An Introduction to Homological Algebra

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1 Introduction

While it began as a tool in algebraic topology, the last fifty years have seen homological algebra grow into an indispensable tool for algebraists, topologists, and geometers.

2 Preliminaries

Before we can truly begin, we must first introduce some basic concepts. Throughout, R will denote a commutative ring (though very little actually depends on commutativity). For the sake of discussion, one may assume either that R is a field (in which case we will have chain complexes of vector spaces) or that $R = \mathbb{Z}$ (in which case we will have chain complexes of abelian groups).

2.1 Chain Complexes

Definition 2.1. A **chain complex** is a collection of $\{C_i\}_{i\in\mathbb{Z}}$ of R-modules and maps $\{d_i : C_i \to C_{i-1}\}$ called differentials such that $d_{i-1} \circ d_i = 0$. Similarly, a **cochain complex** is a collection of $\{C^i\}_{i\in\mathbb{Z}}$ of R-modules and maps $\{d^i : C^i \to C^{i+1}\}$ such that $d^{i+1} \circ d^i = 0$.

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots$$

Remark 2.2. The only difference between a chain complex and a cochain complex is whether the maps go up in degree (are of degree 1) or go down in degree (are of degree -1). Every chain complex is canonically a cochain complex by setting $C^i = C_{-i}$ and $d^i = d_{-i}$.

Remark 2.3. While we have assumed complexes to be infinite in both directions, if a complex begins or ends with an infinite number of zeros, we can suppress these zeros and discuss finite or **bounded** complexes. Additionally, if $C_i = 0$ for all sufficiently large or sufficiently small values of i, then we say that the complex is **bounded above** or **bounded below**.

To ease notation, the subscripts and superscripts on differentials will be suppressed. For example, the condition that one has a chain complex becomes $d^2 = 0$.

Definition 2.4. Given two chain complexes $C = (C_i, d)$ and $C' = (C'_i, d')$, chain map between them is a collection of maps $f = \{f_i : C_i \to C'_i\}$ such that $d' \circ f_i = f_{i-1} \circ d$, i.e., the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r] \arrow[d, "f_{-i1}"] & \cdots \\
\cdots \arrow[r] & C'_{i+1} \arrow[r, "d_{i+1}"] & C'_i \arrow[r, "d_i"] & C'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Given a ring R, the collection of chain complexes of R-modules and chain maps between them forms a category, which we shall denote $\mathbf{Ch}(R)$.

Let C be a chain complex. Let $Z_i = \ker d_i$ be the **cycles** of C_i, and $B_i = \operatorname{im} d_{i+1}$ be the **boundaries** of C_i. Since $d^2 = 0$, we have that, for each i, $B_i \subset Z_i$. Call the quotient by $H_i(C) = Z_i/B_i$, the ith homology of C. Similarly, for a cochain complex, we define the ith cohomology $H^i(C)$.

Remark 2.5. As we shall see later, there is a nice way to associate a chain complex to a space with a given triangulation. While two different triangulations of a space usually give rise to different chain complexes, the homology of these chain complexes will be isomorphic. This observation, one of the first applications of homology, created a powerful family of algebraic invariants for a topological space. In general, most homology theories follow a similar pattern. Given an object (e.g., a topological space, a module, a pair of modules, a graph, a cow, a herd of cattle, etc.), we have a way to generate a chain complex. Unfortunately the chain complex isn't what we want: either it is too unwieldy to work with, there isn't a canonical way to create it, similar objects will have dissimilar chain complexes, or something else will go wrong. However, when we pass to homology, our problems go away and we get an easy to compute algebraic invariant of our object from which we can easily read useful information.

Given a map $f : C \to C'$ between two chain complexes, f maps cycles to cycles and boundaries to boundaries, and thus f induces a map $f_* : H(C) \to H(C')$. It often happens that two different chain maps induce the same maps on homology. The following is a useful sufficient condition for this to occur.

Definition 2.6. Two chain maps, $f, g : C \to C'$ are **chain homotopic**, written $f \sim g$, if there exist $s_i : C_i \to C'_{i+1}$ such that $f = g + d's + sd$.

