Cluster analysis of repeated ordinal data: A model approach based on finite mixtures

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Outline

Aim

To develop probability models based on finite mixture models for cluster analysis of ordinal data that arises repeated measures settings

Why? Traditionally, cluster methods for ordinal data

- Assume continuous data/cardinality
- Do not often use statistical inference to compare models
- Allocate subjects to clusters deterministically.

Introduction

Models for Ordinal Data

- Cumulative Logit Models (Multivariate GLM)
- Proportional Odds Model (POM) by McCullagh (1980)
- For a ordinal response Y with q ordered categories and a set of predictors x

$$\text{Logit}[P(Y \leq k|x)] = \mu_k - \beta' x \qquad k = 1, \ldots, q-1$$

Where $\mu_1 < \mu_2 \cdots < \mu_{q-1}$.

Introduction

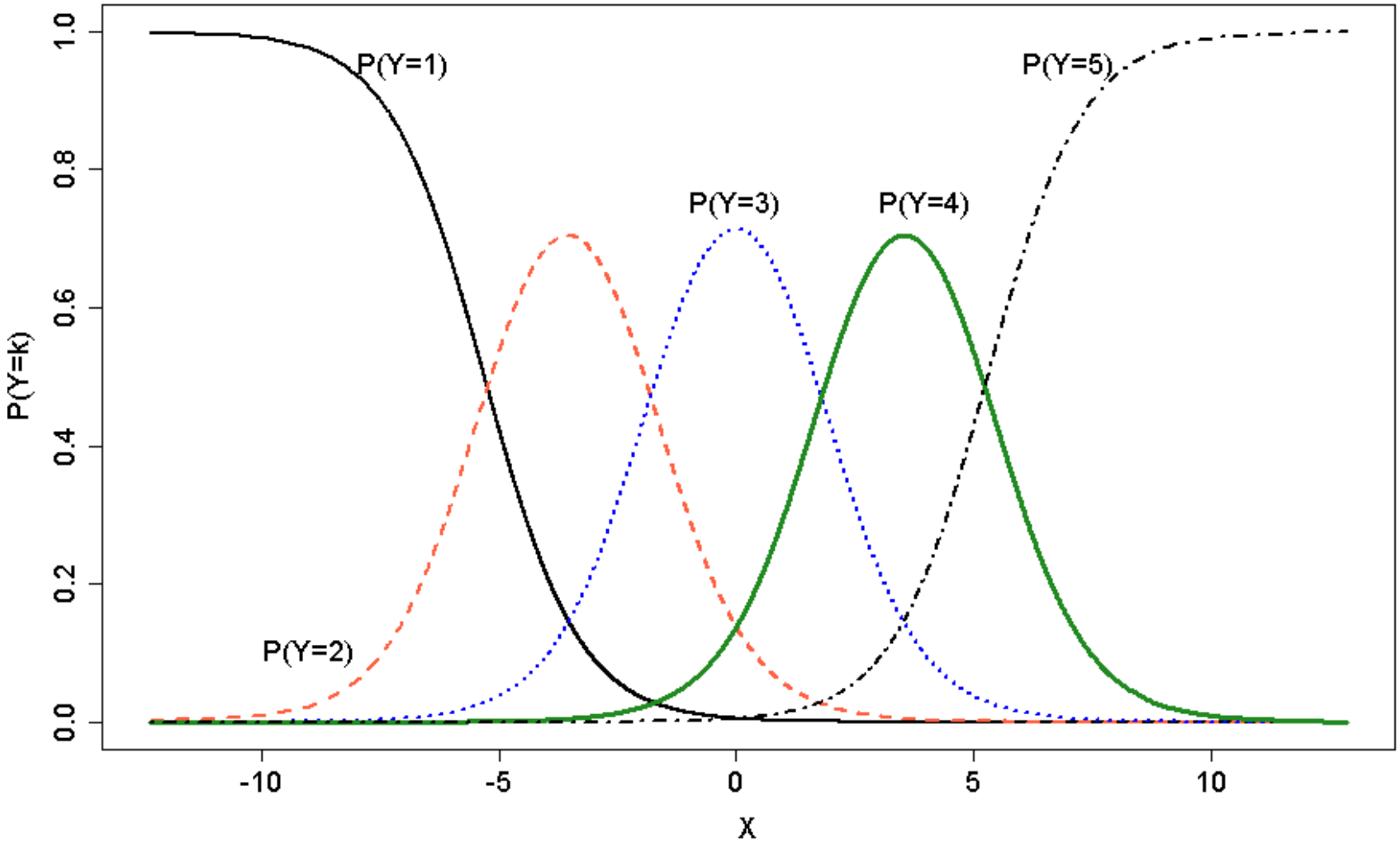


Figure : Individual category probabilities for the POM with five response

Introduction

Models for Ordinal Data

- Trend Odds Model (TOM) by Capuano and Dawson (2012)
- Setting an arbitrary scalar t_k that varies by ordinal outcome (k)

$$\text{Logit}[P(Y \leq k|x)] = \mu_k - (\beta + \gamma t_k)'x \qquad t_k \leq t_{k+1}; k = 1, \ldots, q-1$$

Where

- $\mu_k - \mu_{k-1} \geq \gamma(t_k - t_{k-1})x, \forall x$
- non-decreasing cumulative probabilities constraint

