An Efficient and Flexible Evidence-Providing Polynomial Solver in Agda

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Abstract

We present a new implementation of a ring solver in the programming language Agda [12]. The efficiency is improved over the version included in the standard library [3] by including optimizations described in [5], among others.

We demonstrate techniques for constructing proofs based on the theory of lists, show how Agda's reflection system can be used to provide a safe and simple interface to the solver, and compare the "correct by construction" approach to that of auxiliary proofs.

We also show that, as a by-product of proving equivalences rather than equalities, the generated proofs can be instantiated into a number of exotic settings, including:

1. Pretty-printing step-by-step solutions.
2. Providing human-readable counterexamples when a proof fails.
3. Constructing isomorphisms between types represented as polynomials.

Contents

1 Introduction

Dependently typed programming languages allow programmers and mathematicians alike to write proofs which can be executed. For programmers, this often means being able to formally verify the properties of their programs; for mathematicians, it provides a system of machine-checked verification not available to handwritten proofs.

Naïve usage of these systems can be tedious: the typechecker is often over-zealous in its rigor, demanding justification for every minute step in a proof, no matter how obvious or trivial it may seem to a human. For algebraic proofs, this kind of thing usually consists of long chains of rewrites, of the style "apply commutativity of +, then associativity of +, then at this position apply distributivity of $*$ over +" and so on, when really the programmer wants to say "rearrange the expression into this form, checking it's correct".

However, since our proof assistant is also a programming language, we can provide the desired capability by writing a function which converts expressions into a canonical form, and a proof that the conversion preserves the semantics of the expression. This can then be used to automatically construct equivalence proofs for equivalent expressions.

2 A Case Study in Monoids

Before describing the ring solver, first we will explain the technique of writing a solver in Agda in the simpler setting of monoids.

Definition 2.1. Monoids A monoid is a set equipped with a binary operation, $\bullet$, and a distinguished element ϵ, such that the following equations hold:

```
record Monoid c ℓ : Set (suc (c ⊔ ℓ)) where
  infixl 7 _•_
  infix 4 _≈_
  field
    Carrier   : Set c
    _≈_       : Rel Carrier ℓ
    _•_       : Op₂ Carrier
    ε         : Carrier
    isMonoid  : IsMonoid _≈_ _•_ ε
```

Figure 1: The definition of Monoid in the Agda Standard Library [3]

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \qquad \text{(Associativity)}$$
$$x \bullet \epsilon = x \qquad \text{(Left Identity)}$$
$$\epsilon \bullet x = x \qquad \text{(Right Identity)}$$

Addition and multiplication (with 0 and 1 being the respective identity elements) are perhaps the most obvious instances of the algebra. In computer science, monoids have proved a useful abstraction for formalizing concurrency (in a sense, an associative operator is one which can be evaluated in any order).

Monoids can be represented in Agda in a straightforward way, as a record (see Figure 1). Immediately it should be noted that we're no longer talking about a monoid over a set, but rather one over a setoid. In other words, rather than using propositional equality (indicated by the $\equiv$ symbol), we will use a user-supplied equivalence relation ($\approx$ in Figure 1) in our proofs.

2.1 Equivalence Proofs

Propositions are stated in type signatures in dependently typed languages. Figure 2 is an example of such a proposition. To a human, the fact that the identity holds may well be obvious: $\bullet$ is associative, so scrub out all the parentheses, and ε is the identity element, so scrub it out too. After that, both sides are equal, so voilà!

```
ident : ∀ w x y z
  → ((w • ε) • (x • y)) • z ≈ (w • x) • (y • z)
```

Figure 2: Example Identity

Unfortunately, to convince the compiler we need to specify every instance of associativity and identity, rewriting the left-hand-side repeatedly until it matches the right:

```
1  ident w x y z =
2    begin
3      ((w • ε) • (x • y)) • z
4    ≈⟨ assoc (w • ε) (x • y) z ⟩
5      (w • ε) • ((x • y) • z)
6    ≈⟨ identityʳ w ⟨ •-cong ⟩ assoc x y z ⟩
7      w • (x • (y • z))
8    ≈⟨ sym (assoc w x (y • z)) ⟩
9      (w • x) • (y • z)
10   ■
```

The syntax is designed to mimic that of a hand-written proof: line 3 is the expression on the left-hand side of ≈ in the type, and line 9 the right-hand-side. In between, the expression is repeatedly rewritten into equivalent forms, with justification provided inside the angle brackets. For instance, to translate the expression from the form on line 3 to that on line 5, the associative property of • is used on line 4.

Because we're not using propositional equality, some familiar tools are unavailable, like Agda's rewrite mechanism, or function congruence (this is why we have to explicitly specify the congruence we're using on line 6). The purpose of this particular hair shirt is flexibility: users can still use the solver even if their type only satisfies the monoid laws modulo some equivalence relation (perhaps they are have an implementation of finite, mergeable sets as balanced trees, and want to treat two sets as equivalent if their elements are equal, even if their internal structures are not). Beyond flexibility, we get some other interesting applications, which are explored in section 8.

Despite the pleasant syntax, the proof is mechanical, and it's clear that similar proofs would become tedious with more variables or more complex algebras (like rings). To avoid the tedium, then, we automate the procedure.

2.2 Canonical Forms

Automation of equality proofs like the one above can be accomplished by first rewriting both sides of the equation into a canonical form. Not every algebra has a canonical form: monoids do, though, and it's the simple list.

```
infixr 5 _::_
data List (i : ℕ) : Set where
  [] : List i
  _::_ : Fin i → List i → List i
```

We're going to treat this type like an AST for a simple "language of lists". This language supports two functions: the empty list, and concatenation.

```
infixr 5 _++_
_++_ : ∀ {i} → List i → List i → List i
[] ++ ys = ys
(x :: xs) ++ ys = x :: xs ++ ys
```

The type itself parameterized by the number of variables it contains. Users can refer to variables by their index:

```
infix 9 η_
η_ : ∀ {i} → Fin i → List i
η x = x :: []
```

And we can interpret this language with values for each variable supplied in a vector:

```
_μ_ : ∀ {i} → List i → Vec Carrier i → Carrier
[] μ ρ = ε
(x :: xs) μ ρ = lookup x ρ • xs μ ρ
```

