

SOLUTIONS AND ELABORATIONS FOR “AN INTRODUCTION TO HOMOLOGICAL ALGEBRA” BY CHARLES WEIBEL

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1. INTRODUCTION

Weibel's *homological algebra* is a text with a lot of content but also a lot left to the reader. This document is intended to cover what's left to the reader: I try to fill in gaps in proofs, perform checks, make corrections, and do the exercises. It is very much in progress, covering only chapters 3 and 4 at the moment.

I will assume roughly the same pre-requisites as Weibel: knowledge of some category theory and graduate courses in algebra covering modules, tensor products, and localizations of rings. I will assume less mathematical maturity.

2. MISSING DETAILS

2.1. Chapter 1.

2.1.1. Section 1.6.

Example 1.6.8. It is not quite true that $\mathcal{A}^{\mathcal{U}^{op}} = \text{Presheaves}(X)$, since there are functors that do not take the empty set to 0. In fact, some sources do not require that presheaves take the empty set to zero. With or without this requirement, the resulting category of presheaves is an abelian category.

2.2. Chapter 2.

2.2.1. Section 2.1.

2.2.2. Section 2.2.

2.2.3. Section 2.3.

2.2.4. Section 2.4.

Theorem 2.4.6. It is asserted that the image of a split exact sequence under an additive functor is exact. See Theorem 4 in “Additional Math” below for a (lengthy, but I think neat) proof.

2.3. Chapter 3.

2.3.1. Section 3.1.

Proposition 3.1.2. This is the way that a group is a direct limit of its finitely generated subgroups. Let A be an abelian group.

First, we set up a diagram \mathcal{D} whose objects are the finitely generated subgroups of A , and whose arrows are the inclusion maps $\varphi : G \rightarrow H$ whenever $G \leq H$ for finitely generated subgroups G, H of A . This diagram is indexed by a filtered category, since for any pair G, H of finitely generated subgroups of A , GH is also a finitely generated group, so $G \rightarrow GH$ and $H \rightarrow GH$ are arrows in the diagram (i.e. we satisfy axiom 1 of a filtered category in Definition 2.6.13) and there are no parallel arrows (i.e. we satisfy axiom 2 of a filtered category vacuously.)

The claim is that the colimit of this diagram is A . To see this, first of all, we have a map $\varphi_G : G \rightarrow A$ for each f.g. subgroup G defined by the inclusion of G into A . This commutes with all maps in the diagram since the inclusion of G into H followed by the inclusion of H into A is just the inclusion of G into A .

It remains to show that A is a universal. Select another family of maps $\mathcal{D} \rightarrow B$. We will define a map $\psi : A \rightarrow B$ commuting with everything. Let $g \in A$. The element g is contained in some f.g. subgroup G of A . We define $\psi(g)$ to be the image of g under the map $G \rightarrow B$. This is the only way to define ψ so that

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ A & \xrightarrow{\psi} & B \end{array}$$

commutes, so if ψ defines a map that commutes with the maps $\mathcal{D} \rightarrow A$ and $\mathcal{D} \rightarrow B$, it is unique. On the other hand, if H is another f.g. subgroup of A containing g , then since

$$\begin{array}{ccccc} G & \longrightarrow & GH & \longleftarrow & H \\ & \searrow & \downarrow & \swarrow & \\ & & B & & \end{array}$$

commutes, the image of g under $H \rightarrow B$ must be the same as its image under the map $G \rightarrow B$. This shows that ψ is well defined. Finally, ψ is a group homomorphism, since for any $g, h \in A$, we may select a f.g. subgroup G so that $g, h \in G$. Let φ be the map $G \rightarrow B$. Then $\psi(g+h) = \varphi(g+h) = \varphi(g) + \varphi(h) = \psi(g) + \psi(h)$ since $\varphi : G \rightarrow B$ is a group homomorphism. So there exists a unique group homomorphism $A \rightarrow B$ so that everything commutes, as desired, and A is the colimit.

Intuitively, the direct limit is a fancy way to express a union, and of course A is a union of its finitely generated subgroups, since each element of A is contained in some finitely generated subgroup. The argument above shows that the formal definition agrees with this intuition.

Proposition 3.1.4. The reason that $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}^m, B) = 0$ is that \mathbb{Z}^m is a free \mathbb{Z} -module, hence projective, and Tor vanishes on projective modules.

2.3.2. Section 3.2.

Definition 3.2.1. The target category of $- \otimes_R B$ is intended to be **Ab**. If B is an R - S bimodule, we can also view $- \otimes_R B$ as a functor from **mod- R** to **mod- S** . Again, this is always right exact. It is equivalent for $- \otimes B$ to be exact as a functor to **Ab** and as a functor to **mod- S** , since in each case exactness is the same as taking every injection $A' \rightarrow A$ of R -modules to an injective map $A' \otimes B \rightarrow A \otimes B$: we just view the map as a group homomorphism if **Ab** is the target and as an S -module homomorphism if **mod- S** is the target. This means that flatness of B does not depend on which category we take to be the target of $- \otimes_R B$, although it does depend on R .

Theorem 3.2.2. For an alternate proof without colimits, see [2, Proposition 2.5].

We show that the colimit indicated in the text is the localization, as claimed. Our technique will be to use an explicit description of filtered colimits. Namely, if I is a filtered category, and $F : I \rightarrow R\text{-}\mathbf{mod}$ is an additive functor, the colimit of F is the module C whose additive group has underlying set

$$C = \left(\bigsqcup_{i \in I} F(i) \right) / \sim$$

where $a \in F(i)$ and $b \in F(j)$ are equivalent $a \sim b$ if there exist arrows $\varphi : i \rightarrow k$ and $\psi : j \rightarrow k$ so that $F(\varphi)(a) = F(\psi)(b)$. The addition of $a \in F(i)$ and $b \in F(j)$ is defined by taking a pair of arrows $\varphi : i \rightarrow k$ and $\psi : j \rightarrow k$ (whose existence is assured since I is a filtered category) and letting $a + b$ be the sum $F(\varphi)(a) + F(\psi)(b)$ in k . For $r \in R$ we take the scalar product of r and $a \in F(i)$ to be $ra \in F(i)$.

In our particular case, $a \in F(s_1)$ and $b \in F(s_2)$ are equivalent if and only if there exists a number $c \in S$ so that $cs_2a = cs_1b$. Next, note that $a \sim b$ if and only if $\frac{a}{s_1} = \frac{b}{s_2}$ in $S^{-1}R$. Therefore the map $\varphi : C \rightarrow S^{-1}R$ defined by $a \in F(s_1) \mapsto \frac{a}{s_1}$ is a well defined bijection. Next, the sum of $a \in F(s_1)$ and $b \in F(s_2)$ is $s_2a + s_1b \in F(s_1s_2)$, so φ preserves addition. Finally, the scalar product of $r \in R$ and $a \in F(s_1)$ is $ra \in F(s_1)$, so φ preserves scalar multiples. Altogether, the colimit C is isomorphic to the localization $S^{-1}R$, as claimed.

Lemma 3.2.6. When $M = R$, the map σ is from $A^* \otimes_R R \rightarrow \text{Hom}_R(R, A)^*$, with σ defined by $\sigma(f \otimes r) : h \mapsto f(h(r))$.

There is an isomorphism $A^* \rightarrow A^* \otimes R$ defined by $f \mapsto f \otimes 1$ and there is an isomorphism

$$\text{Hom}_R(R, A)^* = \text{Hom}_{\mathbf{Ab}}(\text{Hom}_R(R, A), \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) = A^*$$

induced by the isomorphism $A \rightarrow \text{Hom}_R(R, A)$ that takes a to the map $\varphi_a : R \rightarrow A$ determined by $\varphi_a(1) = a$. More explicitly, the map $\text{Hom}_R(R, A)^* \rightarrow A^*$ is defined by

$$\begin{aligned} h &\mapsto h' : A \rightarrow \mathbb{Q}/\mathbb{Z} \\ a &\mapsto h \circ \varphi_a \end{aligned}$$

where φ_a is the element of $\text{Hom}(R, A)$ so that $\varphi_a(1) = a$. Composing with these isomorphisms, we may regard σ as a map $\sigma : A^* \rightarrow A^*$ defined by $\sigma(f) : m \mapsto f(\varphi_m(1)) = f(m)$. That is, σ takes $f \in A^*$ to $f \in A^*$, so σ is an isomorphism, as claimed.

The use of the Five Lemma is with this diagram:

$$\begin{array}{ccccccccc} A^* \otimes R^m & \longrightarrow & A^* \otimes R^n & \longrightarrow & A^* \otimes M & \longrightarrow & 0 & \longrightarrow & 0 \\ \sigma \downarrow \cong & & \sigma \downarrow \cong & & \sigma \downarrow & & 0 \downarrow & & 0 \downarrow \\ \text{Hom}(R^m, A)^* & \xrightarrow{\alpha^*} & \text{Hom}(R^n, A)^* & \longrightarrow & \text{Hom}(M, A)^* & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Proposition 3.2.9. We will prove that if T is R -flat and P is a projective R -module, then $P \otimes_R T$ is a projective T -module.

Recall that an R -module P is projective if and only if P is the direct summand of a free R -module (Proposition 2.2.1). So let $F = P \oplus Q$ be a free module with direct summand P . Since F is free, we also have that $F \cong \bigoplus_{s \in S} R$ for some set S . This gives us the split exact sequence

$$0 \longrightarrow P \longrightarrow F \longrightarrow Q \longrightarrow 0.$$

Since T is R -flat, $-\otimes_R T$ is exact as a functor from $\mathbf{mod}\text{-}R$ to $\mathbf{mod}\text{-}T$. It therefore takes split exact sequences to split exact sequences (see comment on Theorem 2.4.6), so $P \otimes T$ is a direct summand of $F \otimes T$. Also, $-\otimes_R T$ as a functor from $\mathbf{mod}\text{-}R$ to $\mathbf{mod}\text{-}T$ has the right adjoint $\text{Hom}(T, -)$ (Proposition 2.6.3), so $-\otimes_R T$ preserves colimits (Theorem 2.6.10). In particular, $F \otimes T = (\bigoplus_{s \in S} R) \otimes T = \bigoplus_{s \in S} (R \otimes T) = \bigoplus_{s \in S} T$, so $-\otimes_R T$ takes free R -modules to free T -modules. Together, $P \otimes T$ is a direct summand of a free T -module, so $P \otimes T$ is a projective T -module.

Lemma 3.2.11. To see why we require that r is central, let $\mu_i : P_i \rightarrow P_i$ be the map $p \mapsto rp$. Then μ_i is an R -module homomorphism if and only if for all $s \in R, p \in P$

$$srp = s\mu(p) = \mu(sp) = rsp.$$

This holds with no conditions on P_i if $sr = rs$ for all $s \in R$, that is, if r is central. Had we not required that r were central, we would then have to impose conditions on the projective resolution $P \rightarrow A$ to maintain that $\tilde{\mu} : P \rightarrow P$ is a chain map.

Corollary 3.2.13. The equality $R_p \otimes_R M = \text{Tor}_n^{R_p}(A_p, B_p)$ follows by first noting

$$R_p \otimes_R M = R_p \otimes_R \text{Tor}_n^R(A, B) = \text{Tor}_n^{R_p}(A \otimes_R R_p, R_p \otimes_R B)$$

from Corollary 3.2.10, and then using that $A \otimes R_p$ and $R_p \otimes B$ are equal to the localizations of A and B at p , respectively.

2.3.3. Section 3.3.

Example 3.3.3. $\text{Ext}_{\mathbb{Z}}^0(A, \mathbb{Z})$ is zero since we have assumed that A is a torsion group: torsion elements must be sent to torsion elements, so the only homomorphism $A \rightarrow \mathbb{Z}$ is the zero homomorphism.

The comment that $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) = \varprojlim (\mathbb{Z}/p^n)$ is proved in Application 3.5.10.

Example 3.3.5.

(1) This is because $A \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/p$, so

$$\text{Ext}_{\mathbb{Z}/m}^1(A, \mathbb{Z}/p) = \prod_{i=1}^{\infty} \text{Ext}_{\mathbb{Z}/m}^1(\mathbb{Z}/p, \mathbb{Z}/p) = \prod_{i=1}^{\infty} \mathbb{Z}/p.$$

where the first equality follows from Proposition 3.3.4 part 1 and the second follows from Exercise 3.3.2. The last is a vector space of uncountably infinite dimension.

(2) The displayed equation follows from Proposition 3.3.4 part 2 and Calculation 3.3.2. ($\text{Ext}_{\mathbb{Z}}^1$ is intended.) The comment on divisibility follows since A is divisible if and only if $A = pA$ for all $p \geq 2$ if and only if $A/pA = 0$ for all $p \geq 2$.

Heuristically, there seems to be a duality between being torsionfree and divisible. $\text{Tor}_1^{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ tests for torsion and $\text{Ext}_{\mathbb{Z}}^1(-, \prod_{p=2}^{\infty} \mathbb{Z}/p)$ tests for divisibility.

