i(n-1)}{n}} & e^{\frac{2\pi i 2(n-1)}{n}} & \cdots & e^{\frac{2\pi i(n-1)(n-1)}{n}} \end{bmatrix} \tag{1}$$

You have a time-domain signal $\vec{s}$ and its frequency domain representation $\vec{w}$ that satisfies the relation $\vec{s} = P\vec{w}$. Both vectors have $n = 16$ elements. Someone tells you that

$$\vec{w} = \left[1, 0, 0, e^{j\frac{\pi}{2}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T$$

What is $\|\vec{s}\|^2$?

The actual exam won't be this heavy on OMP and Least-Squares Problems. However, we had requests for more exam-like practice problems in this area because most of the HWs were IPython oriented.

6. OMP

(a) Suppose we have a vector $\vec{x} \in \mathbb{R}^4$. We take 3 measurements of it, $b_1 = \vec{m}_1^T\vec{x} = 4$, $b_2 = \vec{m}_2^T\vec{x} = 6$ and $b_3 = \vec{m}_3^T\vec{x} = 3$, where $\vec{m}_1$, $\vec{m}_2$ and $\vec{m}_3$ are some measurement vectors. In the general case when there are 5 unknowns in $\vec{x}$ and we only have 3 measurements, it is not possible to solve for $\vec{x}$. Furthermore, there could be noise in the measurements. However in this case, we are given that $\vec{x}$ **is sparse, and only has 2 non-zero entries**. In particular,

$$M\vec{x} \approx \vec{b} \tag{2}$$

$$\begin{bmatrix} - & \vec{m}_1^T & - \\ - & \vec{m}_2^T & - \\ - & \vec{m}_3^T & - \end{bmatrix} \vec{x} \approx \vec{b} \tag{3}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} \tag{4}$$

Where exactly 2 of x_1 to x_4 are non-zero. **Use Orthogonal Matching Pursuit to estimate x_1 to x_4.** Do this by hand.

Solutions: Let $\vec{c}_1$ to $\vec{c}_4$ be the column vectors of M. We first find the column vector in M that correlates most with $\vec{b}$:

$$\begin{bmatrix} \left\langle \vec{c}_1, \vec{b} \right\rangle & \left\langle \vec{c}_2, \vec{b} \right\rangle & \left\langle \vec{c}_3, \vec{b} \right\rangle & \left\langle \vec{c}_4, \vec{b} \right\rangle \end{bmatrix} = \begin{bmatrix} 10 & 7 & -3 & -1 \end{bmatrix} \tag{5}$$

Thus the $\vec{c}_1$ is the best matching vector, and we compute

$$\vec{b}' = \vec{b} - \vec{c}_1 \frac{\left\langle \vec{b}, \vec{c}_1 \right\rangle}{\langle \vec{c}_1, \vec{c}_1 \rangle} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \tag{6}$$

which is the subtraction of the projection of $\vec{b}$ onto $\vec{c}_1$ from $\vec{b}$. Then we find the largest correlation again:

$$\begin{bmatrix} \left\langle \vec{c}_1, \vec{b}' \right\rangle & \left\langle \vec{c}_2, \vec{b}' \right\rangle & \left\langle \vec{c}_3, \vec{b}' \right\rangle & \left\langle \vec{c}_4, \vec{b}' \right\rangle \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 4 \end{bmatrix} \tag{7}$$

Thus we know that $\vec{c}_1$ and $\vec{c}_4$ contributes most to $\vec{b}$. However no linear combination of $\vec{c}_1$ and $\vec{c}_4$ can form $\vec{b}$, because of noise in the measurements. Thus we need to find the least-squares solution. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the least-squares formula gives:

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 6\frac{1}{3} \\ 2\frac{2}{3} \end{bmatrix} \tag{8}$$

Thus $\vec{x} \approx \begin{bmatrix} 6\frac{1}{3} \\ 0 \\ 0 \\ 2\frac{2}{3} \end{bmatrix}$.