Compare this language to the language of monoid expressions that Figure 2 uses: both have identity elements and a binary operator, and both refer to variables. Our language of lists, however, has one significant advantage: the monoid operations don't depend on the contents of the lists, only the structure. In other words, an expression in the language of lists

will reduce to a flat list even if it has elements which are abstract variables. As a result, the identity from Figure 2 is *definitionally* true when written in the language of lists:

```
obvious
  : (List 4 ∋
  ((η # 0 ∔ []) ∔ (η # 1 ∔ η # 2)) ∔ η # 3)
  ≡ (η # 0 ∔ η # 1) ∔ (η # 2 ∔ η # 3)
obvious = ≡.refl
```

2.3 Extracting Evidence

While this is beginning to look like a solver, it's still not entirely clear how we're going to join up the pieces. The first step is to get a concrete representation of expressions which we can manipulate and pattern-match on.

```
data Expr (i : ℕ) : Set c where
  _⊕_ : Expr i → Expr i → Expr i
  e   : Expr i
  ν_  : Fin i → Expr i
```

It has constructors for each of the monoid operations (⊕ and e are • and ε, respectively), and it's indexed by the number of variables it contains, which are constructed with ν.

We can convert from this AST to an unnormalized expression like so[1]:

```
⟦_⟧ : ∀ {i} → Expr i → Vec Carrier i → Carrier
⟦ x ⊕ y ⟧ ρ = ⟦ x ⟧ ρ • ⟦ y ⟧ ρ
⟦ e ⟧ ρ     = ε
⟦ ν i ⟧ ρ   = lookup i ρ
```

Because this function performs no normalization or transformation, the output is definitionally equal to its equivalent expression. This means that we can discharge the proof obligation with refl:

```
definitional
  : ∀ {w x y z}
  → (w • x) • (y • z)
    ≈ ⟦ (ν # 0 ⊕ ν # 1) ⊕ (ν # 2 ⊕ ν # 3) ⟧
      (w :: x :: y :: z :: [])
definitional = refl
```

The AST might look ugly, but it gives us a link between the expressions we had in Figure 2 and a concrete representation. We'll see in Section 2.6 how to improve the syntax.

The next link is between the AST and the normalized expression. We can convert from the AST to a normal form like so:

```
norm : ∀ {i} → Expr i → List i
norm (x ⊕ y) = norm x ∔ norm y
norm e = []
norm (ν x) = η x
```

And since we already defined how to "evaluate" the list normal form, we can combine both steps into one:

```
⟦_⇓⟧ : ∀ {i}
     → Expr i
     → Vec Carrier i
     → Carrier
⟦ x ⇓⟧ ρ = norm x μ ρ
```

Now we have a concrete way to link the normalized and non-normalized forms of the expressions.

2.4 Homomorphism

Our obvious proof has the form:

$$\text{lhs}_{list} = \text{rhs}_{list} \tag{1}$$

What we want, though, is the following:

$$\text{lhs}_{mon} = \text{rhs}_{mon} \tag{2}$$

Equation 1 can be used to build Equation 2, if we supply two extra proofs:

$$\text{lhs}_{mon} \overset{a}{=} \text{lhs}_{list} = \text{rhs}_{list} \overset{b}{=} \text{rhs}_{mon} \tag{3}$$

The proofs labeled a and b are the homomorphism proofs, and it's what we tackle next.

It's unclear how this gets us to the desired proof in the end. Maybe add a diagram like the one in [5]?

[1] The type of the unnormalized expression has changed slightly: instead of being a curried function of n arguments, it's now a function which takes a vector of length n. The final solver has an extra translation step for going between these two representations, but it's a little fiddly, and not directly relevant to what we're doing here, so we've glossed over it. We refer the interested reader to the Relation.Binary.Reflection module of Agda's standard library [3] for an implementation.

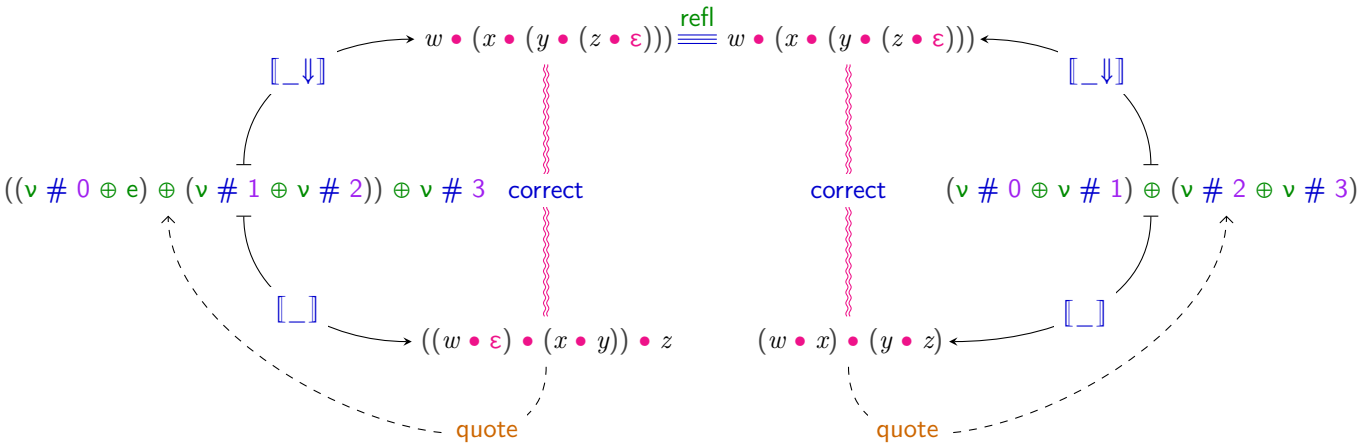

Figure 3: Diagram of Reflexive Proof Process

First, we'll define a concrete AST for the monoid language: interpret it with a vector of values for the variables. This interpreter will result in an unnormalized expression like the one in Figure 2; in other words, the outer sides of Equation 3. To get the normalized expressions, we'll first need to convert a monoid AST to a list AST: Combining that with the fact that we know how to interpret the list AST, we get a normalizing interpreter for the monoid AST: With this, we can see that the proofs needed by a and b are actually proofs of equivalence of these two interpreter functions.