Introduction

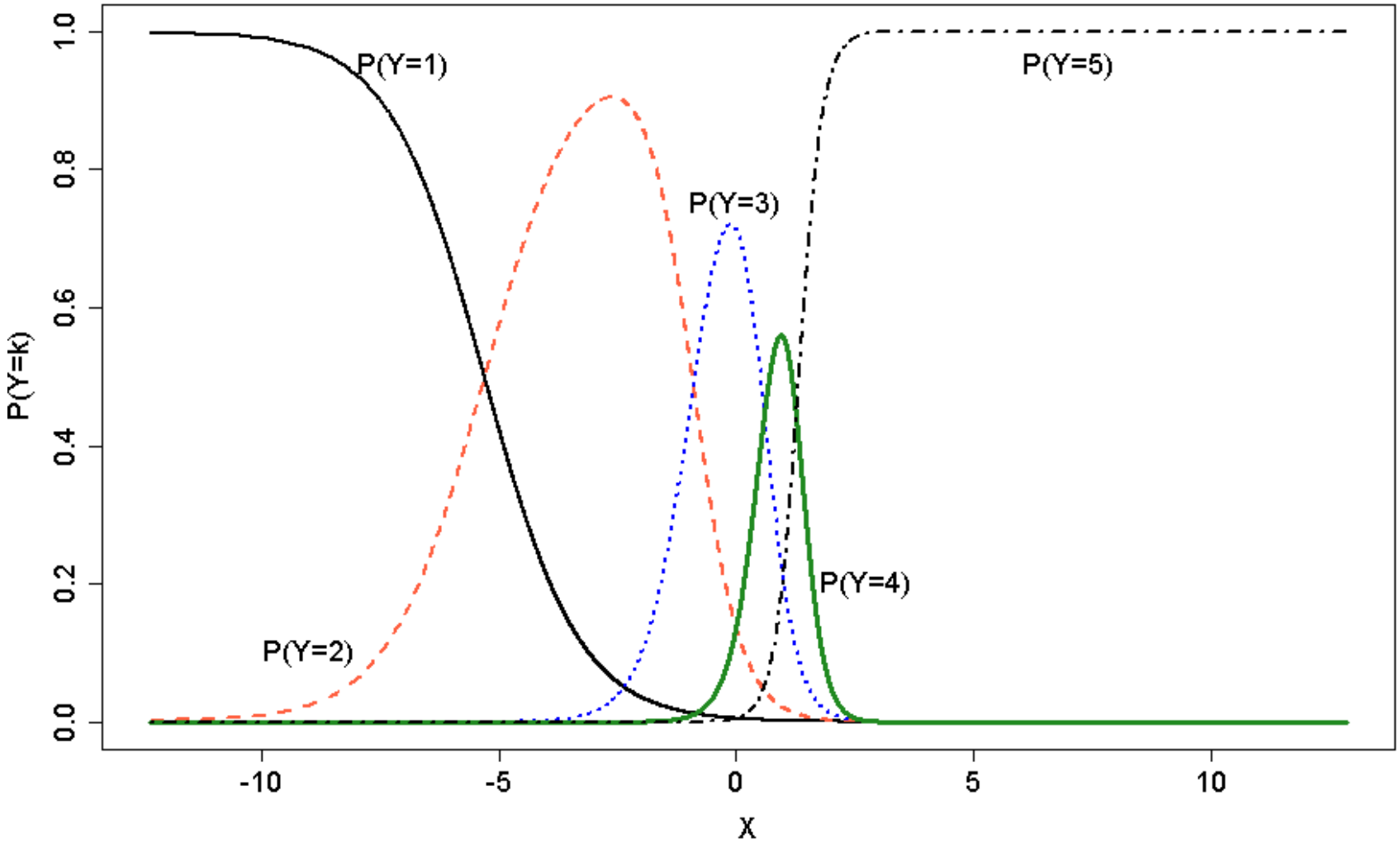


Figure : Individual category probabilities for the TOM with five response

Methods

Model Setup

- Extension of Pledger and Arnold (2013) to ordinal data
- Data **Y** is as n x p matrix where each cell y_{ij} is equal to any of the q ordinal categories
  $i = 1, ..., n$ subjects
  $j = 1, ..., p$ occasions
  $k = 1, ..., q$ ordinal categories

- **Row-clustering**
  - Rows i come from a finite mixture with R components
  - R and row-cluster proportions π_r are unknown.
  - $R < n$ and $\sum_{r=1}^{R} \pi_r = 1$

Methods: TOM row-clustering

- Let $P(y_{ij} = k|i \in r) = \theta_{rjk}$. TOM with row-clustering has the form

$$\text{Logit}[P(y_{ij} \leq k|i \in r)] = \mu_k - \alpha_r - \gamma_r t_k \tag{1}$$

or equivalently

$$\theta_{rjk} = \frac{\exp(\mu_k - \alpha_r - \gamma_r t_k)}{1 + \exp(\mu_k - \alpha_r - \gamma_r t_k)} - \frac{\exp(\mu_{k-1} - \alpha_r - \gamma_r t_{k-1})}{1 + \exp(\mu_{k-1} - \alpha_r - \gamma_r t_{k-1})} \tag{2}$$

Where

- $\alpha_1 = \gamma_1 = 0$
- $\mu_k - \mu_{k-1} \geq \gamma_r \qquad \forall r, k$ when setting $t_k = k - 1$
- non-decreasing cumulative probabilities constraint

Methods: TOM row-clustering

Likelihood

Assuming independence over the rows and, conditional on the rows, independence over the columns, the likelihood is

$$L(\phi, \pi|\mathbf{Y}) = \prod_{i=1}^{n} \sum_{r=1}^{R} \pi_r \prod_{j=1}^{p} \prod_{k=1}^{q} \theta_{rjk}^{I(y_{ij}=k)} \tag{3}$$

where

- ϕ set of model parameters (μ, α, γ)
- $I(y_{ij} = k)$ indicator function equal to 1 if the condition $y_{ij} = k$ is satisfied and 0 otherwise.
- $L(\phi, \pi|\mathbf{Y})$ aka incomplete data likelihood (unknown cluster memberships, π_r).
- number of model parameters $v = (q - 1) + 3(R - 1)$

Methods: TOM row-clustering

Estimation

ϕ and π estimated using the Expected-Maximisation algorithm (EM)

E step Estimate unknown cluster memberships

$$\hat{z}_{ir} = \frac{\pi_r \prod_{j=1}^{p} \prod_{k=1}^{q} \theta_{rjk}^{I(y_{ij}=k)}}{\sum_{a=1}^{R} \pi_a \prod_{j=1}^{p} \prod_{k=1}^{q} \theta_{ajk}^{I(y_{ij}=k)}}$$

M step Numerically maximise complete data log-likelihood

$$\ell_c(\phi, \pi|\mathbf{Y}) = \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{r=1}^{R} \sum_{k=1}^{q} \hat{z}_{ir} I(y_{ij} = k) \log(\theta_{rjk}) + \sum_{i=1}^{n} \sum_{r=1}^{R} \hat{z}_{ir} \log(\pi_r)$$

Methods: TOM row-clustering

Model Selection for finite mixture models

- Comparing models with different number of clusters violates regularity conditions
  1. estimates under the null hypothesis are on boundary of the parameter space
  2. null hypothesis corresponds to a non-identifiable subset of the parameter space