The terminology comes from topology, where two maps which are homotopic at the level of topological spaces induce maps on corresponding chain complexes which are chain homotopic.

Proposition 2.7. *If $f, g : C \to C'$ and $f \sim g$, then $f_* = g_*$.*

Proof. It suffices to show that if $f = d's + sd$, then $f_* = 0$. First note that $d'f = fd = d'sd$, and so f is actually a chain map. Let $[x] \in Z_n/B_n$. Then $f_*([x]) = [d's(x) + sd(x)] = [d'(sx) + s(0)] = [0]$. □

There are certain kinds of chain complexes and chain maps which, due to their usefulness, have names. A map is $f : C \to C'$ a **quasi-isomorphism** if f_* is an isomorphism, and in this event, C and C' are said to be **quasi-isomorphic**. If there is some $g : C' \to C$ such that $fg \sim \mathrm{id}_{C'}$ and $gf \sim \mathrm{id}_C$, then f is said to be a **homotopy equivalence** and C and C' are **homotopy equivalent**. If f and g are inverse chain homotopy equivalences, then f_* and g_* are inverses, and thus f and g are quasi-isomorphisms. Not all quasi-isomorphisms are chain homotopy equivalences. If $\mathrm{id}_C \sim 0$, then C is said to be **contractible**. If C is contractible, then at the level of homology, the identity map and the zero map are the same, and thus all homology groups are zero. This is not a necessary condition for the homology groups to vanish (see 3.2). The vanishing of homology is so important that it occupies it's own section of these notes.

2.2 Exact Sequences

If C is a complex such that $H_i(C) = 0$, then C is said to be **exact** at position i. If C is exact at all positions, then C is said to be an **exact sequence**. Note that this means that at every stage, the image of one map is equal to the kernel of the next. We make the convention of saying that a sequence

$$A_n \longrightarrow A_{n-1} \longrightarrow \dots \longrightarrow A_1 \longrightarrow A_0$$

is exact if $H_i(A) = 0$ for $0 < i < n$. If we mean to suggest that the sequence is still exact when the implied zero maps are added at the ends, we will write

$$0 \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \dots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

.

Example 2.8. A complex of the form $0 \longrightarrow A \longrightarrow 0$ is exact if and only if $A \cong 0$. A complex of the form $0 \longrightarrow A \longrightarrow B \longrightarrow 0$ is exact if and only if the middle map is an isomorphism. These exact sequences are almost too short to be of note.

We make the following observations.

- Since the image of the zero map is zero, in an exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \longrightarrow \dots$, the map f is an injection.
- Since the kernel of the zero map is its entire domain, it follows that in an exact sequence of the form $\dots \longrightarrow B \xrightarrow{g} C \longrightarrow 0$, the map g is a surjection.
- In an exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, viewing A as a submodule of B, the first isomorphism theorem yields that $C \cong B/A$.

An exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$.is said to be a **short exact sequence**, and is just long enough to be interesting.

We can consider not only short exact sequences of modules, but also short exact sequences of chain complexes, namely a commutative diagrams of the form

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & A_i \arrow[r] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{-i1}"] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r] \arrow[d, "g_{i+1}"] & B_i \arrow[r] \arrow[d, "g_i"] & B_{i-1} \arrow[r] \arrow[d, "g_{-i1}"] & \cdots \\
\cdots \arrow[r] & C_{i+1} \arrow[r] & C_i \arrow[r] & C_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that rows are chain complexes and $0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ is an exact sequence for every i. Phrased differently, given a chain map, the image and kernel are both chain complexes, and a sequence of chain maps is exact if at every stage, the image of one map is equal to the kernel of the next.

When we look at derived functors, one of their key features is that short exact sequences give rise to long exact sequences of of homology groups. This will follow from the following proposition, whose proof follows from the snake lemma (4.2). The proposition also gives rise to other long exact sequences, including the long exact sequence of relative homology and the Mayer-Vietoris sequence.

