Portfolio optimization approaches stock trading assets

Felix Vo (7924848)

Ha Uyen Tran (7901093)

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Abstract

When addressing optimization problems in the context of stock trading, a variety of models and algorithms are utilized to optimize assets within the stock market. Given that risks and inflation are key factors influencing stock portfolios, this paper/project introduces two models to address the portfolio optimization problem. Firstly, we present the mean-variance analysis and the Capital Asset Pricing Model (CAPM), a well-established financial optimization model that employs quadratic and linear programming to optimize asset allocation. Additionally, we present

the Enhanced Index Tracking Portfolio, a strategy designed to track inflation rates while accounting for portfolio turnover and expected returns during the previous inflation period. This optimization problem is formulated as a convex programming problem. Ultimately, this paper provides valuable insights for investors seeking to optimize their stock portfolios using finance mathematical models and optimization techniques.

1 Introduction

Inflation refers to the increase in the price of goods or services over a period of time. (Nakagawa & Suimon, 2022) As inflation occurs, the value of a currency decreases relative to its nominal value. In 2023, the inflation rate in the US is recorded to be the highest in 40 years. (Forbes, 2023) To prevent the loss due to the decrease in purchasing power, it is important for us to make an investment or create a financial strategy that will help us exceed the rate of inflation, hence, reducing the risk of losing asset evaluability. There are several financial strategies that help investors gain excess income facing the problem of rising inflation. In this study, we focus on portfolio optimization and its different approaches.

Portfolio optimization is a fundamental problem in finance and investment management, aimed at constructing an optimal portfolio of assets to maximize returns while minimizing risk. (Nakagawa & Suimon, 2022) The problem of portfolio optimization has received extensive attention in the literature, and various approaches have been proposed to tackle it. One of the most widely used approaches is the Capital Asset Pricing Model (CAPM), which provides a framework for asset pricing and portfolio optimization based on the trade-off between risk and return.

Investors are constantly seeking ways to optimize their stock portfolios, especially in the context of the dynamic and ever-changing stock market. With risks and inflation being critical factors that can greatly influence stock portfolios, it becomes increasingly important to adopt effective financial optimization models that can help investors make informed decisions about asset allocation. This paper/project focuses on addressing the portfolio optimization problem by introducing two models, namely mean-variance analysis and the Capital Asset Pricing Model (CAPM). (Haugh, 2016)

Mean-variance analysis and CAPM are two well-established financial optimization models that have been widely used by investors to optimize asset allocation. These models employ quadratic and linear programming to calculate the optimal portfolio allocation based on a set of input variables such as risk, return, and correlation. (Haugh, 2016) By incorporating these models into the investment decision-making process, investors can make more informed and effective decisions about asset allocation.

The Capital Asset Pricing Model (CAPM) and mean-variance analysis are two critical concepts in portfolio management that have had a significant impact on the field of finance. The CAPM, developed in the 1960s by William Sharpe, John Lintner, and Jan Mossin, provides a means of estimating the expected return on investment by factoring in the risk-free rate, the expected return of the market, and the asset's beta. (Haugh, 2016) The model assumes that investors are rational and seek to maximize their expected return for a given level of risk. On the other hand, mean-variance analysis was introduced by Harry Markowitz (1952) and provides a theoretical framework for understanding the relationship between risk and return in portfolio management. (Cochrane, 2000) The core concept of mean-variance analysis is that an investor can construct a portfolio of assets that will maximize expected return for a given level of risk or minimize risk for a given level of expected return. (Cochrane, 2000) The CAPM builds on Markowitz's mean-variance analysis by incorporating the concept of systematic risk, which is measured by beta. (Haugh, 2016) Although the CAPM and mean-variance analysis are distinct concepts, they are closely related and have made significant contributions to the field of portfolio management.

In this research paper, we focus on the application of CAPM to portfolio optimization in the context of stock trading. We examine how the CAPM model can be employed to optimize asset allocation through quadratic and linear programming. Furthermore, we compare the performance of CAPM with the traditional mean-variance analysis, which is a cornerstone of modern portfolio theory.

To expand the portfolio optimization landscape, we introduce a novel approach, the Enhanced Index Tracking Portfolio. Index tracking is a kind of passive income strategy that aims to reproduce the performance of a stock market. The portfolio created is called a tracking portfolio and the index being tracked is called a benchmark. In this approach, the tracking error is used to measure how the optimal portfolio return follows the benchmark index return. A similar approach is called enhanced index-tracking which aims to find returns above the reference index (excess return) while minimizing the tracking error.

(Paulo et al., 2016) This approach considers portfolio turnover and expected returns during the previous inflation period and is formulated as a quadratic programming problem. (Paulo et al., 2016) Our research aims to evaluate the effectiveness of the Enhanced Index Tracking Portfolio compared to the non-tracking portfolio.

The remainder of the paper is structured as follows: Section 2 provides a comprehensive literature review of portfolio optimization and CAPM. In Section 3, we present the methodology for the enhanced index tracking portfolio as well as a numerical example, including the data used and the optimization algorithms employed. In Section 4, we present and discuss the results of our study. Finally, in Section 5, we provide our conclusions and future research directions.

2 Mean-Variance and Capital Asset Pricing Model (CAPM)

2.1 Modern portfolio theory or Markowitz's Mean-Variance Analysis

The modern portfolio theory or the Mean-variance frontier is known as the boundary of the mean-variance distribution of all portfolio returns for a given set of assets. The boundary can be found or defined by minimizing the return variance for a specific mean return(Haugh, 2016).

Figure 1 below illustrates the typical mean-variance frontier of all risky assets, which is graphed as the quadratic hyperbolic region and the risk-free line. This can be called the minimum variance frontier. The hyperbolic risky asset frontier is bounded by two asymptotes, indicated by dotted lines. The risk-free rate is usually depicted below where the asymptotes intersect the vertical axis, or at the minimum variance point on the risky frontier. If the risk-free rate were positioned above this point, investors pursuing a mean-variance strategy would attempt to short the risky assets, which would not result in a stable equilibrium.