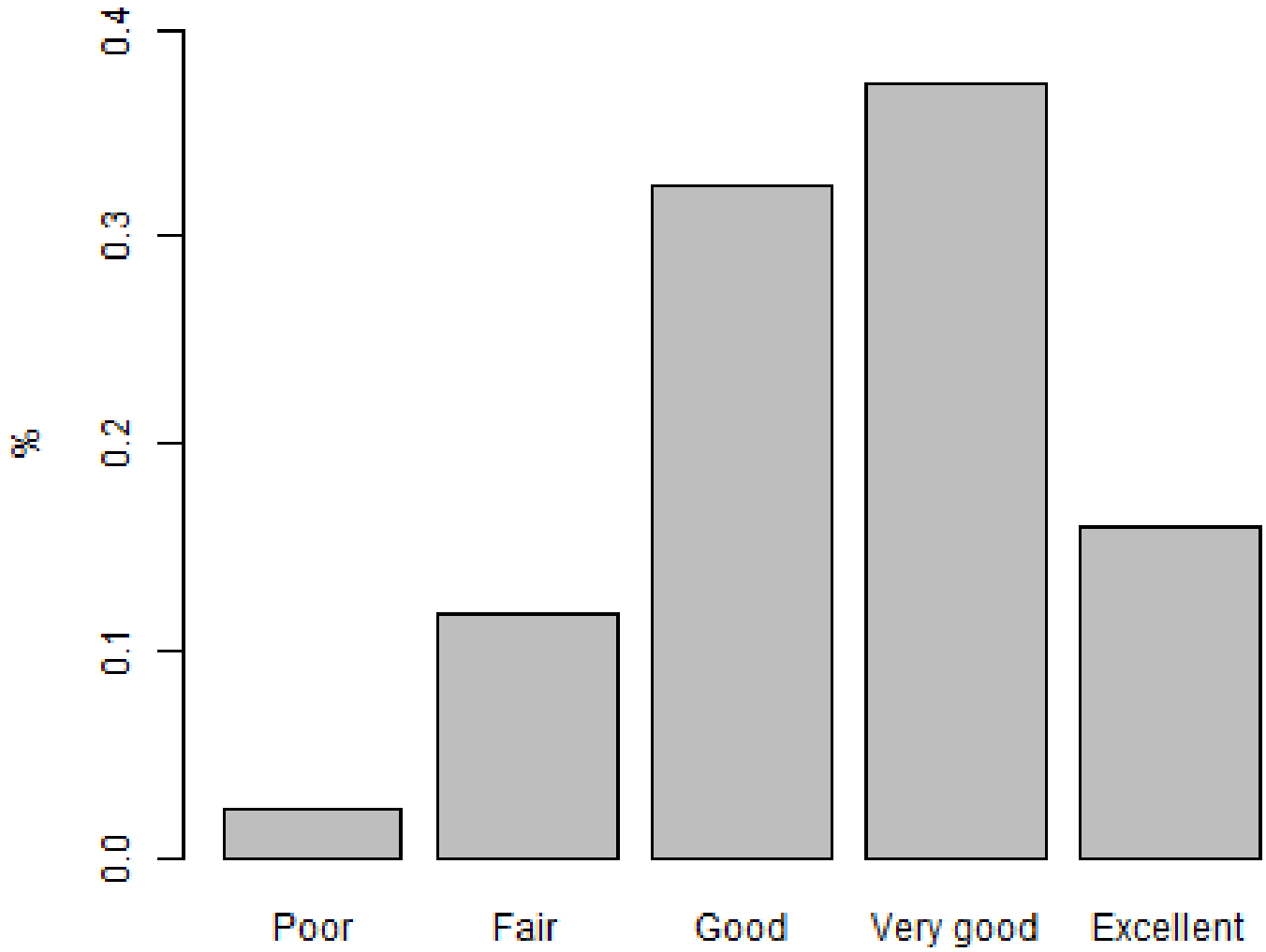
- Simulation studies shown AIC and BIC overestimate number of mixture components
- For this application, we used the ICL-BIC (Biernacki et al 2000)
  - classification-based information criteria
  - also takes into account the degree of separation of the mixture components (entropy).

Applications: Case study

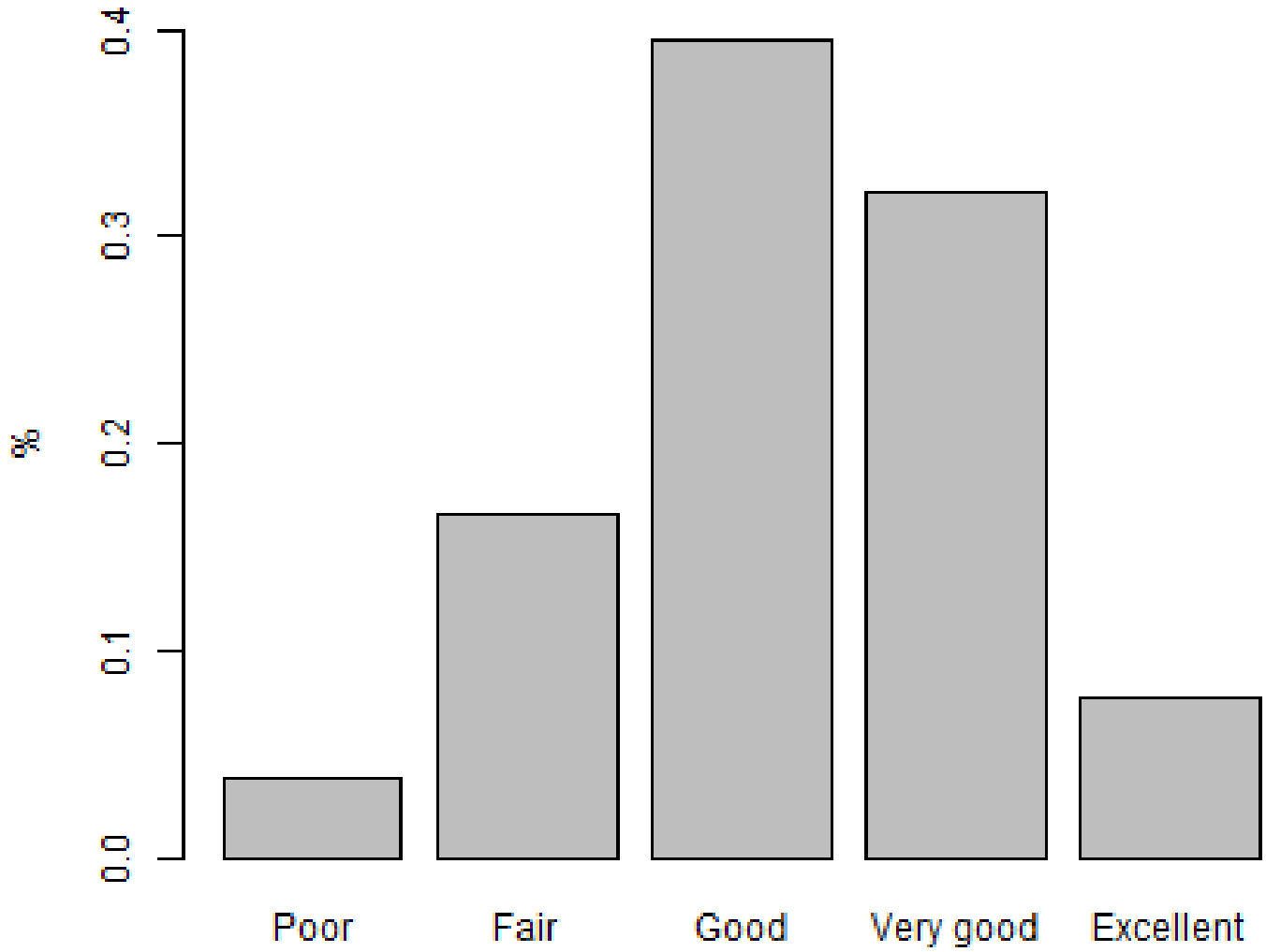
Clustering health status in Australia over 2001-2011

- Household, Income and Labour Dynamics in Australia survey-HILDA
- Self-reported health status (poor, fair, good, very good and excellent)
- 4660 individuals with complete observations over 2001-2011
- POM and TOM results for a subsample of 136

