EE16A - Lecture 20 Notes

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Positioning

Time of Flight

- Beacon sends signal of certain shape with signal a(t) at time t
- Receiver gets signal of same general shape y(t) some time later
- ToF = time it takes from signal sent to signal received
- Derive distance using $d = v\Delta t$
- Convert continuous time signal to a discrete time signal (with sampling period of $T_s = sec/sample$) where $a_{DT}[n] = a_{CT}(NT_s)$ and the sampling frequency is $f_s = \frac{1}{T_s} = Hz(cycles/sec)$

Discrete Time Delay

> 1. To find N_a, look at all possible shifts $S^k a$ where $k = 0, 1, \cdots, N-1$ and find the value of k that gives the closest match between received signal and very shifted version of original signal a
>   - compare y w/ $a, Sa, S^2a, \cdots, S^{N-1}a$
> 2. Compute k that maximizes the absolute value of the inner product between y and $S^k a$ (Max($|\langle y, S^k a, |\rangle$)
> 3. $k = N_a$ or the delay shift between the received and original signal

- Receive delayed version of original signal $a[n]$ (discrete time): $y[n] = \alpha a[n - N_a]$ where delay is N_a
- Delay is N_a samples so get delay time Δt by multiplying by period T_s ($\Delta t = N_a T_s$)
- It follows that the distance: $d = v\Delta t = vN_a T_s$
- Do not know N_a, so look at all possible shifts $S^k a$ where $k = 0, 1, \cdots, N-1$ and find the value of k that gives the closest match between received signal and very shifted version of original signal a
  - y compare w/ $a, Sa, S^2a, \cdots, S^{N-1}a$
  - Compute k that maximizes the absolute value of the inner product between y and $S^k a$ (Max($|\langle y, S^k a, |\rangle$)
  - $k = N_a$ or the delay shift between the received and original signal (not perfect, but close)

Periodic Signals

- Beacon signal a is periodic w/ period N (N-periodic) ($a[n-N] = a[n], \forall n \in \mathbb{Z}$)
- If $y[n] = \alpha a[n - N_A]$, y is also $N-periodic$ because shifting does not change periodicity (circular shift)

Circular Shift:

$$a[n] = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, a[n-1] = \begin{bmatrix} a_2 \\ a_0 \\ a_1 \end{bmatrix} \tag{1}$$

Shift Matrix Example:

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, a[n-1] = Sa[n] \tag{2}$$

Shift Matrix General:

$$\begin{bmatrix} a_{N-1} \\ a_0 \\ \vdots \\ a_{N-2} \end{bmatrix} = \begin{bmatrix} 0^T_{N-1} & 1 \\ I_{N-1} & 0_{N-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \tag{3}$$

Model Delay of Signal a by Shift N_a

$$y = S^{N_A} a \tag{4}$$

Circular Shift of N-periodic signl by N samples $\implies I$

$$S^N = I \tag{5}$$

Shift Transpose is its Inverse

$$S^T = S^{-1} \tag{6}$$

Inner Product

> 1. Scope of the class: Inner products are in $\mathbb{R}^n$, so $\langle x, y \rangle = x^T y = x \cdot y$
> 2. Inner product = dot product

- Elements $x, y, z \in \mathscr{V}$ where $\mathscr{V}$ = vector space in $\mathbb{R}^n$ or $\mathbb{C}^n$
- Inner product is a function that has a domain that is a pair of elements in vector space $\langle x, y \rangle : \mathscr{V} \times \mathscr{V}$ and maps these values from $\mathbb{R}^n$ or $\mathbb{C}^n$ to $\mathbb{R}$ or $\mathbb{C}$
- In $\mathbb{C}^n, \langle x, y \rangle = x^T y^*$
- In $\mathbb{R}^n, \langle x, y \rangle = x^T y = x \cdot y$

Inner Product Properties
Hermitian/Conjugate Symmetry (*=complex conjugate)

$$\langle x, y \rangle = \langle y, x^* \rangle \tag{7}$$

Symmetry/Commutativity (If vector space $\in \mathbb{R}$)

$$\langle x, y \rangle = \langle y, x \rangle \tag{8}$$

Distributive

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \tag{9}$$

Scaling

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \tag{10}$$

Non-negativity

$$\langle x, x \rangle \geq 0 \text{ w/ equality } \iff x = 0 \tag{11}$$

Norm

> 1. 2-Norm Definition ($\|x\|_2$)
>   - $\|x\| = \sqrt{\langle x, x\rangle}$
>   - $\|x\|^2 = \langle x, x\rangle$
> 2. Norm and inner product: $\langle a, b\rangle = \|a\|\|b\| \cos\theta$ where $\theta =$ angle between a, b
> 3. For comparison between signals, find shift that will maximize: $\cos\theta = \frac{\langle a,b\rangle}{\|a\|\|b\|}$
> 4. Norm = magnitude

- Gives sense of size to elements in vector space
- Relationship with cosine of angle between vectors
- Vector a, b with angles w.r.t x-axis of α, β respectively
- Angle between the vectors is θ
- $a = [a_1, a_2], b = [b_1, b_2]$
- $a_1 = \|a\| \cos\alpha, a_2 = \|a\| \sin\alpha; b_1 = \|b\| \cos\beta, b_2 = \|b\| \sin\beta$
- $\langle a, b\rangle = [a_1, a_2][b_1, b_2]^T = a_1b_1 + a_2b_2 = \|a\| \cos\alpha\|b\| \cos\beta + \|a\| \sin\alpha\|b\| \sin\beta$
- $= \|a\|\|b\|(\cos\alpha \cos\beta + \sin\alpha \sin\beta)$
- $= \|a\|\|b\|(\cos(\beta - \alpha)$ because $(\cos(\beta - \alpha) = \cos\alpha \cos\beta + \sin\alpha \sin\beta)$
- $= \|a\|\|b\| \cos\theta$

Non-Negativity

$$\|x\| \geq 0 \text{ w/ equality } \iff x = 0 \tag{12}$$

Scaling Property

$$\|\alpha x\| = |\alpha|\|x\| \tag{13}$$

Triangle Inequality

$$\|x + y\| \leq \|x\| + \|y\| \text{ w/ equality } \iff x = \alpha y, \alpha > 0 \tag{14}$$

Norm and Inner Product:

$$\langle a, b\rangle = \|a\|\|b\| \cos\theta \text{ where } \theta = \text{ angle between } a, b \tag{15}$$

Angle between Cartesian Vectors

$$\cos\theta = \frac{\langle a, b\rangle}{\|a\|\|b\|} \tag{16}$$