2.2 The Mean-Variance Analysis WITHOUT a Risk-free Asset

Problem:

Let's start with a vector R of asset returns. Then denote by the vector of mean returns E, s.t. E $\equiv$ E(R), denote by Σ the variance-covariance of matrix $\Sigma = E[(R-E)(R-E)^{\top}]$

A collection of securities in a portfolio can be characterized by the proportions assigned to each security, represented by the weights w. The return of the portfolio is calculated by taking the weighted average of the returns of the individual securities, denoted by $w^{\top}R$, where $w^{\top}1 = 1$, meaning the sum of the weights equals one. The objective of minimizing the portfolio's variance for a given average return can be framed as a problem

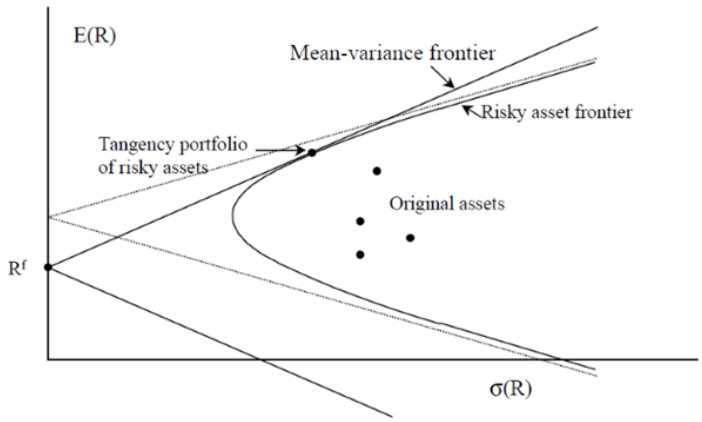


Figure 2.1

Note that the mean portfolio return μ depends on the risk aversion of the investor. This is a standard quadratic optimization problem that can be solved via the Lagrange multiplier method.

Then the mean-variance portfolio optimization problem is introduced as the formula:

$$
\begin{aligned}
& min_w \quad w^\top \Sigma w \\
& \text{s.t} \quad w^\top R = \mu \\
& \text{and} \quad w^\top 1 = 1
\end{aligned}
$$

(Problem 2.1)

Let $\quad A = E^\top \Sigma^{-1} E; \quad B = E^\top \Sigma^{-1} 1; \quad C = 1^\top \Sigma^{-1} 1.$

Then for the given value μ which is the mean portfolio return, the minimum variance portfolio has the variance and the portfolio weight was formed from variance:

$$var(R^p) = \frac{C\mu^2 - 2B\mu + A}{AC + B^2} \quad (2.1) \qquad w = \Sigma^{-1}\frac{E(C\mu - B) + 1(A - B\mu)}{AC + B^2}$$

Formula (2.1) indicates that the variance function is a quadratic function of the mean. The square root of a quadratic parabola is a hyperbolic curve, hence we represent areas bounded by hyperbolic curves in the space of mean and standard deviation.

The minimum-variance portfolio can be found by minimizing equation (2.1) over μ which gives $\mu^{min,var} = B/C$ Therefore, the weight of the minimum-variance portfolio is as:

$$w = \frac{\Sigma^{-1}1}{1^\top \Sigma^{-1}1}$$

By starting with two returns on the forming portfolio and frontier, we can get to any point on the mean-variance frontier. The frontier is spanned by any two frontier returns. Notice that w is a linear function of μ. Therefore, when you take portfolios in respect of any to distinct mean return μ_1 and μ_2 we can imply the weight of the third portfolio with the mean μ_3 is given by $w_3 = \lambda w_1 + (1-\lambda)w_2$ then the mean follows the formula $\mu_3 = \lambda\mu_1 + (1-\lambda)\mu_2$

First, we introduce Lagrange multiplier 2λ and 2δ to the constraints then apply the first order condition to the problem (2.0):

$$\Sigma w - \lambda E - \delta = 0$$

$$w = \Sigma^{-1}(\lambda E + \delta 1).$$

$$(2.2)$$

We find the Lagrangian multiplier from the constraints above

$$E^\top w = E^\top \Sigma^{-1}(\lambda E + \delta 1)$$

$$1^\top = 1^\top \Sigma^{-1}(\lambda E + \delta 1)$$

or

$$\begin{bmatrix} E^\top \Sigma^{-1} E & E^\top \Sigma^{-1} 1 \\ 1\Sigma^{-1} E & 1\Sigma^{-1} 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

Recall the matrix A, B, C. Then the matrix: Let $D = AC - B^2$

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

$$\lambda = \frac{C\mu - B}{D} \tag{1}$$

$$\delta = \frac{A - B\mu}{D} \tag{2}$$

Putting (1) and (2) into equation (2.2), we get the portfolio weights and variance. Then, it leads to the following linear system of equations.

$$\begin{bmatrix} 2\Sigma & -E & -1 \\ E^\top & 0 & 0 \\ 1^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ \mu \\ 1 \end{bmatrix}$$

Hence the equation (2.2): $w^* = \Sigma^{-1}(\lambda E + \delta 1)$. becomes: $w^* = \lambda\Sigma^{-1}E + \delta\Sigma^{-1}1)$.

$$w^* = \frac{C\mu - B}{D}\Sigma^{-1}E + \frac{A - B\mu}{D}\Sigma^{-1}1$$

$$= \frac{1}{D}[B(\Sigma^{-1}1) - B(\Sigma^{-1}E)] + \frac{1}{D}[C(\Sigma^{-1}E) - A(\Sigma^{-1}1)]\mu$$

Result: The optimal solution to the problem (2.0) is given by the portfolio weights which are linear in expected portfolio return such that:

$$w^* = g + h\mu$$

where $g = \frac{1}{D}[B(\Sigma^{-1}1) - B(\Sigma^{-1}E)]$ $\qquad h = \frac{1}{D}[C(\Sigma^{-1}E) - A(\Sigma^{-1}1)]$

$$\text{If } \mu = 0, \qquad w^* = g$$

$$\text{If } \mu = 1, \qquad w^* = g + h$$

Hence, g and g + h are portfolios on the frontier. we can observe the optimal

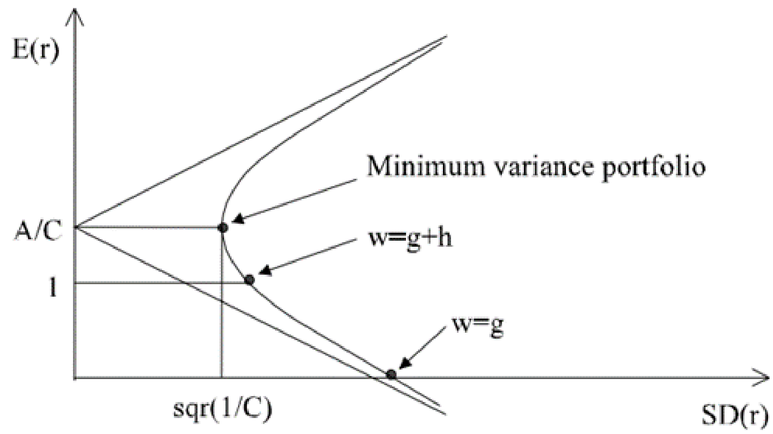


Figure 2.2:

point in Figure (2.2)

2.3 The Mean-Variance WITH a Risk-free Asset

Introduce Problem (2.1) which is based on Problem (2.0), but we add a constraint which is the risk-free asset constraint. Assume there is risk-free security available with r_f the risk-free rate. Let n be the risk assets for the portfolio weight vector: $w := (w_1, w_2, ..., w_n)$, so that $1 - \sum_{i=1}^{n} w_i$ be the weights on that risk-free security.