Lemma 3.3.6. Weibel asserts that $\text{Hom}_{\mathbf{Ab}}(A, B)$ is a (left) R -module when R is commutative. Here is why. We define $rf : A \rightarrow B$ for $r \in R$ and $f \in \text{Hom}_{\mathbf{Ab}}(A, B)$ by $(rf)(b) = f(rb)$. In order for this action to make $\text{Hom}_{\mathbf{Ab}}(A, B)$ an R -module, we need rf to be an R -module homomorphism for all r . Now rf is an R -module homomorphism if and only if for all $s \in R$ and $b \in B$,

$$f(srb) = sf(rb) = s(rf)(b) = (rf)(sb) = f(rsb).$$

If we assume R is commutative, then $srb = rsb$, so we get the equality above for free. Otherwise, we would have to impose conditions on A and B to maintain that $\text{Hom}_{\mathbf{Ab}}(A, B)$ is an R -module.

The assertion that multiplication by r gives a chain map when r is central is addressed in my comment on Lemma 3.2.11.

Lemma 3.3.8. Let's check the isomorphism for $A = R$. Then

$$S^{-1}\text{Hom}_R(A, B) = S^{-1}\text{Hom}_R(R, B) \cong S^{-1}B.$$

On the other hand,

$$\mathrm{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B) \cong S^{-1}B.$$

So both are equal to $S^{-1}B$ as R -modules, as promised.

The implication that the same holds for R^m is due to the fact that additive functors preserve direct sums. This is a corollary of Theorem 4 and Proposition 3 in “Additional Math” above.

2.3.4. Section 3.4.

Theorem 3.4.3. According to the errata, the theorem should be stated with $\Theta : \xi \mapsto \partial(\mathrm{id}_B)$, instead of id_A . We will see why as we fill in the details of this proof.

When Weibel writes that $X \rightarrow A$ is the map induced by the maps $B \xrightarrow{0} A$ and $P \rightarrow A$, he is referring to the universal property of pushouts: since

$$\begin{array}{ccc} M & \xrightarrow{j} & P \\ \downarrow \beta & & \downarrow \\ B & \xrightarrow{0} & A \end{array}$$

commutes (we get zero when composing $M \rightarrow P \rightarrow A$ by exactness of $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$) there is a unique map $X \rightarrow A$ so that the following diagram commutes:

$$(1) \quad \begin{array}{ccccc} M & \xrightarrow{j} & P & & \\ \downarrow \beta & & \downarrow \sigma & \searrow \varphi & \\ B & \xrightarrow{i} & X & \xrightarrow{\varphi'} & A \\ & \searrow 0 & & & \end{array}$$

That map is the map we are to use in the sequence $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$. More explicitly, if φ is the map $P \xrightarrow{\varphi} A$, then the dotted map, $\varphi' : X \rightarrow A$, is defined by taking the equivalence class of (p, b) in X to the element $\varphi(p)$ in A . This is well defined since if (p, b) and (p', b') are two representatives of the same element of X , then $\varphi(p) - \varphi(p') = \varphi(p - p') = \varphi(j(m))$ for some $m \in M$. By exactness of $0 \rightarrow M \xrightarrow{j} P \xrightarrow{\varphi} A \rightarrow 0$, $\varphi(j(m)) = 0$.

Now we show that the bottom row of the diagram

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \xrightarrow{\varphi} & A \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \sigma & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{i} & X & \xrightarrow{\varphi'} & A \longrightarrow 0 \end{array}$$

is exact. (The left square commutes by definition of pushout. The right square commutes since the right triangle of (1) commutes.)

Since φ is surjective, and $\varphi = \varphi' \circ \sigma$, it follows φ' is surjective. This shows exactness at A .

The map $i : B \rightarrow X$ is defined by $b \mapsto [(0, b)]$. Suppose that $[(0, b)] = [(0, 0)]$. Then, by the definition of X , $(0, b) = (0, 0) + (j(m), -\beta(m))$ for some $m \in M$. Since j is injective, we must have that $m = 0$, so $[(0, b)] = [(0, 0)]$. Therefore i is injective. This shows exactness at B .

Finally, suppose (p, b) is the equivalence class of some element of the kernel of φ' . Then $\varphi(p) = 0$, so by exactness of the first row, $p = j(m)$ for some $m \in M$. Therefore, $[(p, b)] = [(j(m), b)] = [(0, b + \beta(m))]$ is in the image of i . This shows exactness at X , and we are done.

The reference to naturality of ∂ refers to the fact that this diagram, induced from the morphism of short exact sequences (2) by the contravariant cohomological δ -functor $\text{Ext}^*(-, B)$, commutes:

$$\begin{array}{ccc} \text{Hom}(M, B) & \xrightarrow{\partial} & \text{Ext}^1(A, B) \\ \beta^* \uparrow & & \uparrow \text{id} \\ \text{Hom}(B, B) & \xrightarrow{\partial} & \text{Ext}^1(A, B) \end{array}$$

By commutativity of the diagram, $\partial(\text{id}_B) = \partial(\beta^*(\text{id}_B)) = \partial(\beta) = x$. This gives surjectivity of Θ .

Now we will check that the maps $i : B \rightarrow X$ and $\sigma + i \circ f : P \rightarrow X$ induce an isomorphism $X' \cong X$ and an equivalence between ξ' and ξ . Recall that we have picked two lifts, β and β' of some element $x \in \text{Ext}^1(A, B)$ under the map ∂ in the diagram

$$\text{Hom}(P, B) \xrightarrow{j^*} \text{Hom}(M, B) \xrightarrow{\partial} \text{Ext}^1(A, B) \longrightarrow 0,$$

which we arrived at by taking the long exact sequence associated to $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$ under $\text{Ext}^*(-, B)$. By exactness, the maps β and β' differ by an element of the image of j^* , so we write $\beta' = \beta + f \circ j$, where $f : P \rightarrow B$.

First, we check that the diagram

$$(3) \quad \begin{array}{ccc} M & \xrightarrow{j} & P \\ \beta' \downarrow & & \downarrow \sigma + i \circ f \\ B & \xrightarrow{i} & X \end{array}$$

commutes. It does:

$$\begin{aligned} i \circ \beta' &= i \circ (\beta + f \circ j) \\ &= i \circ \beta + i \circ f \circ j \\ &= \sigma \circ j + i \circ f \circ j \\ &= (\sigma + i \circ f) \circ j. \end{aligned}$$

To get the third equality, we used the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{j} & P \\ \beta \downarrow & & \downarrow \sigma \\ B & \xrightarrow{i} & X \end{array}$$

Now $\begin{array}{c} P \\ \downarrow \sigma + i \circ f \\ B \xrightarrow{i} X \end{array}$ is a cocone of the diagram $M \xrightarrow{j} P$ so by the universal property

of pushouts, there exists a unique map so that the following diagram commutes:

$$(4) \quad \begin{array}{ccc} M & \xrightarrow{j} & P \\ \downarrow \beta' & & \downarrow \sigma' \\ B & \xrightarrow{i'} & X' \end{array} \quad \begin{array}{c} \searrow \sigma + i \circ f \\ \searrow i \\ \searrow i' \end{array} \quad \begin{array}{c} \\ \\ X \end{array}$$

By a symmetric argument, there exists a unique map so that

$$\begin{array}{ccc} M & \xrightarrow{j} & P \\ \downarrow \beta & & \downarrow \sigma \\ B & \xrightarrow{i} & X \end{array} \quad \begin{array}{c} \searrow \sigma' - i' \circ f \\ \searrow i' \\ \searrow i \end{array} \quad \begin{array}{c} \\ \\ X' \end{array}$$

commutes. Adding $f \circ j$ to β , $i \circ f$ to σ , and $i' \circ f$ to σ' , we obtain the commutative diagram

$$(5) \quad \begin{array}{ccc} M & \xrightarrow{j} & P \\ \downarrow \beta' & & \downarrow \sigma + i \circ f \\ B & \xrightarrow{i} & X \end{array} \quad \begin{array}{c} \searrow \sigma' \\ \searrow i' \\ \searrow i \end{array} \quad \begin{array}{c} \\ \\ X' \end{array}$$

with the same arrow from $X \rightarrow X'$.

Now, composing the arrows $X' \rightarrow X$ and $X \rightarrow X'$ in diagrams (4) and (5), we get a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{j} & P \\ \downarrow \beta' & & \downarrow \sigma' \\ B & \xrightarrow{i'} & X' \end{array} \quad \begin{array}{c} \searrow \sigma' \\ \searrow i' \\ \searrow i \end{array} \quad \begin{array}{c} \\ \\ X' \end{array}$$

By the uniqueness in universal property of pushouts, the dashed arrow must be the identity on X' . By a symmetric argument, the other composition $X \rightarrow X' \rightarrow X$ is the identity on X . Therefore X and X' are isomorphic.

Finally, we show that the isomorphism $X' \rightarrow X$ induces an equivalence of extensions. Consider the diagram

$$\begin{array}{ccccccc} \xi' : 0 & \longrightarrow & B & \xrightarrow{i'} & X' & \longrightarrow & A \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ \xi : 0 & \longrightarrow & B & \xrightarrow{i} & X & \longrightarrow & A \longrightarrow 0 \end{array}$$

where the middle arrow is the isomorphism $X' \rightarrow X$. The left square commutes since the lower triangle of (4) commutes. The right square commutes because the right squares of

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \xrightarrow{\varphi} & A \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \sigma & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{i} & X & \longrightarrow & A \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \xrightarrow{\varphi} & A \longrightarrow 0 \\ & & \downarrow \beta' & & \downarrow \sigma' & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{i'} & X & \longrightarrow & A \longrightarrow 0 \end{array}$$

commute. Therefore, ξ and ξ' are equivalent extensions, as claimed.

Definition 3.4.4. I think it is helpful to compute the Baer sum of a simple example. Let us take the Baer sum of the extensions

$$\xi : 0 \longrightarrow \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{1} \mathbb{Z}/p \longrightarrow 0,$$

$$\xi' : 0 \longrightarrow \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{2} \mathbb{Z}/p \longrightarrow 0.$$

Now the pullback X'' of $\mathbb{Z}/p^2 \xrightarrow{1} \mathbb{Z}/p$ and $\mathbb{Z}/p^2 \xrightarrow{2} \mathbb{Z}/p$ is the submodule $\{(x, y) \in \mathbb{Z}/p^2 \times \mathbb{Z}/p^2 \mid x \equiv 2y \pmod{p}\}$ of $\mathbb{Z}/p^2 \times \mathbb{Z}/p^2$. The pullback diagram is

$$\begin{array}{ccc} X'' & \xrightarrow{\pi_2} & \mathbb{Z}/p^2 \\ \pi_1 \downarrow & & \downarrow 2 \\ \mathbb{Z}/p^2 & \xrightarrow{1} & \mathbb{Z}/p \end{array}$$

We let Y be $X''/\{(px, -px) \mid x \in \mathbb{Z}/p\}$. Then the Baer sum of ξ and ξ' is the sequence

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{f} Y \xrightarrow{g} \mathbb{Z}/p \longrightarrow 0$$

where f takes x to $[(px, 0)] = [(0, px)]$ and g takes $[(x, y)]$ to $x \pmod{p}$ or equivalently to $2y \pmod{p}$.

If $p \neq 3$, this is equivalent with the extension associated to $i = \frac{2}{3} \pmod{p}$, via the following equivalence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{f} & Y & \xrightarrow{g} & \mathbb{Z}/p \longrightarrow 0 \\ & & \parallel & & \downarrow \sigma & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{p} & \mathbb{Z}/p^2 & \xrightarrow{\frac{2}{3}} & \mathbb{Z}/p \longrightarrow 0 \end{array}$$

where $\sigma : [(x, y)] \mapsto x + y \pmod{p^2}$. This is well defined, since any other representative of (x, y) is of the form $(x - pz, y + pz)$, and $(x - pz, y + pz) \mapsto (x - pz) + y + pz = x + y$. To see that the bottom right map is $\frac{2}{3}$, i.e. multiplication by $2 \cdot 3^{-1} \pmod{p}$, note that $(2, 1) \xrightarrow{g} 2$, while $(2, 1) \xrightarrow{\sigma} 3 \pmod{p^2}$, and a morphism between cyclic groups must be multiplication by some number.

Corollary 3.4.5. When Weibel refers to using the notation of the previous theorem, he means that we have constructed the following diagrams from the extensions ξ and ξ' :

$$(6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \gamma & & \downarrow \tau & & \parallel \\ \xi : \quad 0 & \longrightarrow & B & \xrightarrow{i} & X & \longrightarrow & A \longrightarrow 0 \end{array}$$

and

$$(7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \gamma' & & \downarrow \tau' & & \parallel \\ \xi' : \quad 0 & \longrightarrow & B & \xrightarrow{i'} & X' & \longrightarrow & A \longrightarrow 0. \end{array}$$

We construct them by first using the lifting property of the projective module P to obtain a map $\tau : P \rightarrow X$ and a map $\tau' : P \rightarrow X'$. Then by commutativity of the right squares, $\tau(j(M)) \subseteq \ker\{X \rightarrow A\}$ and $\tau'(j(M)) \subseteq \ker\{X' \rightarrow A\}$. The exactness of the bottom rows allows us to lift τ and τ' to maps $\gamma = i^{-1} \circ \tau \circ j$ and $\gamma' = (i')^{-1} \circ \tau' \circ j$ respectively.

