Extensions and Cohomology of Groups

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Zusammenfassung: In dieser Arbeit werden grundlegende Konzepte aus dem Bereich der homologischen Algebra vorgestellt. Ziel ist es einen Bezug zwischen Gruppenerweiterungen und der 2-dimensionalen Kohomologiegruppe herzustellen.

In Kapitel 2 wird der Bifunktor Ext_R^n eingeführt. Dies geschieht mittels Erweiterungen von Moduln. Zur Berechnung von Ext_R^n wird in Kapitel 2.4 ein Isomorphismus zwischen $\mathrm{Ext}_R^n(C, A)$ und der n-dimensionalen Kohomologiegruppe $H^n(X, A)$ für eine projektive Auflösung X des R-Moduls C gegeben. Außerdem wird in Kapitel 2.5 eine lange exakte Sequenz für Ext_R^n konstruiert.

Kapitel 3 beginnt mit Erweiterungen von Gruppen. Für eine abelsche Gruppe A und eine beliebige Gruppe Π, wird eine Menge an Kongruenzklassen $Opext(\Pi, A, \phi)$ eingeführt. Einer Erweiterung ordnen wir in Kapitel 3.2 eine Funktion zu, die Faktor System (engl. factor set) genannt wird. Die 2-dimensionale Kohomologiegruppe $H_\phi^2(\Pi, A)$ ist als Quotient der Menge der Faktor Systeme definiert. In Kapitel 3.3 bilden wir $H_\phi^2(\Pi, A)$ bijektiv auf $Opext(\Pi, A, \phi)$ ab. Zudem wird in Kapitel 3.4 eine Definition der n-dimensionalen Kohomologiegruppe $H^n(\Pi, A)$ mittels Auflösungen gegeben. Die Arbeit aus dem Kapitel 2 zeigt abschließend, dass die Kohomologie von Gruppen ein Spezialfall von Ext_R^n mit $R = \mathbb{Z}(\Pi)$, dem Gruppenring, ist.

1 Introduction

The aim of this thesis is to show the connection between the 2-dimensional cohomology group of a group Π over an abelian group A and group extensions of A by Π. Furthermore the n-dimensional cohomology group of a group Π with coefficients in an abelian group A is shown to be a special case of Ext_R^n with $R = \mathbb{Z}(\Pi)$ the group ring.

The thesis closely follows Mac Lane [Mac63].

2 Extensions and Resolutions

Throughout this thesis R will denote a ring with identity. We only consider left modules over rings. If not stated otherwise, all modules are R-modules and all module homomorphisms are R-module homomorphisms. The notation for an identity map of a set X into itself is 1_X. Image and kernel of a map α are denoted $\operatorname{im}\alpha$ and $\ker\alpha$ respectively.

2.1 Ext

Let A and C be R-modules. An extension E of A by C is a short exact sequence

$$E : 0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \longrightarrow 0 \tag{1}$$

of R-modules. When speaking of extensions we always mean the associated modules and homomorphisms. We write $E = (\chi, \sigma)$ for a sequence (1). A morphism Γ from E to E' is a triple of module homomorphisms (α, β, γ) such that the diagram

$$\begin{array}{ccccccc}
E: & & A & \longrightarrow & B & \longrightarrow & C \\
\downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
E': & & A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}$$

commutes. For a morphism of the form

$$(1_A, \beta, 1_C) : E \to E' \tag{2}$$

the Short Five Lemma [Mac63, Lemma I.3.1.] assures us that β is an isomorphism. Therefore the existence of a morphism (2) defines an equivalence relation which we denote $E \equiv E'$. Define $\mathrm{Ext}_R(C, A)$ to be the set of congruence classes of extensions of A by C. Given an extension E as in (1) and a module homomorphism $\alpha : A \to A'$, we can construct an extension αE of A' by C and a morphism $\Gamma = (\alpha, \beta, 1_C) : E \to \alpha E$ as the push out along α. We can also pull back along a module homomorphism $\gamma : C' \to C$ and thereby construct an extension $E\gamma$ of A by C' and a morphism $\Gamma_1 = (1_A, \beta_1, \gamma) : E\gamma \to E$. The pairs $(\Gamma, \alpha E)$ and $(\Gamma_1, E\gamma)$ are unique up to congruence. For details on how to push out, pull back and a proof see [Mac63, Chapter III.1.]. We denote the category of left R-modules by $\mathbf{M}$ and the category of abelian groups by $\mathbf{Ab}$. The bifunctor $\mathrm{Ext}_R(C, A)$ from $\mathbf{M} \times \mathbf{M}$ to $\mathbf{Ab}$ is contravariant in its first and covariant in its second argument [Mac63, see Chapters III.1. and III.2.]. The group operation on $\mathrm{Ext}(C, A)$ will be explained in subsection 2.2.3.

Definition 1. *Let A and C be modules. We denote their direct sum by $A \oplus C$. The short exact sequence*

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

is called direct sum extension of A by C.

An extension congruent to the direct sum extension is called split.

Example 1. *[Wei94, Exercise 3.4.1.] Let p be prime and $\mathbb{Z}/p$ be the cyclic group of order p. Every extension E of $\mathbb{Z}/p$ by $\mathbb{Z}/p$ is either split or congruent to a sequence of the form $0 \to \mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p^2 \xrightarrow{\cdot i} \mathbb{Z}/p \to 0$, for $i = 1, \ldots, p-1$.*

Proof. Let $E = (\chi, \sigma) : 0 \to \mathbb{Z}/p \to B \to \mathbb{Z}/p \to 0$ be a short exact sequence. Because $\frac{B}{\mathbb{Z}/p} \cong \mathbb{Z}/p$ the group B must be of order p^2. There are two groups of order p^2, the cyclic group $\mathbb{Z}/p^2$ and the direct sum $\mathbb{Z}/p \oplus \mathbb{Z}/p$.

If $B \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, the monomorphism χ must send the generator s of $\mathbb{Z}/p$ to (i, j) with $i, j \in \mathbb{Z}/p$ and not both zero. Suppose $i \neq 0$ then $(i, 0) \mapsto s$ is a left inverse of χ and by [Mac63, Proposition I.4.3.] E is isomorphic to the direct sum sequence.

If $B \cong \mathbb{Z}/p^2$, the monomorphism χ must send the generator to a multiple of p. Then σ can send the generator of $\mathbb{Z}/p^2$ to any integer from $1, \ldots, p-1$. Let i and $j \in 1, \ldots, p-1$. Notice that for $E = (ip, j)$ we have a congruence

$$\begin{tikzcd}
E: & \mathbb{Z}/p \arrow[r, "\cdot ip"] \arrow[d, equal] & \mathbb{Z}/p^2 \arrow[r, "\cdot j"] \arrow[d, "\cdot i^{-1}"] & \mathbb{Z}/p \arrow[d, equal] \\
E': & \mathbb{Z}/p \arrow[r, "\cdot p"] & \mathbb{Z}/p^2 \arrow[r, "\cdot ij"] & \mathbb{Z}/p
\end{tikzcd}$$

to $E' = (p, ij) = (p, ij \bmod p)$. And two extensions (p, i') and (p, j') are congruent only if $i' = j'$. ☐