The portfolio optimization problem is given as:

$$min_w \ \frac{w^\top \Sigma w}{2}$$

$$\text{s.t } (1 - \textstyle\sum_{i=1}^{n} w_i) r_f + w^\top \mu = E[r_p]$$

(Problem 2.1)

The first-order necessary condition for equality constraints: Let w^* be the optimal point for the optimization problem above and for the equality constraints $(1 - \sum_{i=1}^{n} w_i)r_f + w^\top \mu - E[r_p] = 0$. There exists λ which is the Lagrange multiplier. Therefore, Problem (2.1) can imply as:

$$w^* = \lambda \Sigma^{-1}(\mu + r_f 1) \tag{2.3}$$

Multiply by $(e - r_f)^T$ and the Lagrange multiplier λ is:

$$\lambda = \frac{E[r_p] - r_f}{(\mu - r_f)^T \Sigma^{-1}(\mu - r_f)}$$

However, $\lambda = \sigma^2_{min}/(E(r_p) - r_f)$ where σ^2_{min} be the minimized variance, i.e the resulting solution yields an objective function value that is twice as large as the optimal value obtained in the original problem as:

$$\sigma^2_{min} = \frac{(E[r_p] - r_f)^2}{(\mu - r_f 1)^T \Sigma^{-1}(\mu - r_f 1)}$$

where 1 is a n x 1 vector.

The equation of the Lagrange multiplier parameter λ is an essential component of portfolio optimization that captures the extent of an investor's risk aversion. Although the level of risk aversion of a given investor significantly influences this parameter, it is usually estimated based on the market portfolio. Specifically, the market value of λ can be determined by dividing the variance of the market return by the difference between the average excess market return $E(r_p)$ and the risk-free rate (rf).

Including the risk-free asset when the risk-free rate is lower than the expected return of the minimum variance portfolio results in a straight-line efficient frontier, called the tangency portfolio. This portfolio combines the risk-free asset and a single risky portfolio, offering the highest expected return for a given risk level. The weights of this portfolio, w^*, are determined by solving equation (2.3) and then normalized to sum to one. This approach is known as a one-fund theorem, indicating that all investors with different risk preferences will invest in this portfolio.

The linearity of the efficient frontier is due to the linear relationship between the minimum portfolio variance (σ_{min}) and the expected return of the portfolio $E(r_p)$. The slope of the tangency portfolio is defined by the ratio of excess return and portfolio variance, and the y-intercept is the risk-free rate. Overall, the tangency portfolio is a widely accepted method for balancing risk and return in portfolio optimization. Its mathematical framework provides a sound basis for portfolio selection and is commonly employed in financial modeling and analysis.(Haugh, 2016)

2.4 The Capital Asset Pricing Model (CAPM)

The Capital Asset Pricing Model (CAPM) is a financial model used to determine the expected return on investment. It considers the risk-free rate of return, the expected return of the market, and the beta coefficient of the investment to calculate the expected return. The β coefficient measures the volatility of an investment relative to the market. CAPM is used in portfolio optimization to construct efficient portfolios that balance expected return with risk.

In optimization math, CAPM is used as an input to portfolio optimization models. Portfolio optimization models use CAPM to construct efficient portfolios by balancing the expected return of the portfolio with the risk associated with the portfolio. These models use optimization techniques to find the portfolio that maximizes the expected return for a given level of risk or minimizes the risk for a given level of expected return.

Assuming that all investors use mean-variance optimization, it can be observed from Figure 2 and previous discussions that they will all hold the same tangency portfolio consisting of risky securities, along with a position in the risk-free asset. Since this tangency portfolio is held by all investors and market equilibrium is maintained, it can be identified as the market portfolio. The efficient frontier is then referred to as the capital market line.

We now denote the return and the expected return of the market which is the tangency portfolio, respected by R_m and $E(R_m)$ The fundamental concept of the Capital Asset Pricing Model is that, in a state of equilibrium, the risk associated with an asset is not gauged by the standard deviation of its return, but by its beta coefficient.

There exists a linear relationship between the expected return $E(R_i)$ of a portfolio of a particular security and the expected return of the market portfolio. Then, CAPM is usually expressed as:

$$E(R_i) = r_f + \beta(E(R_m) - r_f) \tag{2.4}$$

where $\beta = \frac{Cov(R,R_m)}{Var(R_m)}$

Proof: Consider a portfolio on the risky security and market portfolio, respectively with weights α and weight $1 - \alpha$. Let R_α denote the return of this portfolio.

$$E(R_\alpha) = \alpha E(R_i) + (1 - \alpha)E(R_m)$$

$$\sigma^2_{R_\alpha} = \alpha^2\sigma^2_R + (1 - \alpha)\sigma^2_{R_m} + 2\alpha(1 - \alpha)\sigma_{RR_m} \tag{2.5}$$

As we remember from above, the return variances of the portfolio, security and market portfolio respectively, $\sigma^2_\alpha \quad \sigma^2_R \quad \sigma^2_{R_m}$. the covariance of the portfolio and the market portfolio is denoted as $Cov(R, R_m)$ which used σ_{RR_m} to denote the covariance.

Considering α , the dashed curve in Figure (2.2) the mean and standard deviation of the model which is $E(R_\alpha), \sigma^2_{R_\alpha}$ will draw a curve when α varies, however, it cannot cross the efficient frontier.

See the scenario such that $\alpha = 0$ then the curve tangent to the capital market line. Slope when $\alpha = 0$, therefore, equal to the slope of the capital market line. Then apply to the formula (2.5) we have:

$$\begin{aligned}
\frac{dE(R_\alpha)}{d\sigma_{R_\alpha}}|_{\alpha=0} &= \frac{dE(R_\alpha)/d\alpha}{d\sigma_{R_\alpha}/d\alpha}|_{\alpha=0} \\
&= \frac{\sigma_{R_\alpha}(E(R_i) - E(R_m))}{\alpha\sigma^2_R - (1-\alpha)\sigma^2_{R_m} + (1-2\alpha)\sigma_{RR_m}}|_{\alpha=0} \\
&= \frac{\sigma_{R_\alpha}(E(R_i) - E(R_m))}{(-\sigma^2_{R_m} + \sigma_{RR_m}}
\end{aligned}$$

Recall the slope of the capital market line, said above, slope when $\alpha = 0$, equal to the slope of the capital market line:

$$\frac{E(R_m) - r_F}{\alpha_{R_m}} = \frac{\sigma_{R_\alpha}(E(R_i) - E(R_m))}{(-\sigma^2_{R_m} + \sigma_{RR_m})}$$

which upon simplification gives the original problem, hence, (2.4) proven.