Recall that X'' is the pullback of the diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & A. \end{array}$$

Now

$$\begin{array}{ccc} P & \xrightarrow{\tau} & X \\ \downarrow \tau' & & \downarrow \\ X' & \longrightarrow & A \end{array}$$

commutes, since each composition is the map $P \rightarrow A$, by the diagrams (6) and (7) above. Accordingly, by the universal property of pullbacks, we get a unique map $P \rightarrow X''$ so that

$$\begin{array}{ccccc} P & & & & \\ & \searrow \tau & & & \\ & & X'' & \longrightarrow & X \\ & \searrow \tau' & \downarrow & & \downarrow \\ & & X' & \longrightarrow & A \end{array}$$

commutes. This is the induced map $\tau'' : P \rightarrow X''$. More explicitly, τ'' is the map $P \rightarrow X'' \subseteq X \times X'$ defined by $\tau''(p) = (\tau(p), \tau'(p))$. The map $\bar{\tau} : P \rightarrow Y$ is obtained from τ'' by composition with the quotient map $X'' \rightarrow Y$.

Next, Weibel claims that

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \gamma + \gamma' & & \downarrow \bar{\tau} & & \parallel \\ \xi + \xi' : \quad 0 & \longrightarrow & B & \xrightarrow{i''} & Y & \longrightarrow & A \longrightarrow 0. \end{array}$$

commutes. To see this, we first check that the left square commutes. Recall that the map $i'' : B \rightarrow Y$ is defined by $b \mapsto [(i(b), 0)] = [(0, i'(b))]$. Then composing $i'' \circ (\gamma + \gamma')$, we get

$$\begin{aligned} (i'' \circ (\gamma + \gamma'))(m) &= i''(\gamma(m) + \gamma'(m)) \\ &= [(i(\gamma(m)), 0)] + [(0, i'(\gamma'(m)))] \\ &= [(\tau(j(m)), 0)] + [(0, \tau'(j(m)))] \\ &= [(\tau(j(m)), \tau'(j(m)))] \\ &= \bar{\tau}(j(m)), \end{aligned}$$

so the square commutes. **To do: the right square commutes because...**

2.3.5. Section 3.5.

Definition 3.5.1. We check that the kernel of Δ is $\varprojlim A_i$. **To do**

Lemma 3.5.2. The snake lemma mentioned only gives the map $\varprojlim C_i \xrightarrow{\partial} \varprojlim^1 A_i$.

To see why we may take the higher derived functors of \varprojlim to vanish, note that the induced map on cokernels $\varprojlim^1 B_i \rightarrow \prod \varprojlim^1 C_i$ is onto, since the map $\prod B_i \rightarrow \prod C_i$ is onto. Accordingly, the sequence

$$0 \rightarrow \varprojlim A_i \rightarrow \varprojlim B_i \rightarrow \varprojlim C_i \xrightarrow{\partial} \varprojlim^1 A_i \rightarrow \varprojlim^1 B_i \rightarrow \varprojlim^1 C_i \rightarrow 0$$

is exact.

Example 3.5.5. Applying the δ -functor $\varprojlim^* -$ to the short exact sequence $0 \rightarrow \{p^i \mathbb{Z}\} \rightarrow \{\mathbb{Z}\} \rightarrow \{\mathbb{Z}/p^i\} \rightarrow 0$, we get the long exact sequence

$$\varprojlim p^i \mathbb{Z} \longrightarrow \varprojlim \mathbb{Z} \longrightarrow \varprojlim \mathbb{Z}/p^i \mathbb{Z} \xrightarrow{\partial} \varprojlim^1 p^i \mathbb{Z} \longrightarrow \varprojlim^1 \mathbb{Z}.$$

This simplifies to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \hat{\mathbb{Z}}_p \xrightarrow{\partial} \varprojlim^1 p^i \mathbb{Z} \longrightarrow 0.$$

To see this, note that since each map $p^i\mathbb{Z} \rightarrow p^{i-1}\mathbb{Z}$ is an inclusion, $\varprojlim p^i\mathbb{Z} = \bigcap_i p^i\mathbb{Z} = 0$. It is clear that $\varprojlim \mathbb{Z} = \mathbb{Z}$. Finally, $\varprojlim^1 \mathbb{Z} = 0$ by Lemma 3.5.3. By exactness, we conclude that $\varprojlim^1 p^i\mathbb{Z} = \hat{\mathbb{Z}}_p/\mathbb{Z}$. **(To do: explain why the image of \mathbb{Z} is the usual subgroup identified with \mathbb{Z} ?)**

Theorem 3.5.8. It is helpful to know that $C = \varprojlim C_i$ is the complex whose j th degree component is $C_j = \varprojlim_i C_{i,j}$ and whose differentials $C_j \xrightarrow{d} C_{j-1}$ are induced by the collection of maps $C_{i,j} \xrightarrow{d^{(i)}} C_{i,j-1}$. **(To do? Prove this statement.)**

We apply the left exact functor \varprojlim to $0 \rightarrow \{Z_i\} \rightarrow \{C_i\} \xrightarrow{d} \{C_i[-1]\}$, and obtain that

$$0 \rightarrow \varprojlim Z_i \rightarrow C \xrightarrow{d} C[-1]$$

is exact. The last map is that induced by the collection of maps $d^{(i)} : C_{i,j} \rightarrow C_{i,j-1}$, so its maps in each degree are the differentials of C . Then by exactness, the degree j part of $\varprojlim Z_i$ is the kernel of the degree j part of $C \xrightarrow{d} C[-1]$, so it really consists of the cycles of C , as claimed.

Next, from the exact sequence $0 \rightarrow \{Z_i\} \rightarrow \{C_i\} \xrightarrow{d} \{B_i[-1]\} \rightarrow 0$, we get the long exact sequence

$$0 \longrightarrow Z \longrightarrow C \xrightarrow{d} \varprojlim B_i[-1] \longrightarrow \varprojlim^1 Z_i \longrightarrow 0 \longrightarrow \varprojlim^1 B_i[-1] \longrightarrow 0$$

where we have $\varprojlim^1 C_i = 0$ by the Mittag-Leffler condition. By exactness, $\varprojlim^1 B_i[-1] = 0$, so the same holds of its shift, $\varprojlim^1 B_i = 0$, as claimed. Also by exactness, the kernel of $\varprojlim^1 B_i[-1] \rightarrow \varprojlim^1 Z_i$ is the image of $C \xrightarrow{d} \varprojlim B_i$, which is B . This gives us that the sequence

$$0 \longrightarrow B \longrightarrow \varprojlim B_i[-1] \longrightarrow \varprojlim^1 Z_i \longrightarrow 0$$

is exact, as claimed. Next, the long exact sequence associated to $0 \rightarrow \{B_i\} \rightarrow \{Z_i\} \rightarrow \{H_*(C_i)\} \rightarrow 0$ is

$$0 \longrightarrow \varprojlim B_i \longrightarrow Z \longrightarrow \varprojlim H_*(C_i) \longrightarrow 0 \longrightarrow \varprojlim^1 Z_i \longrightarrow \varprojlim^1 H_*(C_i) \longrightarrow 0.$$

since $\varprojlim^1 B_i = 0$. This gives the exact sequence $0 \rightarrow \varprojlim B_i \rightarrow Z \rightarrow \varprojlim H_*(C_i) \rightarrow 0$ and the isomorphism $\varprojlim^1 Z_i \cong \varprojlim^1 H_*(C_i)$. The chain of inclusions

$$0 \subseteq B \subseteq \varprojlim^1 B \subseteq Z \subseteq C$$

follows from the beginnings of the various exact sequences we have produced. The quotients of these come from those same exact sequences. The final statement follows from the third isomorphism theorem: $H_*(C) = Z/B$ has the subgroup $\varprojlim^1 H_*(C_i)[1] = \varprojlim B_i/B$, and the quotient of these two is $Z/\varprojlim B_i = \varprojlim H_*(C_i)$. Evaluating the resulting short exact sequence

$$0 \longrightarrow \varprojlim^1 H_*(C_i)[1] \longrightarrow H_*(C) \longrightarrow \varprojlim H_*(C_i) \longrightarrow 0$$

at a given degree gives the result.

Application 3.5.10. These towers of cochain complexes can be a bit difficult to visualize at first, so we go ahead and write out some diagrams. We have the chain of R -modules

$$A_0 \longrightarrow \cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

where each arrow is an inclusion map. The colimit of this chain of modules is the module A . For some R -module B we select an injective resolution

$$0 \longrightarrow B \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

Applying the functors $\text{Hom}(A_i, -)$ to the cocomplex E_\bullet , we get the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(A_0, E_0) & \longrightarrow & \text{Hom}(A_0, E_1) & \longrightarrow & \text{Hom}(A_0, E_2) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \text{Hom}(A_{i-1}, E_0) & \longrightarrow & \text{Hom}(A_{i-1}, E_1) & \longrightarrow & \text{Hom}(A_{i-1}, E_2) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \text{Hom}(A_i, E_0) & \longrightarrow & \text{Hom}(A_i, E_1) & \longrightarrow & \text{Hom}(A_i, E_2) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \text{Hom}(A_{i+1}, E_0) & \longrightarrow & \text{Hom}(A_{i+1}, E_1) & \longrightarrow & \text{Hom}(A_{i+1}, E_2) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

The vertical arrows have reversed since $\text{Hom}(-, E_i)$ is contravariant. Each row is a cochain complex and each collection of arrows from row to row forms a chain map, so this is actually a tower of cochain complexes,

$$\text{Hom}(A_{i+1}, E) \longrightarrow \text{Hom}(A_i, E) \longrightarrow \cdots \longrightarrow \text{Hom}(A_0, E).$$

To get $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \hat{\mathbb{Z}}_p$, we substitute $A_i = \mathbb{Z}/p^i$, $A = \mathbb{Z}_{p^\infty}$, and $q = 1$ to the first sequence:

$$0 \longrightarrow \varprojlim^1 \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, \mathbb{Z}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \longrightarrow \varprojlim \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p^i, \mathbb{Z}) \longrightarrow 0$$

which simplifies to

$$0 \longrightarrow 0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \longrightarrow \varprojlim \mathbb{Z}/p^i \longrightarrow 0$$

since $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, \mathbb{Z}) = 0$ and $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p^i, \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}/p^i, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/p^i$. Therefore $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \varprojlim \mathbb{Z}/p^i = \hat{\mathbb{Z}}_p$.

Application 3.5.11. $\text{Tot}^\Pi(C)$ is intended.

2.4. Chapter 4.

2.4.1. Section 4.1.

2.4.2. *Section 4.2.*

Proposition 4.2.6. R and $\text{Hom}_k(R, k)$ have the same dimension since $\text{Hom}_k(R, k)$ is the dual space to R , and R is finite dimensional.

Vista 4.2.7. Not every quasi-Frobenius ring is a Gorenstein ring since some quasi-Frobenius rings are non-commutative. For example, $k[G]$, for any non-abelian group G .

Lemma 4.2.8. The equality $bR = abR + (1 - a)bR$ follows from the following result: if $e \in R$ is idempotent, and M is an R -module, then $M = eM \oplus (1 - e)M$. Proof:

Every element $x \in M$ may be written as $ex + (1 - e)x$, so $eM + (1 - e)M = M$. To see that the sum is direct, suppose $ex = (1 - e)y \in eM \cap (1 - e)M$. Then, multiplying by e ,

$$ex = e^2x = e(1 - e)y = (e - e^2)y = (e - e)y = 0.$$

Since $eM + (1 - e)M = M$ and $eM \cap (1 - e)M = 0$, we have that $M = eM \oplus (1 - e)M$.

2.4.3. *Section 4.3.*

Theorem 4.3.3. To show that A cannot be projective, note that A would then be a direct summand of $\bigoplus_I R$, but x acts as a nonzerodivisor on $\bigoplus_I R$, contradicting that x annihilates A .

To justify the second sentence, notice first that

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/x \longrightarrow 0$$

is a projective resolution of R -modules, so R/x has projective dimension less than or equal to 1. On the other hand, $x(R/x) = 0$, so R/x cannot be a projective module, for the same reason that A cannot, so R/x has projective dimension 1. Then by the general change of rings theorem 4.3.1,

$$0 < \text{pd}_R(A) \leq \text{pd}_{R/x}(A) + \text{pd}_R(R/x) = 1,$$

so $\text{pd}_R(A) = 1$.

We have that $\text{pd}_R(A) = \text{pd}_{R/x}(M) + 1$ by dimension shifting, and the induction is on $\text{pd}_{R/x}(A)$. The next sentence comes from Exercise 4.1.2, plus applying the general change of rings theorem to show $\text{pd}_R(P) \leq \text{pd}_{R/x}(P) + \text{pd}_R(R/x) = 1$.

Theorem 4.3.7. It is claimed that $\text{pd}_T(M) \leq 1 + \text{pd}_T(T \otimes_R U(M))$. This follows from exercise 4.1.2, as follows:

$$\text{pd}_T(T \otimes_R U(M)) \leq \max\{\text{pd}_T(T \otimes_R U(M)), \text{pd}_T(M)\}$$

with equality unless $\text{pd}_T(M) = \text{pd}_T(T \otimes_R U(M)) + 1$. If so, then trivially

$$\text{pd}_T(M) \leq \text{pd}_T(T \otimes_R U(M)) + 1.$$

Otherwise, $\text{pd}_T(M) \leq \text{pd}_T(T \otimes_R U(M))$, so $\text{pd}_T(M) \leq \text{pd}_T(T \otimes_R U(M)) + 1$.