Lemma 1. *Let E be as in* (1)*. Then χE and $E\sigma$ are split extensions.*

Proof. By uniqueness of χE and $E\sigma$ up to congruence it is enough to push out and pull back the sequence E to split extensions. Let ι, ι' denote inclusions into the first and ρ, ρ' projections onto the second factor. The diagrams

$$\begin{tikzcd}
A \arrow[r, "\chi"] \arrow[d, "\chi"] & B \arrow[r, "\sigma"] \arrow[d, "(1_B,\sigma)"] & C \arrow[d, equal] \\
B \arrow[r, "\iota"] & B \oplus C \arrow[r, "\rho"] & C
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
A \arrow[r, "\iota'"] \arrow[d, equal] & A \oplus B \arrow[r, "\rho'"] \arrow[d, "\chi+1_B"] & B \arrow[d, "\sigma"] \\
A \arrow[r, "\chi"] & B \arrow[r, "\sigma"] & C
\end{tikzcd}$$

are commutative. $\square$

2.2 Ext$^{\mathbf{n}}$

Fix a positive integer n. We call an exact sequence of modules

$$S = (\chi, \lambda_{n-1}, \dots, \lambda_1, \sigma) : 0 \to A \to B_{n-1} \to \dots \to B_0 \to C \to 0 \quad (3)$$

an n-fold extension of A by C. Let S and S' be n-fold extensions. A morphism $\Gamma : S \to S'$ is a tuple of $n+2$ module homomorphisms $(\alpha, \beta_{n-1}..., \beta_0, \gamma)$ such that the diagram

$$\begin{tikzcd}
S: \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & B_{n-1} \arrow[r] \arrow[d, "\beta_{n-1}"] & \dots \arrow[r] & B_0 \arrow[r] \arrow[d, "\beta_0"] & C \arrow[d, "\gamma"] \\
S': & A' \arrow[r] & B'_{n-1} \arrow[r] & \dots \arrow[r] & B'_0 \arrow[r] & C'
\end{tikzcd}$$

commutes. Let S ad S' be two n-fold extensions of A by C. We say S is congruent to S' if there is a positive integer k, modules $S_0, S_1 \dots, S_{2k}$ with $S = S_0$ and $S_{2k} = S'$ and morphisms

$$S_0 \to S_1 \leftarrow S_2 \to S_3 \leftarrow \dots \to S_{2k-1} \leftarrow S_{2k}$$

all starting in 1_A and ending in 1_C. This defines an equivalence relation on the set of all n-fold extensions of A by C. Notice that the definition for congruence on Ext_R agrees with the one defined here for $n = 1$. Let $\mathrm{Ext}^n_R(C, A)$ denote the set of congruence classes of n-fold extensions of A by C. When the ring is clear we write $\mathrm{Ext}^n(C, A)$. A congruence class $\sigma \in \mathrm{Ext}^n(C, A)$ consists of n-fold extensions of A by C. If we are not interested in the congruence class of an extension, we use the notation $S \in\in \mathrm{Ext}^n(C, A)$ for an element $S \in \sigma \in \mathrm{Ext}^n(C, A)$.

2.2.1 Splicing and Factorizing

Given exact sequences

$$S : 0 \to A \to B_{n-1} \to \dots \to B_0 \xrightarrow{\sigma} K \to 0$$
$$S' : 0 \to K \xrightarrow{\chi} B'_{m-1} \to \dots \to B'_0 \to C \to 0$$

we assign to them an exact sequence $S \circ S'$ defined as

$$0 \to A \to \dots \to B_0 \xrightarrow{\lambda} B'_{m-1} \to \dots \to C \to 0$$

where $\lambda := \chi\sigma$. This process is called splicing and $S \circ S'$ is called the Yoneda composite of S and S'. Furthermore any exact sequence can be factorized into short exact sequences. Let S be as in (3). Let ρ be the restriction of $B_{n-1} \to B_{n-2}$ onto its image and $\iota : \ker(B_{n-2} \to B_{n-3}) \to B_{n-2}$ the inclusion. Then $S =: S_n = E_n \circ S_{n-1}$ where

$$E_n : 0 \to A \to B_{n-1} \xrightarrow{\rho} \mathrm{im}(B_{n-1} \to B_{n-2}) \to 0$$
$$S_{n-1} : 0 \to \ker(B_{n-2} \to B_{n-3}) \xrightarrow{\iota} B_{n-2} \to \cdots \to B_0 \to C \to 0$$

are both exact. Iterating this process for S_{n-1} and so on gives us a factorization of S into short exact sequences $S = E_n \circ \cdots \circ E_1$ where $E_1 := S_1$.

2.2.2 Ext$^{\mathbf{n}}$ as Bifunctor

Let S be as in (3) with factorization $S = E_n \circ \cdots \circ E_1$. Let $\alpha : A \to A'$ and $\gamma : C' \to C$ be module homomorphisms. We assign extensions $\alpha S := (\alpha E_n) \circ \cdots \circ E_1$ of A' by C and $S\gamma := E_n \circ \cdots \circ (E_1 \gamma)$ of A by C'. By this definition we get morphisms $\Gamma : S \to \alpha S$ and $\Gamma_1 : S\gamma \to S$.[1] The bifunctor $\mathrm{Ext}^n_R(C, A) : \mathbf{M} \times \mathbf{M} \to \mathbf{Ab}$ is contravariant in its first and covariant in its second argument [Mac63, p. 85]. The group operation is presented in the succeeding subsection.

We will later need

Lemma 2. *Every morphism $\Gamma = (\alpha, \dots, \gamma) : S \to S'$ between two n-fold extensions S and S' yields a congruence $\alpha S \equiv S'\gamma$.*

Proof. See [Mac63, Proposition III.5.1.]. □

2.2.3 Addition in Ext$^{\mathbf{n}}$

We will sketch how to define an addition on Ext^n_R. Define the diagonal $\Delta : C \to C \oplus C$ by $c \mapsto (c, c)$ and the codiagonal $\nabla : A \oplus A \to A$ by $(a, a_1) \mapsto a + a_1$. Given two n-fold extensions S, S' of A by C we define $S \oplus S'$ to be the component wise direct sum of their modules and homomorphisms. The resulting sequence is exact, starts in $A \oplus A$ and ends in $C \oplus C$. We define $S + S'$ by $\nabla(S \oplus S')\Delta$. Note that $(\nabla(S \oplus S'))\Delta = \nabla((S \oplus S')\Delta)$ so we can omit the parentheses. The element $S + S'$ is called the Baer sum of S and S'. Together with this operation $\mathrm{Ext}^n_R(C, A)$ is an abelian group [Mac63, p. 85]. The congruence class of the sequence

$$S_0 = (1_A, 0, \dots, 0, 1_C) : 0 \to A \to A \to 0 \to \cdots \to 0 \to C \to C \to 0 \quad (4)$$

is the zero element of $\mathrm{Ext}^n_R(C, A)$ under the Baer sum. Let $\alpha : A \to A'$, $\gamma : C' \to C$ be homomorphisms and $S, S' \in\in \mathrm{Ext}^n_R(C, A)$. The Baer sum is distributive [Mac63, see Theorem 5.3.]. That is:

$$\alpha(S + S') \equiv \alpha S + \alpha S', \quad (S + S')\gamma \equiv S\gamma + S'\gamma$$

We will later need

[1] Pulling back and pushing out gives morphisms between the short exact sequences as mentioned in subsection 2.1. Use the identity everywhere else.

Lemma 3. *If S is as in* (3)*, the composite extensions χS and $S\sigma$ are congruent to the zero element S_0 as defined in* (4)*.*

Proof. Consider the morphisms $S \to \chi S \to S_0$ in the following diagram

$$\begin{tikzcd}
A \arrow[r, "\chi"] \arrow[d, "\chi"] & B_{n-1} \arrow[r, "\lambda_{n-1}"] \arrow[d, "{(1,\lambda_{n-1})}"] & B_{n-2} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & B_0 \arrow[r] \arrow[d, equal] & C \arrow[d, equal] \\
B_{n-1} \arrow[r] \arrow[d, equal] & B_{n-1} \oplus \lambda_{n-1} B_{n-1} \arrow[r, "\rho_2"] \arrow[d, "\rho_1"] & B_{n-2} \arrow[r] \arrow[d] & \dots \arrow[r] & B_0 \arrow[r] \arrow[d] & C \arrow[d, equal] \\
B_{n-1} \arrow[r, "1"] & B_{n-1} \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & C \arrow[r, "1"] & C
\end{tikzcd}$$

where ρ_1 is the projection onto the first and ρ_2 the projection onto the second factor. This shows that $\chi S \equiv S_0$. One proves $S\sigma \equiv S_0$ via construction of morphisms $S_0 \leftarrow S\sigma \to S$ in a similar fashion. $\square$

2.3 Resolutions

We recall some basic definitions from homological algebra.