Taking the non-normalizing interpreter as a template, the three cases are as follows[2]:

$$\llbracket x \oplus y \rrbracket \rho \approx \llbracket x \oplus y \Downarrow \rrbracket \rho \tag{4}$$

$$\llbracket \mathsf{e} \rrbracket \rho \approx \llbracket \mathsf{e} \Downarrow \rrbracket \rho \tag{5}$$

$$\llbracket \nu i \rrbracket \rho \approx \llbracket \nu i \Downarrow \rrbracket \rho \tag{6}$$

Proving each of these cases in turn finally verifies the correctness of our list language.

```
+-hom : ∀ {i} (x y : List i)
      → (ρ : Vec Carrier i)
      → (x ++ y) μ ρ ≈ x μ ρ • y μ ρ
+-hom [] y ρ = sym (identityˡ _)
+-hom (x :: xs) y ρ =
  begin
    lookup x ρ • (xs ++ y) μ ρ
  ≈⟨ refl ⟨ •-cong ⟩ +-hom xs y ρ ⟩
    lookup x ρ • (xs μ ρ • y μ ρ)
  ≈⟨ sym (assoc _ _ _) ⟩
    lookup x ρ • xs μ ρ • y μ ρ
  ■

correct : ∀ {i}
        → (x : Expr i)
        → (ρ : Vec Carrier i)
        → ⟦ x ⇓⟧ ρ ≈ ⟦ x ⟧ ρ
correct (x ⊕ y) ρ =
  begin
    (norm x ++ norm y) μ ρ
  ≈⟨ +-hom (norm x) (norm y) ρ ⟩
    norm x μ ρ • norm y μ ρ
  ≈⟨ correct x ρ ⟨ •-cong ⟩ correct y ρ ⟩
    ⟦ x ⟧ ρ • ⟦ y ⟧ ρ
  ■
correct e ρ = refl
correct (ν x) ρ = identityʳ _
```

[2] Equations 4 and 5 comprise a monoid homomorphism.

2.5 Usage

Combining all of the components above, with some plumbing provided by the Relation.Binary.Reflection module, we can finally automate the solving of the original identity in figure 2:

```
ident′ : ∀ w x y z
       → ((w • ε) • (x • y)) • z
       ≈ (w • x) • (y • z)
ident′ = solve 4
  ( λ w x y z
    → ((w ⊕ e) ⊕ (x ⊕ y)) ⊕ z
    ⊜ (w ⊕ x) ⊕ (y ⊕ z))
  refl
```

2.6 Reflection

One annoyance of the automated solver is that we have to write the expression we want to solve twice: once in the type signature, and again in the argument supplied to solve. Agda can infer the type signature:

```
ident-infer : ∀ w x y z → _
ident-infer = solve 4
  ( λ w x y z
    → ((w ⊕ e) ⊕ (x ⊕ y)) ⊕ z
    ⊜ (w ⊕ x) ⊕ (y ⊕ z))
  refl
```

But we would prefer to write the expression in the type signature, and have it infer the argument to solve, as the expression in the type signature is the desired equality, and the argument to solve is something of an implementation detail.

Fill in reflection section

This inference can be accomplished using Agda's reflection mechanisms.

3 A Polynomial Solver

We now know the components required for an automatic solver for some algebra: a canonical form, a concrete representation of expressions, and a proof of correctness. We now turn our focus to polynomials.

Prior work in this area includes [14], [10], [16], [2], and [13], but perhaps the state-of-the-art (at least in terms of efficiency) is Coq's `ring` tactic [15], which is based on an implementation described in [5].

3.1 Choice of Algebra

The choice of algebra has been glossed over thus far, but it is an important design decision: choose one with too many laws, and the solver becomes unusable for several types; too few, and we may miss out on normalization opportunities.

The algebra defined in [5] is that of an *almost-ring*. It's very close in definition too a ring, but it discards the requirement that negation is an inverse ($x + (-x) = 0$). Instead, it merely requires that negation distribute over addition and multiplication appropriately. This allows the solver to be used with non-negative types, like $\mathbb{N}$, where negation is simply the identity function.

A potential worry is that because we don't require $x + (-x) = 0$ axiomatically, it won't be provable in our system. This is not so: as long as $1 - 1$ reduces to 0 in the coefficient set, the solver will verify the identity.

4 Horner Normal Form

The canonical representation of polynomials is a list of coefficients, least significant first ("Horner Normal Form"). Our initial attempt at encoding this representation will begin like so:

```
open import Algebra

module Dense {ℓ} (coeff : RawRing ℓ) where
  open RawRing coeff
```

The entire module is parameterized by the choice of coefficient. This coefficient should support the ring operations, but it is "raw", i.e. it doesn't prove the ring laws. The operations on the polynomial itself are defined like so[3]:

```
Poly : Set ℓ
Poly = List Carrier

_⊞_ : Poly → Poly → Poly
```

```
[] ⊞ ys = ys
(x :: xs) ⊞ [] = x :: xs
(x :: xs) ⊞ (y :: ys) = x + y :: xs ⊞ ys

_⊠_ : Poly → Poly → Poly
[] ⊠ ys = []
(x :: xs) ⊠ [] = []
(x :: xs) ⊠ (y :: ys) =
  x * y :: (map (x *_) ys ⊞ (xs ⊠ (y :: ys)))
```

Finally, evaluation of the polynomial uses "Horner's rule" to minimize multiplications:

```
⟦_⟧ : Poly → Carrier → Carrier
⟦ x ⟧ ρ = foldr (λ y ys → y + ρ * ys) 0# x
```

4.1 Sparse Horner Normal Form

As it stands, the above representation has two problems:

Redundancy The representation suffers from the problem of trailing zeroes. In other words, the polynomial $2x$ could be represented by any of the following:

$$0, 2$$
$$0, 2, 0$$
$$0, 2, 0, 0$$
$$0, 2, 0, 0, 0, 0, 0$$

This is a problem for a solver: the whole *point* is that equivalent expressions are represented the same way.

Inefficiency Expressions will tend to have large gaps, full only of zeroes. Something like x^5 will be represented as a list with 6 elements, only the last one being of interest. Since addition is linear in the length of the list, and multiplication quadratic, this is a major concern.

In [5], the problem is addressed primarily from the efficiency perspective: they add a field for the "power index". For our case, we'll just store a list of pairs, where the second element of the pair is the power index[4].