Proposition 4.3.11. $R^n/\mathfrak{m}R^n = k^n$ for obvious reasons, and $P/\mathfrak{m}P \cong k^n$ by the hypothesis that $P/\mathfrak{m}P$ is generated by n elements. Now

$$\frac{P}{\mathfrak{m}P} \cong k^n = \frac{R}{\mathfrak{m}R} = \frac{P \oplus \ker \epsilon}{\mathfrak{m}(P \oplus \ker \epsilon)} = \frac{P}{\mathfrak{m}P} \oplus \frac{\ker \epsilon}{\mathfrak{m} \ker \epsilon}$$

so $\ker \epsilon / \mathfrak{m} \ker \epsilon = 0$ (this works since k^n is a finite dimensional vector space), whence $\ker \epsilon \subseteq \mathfrak{m}R$.

2.4.4. *Section 4.4.*

Proposition 4.4.5. The Krull intersection theorem (see [2, Corollary 5.4]) tells us that $\cap x^n R = 0$. This is useful since 0 is a prime ideal if and only if R is a domain.

There are finitely many prime ideals of R by a theorem of Noether (see [2, Exercise 1.2]). We include a statement and a proof since I should do the exercise anyway:

If R is Noetherian, and $I \subseteq R$ is an ideal then among the primes of R containing I , there are only finitely many that are minimal with respect to inclusion.

Write $P(I)$ for the set of minimal primes containing an ideal I of R . Select an ideal I maximal among those so that $|P(I)| = \infty$. I is not prime, since if it were, we would have $P(I) = \{I\}$. Therefore there exist $f, g \in R$ so that $f, g \notin I$ but $fg \in I$.

Consider a prime $\mathfrak{p} \in P(I)$. Since $fg \in I \subseteq \mathfrak{p}$, either $f \in \mathfrak{p}$ so $(I, f) \subseteq \mathfrak{p}$ or else $g \in \mathfrak{p}$ so $(I, g) \subseteq \mathfrak{p}$. Since \mathfrak{p} was a minimal prime containing I , \mathfrak{p} is also a minimal prime containing (I, f) or (I, g) . Then $P(I) \subseteq P((I, f)) \cup P((I, g))$, so either $|P((I, f))| = \infty$ or $|P((I, g))| = \infty$, contradicting the maximality of I . The result follows.

The implication that $\mathfrak{m} \subseteq P_i$ for some i is from “prime avoidance,” see [2, Lemma 3.3]. Since \mathfrak{m} is maximal, we actually have that $\mathfrak{m} = P_i$, and since we have assumed that \mathfrak{m} contains no prime ideals, this shows that there are no proper inclusions of prime ideals. It follows R has dimension 0.

Standard Facts 4.4.6. An **associated prime ideal** of R is a prime ideal of the form $\text{Ann}_R(M)$ for a prime R -module M . A **prime R -module** M is one so that $\text{Ann}_R(M) = \text{Ann}_R(N)$ for any nonzero submodule N of M .

Corollary 4.4.12. The inequality $\sup\{\text{pd}(R/I)\} \leq \text{fd}(R/\mathfrak{m})$ comes from the previous proposition. Each module R/I is finitely generated (actually generated by one element) so the lemma tells us that $\text{pd}(R/I)$ is the maximum integer d so that $\text{Tor}_d^R(R/I, k) \neq 0$. Thus each $\text{pd}(R/I) \leq \text{fd}(k) = \text{fd}(R/\mathfrak{m})$, giving the inequality.

2.4.5. *Section 4.5.*2.5. **Chapter 5.**2.6. **Chapter 6.**2.7. **Chapter 7.**2.8. **Chapter 8.**

2.8.1. *Section 8.1.* Although there are upper and lower indices used throughout this section, it does not appear that there is a convention for which go with simplicial objects and which go with cosimplicial objects. This appears to agree with other sources.

Lemma 8.1.2. $0 \leq i_s \leq \dots \leq i_1 \leq m$ should be replaced by $0 \leq i_s < \dots < i_1 \leq m$ in the statement of the lemma.

To do: explain why the factorization is equal to α

This expression for α is unique since a different factorization into face maps and degeneracies would imply that different elements in the image are skipped or that different elements in the domain are mapped to the same place.

Example 8.1.8. The words “combinational” and “combinatorial” are used interchangeably here and later. According to the errata, the word “combinatorial” was intended, so we will use the name combinatorial in these notes.

2.8.2. *Section 8.2.* We check that $d^2 = 0$:

$$\begin{aligned}
d^2 &= \left(\sum_{i=0}^{n-1} (-1)^i \partial_i \right) \left(\sum_{j=0}^n (-1)^j \partial_j \right) \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^n (-1)^{i+j} \partial_i \partial_j \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^{i+j} \partial_i \partial_j + \sum_i^n \sum_{j=i+1}^n (-1)^{i+j} \partial_i \partial_j \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^{i+j} \partial_i \partial_j + \sum_i^n \sum_{j=i+1}^n (-1)^{i+j} \partial_{j-1} \partial_i \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^{i+j} \partial_i \partial_j - \sum_i^n \sum_{j=i}^{n-1} (-1)^{i+j} \partial_j \partial_i \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^i [(-1)^{i+j} \partial_i \partial_j - (-1)^{i+j} \partial_i \partial_j] \\
&= 0.
\end{aligned}$$

Lemma 8.2.8. We verify a few equalities asserted in the course of the lemma.

Recall that we assume inductively that we have found $g = g_{r-1} \in G_{n+1}$ so that $\partial_i(g) = x_i$ for all $i \leq r-1$, $i \neq k$. In the case that $r \neq k$, we set $u = x_r^{-1} \partial_r(g)$. Then $\partial_i(u) = 1$ for $i < r$ since

$$\begin{aligned}
\partial_i(u) &= \partial_i(x_r)^{-1} \partial_i(\partial_r g) \\
&= \partial_i(x_r)^{-1} \partial_{r-1}(\partial_i(g)) && \text{(simplicial identities)} \\
&= \partial_i(x_r)^{-1} \partial_{r-1}(x_i) && \text{(induction hypothesis)} \\
&= \partial_i(x_r)^{-1} \partial_i(x_r) && \text{(hypothesis)} \\
&= 1.
\end{aligned}$$

This implies $\partial_i(\sigma_r u) = 1$ since $\partial_i(\sigma_r u) = \sigma_{r-1}(\partial_i(u)) = \sigma_{r-1}(1) = 1$, using the simplicial identities. The formula $g_r = g(\sigma_k u)^{-1}$ should be $g_r = g(\sigma_r u)^{-1}$. This satisfies the induction hypothesis since, for $i < r$, $i \neq k$,

$$\begin{aligned}
\partial_i(g_r) &= \partial_i(g(\sigma_r(u))^{-1}) \\
&= x_i \partial_i(\sigma_r(u))^{-1} \\
&= x_i.
\end{aligned}$$

And for $i = r$,

$$\begin{aligned}
 \partial_r(g_r) &= \partial_r(g(\sigma_r(u))^{-1}) \\
 &= \partial_r(g)\partial_r(\sigma_r(u))^{-1} \\
 &= \partial_r(g)u^{-1} && \text{(simplicial identities)} \\
 &= \partial_r(g)(x_r^{-1}\partial_r(g))^{-1} \\
 &= x_r.
 \end{aligned}$$

as required.

2.8.3. Section 8.3.

Lemma 8.2.9. We check that $y = \partial_{n+2}z$ is a homotopy from x' to x'' . First, for $i < n$,

$$\partial_i y = \partial_i(\partial_{n+2}z) = \partial_{n+1}(\partial_i z) = \partial_{n+1}(*) = *.$$

Next,

$$\partial_n(y) = \partial_n(\partial_{n+2}z) = \partial_{n+1}(\partial_n z) = \partial_{n+1}y' = x'.$$

and finally,

$$\partial_{n+1}(y) = \partial_{n+1}(\partial_{n+2}z) = \partial_{n+1}(\partial_{n+1}z) = \partial_{n+1}y'' = x''.$$

3. SOLUTIONS TO EXERCISES

3.1. Chapter 1.

3.2. Chapter 2.

3.2.1. Section 2.1.

3.2.2. Section 2.2.

3.2.3. Section 2.3.

3.2.4. Section 2.4.

Exercise 2.4.5.

3.3. Chapter 3.

3.3.1. Section 3.1.

3.3.2. Section 3.2.

Exercise 3.2.1. Suppose that B is a flat R module and that A is any R -module. Let $P_\bullet \rightarrow A \rightarrow 0$ be a projective resolution of A . Then $P_\bullet \otimes B$ is exact except at $P_0 \otimes B$, since $-\otimes B$ is an exact functor and P_\bullet is exact except at P_0 . Then $\text{Tor}_n^R(A, B) = H_n(P_\bullet \otimes B) = 0$ for $n \neq 0$. This gives that $1 \implies 2$.

The implication $2 \implies 3$ is trivial.

Finally, suppose $\text{Tor}_1^R(A, B) = 0$ for all A . Let $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ be any exact sequence. Then, from the long exact sequence for $\text{Tor}_*^R(-, B)$, we get that

$$\text{Tor}_1^R(C', B) \xrightarrow{\partial} C' \otimes B \longrightarrow C \otimes B \longrightarrow C'' \otimes B \longrightarrow 0$$

is exact. By hypothesis, $\text{Tor}_1^R(C', B) = 0$, so in fact

$$0 \longrightarrow C' \otimes B \longrightarrow C \otimes B \longrightarrow C'' \otimes B \longrightarrow 0$$

is exact. Since $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ was arbitrary, this shows $- \otimes B$ is exact, so B is flat. This gives the implication $3 \implies 1$.

Exercise 3.2.2. Suppose that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of R -modules and A and C are flat R -modules. We take the long exact sequence associated to $\text{Tor}_*^R(-, D)$ for an R -module D and obtain that

$$\cdots \xrightarrow{\partial} \text{Tor}_1^R(A, D) \longrightarrow \text{Tor}_1^R(B, D) \longrightarrow \text{Tor}_1^R(C, D) \longrightarrow \cdots$$

is exact. Since A and C are flat, $\text{Tor}_1^R(A, D) = 0$ and $\text{Tor}_1^R(C, D) = 0$. By exactness, $\text{Tor}_1^R(B, D) = 0$. Since D was arbitrary, B is flat, by the implication $3 \implies 1$ in the preceding exercise.

Exercise 3.2.3. As hinted, we use the projective resolution of k given by

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R \longrightarrow k \rightarrow 0.$$

From the projective resolution, we deduce that k is intended to be an R module by having x and y act on k trivially, and the constants act by the usual multiplication.

Now we compute $\text{Tor}_2^R(k, k)$ as the cohomology at the second location of the complex

$$0 \longrightarrow R \otimes k \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix} \otimes 1} R^2 \otimes k \xrightarrow{\begin{bmatrix} x & y \end{bmatrix} \otimes 1} R \otimes k \longrightarrow 0,$$

which may be rewritten

$$0 \longrightarrow k \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} k \longrightarrow 0.$$

This is the same as the kernel of the map $\begin{bmatrix} -y \\ x \end{bmatrix}$. But x and y both act as zero on k , so this kernel is k . Thus $\text{Tor}_2^R(k, k) = k$.

Next, we establish the isomorphism $\text{Tor}_1^R(I, k) \cong \text{Tor}_2^R(k, k)$ by a long exact sequence argument. Consider the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow k \longrightarrow 0.$$

A portion of the associated long exact sequence under the homological δ -functor $\text{Tor}_*^R(-, k)$ is

$$\text{Tor}_2^R(R, k) \longrightarrow \text{Tor}_2^R(k, k) \longrightarrow \text{Tor}_1^R(I, k) \longrightarrow \text{Tor}_1^R(R, k).$$

Since R is projective, each of the modules on the ends are zero. By exactness, $\text{Tor}_1^R(I, k) \cong \text{Tor}_2^R(k, k)$. Combining this with the previous paragraph $\text{Tor}_1^R(I, k) = k$, so I is not flat, despite being torsionfree.

Exercise 3.2.4. Let $A \rightarrow B \rightarrow C$ be an exact sequence of R -modules. Then since \mathbb{Q}/\mathbb{Z} is an injective abelian group (since \mathbb{Q}/\mathbb{Z} is divisible: see Corollary 2.3.2), $\text{Hom}_{\mathbf{Ab}}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor from \mathbf{Ab} to \mathbf{Ab} . Since the forgetful functor F from $R\text{-mod}$ to \mathbf{Ab} is also exact, the composition $\text{Hom}_{\mathbf{Ab}}(F(-), \mathbb{Q}/\mathbb{Z})$ is an exact functor. The image of $A \rightarrow B \rightarrow C$ under this composition is $A^* \rightarrow B^* \rightarrow C^*$, so $A^* \rightarrow B^* \rightarrow C^*$ is exact.

Conversely, suppose **To do**

3.3.3. Section 3.3.

Exercise 3.3.1. We use the long exact sequence of the functor $\text{Ext}_{\mathbb{Z}}^*(-, \mathbb{Z})$ associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[\frac{1}{p}] \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0,$$

then hijack the calculation given in Example 3.3.3. Since Hom is contravariant in its first input, Ext is a contravariant cohomological δ -functor in its first input, which means that the order of the parts of the short exact sequence is reversed in the long exact sequence and indices on Ext are increasing. The relevant portion of the long exact sequence is

$$\text{Hom}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \longrightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}).$$

First, we show $\text{Hom}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) = 0$. Suppose $1 \mapsto a$ under a homomorphism $\varphi : \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}$. Then $\varphi(\frac{1}{p^n}) \mapsto \frac{a}{p^n}$ for all n ; that is, $\frac{1}{p^n}$ is mapped to an element of \mathbb{Z} whose p^n th multiple is a . This is impossible unless $a = 0$, so $\text{Hom}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) = 0$.