The result of the Capital Asset Pricing Model (CAPM) is widely recognized in finance and provides valuable insight into asset pricing despite originating from a simple one-period model. One significant aspect is that riskier securi-

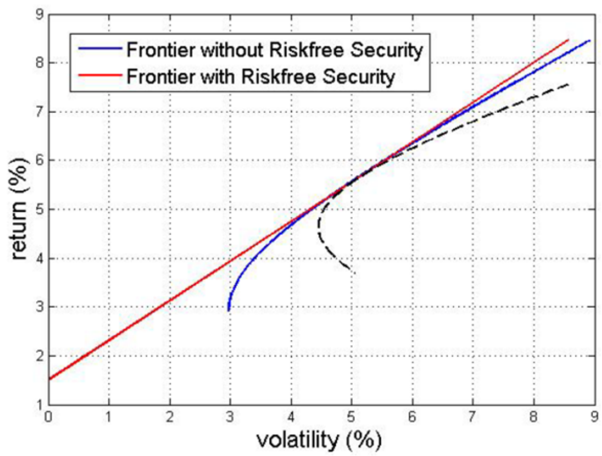


Figure 2.3:

ties should have higher expected returns to compensate investors for holding them. However, it challenges the conventional wisdom of measuring risk by return volatility. Instead, risk is measured by beta, which is proportionate to the covariance with the market portfolio. This is a crucial observation and does not contradict Markowitz's mean-variance formulation where investors are concerned about return variance.(Haugh, 2016) In fact, the CAPM is derived from mean-variance analysis.

2.5 Example

In order to make clear and apply the above model to real-world projects and applications, We use the Mean-Variance Analysis to give out some graphs and MATLAB examples to find out the recommended weights of the stocks portfolio. The purpose is to demonstrate how weights can be calculated for the optimal portfolio together with risks and return

2.5.1 Dataset

In this study, we set the dataset based on daily prices from 3rd January 2022 until the latest 26th April 2023. We use the dataset which constructed using 14 stocks of the US markets in 2023. The data is collected from the Yahoo Finance and the data is up-to-date and the implementation was approached using MATLAB.

We introduce the stocks that we are using in the portfolio, such as: 'AAPL', 'AMZN, 'GOOGLE', 'MSFT', 'NFLX', 'SNAP', 'BYND', 'ROKU', 'SQ', 'UBER', 'PINS', 'ZM', 'META' and 'TSLA'.

2.5.2 Method:

Start by setting up the data for the code, The instances demonstrate the utilization of the Portfolio entity in illustrating the configuration of optimization problems for the mean-variance portfolio, which concentrates on the two-fund principle, the influence of expenses incurred during transactions, and limitations imposed on turnover. They also explain the procedure for obtaining portfolios that maximize the Sharpe ratio.

First of all, Set up the portfolio in standard form by including the roster of assets, the rate of risk-free return, and the statistical measures of returns on assets within the object. Then two objects "AssetList" and "RiskFreeRate" is created as the name of the stock and the $\frac{0.01}{\text{Total days}}$ respectively.

Setup the data set

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	4.6496e-04	4.5103e-04	4.0317e-04	3.6542e-04	4.2941e-04	5.8304e-04	7.0761e-04	5.8958e-04	6.7042e-04	3.6810e-04	5.2198e-04	4.8661e-04	5.0517e-04	5.3144e-04
2	4.5103e-04	9.0615e-04	5.4516e-04	4.9415e-04	6.2566e-04	7.9534e-04	9.6584e-04	8.2775e-04	0.0010	6.1528e-04	7.3715e-04	6.8816e-04	7.0855e-04	6.6118e-04
3	4.0317e-04	5.4516e-04	6.0108e-04	4.3422e-04	4.5071e-04	7.9269e-04	7.2555e-04	7.7066e-04	7.6798e-04	4.1195e-04	6.9047e-04	5.6850e-04	6.5316e-04	4.8824e-04
4	3.6542e-04	4.9415e-04	4.3422e-04	4.7645e-04	3.9805e-04	5.1632e-04	5.9480e-04	5.5295e-04	6.6157e-04	3.4829e-04	4.6654e-04	4.7662e-04	5.1352e-04	4.1251e-04
5	4.2941e-04	6.2566e-04	4.5071e-04	3.9805e-04	0.0011	8.5976e-04	9.8721e-04	9.9387e-04	9.5408e-04	6.5141e-04	8.1749e-04	7.1405e-04	7.2331e-04	7.1599e-04
6	5.8304e-04	7.9534e-04	7.9269e-04	5.1632e-04	8.5976e-04	0.0037	0.0016	0.0019	0.0016	0.0011	0.0019	0.0010	0.0012	0.0010
7	7.0761e-04	9.6584e-04	7.2555e-04	5.9480e-04	9.8721e-04	0.0016	0.0035	0.0018	0.0017	9.5520e-04	0.0014	0.0013	0.0011	0.0012
8	5.8958e-04	8.2775e-04	7.7066e-04	5.5295e-04	9.9387e-04	0.0019	0.0018	0.0027	0.0017	0.0012	0.0016	0.0013	0.0011	0.0011
9	6.7042e-04	0.0010	7.6798e-04	6.6157e-04	9.5408e-04	0.0016	0.0017	0.0017	0.0023	0.0012	0.0014	0.0013	0.0011	0.0011
10	3.6810e-04	6.1528e-04	4.1195e-04	3.4829e-04	6.5141e-04	0.0011	9.5520e-04	0.0012	0.0012	0.0013	0.0010	7.3833e-04	7.0545e-04	7.4364e-04
11	5.2198e-04	7.3715e-04	6.9047e-04	4.6654e-04	8.1749e-04	0.0019	0.0014	0.0016	0.0014	0.0010	0.0020	8.9811e-04	0.0010	8.7211e-04
12	4.8661e-04	6.8816e-04	5.6850e-04	4.7662e-04	7.1405e-04	0.0010	0.0013	0.0013	0.0013	7.3833e-04	8.9811e-04	0.0014	7.5105e-04	7.9150e-04
13	5.0517e-04	7.0855e-04	6.5316e-04	5.1352e-04	7.2331e-04	0.0012	0.0011	0.0011	0.0011	7.0545e-04	0.0010	7.5105e-04	0.0015	5.9133e-04
14	5.3144e-04	6.6118e-04	4.8824e-04	4.1251e-04	7.1599e-04	0.0010	0.0012	0.0011	0.0011	7.4364e-04	8.7211e-04	7.9150e-04	5.9133e-04	0.0017

Take the name of the variables which is the name of the stock. Create a portfolio object by using 'Portfolio()' Function. The Portfolio object implements mean-variance portfolio optimization. Every property and function of the Portfolio object is public, although some properties and functions are hidden.

