EE16A - Lecture 24 Notes

Name: Felix Su SID: 25794773

Spring 2016 GSI: Ena Hariyoshi

Review

- Redundant (Overcomplete) Dictionary of vectors $D = \{\phi_l\}, \phi_l \in \mathbb{R}^N$
- N = length of each player's message
- $\text{span}\{\phi_l\} = \mathbb{R}^N$
- D contains no fewer than N lin. ind. vectors
- $y = \sum_{l=1}^{m-1} \alpha_l \vec{z}_l$
  - $\vec{z}_l$: message form the lth player
  - α_l: scalars
  - Simplified model: ignoring time delay S^{N_k}
  - of the $L = 2000$ players, very few are talking to us at any given time window
  - Assume: All $\|\phi_l\| = 1$, if not, normalize
- Use Matching Pursuit Algorithm to solve for y
  - Continuously match with the best vector in the dictionary
  - Minimize residual vector each time using local optimization

Another View of Matching Pursuit Algorithm

- Initialize:
  - Residual= y at iteration 0: $r_0 = y$
  - $A_0 = [\quad]$, placeholder matrix for the matched distionary vectors
- While $r_m \geq \varepsilon$, continue (Here: $\varepsilon = 0$)
  1. Look for the vector that gives the best match: $k = \text{argmax}_i |\langle r_m, \vec{z}_i, \rangle|$
  2. k is the largest projection between the current residual and all the vectors in the dictionary D.
  3. $\vec{v}_m = \vec{z}_k$ is the vector in the dictionary the gives the projection k
  4. Augment matrix A with $\vec{v}_m$ as a new column: $A_m = \begin{bmatrix} A_{m-1} & \vec{v}_m \end{bmatrix}$
  5. Model received signal as a linear combination of the columns of the A matrix up to this point + error term:
     - $\vec{y} = A_m \vec{\alpha}_m + \vec{r}_m$
     - $A_1 \alpha_1$ is the best estimate at iteration 1 of the received signal $\vec{y}$: $A_m \alpha_M = \vec{y}_m$
     - Best α for answer is given by the Least Squares formula: $\alpha_m = (A_M^T A_m)^{-1} A_m^T \vec{y}$
  6. $\vec{r}_m = \vec{y} - \vec{y}_m = y - A_m \alpha_m$
- **Problem:**
  - Matrix A keeps getting wider on every iteration

- Eacch iteration you have to compute $(A_m^T A_m)^{-1}$, which is too costly

- **Solution:**
  - Try to construct matrix A at each iteration that has *orthonormal* columns
  - $\alpha_m = (A_M^T A_m)^{-1} A_m^T \vec{y}$
  - $\vec{y}_m = A_m (A_M^T A_m)^{-1} A_m^T \vec{y}$
  - If A_m has orthonormal cols, (if all $\vec{v}_k$'s are orthonormal): $A_m^T A_m = T$
  - Thus, $\vec{y}_m = A_m(A_m^T y) = \sum_{i=1}^{m} \langle \vec{v}_i, \vec{y} \rangle \vec{v}_i$

Gram-Schmidt Orthogonlization (Orthonormalization)