As an example, the polynomial:

$$3 + 2x^2 + 4x^5 + 2x^7$$

Will be represented as:

$$(3, 0), (2, 1), (4, 2), (2, 1)$$

Or, mathematically:

$$x^0(3 + xx^1(2 + xx^2 * (4 + xx^1(2 + x0))))$$

4.1.1 Uniqueness

While this form solves our efficiency problem, we still have redundant representations of the same polynomials. In [5], care is taken to ensure all operations include a normalizing step, but this is not verified: in other words, it is not proven that the polynomials are always in normal form.

Expressing that a polynomial is in normal form turns out to be as simple as disallowing zeroes: without them, there can be no trailing zeroes, and all gaps must be represented by power indices. To check for zero, we require the user supply a decidable predicate on the coefficients. This changes the module declaration like so:

```
module Sparse
  {a ℓ}
  (coeffs : RawRing a)
  (Zero : Pred (RawRing.Carrier coeffs) ℓ)
```

[3] Symbols chosen for operators use the following mnemonic:

1. Operators preceded with "ℕ." are defined over ℕ; e.g. ℕ.+, ℕ.*.
2. Plain operators, like + and *, are defined over the coefficients.
3. Boxed operators, like ⊞ and ⊠, are defined over polynomials.
4. Operators which are boxed on one side are defined over polynomials on the corresponding side, and the coefficient on the other; e.g. ⋉, ⋊.

[4] In [5], the expression $(c, i) :: P$ represents $P \times X^i + c$. We found that $X^i \times (c + X \times P)$ is a more natural translation, and it's what we use here. A power index of i in this representation is equivalent to a power index of $i + 1$ in [5].

```
  (zero? : Decidable Zero)
  where
  open RawRing coeffs
```

Finally, we can define a sparse encoding of Horner Normal Form:

```
infixl 6 _≠0
record Coeff : Set (a ⊔ ℓ) where
  inductive
  constructor _≠0
  field
    coeff : Carrier
    .{coeff≠0} : ¬ Zero coeff
open Coeff

Poly : Set (a ⊔ ℓ)
Poly = List (Coeff × ℕ)
```

The proof of nonzero is marked irrelevant (preceded with a dot) to avoid computing it at runtime.

We can wrap up the implementation with a cleaner interface by providing a normalizing version of _::_:

```
infixr 8 _△_
_△_ : Poly → ℕ → Poly
[] △ i = []
((x , j) :: xs) △ i = (x , j ℕ.+ i) :: xs

infixr 5 _::↓_
_::↓_ : Carrier × ℕ → Poly → Poly
(x , i) ::↓ xs with zero? x
... | yes p = xs △ suc i
... | no ¬p = (_≠0 x {¬p} , i) :: xs
```

4.1.2 Comparison

Our addition and multiplication functions will need to properly deal with the new gapless formulation. First things first, we'll need a way to match the power indices. We can use a function from [8] to do so.

```
data Ordering : ℕ → ℕ → Set where
  less    : ∀ m k
          → Ordering m (suc (m ℕ.+ k))
  equal   : ∀ m
          → Ordering m m
  greater : ∀ m k
          → Ordering (suc (m ℕ.+ k)) m

compare : ∀ m n → Ordering m n
compare zero    zero    = equal zero
compare (suc m) zero    = greater zero m
compare zero    (suc n) = less zero n
compare (suc m) (suc n) with compare m n
compare (suc .m) (suc .(suc m ℕ.+ k))
  | less m k = less (suc m) k
compare (suc .m) (suc .m)
  | equal m = equal (suc m)
compare (suc .(suc m ℕ.+ k)) (suc .m)
  | greater m k = greater (suc m) k
```

This is a classic example of a "leftist" function: after pattern matching on one of the constructors of Ordering, it gives you information on type variables to the *left* of the pattern. In other words, when you run the function on some variables, the result of the function will give you information on its arguments.

4.1.3 Efficiency

The implementation of compare may raise suspicion with regards to efficiency: if this encoding of polynomials improves time complexity by skipping the gaps, don't we lose all of that when we encode the gaps as Peano numbers?

The answer is a tentative no. Firstly, since we are comparing gaps, the complexity can be no larger than that of the dense implementation. Secondly, the operations we're most concerned about are those on the underlying coefficient; and, indeed, this sparse encoding does reduce the number of those significantly. Thirdly, if a fast implementation of compare is really and truly demanded, there are tricks we can employ.

Agda has a number of built-in functions on the natural numbers: when applied to closed terms, these call to an implementation on Haskell's `Integer` type, rather than the unary implementation. For our uses, the functions of interest are -, +, <, and ==. The comparison functions provide booleans rather than evidence, but we can prove they correspond to the evidence-providing versions. Combined with judicious use of erase, we get the following:

```
less-hom : ∀ n m
         → ((n < m) ≡ true)
         → m ≡ suc (n + (m - n - 1))
less-hom zero zero ()
less-hom zero (suc m) _ = refl
less-hom (suc n) zero ()
less-hom (suc n) (suc m) n<m =
  cong suc (less-hom n m n<m)

eq-hom : ∀ n m
       → ((n == m) ≡ true)
       → n ≡ m
eq-hom zero zero _ = refl
eq-hom zero (suc m) ()
eq-hom (suc n) zero ()
eq-hom (suc n) (suc m) n≡m =
  cong suc (eq-hom n m n≡m)

gt-hom : ∀ n m
       → ((n < m) ≡ false)
       → ((n == m) ≡ false)
       → n ≡ suc (m + (n - m - 1))
gt-hom zero zero n<m ()
gt-hom zero (suc m) () n≡m
gt-hom (suc n) zero n<m n≡m = refl
gt-hom (suc n) (suc m) n<m n≡m =
  cong suc (gt-hom n m n<m n≡m)

compare : (n m : ℕ) → Ordering n m
compare n m with n < m | inspect (_<_ n) m
... | true  | [ n<m ]
  rewrite erase (less-hom n m n<m) =
    less n (m - n - 1)
... | false | [ n≮m ]
  with n == m | inspect (_==_ n) m
... | true  | [ n≡m ]
  rewrite erase (eq-hom n m n≡m) =
    equal m
... | false | [ n≠m ]
  rewrite erase (gt-hom n m n≮m n≠m) =
    greater m (n - m - 1)
```

4.1.4 Termination

Unfortunately, using compare in the most obvious way won't pass the termination checker.

```
{-# NON_TERMINATING #-}
_⊞_ : Poly → Poly → Poly
[] ⊞ ys = ys
(x :: xs) ⊞ [] = x :: xs
((x , i) :: xs) ⊞ ((y , j) :: ys) with compare i j
... | less .i k = (x , i) :: xs ⊞ ((y , k) :: ys)
... | greater .j k = (y , j) :: ((x , k) :: xs) ⊞ ys
... | equal .i =
  (coeff x + coeff y , i) ::↓ (xs ⊞ ys)
```

Agda needs to be able to see that one of the numbers returned by compare always reduces in size: however, since the difference is immediately packed up in a list in the recursive call, it's buried too deeply in constructors for the termination checker to see it.