Applications: Self-Reported Health Status 2001



Applications: Self-Reported Health Status 2011



Applications: Case study

Table : Model comparison in the case study: SRHS from HILDA

Model	Linear Predictor	Rows	Cols	Param	AIC	BIC	ICL-BIC
Null	μ_k	1	1	4	4089	4139	
Row fixed effects	$\mu_k - \alpha_i$	136	1	139	2446	4200	
Col fixed effects	$\mu_k - \beta_j$	1	11	14	4094	4271	
Row and Col fixed effects	$\mu_k - \alpha_i - \beta_j$	136	11	149	**2424**	4305	
POM row clustering	$\mu_k - \alpha_r$	2	1	6	3313	3345	3355
		3	1	8	3029	3071	3095
		4	1	10	2920	2973	2999
		5	1	12	2829	2893	2927
		6	1	14	2809	2883	2947
POM col clustering	$\mu_k - \beta_c$	1	2	6	4092	4124	4134
POM biclustering	$\mu_k - \alpha_r - \beta_c$	2	2	8	3313	3355	3376
		3	2	10	3024	3077	3111
		4	2	12	2914	2978	3015
		5	2	14	2827	2901	2951
		6	2	16	2803	2888	2967
TOM row clustering	$\mu_k - \alpha_r - \gamma_r t_k$	2	1	7	3313	3351	3360
		3	1	10	3025	3078	3098
		4	1	13	2912	2981	3006
		5	1	16	2808	2893	**2922**
		6	1	19	2774	**2875**	2925
TOM col clustering	$\mu_k - \beta_c - \delta_c t_k$	1	2	7	4095	4132	4132
TOM biclustering	$\mu_k - \alpha_r - \beta_c - (\gamma_r + \delta_c)t_k$	2	2	10	4033	4087	4078
		3	2	13	3882	3951	3938
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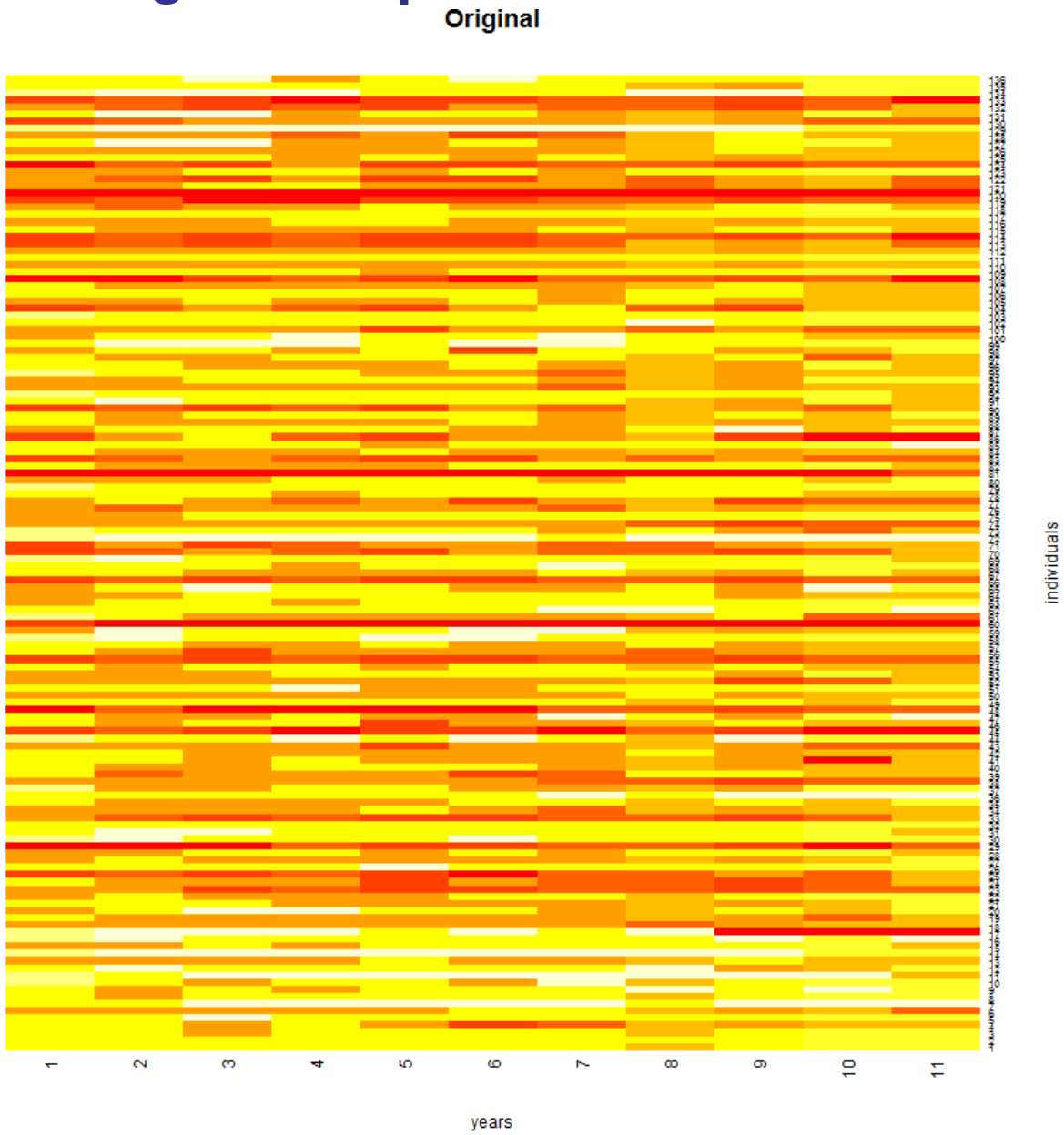
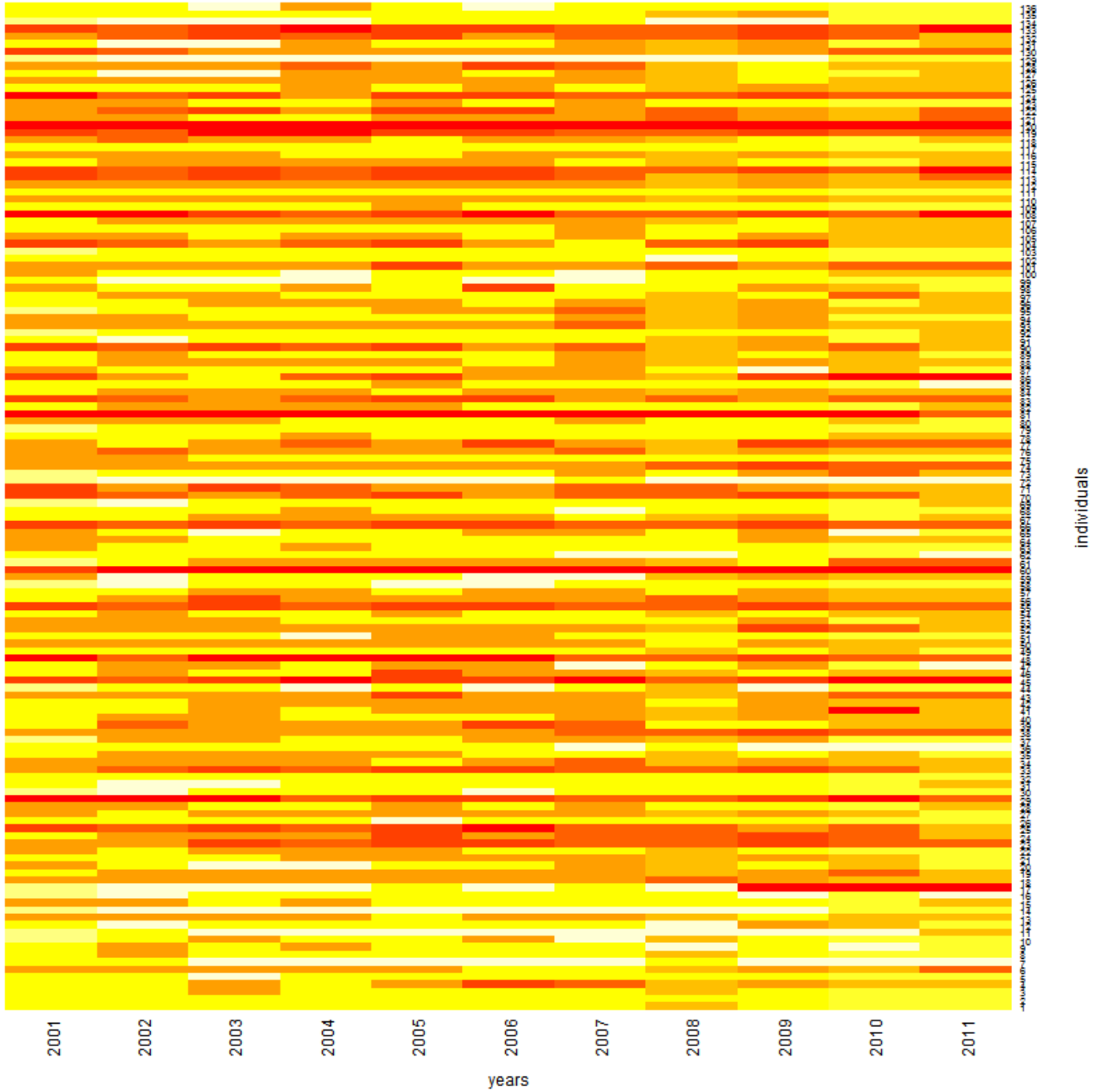
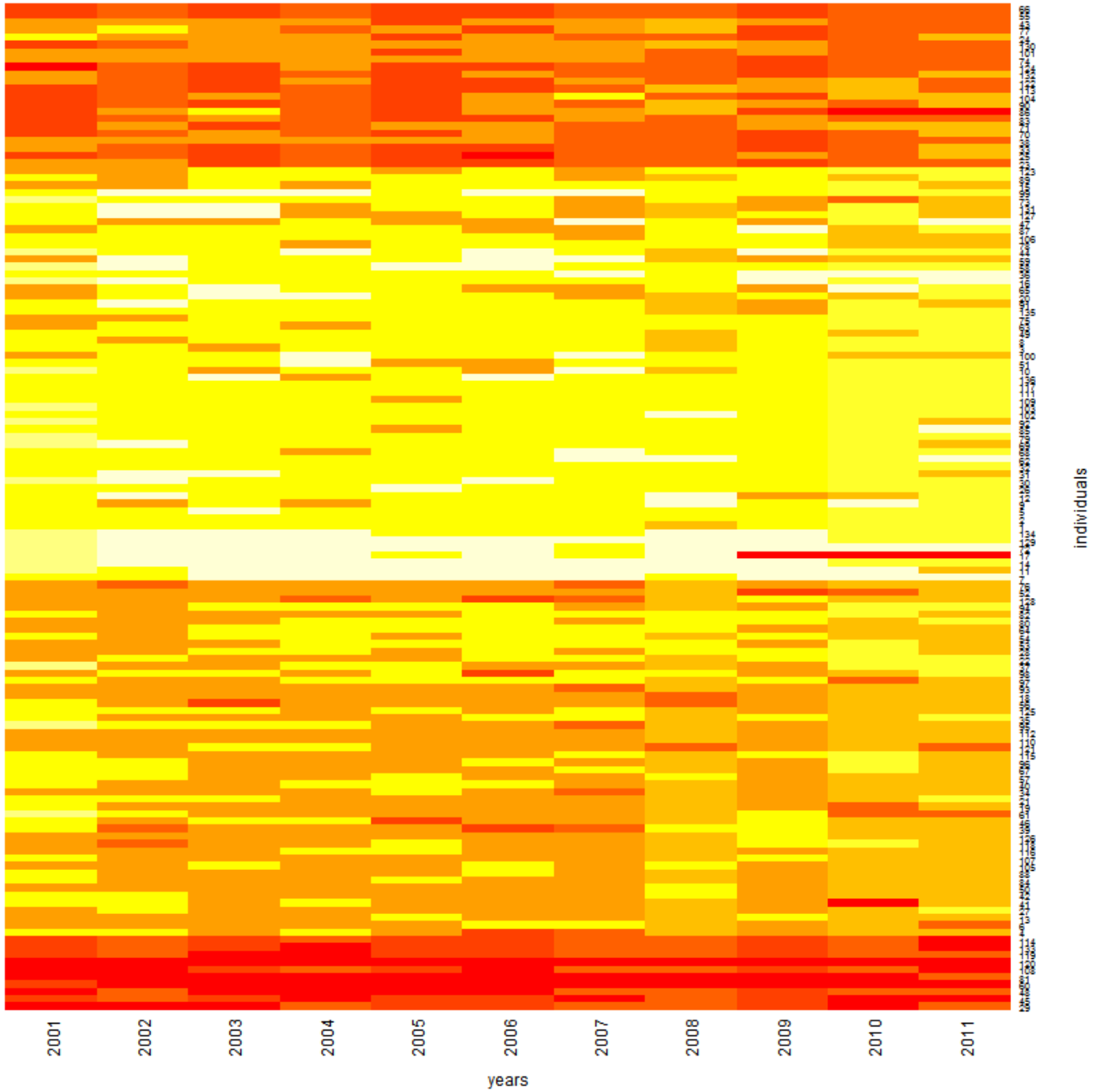
Visualisation using Heatmaps