Next, $\text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. By the universal property of free abelian groups, any group homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ is determined by the image of 1, and conversely, the image of 1 may be chosen freely. Then if $\varphi_a : \mathbb{Z} \rightarrow \mathbb{Z}$ denotes the group homomorphism such that $1 \mapsto a$, the map $a \mapsto \varphi_a$ is clearly a group isomorphism $\mathbb{Z} \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z})$.

Next, $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) = \hat{\mathbb{Z}}_p$ by Exercise 3.3.3.

Next, $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}) = 0$, since \mathbb{Z} is projective.

Our exact sequence now looks like

$$0 \longrightarrow \mathbb{Z} \longrightarrow \hat{\mathbb{Z}}_p \longrightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \longrightarrow 0$$

So $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}[\frac{1}{p}], \mathbb{Z})$ is the quotient of $\hat{\mathbb{Z}}_p$ by some subgroup isomorphic to \mathbb{Z} . I have not checked that this subgroup is the usual subgroup identified with \mathbb{Z} .

The next isomorphism we were given to prove, $\hat{\mathbb{Z}}_p/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$, is false. This can be checked by noting that $\hat{\mathbb{Z}}_p/\mathbb{Z}$ has an element of order m where m is relatively prime to p (namely $[\frac{1}{m}]$) while \mathbb{Z}_{p^∞} does not. The error is acknowledged in the errata online.

Exercise 3.3.2. Let $R = \mathbb{Z}/m$ and $B = \mathbb{Z}/p$ with $p \mid m$, so that B is an R -module. We must show that

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{i} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \dots$$

is an infinite periodic injective resolution of B . The first arrow is the inclusion map, p denotes multiplication by $p \pmod{m}$, and m/p denotes multiplication by $m/p \pmod{m}$. Multiplication by p in \mathbb{Z}/m vanishes precisely on $\mathbb{Z}/p = (m/p)\mathbb{Z}/m$ and multiplication by m/p in \mathbb{Z}/m vanishes precisely on $p\mathbb{Z}/m$, so this is an exact sequence. Each of the modules \mathbb{Z}/m are injective by Exercise 2.3.1, so this is actually an injective resolution of B .

To compute $\text{Ext}_{\mathbb{Z}/m}^*(A, \mathbb{Z}/p)$ we take the cohomology of the image of the injective resolution above under the functor $\text{Hom}_{\mathbb{Z}/m}(A, -)$, namely

$$0 \longrightarrow A^* \xrightarrow{p^*} A^* \xrightarrow{(m/p)^*} A^* \xrightarrow{p^*} A^* \xrightarrow{(m/p)^*} \dots$$

where A^* is shorthand for $\text{Hom}_{\mathbb{Z}/m}(A, \mathbb{Z}/m)$ as suggested in the problem statement. Then we get that

$$\text{Ext}_{\mathbb{Z}/m}^n(A, \mathbb{Z}/m) = \begin{cases} A^* & \text{if } n = 0 \\ \ker(m/p)^*/\text{im } p^* & \text{if } n > 0 \text{ is odd} \\ \ker p^*/\text{im } (m/p)^* & \text{if } n > 0 \text{ is even} \end{cases}$$

Consider the case that $A = \mathbb{Z}/p$ and $p^2 \mid m$. Then we may identify $A^* = \text{Hom}_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/m)$ with $(m/p)\mathbb{Z}/m \cong \mathbb{Z}/p$ via the isomorphism $\varphi \mapsto \varphi(1)$. This is injective since each homomorphism is determined by the image of 1 and surjective since each homomorphism must take 1 to $(p/m)\mathbb{Z}/m \subseteq \mathbb{Z}/m$ by order considerations. Now note that p and m/p both act as zero on \mathbb{Z}/p since both are divisible by p . It follows that each kernel in the chain complex is \mathbb{Z}/p and each image is 0. Taking quotients,

$$\text{Ext}_{\mathbb{Z}/m}^n(\mathbb{Z}/p, \mathbb{Z}/m) = \mathbb{Z}/p,$$

for all $n \geq 0$.

3.3.4. Section 3.4.

Exercise 3.4.1. Suppose

$$(8) \quad 0 \longrightarrow \mathbb{Z}/p \longrightarrow X \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

is an extension of \mathbb{Z}/p by \mathbb{Z}/p in **Ab**. We know from finite group theory that any extension of a group A by a group B has order $|A||B|$. In this case, the group X must have order p^2 . By the fundamental theorem of abelian groups, $X \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$ or $X \cong \mathbb{Z}/p^2$. We will show that in each case the extension above is equivalent to one of the given extensions. Then we will show that each of the given extensions are distinct.

Suppose first that $X \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$. Let $(a, b) \in \mathbb{Z}/p \oplus \mathbb{Z}/p$ be the image of 1 under the map $\mathbb{Z}/p \rightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p$ and let (c, d) be a preimage of 1 under the map $\mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow \mathbb{Z}/p$. Then (c, d) and (a, b) form a basis of $\mathbb{Z}/p \oplus \mathbb{Z}/p$, viewing $\mathbb{Z}/p \oplus \mathbb{Z}/p$ as a \mathbb{Z}/p vector space, since $(c, d) \notin \ker\{\mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow \mathbb{Z}/p\} = \text{Span}\{(a, b)\}$ and $\mathbb{Z}/p \oplus \mathbb{Z}/p$ has dimension two. Let $\varphi : \mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p$ be the \mathbb{Z}/p -linear (hence \mathbb{Z} -linear) map that takes (a, b) to $(1, 0)$ and (c, d) to $(0, 1)$. This is an isomorphism since we take a basis to a basis. Then the following diagram commutes, where the first row is the extension (8) and the second row is the usual split extension:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathbb{Z}/p \oplus \mathbb{Z}/p & \longrightarrow & \mathbb{Z}/p \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{i} & \mathbb{Z}/p \oplus \mathbb{Z}/p & \xrightarrow{\pi} & \mathbb{Z}/p \longrightarrow 0 \end{array}$$

Next suppose that $X \cong \mathbb{Z}/p^2$. The image of $\mathbb{Z}/p \rightarrow \mathbb{Z}/p^2$ must be $p\mathbb{Z}/p^2$, as it is the unique subgroup of \mathbb{Z}/p^2 of order p . Then 1 is mapped by $\mathbb{Z}/p \rightarrow \mathbb{Z}/p^2$ to some element of the form ap . Let b be the image of 1 under the map $\mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p$. Let $\varphi : \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p^2$ be the map taking 1 to a^{-1} . This is an isomorphism since a is relatively prime to p . Then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathbb{Z}/p^2 & \longrightarrow & \mathbb{Z}/p \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{p} & \mathbb{Z}/p^2 & \xrightarrow{ab} & \mathbb{Z}/p \longrightarrow 0 \end{array}$$

commutes, where the top row is extension (8) and the bottom row is one of the given extensions, where $i = ab$.

Now we have to show that the given extensions are distinct. Let

$$\xi_i : 0 \longrightarrow \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{i} \mathbb{Z}/p \longrightarrow 0.$$

Since $\mathbb{Z}/p \otimes \mathbb{Z}/p \not\cong \mathbb{Z}/p^2$, we really only have to show that the ξ_i are distinct. Suppose

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{p} & \mathbb{Z}/p^2 & \xrightarrow{i} & \mathbb{Z}/p \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{p} & \mathbb{Z}/p^2 & \xrightarrow{j} & \mathbb{Z}/p \longrightarrow 0 \end{array}$$

is an equivalence of extensions. In order that the right square commute, we must have that $1 \in \mathbb{Z}/p^2$ is mapped by φ to some $k \bmod p^2$ so that $k \equiv ij^{-1} \pmod{p}$. We must also have that px is mapped by φ to px for all $x \in \mathbb{Z}/p$ so that the left square commutes. So $px = kpx \pmod{p}$ for all x . This implies that $k \equiv 1 \pmod{p}$, so $ij^{-1} \equiv 1 \pmod{p}$, so $i \equiv j \pmod{p}$, which completes the proof.

3.3.5. Section 3.5.

Exercise 3.5.1. Suppose that $\{A_i\}$ is a tower of abelian groups with injective maps $A_{i+1} \rightarrow A_i$. We get a topology on $A = A_0$ out of the sets $a + A_i$ ($a \in A$, $i \geq 0$) by using these as a basis for a topology. To see this is a basis, let $a + A_i$ and $b + A_j$ be basic open sets. Then

$$(a + A_i) \cap (b + A_j) = \begin{cases} a + A_i & a \in b + A_j, i \geq j \\ b + A_j & b \in a + A_i, j \geq i \\ \emptyset & \text{else} \end{cases}$$

So we have a basis.

Suppose first that A is Hausdorff. Let a be any nonzero element of A . Since A is Hausdorff, there exists an open set U containing 0 but not containing a . Since the sets $a' + A_i$ form a basis, there is a set $a' + A_i \subseteq U$ containing 0. Since $0 \in a' + A_i$, $a' + A_i = A_i$. Altogether, there is, for any nonzero element $a \in A$, an integer i so that $a \notin A_i$. Therefore $\varprojlim A_i = \bigcap_i A_i = 0$.

Suppose conversely that $\varprojlim A_i = \bigcap_i A_i = 0$. Then, given any two distinct points a, b , choose i so that $a - b \notin A_i$. Then $a + A_i$ and $b + A_i$ are disjoint open neighborhoods of a and b respectively, so A is a Hausdorff topological space.

For the second part, we need a definition of completeness that makes sense in a topological group. We say A is complete if every Cauchy sequence in A converges, where by a Cauchy sequence (a_j) we mean a sequence such that for all $i \geq 0$, there exists an integer N_i so that for all $j, k \geq N_i$, $a_j - a_k \in A_i$. (Thanks to Sebastian Casalaina-Martin for the definition.)

We take the long exact sequence associated to the short exact sequence of towers $0 \rightarrow \{A_i\} \rightarrow \{A\} \rightarrow \{A/A_i\} \rightarrow 0$, which includes

$$\varprojlim A_i \longrightarrow A \longrightarrow \varprojlim A/A_i \longrightarrow \varprojlim^1 A_i \longrightarrow 0$$

where we have used that $\varprojlim A = A$ and that $\varprojlim^1 A = 0$, since the maps $A \rightarrow A$ are onto (Lemma 3.5.3). Now $A \rightarrow \varprojlim A/A_i$ is an isomorphism if and only if $\varprojlim A_i = 0$ and $\varprojlim^1 A_i = 0$. This only shows that if A is Hausdorff, then A is complete if and only if $\varprojlim^1 A_i = 0$.

To complete the proof, we show that if A is Hausdorff, A is complete if and only if the induced map $A \rightarrow \varprojlim A/A_i$ is an isomorphism. (The problem statement is adjusted in the errata.)

First, suppose (a_j) is a Cauchy sequence. For each i , let N_i be as in the definition of Cauchy sequence. Note that then for all $i, j \geq N_i$, $a_j \equiv a_i \pmod{A_i}$. Let $\varphi((a_j))_i = a_{N_i} \pmod{A_i}$. Then $\varphi((a_j))$ is an element of $\varprojlim A/A_i$. Now suppose that $\varphi((a_j)) = \varphi((b_j))$ for two Cauchy sequences (a_j) and (b_j) . Then (a_j) and (b_j) are equivalent Cauchy sequences. Conversely, given $(b_j) \in \varprojlim A/A_i$, we can find a Cauchy sequence (a_j) so that $\varphi((a_j)) = (b_j)$ by choosing each a_j to be a lift of b_j in A . This tells us that $\varprojlim A/A_i$ is the completion of A ; that is, it is the space of Cauchy sequences in A modulo equivalence of Cauchy sequences.

Now the induced map $\varprojlim A \rightarrow \varprojlim A/A_i$ is defined by $(a) \mapsto ([a])$. That is, its image is the space of equivalence classes of Cauchy sequences that are equivalent to a constant sequence. This map is an isomorphism if and only if all Cauchy sequences converge if and only if A is complete.

Exercise 3.5.2. Suppose that $\{A_i\}$ is a tower of finite abelian groups. Denote by $\varphi_{j,k}$ the map $A_j \rightarrow A_k$. Then for any k , the sequence of subgroups $\varphi_{j,k}(A_j)$ is decreasing in j (i.e. $\varphi_{j+1,k}(A_{j+1}) \subseteq \varphi_{j,k}(A_j)$). Since the collection of subgroups is finite, $\varphi_{j,k}(A_j)$ must eventually be constant; that is, $\{A_i\}$ must satisfy the Mittag-Leffler condition. By Proposition 3.5.7, $\varprojlim^1 A_i = 0$.

Similarly, suppose that $\{A_i\}$ is a tower of finite-dimensional vector spaces. Then for any k , the sequence $\varphi_{j,k}(A_j)$ is decreasing in j . In particular, the dimension of $\varphi_{j,k}(A_j)$ is decreasing. Since the dimension is strictly positive, this implies that the dimension of $\varphi_{j,k}(A_j)$ is eventually constant. Then since $\varphi_{j,k}(A_j)$ is also decreasing, this implies that $\varphi_{j,k}(A_j)$ is also eventually constant. Therefore $\{A_i\}$ satisfies the Mittag-Leffler condition, so by the proposition $\varprojlim^1 A_i = 0$.

