

Programming Mathematics in Agda

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What do Programming Languages Have to Say About Mathematics?

Programming is Proving

A Polynomial Solver

The p -Adics

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Semantics that are

- Small
- Powerful
- Consistent

Why not use a programming language as
our proof language?

Benefits For Programmers

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- *Prove* things about code

```
assert(list(reversed([1,2,3]))) == [3,2,1])
```

vs

```
reverse-involution :  $\forall xs \rightarrow \text{reverse} (\text{reverse } xs) \equiv xs$ 
```

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- *Prove* things about code
- Use ideas and concepts from maths—why reinvent them?
- Provide coherent *justification* for language features

Benefits For Mathematicians

- Have a machine check your proofs

Currently, though, this is *tedious*

Benefits For Mathematicians

- Have a machine check your proofs
- Run your proofs

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Wait—Isn't this impossible?

Lawrence C Paulson. The Future of Formalised Mathematics, 2016

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We don't need universal systems

Lawrence C Paulson. The Future of Formalised Mathematics, 2016

What About Automated Theorem Provers?

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Georges Gonthier. Formal Proof—The Four-Color Theorem.

Notices of the AMS, 55(11):12, 2008

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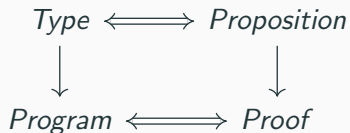
Georges Gonthier. Formal Proof—The Four-Color Theorem.

***Notices of the AMS*, 55(11):12, 2008**

Since our proof language is *executable*, maybe we can write *verified* automated theorem provers?

Programming is Proving

The Curry-Howard Correspondence



Philip Wadler. Propositions As Types.

Commun. ACM, 58(12):75–84, November 2015

Types are Propositions

Types are (usually):

- `Int`
- `String`
- ...

How are these propositions?

So when you see:

$$x : \mathbb{N}$$

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Think:

$$\exists . \mathbb{N}$$

So when you see:

$x : \mathbb{N}$

Think:

$\exists. \mathbb{N}$

NB

We'll see a more powerful and precise version of \exists later.

Existential Proofs

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Proof is “by example”:

So when you see:

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NB

We'll see a more powerful and precise version of \exists later.

Proof is “by example”:

$$x = 1$$

Programs are Proofs

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```
>>> head [1,2,3]
```

```
1
```

Programs are Proofs

```
>>> head [1,2,3]  
1
```

Here's the type:

`head` : $\{A : \text{Set}\} \rightarrow \text{List } A \rightarrow A$

Equivalent in other languages:

Haskell

```
head :: [a] -> a
```

Swift

```
func head<A>(xs : [A]) -> A {
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`func head<A>(xs : [A]) -> A {`

`head : {A : Set} → List A → A` “Takes a list of things, and
returns one of those things”.

The Proposition is False!

```
>>> head []  
error "head: empty list"
```

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False

If Agda is correct (as a formal logic):

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We shouldn't be able to prove this using Agda

If Agda is correct (as a formal logic):

We shouldn't be able write this function in Agda

Function definition syntax

$\text{fib} : \mathbb{N} \rightarrow \mathbb{N}$

$\text{fib } 0 = 0$

$\text{fib } (1 + 0) = 1 + 0$

$\text{fib } (1 + (1 + n)) = \text{fib } (1 + n) + \text{fib } n$

But Let's Try Anyway! ii

$\text{length} : \{A : \text{Set}\} \rightarrow \text{List } A \rightarrow \mathbb{N}$

$\text{length } [] = 0$

$\text{length } (x :: xs) = 1 + \text{length } xs$

Here's a definition for `head`:

$$\text{head } (x :: xs) = x$$

No!

For correct proofs, partial functions aren't allowed

We're not out of the woods yet:

```
head [] = head []
```

No!

For correct proofs, all functions must be total

For the proofs to be correct, we have two extra conditions that you usually don't have in programming:

- No partial programs
- Only total programs

Can we *prove* that **head** doesn't exist?

Principle of Explosion

“Ex falso quodlibet”

If you stand for nothing, you'll
fall for anything.

$\neg : \forall \{ \ell \} \rightarrow \text{Set } \ell \rightarrow \text{Set } _$
 $\neg A = A \rightarrow \{ B : \text{Set} \} \rightarrow B$

Principle of Explosion

"Ex falso quodlibet"

If you stand for nothing, you'll fall for anything.

head-doesn't-exist : $\neg (\{A : \text{Set}\} \rightarrow \text{List } A \rightarrow A)$

head-doesn't-exist *head* = *head* []

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