Raw data


2001
2002
2003
2004
2005
2006
2007
2008
2009
2010
2011
years
individuals

Best model (R=5)

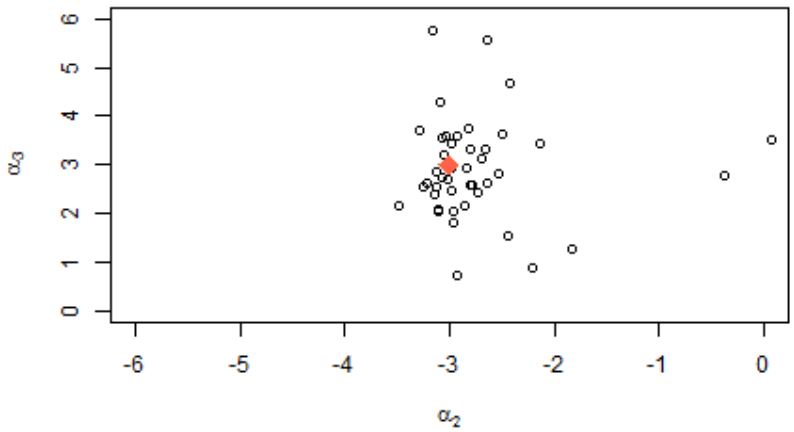


Applications: Simulation

- We simulated data and estimated the model 50 times for each scenario.
- TOM model with three row and two column clusters ($R = 3$ and $C = 2$) and $\alpha = (0, -3, 3)$, $\gamma = (0, -0.2, 0.5)$, $\beta = (0, 2)$, $\delta = (0, 0.5)$
- Same mixture proportions $\pi = (0.3\overline{3}, 0.3\overline{3}, 0.3\overline{3})$ and $\kappa = (0.5, 0.5)$.
- Five ordinal categories ($q = 5$), ten occasions $p = 10$
- Increasing sample size (n)

