EE16A Sample Final Solutions

Google Stalking

1. Linear model would look like this: given demographic information, use a set of weights to linearly combine those numbers.

I.e., $c_N = k_1 \cdot age_N + k_2 \cdot income_N + k_3 \cdot ethnicity_N + k_4 \cdot politics_N + k_5 \cdot query_N$

$\uparrow$ user N will click on the ad or not

(In practice you would you probably take $\text{sign}(c_N)$ to get the I 1 values in the problem description.)

2. Basically we need to perform a linear regression to find the appropriate values of $k_1 \dots k_5$ to use in order to predict a new user's behavior.

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{1000} \end{bmatrix} \approx \begin{bmatrix} age_1 & income_1 & ethnicity_1 & politics_1 & query_1 \\ age_2 & income_2 & ethnicity_2 & politics_2 & query_2 \\ & & \vdots & & \\ age_{1000} & income_{1000} & ethnicity_{1000} & politics_{1000} & query_{1000} \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix}$$

$\vec{c} = A \cdot \vec{k} + \varepsilon \longrightarrow$ use least squares to find best $\vec{k}$:

$$\vec{k} = (A^T A)^{-1} A^T \cdot \vec{c}$$

Limited Happiness

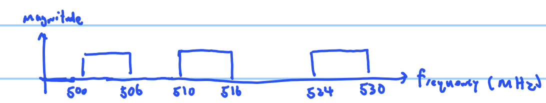
Just like in the homework, we can compress the song by finding the frequencies with the largest magnitude and dropping all of the rest. We need to keep in mind however that your roommate needs to know not only the complex coefficients associated with the remaining frequencies, but also what those frequencies actually are. I.e., when we send the compressed version of the song, we need to send both the frequency component and the index of the frequency that component corresponds to.

So, we would first use the first function (time_to_freq) to find the spectrum of the song. Next, we would use the third (magnitude) and second (find largest) functions to select the $\frac{10000}{3} \approx 3333$ frequencies with largest magnitude. Note that the factor of 3 comes from the fact that we need to send three numbers associated with each frequency component: its real part, its imaginary part, and its index (so that you know what frequency it actually corresponds to).

Note that if a complex number counts as only one number, or if the audio quality is unaffected by sending only the magnitude (and not the phase) of each frequency component, you would then pick the 10000/2 largest magnitude components.

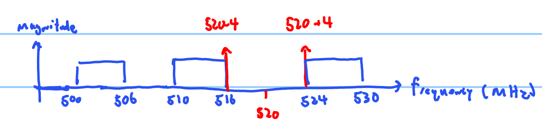
Spectrum allocation

1. Existing TV spectrum looks like this (drawing only positive f's; negative f's are symmetric):



So, the bands of frequencies we can potentially use are in the unused regions in between – i.e., 506 MHz – 510 MHz, and 516 MHz – 524 MHz.

2. The maximum W you can use is 4 MHz, and would have to be modulated on to a 520 MHz carrier frequency; the spectrum associated with this is shown below:



Missing Women

1. $g[n+1] = \frac{1}{4}\cdot\frac{1}{10}\cdot\frac{100}{211}\cdot g[n] - \frac{m_g}{100}\cdot g[n]$

   $b[n+1] = \frac{1}{4}\cdot\frac{1}{10}\cdot\frac{111}{211}\cdot g[n] - \frac{m_b}{100}\cdot b[n]$

2. Girls born each year is $\frac{1}{4}\cdot\frac{1}{10}\cdot\frac{100}{211}\cdot g[n]$, or $\sim 1.18\%$ of current population of girls. So, if $m_g = 1.3\%$, total number of women is decreasing (dying out).

3. $$\begin{bmatrix} g[n+1] \\ b[n+1] \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{40}\cdot\frac{100}{211} - m_g & 0 \\ \frac{1}{40}\cdot\frac{111}{211} & 1-m_b \end{bmatrix} \begin{bmatrix} g[n] \\ b[n] \end{bmatrix}$$

4. Let's rewrite 3 as: $\begin{bmatrix} g[n+1] \\ b[n+1] \end{bmatrix} = A \cdot \begin{bmatrix} g[n] \\ b[n] \end{bmatrix}$

   For $m_g = m_b = \frac{1}{40}\cdot\frac{100}{211}$, $A = \begin{bmatrix} 1 & 0 \\ \frac{1}{40}\cdot\frac{111}{211} & 1-\frac{1}{40}\cdot\frac{100}{211} \end{bmatrix}$

