EE16A - Lecture 26 Notes

Name: Felix Su SID: 25794773

Spring 2016 GSI: Ena Hariyoshi

Review Eigenvalues/Eigenvectors

> 1. $A\vec{v}_n = \lambda_n \vec{v}_n$
>   - Scaling factor λ: **Eigenvalue**
>   - Vector $\vec{v}_n$: **Eigenvector**
> 2. For a matrix $M_{n\times n}$ you can have at most n eigenvalues
> 3. $(A - \lambda I)\vec{v} = 0, x \neq 0 \implies A - \lambda I$ has a non trivial null space and $\det(A - \lambda I) = 0$
>   - M is non-invertible
>   - M has a non-trivial null space
>   - $\det M = 0$
> 4. If matrix A is symmetric ($A^T = A$), eigenvectors are independent and orthogonal

- Given $\vec{v}_1$ and λ_1 of matrix M_1, find another eigenvalue/eigenvector pair
  - $\vec{v}_2$ is also an eigenvector of M_1 if $\vec{v}_2 = \alpha\vec{v}_1$
    * $M_1\vec{v}_2 = M_1(\alpha\vec{v}_1) = \alpha(M\vec{v}_1) = \alpha(\lambda_1\vec{v}_1) = \lambda_1(\alpha\vec{v}_1) = \lambda_1(\vec{v}_2)$
    * So, $\lambda_2 = \lambda_1$
  - Any eigenvalue $\implies$ an eigenspace (linear combinations of any valid eigenvector works for that eigenvalue)
  - $(A - \lambda I)\vec{v} = 0 \implies A - \lambda I$ has a non-trivial null space
    * $A - \lambda I$ is non-invertible, so the transformation destroys some information (lose a dimension)
    * Invertible matrix (trivial null space) performs a transformation (scale x value)

Linear Transformations

- Given a linear transformation $y = ax + b = 3x$, find eigenvectors and eigenvalues for this transformation
  - Vector in the same direction (**colinear**) with the transformation vector, $\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
  - Vector **orthogonal** to the transformation vector $\lambda_2 = -1, \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Determinant

> 1. The "oriented" volume of the polygon obtained by applying the matrix M onto a unit hypercube.

- Because $A - \lambda I$ is non-invertible, the transformation destroys some information (lose a dimension), so the volume is smaller

Inverse Formula

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1}$$

Solve for Eigenvalues/Eigenvectors

Solve for λ

$$\det(A - \lambda I) = 0 \tag{2}$$

Determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \tag{3}$$

Determinant of a $n \times n$ matrix

Mult. each elem. in the first row by the det. of the $n-1 \times n-1$ matrix not in that element's row or column

Take an **alternating sum** of the products from the previous step

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \times \det \begin{bmatrix} e & i \\ f & h \end{bmatrix} - b \times \det \begin{bmatrix} d & i \\ f & g \end{bmatrix} + c \times \det \begin{bmatrix} d & h \\ e & g \end{bmatrix} \tag{4}$$

Plug resulting Eigenvalues λ_i into $A - \lambda_i I$

$$B_i = A - \lambda_i I = \begin{bmatrix} a_{11} - \lambda_i & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda_i & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda_i \end{bmatrix} \tag{5}$$

Solve for Eigenvectors $\vec{v}_i$ s.t.

This will be the linear combination of the columns of B_i that cause $B_i \vec{v}_i = 0$

$$(A - \lambda_i I)\vec{v}_i = B_i \vec{v}_i = 0 \tag{6}$$

Properties of Determinants

1. If you **scale a row/col** of a matrix by α, the determinant of the matrix is **multiplied by** α
2. If you **add a scalar multiple of a row/col to any other row/col**, the determinant **doesn't change**
3. If you **swap rows**, the determinant is **multiplied by** -1
4. Determinant of an **upper triangular matrix** is the **product of its pivots**
   - Take unit hypercube, multiply first dimension by a_1, second dimension by a_2...
   - Volum of a hypercube is the product of all its dimensions
5. Generic determinant of $\det(A - \lambda I)$ is $\lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \lambda_n = 0$
   - The max number of roots is n
6. If $\lambda_1 \neq \lambda_2$: eigenspace(λ_1)$\cap$ eigenspace(λ_2)$= \vec{0}$
7. If eigenvectors of A span $\mathbb{R}^n$,$\vec{x} \in \mathbb{R}^n$ can be expressed as $\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$
8. $A\vec{x} = \sum_{i=1}^{n} \alpha_i \lambda_i \vec{v}_i$
9. $A^n \vec{x} = \sum_{i=1}^{n} \alpha_i \lambda_i^n \vec{v}_i$ where $\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$

- $\vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$
- $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}^{-1} \vec{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$