(b) We know that OMP works only when the vector $\vec{x}$ is sparse, which means that it has very few non-zero entries. What if $\vec{x}$ is not sparse in the standard basis but is only sparse in a different basis? What we can do is to change to the basis where $\vec{x}$ is sparse, run OMP in that basis and transform the result back into the standard basis. Suppose we have a $m \times n$ measurement matrix M and a vector of measurements $\vec{b} \in \mathbb{R}^m$ where $M\vec{x} = \vec{b}$ and we want to find $\vec{x} \in \mathbb{R}^n$. The basis that $\vec{x}$ is sparse in is defined by basis vectors $\vec{a}_1 \cdots \vec{a}_n$, and we define:

$$A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$$

So that the true $\vec{x} = A\vec{x}'$ and $\vec{x}'$ is sparse.

Suppose we have an OMP function which will compute $\vec{x}'$ given M' and $\vec{b}'$ **only when $\vec{x}'$ is sparse**: $\vec{x}' = \text{OMP}(M', \vec{b}')$. Assuming that the change of basis does not significantly affect the orthogonality of vectors, describe how you would compute $\vec{x}$ using the function OMP.

Solutions: Since

$$M\vec{x} = \vec{b} \tag{9}$$
$$MA\vec{x}' = \vec{b} \tag{10}$$

We get:

$$\vec{x} = A \cdot \text{OMP}(MA, \vec{b}) \tag{11}$$

7. Least Squares

Suppose we are trying to solve the following least-squares problem, where the matrix A is $n \times 2$, and the vector $\vec{b}$ is length n.

$$\min_{\vec{x}} ||A\vec{x} - \vec{b}||^2$$

Let the columns of matrix A be $\vec{a}_1, \vec{a}_2$, such that

$$A = \left[\vec{a}_1, \vec{a}_2\right].$$

Now, suppose we don't know the entries of matrix A and vector $\vec{b}$ explicitly, but we do know all the inner-products:

- $\langle \vec{a}_i, \vec{a}_j \rangle$ for all $i, j \in \{1, 2\}$.
- $\left\langle \vec{a}_i, \vec{b} \right\rangle$ for all $i \in \{1, 2\}$.
- $\left\langle \vec{b}, \vec{b} \right\rangle$

(a) Given these inner-products, can we solve for the least-squares solution $\hat{x}$? Express $\hat{x}$ in terms of the inner-products above.

(b) Using $\hat{x}$ as above, can we compute the norm of the residual $\vec{e} = A\hat{x} - \vec{b}$ using only the provided inner-products? Express

$$\|A\hat{x} - \vec{b}\|^2$$

in terms of the inner-products above.

8. What should Maximum Correlation really mean?

Recall that in OMP we have a vector $\vec{b}$ representing our observation, which is a (noisy) linear combination of a few of the vectors $\vec{c}_1, \cdots, \vec{c}_n$. The first step of OMP is to find the vector $\vec{c}_i$ ($1 \le i \le n$) that has the largest "correlation" with $\vec{b}$. More precisely, subtract the projection of $\vec{b}$ onto $\vec{c}_i$ from $\vec{b}$ and call that the residue. We want to find the index i such that $\vec{c}_i$ minimizes the length of the residue:

$$\min_i \left\| \vec{b} - \vec{c}_i \frac{\left\langle \vec{b}, \vec{c}_i \right\rangle}{\langle \vec{c}_i, \vec{c}_i \rangle} \right\|^2$$

Show that this is equivalent to:

