# CS70 - Combinatorial Arguments

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Spring 2016    GSI: Gerald Zhang

## Strategy

1. Define equivalent quantity $Q$

2. Count $Q$ in a way to determine $LHS = Q$

3. Count $Q$ in a way to determine $RHS = Q$

4. Conclude $LHS = RHS$

## Examples

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \tag{1}$$

**LHS:** Using counting product rule, this is the number subsets of $n$ decided by either choosing or not choosing each item. There are a total of $2^n$ possible subsets of size $k \in (0, n)$ from a total of $n$ items.

**RHS:** The summation of all possible ways to choose $k \in (0, n)$ items from $n$, which is the same as left.

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1} \tag{2}$$

**LHS:**

**RHS:**

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n} \tag{3}$$

**LHS:** Specify the smallest element from $\{1, \cdots, n+m+1\}$ not in the selected subset. If it is 1, you must choose the remaining $n$ items from $\{2, \cdots, n+m+1\}$, $\binom{m+n}{n}$, which is the $k = n$ term. If it is 2, 1 is included in the subset, so you only need to choose from $n-1$ remaining items from $\{2, \cdots, n+m+1\}$, $\binom{m+n-1}{n-1}$. This continues down to having the smallest number not in the set be $n+1$, which would cause, $\binom{m+0}{0}$. This equates to all possible ways to select select $n$ items from $n+m+1$ choices

**RHS:** All possible ways to select select $n$ items from $n+m+1$ choices.

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{4}$$

**LHS:** Split $\{\pm 1, \cdots, \pm n\}$ into 2 sets, 1 of positive and 1 of negative elements. Choose $k$ positive items and $n-k$ negative items (which is still equivalent to $\binom{n}{k}$). This is equivalent to choosing $n$ items out of $\{\pm 1, \cdots, \pm n\}$.

**RHS:** All possible ways to directly choose $n$ items out of $\{\pm 1, \cdots, \pm n\}$.

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r} \tag{5}$$

**LHS:** Split the collection of books into a set of textbooks and one of comic books. Select $k$ books from the set of textbooks and $r-k$ books from the set of $m$ comic books. This is the same as selecting $r$ total books from a collection of $n$ textbooks and $m$ comic books.

**RHS:** Choose $r$ books from a collection of $n$ textbooks and $m$ comic books.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{6}$$

**LHS:** All possible ways to select $k$ items from a collection of $n$ choices.

**RHS:** Set aside one element. Assuming this item must be in the selected subset of $k$ items, select the remaining $k-1$ items from the remaining $n-1$ choices $\binom{n-1}{k-1}$. Then assume the item set aside is not in the selected subset of $k$ items. Select the rest of the $k$ items from the remaining $n-1$ choices $\binom{n-1}{k}$. These two disjoint sets add up to the original selection of $k$ items from $n$ choices

$$\binom{2n}{2} = 2\binom{n}{2} + n^2 \tag{7}$$

**LHS:** All possible ways to choose 2 items from a set of $2n$ choices.

**RHS:** There are 3 ways to choose 2 items from a set of $2n$ choices if you split the $2n$ choices into 2 sets of $n$ choices. You can choose both from the first set of $n$ choices $\binom{n}{2}$, choose both from the second set of $n$ choices $\binom{n}{2}$, or you can choose 1 from each set $\binom{n}{1}^2 = n^2$. Add these disjoint sets together to get LHS.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \tag{8}$$

**LHS:**

**RHS:**