Programming Mathematics in Agda

Donnacha Oisín Kidney

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Programming is Proving

A Polynomial Solver

The p-Adics

Programming is Proving

The Curry-Howard Correspondence

$$
\begin{tikzcd}
\mathit{Type} \arrow[r, Leftrightarrow] \arrow[d] & \mathit{Proposition} \arrow[d] \\
\mathit{Program} \arrow[r, Leftrightarrow] & \mathit{Proof}
\end{tikzcd}
$$

Philip Wadler. Propositions As Types.

***Commun. ACM*, 58(12):75–84, November 2015**

The Curry-Howard Correspondence

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Central idea of dependent types for proofs.We start by showing that the left-hand-side translates to the right.

Types are Propositions

Types are (usually):

- Int
- String
- ...

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Propositions are things like "there are infinite primes", etc. Int certainly doesn't *look* like a proposition.

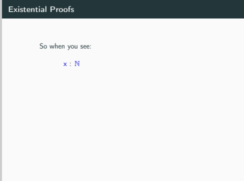
Existential Proofs

We use a trick to translate: put a "there exists" before the type.

Existential Proofs

So when you see:

$$x : \mathbb{N}$$



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Think:

$\exists.\mathbb{N}$

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Programs are Proofs

Let's start working with a function as if it were a proof. The function we'll choose gets the first element from a list. It's commonly called "head" in functional programming.

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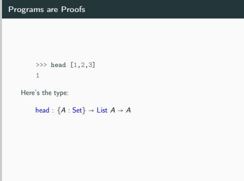
Here's the type:

$$\mathsf{head} : \{A : \mathsf{Set}\} \to \mathsf{List}\ A \to A$$



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Basic Agda Syntax

head is what would be called a “generic” function in languages like Java.In other words, the type A is not specified in the implementation of the function.In Agda, you must supply the type to the function: the curly brackets mean the argument is implicit.

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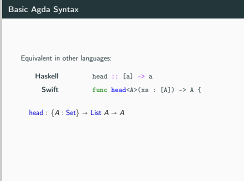

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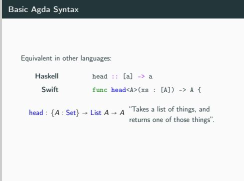
$\mathsf{head} : \{A : \mathsf{Set}\} \to \mathsf{List}\ A \to A$ “Takes a list of things, and returns one of those things”.



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The Proposition is False!

```
>>> head []
error "head: empty list"
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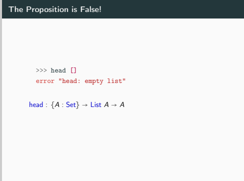

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False

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```
head : {A : Set} → List A → A
```

False

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If Agda is correct (as a formal logic):

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We shouldn't be able to prove this using Agda

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We shouldn't be able write this function in Agda

But Let's Try Anyway! i

Function definition syntax

$$
\begin{aligned}
&\mathsf{fib} : \mathbb{N} \to \mathbb{N} \\
&\mathsf{fib}\ 0 &&= 0 \\
&\mathsf{fib}\ (\mathsf{1+}\ 0) &&= \mathsf{1+}\ 0 \\
&\mathsf{fib}\ (\mathsf{1+}\ (\mathsf{1+}\ n)) &&= \mathsf{fib}\ (\mathsf{1+}\ n) + \mathsf{fib}\ n
\end{aligned}
$$

But Let's Try Anyway! i

Function definition syntax

```
fib : ℕ → ℕ
fib 0              = 0
fib (1+ 0)         = 1+ 0
fib (1+ (1+ n)) = fib (1+ n) + fib n
```

Agda functions are defined (usually) with *pattern-matching*.For the natural numbers, we use the Peano numbers, which gives us 2 patterns: zero, and successor.For lists, we also have two patterns: the empty list, and the head element followed by the rest of the list.We need to disallow functions which don't match all patterns. Array access out-of-bounds, etc., also not allowed.To disallow *this* kind of thing, we must ensure all functions are *total*. For now, assume this means "terminating".

$$
\begin{aligned}
&\mathsf{length} : \{A : \mathsf{Set}\} \to \mathsf{List}\ A \to \mathbb{N} \\
&\mathsf{length}\ [] = 0 \\
&\mathsf{length}\ (x :: xs) = 1 + \mathsf{length}\ xs
\end{aligned}
$$

Here's a definition for head:

$$\text{head}\ (x :: xs) = x$$

No!
For correct proofs, partial functions aren't allowed

But Let's Try Anyway! iv

We're not out of the woods yet:

$$\text{head } [] = \text{head } []$$

No!
For correct proofs, all functions must be total

Correctness

For the proofs to be correct, we have two extra conditions that you usually don't have in programming:

- No partial programs
- Only total programs

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Without these conditions, your proofs are still correct *if they run.*

Enough Restrictions!

That's a lot of things we *can't* prove.

How about something that we can?

How about the converse? After all, all we have so far is "proof by trying really hard".

Can we *prove* that head doesn't exist?

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Falsehood

First we'll need a notion of "False"Often it's said that you can't prove negatives in dependently typed programming: not true!We'll use the principle of explosion: "A false thing is one that can be used to prove anything"

Principle of Explosion
"Ex falso quodlibet"

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Falsehood

$$\neg : \forall \{\ell\} \to \mathsf{Set}\ \ell \to \mathsf{Set}\ \_$$
$$\neg\ A = A \to \{B : \mathsf{Set}\} \to B$$

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Falsehood

```
¬ : ∀ {ℓ} → Set ℓ → Set _
¬ A = A → {B : Set} → B
```

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First we'll need a notion of "False"Often it's said that you can't prove negatives in dependently typed programming: not true!We'll use the principle of explosion: "A false thing is one that can be used to prove anything"

$$\text{head-doesn't-exist} : \neg\, (\{A : \mathsf{Set}\} \to \mathsf{List}\ A \to A)$$
$$\text{head-doesn't-exist}\ \mathit{head} = \mathit{head}\ []$$

```
head-doesn't-exist : ¬ ({A : Set} → List A → A)
head-doesn't-exist head = head []
```

Here's how the proof works: for falsehood, we need to prove the supplied proposition, no matter what it is. If head exists, this is no problem! Just get the head of a list of proofs of the proposition, which can be empty.

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