   Let's find the eigenvalues by setting $\det(A-\lambda I)=0$:

   $$(1-\lambda)\left(1-\frac{1}{40}\cdot\frac{100}{211}-\lambda\right) - 0 = 0$$

   $$\lambda^2 - \left(1-\underbrace{\frac{1}{40}\cdot\frac{100}{211}}_{m_b}+1\right)\lambda + 1-\underbrace{\frac{1}{40}\cdot\frac{100}{211}}_{m_b} = 0$$

$$\lambda^2 - (2-m_b)\lambda + (1-m_b) = 0$$

$$\lambda = \frac{2-m_b \pm \sqrt{4-4m_b+m_b^2-4+4m_b}}{2}$$

$$\lambda = \frac{2-m_b \pm m_b}{2}$$

$\lambda_1 = 1$ — This component represents a steady state

$\lambda_2 = 1-m_b$ — This component dies out over time

So: $A \cdot \begin{bmatrix} g \\ b \end{bmatrix} = \begin{bmatrix} g \\ b \end{bmatrix}$

$$g = g$$

$$\frac{1}{40}\cdot\frac{111}{211}\cdot g + \left(1 - \frac{1}{40}\cdot\frac{100}{211}\right)\cdot b = b$$

$$\frac{1}{40}\cdot\frac{111}{211}\cdot g = \frac{1}{40}\cdot\frac{100}{211}\cdot b$$

$$\hookrightarrow \boxed{\frac{b}{g} = \frac{111}{100}}$$

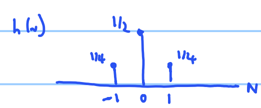
5. For population of men to equal population of women $\begin{bmatrix} g \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ needs to be an eigenvector with $\lambda = 1$ for the transition matrix A:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{40}\cdot\frac{111}{211} & 1-m_b \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \frac{1}{40}\cdot\frac{111}{211} + 1 - m_b = 1$$

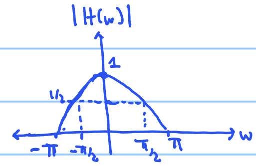
$$\boxed{m_b = \frac{1}{40}\cdot\frac{111}{211}}$$

DT-LTI System

1. We can recover $\delta(n)$ by realizing that $\delta(n) = u(n) - u(n-1)$. Thus, given $y(n)$ for $x(n) = u(n)$, $h(n) = y(n) - y(n-1)$:



2. $H(w) = \frac{1}{4}e^{-jw} + \frac{1}{2}e^{0} + \frac{1}{4}e^{jw} = \frac{1}{2}\left(1 + \frac{e^{jw}+e^{-jw}}{2}\right) = \frac{1}{2}(1+\cos(w))$



3. The system is LTI, so let's find the response to each portion of the input individually and then add those responses together:

For $x(n) = A$, $y(n) = H(0)\cdot A = A$

For $x(n) = C\cos\left(\frac{\pi}{2}n\right)$, $y(n) = H\left(\frac{\pi}{2}\right)\cdot C\cos\left(\frac{\pi}{2}n\right) = \frac{1}{2}C\cos\left(\frac{\pi}{2}n\right)$

For $x(n) = B(-1)^n$, we need to realize that $(-1)^n = \cos(\pi n)$

Therefore, $y(n) = H(\pi)\cdot B\cdot\cos(\pi n) = 0$

So, the complete response $\boxed{y(n) = A + \frac{1}{2}(\cos(\frac{\pi}{2}n)}$

EWMA Filter

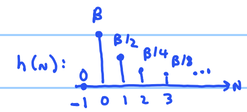
1. $y(n) = \beta \sum_{m=0}^{M-1} \alpha^m \underbrace{x(n-m)}_{1} \longrightarrow y(n) = \beta \sum_{m=0}^{M-1} \alpha^m = \beta \frac{(1-\alpha^M)}{(1-\alpha)}$

To make $y(n) = x(n)$, $y(n) = 1 \ \forall n$:

$$\beta \frac{(1-\alpha^M)}{(1-\alpha)} = 1 \longrightarrow \boxed{\beta = \frac{(1-\alpha)}{(1-\alpha^M)}}$$

2. Let's not worry about the value of β for now since we haven't specified M.

The shape of the impulse response with $\alpha = 1/2$ is:



I.e., $h(n) = \begin{cases} \frac{1/2}{1-(1/2)^M}\cdot(1/2)^n & \text{for } 0 \le n \le M-1 \\ 0 & \text{for } n < 0 \end{cases}$

3. $H(\omega) = \sum_{m=0}^{M-1} \beta\alpha^m e^{-j\omega m} = \boxed{\beta \cdot \frac{(1-\alpha^M e^{-j\omega M})}{(1-\alpha e^{-j\omega})}}$

4. We found β in part (1) so that $y(n) = x(n)$ for $x(n) = 1 \ \forall n$, and thus the DC gain must be 1. We can also find this by plugging in $\omega = 0$ to the expression we found in part (3) and using the β we found in part (1).