Applications: Simulation

n=60



n=1200

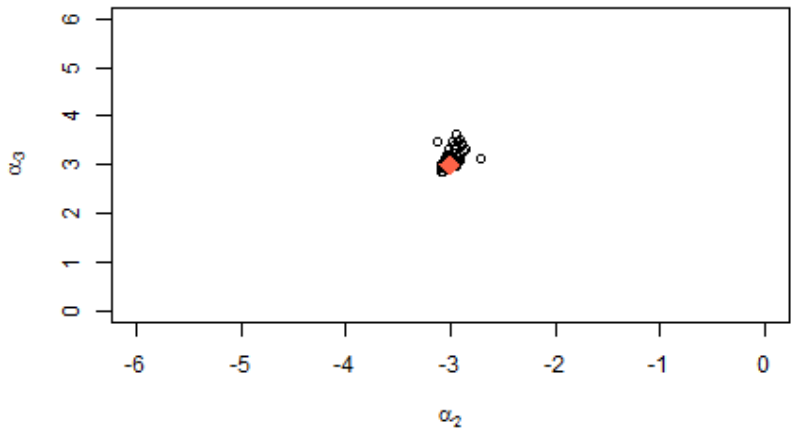
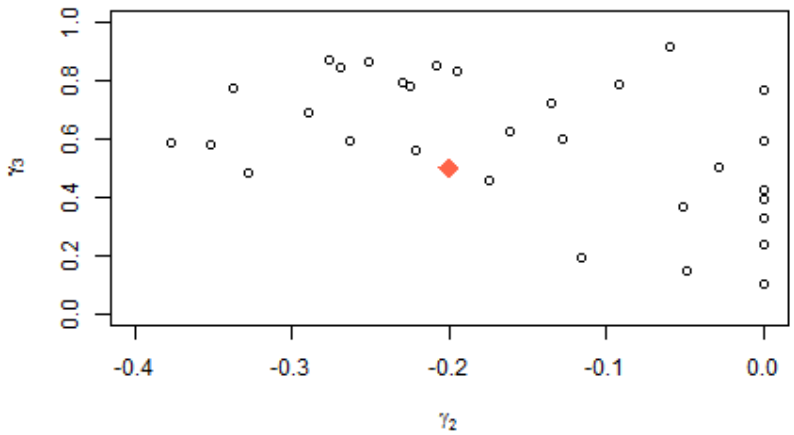


Figure : Simulation results $\alpha_2 = -3$ and $\alpha_3 = 3$

Applications: Simulation

n=60



n=1200

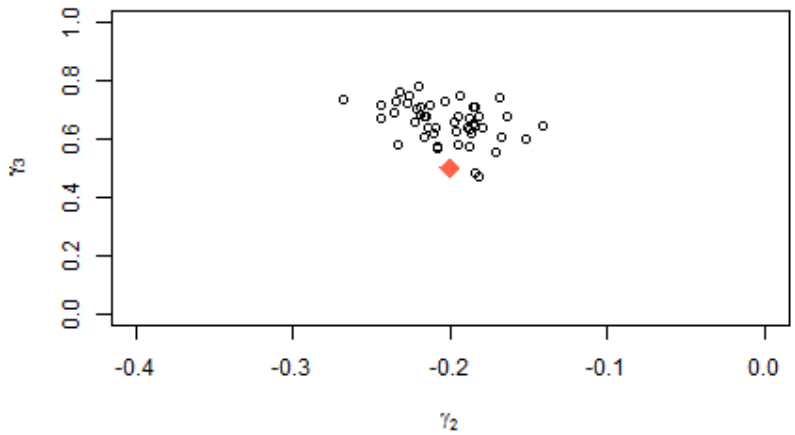
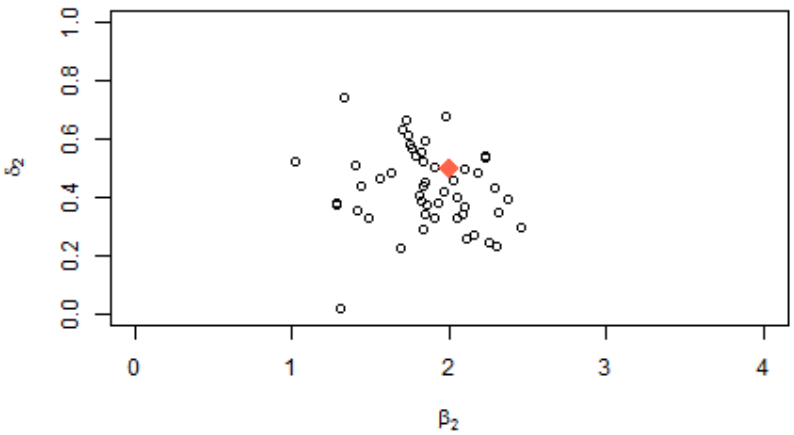


Figure : Simulation results $\gamma_2 = -0.2$ and $\gamma_3 = 0.5$

Applications: Simulation

n=60



n=1200

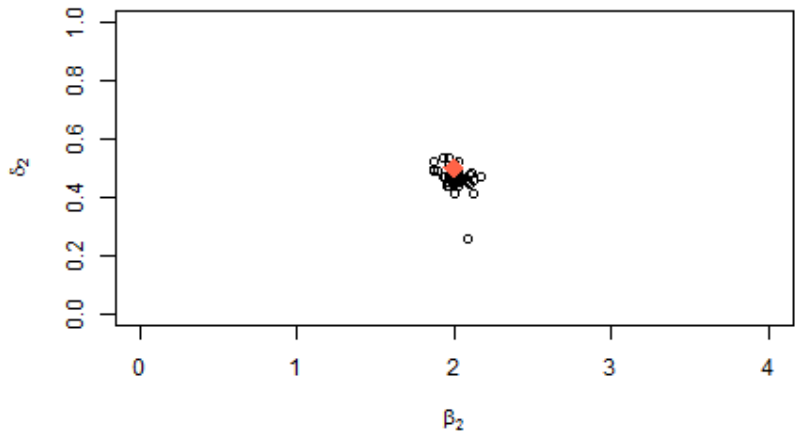


Figure : Simulation results $\beta_2 = 2$ and $\delta_2 = 0.5$

Research Goals (RG)

For the next two years:

- RG1: To determine the impact of different correlation structures on estimated clusters for repeated ordinal data
- RG2: to compare the performance of different model selection criteria
- RG3: To explore new visualisation methods for data, parameters and predictions
- RG4: To jointly model repeated ordinal and continuous data

RG1: To determine the impact of correlation structures on estimated clusters for repeated ordinal data

- Models for two-way data (unit by time period)
  - random effects by cluster and occasion
  - autoregressive order of order one (AR1)
- Models for three-way data (unit by question by time period)
  - bi-clustering in individuals and questions and an AR1 temporal correlation

RG1: Correlation structure example

- Random effects by cluster and occasion
- Repeated measures Proportional Odds Model with clustering

$$\text{Logit}[P(y_{ij} \leq k | i \in r)] = \mu_k - \alpha_r - \beta_{rj} \tag{4}$$

- Where $\beta_{rj} \sim N(0, \sigma_r^2)$

RG2: to compare the performance of different model selection criteria

- Simulation and case studies using models from RG1
- Frequentist: AIC, BIC, and ICL
- Bayesian
  - DIC and Bayes Factors, "fixed" number of mixture components
  - Reversible Jump Markov Chain Montecarlo (RJMCMC), "variable" number of mixture components

RG3: To explore new visualisation methods for data, parameters and predictions

- More heatmaps!
- Transition probability plots (initial and final distributions)
- Others?

