

# CS70 - Lecture 16 Notes

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## Set Notation Review

- Set  $A$ , Complement  $\bar{A}$
- Union (In either: or):  $A \cup B$
- Intersection (In both: and):  $A \cap B$
- Difference (In  $A$ , not  $B$ )  $A \setminus B$
- Symmetric Difference (In only one: xor)  $A \Delta B$

## Probability

- event  $E$  = subset of outcome:  $E \subset \Omega$
- Any Sample Space:  $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$
- Uniform Space:  $\Pr[E] = \frac{|E|}{|\Omega|}$
- $p_n := \Pr[E_n] = \frac{|E_n|}{|\Omega|}$ 
  - $p_n := \frac{\binom{n}{k}}{|\omega|^n}$  if  $E = n$  coin tosses with exactly  $k$  heads

**Stirling Formula:** (for large  $n$ )

- $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$
- $\Pr[E] = \frac{|E|}{|\Omega|}$ 
  - Can apply Stirling Formula because  $|E|$  and  $|\Omega|$  are defined by combinations (factorials)

## Probability is Additive

- If events  $A$  and  $B$  are disjoint, then sum probabilities
- Non-disjoint sets, use Inclusion/exclusion property:  $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$
- **Union bound:**  $\Pr[A_1 \cup \dots \cup A_n] \leq \Pr[A_1] + \dots + \Pr[A_n]$
- If  $A_1, \dots, A_N$  are a pairwise disjoint partition of  $\Omega$  and  $\cup_{m=1}^N A_m = \Omega$ , then  $\Pr[B] = \Pr[B \cap A_1] + \dots + \Pr[B \cap A_N]$

**Inclusion/Exclusion Property:**

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

**Union Bound:**

$$\Pr[A_1 \cup \dots \cup A_n] \leq \Pr[A_1] + \dots + \Pr[A_n]$$

**Law of Total Probability:**

If  $A_1, \dots, A_N$  are a pairwise disjoint partition of  $\Omega$  and  $\cup_{m=1}^N A_m = \Omega$  then,

$$\Pr[B] = \Pr[B \cap A_1] + \dots + \Pr[B \cap A_N]$$

## Conditional Probability

- Probability of  $A$  given  $B$
- $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$

### Product Rule

$$\Pr[A_1 \cap \dots \cap A_n] = \Pr[A_1] \Pr[A_2|A_1] \dots \Pr[A_n|A_1 \cap \dots \cap A_{n-1}] \quad (1)$$

### Total Probability $\times$ Product Rule

$$\Pr[B] = \Pr[A_1] \Pr[B|A_1] + \dots + \Pr[A_N] \Pr[B|A_N] \quad (2)$$

## Causality vs. Correlation

- Events  $A$  and  $B$  are **positively correlated** if  $\Pr[A \cap B] > \Pr[A] \Pr[B]$ , but this does not imply causation
- Eliminate external/common causes to test causality

## Bayes Rule

- Let  $m$  = number of situations where  $A$  and  $B$  occurred, and  $n$  = number of situations where  $\bar{A}$  and  $B$  occurred.
- Therefore:  $\Pr[A|B] = \frac{m}{m+n}$

### Bayes Rule (Simplified using Law of Total Probability)

$$\Pr[A_n|B] = \frac{\Pr[A_n] \Pr[B|A_n]}{\sum_m \Pr[A_m] \Pr[B|A_m]} = \frac{\Pr[A_n] \Pr[B|A_n]}{\Pr[B]} \quad (3)$$

## Independence

- Two events  $A$  and  $B$  are independent if  $\Pr[A \cap B] = \Pr[A] \Pr[B]$
- Two events  $A$  and  $B$  are independent if and only if  $\Pr[A|B] = \Pr[A]$
- $\Pr[A]$  decreases/increases given  $B$

If  $A$  and  $B$  are **independent** sets:

$$\Pr[\bar{A} \cap \bar{B}] = 1 - \Pr[A \cup B] \quad (4)$$

$$\Pr[A \cap B] = \Pr[A] + \Pr[B] - \Pr[A \cup B] \quad (5)$$