5. The key is to realize that the input can be broken in to two components at two frequencies:

$$x(n) = \$0.75 \cdot \cos(0) + \$0.25 \cdot \cos(\pi n)$$

We already know that $H(0) = 1$, so really we just need to find $H(\pi)$ for $\alpha = 1/2$ and $M = 10$

$$H(\pi) = \frac{1-1/2}{1-(1/2)^{10}} \cdot \frac{(1-(\frac{1}{2})^{10}\cdot(e^{-j10\pi}))}{1-\frac{1}{2}\cdot e^{-j\pi}}$$

$$= \frac{1/2}{1-(1/2)^{10}} \cdot \frac{(1-(1/2)^{10})}{1-\frac{1}{2}(-1)} = \frac{1/2}{3/2} = 1/3$$

Therefore, $y(n) = \$\frac{3}{4} + \$\frac{1}{12}\cdot\cos(\pi n)$, or $y(n) = \begin{cases} \$5/6 & \text{for } n \text{ odd} \\ \$2/3 & \text{for } n \text{ even} \end{cases}$

Quadrature Multiplexing

(a) $y(t) = x_1(t)\cdot\cos(w_o t) + x_2(t)\sin(w_o t) = A(t)\cos(w_o t + \phi(t))$

First let's convert to complex exponentials:

$$\frac{1}{2}x_1(t)\left(e^{jw_o t} + e^{-jw_o t}\right) + \frac{1}{2j}x_2(t)\left(e^{jw_o t} - e^{-jw_o t}\right) = \frac{A(t)}{2}\left(e^{j(w_o t+\phi(t))} + e^{-j(w_o t+\phi(t))}\right)$$

$$\frac{1}{2}\left[x_1(t) + \frac{1}{j}x_2(t)\right]e^{jw_o t} + \frac{1}{2}\left[x_1(t) - \frac{1}{j}x_2(t)\right]e^{-jw_o t} = \frac{1}{2}\left[A(t)e^{j\phi(t)}\right]e^{jw_o t} + \frac{1}{2}\left[A(t)e^{-j\phi(t)}\right]e^{-jw_o t}$$

So: $\frac{1}{2}\left[x_1(t) + \frac{1}{j}x_2(t)\right] = \frac{1}{2}A(t)e^{j\phi(t)}$ (1)

$\frac{1}{2}\left[x_1(t) - \frac{1}{j}x_2(t)\right] = \frac{1}{2}A(t)e^{-j\phi(t)}$ (2)

$2(1)\cdot 2(2) \longrightarrow x_1^2(t) + x_2^2(t) = A^2(t) \Longrightarrow \boxed{A(t) = \sqrt{x_1^2(t) + x_2^2(t)}}$

To find $\phi(t)$, expand $e^{j\phi(t)}$ back into $\cos(\phi(t)) + j\sin(\phi(t))$

(1) $x_1(t) + \frac{1}{j}x_2(t) = A(t)\left[\cos(\phi(t)) + j\sin(\phi(t))\right]$

$\hookrightarrow x_1(t) = \cos(\phi(t))$ (a)

$\frac{1}{j}x_2(t) = j\sin(\phi(t)) \longrightarrow x_2(t) = -\sin(\phi(t))$ (b)

$\frac{(b)}{(a)} \rightarrow \frac{x_2(t)}{x_1(t)} = -\frac{\sin(\phi(t))}{\cos(\phi(t))}$

$\hookrightarrow -\frac{x_2(t)}{x_1(t)} = \tan(\phi(t))$

$$\boxed{\phi(t) = -\tan^{-1}\left(\frac{x_2(t)}{x_1(t)}\right)}$$

(b) $r_1(t) = \left[x_1(t)\cos(w_o t) + x_2(t)\sin(w_o t)\right]\cos(w_o t + \theta)$

$= \frac{1}{2}\left[x_1(t)\cos(\theta) + x_1(t)\cos(2w_o t + \theta) + x_2(t)\sin(-\theta) + x_2(t)\sin(2w_o t + \theta)\right]$

After going through the low pass filter and hence dropping high frequency terms:

$$\boxed{q_1(t) = x_1(t)\cos(\theta) - x_2(t)\sin(\theta)}$$

$r_2(t) = \left[x_1(t)\cos(w_o t) + x_2(t)\sin(w_o t)\right]\sin(w_o t + \theta)$