RG4: To jointly model repeated ordinal and continuous data

- A common latent variable that could be causing both
- For instance, jointly modelling of household work status and household income overtime
  - household work status: ordinal (none member working, female working, male working, both working)
  - household income (continuous)
- Potential colaboration with SEF-VUW (who will be using simulated methods of moments)

Mosaics by Gaudi, Park Güell, Barcelona

Gracias por su atencion!!

References

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Column-clustering

$$\theta_{ick} = \frac{\exp(\mu_k - \beta_c - \delta_c t_k)}{1 + \exp(\mu_k - \beta_c - \delta_c t_k)} - \frac{\exp(\mu_{k-1} - \beta_c - \delta_c t_{k-1})}{1 + \exp(\mu_{k-1} - \beta_c - \delta_c t_{k-1})} \tag{5}$$

E step:

$$\hat{x}_{jc} = \frac{\kappa_c \prod_{i=1}^{n} \prod_{k=1}^{q} \theta_{ick}^{I(y_{ij}=k)}}{\sum_{a=1}^{C} \kappa_a \prod_{i=1}^{n} \prod_{k=1}^{q} \theta_{iak}^{I(y_{ij}=k)}}$$

M step: Numerically maximise:

$$\ell_c = \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{c=1}^{C} \sum_{k=1}^{q} \hat{x}_{jc} I(y_{ij} = k) \log(\theta_{ick}) + \sum_{j=1}^{p} \sum_{c=1}^{C} \hat{x}_{jc} \log(\kappa_c)$$

Bi-clustering

$$\theta_{rck} = \frac{\exp(\mu_k - \alpha_r - \gamma_r t_k - \beta_c - \delta_c t_k)}{1 + \exp \mu_k - \alpha_r - \gamma_r t_k - \beta_c - \delta_c t_k)} - \frac{\exp \mu_k - \alpha_r - \gamma_r t_k - \beta_c - \delta_c t_k)}{1 + \exp(\mu_k - \alpha_r - \gamma_r t_k - \beta_c - \delta_c t_k)}$$

$$\hat{z}_{ir} = \frac{\pi_r \prod_{j=1}^{p} \left\{ \sum_{c=1}^{C} \kappa_c \prod_{k=1}^{q} \theta_{rck}^{I(y_{ij}=k)} \right\}}{\sum_{a=1}^{R} \pi_a \prod_{j=1}^{p} \left\{ \sum_{b=1}^{C} \kappa_b \prod_{k=1}^{q} \theta_{abk}^{I(y_{ij}=k)} \right\}}$$

$$\hat{x}_{jc} = \frac{\kappa_c \prod_{i=1}^{n} \left\{ \sum_{r=1}^{R} \pi_r \prod_{k=1}^{q} \theta_{rck}^{I(y_{ij}=k)} \right\}}{\sum_{b=1}^{C} \kappa_b \prod_{i=1}^{n} \left\{ \sum_{a=1}^{R} \pi_a \prod_{k=1}^{q} \theta_{abk}^{I(y_{ij}=k)} \right\}}$$

Bi-clustering

$$\begin{aligned}\ell_c &= \sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{r=1}^{R}\sum_{c=1}^{C}\sum_{k=1}^{q}\hat{z}_{ir}\hat{x}_{jc}I(y_{ij}=k)\log(\theta_{rck}) \\ &\quad + \sum_{i=1}^{n}\sum_{r=1}^{R}\hat{z}_{ir}\log(\pi_r) + \sum_{j=1}^{p}\sum_{c=1}^{C}\hat{x}_{jc}\log(\kappa_c).\end{aligned}$$

Repeated measures Proportional Odds Model with row-clustering

Estimation Bayesian hierarchical model with three levels: row groups (latent), individuals and occasions by individual. Given a set of parameters ϕ, an response $\mathbf{Y}$, and prior $\omega(\phi|\mathbf{Y})$, the posterior is

$$\omega(\phi|\mathbf{Y}) = \frac{\ell(\phi|\mathbf{Y})\omega(\phi)}{\int \ell(\phi|\mathbf{Y})\omega(\phi)d\phi}$$

Repeated measures Proportional Odds Model with row-clustering

$$
\begin{aligned}
y_{ij} \mid \mu, \alpha_{r_i}, \beta_{r_i j}, r_i &\sim \text{POM}(\mu, \alpha_{r_i}, \beta_{r_i j}) && i = 1 \ldots n,\ j = 1 \ldots p,\ r = 2 \ldots \\
\mu'_k \mid \sigma^2_\mu &\overset{iid}{\sim} \text{Normal}(0, \sigma^2_\mu) && \mu_k = \mu'_{(k)},\ \text{if } \mu_k > \mu_{k-1},\ k = 1 \ldots \\
\alpha_r \mid \sigma^2_\alpha &\sim \text{Normal}(0, \sigma^2_\alpha) && r = 2 \ldots R,\ \alpha_1 = 0 \\
\beta_{rj} &\sim \text{Normal}(0, \sigma^2_r) && r = 2 \ldots R,\ j = 1 \ldots p,\ \beta_{1j} = 0\ \forall \\
r_i \mid \pi &\sim \text{Categorical}(\pi) && i = 1 \ldots n \\
\sigma^2_\mu &\sim \text{Inverse Gamma}(a_\mu, b_\mu) && \\
\sigma^2_\alpha &\sim \text{Inverse Gamma}(a_\alpha, b_\alpha) && \\
\sigma^2_r &\sim \text{Inverse Gamma}(a_r, b_r) && r = 2 \ldots R \\
\pi &\sim \text{Dirichlet}(\psi) &&
\end{aligned}
$$

With (fixed known) hyperparameters: $\psi, a_\alpha, b_\alpha, a_\mu, b_\mu, a_r$, and b_r.