Definition 2. *A module P is projective if for all epimorphisms $\alpha : B \twoheadrightarrow C$ and every module homomorphism $\gamma : P \to C$ there is a homomorphism $\beta : P \to B$ such that $\alpha\beta = \gamma$.*

Definition 3. *A chain complex X is a family $(X_n, \partial_n)_{n\in\mathbb{Z}}$ of modules X_n and homomorphisms $\partial_n : X_n \to X_{n-1}$ satisfying $\partial_n \, \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.*

Definition 4. *The homology $H(X)$ of a complex $X = (X_n, \partial_n)$ is the family of modules $H_n(X) := \ker \partial_n \, / \, \mathrm{im} \, \partial_{n+1}$.*

Let C be a module, $X = (X_n, \partial_n)$ a chain complex, trivial in negative degrees, and $\epsilon : X_0 \to C$ a module homomorphism with $\epsilon \, \partial_1 = 0$. The pair (X, ϵ) is called a complex over C. If X has trivial homology $H_n(X)$ for every $n > 0$ and $\partial_1 X_1 = \ker \epsilon$, then (X, ϵ) is called a resolution of C. If all X_n are projective modules, then (X, ϵ) is called projective.

Definition 5. *Let $X = (X_n, \partial_n)$ and $X' = (X'_n, \partial'_n)$ be chain complexes. A chain transformation $f : X \to X'$ is a family of module homomorphisms $f_n : X_n \to X'_n$ satisfying $\partial'_n \, f_n = f_{n-1} \, \partial_n$ for all $n \in \mathbb{Z}$.*

Theorem 1. *[Mac63, Theorem II.6.1.] Let C and C' be modules. Given a projective complex (X, ϵ) over C, a resolution (X', ϵ') of C' and a module homomorphism $\gamma : C \to C'$ there is a chain transformation $f : X \to X'$ lifting γ. That is, there are homomorphisms $f_0, f_1, \dots$ such that the diagram*

$$\begin{tikzcd}
\dots \arrow[r] & X_2 \arrow[r, "\partial_2"] \arrow[d, dotted, "f_2"] & X_1 \arrow[r, "\partial_1"] \arrow[d, dotted, "f_1"] & X_0 \arrow[r, "\epsilon"] \arrow[d, dotted, "f_0"] & C \arrow[d, "\gamma"] \\
\dots \arrow[r] & X'_2 \arrow[r, "\partial'_2"] & X'_1 \arrow[r, "\partial'_1"] & X'_0 \arrow[r, two heads, "\epsilon'"] & C'
\end{tikzcd}$$

commutes.

Proof. Because X_0 is projective and $\epsilon' : X_0' \to C'$ is surjective there is a map $f_0 : X_0 \to X_0'$ with $\epsilon' f_0 = \gamma\epsilon$. Suppose we constructed maps $f_0, \dots, f_n$. Because $\partial_n \partial_{n+1} = 0$ and $f_{n-1} \partial_n = \partial_n' f_n$ we have $0 = f_{n-1} \partial_n \partial_{n+1} = \partial_n' f_n \partial_{n+1}$. Hence $f_n \partial_{n+1} X_{n+1} \subset \ker \partial_n' = \partial_{n+1}' X_{n+1}'$ by exactness at X_n'. As X_{n+1} is projective there is a map f_{n+1} with $\partial_{n+1}' f_{n+1} = f_n \partial_{n+1}$. $\square$

Definition 6. *Let $f, f' : X \to X'$ be chain transformations. A chain homotopy s between f and f' is a familiy of maps $s_n : X_n \to X_{n+1}'$ satisfying $f_n - f_n' = s_{n-1} \partial_n + \partial_{n+1}' s_n$ for all $n \in \mathbb{Z}$.*

Lemma 4. *Under the assumptions of Theorem 1, two chain transformations f, f' lifting the same γ are chain homotopic.*

Proof. For convenience we use the same symbol ∂ for all boundary maps ∂_n and ∂_n'. We want to construct maps $s_n : X_n \to X_{n+1}'$ satisfying

$$f_0 - f_0' = \partial s_0 \tag{5}$$

$$f_{n+1} - f_{n+1}' = \partial s_{n+1} + s_n \partial \tag{6}$$

for $n \geq 0$. We observe that by commutativity $\epsilon'(f_0 - f_0') = 0$. By exactness of the bottom row we know $(f_0 - f_0')X_0 \subset \partial X_1'$. Projectivity of X_0 gives us a map $s_0 : X_0 \to X_1'$ satisfying (5). Suppose we have maps $s_0, \dots, s_n$ satisfying (5) and (6). Then $\partial s_n = f_n - f_n' - s_{n-1} \partial$ and therefore $\partial(f_{n+1} - f_{n+1}' - s_n \partial) = (f_n - f_n') \partial - (f_n - f_n' - s_{n-1} \partial) \partial = 0$. Exactness of the bottom row implies $(f_{n+1} - f_{n+1}' - s_n \partial)X_{n+1} \subset \partial X_{n+2}'$. Finally, because X_{n+1} is projective, we get a map $s_{n+1} : X_{n+1} \to X_{n+2}'$ satisfying $\partial s_{n+1} = f_{n+1} - f_{n+1}' - s_n \partial$. $\square$

2.4 $\mathbf{Ext^n}$ and Resolutions

We have to recall some more definitions.

Definition 7. *Let G be a module and $X = (X_n, \partial_n)$ a complex. The cohomology $H^*(X, G)$ of X with coefficients in G is defined as the homology of the complex* $\mathrm{Hom}(X, G)$. *An element of* $\mathrm{Hom}(X_n, G)$ *is called n-cochain. The coboundary for an n-cochain f is defined by $\delta^n f = (-1)^{n+1} f \partial_{n+1}$. An element of* $\ker \delta^n$ *is called n-cocycle. Two n-cocycles are cohomologous if their difference is the coboundary of an (n-1)-cochain.*

For $n > 0$ the groups $\mathrm{Ext}^n(C, A)$ can be computed as cohomology groups $H^n(X, A)$ of a projective resolution of C.

Regard an n-fold extension S of A by C as a resolution of C with zeros beyond the nth term. By Theorem 1 we can lift the identity 1_C to a chain transformation $g : X \to S$:

$$\begin{tikzcd} X_{n+1} \arrow[r, "\partial_{n+1}"] & X_n \arrow[r, "\partial_n"] \arrow[d, dotted, "g_n"] & X_{n-1} \arrow[r] \arrow[d, dotted, "g_{n-1}"] & \dots \arrow[r] & X_0 \arrow[r] \arrow[d, dotted, "g_0"] & C \arrow[d, equal] \\ 0 \arrow[r] & A \arrow[r] & B_{n-1} \arrow[r] & \dots \arrow[r] & B_0 \arrow[r] & C \end{tikzcd}$$

Note that by commutativity $g_n \, \partial_{n+1} = 0$. This means that g_n is a cocycle. Define a map $\zeta : \operatorname{Ext}^n(C, A) \to H^n(X, A)$ by $\zeta(\operatorname{cls}(S)) := \operatorname{cls}(g_n)$, where cls assigns a congruence class to an extension (or a cohomology class in the case of a cocycle) to the respective element. Now we show that ζ is well defined.

1. We need to show that any two chain transformations lifting 1_C yield cohomologous elements. Suppose $g' : X \to S$ is a seconds chain transformation lifting 1_C. Let s denote a chain homotopy between g and g' given by Lemma 4. Note that $s_n : X_n \to 0$ is zero. The chain homotopy there gives $g_n - g'_n = s_{n-1} \, \partial_n$, so g_n and g'_n are cohomologous.