Exercise 3.5.5. Suppose

$$\begin{array}{ccc} & & A_x \\ & & \downarrow \varphi \\ A_y & \xrightarrow{\psi} & A_z \end{array}$$

is an element of \mathcal{A}^I . So that we can deal with elements, I will let $\mathcal{A} = R\text{-mod}$.

The only nonzero terms of the associated cochain complex are C_0 and C_1 , so the only differential we have to worry about is the differential $d : C_0 \rightarrow C_1$. By definition, the object C_0 is $A_x \times A_y \times A_z$, with the terms corresponding to the chains x, y , and z respectively, and the object C_1 is $A_z \times A_z$ with the first term of the product corresponding to the chain $x < z$ and the other term corresponding to the chain $y < z$. Following definitions again, $d^0 : C_0 \rightarrow C_1$ is defined by $(a, b, c) \mapsto (\varphi(a), \psi(b))$ and $d^1 : C_0 \rightarrow C_1$ is defined by $(a, b, c) \mapsto (c, c)$. All other d^p are zero, so the differential $d : C_0 \rightarrow C_1$ has the formula $d^0 - d^1$ which takes $(a, b, c) \mapsto (\varphi(a) - c, \psi(b) - c)$.

The kernel of d is then

$$\{(a, b, c) \in C_0 \mid \varphi(a) = \varphi(b) = c\}$$

which is isomorphic to

$$\{(a, b) \in A_x \times A_y \mid \varphi(a) = \varphi(b)\},$$

the traditional concrete form given to the pullback of A_x and A_y over A_z . This gives that $H^0(C_*)$ is the pullback, as desired.

Next, since $d : C_1 \rightarrow C_2$ is zero, $H^1(C_*)$ is the cokernel of $d : C_0 \rightarrow C_1$. Since $\Delta = \{(c, c) : c \in A_z\} = d(0 \times 0 \times A_z)$ is contained in $\text{im } d$, we may quotient out by it first. More explicitly, we have the diagram with exact rows

$$\begin{array}{ccccccc} C_0 & \xrightarrow{d} & C_1 & \longrightarrow & H^1(C_*) & \longrightarrow & 0 \\ \downarrow i & & \downarrow j & & \parallel & & \\ A_x \times A_y & \xrightarrow{\bar{d}} & A_z & \longrightarrow & H^1(C_*) & \longrightarrow & 0 \end{array}$$

where $i : (a, b, c) \mapsto (a, b)$, $j : (c, c') \mapsto c - c'$ and $\bar{d} : (a, b) \mapsto (\varphi(a) - \psi(b))$. This commutes since $\ker i = 0 \times 0 \times A_z$, $\ker j = \Delta = d(\ker i)$, and $\Delta \subseteq \ker\{C_1 \rightarrow H^1(C_*)\}$. This makes $H^1(C_*)$ the cokernel of the difference map, as claimed.

3.4. Chapter 4.

3.4.1. Section 4.1.

Exercise 4.1.1.

Exercise 4.1.2. The proof is by taking long exact sequences. We'll tackle the projective dimension part, then claim that the other parts have analogous proofs.

Suppose that $d > \text{pd}(A)$ and $d > \text{pd}(B)$. Then for any R -module D ,

$$\text{Ext}^d(C, D) \longrightarrow \text{Ext}^d(B, D) \longrightarrow \text{Ext}^d(A, D)$$

is exact, and $\text{Ext}^d(C, D) = \text{Ext}^d(A, D) = 0$, so $\text{Ext}^d(B, D) = 0$. By the pd Lemma, $\text{pd}(B) \leq \max\{\text{pd}(A), \text{pd}(C)\}$ as desired. For the other inequality, suppose first that $\infty > \text{pd}(C) > \text{pd}(A) + 1$. By the pd Lemma, there is some R -module D so that $\text{Ext}^{\text{pd}(C)}(C, D) \neq 0$. The long exact sequence associated to $\text{Ext}^*(-, D)$ contains the sequence

$$\text{Ext}^{\text{pd}(C)-1}(A, D) \longrightarrow \text{Ext}^{\text{pd}(C)}(C, D) \longrightarrow \text{Ext}^{\text{pd}(C)}(B, D) \longrightarrow \text{Ext}^{\text{pd}(C)}(A, D).$$

By hypothesis, the terms on the ends are zero, so $\text{Ext}^{\text{pd}(C)}(B, D) \neq 0$, which proves $\text{pd}(B) \geq \text{pd}(C) = \max\{\text{pd}(A), \text{pd}(C)\}$. Next, suppose $\text{pd}(C) < \text{pd}(A) + 1 < \infty$. There is an R -module D so that $\text{Ext}^{\text{pd}(A)}(A, D) \neq 0$. This gives us the exact sequence

$$\text{Ext}^{\text{pd}(A)}(B, D) \longrightarrow \text{Ext}^{\text{pd}(A)}(A, D) \longrightarrow \text{Ext}^{\text{pd}(A)+1}(C, D)$$

By assumption, the term on the right vanishes, so by exactness, $\text{Ext}^{\text{pd}(A)}(B, D) \neq 0$. This proves $\text{pd}(B) \geq \text{pd}(A) = \max\{\text{pd}(A), \text{pd}(C)\}$, when both $\text{pd}(A)$ and $\text{pd}(C)$ are finite.

When $\text{pd}(A)$ is infinite but $\text{pd}(C)$ is not, the long exact sequence associated to $\text{Ext}^*(-, D)$ for any D eventually becomes a sequence of isomorphisms of $\text{Ext}^p(B, D)$ with $\text{Hom}^p(A, D)$. By choosing D_p so that $\text{Hom}^p(A, D_p)$ is nonzero for each p , we get that $\text{pd}(B) = \infty$, as desired. The same argument works in the case that $\text{pd}(A)$ is finite while $\text{pd}(C)$ is infinite, which completes the proof.

When $\text{pd}(A) = \text{pd}(C) = \infty$ (in which case $\text{pd}(C) = \text{pd}(A) + 1$ could be interpreted as true) we need not have equality, as is shown by the short exact sequence $0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0$ in the category of \mathbb{Z}/p^2 modules. Here $\text{pd}(\mathbb{Z}/p) = \infty$ (see Calculation 3.1.6) while $\text{pd}(\mathbb{Z}/p^2) = 0$, since \mathbb{Z}/p^2 is projective.

It's interesting to see that the result cannot be strengthened to include $\text{pd}(C) = \text{pd}(A) + 1$ even with finite projective dimensions, since $0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$ is a counterexample over \mathbb{Z} -modules. More generally, for any non-projective C , we can find a short exact sequence $0 \rightarrow M \rightarrow F \rightarrow C \rightarrow 0$ where F is free, so we have a counterexample for every non-projective object C .

Parts 2 and 3 have exactly analogous proofs. For the second part, we take long exact sequences with respect to $\text{Ext}^*(D, -)$ instead of $\text{Ext}^*(-, D)$, and in the third part we take long exact sequences with respect to $\text{Tor}^*(-, D)$. The short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ is an example that shows the second part's result cannot be strengthened to include the case that $\text{id}(A) = \text{id}(C) + 1$.

Exercise 4.1.3.

- (1) First, choose a projective resolution $P_i \rightarrow A_i$ of minimal length for each i . Then $\bigoplus_i P_i \rightarrow \bigoplus_i A_i$ is a projective resolution of $\bigoplus_i A_i$. The length of $\bigoplus_i P_i$ is the given supremum, so

$$\text{pd}(\bigoplus_i A_i) \leq \sup\{\text{pd}(A_i)\}.$$

To do: other inequality.

- (2)
(3) Choose a sequence of R -modules A_i such that $\text{pd}(A_i) \geq i$ for each $i \in \mathbb{N}$. Then by part 1, $A = \bigoplus_i A_i$ is a module with infinite projective dimension.

3.4.2. Section 4.2.

Exercise 4.2.1. \mathbb{Z}/m is Noetherian because it is finite. To show that \mathbb{Z}/m is injective, we use Baer's criterion (2.3.1). Each ideal of \mathbb{Z}/m is of the form (n) . By order considerations, any map $(n) \rightarrow \mathbb{Z}/m$ must take $n \mapsto k \cdot n$, and since (n) is cyclic this determines the map. This extends to the map $\mathbb{Z}/m \rightarrow \mathbb{Z}/m$ determined by $1 \mapsto k$. Therefore \mathbb{Z}/m is also injective as a module over itself, and \mathbb{Z}/m is quasi-Frobenius.

Exercise 4.2.5. Let $a \in R$. We want to show that there exists $x \in R$ so that $axa = a$. By hypothesis,

$$aR = eR$$

for some idempotent e . From this equality, there is some $x \in R$ so that $ax = e$ and some $y \in R$ so that $a = ey$. Then $axa = eey = ey = a$, as desired.

Exercise 4.2.6.

3.4.3. Section 4.3.

Exercise 4.3.1. (Thanks to Jonathan Wise for the suggestion to use the local criterion for flatness.)

By the first change of rings theorem,

$$\text{pd}_{k[[x_1, \dots, x_n]]}(k) = 1 + \text{pd}_{k[[x_1, \dots, x_{n-1}]]}(k).$$

By induction and noting that $\text{pd}_k(k) = 0$, $\text{pd}(k) = n$, so the global dimension of R is greater than or equal to n .

Since R is Noetherian, the flat and global dimensions of R coincide, so we now show that the flat dimension of M over R is less than n for all R -modules M . By the fd lemma, it is enough to show that $\text{Tor}_{n+1}(M, N) = 0$ for all R -modules N . Assume first that M is

finitely generated. Then $\text{Tor}_{n+1}(M, k) = 0$, since $\text{fd}(k) = \text{pd}(k) = n$. By the local criterion for flatness [2, Theorem 6.8], M is flat. If M is not finitely generated, write M as a colimit of its finitely generated submodules, $\varinjlim M_I$. Then $\text{Tor}_{n+1}(M, N) = \varinjlim \text{Tor}_{n+1}(M_I, N) = 0$. Therefore R has global dimension n .

Exercise 4.3.2. By the first change of rings theorem,

$$\text{pd}_R(A/xA) = 1 + \text{pd}_{R/x}(A/xA).$$

We need to relate $\text{pd}_R(A)$ with $\text{pd}_R(A/xA)$, so we consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{x} A \longrightarrow A/xA \longrightarrow 0$$

By Exercise 4.1.2, $\text{pd}_R(A) \leq \max\{\text{pd}_R(A), \text{pd}_R(A/xA)\}$, with equality unless $\text{pd}_R(A/xA) = 1 + \text{pd}_R(A)$. If $\text{pd}_R(A) + 1 = \text{pd}_R(A/xA)$, then $\text{pd}_R(A) = \text{pd}_{R/x}(A/xA)$, and the theorem holds. Otherwise, we have that $\text{pd}_R(A) \geq \text{pd}_R(A/xA)$, so $\text{pd}_R(A) \geq \text{pd}_{R/x}(A/xA) + 1$, and the theorem holds.

Exercise 4.3.3.

3.4.4. Section 4.4.

Exercise 4.4.1. (Thanks again to Jonathan Wise for noting that the remark from the “Standard Facts” was applicable.)

Since R is a regular local ring, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(R) = d$. By the fourth isomorphism theorem for rings, $R/(x_1, \dots, x_i)R$ is local with maximal ideal $\mathfrak{m}/(x_1, \dots, x_i)R$. Now

$$(\mathfrak{m}/(x_1, \dots, x_i))^2 = \mathfrak{m}^2/(x_1, \dots, x_i)$$

so the quotient in question is

$$\left(\frac{\mathfrak{m}}{(x_1, \dots, x_i)} \right) / \left(\frac{\mathfrak{m}^2}{(x_1, \dots, x_i)} \right) \cong \frac{\mathfrak{m}}{\mathfrak{m}^2 + (x_1, \dots, x_i)} \cong \left(\frac{\mathfrak{m}}{\mathfrak{m}^2} \right) / \left(\frac{(x_1, \dots, x_i)}{\mathfrak{m}^2} \right)$$

which has dimension $d - i$.

It remains to check that the Krull dimension of $R/(x_1, \dots, x_d)R$ is as claimed. By the remark in the “Standard Facts” following this exercise, if x is a nonzerodivisor on R , then R/x has dimension $\dim R - 1$. Inductively, $R/(x_1, \dots, x_d)$ has dimension $d - i$ as desired.

Exercise 4.4.2.

3.4.5. Section 4.5.

Exercise 4.5.1.

- (1) We define the product of $[a] \in \ker d_p / \text{im } d_{p+1}$ and $[b] \in \ker d_q / \text{im } d_{q+1}$ to be the element $[ab] \in \ker d_{p+q} / \text{im } d_{p+q+1}$. To show this is well defined, suppose $a = a' + d(\alpha)$ for some $\alpha \in K_{p+1}$ and $b = b' + d(\beta)$ for some $\beta \in K_{q+1}$. Now

$$ab - a'b' = d(\alpha)b + ad(\beta) + d(\alpha)d(\beta).$$

I claim the right side is equal to $d(\alpha b + a\beta + \alpha d(\beta))$:

$$\begin{aligned} d(\alpha b + a\beta + \alpha d(\beta)) &= d(\alpha)b + d(\alpha)d(b) + d(a)\beta + ad(\beta) + d(\alpha)d(\beta) + \alpha d^2(\beta) \\ &= d(\alpha)b + ad(\beta) + d(\alpha)d(\beta), \end{aligned}$$

where the second equality follows from the assumption that $a, b \in \ker d$ and $d^2 = 0$. This shows that the products of different representatives differ by an element of $\text{im } d_{p+q+1}$, so the product is well defined.