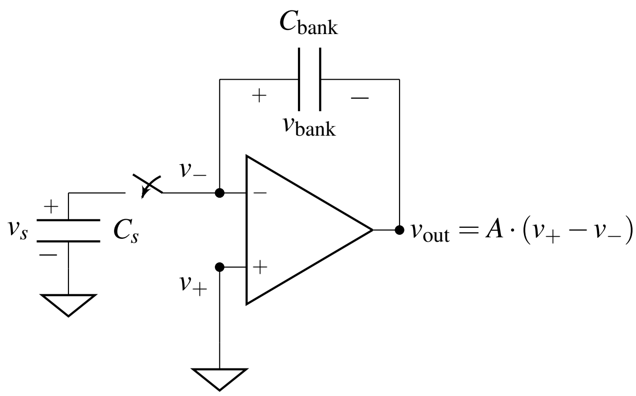
$$\max_i \frac{\left\langle \vec{b}, \vec{c}_i \right\rangle^2}{\langle \vec{c}_i, \vec{c}_i \rangle}$$

Note in particular that when all $\vec{c}_i$ have the same length, that is $||\vec{c}_i|| = ||\vec{c}_j||$ for all i and j, then this is equivalent to $\max_i \left\langle \vec{b}, \vec{c}_i \right\rangle$.

9. Least Squares By Circuit

In this problem, you can assume that the charges settle immediately after switching.

(a) Consider the circuit below. Initially, C_s is charged with Q_s charge and C_{bank} is charged with 0C charge (that is, uncharged). What are the charges on C_s and C_{bank} after switching as a function of A?



(b) What are the voltages v_s and v_{bank} after switching as a function of the op-amp gain, A?

(c) What are the charges on C_s and C_{bank} as $A \to \infty$?

(d) What are the voltage drops v_s and v_{bank} as $A \to \infty$?

(e) Assume you are given a system of linear equations

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} x \approx \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{12}$$

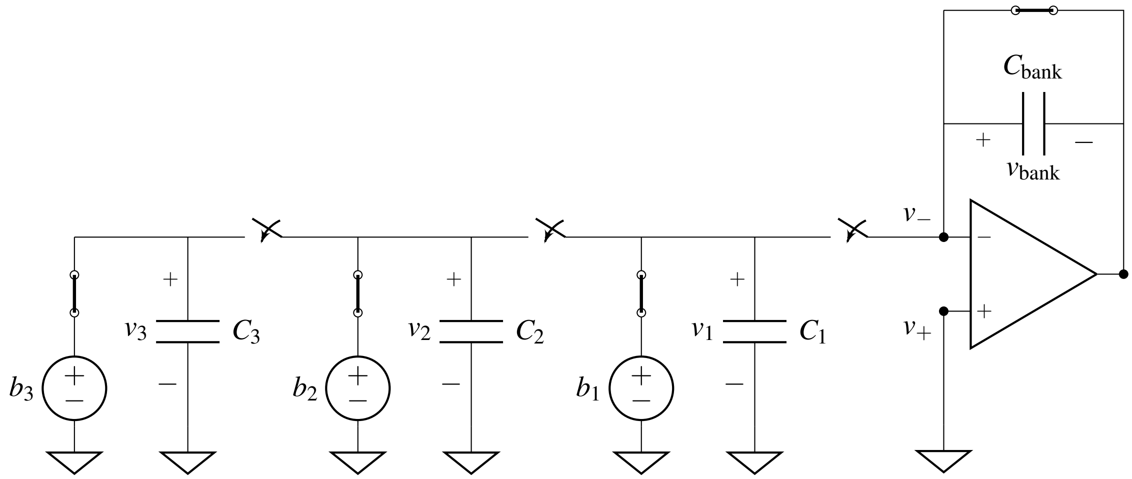
Show that the linear least squares solution to the system in Equation **??** is

$$\widehat{x} = \frac{3b_1 + 2b_2 + b_3}{9+4+1} = \frac{3b_1 + 2b_2 + b_3}{14}$$

Hint: Write down A, $\vec{x}$ and $\vec{b}$; write down the solution for $\widehat{x}$ and simplify.