2. We need to prove that the class of g_n does not depend on the representative of $\operatorname{cls}(S)$. By our definition of congruence of n-fold exact sequences it is sufficient to check two cases.

   First case. Suppose $\Gamma : S \to S'$ is a morphism that starts with 1_A and ends with 1_C. Then $(\Gamma g)_n = g_n$ and Γg is a chain transformation.

   Second case. Suppose $\Gamma : S'' \to S$ is a morphism that starts with 1_A and ends with 1_C. Now we construct a chain transformation $f : X \to S''$ lifting 1_C as in Theorem 2. Then Γf and g are chain transformations lifting 1_C. We saw in 1 that they therefore yield cohomologous elements f_n and g_n.

Theorem 2. *[Mac63, Theorem III.6.4.] Let A, C be modules and (X, ϵ) be a projective resolution of C. Then the map $\zeta : \operatorname{Ext}^n(C, A) \to H^n(X, A)$ is an isomorphism for n>0. ζ is natural in A.*

Proof. The function η defined below will be the inverse of ζ. We factor $\partial_n : X_n \to X_{n-1}$ as $(\partial', \chi) : X_n \to \partial_n \, X_n \to X_{n-1}$ with χ the inclusion. Let $g : X_n \to A$ be an n-cocyle, i.e. $g \, \partial_{n+1} = 0$. Because $\ker(\partial') = \ker(\partial_n) = \partial_{n+1} \, X_{n+1} \subset ker(g)$ we can factor g as $h \, \partial'$ by the universal property of the quotient.

$$\begin{tikzcd} & & X_{n+1} \arrow[r] & X_n \arrow[d, "\partial'"] \arrow[dr, "\partial_n"] & & & \\ S_n : & & 0 \arrow[r] & \partial \, X_n \arrow[r, "\chi"] \arrow[d, "h"] & X_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & C \arrow[d, equal] \\ hS_n : & & 0 \arrow[r] & A \arrow[r] & B_{n-1} \arrow[r] & \dots \arrow[r] & C \end{tikzcd}$$

In the above diagram from [Mac63, p. 89] S_n is an n-fold exact sequence and hS_n is the push out along h. We define $\eta : H^n(X,A) \to \operatorname{Ext}^n(C,A)$ by $\eta\,\mathrm{cls}(g) := \mathrm{cls}(hS_n)$. Using the distributive law in Ext we show that η is well defined. Consider a coboundary $h\,\partial'$ with $\delta k = h\,\partial'$ for some $k : X_{n-1} \to A$. By definition of the coboundary $\delta k = (-1)^n k\,\partial = (-1)^n k\chi\,\partial'$. Therefore $h = (-1)^n k\chi$ and $hS_n = ((-1)^n k\chi)S_n$. By Lemma 3 the composite extension χS_n is congruent to S_0 as defined in (4). Because S_0 is the zero element of addition in Ext^n the distributive law implies that cohomologous elements are assigned the same element. This shows that η is well defined. Now we show that the maps are inverses of each other.

Let $S \in\in \operatorname{Ext}^n(C,A)$ be an extension and $g : X \to S$ a chain transformation lifting 1_C. Denote the factorization of g_n by $h\,\partial'$. Notice that $(h, g_{n-1}, \ldots, g_0, 1_C) : S_n \to S$ is a morphism. Lemma 2 implies $hS_n \equiv S$. This shows $\eta\zeta = 1$.

Consider a cocycle $g : X_n \to A$ with factorization $g = \partial'\,h$. Constructing the sequence hS_n yields a chain transformation $X \to hS_n$ via composition of $X \to S_n \to hS_n$. The nth homomorphism of this chain transformation is exactly g. Therefore $\zeta\eta = 1$.

It still remains to prove that ζ is natural. Let $\alpha : A \to A'$ be a module homomorphism. We need to show that the diagram

$$\begin{array}{ccc}
\operatorname{Ext}^n(C,_)(A) & \xrightarrow{\zeta} & H^n(X,_)(A) \\
\downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\
\operatorname{Ext}^n(C,_)(A') & \xrightarrow{\zeta} & H^n(X,_)(A')
\end{array}$$

commutes.

Suppose $S \in\in \operatorname{Ext}^n(C,A)$.

Let $g : X \to S$ be a chain transformation lifting 1_C. Then $\alpha_*\zeta\,\mathrm{cls}(S) = \mathrm{cls}(\alpha g_n)$.

Let $h : X \to \alpha S$ be a chain transformation lifting 1_C. Then $\zeta\alpha_*\,\mathrm{cls}(S) = \zeta\,\mathrm{cls}(\alpha S) = \mathrm{cls}(h_n)$.

By definition of αS we have a morphism $\Gamma : S \to \alpha S$. The composition $\Gamma g : X \to \alpha S$ also lifts 1_C. Thus Γg and h are chain homotopic by Lemma 4, so $\mathrm{cls}(h_n) = cls((\Gamma g)_n) = cls(\alpha g_n)$. ☐

Theorem 2 is a useful tool for computing Ext groups.

Application 1. *Let A be an abelian group and n, m positive integers. Then*

$$\operatorname{Ext}^m_{\mathbb{Z}}(\mathbb{Z}/n, A) \cong \begin{cases} A/nA, & m = 1 \\ 0, & m \geq 2 \end{cases}$$

Proof. Let $\rho : \mathbb{Z} \to \mathbb{Z}/n$ be the projection. Consider the exact sequence

$$(X,\rho) : \qquad \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}/n \longrightarrow 0.$$

Because $\mathbb{Z}$ is a projective module over itself, the pair (X, ρ) is a projective resolution of $\mathbb{Z}/n$. Theorem 2 implies $\operatorname{Ext}^m(\mathbb{Z}/n, A) \cong H^m(X, A)$ for $m > 0$. Denote the induced maps on Hom-groups by n_* and p_*. We calculate

$$\operatorname{im} n_* = \{nf : \mathbb{Z} \to A | f \in \operatorname{Hom}(\mathbb{Z}, A)\} \cong \{na | a \in A\} =: nA.$$

This gives us

$$\operatorname{Ext}^1(\mathbb{Z}/n, A) \cong \ker(\operatorname{Hom}(X_1, A) \to 0)/\operatorname{im} n_* = A/nA$$

and clearly the cohomology groups $H^m(X, A)$ are trivial for $m > 2$. □

From here on we will denote $\operatorname{Hom}(C, A)$ by $\operatorname{Ext}^0(C, A)$. Given a resolution $\cdots \to X_1 \to X_0 \xrightarrow{\epsilon} C \to 0$, the induced sequence $\operatorname{Hom}(X_1, A) \leftarrow \operatorname{Hom}(X_0, A) \xleftarrow{\epsilon^*} \operatorname{Hom}(C, A) \leftarrow 0$ is exact [Mac63, Theorem II.6.1.]. Hence $\ker(\operatorname{Hom}(X_0, A) \to \operatorname{Hom}(X_1, A)) \cong \operatorname{im} \epsilon^*$ and therefore $\epsilon^* : \operatorname{Hom}(C, A) = \operatorname{Ext}^0(C, A) \cong H^0(X, A)$. For convenience we will denote this isomorphism by ζ.

2.5 A Long Exact Sequence for $\mathbf{Ext^n}$

Definition 8. *A module F is called free if it has a basis.*

Lemma 5. *Every free module is projective.*

Proof. Let F be free with basis $T \subset F$. Given an epimorphism $\alpha : B \twoheadrightarrow C$ and a homomorphism $\gamma : F \to C$, we can choose elements $b_t \in B$ with $\alpha b_t = \gamma t$ for each $t \in T$. This defines a homomorphism $\beta : F \to B$ satisfying $\alpha\beta = \gamma$. □

In the main proof of this subsection we will use some notation of

Theorem 3. *[Mac63, Theorem II.4.5.] If X is a projective complex of R-modules and if $E = (\chi, \sigma) : 0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, there is a connecting homomorphism $\delta_E : H^n(X, C) \to H^{n+1}(X, A)$. Explicitly δ_E is defined by $\delta_E = \operatorname{cls} \chi^{-1} \delta \sigma^{-1} \operatorname{cls}^{-1}$ where δ is the coboundary. The connecting homomorphism yields a long exact sequence*

$$\cdots \to H^n(K, A) \xrightarrow{\chi_*} H^n(K, B) \xrightarrow{\sigma_*} H^n(K, C) \xrightarrow{\delta_E} H^{n+1}(K, A) \to \ldots.$$

The maps χ_* and σ_* are the induced maps on cohomology classes.