- Have $\vec{v}_1, \ldots, \vec{v}_m$ lin. ind. vectors $V_m =$span$\{\vec{v}_1, \ldots, \vec{v}_m\}$
- **Claim:** We can find another set of vectors $\vec{z}_1, \ldots, \vec{z}_m$ that are mutually orthogonal and for which $Z_m =$ span$\{\vec{z}_1, \ldots, \vec{z}_m\}$ =span$\{\vec{v}_1, \ldots, \vec{v}_m\} = V_m$
  - Once we have $\vec{z}_1, \ldots, \vec{z}_m$, we can construct vectors $\vec{q}_i = \frac{\vec{z}_i}{\|\vec{z}_i\|}, i = 1, \ldots, m$
  - $Q_m =$span$\{\vec{q}_1, \ldots, \vec{q}_m\} = V_m = Z_m$ where $\langle \vec{q}_l, \vec{q}_k \rangle = 1$ when $k = l$ and $\langle \vec{q}_l, \vec{q}_k \rangle = 0$ when $k \neq l$
  - All $\vec{q}_i$'s are orthonormal
- **Proof** (By construction):
  - $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent
  - Let $\vec{z}_1 = \vec{v}_1$: 1st principal direction of the orthogonal set
  - $\vec{z}_2 = \vec{v}_2 - \alpha \vec{z}_1$ s.t. $\vec{z}_2 \perp \vec{z}_1$
    * $\langle \vec{z}_2, \vec{z}_1 \rangle = \langle \vec{v}_2 - \alpha \vec{z}_1, \vec{z}_1 \rangle = 0$
    * $\langle \vec{v}_2, \vec{z}_1 \rangle - \alpha \langle \vec{z}_1, \vec{z}_1 \rangle = 0 \implies \frac{\langle \vec{v}_2, \vec{z}_1 \rangle}{\langle \vec{z}_1, \vec{z}_1 \rangle}$
    * $\vec{z}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{z}_1 \rangle}{\langle \vec{z}_1, \vec{z}_1 \rangle} \vec{z}_1$
    * $\vec{z}_2$ is the 2nd principal direction in the orthogonal set
  - $\vec{z}_2 = \vec{z}_3 - \alpha_1 \vec{z}_1 - \alpha_2 \vec{z}_2 = \vec{v}_3 - \sum_{l=1}^{2} \alpha_l \vec{z}_l$ s.t. $\vec{z}_3 \perp \vec{z}_1 \& \vec{z}_3 \perp \vec{z}_2$
    * $\langle \vec{z}_3, \vec{z}_1 \rangle = \langle \vec{v}_3 - \alpha_1 \vec{z}_1 - \alpha_2 \vec{z}_2, \vec{z}_1 \rangle = 0$
    * $\langle \vec{z}_3, \vec{z}_2 \rangle = \langle \vec{v}_3 - \alpha_1 \vec{z}_1 - \alpha_2 \vec{z}_2, \vec{z}_2 \rangle = 0$
    * Solve for $\vec{z}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{z}_1 \rangle}{\langle \vec{z}_1, \vec{z}_1 \rangle} \vec{z}_1 - \frac{\langle \vec{v}_3, \vec{z}_2 \rangle}{\langle \vec{z}_2, \vec{z}_2 \rangle} \vec{z}_2$
    * $\vec{z}_3$ is the 3nd principal direction in the orthogonal set
  - Continuing in this way: $\vec{z}_n = \vec{v}_n - \sum_{i=1}^{n-1} \frac{\langle \vec{v}_n, \vec{z}_i \rangle}{\langle \vec{z}_i, \vec{z}_i \rangle} \vec{z}_i$
  - Gets the Orthogonal set, but still need to normalize
    * Get orthonormalized set $\vec{q}_1, \ldots, \vec{q}_m$ where $\vec{q}_l = \frac{\vec{z}_l}{\|\vec{z}_l\|}$
    * Orthonormal Representation of Principal Direction:
      · $\alpha_l = \frac{\langle \vec{v}_m, \vec{z}_l \rangle}{\langle \vec{z}_l, \vec{z}_l \rangle}$
      · $\vec{z}_l = \|\vec{z}_l\| \vec{q}_l$
      · $\vec{v}_m = \vec{z}_m + \sum_{l=1}^{m-1} \alpha_l \vec{z}_l = \|\vec{z}_m\| \vec{q}_m + \sum_{l=1}^{m-1} \alpha_l \|\vec{z}_l\| \vec{q}_l = \sum_{l=1}^{m} \alpha_l \|\vec{z}_l\| \vec{q}_l = \sum_{l=1}^{m} r_{lm} \vec{q}_l$
  - This gets $A = QR$ where:
    * R is composed of elements r_{lm} and is upper triangular
    * Q is the orthonormalized matrix $\begin{bmatrix} \vec{q}_1 & \ldots & \vec{q}_m \end{bmatrix}$
    * A is the original principal direction matrix $\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_m \end{bmatrix}$
    * $Q^T Q = I$

Gram Schmidt Orthogonalization:

1st principal direction in the orthogonal set

$$\vec{z}_1 = \vec{v}_1 \tag{1}$$

n-th principal direction in the orthogonal set

$$\vec{z}_n = \vec{v}_n - \sum_{i=1}^{n-1} \frac{\langle \vec{v}_n, \vec{z}_i \rangle}{\langle \vec{z}_i, \vec{z}_i \rangle} \vec{z}_i \tag{2}$$

Orthonormal Representation of Principal Direction
R is composed of elements r_{lm} and is upper triangular
Q is the orthonormalized matrix $\begin{bmatrix} \vec{q}_1 & \dots & \vec{q}_m \end{bmatrix}$
A is the original principal direction matrix $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_m \end{bmatrix}$

$$A = QR \tag{3}$$

$$\vec{v}_m = \sum_{l=1}^{m} r_{lm} \vec{q}_l \tag{4}$$