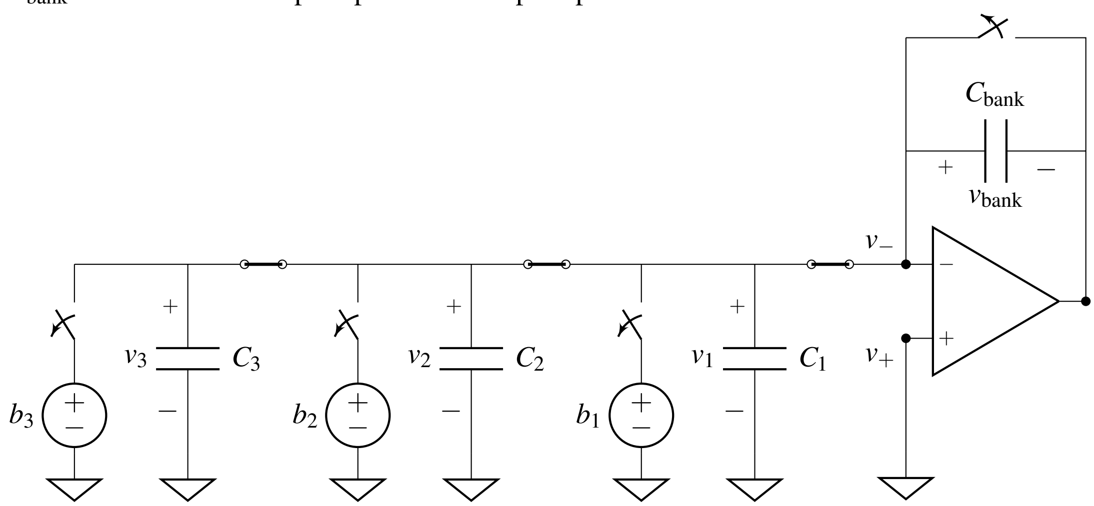
Solutions:

(f) We will now design a circuit that takes the values of b_1, b_2 and b_3 as voltages, and outputs the solution $\widehat{x}$ as voltage. The circuit will operate in two phases. In the first phase, we will charge a set of capacitors so that the capacitors would hold charge $Q_1 = 3b_1, Q_2 = 2b_2$ and $Q_3 = b_3$. Set the values of the capacitors C_1, C_2 and C_3 in the circuit below to achieve this.



Solutions:

(g) In the second phase, we flip all the switches, as shown below. Calculate the charge on the capacitor C_{bank}. Assume that the op-amp is an ideal op-amp with $A \to \infty$.



Solutions:

(h) Set the value of the capacitor C_{bank} so that $v_{\text{bank}} = \widehat{x} = \frac{3b_1+2b_2+b_3}{14}$. Assume that the op-amp is an ideal op-amp with $A \to \infty$.

Solutions:

10. Digital to Analog Converter

In this problem, you will design a 3-bit digital to analog converter (DAC), that converts a three bit digital input to an analog value.

A 3-bit binary number $b = [b_1, b_2, b_3]$ for $b_i \in \{0, 1\}$ can be used to represent an integer V_b in the range $\{0, 1, \ldots, 7\}$, as:

$$V_b = 4b_1 + 2b_2 + b_3 \tag{13}$$

For example $b = [1, 0, 1]$ represents the number $4(1) + 2(0) + (1) = 5$.

Suppose you are given inputs b_i as **current sources**, which supply either $0mA$ or $1mA$ of current depending on if bit b_i is 0 or 1. For example, if the input number was $b = [1, 0, 1]$, then the first and third current sources would supply $1mA$ of current, while the second source would supply $0mA$.

Design a circuit to output the **voltage** V_b (in Volts) corresponding to the binary number $b = [b_1, b_2, b_3]$, as in Equation (**??**).

Use only resistors and op-amps in your design. Assume that the current sources will supply their current at any required voltage.

Warning: Be careful of loading in your design. You may use as many op-amps are you want.