The solution is twofold: unpack any constructors into function arguments as soon as possible, and eliminate any redundant pattern matches in the offending functions. Taken together, these form an transformation known as "call pattern specialization" [6][5]. Happily, this transformation both makes termination more obvious *and* improves performance.

```
mutual
  infixl 6 _⊞_
  _⊞_ : Poly → Poly → Poly
  [] ⊞ ys = ys
  ((x , i) :: xs) ⊞ ys = ⊞-zip-r x i xs ys

  ⊞-zip-r : Coeff → ℕ → Poly → Poly → Poly
  ⊞-zip-r x i xs [] = (x , i) :: xs
  ⊞-zip-r x i xs ((y , j) :: ys) =
    ⊞-zip (compare i j) x xs y ys

  ⊞-zip : ∀ {p q}
        → Ordering p q
        → Coeff
        → Poly
        → Coeff
        → Poly
        → Poly
  ⊞-zip (less i k) x xs y ys =
    (x , i) :: ⊞-zip-r y k ys xs
```

[5] This transformation is performed automatically by GHC as an optimization: perhaps a similar transformation could be performed by Agda's termination checker to reveal more terminating programs.

```
⊞-zip (greater j k) x xs y ys =
  (y , j) :: ⊞-zip-r x k xs ys
⊞-zip (equal i) x xs y ys =
  (coeff x + coeff y , i) ::↓ (xs ⊞ ys)
```

Every helper function in the mutual block matches on exactly one argument, eliminating redundancy. This also simplifies some of the homomorphism proofs later on.

5 Binary

Before continuing with polynomials, we'll take a short detour to look at binary numbers. These have a number of uses in dependently typed programming: as well as being a more efficient alternative to Peano numbers, their structure informs that of many data structures, such as binomial heaps, and as such they're used in proofs about those structures.

Similarly to polynomials, though, the naïve representation suffers from redundancy in the form of trailing zeroes. There are a number of ways to overcome this (see [9] and [4], for example); yet another is the repurposing of our sparse polynomial from above.

```
Bin : Set
Bin = List ℕ
```

We don't need to store any coefficients, because 1 is the only permitted coefficient. Effectively, all we store is the distance to another 1.

Addition (elided here for brevity) is linear in the number of bits, as expected, and multiplication takes full advantage of the sparse representation:

```
pow : ℕ → Bin → Bin
pow i [] = []
pow i (x :: xs) = (x ℕ.+ i) :: xs

infixl 7 _*_
_*_ : Bin → Bin → Bin
_*_ [] _ = []
_*_ (x :: xs) =
  pow x ∘ foldr (λ y ys → y :: xs + ys) []
```

6 Multivariate

Up until now our polynomial has been an expression in just one variable. For it to be truly useful, though, we'd like to be able to extend it to many: luckily there's a well-known isomorphism we can use to extend our earlier implementation. For a polynomial with two variables, we will represent it as a polynomial whose coefficients are themselves polynomials of one variable. For three, it'll be a polynomial whose coefficients are polynomials in two variables, and so on. Generally speaking, a multivariate polynomial is one where its coefficients are polynomials with one fewer variable [2].

Before jumping to implement this, though, we should notice an opportunity for optimization (also pointed out in [5]). Just like how the monomials had gaps of zeroes between non-zero terms, this type will have gaps of constant polynomials between non-constant polynomials. In other words, in a expression of n variables, information about the last variable will be hidden behind n layers of nesting. If those n layers don't contain any information (i.e. they're all constant), we want to skip all that nesting with an *injection* index (like the power index).

6.1 Sparse Nesting

It's immediately clear that removing the gaps from the nesting will be more difficult than it was for the exponents: the Poly type is *indexed* by the number of variables it contains, so any manipulation of the injection index will have to correspond carefully to the Poly's index.

Our first approach might mimic the structure of Ordering, with an indexed type:

```
data Poly : ℕ → Set (a ⊔ ℓ) where
  _Π_ : ∀ i {j}
      → FlatPoly j
      → Poly (suc (i ℕ.+ j))
```

Where FlatPoly is effectively the gappy type we had earlier. If you actually tried to use this type, though, you'd run into issues: because it's an indexed type, pattern matching on it will force unification of the index with whatever type variable it was bound to.

This is problematic because the index is defined by a function: pattern match on a pair of Polys and you're asking Agda to unify $i_1 + j_1$ and $i_2 + j_2$, a task it will likely find too difficult. How do we avoid this? "Don't touch the green slime!" [7]:

> When combining prescriptive and descriptive indices, ensure both are in constructor form. Exclude defined functions which yield difficult unification problems.

We'll have to take another route.

6.1.1 Inequalities

First, we'll define our polynomial like so:

```
infixl 6 _Π_
record Poly (n : ℕ) : Set (a ⊔ ℓ) where
  inductive
  constructor _Π_
  field
    {i} : ℕ
    flat : FlatPoly i
    i≤n : i ≤ n
```

The type is now parameterized, rather than indexed: our pattern-matching woes have been solved. Also, the gap is now implicit; instead, we store a proof that the nested polynomial has no more variables then the outer. Next, FlatPoly:

```
data FlatPoly : ℕ → Set (a ⊔ ℓ) where
  K : Carrier → FlatPoly zero
  Σ : ∀ {n}
    → (xs : Coeffs n)
    → .{xn : Norm xs}
    → FlatPoly (suc n)
```

We're back to an indexed type here, so you may be concerned about similar unification problems like the ones we had above: not to worry, though, as these indices are in constructor form, which prove much easier to unify than functions.