Proof. See [Mac63, Theorem II.4.5.]. □

Definition 9. *Given a short exact sequence E from A to C we define the connecting homomorphisms $E_* : \operatorname{Ext}^n(G, C) \to \operatorname{Ext}^{n+1}(G, A)$ for each $n \geq 0$ by $E_*(cls(S)) := cls(E \circ S)$.*

To see that $E*$ is well defined, suppose we have representatives S and S' of an element $\sigma \in \operatorname{Ext}^n(G, C)$. By definition of congruence we have morphisms

$$S \to S_1 \leftarrow S_2 \to \cdots \to S_{2k-1} \leftarrow S'$$

all starting in 1_C and ending in 1_G. We use this sequence to construct morphisms

$$E \circ S \to E \circ S_1 \leftarrow E \circ S_2 \to \cdots \to E \circ S_{2k-1} \leftarrow E \circ S'$$

by filling up the missing module homomorphisms with identities.

We use the notation $E_*\tau = E\tau$ for $\tau \in \operatorname{Ext}^n(G, C)$. We can regard $\operatorname{Ext}^n(G)(E) := \operatorname{Ext}^n(G, C)$ and $\operatorname{Ext}^{n+1}(G)(E) := \operatorname{Ext}^{n+1}(G, A)$ as covariant functors of E. Then the connecting homomorphism is a natural transformation between $\operatorname{Ext}^n(G)(\_)$ and $\operatorname{Ext}^{n+1}(G)(\_)$ [Mac63, p. 97].

Theorem 4. *[Mac63, Theorem III.9.1.] Consider a short exact sequence $E = (\chi, \sigma) : 0 \to A \to B \to C \to 0$ of R-modules and an R-module G. Then*

$$\cdots \to \operatorname{Ext}^n(G, A) \xrightarrow{\chi_*} \operatorname{Ext}^n(G, B) \xrightarrow{\sigma_*} \operatorname{Ext}^n(G, C) \xrightarrow{E_*} \operatorname{Ext}^{n+1}(G, A) \to \ldots$$

is exact. It starts with $0 \to \operatorname{Ext}^0(G, A)$. The maps involved are defined as

$$\chi_*\rho = \chi\rho, \quad \sigma_*\omega = \sigma\omega, \quad E_*\tau = E\tau \tag{7}$$

for elements $\rho \in \operatorname{Ext}^n(G, A)$, $\omega \in \operatorname{Ext}^n(G, B)$ and $\tau \in \operatorname{Ext}^n(G, C)$.

Proof. Because every module is isomorphic to a quotient of a free module [Mac63, Proposition I.5.3.] we can construct free resolutions for any module.[2] Let X be a free resolution of G. Theorems 2 and 3 yield a long exact sequence for Ext:

$$\begin{array}{ccccccc}
\operatorname{Ext}^n(G, A) & \xrightarrow{\chi_*} & \operatorname{Ext}^n(G, B) & \xrightarrow{\sigma_*} & \operatorname{Ext}^n(G, C) & \xrightarrow{E_*} & \operatorname{Ext}^{n+1}(G, A) \\
\downarrow \zeta & & \downarrow \zeta & & \downarrow \zeta & & \downarrow \zeta \\
H^n(X, A) & \xrightarrow{\chi_*} & H^n(X, B) & \xrightarrow{\sigma_*} & H^n(X, C) & \xrightarrow{(-1)^{n+1}\delta_E} & H^{n+1}(X, A)
\end{array} \tag{8}$$

It suffices to check that the maps defined in (7) make the diagram commutative for every $n \geq 0$. Commutativity of the first two squares of (8) follows by naturality of ζ for $n > 0$ and by recalling the definition of the isomorphism for $n = 0$.

To prove commutativity of the square on the right in (8) in the case $n = 0$, we have to show that $(-1)\delta_E\zeta = \zeta E_*$. Let $\gamma : G \to C$ be a

[2] Suppose we have a module $A \cong F_0/G_0$ for F_0 free. Then $G_0 \cong F_1/G_1$ with F_1 free and so on. Then $\cdots \to F_1 \to F_0 \to A \to 0$ is a free resolution of A.

homomorphism and $(1_A, \beta, \gamma) : E\gamma \to E$ a morphism. The diagram

$$\begin{array}{ccccccc}
X: & & X_1 & \xrightarrow{\partial} & X_0 & \xrightarrow{\epsilon} & G \\
\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| \\
E\gamma: & & A & \longrightarrow & B' & \longrightarrow & G \\
\downarrow & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
E: & & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C
\end{array}$$

shows chain transformations $X \to E\gamma \to E$. By definition of δ_E and commutativity

$$\begin{aligned}
\delta_E \zeta \gamma = \delta_E \operatorname{cls}(\gamma\epsilon) &= \operatorname{cls} \chi^{-1} \delta \sigma^{-1} \gamma \epsilon \\
&= \operatorname{cls} \chi^{-1} \delta(\beta f_0) = (-1) \operatorname{cls} \chi^{-1} \beta f_0 \, \partial = (-1) \operatorname{cls}(f_1).
\end{aligned}$$

On the other hand $\zeta E_* \gamma = \zeta \operatorname{cls}(E\gamma) = cls(f_1)$. Thus the case is proven.

Let $S \in\in \operatorname{Ext}^n(G, C)$ for $n > 0$. Regard $E \circ S$ as a resolution of G. Let $f : X \to E \circ S$ be a chain transformation lifting 1_G. The diagram

$$\begin{array}{ccccccccccccc}
X: & & X_{n+1} & \xrightarrow{\partial_{n+1}} & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \dots & \longrightarrow & X_0 & \longrightarrow & G \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_0} & & \| \\
E \circ S: & & A & \xrightarrow{\chi} & B & \xrightarrow{\lambda\sigma} & B_{n-1} & \longrightarrow & \dots & \longrightarrow & B_0 & \longrightarrow & G \\
\downarrow{\scriptstyle \Gamma} & & & & \downarrow{\scriptstyle \sigma} & & \| & & & & \| & & \| \\
S: & & & & C & \xrightarrow{\lambda} & B_{n-1} & \longrightarrow & \dots & \longrightarrow & B_0 & \longrightarrow & G
\end{array}$$

shows chain transformations $X \to E \circ S \to S$. By definition of ζ we get $\zeta E_* \operatorname{cls}(S) = \operatorname{cls}(f_{n+1})$. Composing Γf results in a chain transformation $X \to S$ lifting 1_G. Therefore $\zeta(S) = \operatorname{cls}((\Gamma f)_n) = \operatorname{cls}(\sigma f_n)$. Hence $\delta_E \zeta(S) = \operatorname{cls} \chi^{-1} \delta f_n = (-1)^{n+1} \operatorname{cls} \chi^{-1} f_n \, \partial_{n+1} = (-1)^{n+1} \operatorname{cls}(f_{n+1})$. The last equation follows by commutativity of the upper left square in the diagram. Signs cancel out with those in our exact sequence (8). This shows commutativity for $n > 0$. □