$= \frac{1}{2}\left[x_1(t)\sin(\theta) + x_1(t)\sin(2w_o t + \theta) + x_2(t)\cos(\theta) - x_2(t)\cos(2w_o t + \theta)\right]$

After the low pass filter:

$$\boxed{q_2(t) = x_1(t)\sin(\theta) + x_2(t)\cos(\theta)}$$

(c) (i) If $\theta \approx 0$, then $\sin(\theta) \approx 0$ and $\cos(\theta) \approx 1$. Thus:

$\boxed{q_1(t) \approx x_1(t)}$ & $\boxed{q_2(t) \approx x_2(t)}$

(ii) We're looking for θ such that $\frac{|x_2(t)\sin(\theta)|}{|x_1(t)\cos(\theta)|} = 0.01$

Since $|x_1(t)| = |x_2(t)|$, we're looking for $\frac{|\sin(\theta)|}{|\cos(\theta)|} = 0.01$

$\hookrightarrow |\tan(\theta)| = 0.01$

$\theta = \tan^{-1}(0.01) \approx 0.57°$

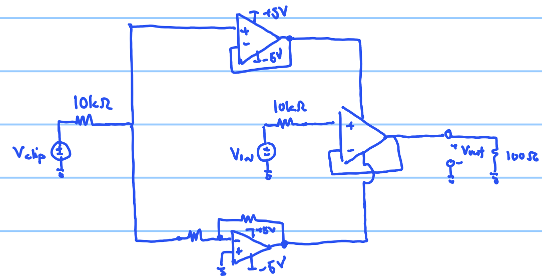
≈ 0.01 rad

Audio Distortion Generator

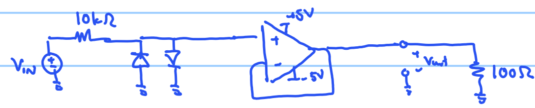
(a)

$$V_{out} = \begin{cases} V_{clip} & \text{for } V_{in} \geq V_{clip} \\ V_{in} & \text{for } -V_{clip} < V_{in} < V_{clip} \\ -V_{clip} & \text{for } V_{in} \leq -V_{clip} \end{cases}$$

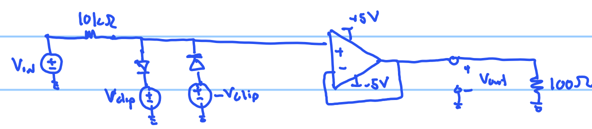
(b)



(c)



(d)



Scaling of Wires

(a) $R = \rho \frac{L}{A} = 17e\text{-}9\ \Omega \cdot m \cdot \frac{1e\text{-}3\ m}{1e\text{-}6\ m \cdot 1e\text{-}6\ m}$

$$\boxed{R = 17\Omega}$$

(b) $C = \varepsilon \cdot \frac{A}{d} = 3.9 \cdot 8.85e\text{-}12 \cdot \frac{1e\text{-}3 \cdot 1e\text{-}6}{1e\text{-}6}$

$$\boxed{C \approx 34.52\ fF}$$

(c) $R_{scaled} = R_{old} \cdot \frac{1/\sqrt{2}}{1/\sqrt{2} \cdot 1/\sqrt{2}} = R_{old} \cdot \sqrt{2} \approx \boxed{24\Omega}$

$C_{scaled} = C_{old} \cdot \frac{1/\sqrt{2} \cdot 1/\sqrt{2}}{1/\sqrt{2}} = C_{old} \cdot \frac{1}{\sqrt{2}} \approx \boxed{24.4 fF}$

(d) $R_{scaled,global} = R_{old} \cdot \frac{1}{1/\sqrt{2} \cdot 1/\sqrt{2}} = R_{old} \cdot 2 = \boxed{34\Omega}$

$C_{scaled,global} = C_{old} \cdot \frac{1/\sqrt{2} \cdot 1}{1/\sqrt{2}} = C_{old} \approx \boxed{34.52\ fF}$

(e) $V = I \cdot R$

If for each generation we scale all dimensions except length by $S = 1/\sqrt{2}$, and the generations are indexed by n (n=0 is the current gen, n=1 the next gen, etc.):

$$R_{scaled}(N) = R_{old} \cdot \frac{1}{.5^N \cdot .5^N} = 2^N \cdot R_{old}$$

* Therefore, for a fixed current, the voltage dropped across the wire will scale as 2^N - i.e., in two generations, the voltage drop goes up by 4X, in three generations by 8X, etc.