Portfolio optimization is a technique used by quantitative investment managers and risk managers to determine the proportions of different assets to be included in a portfolio. Its objective is to increase a portfolio's return based on a measure of its risk. The toolbox presented here offers a full range of tools for performing capital allocation, asset allocation, and risk assessment using mean-variance analysis. Using the package 'FINANCE TOOLBOX' for MATLAB, to take the daily return for The Portfolio object implements mean-variance portfolio optimization. Every property and function of the Portfolio object is public, although some properties and functions are hidden. Using 'estimateAssetMoments' which gives the estimation mean and covariance of asset returns from data for a Portfolio object.(Mathworks) Use the 'estimateMaxSharpeRa-

Ticker	Weight (%)
AAPL	9.4856e-13
AMZN	1.9737e-13
GOOGLE	2.2044e-12
MSFT	1.1455e-12
NFLX	86.9487
SNAP	2.6358e-13
BYND	2.4091e-13
ROKU	3.9193e-13
SQ	1.9510e-13
UBER	7.5666e-13
PINS	13.0513
ZM	1.7990e-13
META	3.4448e-11
TSLA	1.6504e-13

tio' function to calculate the efficient portfolio with the highest Sharpe ratio, which is achieved by maximizing the Sharpe ratio among all portfolios on the efficient frontier.(Mathworks, 2010)

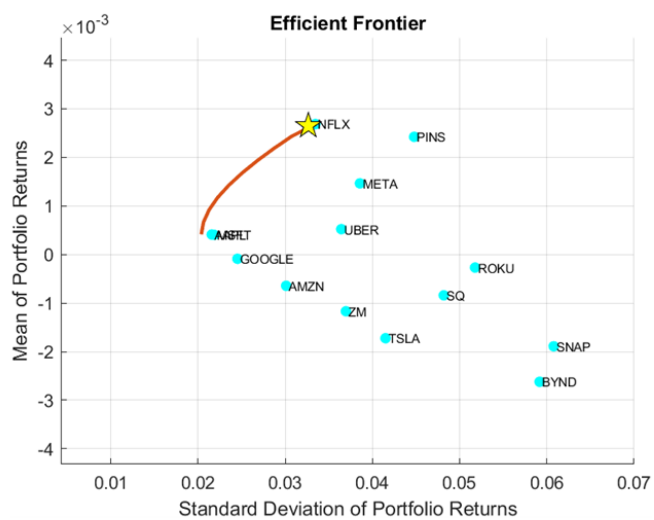
Then the Optimal Portfolio Weight is given out as above :

After collecting all the information and data needed we use the Function: plotFrontier which use to plot the Mean-variance problem. Therefore, we plot the frontier and stocks to the mean and standard deviation plots.

2.5.3 Discussion:

The provided information outlines a process for portfolio optimization using various functions and tools. The estimation of mean and covariance of asset returns using the 'estimateAssetMoments' function is a key step in this process, which provides the necessary data for further analysis. The 'estimateMaxSharpeRatio' function is then used to calculate the efficient portfolio with the highest Sharpe ratio, which is achieved by maximizing the Sharpe ratio among all portfolios on the efficient frontier.

To visualize the results, the 'plotFrontier' function is used, which plots the



mean-variance problem and the stocks to the mean and standard deviation on the plot. The resulting graph helps in analyzing the optimal asset allocation strategy and provides valuable insights into risk management.

Overall, the toolbox offers a comprehensive set of portfolio optimization and analysis tools for quantitative investment managers and risk managers. These tools provide a framework for performing Optimal Portfolio Weight, Efficient Frontier or Mean-Variance Analyse, capital allocation, asset allocation, and risk assessment using mean-variance analysis, which can help investors make informed investment decisions while minimizing risk.

3 Enhanced index-tracking in portfolio optimization

The enhanced index tracking portfolio approach is introduced by Paulo et al. (2016) with the purpose of minimizing the tracking error function based on the difference between the portfolio's return and the benchmark's return. We present the formulation and methodology for the enhanced index tracking problem with the restricted number of assets (Section 3.2) along with an analytical solution for the optimization problem. Additionally, we compare the optimal solution from problem (3.6) with the one attained without the required number of assets in the objective function (Section 3.3).

3.1 Notations and Terminology

i) P: the total return of a tracking portfolio that contains n assets with returns R_1, R_2, $\dots$, R_n ($R = [R_1, ..., R_n]$) and covariance matrix, $\Sigma > 0$. (from Section 2)

ii) w_i: the weight invested in the asset with R_i return.

$$P = \sum_{i=0}^{n} w_i R_i$$

$$\sum_{i=0}^{n} w_i = 1$$

$$w = [w_1, ..., w_n]$$

iii) w_B: the benchmark portfolio containing n assets with return $P_B$and expected return E_B

$$w_B = [w_{B1}, ..., w_{Bn}]$$

iv) P_e: the error between the return from the tracking portfolio and the return from the benchmark portfolio

$$\text{so } P_e = P - P_B = (w - w_B)^T R \tag{3.1}$$

v) μ_e: the excess return (expected value) of tracking portfolio

$$\mu_e = E(R_e) = (w = w_B)^T E(R) \tag{3.2}$$

vi) σ_e^2: the tracking error (variance) of the tracking portfolio

$$\sigma_e^2 = E((R_e - \mu_e)^2) = (w - w_B)^T \Sigma (w - w_B) \tag{3.3}$$

vii) This approach considers the restriction of the number of assets so that the number of assets in the portfolio is $p < n$. With the assumption that the first p of n assets have been selected into the tracking portfolio, the vector of specific asset weights in the tracking portfolio is $w' = [w_1, ..., w_p]$, and the selected return is $R' = [R_1, ..., R_p]$, the covariance matrix of R' will now be $\Gamma = E((R' - r)(R' - r)^T)$ with $r = E(R')$.

viii) Define a $p \times n$ matrix be the covariance matrix, $\Sigma' = E((R' - r)^T (R - E(R)))$

3.2 Methodology

Using the above notation, considering the case where the restriction on the number of assets is applied, we can form P_e and μ_e as follows:

$$
\begin{aligned}
&P_e = w^T R - w_B^T R = w'^T R' - w_B^T R \\
&\mu_e = E(P_e) = w^T r - \mu_B
\end{aligned}
\tag{3.4}
$$

The variance of P_e now becomes:

$$
\begin{aligned}
\sigma_e^2 &= w^T \Sigma w - w^T \Sigma w_B - w_B^T \Sigma w + w_B^T \Sigma w_B \\
=> \sigma_e^2 &= w'^T \Gamma w' - 2 w'^T \Sigma' w_B + \sigma_B^2
\end{aligned}
\tag{3.5}
$$

where σ_B^2 is defined as the variance of the benchmark portfolio, $\sigma_B^2 = w_B^T \Sigma w_B$.