3 Extensions and Cohomology of Groups

Let G, B, Π be groups. An exact sequence

$$E : 0 \longrightarrow G \xrightarrow{\chi} B \xrightarrow{\sigma} \Pi \longrightarrow 1 \tag{9}$$

is called a group extension of G by Π. For convenience G, B are denoted as additive groups and Π as a multiplicative group. We define a homomorphism $\theta : B \to \operatorname{Aut}(G)$ via conjugation

$$\chi(\theta(b)g) = b + \chi g - b$$

for $b \in B$ and $g \in G$. Let $A := G$ be an abelian group. Then for any $b \in \ker \sigma$ by exactness $b = \chi a$ for some $a \in A$, hence $\theta(b) = 1_A$. Therefore $\ker \sigma \subset \ker \theta$. Thus we can define $\phi : \Pi \to Aut(A)$ via the universal property of the quotient as the unique map such that $\phi\sigma = \theta$. Call ϕ the operators of the extension E. We have the following equality

$$\chi(\phi(\sigma(b))a) = b + \chi a - b \tag{10}$$

for $a \in A$ and $b \in B$. A morphism $\Gamma : E \to E'$ is a tuple of group homomorphisms (α, β, γ) such that the diagram

$$\begin{tikzcd} E: \arrow[d, "\Gamma"] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & \Pi \arrow[d, "\gamma"] \\ E': & A' \arrow[r] & B' \arrow[r] & \Pi' \end{tikzcd}$$

commutes. Two group extensions E, E' are congruent if there is a morphism of the form $(1_A, \beta, 1_\Pi) : E \to E'$. We use the familiar notation $E \equiv E'$ for congruent extensions. As in the case of Ext we may speak of congruence classes of group extensions because β is an isomorphism via the Short 5 Lemma.

Lemma 6. *Congruent extensions have the same operators.*

Proof. Consider extensions $E = (\chi, \sigma) : A \to B \to \Pi$ with operators ϕ and $E' = (\chi', \sigma') : A \to B' \to \Pi$ with operators ϕ'. Assume E and E' are congruent, so there is a morphism $(1_A, \beta, 1_\Pi) : E \to E'$. Let $b \in B$ and set $\sigma b = x$. By commutativity $\sigma'\beta b = \sigma b = x$ and $\beta\chi = \chi'$. By the definition of operators for $a \in A$ we have

$$\begin{aligned} \beta\chi(\phi(\sigma b)a) &= \beta b + \beta\chi a - \beta b \\ \implies \beta\chi(\phi(x)a) &= \chi'(\phi'(\sigma'\beta b)a) \\ \implies \chi'(\phi(x)a) &= \chi'(\phi'(x)a). \end{aligned}$$

Because χ' is injective the operators ϕ and ϕ' are equal. $\square$

Denote the set of congruence classes of group extensions of an abelian group A by any group Π with operators ϕ by $\mathrm{Opext}(\Pi, A, \phi)$.

3.1 Group Ring

In this subsection we follow [PS02, Chapter 3.2.]. Let Π be any multiplicative group. Let $\mathbb{Z}(\Pi)$ denote the set of all formal linear combinations $\alpha = \sum_{x\in\Pi} a_x x$ where $a_x \in \mathbb{Z}$ with $a_x = 0$ for all but finitely many x. We define the sum of two elements in $\mathbb{Z}(\Pi)$ by

$$\left(\sum_{x\in\Pi} a_x x\right) + \left(\sum_{y\in\Pi} b_y y\right) = \sum_{x\in\Pi} (a_x + b_x)x.$$

We define the product of two elements $\alpha = \sum_{x\in\Pi} a_x x$ and $\beta = \sum_{y\in\Pi} b_y y$ by

$$\alpha\beta = \sum_{x,y\in\Pi} (a_x b_y)xy.$$

These operations induce a ring structure on $\mathbb{Z}(\Pi)$. A module over $\mathbb{Z}(\Pi)$ is called Π-module. We want to show that a group homomorphism $\Pi \to Aut(A)$ for an abelian group A defines a unique Π-module structure on A.

Define an embedding $\iota : \Pi \to \mathbb{Z}(\Pi)$ by $\iota(x) = 1 \cdot x$. Note that $\iota\Pi$ is a basis of $\mathbb{Z}(\Pi)$.

Lemma 7. *[Mac63, Proposition IV.1.1.] Let $\mu : \Pi \to R$ be a function with $\mu(1) = 1$ and $\mu(xy) = \mu(x)\mu(y)$. There is a unique ring homomorphism $\rho : \mathbb{Z}(\Pi) \to R$ such that the diagram*

$$\begin{tikzcd} & \mathbb{Z}(\Pi) \arrow[d, "\rho"] \\ \Pi \arrow[ur, "\iota"] \arrow[r, "\mu"'] & R \end{tikzcd}$$

commutes.

Proof. Define $\rho(\sum_{x\in\Pi} a_x x) := \sum_{x\in\Pi} a_x \mu(x)$. It is easy to check that this is a ring homomorphism and that the diagram commutes. Since $\iota\Pi$ is a basis, ρ is unique. □

Lemma 8. *[Mac63, Proposition IV.1.2.] Let A be an abelian group. A group homomorphism $\phi : \Pi \to Aut(A)$ gives A a unique Π-module structure.*

Proof. The group homomorphisms $A \to A$ form a ring $End(A)$. The automorphisms $Aut(A)$ are a subset of $End(A)$. Extending the range of ϕ to $End(A)$ allows us to apply Lemma 7. Hence ϕ induces a Π-module structure on A. We say A is a Π-module with operators ϕ. □

3.2 Factor Sets

From here on we identify A with $\chi A \subset B$. For an extension E as in (9) we choose a set function $u : \Pi \to B$ such that $\sigma u = 1_\Pi$ and $u(1) = 0$. We call u representatives. Notice that σ being onto ensure the existence of representatives. We use the notation $xa := \phi(x)a$. The equation (10) with $b = u(x)$ then becomes

$$u(x) + a = xa + u(x). \tag{11}$$

Since $A = \ker\sigma$ we conclude that each coset $b + A$ in B contains exactly one $u(x)$. For suppose $u(x) = u(x') + a$ then $x = \sigma u(x) = \sigma(u(x') + a) = \sigma u(x') + \sigma(a) = x'$.

Because $\sigma u(xy) = xy = \sigma u(x)\sigma u(y) = \sigma(u(x)+u(y))$ there exist unique elements $f(x,y) \in A$ such that $u(x)+u(y) = f(x,y)+u(xy)$. Call f a factor set of the extension E.

The described procedure of assigning a factor set to a given group extension will be used in Theorem 5.

The addition in B is determined by u and f in the following way: Every element in B can be written uniquely as $a+u(x)$ for $a \in A$ and $x \in \Pi$. For elements $a+u(x)$ and $a_1+u(y)$ in B we calculate their sum

$$(a+u(x))+(a_1+u(y)) = a+xa_1+u(x)+u(y) = a+xa_1+f(x,y)+u(xy)$$

by using (11) and the definition of f.

3.3 Opext and the 2-dimensional Cohomology Group

We define $Z^2_\phi(\Pi, A)$ to be the set of functions $f : \Pi \times \Pi \to A$ such that the conditions

$$f(x,1) = 0 = f(1,y) \tag{12}$$

$$xf(y,z) + f(x,yz) = f(x,y) + f(xy,z) \tag{13}$$

are satisfied for all $x,y,z \in \Pi$. Denote by $B^2_\phi(\Pi, A)$ the subset of $Z^2_\phi(\Pi, A)$ containing all functions δg of the form

$$\delta g(x,y) := xg(y) - g(xy) + g(x) \tag{14}$$

for some function $g : \Pi \to A$ with $g(1) = 0$. We define an operation $(f+f')(x,y) = f(x,y)+f'(x,y)$. Together with this operation $Z^2_\phi(\Pi, A)$ is an abelian group with $B^2_\phi(\Pi, A)$ being a subgroup.