Associativity, distributivity, and graded commutativity come directly from the corresponding properties of K after reduction modulo the image of d .

(2) We will need to assume that R is commutative.

To see $d^2 = 0$, it suffices to check that this holds on basis vectors. Suppose $i_1 < \dots < i_n$. Then

$$\begin{aligned} d^2(e_{i_1} \wedge \dots \wedge e_{i_n}) &= d \left(\sum_{k=1}^n (-1)^{k+1} x_k e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \wedge \dots \wedge e_{i_n} \right) \\ &= \sum_{1 \leq j < k \leq n} (-1)^{j+k+1} x_{i_j} x_{i_k} e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \dots \hat{e}_{i_k} \dots \wedge e_{i_n} \\ &\quad + \sum_{1 \leq k < j \leq n} (-1)^{j+k} x_{i_j} x_{i_k} e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \dots \hat{e}_{i_j} \dots \wedge e_{i_n} \\ &= \sum_{1 \leq j < k \leq n} (-1)^{j+k} (x_{i_k} x_{i_j} - x_{i_j} x_{i_k}) e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \dots \hat{e}_{i_k} \dots \wedge e_{i_n} \\ &= 0. \end{aligned}$$

In the second equality, the first sum is over those summands where a smaller index was removed in the second application of d and the second sum is over those summands where a larger index was removed in the second application of d . The third equality follows by pairing summands containing the same basis vector.

We also need to check that the Leibnitz rule holds. For convenience, we will show first that the formula given for d of a standard basis vector is independent of the order in which it is written as a wedge product of the e_i . For this, it suffices to show that swapping two adjacent elements changes the sign of the result. (This will also show that if some e_i is repeated, the formula for d correctly gives zero.) More concretely, we wish to show that if σ is the transposition $(j, j+1)$

$$\sum_{k=1}^n (-1)^{k+1} x_{i_k} e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \wedge \dots \wedge e_{i_n} = - \sum_{k=1}^n (-1)^{k+1} x_{\sigma(k)} e_{i_{\sigma(1)}} \wedge \dots \wedge \hat{e}_{i_{\sigma(k)}} \wedge \dots \wedge e_{i_{\sigma(n)}}.$$

We take a few deep breaths then compute:

$$\begin{aligned} &\sum_{k=1}^n (-1)^{k+1} x_{\sigma(k)} e_{i_{\sigma(1)}} \wedge \dots \wedge \hat{e}_{i_{\sigma(k)}} \wedge \dots \wedge e_{i_{\sigma(n)}} \\ &= \sum_{k=1}^{j-1} (-1)^{k+1} x_k e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \dots e_{i_{j+1}} \wedge e_{i_j} \dots \wedge e_{i_n} \\ &\quad + (-1)^{j+1} x_{i_{j+1}} e_{i_1} \wedge \dots \wedge \hat{e}_{i_{j+1}} \wedge e_{i_j} \dots e_{i_n} \\ &\quad + (-1)^{j+1+1} x_{i_j} e_{i_1} \wedge \dots \wedge e_{i_{j+1}} \wedge \hat{e}_{i_j} \dots e_{i_n} \\ &\quad + \sum_{k=j+2}^n (-1)^{k+1} x_{i_k} e_{i_1} \wedge \dots \wedge e_{i_{j+1}} \wedge e_{i_j} \dots \hat{e}_{i_k} \dots \wedge e_{i_n} \end{aligned}$$

Noting that the 3rd and 4th lines have additional minus signs and swapping $e_{i_{j+1}}$ and e_{i_j} on the 2nd and 5th lines, we get the desired equality.

We may now check the Leibnitz rule. By linearity, it suffices to check on basis vectors $a = e_{i_1} \wedge \cdots \wedge e_{i_p}$ and $b = e_{j_1} \wedge \cdots \wedge e_{j_q}$, where $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$. Then

$$\begin{aligned} d(a \cdot b) &= d(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}) \\ &= \left(\sum_{k=1}^p (-1)^{k+1} x_{i_k} e_{i_p} \wedge \cdots \wedge \hat{e}_{i_k} \cdots \wedge e_{i_p} \right) \wedge e_{j_1} \wedge \cdots \wedge e_{j_q} \\ &\quad + e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \left(\sum_{k=1}^q (-1)^{p+k+1} x_{j_k} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_k} \cdots \wedge e_{j_q} \right) \\ &= d(a)b + (-1)^p ad(b). \end{aligned}$$

Next, the wedge product functions as a product which respects the grading and is graded-commutative. We have now shown that the Koszul complex $K(\mathbf{x})$ is a graded-commutative DG-algebra.

The desired external product is defined by the bilinear map

$$(a(e_{i_1} \wedge \cdots \wedge e_{i_p}), b(e_{j_1} \wedge \cdots \wedge e_{j_q})) \mapsto a \otimes b(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}).$$

The Koszul homology is a graded-commutative R -algebra by the part 1.

- (3) We note that since each $x_i \in I$ acts as zero, each map in the Koszul complex is the zero map. It follows that the homology is isomorphic to the Koszul complex itself, $\Lambda^*(A^n)$.

Exercise 4.5.2. We reproduce the argument from page 13 in this context to show that $\{H_q(\mathbf{x}, -)\}$ is a homological δ -functor.

If we can show that for each short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the diagram

$$\begin{array}{ccccccc} \Lambda^q A^n / d\Lambda^{q+1} A^n & \longrightarrow & \Lambda^q B^n / d\Lambda^{q+1} B^n & \longrightarrow & \Lambda^q C^n / d\Lambda^{q+1} C^n & \longrightarrow & 0 \\ \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & Z^{q-1} \Lambda^* A^n & \longrightarrow & Z^{q-1} \Lambda^* B^n & \longrightarrow & Z^{q-1} \Lambda^* C^n \end{array}$$

has exact rows, then the Snake Lemma will give the required connecting homomorphisms.

The top row can be obtained from the short exact sequence by tensoring over R with the R -module $\Lambda^q R^n / d\Lambda^{q+1} R^n$. Thus the top row is exact by the right exactness of the tensor product.

For exactness of the bottom row, consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z^{q-1} \Lambda^* A^n & \longrightarrow & Z^{q-1} \Lambda^* B^n & \longrightarrow & Z^{q-1} \Lambda^* C^n \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Lambda^q A^n & \longrightarrow & \Lambda^q B^n & \longrightarrow & \Lambda^q C^n \longrightarrow 0 \end{array}$$

The bottom row of this diagram is exact since it is obtained by tensoring the short exact sequence with $\Lambda^q R^n$, which is a free and therefore flat R -module. The map $Z^{q-1} \Lambda^* A^n \rightarrow Z^{q-1} \Lambda^* B^n$ is injective since it is the restriction of an injective map. We have exactness at $Z^{q-1} \Lambda^* B^n$ by a diagram chase. This completes the proof of the existence of the connecting homomorphism.

For naturality of the connecting homomorphism, we can repeat the argument of Proposition 1.3.4, noting that $H_q(\mathbf{x}, -)$ is a functor, since it is the composition of the homology and $\Lambda^* R^n \otimes -$ functors. This completes the proof that $\{H_q(\mathbf{x}, -)\}$ is a homological δ -functor.

To do: $\{H^q(\mathbf{x}, -)\}$.

Exercise 4.5.3.

Exercise 4.5.4. By Exercise 1.4.3, it suffices to show that there is a chain contraction s of the identity map. Suppose without loss of generality that x_1 is the unit. I claim that the maps $\{s_q\}$ defined by

$$s_q : e_{i_1} \wedge \cdots \wedge e_{i_q} \mapsto x_1^{-1} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_q}$$

is such a chain contraction. We check:

$$\begin{aligned} (ds + sd)(e_{i_1} \wedge \cdots \wedge e_{i_q}) &= d(x_1^{-1} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_q}) + s\left(\sum_{k=1}^q (-1)^{k+1} x_k e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \cdots \wedge e_{i_q}\right) \\ &= e_{i_1} \wedge \cdots \wedge e_{i_q} + \sum_{k=1}^q (-1)^k x_k x_1^{-1} e_1 \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \cdots \wedge e_{i_q} \\ &\quad + \sum_{k=1}^q (-1)^{k+1} x_1^{-1} x_k e_1 \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \cdots \wedge e_{i_q} \\ &= e_{i_1} \wedge \cdots \wedge e_{i_q}. \end{aligned}$$

So $\text{id} = ds + sd$, as desired.

The groups $H_*(\mathbf{x}, A) = H^*(\mathbf{x}, A) = 0$ are zero since split exact sequences are preserved by additive functors, namely $- \otimes A$ and $\text{Hom}(-, A)$.

3.5. Chapter 5.

3.6. Chapter 6.

3.7. Chapter 7.

3.8. Chapter 8.

3.8.1. Section 8.1.

Exercise 8.1.1. Recall that $\epsilon_i : [n-1] \rightarrow [n]$ and $\eta_i : [n+1] \rightarrow [n]$ are defined by

$$\epsilon_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

and

$$\eta_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

To get the idea of what's going on, it helps to draw a picture of each simplicial identity for $n = 4$ or so. Of course, this won't do for a proof, so now we plug and chug. Fix an integer

n greater than or equal to zero, and assume i, j are integers so that $0 \leq i < j \leq n$. Then

$$\begin{aligned} \epsilon_j(\epsilon_i(k)) &= \begin{cases} \epsilon_j(k) & \text{if } k < i \\ \epsilon_j(k+1) & \text{if } k \geq i \end{cases} \\ &= \begin{cases} k & \text{if } k < i \text{ and } k < j \\ k+1 & \text{if } k < i \text{ and } k \geq j \\ k+1 & \text{if } i \leq k \text{ and } k+1 < j \\ k+2 & \text{if } i \leq k \text{ and } k+1 \geq j \end{cases} \\ &= \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i \leq k \leq j \\ k+2 & \text{if } j < k \end{cases} \end{aligned}$$

while

$$\begin{aligned} \epsilon_i(\epsilon_{j-1}(k)) &= \begin{cases} \epsilon_i(k) & \text{if } k < j-1 \\ \epsilon_i(k+1) & \text{if } k \geq j-1 \end{cases} \\ &= \begin{cases} k & \text{if } k < j-1 \text{ and } k < i \\ k+1 & \text{if } k < j-1 \text{ and } k \geq i \\ k+1 & \text{if } k \geq j-1 \text{ and } k+1 < i \\ k+2 & \text{if } k \geq j-1 \text{ and } k+1 \geq i \end{cases} \\ &= \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i \leq k \leq j \\ k+2 & \text{if } j < k \end{cases} \end{aligned}$$

so $\epsilon_j \epsilon_i = \epsilon_i \epsilon_{j-1}$. Intutively, $\epsilon_i \epsilon_{j-1}$ inserts a space at the $j-1$ st spot then the i th spot of $[n]$. If we were to do this in the opposite order, we would have to take into account that the $j-1$ st element of $[n]$ is moves one spot to the right after inserting a space at the i th spot, so $\epsilon_j \epsilon_i = \epsilon_i \epsilon_{j-1}$.

Next, with the same hypotheses,

$$\begin{aligned} \eta_j(\eta_i(k)) &= \begin{cases} \eta_j(k) & \text{if } k \leq i \\ \eta_j(k-1) & \text{if } k > i \end{cases} \\ &= \begin{cases} k & \text{if } k \leq i \text{ and } k \leq j \\ k-1 & \text{if } k \leq i \text{ and } k > j \\ k-1 & \text{if } k > i \text{ and } k-1 \leq j \\ k-2 & \text{if } k > i \text{ and } k-1 > j \end{cases} \\ &= \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } i < k \leq j+1 \\ k-2 & \text{if } j+1 < k \end{cases} \end{aligned}$$

while

$$\begin{aligned}
 \eta_i(\eta_{j+1}(k)) &= \begin{cases} \eta_i(k) & \text{if } k \leq j+1 \\ \eta_i(k-1) & \text{if } k > j+1 \end{cases} \\
 &= \begin{cases} k & \text{if } k \leq j+1 \text{ and } k \leq i \\ k-1 & \text{if } k \leq j+1 \text{ and } k > i \\ k-1 & \text{if } k > j+1 \text{ and } k-1 \leq i \\ k-2 & \text{if } k > j+1 \text{ and } k-1 > i \end{cases} \\
 &= \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } i < k \leq j+1 \\ k-2 & \text{if } j+1 < k \end{cases}
 \end{aligned}$$

so $\eta_j\eta_i = \eta_i\eta_{j+1}$.

If $0 \leq i < j \leq n$, then

$$\begin{aligned}
 \eta_j(\epsilon_i(k)) &= \begin{cases} \eta_j(k) & \text{if } k < i \\ \eta_j(k+1) & \text{if } k \geq i \end{cases} \\
 &= \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i \leq k \text{ and } k+1 \leq j \\ k & \text{if } k+1 > j \end{cases} \\
 &= \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i \leq k < j \\ k & \text{if } k \geq j \end{cases}
 \end{aligned}$$

while

$$\begin{aligned}
 \epsilon_i(\eta_{j-1}(k)) &= \begin{cases} \epsilon_i(k) & \text{if } k \leq j-1 \\ \epsilon_i(k-1) & \text{if } k > j-1 \end{cases} \\
 &= \begin{cases} k & \text{if } k < i \\ k-1 & \text{if } k \leq j-1 \text{ and } k \geq i \\ k & \text{if } k > j-1 \end{cases} \\
 &= \begin{cases} k & \text{if } k < i \\ k-1 & \text{if } i \leq k < j \\ k & \text{if } k \geq j \end{cases}
 \end{aligned}$$

so $\eta_j\epsilon_i = \epsilon_i\eta_{j-1}$ in this case.