Proposition 2.9. *Given a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of chain complexes, there are maps δ, natural in the sense of natural transformations such that*

$$\cdots \longrightarrow H_i(A) \xrightarrow{f_*} H_i(B) \xrightarrow{g_*} H_i(C) \xrightarrow{\delta} H_{i-1}(A) \xrightarrow{f_*} H_{i-1}(B) \xrightarrow{g_*} H_{i-1}(C) \longrightarrow \cdots$$

2.2.1 Exact Functors

Short exact sequences are fundamental objects in abelian categories, and one of the most basic ways to study an additive functor from one abelian category to another is to examine what it does to short exact sequences. Let F be a covariant functor, and let G be a contravariant functor.

We say that F (*resp.* G) is **exact** if

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0 \qquad (\textit{resp.}\ \ 0 \longrightarrow G(C) \longrightarrow G(B) \longrightarrow G(A) \longrightarrow 0\ )$$

is an exact sequence whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is. Similarly, we say that F (*resp.* G) is **left exact** if

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \qquad (\textit{resp.}\ \ 0 \longrightarrow G(C) \longrightarrow G(B) \longrightarrow G(A)\ )$$

is an exact sequence whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is. Finally, we say that F (*resp.* G) is **right exact** if

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0 \qquad (\textit{resp.}\ \ G(C) \longrightarrow G(B) \longrightarrow G(A) \longrightarrow 0\ )$$

is an exact sequence whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is.

Example 2.10. Let M be an R-module. Then the functor $- \otimes_R M : R-\text{mod} \to \text{R}-\text{mod}$, $N \mapsto N \otimes M$ is right exact. For some choices of M, tensoring is an exact functor. These modules are called **flat**. Later, we will see how the Tor functor gives a measure of the failure of a module to be flat.

Example 2.11. Let M be an R-module. Then the functors $\hom_R(M, -) : R-\text{mod} \to \text{R}-\text{mod}$, $N \mapsto \hom_R(M, N)$ and $\hom_R(-, M) : R-\text{mod}^{\text{op}} \to \text{R}-\text{mod}$, $N \mapsto \hom_R(N, M)$ are both left exact, the first being covariant and the second being contravariant.

Example 2.12. It happens that $- \otimes_R M$ and $\hom_R(M, -)$ are adjoint functors, i.e., $\hom_R(A \otimes M, B) \cong \hom_R(A, \hom_R(M, B))$. Their exactness properties are not coincidental. Any left adjoint functor is right exact, and any right adjoint functor is left exact. This is often the easiest way to show that a particular functor is exact.

3 Examples

Example 3.1. The following are all examples of complexes.

1. The complex $\ldots 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$ has two nonzero homology groups, both isomorphic to $\mathbb{Z}$. In general, if all the maps in a complex are zero, then $H_i(C) \cong C_i$.
2. The complex $\ldots 0 \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$ is exact. In fact, it is contractible.
3. The complex $\ldots 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$ has $H_0(C) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(C) \cong 0$.
4. The complex $\ldots 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$ has $H_0(C) \cong 0$ and $H_1(C) \cong \mathbb{Z}$.

Example 3.2. The complex $\ldots \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \longrightarrow \ldots$ is exact, but not contractible.

Example 3.3. If C and D are chain complexes, then $C \oplus D$ is a complex in a natural way. If $f : C \to D$ is a chain map, then $\ker f, \operatorname{im} f,$ and $\operatorname{coker} f$ are also all chain complexes in natural ways. As an exercise, define all these objects in the way you think you should and show that they are actually chain complexes.

Example 3.4. Assume that one has a surface X with a triangulation T, namely a collection of (oriented) vertices, edges, and faces such that every point not on an edge is in the interior of a face, every face is bounded by three edges, and no vertex is in the interior of an edge. We can associate a chain complex C^T to this triangulation by denoting C_i^T to be the free abelian group on the i-cells of the triangulation and defining the differential on a generator of C_i^T to be an alternating sum of the $i-1$-cells on its boundary. Given a refinement $T' \supset T$, there is a natural inclusion map $i : C^T \to C^{T'}$ which is a quasi-isomorphism. Given two triangulations T' and T'', we can consider a common refinement T, and since C^T is quasi-isomorphic to both $C^{T'}$ and $C^{T''}$, we see that $H_n(C^{T'}) \cong H_n(C^{T''})$ for every n, and thus $H_*(C^T)$ depends only on X. This is the beginning of **simplicial homology**, which is an important tool in the proof of the classification of surfaces (For details, see [**?**]).