Definition 10. *We define the 2-dimensional cohomology group as the quotient*

$$H^2_\phi(\Pi, A) = Z^2_\phi(\Pi, A)/B^2_\phi(\Pi, A).$$

Lemma 9. *Factor sets satisfy conditions* (12) *and* (13)*.*

Proof. It is easy to see that the equation (12) holds for factor sets. Let f be a factor set of the extension $A \to B \to \Pi$. Using the addition described in subsection 3.2 we calculate

$$\begin{aligned} 1.\ (u(x)+u(y))+u(z) &= (f(x,y)+u(xy))+u(z) \\ &= f(xy)+f(xy,z)+u(xyz) \\ 2.\ u(x)+(u(y)+u(z)) &= u(x)+(f(y,z)+u(yz)) \\ &= xf(y,z)+f(x,yz)+u(xyz) \end{aligned}$$

Since addition in A is associative 1 and 2 are equal. So equation (13) is satisfied. □

When we assigned factor sets to group extensions it involved a choice of representatives. To prove Theorem (5) we will need

Lemma 10. *The factor set of a group extension of A by Π with operators ϕ is well defined modulo $B^2_\phi(\Pi, A)$.*

Proof. Let $u, u' : \Pi \to B$ be representatives. By definition $\sigma u(x) = \sigma u'(x) = x$ for all $x \in \Pi$. Therefore $u(x)$ and $u'(x)$ lie in the same coset of A in B and for all $x \in \Pi$ we can choose some set function $g : \Pi \to A$ such that $u'(x) = g(x) + u(x)$. Using (11) we get

$$\begin{aligned}
u'(x) + u'(y) &= g(x) + u(x) + g(y) + u(y) \\
&= g(x) + xg(y) + u(x) + u(y) \\
&= g(x) + xg(y) + f(x, y) + u(xy) \\
&= g(x) + xg(y) + f(x, y) - g(xy) + u'(xy) \\
&= xg(y) - g(xy) + g(x) + f(x, y) + u'(xy) \\
&= \delta g(x, y) + f(x, y) + u'(xy)
\end{aligned}$$

where f is the factor set for representatives u and δg as defined in (14). So we can define the new factor set as

$$f'(x, y) = \delta g(x, y) + f(x, y)$$

with $\delta g \in B^2_\phi$. □

Theorem 5. *Assigning a factor set to a congruence class of group extensions yields a bijection*

$$\omega : \mathrm{Opext}(\Pi, A, \phi) \to H^2_\phi(\Pi, A)$$

modulo $B^2_\phi(\Pi, A)$.

Proof. Given a congruence class $\tau \in \mathrm{Opext}(\Pi, A, \phi)$ choose a representative $E \in \tau$. Now construct a factor set as described in subsection 3.2. Define $\omega\tau := f + B^2_\phi(\Pi, A)$. It is easy to see that congruent extensions have the same factor sets. Together with Lemma 10, this shows that ω is well defined.

To show that it is injective, let $E : A \to B \to \Pi$ and $E' : A \to B' \to \Pi$ be two group extensions. Choose representatives u and u' with factor sets f and f'. Assume $f' - f = \delta g$ for some set function $g : \Pi \to A$ satisfying $g(1) = 0$. Choosing representatives $g(x) + u'(x)$ for E' shows that f is factor set for E'. As representatives and factor set determine the addition of B and B', the extensions are congruent.

Lemma 11 shows surjectivity. □

Lemma 11. *Every $f \in Z^2_\phi$ is factor set of some group extension E.*

Proof. Define $B := A \times \Pi$. The operation

$$(a, x) + (a_1, y) := (a + xa_1 + f(x, y), xy)$$

defined for elements $(a, x), (a_1, y) \in B$ induces a group structure on B. Associativity is shown by

$$\begin{aligned}((a, x) + (a_1, y)) + (a_2, z) &= (a + xa_1 + f(x, y), xy) + (a_2, z) \\ &= (a + xa_1 + f(x, y) + xya_2 + f(xy, z), xyz) \\ &= (a + x(a_1 + ya_2 + f(y, z)) + f(x, yz), xyz) \\ &= (a, x) + (a_1 + ya_2 + f(y, z), yz) \\ &= (a, x) + ((a_1, y) + (a_2, z)).\end{aligned}$$

The third equation holds because of (13). The short exact sequence (χ, σ) : $A \to B \to \Pi$ with χ inclusion and σ projection has representatives $u(x) = (0, x)$ and factor set f. $\square$

Let A be a Π-module. The semi-direct product $A \times_\phi \Pi$ is defined as the set $A \times \Pi$ together with the addition

$$(a, x) + (a_1, y) := (a + xa_1, xy)$$

for $(a, x), (a_1, y) \in A \times \Pi$. The neutral element is $(0, 1)$ and the inverse of an element (a, x) is given by $-(a, x) := (x^{-1}(-a), x^{-1})$. Let $\iota : A \to A \times_\phi \Pi$ denote the inclusion and $\rho : A \times_\phi \Pi \to \Pi$ the projection. An extension that is congruent to (ι, ρ) is called semi-direct product extension.

The proof of Lemma 11 shows that the image of the zero element of $H^2_\phi(\Pi, A)$ is the semi-direct product extension.

3.4 Bar Resolution

Given a group Π we will construct a chain complex $B(\mathbb{Z}(\Pi))$ of free Π-modules.

Definition 11. *Let $n \geq 0$. Define $B_n(\mathbb{Z}(\Pi))$ to be the free Π-module generated by all tuples $[x_1|x_2|\dots|x_n]$ with $x_i \in \Pi$ and $x_i \neq 1$ for $1 \leq i \leq n$.*

Notation wise we set $[x_1|\dots|x_n] = 0$ if any $x_i = 1$. The module B_0 is generated by a single generator $[\ ]$. Regard $\mathbb{Z}$ as trivial Π-module, that is $\alpha m = m$ for all $\alpha \in \mathbb{Z}(\Pi)$ and $m \in \mathbb{Z}$. We have a module homomorphism $\epsilon : B_0 \to \mathbb{Z}$ defined by $\epsilon[\ ] := 1$. Define module homomorphisms $\partial : B_n \to B_{n-1}$ by

$$\begin{aligned}\partial[x_1|\dots|x_n] := x_1[x_2|\dots|x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1|\dots|x_i x_{i+1}|\dots|x_n] + \\ + (-1)^n[x_1|\dots|x_{n-1}]\end{aligned}$$

for $n > 0$. The definition includes the case where $x_j = 1$ for some $1 \leq j \leq n$. Regarding the B_n as abelian groups generated by $x[x_1|\dots|x_n]$ we define group homomorphisms $s_{-1} : \mathbb{Z} \to B_0$ and $s_n : B_n \to B_{n+1}$ by

$$s_{-1}1 := [\,] \quad \text{and} \quad s_n x[x_1|\dots|x_n] := [x|x_1|\dots|x_n].$$

Lemma 12. *The equations*

$$\epsilon s_{-1} = 1_{\mathbb{Z}}, \quad \partial\, s_0 + s_{-1}\epsilon = 1_{B_0}, \quad \partial\, s_n + s_{n-1}\,\partial = 1_{B_n} \tag{15}$$

are satisfied.

Proof. The first equation is clear. We compute the two summands in the second equation:

$$\begin{array}{ll} 1. & \partial\, s_0 x[\,] = \partial[x] = x[\,] - [\,] \\ 2. & s_{-1}\epsilon x[\,] = s_{-1}x1 = s_{-1}1 = [\,] \end{array}$$

Adding 1 and 2 proves the second equation. Now we calculate the summands of the remaining equation.

