EE16A - Lecture 27 Notes

Name: Felix Su SID: 25794773

Spring 2016 GSI: Ena Hariyoshi

Review

1. **Eigenvalue-Eigenvector Eq.**: $A\vec{v} = \lambda\vec{v}$

Diagonlization

- Set of eigenvectors ($\vec{v}_i$) and eigenvalues (λ_i) for $i = 1, \ldots, N$ s.t. $\begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_N \end{bmatrix} = \begin{bmatrix} \lambda\vec{v}_1 & \cdots & \lambda\vec{v}_N \end{bmatrix}$
  - Left side: Take out A
  - Right side: Take out diagonal eigenvalue matrix (λ's)
  - $\implies A \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_N \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_N \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$
  - So, $AV = V\Lambda$
  - Postmultiply by V^{-1} on both sides to get $A = V\Lambda V^{-1}$

Summary Equation for Eigen Decomposition of Matrix A:

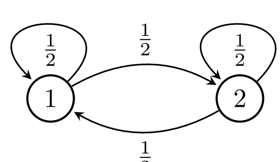
$$AV = V\Lambda \text{ where } V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_N \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tag{1}$$

$$A = V\Lambda V^{-1} \tag{2}$$

Pagerank

- $A = V\Lambda V^{-1}$
- $s[n] = A^n s[0]$
- $A^n = V\Lambda^n V^{-1}$
  - If $|\lambda_i| > 1 \rightarrow \lambda_i^n$ keeps growing
  - If $|\lambda_i| < 1 \rightarrow \lambda_i^n$ decays towards 0
  - If $|\lambda_i| = 1 \rightarrow \lambda_i^n$ stays the same

Example:
Pagerank Example



$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

1 linearly independent col/row $\implies$ rank(A) = 1

One of the eigenvalues has to be zero: $\lambda_1 = 0$, the other is non-zero: $\lambda_2 = n$

0 eigenvalue corresponds to singular matrix

A is a projection matrix onto vector $\vec{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Projection matrix $P = \frac{\vec{r}\vec{r}^T}{\vec{r}^T\vec{r}}$

A is symmetric

A is column and row-stochastic (doubly-stochastic)

column-stochastic matrix = Markoc matrix $A \geq 0$ (non-negative)

$A - \lambda I = \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix}$

$\det(A - \lambda I) = (\frac{1}{2} - \lambda)^2 - (\frac{1}{2})^2 = 0 \implies \lambda(1-\lambda)0$, so, $\lambda = 0, 1$

Markov matrices always have 1 as an eigenvalue (All other eigenvalues are smaller), so, as teh system continues to run, the other eignvalues decay to 0 and the eigenvector corresponding to the $\lambda_i = 1$ will be the **important score** for the website

$\lambda_1 = 0, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 0 \implies \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 1, \begin{bmatrix} \frac{1}{2} - 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = 0 \implies \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$V^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \vec{w}_1^T \\ \vec{w}_2^T \end{bmatrix} = \lambda_1\vec{v}_1\vec{w}_1^T + \lambda_2\vec{v}_2\vec{w}_2^T$

$A^n = V\Lambda^n V^{-1} = \sum_{i=1}^{N} \lambda_i^n \vec{v}_i\vec{w}_i^T$

General Form of nth Power of a Matrix

$$A^n = V\Lambda^n V^{-1} = \sum_{i=1}^{N} \lambda_i^n \vec{v}_i\vec{w}_i^T \tag{3}$$

Importance Score for Pagerank

Pagerank Importance Score

Converges to the value $A^n s[0]$ where:

$$\lim_{n\to\infty} A^n = \lambda_i\vec{v}_i\vec{w}_i^T = \vec{v}_i\vec{w}_i^T \text{ where } \lambda_i = 1 \text{ if } |\lambda_i| < 1 \forall i = 1, \ldots, N-1 \tag{4}$$

Preview of Next Lecture

- If $s[n+1] = As[n]$, and $\vec{q}[n] = V^{-1}s[n] \implies \vec{s}[n] = V\vec{q}[n]$
- Then, $V\vec{q}[n+1] = AV\vec{q}[n] \implies \vec{q}[n+1] = V^{-1}AV\vec{q}[n]$

- Because $A = V\Lambda V^{-1} \implies V^{-1}\Lambda V = \Lambda$
- $\therefore \vec{q}[n+1] = \Lambda\vec{q}[n]$
  - Decouples dynamic system