4 Homological Algebra

Several of the results of this section will not be given complete proofs. However, they are all standard and can be found on any text on homological algebra, including [**?**].

4.1 The Long Exact Sequence

The long exact sequence associated to a short exact sequence is one of the main computational aspects of homology. One use is that, by knowing what particular homology groups and particular maps are in the sequence, one can use exactness to deduce what other groups must be. Another use of long exact sequences comes from the following.

Lemma 4.1 (The Five Lemma)**.** *Suppose one has a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & D \arrow[r] \arrow[d] & E \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

such that each row is exact, and all the vertical maps except for f are isomorphisms. Then f is an isomorphism.

Proof. This is an exercise in diagram chasing. We will show that f is injective. The proof that f is surjective is similar. Let $c \in C$ such that $f(c) = 0$. Since the vertical map from D is injective and the image of c in D' is zero, we must have that the image of c in D is zero. Thus c is in the image of the horizontal map from B. Let $b \in B$ map to c, and let $b' \in B'$ be the image of b. Since b' is in the kernel of the horizontal map from B' and the bottom row is exact, we can find $a' \in A'$ which maps to b'. Since the vertical map from A is surjective, we can find $a \in A$ which maps to a'. Let $b'' \in B$ be the image of a. Then $b - b''$ maps to $b' - b' = 0$, and since the vertical map from B is injective, this implies that $b = b''$. Thus, a maps to b maps to c. However, since the top row is exact, we must have that $c = 0$. $\square$

Lemma 4.2 (The Snake Lemma)**.** *Given a map of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r, "p"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i"] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

there is a long exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0$$

where $\delta : \ker h \to \operatorname{coker} g$ is the (well defined!) map $c \mapsto i^{-1}gp^{-1}(c)$.

Additionally, if the map $A \to B$ is not injective or the map $B' \to C'$ is not surjective, the result still holds, except without the map $\ker f \to \ker g$ being injective or without the map $\operatorname{coker} g \to \operatorname{coker} h$ being surjective, respectively.

The proof of the snake lemma is diagram chasing, and will be omitted here. With the snake lemma in hand, we can finally prove (2.9). Addopting the notation $B_n(C) \subset Z_n(C) \subset C_n$ for the n-boundaries and n-cycles of C, we apply the snake lemma to diagrams of the form

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

to see that we have exact rows in the commutative diagram

$$\begin{tikzcd}
& A_n/B_n(A) \arrow[r] \arrow[d, "d"] & B_n/B_n(B) \arrow[r] \arrow[d, "d"] & C_n/B_n(C) \arrow[r] \arrow[d, "d"] & 0\,. \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f"] & Z_{n-1}(B) \arrow[r, "g"] & Z_{n-1}(C) \arrow[r] & {}
\end{tikzcd}$$

(The diagram itself does not come from the snake lemma.) Noting that the kernel and cokernel of the vertical maps are the n and $n-1$ homology groups, we merely splice together the exact sequences we get from applying the snake lemma to these diagrams to complete the proof.

4.2 Projective Resolutions and the Horseshoe Lemma

Consider the following scenario. Let M be an R-module with a set $\{m_\alpha\}$ of generators. To understand M, one might try to understand the generators by looking at the natural surjection $\bigoplus_\alpha R \to M$ sending the generators of the free group to the generators of M. The relations between the generators are naturally given by the kernel of this map. We can study the relations in the same way by finding a free group and a surjection onto the kernel. If we are lucky, we might not have any relations between our relations. For example, if $R = \mathbb{Z}$ so that we are dealing with abelian groups (or more generally, if R is a PID), then since subgroups of free groups are free, we would have that our relations have the structure of a free group. If we are not lucky, or we chose our generators for our relations wrong, we continue the process and create an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the F_i are free modules which describe relations between relations between relations. Note that if we remove M from the sequence, we have a chain complex which has homology M in degree 0 and is exact everywhere else (a chain complex which is bounded below 0 and is exact except in degree 0 is called **acyclic**). An acyclic complex which contains only free modules is called a **free resolution**.