Also new here is the Norm function. It serves the same purpose that Zero did in the monomial, ensuring that there are no constant polynomials which could be replaced by incrementing the injection index. Its definition is as follows:

```
Norm : ∀ {i} → Coeffs i → Set
Norm []                    = ⊥
Norm (_ Δ zero  :: [])     = ⊥
Norm (_ Δ zero  :: _ :: _) = ⊤
Norm (_ Δ suc _ :: _)      = ⊤
```

The rest of types are similar to what they were before:

```
infixl 6 _Δ_
record CoeffExp (i : ℕ) : Set (a ⊔ ℓ) where
  inductive
  constructor _Δ_
  field
    coeff : Coeff i
    pow : ℕ

Coeffs : ℕ → Set (a ⊔ ℓ)
Coeffs n = List (CoeffExp n)

infixl 6 _≠0
record Coeff (i : ℕ) : Set (a ⊔ ℓ) where
  inductive
  constructor _≠0
  field
    poly : Poly i
    .{poly≠0} : ¬ Zero poly

Zero : ∀ {n} → Poly n → Set ℓ
Zero (K x       Π _) = Zero-C x
Zero (Σ []      Π _) = Lift ℓ ⊤
Zero (Σ (_ :: _) Π _) = Lift ℓ ⊥
```

Again, similarly to the sparse exponent encoding, we provide a smart constructor which ensures normalization.

6.1.2 Choosing an Inequality

Conspicuously missing above is a definition for ≤.

Option 1: The Standard Way The most commonly used definition of ≤ is as follows:

```
data _≤_ : ℕ → ℕ → Set where
  z≤n : ∀ {n} → zero ≤ n
```

```
s≤s : ∀ {m n}
    → (m≤n : m ≤ n)
    → suc m ≤ suc n
```

For our purposes, though, this type is dangerous: it actually *increases* the complexity from the dense encoding. To understand why, remember the addition function above with the gapless exponent encoding. For it to work, we needed to compare the gaps, and proceed based on that. We'll need to do a similar comparison on variable counts for this gapless encoding. However, we don't store the *gaps* now, we store the number of variables in the nested polynomial. Consider the following sequence of nestings:

$$(5 \le 6), (4 \le 5), (3 \le 4), (1 \le 3), (0 \le 1)$$

The outer polynomial has 6 variables, but it has a gap to its inner polynomial of 5, and so on. The comparisons will be made on 5, 4, 3, 1, and 0. Like repeatedly taking the length of the tail of a list, this is quadratic. There must be a better way.

Option 2: With Propositional Equality Once you realize we need to be comparing the gaps and not the tails, another encoding of ≤ in Data.Nat seems the best option:

```
record _≤_ (m n : ℕ) : Set where
  constructor less-than-or-equal
  field
    {k} : ℕ
    proof : m + k ≡ n
```

It stores the gap *right there*: in k!

Unfortunately, though, we're still stuck. While you can indeed run your comparison on k, you're not left with much information about the rest. Say, for instance, you find out that two respective ks are equal. What about the ms? Of course, you *can* show that they must be equal as well, but it requires a proof. Similarly in the less-than or greater-than cases: each time, you need to show that the information about k corresponds to information about m. Again, all of this can be done, but it all requires propositional proofs, which are messy, and slow. Erasure is an option, but I'm not sure of the correctness of that approach.

Option 3 What we really want is to *run* the comparison function on the gap, but get the result on the tail. Turns out we can do exactly that with the following:

```
infix 4 _≤_
data _≤_ (m : ℕ) : ℕ → Set where
  m≤m : m ≤ m
  ≤-s : ∀ {n}
      → (m≤n : m ≤ n)
      → m ≤ suc n
```

While this structure stores the inequality by induction on the gap. That structure can be used to write a comparison function which was linear in the size of the gap (even though it compares the length of the tail).

Explain more the path from this to the final version. Intermediate comparison function and axiom K, especially

6.1.3 Indexed Ordering

Now that the inequality is an inductive type, which mimics a Peano number stored in the gap, the parallels with the sparse exponent encoding should be even more clear. To write a comparison function, then, we should first look for an equivalent to addition. This turns out to be transitivity:

```
infixl 6 _⋈_
_⋈_ : ∀ {x y z} → x ≤ y → y ≤ z → x ≤ z
xs ⋈ m≤m = xs
xs ⋈ (≤-s ys) = ≤-s (xs ⋈ ys)
```

With this defined, the Ordering type is obvious:

```
data Ordering {n : ℕ} : ∀ {i j}
                      → (i≤n : i ≤ n)
                      → (j≤n : j ≤ n)
                      → Set
                      where
  _<_ : ∀ {i j-1}
      → (i≤j-1 : i ≤ j-1)
      → (j≤n : suc j-1 ≤ n)
      → Ordering (≤-s i≤j-1 ⋈ j≤n) j≤n
```

```
_>_ : ∀ {i-1 j}
      → (i≤n : suc i-1 ≤ n)
      → (j≤i-1 : j ≤ i-1)
      → Ordering i≤n (≤-s j≤i-1 ⋈ i≤n)
eq : ∀ {i} → (i≤n : i ≤ n) → Ordering i≤n i≤n
```

6.2 Operations

After adding the injection index optimization, we use a normalizing version of Π, like we did for the power indices (Figure 4). Operations now perform two "match-up" operations: one for the injection indices and one for the power indices (Figure 5).

Maybe unify the two match-up functions?

6.3 Semantics

When it comes to "interpreting" the polynomials, we don't need to use the same ring as we did for the coefficients: all we need is a homomorphism between the coefficients and the target. This allows the user to supply (potentially) a more efficient implementation for the coefficients, and then use it to prove things about some slower target type. The declaration for the semantics module, then, looks like this:

```
module Semantics
  {r₁ r₂ r₃ r₄}
  (coeffs : RawRing r₁)
  (Zero : Pred (RawRing.Carrier coeffs) r₂)
  (zero? : Decidable Zero)
  (ring : AlmostCommutativeRing r₃ r₄)
  (morphism :
    coeffs -Raw-AlmostCommutative⟶ ring)
  where

  open AlmostCommutativeRing ring
  open SparseNesting coeffs Zero zero?
  open _-Raw-AlmostCommutative⟶_
    morphism
    using ()
    renaming (⟦_⟧ to ⟦_⟧ᵣ)
```