model's DAG

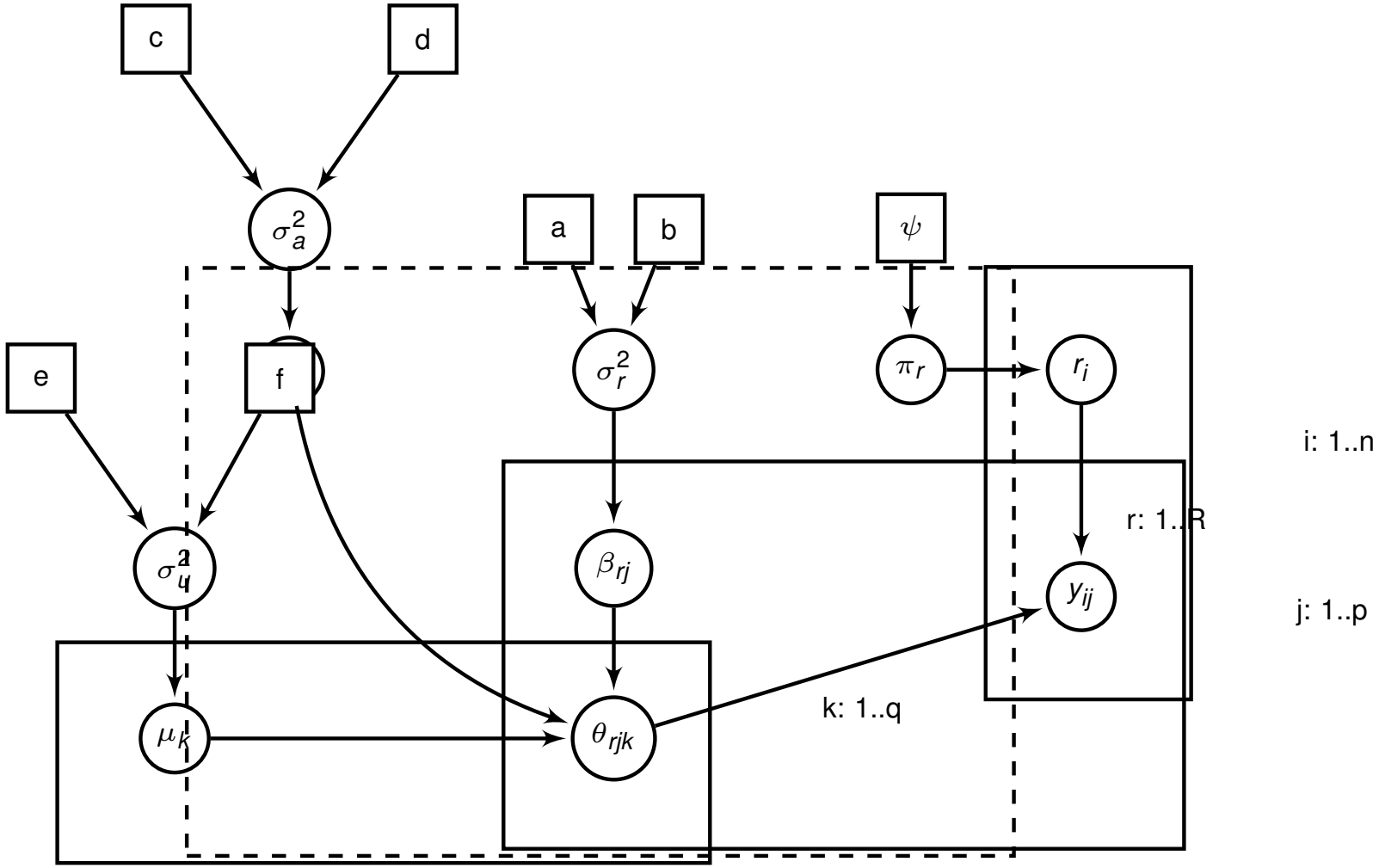
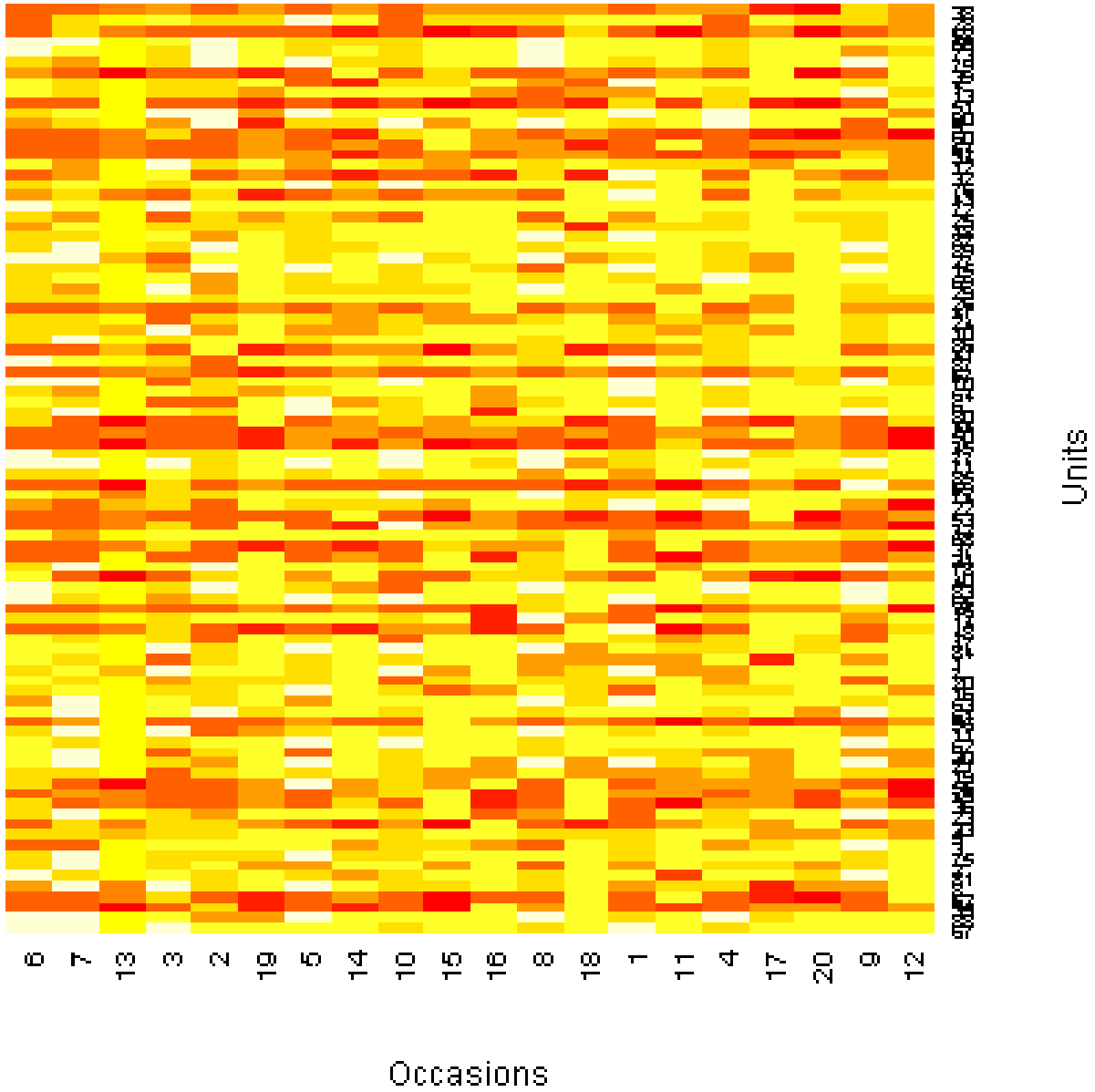


Figure : Graphical representation of REPOMC with row-clustering

Heatmap simulated bi-clustered data (R=3, C=2)



Heatmap simulated bi-clustered data (R=3, C=2)