At this point, there are several questions that one might ask. What is the smallest length of free resolution of M? What is the longest a minimal free resolution of an R-module must be? What kind of information about M can a free resolutions give? The first two questions, which deal with homological dimension, will not be dealt with here. The third question has a beautiful answer which we can only touch on. However, first, we need to summarize some results on projective modules.

Definition 4.3. A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is said to be **split** if any of the following equivalent conditions hold

1. There is a map $p : B \to A$ such that $pf = \mathrm{id}_A$.
2. There is a map $i : C \to B$ such that $gi = \mathrm{id}_C$.
3. There is an isomorphism $B \to A \oplus C$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[dr] & B \arrow[r, "g"] \arrow[d, "\cong"] & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

where the bottom maps are $a \mapsto (a, 0)$ and $(a, c) \mapsto c$.

Definition 4.4. We say that a module P is **projective** if for any surjection $f : M \to N$ and any map $\phi : P \to M$, there exists a map $\psi : P \to N$ such that $\phi = f\psi$.

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[d, "\phi"] & \\ M \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

Proposition 4.5. *The following are equivalent*

1. *P is projective.*
2. *If for any surjective map $M \to N$, then the induced map $\hom(P, M) \to \hom(P, N)$ is surjective.*
3. *The functor $\hom(P, -)$ is exact, i.e., if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence, then so is $0 \longrightarrow \hom(P, A) \longrightarrow \hom(P, B) \longrightarrow \hom(P, C) \longrightarrow 0$*
4. *Any short exact sequence ending with a surjection to P is split.*
5. *P is a direct summand of a free module.*

We see that being projective is a generalization of being free. Just as we defined free resolutions, we can define projective resolutions. While free resolutions would suffice for our purposes, not only are there situations where one prefers projective objects to free ones (or where there are not enough free objects), but by exposing the property of free modules that we are going to use, the poofs (which will not be provided) are easier to construct. One of the reasons projective objects are useful is the following.

Proposition 4.6. *Suppose that P_* and C_* are two chain complexes bounded below 0. Suppose further that C_* is acyclic and that P_i is projective for every i. Then any map $f : H_0(P) \to H_0(C)$ lifts to a map $\tilde{f} : P_* \to C_*$, and this map is unique up to chain homotopy.*

Using this proposition to lift the identity map, we see that any two projective resolutions of a module M are homotopy equivalent. Let F be an additive functor from chain complexes to chain complexes. If $f \sim g$, then $F(f) \sim F(g)$, and since $F(\mathrm{id}) = \mathrm{id}$, we must have that F preserves homotopy equivalence. Therefore, if P and Q are two projective resolutions of M, then $F(P)$ and $F(Q)$ are homotopy equivalent. Thus, taking a projective resolution of M, applying F, and taking homology is a well defined operation which depends only on M and F, and not on our choice of resolution. Furthermore, by using the lifting property to lift module maps to maps between projective resolutions we see that this process actually defines a functor.

Definition 4.7. Suppose that F is a right exact covariant functor. Then the process defined above defines a family of functors (one for every homology group) called the **left derived functors** of F and denoted by $(L_iF)i \in \mathbb{Z}_+$.

Proposition 4.8. *If F is a right exact covariant functor, then $L_0F = F$.*

In fact, we do not need right exactness to define the left derived functors. However, the proposition would not hold without it.

The important computational tool of derived functors is the long exact sequence associated to a short exact sequence. Explicitly, let F be an additive functor, and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be exact. Then we have a long exact sequence

$$\begin{aligned} \cdots \longrightarrow L_{i+1}F(C) \xrightarrow{\delta} L_iF(A) \longrightarrow L_iF(B) \longrightarrow L_iF(C) \xrightarrow{\delta} L_{i-1}(A) \longrightarrow \cdots \\ \longrightarrow \cdots \longrightarrow L_1(C) \xrightarrow{\delta} L_0F(A) \longrightarrow L_0F(B) \longrightarrow L_0F(C) \longrightarrow 0 \end{aligned}$$