Based on the approach provided by Paulo et al. (2016), the objective function represents the trade-off between tracking error and excess return given by $F = g\sigma_e^2 - \epsilon\mu_e$, where $g > 0$ and $\epsilon \geq 0$ are real numbers. Substitute (3.4) and (3.5) into the function of F, we get the enhanced index tracking optimization problem with the limited assets as follows:

$$\min\ g(w'^T\Gamma w - 2w'\Sigma' w_B) - \epsilon(w'^T r - \mu_B)$$

subject to

$$w'^T\mathbf{1} = 1$$

$$w' \in \mathbb{R}^p \tag{3.6}$$

where $\mathbf{1}$ is a vector of 1s of suitable dimension.

Paulo et al. (2016) indicate another motivation for the problem (3.6) is to maximize the expected value of the negative exponential utility function for the normal case. The negative exponential utility function is given by $G = -e^{-\eta P_e}$, where P_e is a normal random variable, μ_e and σ_e^2 are the mean and variance of the tracking portfolio respectively, and $\eta > 0$. The excess of the return optimization problem is:

$$\max\ E(G)$$

$$\text{subject to } w'^T e = 1 \text{ and } w' \in \mathbb{R}^p \tag{3.7}$$

As G is a lognormal random variable, $E(G)$ can be written as $E(G) = -e^{-\eta(\mu_e - \frac{1}{2}\eta\sigma_e^2)}$. By monotonicity, (3.7) is equivalent to

$$\max \quad \mu_e - \tfrac{1}{2}\eta\sigma_e^2$$

$$\text{subject to } \mathbf{1}^T w' = 1$$

The above problem can be written as

$$\min \quad \tfrac{1}{2}\eta\sigma_e^2$$

subject to $\mathbf{1}^T w' = 1$

This turns out to be equivalent to the problem (3.6) with $g = \frac{\eta}{2}$ and $\epsilon = 1$.

3.3 Analytical Solution

This section presents the analytical solution for the enhanced index tracking problem with a finite number of assets (from Section 3.2). Using the method of Lagrange multipliers, the Lagrangian function considering the problem (3.6) is:

$$\mathcal{L}(w', \lambda) = g(w'^T \Gamma w - 2w'\Sigma' w_B) - \epsilon(w'^T r - \mu_B) + \lambda(w'^T \mathbf{1} - 1) \tag{3.8}$$

where λ is the Lagrangian multiplier to the equality constraint.

Apply the first-order conditions, we get:

$$\begin{cases} 2g(\Gamma w^* - \Sigma' w_B) - \epsilon r + \lambda \mathbf{1} = 0 \\ \mathbf{1}^T w^* = 1 \end{cases} \tag{3.9}$$

where w^* is the optimal solution for the problem. So

$$\begin{aligned} \Gamma w^* - \Sigma' w_B &= \frac{\epsilon r - \lambda \mathbf{1}}{2g} \\ \Gamma w^* &= (\Sigma' w_B + \frac{\epsilon r - \lambda \mathbf{1}}{2g}) \\ w^* &= \Gamma^{-1}(\Sigma' w_B + \frac{\epsilon r - \lambda \mathbf{1}}{2g}) \end{aligned}$$

$$(3.10)$$

Substitute (3.10) into the second equation in (3.9), we get:

$$\mathbf{1}^T\Gamma^{-1}(\Sigma' w_B + \tfrac{\epsilon r - \lambda \mathbf{1}}{2g}) = 1$$

The Lagrange multiplier, λ can be expressed as the following steps:

$$\begin{aligned} \mathbf{1}\Gamma^{-1}\Sigma' w_B + \mathbf{1}\Gamma^{-1}(\frac{\epsilon}{2g}r) - \mathbf{1}\Gamma^{-1}(\frac{\lambda\mathbf{1}}{2g}) = 1 \\ -\lambda(\frac{\mathbf{1}\Gamma^{-1}\mathbf{1}}{2g}) = 1 - \mathbf{1}\Gamma^{-1}\Sigma' w_B - \epsilon(\frac{\mathbf{1}\Gamma^{-1}r}{2g}) \end{aligned}$$

Let $\mathcal{T} = 1 - \mathbf{1}\Gamma^{-1}\Sigma' w_B - \epsilon(\frac{\mathbf{1}\Gamma^{-1}r}{2g})$ and $\alpha = \mathbf{1}\Gamma^{-1}\mathbf{1}$. We have:

$$\begin{aligned} -\lambda\frac{\alpha}{2g} = \mathcal{T} \\ \lambda = -\frac{2g}{\alpha}\mathcal{T} \end{aligned} \tag{3.11}$$

Substitute (3.11) into (3.10), we get the optimal solution as:

$$w^* = \Gamma^{-1}(\Sigma' w_B + \frac{\epsilon}{2g}r + \frac{\mathcal{T}}{\alpha}\mathbf{1}) \tag{3.12}$$

Apply the second-order conditions, we get the second derivative of (3.8), $\mathbf{L}(w)$, is:

$$\mathbf{L}(w) = 2g\Gamma > 0$$

Since $g > 0$ and $\Gamma > 0$, $\mathbf{L}(w) > 0$

Therefore, the optimal solution, w^* is a local minimum.

Because the objective function is convex, and the feasible set is a convex set, the local minimum is also the global minimum.