1. $$\begin{aligned} \partial\, s_n x_0[x_1|\dots|x_n] &= \partial[x_0|x_1|\dots|x_n] \\ &= x_0[x_1|\dots|x_n] \\ &\quad + \sum_{i=0}^{n-1}(-1)^{i+1}[x_0|x_1|\dots|x_i x_{i+1}|\dots|x_n] \\ &\quad + (-1)^{n+1}[x_0|x_1|\dots|x_{n-1}]) \end{aligned}$$

2. $$\begin{aligned} s_{n-1}\,\partial\, x_0[x_1|\dots|x_n] &= s_{n-1}x_0(x_1[x_2|\dots|x_n] \\ &\quad + \sum_{i=1}^{n-1}(-1)^{i}[x_1|\dots|x_i x_{i+1}|\dots|x_n] \\ &\quad + (-1)^{n}[x_1|\dots|x_{n-1}]) \\ &= [x_0 x_1|x_2|\dots|x_n] \\ &\quad + \sum_{i=1}^{n-1}(-1)^{i}[x_0|x_1|\dots|x_i x_{i+1}|\dots|x_n] \\ &\quad + (-1)^{n}[x_0|x_1|\dots|x_{n-1}]) \\ &= \sum_{i=0}^{n-1}(-1)^{i}[x_0|x_1|\dots|x_i x_{i+1}|\dots|x_n] \\ &\quad + (-1)^{n}[x_0|x_1|\dots|x_{n-1}]) \end{aligned}$$

The two sums cancel, as the $[x_0|x_1|\dots|x_{n-1}]$ terms do. This proves the last equation. $\square$

Lemma 13. (B_n, ∂_n) *is a complex.*

Proof. Observe that $\epsilon\,\partial_1([x]) = \epsilon(x[\;] - [\;]) = 0$. We can rewrite the third equation in (15) as $\partial_{n+1}\, s_n = 1 - s_{n-1}$. Applying this twice results in

$$\begin{aligned}\partial_n\,\partial_{n+1}\, s_n = \partial_n(1 - s_{n-1}\,\partial_n) = \partial_n - \partial_n\, s_{n-1}\,\partial_n \\ = \partial_n - (1 - s_{n-2}\,\partial_{n-1})\,\partial_n = s_{n-2}\,\partial_{n-1}\,\partial_n\,.\end{aligned} \tag{16}$$

The module B_{n+1} is generated by $s_n B_n$. By induction it follows that $\partial_n\,\partial_{n+1} = 0$. □

Let $x \in \ker\partial_n$. By (15) $\partial_{n+1}\, s_n x = x$. This means $x \in \operatorname{im}\partial_{n+1}$. Therefore $\ker\partial_n \subset \partial_{n+1}\, B_{n+1}$ and $B(\mathbb{Z}(\Pi)) := (B_n, \partial_n)$ is a resolution of $\mathbb{Z}$.

Define the n-dimensional cohomology group of Π with coefficients in the Π-module A by $H^n(\Pi, A) := H^n(B(\mathbb{Z}(\Pi)), A)$.

An element of $\operatorname{Hom}(B_2, A)$ is a module homomorphisms $f : B_2 \to A$. It is determined by the images of the module generators $[x|y]$. Hence we can view f as a function from $\Pi \times \Pi$ to A with $f(x, 1) = f(1, y) = 0$. If f is a cocycle, it satisfies (13). Thus f can be viewed as an element of $Z^2_\phi(\Pi, A)$. The subgroup $\partial\, B_3$ can be identified with $B^2_\phi(\Pi, A)$ in the same manner. This means that given a Π-module A, with module structure recorded by the operators ϕ, the assigned groups $H^2_\phi(\Pi, A)$ and $H^2(\Pi, A)$ are isomorphic.

The n-dimensional cohomology of groups is a special case of Ext^n. This is shown by

Theorem 6. *[Mac63, Corollary IV.5.2.] Let A be a Π-module. There is an isomorphism*

$$\operatorname{Ext}^n_{\mathbb{Z}(\Pi)}(\mathbb{Z}, A) \cong H^n(\Pi, A),$$

which is natural in A.

Proof. We proved that $B(\mathbb{Z}(\Pi))$ is a resolution of $\mathbb{Z}$ as a trivial Π-module and the B_n are free by construction. Since free modules are projective, the isomorphism and its naturality follow from Theorem 2. □

We saw in subsection 2.5 that for any short exact sequence $0 \to A \to B \to C \to 0$ there is a long exact sequence for Ext^n. Theorem 6 therefore yields a long exact sequence

$$\cdots \to H^n(\Pi, A) \to H^n(\Pi, B) \to H^n(\Pi, C) \to H^{n+1}(\Pi, A) \to \ldots$$

We end this section with two calculations using Theorem 6.

Application 2. *[Mac63, Chapter IV.7.] Let A be an abelian group and m a positive integer. We denote the cyclic group of order m by C_m and its generator by t. We calculate $H^n(C_m, A)$ for $n > 0$.*

First we construct a projective resolution of $\mathbb{Z}$ as a C_m-module. Let Γ denote the group ring $\mathbb{Z}(C_m)$. The elements

$$N := 1 + t + \cdots + t^{m-1} \quad \text{and} \quad D := t - 1$$

induce maps $N_*, D_* : \Gamma \to \Gamma$ via multiplication. It is easy to see that $ND = 0$. Therefore

$$W : \quad \cdots \xrightarrow{N_*} \Gamma \xrightarrow{D_*} \Gamma \xrightarrow{N_*} \Gamma \xrightarrow{D_*} \Gamma.$$

defines a complex. Furthermore the module homomorphism $\epsilon : \Gamma \to \mathbb{Z}$ defined by $\epsilon\left(\sum a_i t_i\right) = \sum a_i$ sends all elements of the form Du for $u \in \Gamma$ to zero. Therefore

$$(W, \epsilon) : \quad \cdots \xrightarrow{N_*} \Gamma \xrightarrow{D_*} \Gamma \xrightarrow{N_*} \Gamma \xrightarrow{D_*} \Gamma \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a complex over $\mathbb{Z}$. To show exactness we have to look at elements $u := \sum_{i=0}^{m-1} a_i t^i \in \Gamma$.

1. Suppose $Du = 0$. Then $Du = \sum_{i=0}^{m-1}(a_{i-1} - a_i)t^i = 0$ (with $a_{-1} := a_{m-1}$) and therefore $a_i = a_{i-1}$ for all i. So we can write $u = a_0 N$.
2. Suppose $Nu = 0$. Then $Nu = \sum_j \left(\sum_i a_i\right) t^j = 0$ which implies $\sum_i a_i = 0$. Hence we can write $u = -D(a_0 + (a_1 + a_0)t + \cdots + (a_{m-1} + \cdots + a_0)t^{m-1})$.
3. Suppose $\epsilon u = 0$. Then $\sum_i a_i = 0$ and as above we see that we can write $u = -D(a_0 + (a_1 + a_0)t + \cdots + (a_{m-1} + \cdots + a_0)t^{m-1})$.

We proved that (W, ϵ) is a resolution over $\mathbb{Z}$. The ring Γ is a projective module over itself. So by Theorem 2 we can compute Ext^n using (W, ϵ).

Note that $\mathrm{Hom}(\Gamma, A) \cong A$ since any homomorphism $f : \Gamma \to A$ is determined by $f(1) \in A$. Thus applying $\mathrm{Hom}(\_, A)$ yields the sequence

$$\ldots \xleftarrow{N^*} A \xleftarrow{D^*} A \xleftarrow{N^*} A \xleftarrow{D^*} A.$$

Calculating the homology of this complex results in

$$H^{2n}(C_m, A) = [a|ta = a]/N^* A, n \geq 0$$
$$H^{2n+1}(C_m, A) = [a|Na = 0]/D^* A, n > 0.$$

Example 2. *With the same situation as in Application 2, but A a trivial C_m-module, the cohomology groups are*

$$H^{2n}(C_m, A) = [a \in A|ta = a]/N^* A = A/mA, n \geq 0$$
$$H^{2n+1}(C_m, A) = [a \in A|Na = 0]/D^* A = [a|\textit{order of } a \textit{ divides } m], n > 0.$$

Proof. All elements of A are invariant under C_m. The image of N^* is all elements of the form ma, $a \in A$. The kernel of N^* are all elements with order dividing m. The image of elements $a \in A$ under D^* is $(1 - t)a = a - a = 0$. Now the results follow by our calculations in Application 2 $\square$

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