Finally, for the interpretation functions, we see again evidence that our choice of inequality was the right one: it naturally works with the vector input, dropping elements in linear time.

```
infixr 8 _^_
_^_ : Carrier → ℕ → Carrier
x ^ zero = 1#
x ^ suc n = x * x ^ n

drop : ∀ {i n}
     → i ≤ n
     → Vec Carrier n
     → Vec Carrier i
drop m≤m P = P
drop (≤-s si≤n) (_ :: P) = drop si≤n P

vec-uncons : ∀ {n}
           → Vec Carrier (suc n)
           → Carrier × Vec Carrier n
vec-uncons (x :: xs) = x , xs

drop-1 : ∀ {i n}
       → suc i ≤ n
       → Vec Carrier n
       → Carrier × Vec Carrier i
drop-1 si≤n xs = vec-uncons (drop si≤n xs)

mutual
  Σ⟦_⟧ : ∀ {n}
       → Coeffs n
       → Carrier × Vec Carrier n
       → Carrier
  Σ⟦ [] ⟧ _ = 0#
  Σ⟦ x ≠0 Δ i :: xs ⟧ (ρ , P) =
    (⟦ x ⟧ P + Σ⟦ xs ⟧ (ρ , P) * ρ) * ρ ^ i

  ⟦_⟧ : ∀ {n}
      → Poly n
      → Vec Carrier n
      → Carrier
  ⟦ K x  Π i≤n ⟧ _ = ⟦ x ⟧ᵣ
  ⟦ Σ xs Π i≤n ⟧ P = Σ⟦ xs ⟧ (drop-1 i≤n P)
```

7 Writing The Proofs

The proofs are long (roughly 1000 lines), albeit mechanical. There are some techniques that made the size manageable.

7.1 Equational Reasoning Techniques

7.1.1 Operators

Because congruence has to be explicitly stated in the reasoning tools, quite often you'll find yourself writing long chains of +-cong or *-cong just to focus into one expression buried in a larger one. To that end, we can use the following operators to make the "focusing" code shooter and easier to read:

```
infixr 1 ≪+_ +≫_ ≪*_ *≫_
≪+_ : ∀ {x₁ x₂ y}
      → x₁ ≈ x₂ → x₁ + y ≈ x₂ + y
≪+ prf = +-cong prf refl

+≫_ : ∀ {x y₁ y₂}
      → y₁ ≈ y₂ → x + y₁ ≈ x + y₂
+≫_ = +-cong refl

≪*_ : ∀ {x₁ x₂ y}
      → x₁ ≈ x₂ → x₁ * y ≈ x₂ * y
≪* prf = *-cong prf refl

*≫_ : ∀ {x y₁ y₂}
      → y₁ ≈ y₂ → x * y₁ ≈ x * y₂
*≫_ = *-cong refl
```

Their fixty means that they can be chained without parentheses. Something like: "≪+ *≫ *≫ ≪*" means "go to the left of +, then to the right of ∗, right again, then to the left of ∗". In other words, it can take a proof of $x \approx y$, and put it in a larger context:

```
example
  : ∀ {a b c d x y}
  → x ≈ y
  → (a * (b * (x * c))) + d ≈
          (a * (b * (y * c))) + d
example prf = ≪+ *≫ *≫ ≪* prf
```

7.1.2 Examples

Expand on the proofs. Maybe include an example proof?

7.2 The Algebra of Programming and List Homomorphisms

I haven't actually been able to apply the "algebra of programming" [11] stuff in

8 Setoid Applications

I mentioned that the notion of equality we were using was more general than propositional, and that we could use it more flexibly in different contexts.

8.1 Traced

One "equivalence relation" is simply a labeled path: a list of rewrite rules or identities, repeatedly applied until the left-hand-side has been changed to the right. Print out the labels when done, and you have a step-by-step computer algebra system à la Wolfram Alpha. The definition of this type is straightforward:

```
infix 4 _≡⋯≡_
infixr 5 _≡⟨_⟩_
data _≡⋯≡_ : A → A → Set a where
  [refl] : ∀ {x} → x ≡⋯≡ x
  _≡⟨_⟩_ : ∀ {x} y {z}
           → String
           → y ≡⋯≡ z
           → x ≡⋯≡ z
  cong₁ : ∀ {x y z} {f : A → A}
          → String
          → x ≡⋯≡ y
          → f y ≡⋯≡ z
          → f x ≡⋯≡ z
  cong₂ : ∀ {x₁ x₂ y₁ y₂ z} {f : A → A → A}
          → String
          → x₁ ≡⋯≡ x₂
          → y₁ ≡⋯≡ y₂
          → f x₂ y₂ ≡⋯≡ z
          → f x₁ y₁ ≡⋯≡ z
```

And it does indeed implement the expected properties of an equivalence relation:

```
trans-≡⋯≡ : ∀ {x y z}
            → x ≡⋯≡ y
            → y ≡⋯≡ z
            → x ≡⋯≡ z
trans-≡⋯≡ [refl] ys = ys
trans-≡⋯≡ (y ≡⟨ x₁ ⟩ xs) ys =
  y ≡⟨ x₁ ⟩ (trans-≡⋯≡ xs ys)
trans-≡⋯≡ (cong₁ e x≡y fy≡z) ys =
  cong₁ e x≡y (trans-≡⋯≡ fy≡z ys)
```

```
trans-≡⋯≡ (cong₂ e x y fxy≡z) ys =
  cong₂ e x y (trans-≡⋯≡ fxy≡z ys)

cong : ∀ {x y}
     → (f : A → A)
     → x ≡⋯≡ y
     → f x ≡⋯≡ f y
cong f xs = cong₁ "cong" xs [refl]

sym-≡⋯≡ : ∀ {x y} → x ≡⋯≡ y → y ≡⋯≡ x
sym-≡⋯≡ {x} {y} = go [refl]
  where
    go : ∀ {z}
       → z ≡⋯≡ x
       → z ≡⋯≡ y
       → y ≡⋯≡ x
    go xs [refl] = xs
    go xs (y ≡⟨ y? ⟩ ys) =
      go (_ ≡⟨ y? ⟩ xs) ys
    go xs (cong₁ e ys zs) =
      go (cong₁ e ys (_ ≡⟨ e ⟩ xs)) zs
    go xs (cong₂ e xp yp zp) =
      go (cong₂ e xp yp (_ ≡⟨ e ⟩ xs)) zp
```

Expand on the traced version, maybe clean it up? Also provide some examples.

8.2 Isomorphisms

8.3 Counterexamples

Use the proof to translate between types. Check out Conor McBride's work on containers for this.

Possible to provide counterexamples if a proof fails?