From (3.12), we can see that the optimal solution to the optimization problem (3.6) depends on the ratio ϵ/g. (Paulo et al., 2016)

3.4 Optimal portfolio approach

This section represents a comparison between the optimal solution from Section 3.3 and the enhanced index tracking portfolio problem without considering the restriction in the objective function. The problem can be written as follows:

$$\begin{aligned} &\min \quad g(w^T\Gamma w) - \epsilon(w^T r) \\ &\text{subject to } w^T\mathbf{1} = 1 \text{ and } w \in \mathbf{R}^p \end{aligned} \tag{3.13}$$

Using the same approach as in Section 3.3, the optimal solution to the problem (3.13), $\tilde{w}$, is proceeded as the following steps:

The Lagrangian function with λ is the Lagrangian multiplier:

$$\mathcal{L}(w, \lambda) = g(w^T\Gamma w) - \epsilon(w^T r) + \lambda(w^T\mathbf{1} - 1)$$

Apply the first-order conditions, we have:

$$\begin{cases} 2g(\Gamma\tilde{w}) - \epsilon r + \lambda\mathbf{1} = 0 \\ \mathbf{1}^T\tilde{w} = 1 \end{cases} \tag{3.14}$$

Therefore,

$$\tilde{w} = \Gamma^{-1}(\frac{\epsilon r - \lambda}{2g}) \tag{3.15}$$

Replacing $\tilde{w}$ into (3.15) we get:

$$\tilde{w} = \Gamma^{-1}(\frac{\epsilon}{2g}r + \frac{\tilde{\mathcal{T}}}{\alpha}\mathbf{1}) \tag{3.16}$$

with $\tilde{\mathcal{T}} = 1 - \epsilon\frac{\mathbf{1}\Gamma^{-1}r}{2g}$

According to Remark 1 from the Finance Research Letter of Paulo et al. (2016), for the case where the restriction of the number of assets is unknown, the return error from (3.1) can be rewritten as $R_e = w^T R' - R_M$ with R_M as the return of the market index. The tracking error, σ_e^2 is then given by:

$$\sigma_e^2 = w'^T \Gamma w' - 2w'^T \Sigma' w_B + \sigma_B^2$$

$$\sigma_e^2 = w'^T \Gamma w' - 2\sigma_M^2 w'^T \beta + \sigma_M^2$$

where β is the p-vector of betas of the R_i asset and σ_M^2 is the variance of the market index.

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

The optimal solution for problem (3.6) can be obtained by replacing $\Sigma' w_B$ with $\sigma_M^2 \beta$:

$$w^* = \Gamma^{-1}(\sigma_M^2\beta + \frac{\epsilon}{2g}r + \frac{\mathcal{T}}{\alpha}\mathbf{1}) \tag{3.17}$$

From (3.16) and (3.17), we have:

$$w^* - \tilde{w} = \Gamma^{-1}(\sigma_M^2\beta + \frac{\epsilon}{2g}r + \frac{\mathcal{T}}{\alpha}\mathbf{1}) - \Gamma^{-1}(\frac{\epsilon}{2g}r + \frac{\tilde{\mathcal{T}}}{\alpha}\mathbf{1})$$
$$w^* - \tilde{w} = \Gamma^{-1}(\sigma_M^2\beta + \frac{\mathcal{T}}{\alpha}\mathbf{1} - \frac{\tilde{\mathcal{T}}}{\alpha}\mathbf{1})$$

Substitute $\tilde{\mathcal{T}}$ and $\mathcal{T}$, we have:

$$w^* - \tilde{w} = \sigma_M^2\Gamma^{-1}(\beta - \frac{\mathbf{1}^T\Gamma^{-1}\beta}{\alpha}\mathbf{1}) \tag{3.18}$$

Let $\beta^* = \beta^T w'^*$ and $\tilde{\beta} = \beta^T\tilde{w}$ be the beta vectors for the portfolio w^* and $\tilde{w}$ respectively.

Set $F^* = g(w^{*T}\Gamma w^* - \sigma_M^{*T}\beta) - \epsilon(w^{*T}r - \mu_B)$ and $\tilde{G} = g(\tilde{w}^T\Gamma\tilde{w} - 2\sigma_M\tilde{w}^T\beta)\epsilon(\tilde{w}^T r - \mu_B)$ be the value functions of the problem (3.6) for the portfolio $\tilde{w}$ and w^* respectively using notation from Remark 1.

Set $H^* = g(w^{*T}\Gamma w^*) - \epsilon(w^{*T}r)$ and $\tilde{H} = g(\tilde{w}^T\Gamma\tilde{w}) - \epsilon(\tilde{w}^T r)$ be the value functions of the problem (3.13) for the portfolio $\tilde{w}$ and w^* respectively.

Define

$$C = (\beta - \frac{\mathbf{1}^T\Gamma^{-1}\beta}{\alpha}\mathbf{1})^T\Gamma^{-1}(\beta - \frac{\mathbf{1}^T\Gamma^{-1}\beta}{\alpha}\mathbf{1}) \tag{3.19}$$

Since $\Gamma > 0$ by definition, C is a positive definite matrix, $C > 0$.

Recall $\alpha = \mathbf{1}^T\Gamma^{-1}\mathbf{1}$, we get:

$$C = \beta^T\Gamma^{-1}\beta - \frac{(\mathbf{1}^T\Gamma^{-1}1\beta)^2}{\alpha} \tag{3.20}$$

We notice that C is not dependent on parameters g and ϵ from the problem (3.6).

From Proposition 3.2 in Edirisinghe (2013) and Proposition 1 in Paulo et al. (2016), we have that:

(a) The tracking optimal portfolio beta increases σ_M^2 with respect to the optimal portfolio beta, that is, $\beta^* - \tilde{\beta} = \sigma_M^2 C$.

(b) The value function of the problem (3.13) for the tracking optimal portfolio increases $g\sigma_M^4 C$ with respect to the one for optimal portfolio, that is, $H^* - \tilde{H} = g\sigma_M^4 C$.

(c) The value function of the problem (3.6) for the tracking optimal portfolio decreases with respect to the one for optimal portfolio, that is, $G^* - \tilde{G} = -g\sigma_M^4 C$.

3.5 Example

This section aims to implement the comparison between the performance of the optimal portfolio with tracking and the one without tracking using the methodology indicated in Section 3.

3.5.1 Dataset

We proceed with an enhanced index-tracking strategy whose tracking portfolio is constructed using the top 5 technology stocks in 2023, AAPL, MSCT, GOOG, AMZN, TSLA, for i index, $i = 1, 2, ..., 5$. The benchmark index chosen for the tracking portfolio of the above stocks is the S&P 500 index (ĜSPC). The data is collected from the Yahoo Finance website with the selected period from Jan 2020 to Jan 2023. The implementation was approached using MATLAB. Note that this approach assumes a given set of assets (N=5) as a tracking portfolio. Considering the number of assets, we arbitrarily choose asset weights in this example, as presented in Table 3.1.

N	Assets	Name	Weight (%)
1	AAPL	Apple Inc.	15.23
2	MSFT	Microsoft Corp.	11.62
3	AMZN	Amazon.com Inc.	25.91
4	GOOGL	Alphabet Inc.	25.46
5	TSLA	Tesla Inc.	21.78

Table 3.1: The composition of the tracking portfolio selected

3.5.2 Method

First, we solve two optimization problems of tracking portfolios with and without asset restriction using the FMINCON function in MATLAB. To start, we compute the covariance matrix, Γ as below:

$$\begin{bmatrix} 0.00054081 & 0.00041501 & 0.00038141 & 0.00037188 & 0.0006029 \\ 0.00041501 & 0.00048058 & 0.00037954 & 0.00039737 & 0.00050935 \\ 0.00038141 & 0.00037954 & 0.00060663 & 0.00036579 & 0.00053847 \\ 0.00037188 & 0.00039737 & 0.00036579 & 0.00047404 & 0.00045615 \\ 0.00056029 & 0.00050953 & 0.00053847 & 0.00045615 & 0.00210000 \end{bmatrix}$$

We obtain the expected return of assets in the benchmark portfolio to be $\mu_B = 0.00021758$ and the expected returns of assets in the tracking portfolio to be $r = [0.00075348, 0.0005629, -0.00016161, 0.00033653, 0.0019]$. For the purpose of this example, we set the values for g and ϵ to be 1 and 0.1 respectively. Note that the two values can be changed if necessary. Plugin all the above values into problems (3.6) and (3.13), we obtain the optimal solutions as below:

$$w^* = \begin{bmatrix} 0.0036 \\ 0.0031 \\ 0.0020 \\ 0.0026 \\ 0.9887 \end{bmatrix}$$

$$\tilde{w} = \begin{bmatrix} 0.2858 \\ 0.2965 \\ 0.0682 \\ 0.3415 \\ 0.0081 \end{bmatrix}$$

In the following, we simulate the cumulative values of the two portfolios over a time period [0,64]. (Set T = 64) Let $X^*(t)$ and $\tilde{X}(t)$ be the value of the tracking portfolio related to the optimal composition w^* and the value of the portfolio without tracking related to the optimal composition $\tilde{w}$ respectively. The values for both optimal portfolios with and without tracking are written as (Paulo et al., 2016):

$$X^*(t+1) = (1 + R'^T(t)w^*)X^*(t)$$

$$\tilde{X}(t+1) = (1 + R^T(t)\tilde{w})\tilde{X}(t)$$

where $R'(t) = (R'_1(t), ..., R'_p(t)$ with $t = 0, 1, 2, ..., T-1$ and $X^*(0) = \tilde{X}(0) = X_0$. Take $X_0 = 100$, Figure 3.2 shows the cumulative portfolio values obtained from the two equations above.

3.5.3 Results

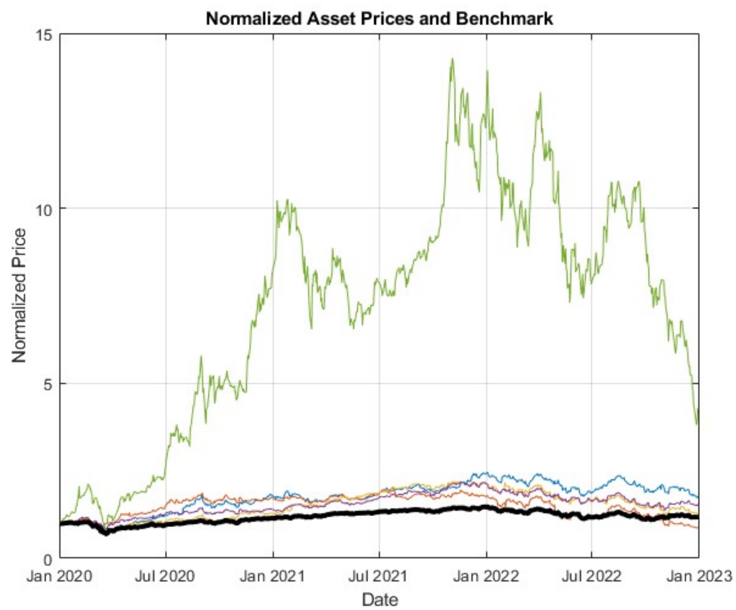


Figure 3.1: Normalized Daily Asset Prices and Benchmark

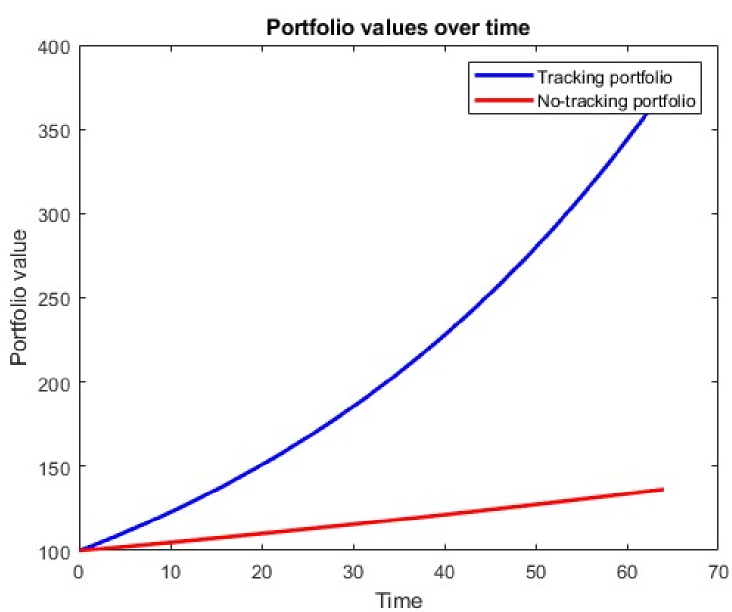


Figure 3.2: Portfolio values over time of the enhanced tracking portfolio and

the non-tracking portfolio

In Figure 3.2, we can see the difference in the performance between the two portfolios (enhanced tracking and non-tracking). The enhanced tracking portfolio values increase significantly faster than the non-tracking portfolio.

3.5.4 Discussion

In Figure 3.2, the occurred major difference in performance between the two portfolios might be the result of our choice of assets since we choose the top stocks from big technology companies. The study of Paulo et al. (2016) indicates there is no possible way to select the best subset of stocks to hold. For future direction, an example with a larger sample size could have more accurate results in comparison of the performance between the non-tracking portfolio and the enhanced index tracking portfolio.

4 Conclusion

Portfolio optimization is a fundamental problem in finance and investment management aimed at constructing an optimal portfolio of assets to maximize returns while minimizing risk. Two well-established financial optimization models, the Capital Asset Pricing Model (CAPM) and mean-variance analysis, have been widely used by investors to optimize asset allocation. The paper focuses on the application of CAPM to portfolio optimization in the context of stock trading and introduces a novel approach, the Enhanced Index Tracking Portfolio, which aims to find returns above the reference index while minimizing the tracking error. The study aims to evaluate the effectiveness of the Enhanced Index Tracking Portfolio compared to the non-tracking portfolio. The paper provides a comprehensive literature review of portfolio optimization and CAPM presents the methodology for the enhanced index tracking portfolio and discusses the results of the study.

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