9 The Correct-by-Construction Approach

The Agda and Coq communities exhibit something of a cultural difference when it comes to proving things. Coq users seem to prefer writing simpler, almost non-dependent code and algorithms, to separately prove properties about that code in auxiliary lemmas. Agda users, on the other hand, seem to prefer baking the properties into the definition of the types themselves, and writing the functions in such a way that they prove those properties as they go (the "correct-by-construction" approach).

There are advantages and disadvantages to each approach. The Coq approach, for instance, allows you to reuse the same functions in different settings, verifying different properties about them depending on what's required. In Agda, this is more difficult: you usually need a new type for every invariant you maintain (lists, and then length-indexed lists, and then sorted lists, etc.). On the other hand, the proofs themselves often contain a lot of duplication of the logic in the implementation: in the Agda style, you avoid this duplication, by doing both at once. Also worth noting is that occasionally attempting to write a function that is correct by construction will lead to a much more elegant formulation of the original algorithm, or expose symmetries between the proof and implementation that would have been difficult to see otherwise.

[5], as an example, is very much in the Coq style: the definition of the polynomial type has no type indices, and makes no requirements on its internal structure:

```
Inductive Pol (C:Set) : Set :=
| Pc : C -> Pol C
| Pinj : positive -> Pol C -> Pol C
| PX : Pol C -> positive -> Pol C -> Pol C.
```

The implementation presented here straddles both camps: we verify homomorphism in separate lemmas, but the type itself does carry information: it's indexed by the number of variables it contains, for instance, and it statically ensures it's always in canonical form.

Performing the same task in a correct-by-construction way is explored in [14] (in Idris [1]).

Expand on this section.

Here we provide a similar implementation:

```
data Poly : Carrier → Set (a ⊔ ℓ) where
  ⟦⟧ : Poly 0#
  ⟦_::_⟧
    : ∀ x {xs}
    → Poly xs
    → Poly (λ ρ → x Coeff.+ ρ Coeff.* xs ρ)

infixr 0 _⇐_
```

```
record Expr (expr : Carrier) : Set (a ⊔ ℓ) where
  constructor _⇐_
  field
    {norm} : Carrier
    poly   : Poly norm
    proof  : expr ≋ norm

infixr 0 _⟸_
_⟸_ : ∀ {x y} → x ≋ y → Expr y → Expr x
x≋y ⟸ xs ⇐ xp = xs ⇐ x≋y ⟨ trans ⟩ xp


_⊞_ : ∀ {x y}
      → Expr x
      → Expr y
      → Expr (x + y)
(x ⇐ xp) ⊞ (y ⇐ yp) =
  xp ⟨ +-cong ⟩ yp ⟸ x ⊕ y
  where
  _⊕_ : ∀ {x y}
        → Poly x
        → Poly y
        → Expr (x + y)
  [] ⊕ ys = ys ⇐ +-identityˡ _
  [ x :: xs ] ⊕ [] = [ x :: xs ] ⇐ +-identityʳ _
  [ x :: xs ] ⊕ [ y :: ys ] with xs ⊕ ys
  ... | zs ⇐ zp = [ x Coeff.+ y :: zs ] ⇐
        (λ ρ → +-distrib ρ)
      ⟨ trans ⟩
        (refl ⟨ +-cong ⟩ (refl ⟨ *-cong ⟩ zp))
```

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A Extra Code Examples

```
_Π↑_ : ∀ {n m} → Poly n → (suc n ≤ m) → Poly m
(xs Π i≤n) Π↑ n≤m = xs Π (≤-s i≤n ⋈ n≤m)

infixr 4 _Π↓_
_Π↓_ : ∀ {i n} → Coeffs i → suc i ≤ n → Poly n
[]                          Π↓ i≤n = K 0#                          Π  z≤n
(x ≠0 Δ zero :: [])         Π↓ i≤n = x                             Π↑ i≤n
(x₁ Δ zero   :: x₂ :: xs)   Π↓ i≤n = Σ (x₁ Δ zero :: x₂ :: xs)    Π  i≤n
(x  Δ suc j  :: xs)         Π↓ i≤n = Σ (x Δ suc j :: xs)           Π  i≤n
```

Figure 4: Normalizing Version of Π

```
mutual
  infixl 7 _⊠_
  _⊠_ : ∀ {n}
      → Poly n
      → Poly n
      → Poly n
  (xs Π i≤n) ⊠ (ys Π j≤n) =
    ⊠-match (i≤n cmp j≤n) xs ys

  ⊠-inj : ∀ {i k}
      → i ≤ k
      → FlatPoly i
      → Coeffs k
      → Coeffs k
  ⊠-inj _ _ [] = []
  ⊠-inj i≤k x (y Π j≤k ≠0 Δ p :: ys) =
    ⊠-match (i≤k cmp j≤k) x y ^ p ::↓
      ⊠-inj i≤k x ys

  ⊠-match : ∀ {i j n}
          → {i≤n : i ≤ n}
          → {j≤n : j ≤ n}
          → Ordering i≤n j≤n
          → FlatPoly i
          → FlatPoly j
          → Poly n
  ⊠-match (eq i&j≤n)    (K x)  (K y)  = K (x * y)          Π  i&j≤n
  ⊠-match (eq i&j≤n)    (Σ xs) (Σ ys) = ⊠-coeffs xs ys     Π↓ i&j≤n
  ⊠-match (i≤j-1 < j≤n) xs     (Σ ys) = ⊠-inj i≤j-1 xs ys Π↓ j≤n
  ⊠-match (i≤n > j≤i-1) (Σ xs) ys     = ⊠-inj j≤i-1 ys xs Π↓ i≤n

  ⊠-coeffs : ∀ {n}
           → Coeffs n
           → Coeffs n
           → Coeffs n
  ⊠-coeffs _ [] = []
  ⊠-coeffs xs (y ≠0 Δ j :: ys) = ⊠-step y ys xs △ j

  ⊠-step : ∀ {n}
         → Poly n
         → Coeffs n
         → Coeffs n
         → Coeffs n
  ⊠-step y ys [] = []
  ⊠-step y ys (x Π j≤n ≠0 Δ i :: xs) =
    (x Π j≤n) ⊠ y ^ i ::↓
      ⊞-coeffs (⊠-inj j≤n x ys) (⊠-step y ys xs)
```


Figure 5: Full Implementation of Multiplication on the Final Encoding of Poly