If $i = j$ or $i+1 = j$, then

$$\begin{aligned}
 \eta_j(\epsilon_i(k)) &= \begin{cases} \eta_j(k) & \text{if } k < i \\ \eta_j(k+1) & \text{if } k \geq i \end{cases} \\
 &= k
 \end{aligned}$$

Finally, if $j < i$,

$$\begin{aligned}\eta_j(\epsilon_i(k)) &= \begin{cases} \eta_j(k) & \text{if } k < i \\ \eta_j(k+1) & \text{if } k \geq i \end{cases} \\ &= \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } j < k < i \\ k & \text{if } i \leq k \end{cases}\end{aligned}$$

while

$$\begin{aligned}\epsilon_{i-1}(\eta_j(k)) &= \begin{cases} \epsilon_{i-1}(k) & \text{if } k \leq j \\ \epsilon_{i-1}(k-1) & \text{if } k > j \end{cases} \\ &= \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } k > j \text{ and } k-1 < i-1 \\ k & \text{if } k-1 \geq i-1 \end{cases} \\ &= \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } j < k < i \\ k & \text{if } i \leq k \end{cases}\end{aligned}$$

so $\eta_j\epsilon_i = \epsilon_{i-1}\eta_j$ in this case, as required.

Exercise 8.1.2. Suppose that K is an ordered combinatorial simplicial complex consisting of subsets of V . Then given any subset $W = \{v_0, \dots, v_n\} \in K$, with $v_0 < v_1 < \dots < v_n$, we get a corresponding element $(v_0, \dots, v_n) \in SS_n(K)$. This cannot be degenerate, as any degenerate n -simplex has a repeated index, while this has none.

Conversely, if $(v_0, \dots, v_n) \in SS_n(K)$ is nondegenerate, then $\{v_0, \dots, v_n\}$ is in K , by definition. This establishes the one-to-one correspondence.

Exercise 8.1.3. Note that each (v_0, \dots, v_k) in $SS_k(K)$ may be interpreted as a morphism $v : [k] \rightarrow [n]$ in Δ defined by $i \mapsto v_i$. This is a one-to-one correspondence. Recall that given $\alpha : [l] \rightarrow [k]$, we have defined $\alpha_*(v_0, \dots, v_k) = (v_{\alpha(0)}, \dots, v_{\alpha(l)})$. The associated morphism takes $i \mapsto v_{\alpha(i)}$, so α_* acts by precomposition on associated morphisms. That is, $SS(K)$ is isomorphic to the functor $\text{Hom}_\Delta(-, [n])$, and we will regard them as interchangeable from here on out.

Now, to each k and $\alpha \in \text{Hom}_\Delta([k], [n])$, we have an associated continuous map $\alpha^* : \Delta^k \rightarrow \Delta^n$ (from Example 8.1.5, although there it is written with a lower star). These induce, by the universal property of coproducts, a map $\varphi : \coprod_k SS(K) \times \Delta^k \rightarrow \Delta^n$ defined by $(\alpha, x) \mapsto \alpha^*(x)$. Going the other direction, we have the map $\psi : \Delta^n \rightarrow \coprod_k SS(K) \times \Delta^k$ that includes Δ^n into the copy of Δ^n corresponding to $\text{id}_{[n]}$. Clearly, $\varphi \circ \psi = \text{id}_{\Delta^n}$.

Next, I claim that $(\alpha, x) \in SS_k(K) \times \Delta^k$ is equivalent (see 8.1.6) to $(\beta, y) \in SS_l(K) \times \Delta^l$ if and only if $\varphi(\alpha, x) = \varphi(\beta, y)$. Suppose first that $\varphi(\alpha, x) = \varphi(\beta, y)$. Then $\alpha^*(x) = \beta^*(y)$, so

$$(\alpha, x) = (\alpha \circ \text{id}_{[n]}, x) \sim (\text{id}_{[n]}, \alpha^*(x)) = (\text{id}_{[n]}, \beta^*(y)) \sim (\beta \circ \text{id}_{[n]}, y) = (\beta, y).$$

It is clear that $\varphi(\alpha, x) = \varphi(\beta, y)$ is an equivalence relation, so for the other implication it suffices to show that if there is $\gamma : [l] \rightarrow [k]$ so that $(\alpha \circ \gamma, x) = (\beta, \gamma^*(y))$, then $\varphi(\alpha, x) =$

$\varphi(\beta, y)$. For this, we note

$$\varphi(\beta, y) = \beta^*(y) = (\alpha \circ \gamma)^*(y) = \alpha^*(\gamma^*(y)) = \alpha^*(x) = \varphi(\alpha, x).$$

Therefore φ respects the equivalence relation. The universal property of quotients then tells us that φ factors uniquely through the quotient map $\pi : \coprod_k SS_k(K) \times \Delta^k \rightarrow |SS(K)|$ as $\bar{\varphi} : |SS(K)| \rightarrow \Delta^n$. In fact, since points are equivalent if and only if their images under φ agree, $\bar{\varphi}$ is a bijection. It remains to see that its inverse function is continuous. For this, we note that

$$\text{id}_{\Delta^n} = \psi \circ \varphi = \psi \circ \pi \circ \bar{\varphi},$$

so $\psi \circ \pi$ is its $\bar{\varphi}$'s inverse, and $\psi \circ \pi$ is continuous. Altogether, $|SS(K)| \cong \Delta^n$.

Exercise 8.1.4.

Exercise 8.1.5.

Exercise 8.1.6. Considering Lemma 8.1.2 in the case that $\alpha : [n] \rightarrow [m]$ is injective, we get that every morphism in Δ_s is a composition of face maps. Therefore a semi-simplicial object $K : \Delta_s \rightarrow \mathcal{A}$ is determined by its values on objects, $K_n = K([n])$, and its values on face maps, $\partial_i : K_n \rightarrow K_{n-1} = K(\epsilon_i : [n-1] \rightarrow [n])$, for $i = 0, \dots, n$. We know from the simplicial identities that $\partial_i \partial_j = \partial_{j-1} \partial_i$ when $i < j$.

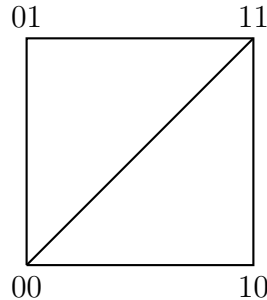
On the other hand, given objects K_n and morphisms ∂_i satisfying $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$, we get a semi-simplicial object since, the relations $\partial_i \partial_j = \partial_{j-1} \partial_i$ determine all relations among the ϵ_i in Δ_s . (**To Do:** This is a bit handwavy. Why is this a complete set of relations?)

3.8.2. Section 8.2.

Exercise 8.2.1. We did this in Exercise 8.1.3.

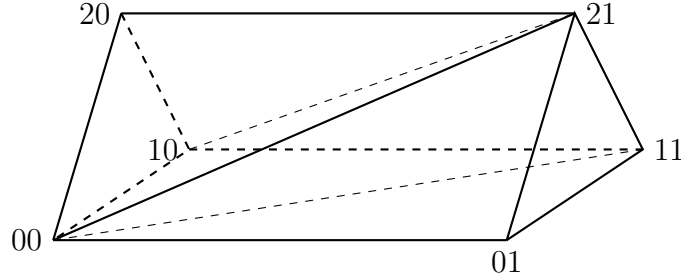
Exercise 8.2.2. By Exercise 8.1.2, it is enough to enumerate the non-degenerate n -simplices of $\Delta[1] \times \Delta[1]$ and $\Delta[2] \times \Delta[1]$. Writing the elements $(v_0, \dots, v_k) \times (w_0, \dots, w_k)$ of each as $((v_0, w_0), (v_1, w_1), \dots, (v_k, w_k))$, we see that the non-degenerate simplices correspond to sequences of pairs of numbers, increasing in each coordinate, without repeating pairs of numbers.

For $\Delta[1] \times \Delta[1]$, we have the non-degenerate 0-simplices $((0, 0))$, $((0, 1))$, $((1, 0))$, and $((1, 1))$, five non-degenerate 1-simplices, $((0, 0), (0, 1))$, $((0, 0), (1, 0))$, $((0, 1), (1, 1))$, $((1, 0), (1, 1))$, and $((0, 0), (1, 1))$, and two non-degenerate 2-simplices, $((0, 0), (0, 1), (1, 1))$ and $((0, 0), (1, 0), (1, 1))$. The geometric realization of the corresponding combinatorial simplicial complex consists of two triangles meeting on one edge.



This is a square, just like $|\Delta[1]| \times |\Delta[1]|$.

For $\Delta[2] \times \Delta[1]$, the list of non-degenerate simplices gets unwieldy. We will just draw a picture:



This is a triangular prism made up of three tetrahedrons, namely one with vertices 00, 01, 11, 21, another with vertices 00, 10, 11, 21, and finally one with vertices 00, 10, 20, 21. $|\Delta[2]| \times |\Delta[1]|$ is also a triangular prism, so $|\Delta[2] \times \Delta[1]| = |\Delta[2]| \times |\Delta[1]|$, as desired.

Exercise 8.2.3. Let $n > 0$. Consider $x_0 = (0, 1)$, $x_1 = (1, 1)$ in $\Delta[n]_1$. We have that $\partial_0 x_1 = 1 = \partial_0 x_0$. In order that $y \in \Delta[n]_2$ satisfies $\partial_0 y = x_0$ and $\partial_1 y = x_1$, we would need $y = (1, 0, 1)$. But this is not an increasing sequence, so $\Delta[n]$ is not fibrant.

To Do: Prove that a non-constant fibrant simplicial set must have non-degenerate n -cells for all n .

Exercise 8.2.4.

Exercise 8.2.5.

3.8.3. *Section 8.3.*

Exercise 8.2.6. Take G_0 to act on G by $g \cdot h = \sigma_0^n(g)h$ for $g \in G_0, h \in G_n$.

4. ADDITIONAL MATH

4.1. Short Exact Sequences.

([3, p.132] and [1] provided the necessary hints to complete this section.)

Definition 1. A **split exact sequence** in an abelian category \mathcal{A} is a sequence

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

such that there exist maps $\varphi : A'' \rightarrow A$ and $\psi : A \rightarrow A'$ so that

- (1) $g \circ \varphi = \text{id}_{A''}$,
- (2) $\psi \circ f = \text{id}_{A'}$,
- (3) $g \circ f = 0$,
- (4) $\psi \circ \varphi = 0$,
- (5) $f \circ \psi + \varphi \circ g = \text{id}_A$.

(I don't think anyone else defines a split exact sequence this way.)

The next proposition shows that a split exact sequence can be thought of as one which is “exact when read forwards as well as backwards,” or as a consequence, as a sequence in which both A' and A'' are subobjects as well as quotient objects of A .

Proposition 2. If a sequence of R -modules

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

is split exact, and we fix associated maps φ and ψ , then both $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$ and $0 \rightarrow A'' \xrightarrow{\varphi} A \xrightarrow{\psi} A' \rightarrow 0$ are exact.

Proof. The map g is an epimorphism since $g \circ \varphi = \text{id}_{A''}$, so the first sequence is exact at A'' . Similarly, f is a monomorphism since $\psi \circ f = \text{id}_{A'}$, so the first sequence is exact at A' .

Next, $\ker g \supseteq \text{im } f$, since $g \circ f = 0$. For the other inclusion, let $x \in \ker g$. Then

$$\begin{aligned} x &= (f \circ \psi + \varphi \circ g)(x) \\ &= f(\psi(x)) + \varphi(0) = f(\psi(x)). \end{aligned}$$

so $x \in \text{im } f$. This proves the first sequence is exact at A .

The second sequence is exact by a symmetric argument, since the definition is symmetric under the exchange of g with ψ and f with φ . \square

Other definitions of split exact sequences are equivalent to the one we have given by the following proposition:

Proposition 3. Let $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$ be a short exact sequence of R -modules. The following are equivalent:

- (1) $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$ is a split exact sequence.
- (2) There exists a map $\varphi : A'' \rightarrow A$ so that $g \circ \varphi = \text{id}_{A''}$.
- (3) There exists a map $\psi : A \rightarrow A'$ so that $\psi \circ f = \text{id}_{A'}$.
- (4) There exists an isomorphism $\tau : A \rightarrow A' \oplus A''$ so that $\tau \circ f$ is the usual inclusion $a' \mapsto (a', 0)$ and $g \circ \tau^{-1}$ is the usual projection $(a', a'') \mapsto a''$.

Proof. The implications (1) \implies (2) and (1) \implies (3) are trivial. We shall show (3) \implies (4), (4) \implies (1), and leave (2) \implies (4) as an adjustment of the proof that (3) \implies (4).

((3) \implies (4)) Suppose that $\psi : A \rightarrow A'$ is a map so that $\psi \circ f = \text{id}_{A'}$. Note that for any $x \in A$, we have that $x - f(\psi(x)) \in \ker \psi$, since

$$\psi(x - f(\psi(x))) = \psi(x) - (\psi \circ f)\psi(x) = \psi(x) - \psi(x) = 0.$$

Thus, any $x \in A$ can be written as a sum $x = f(\psi(x)) + (x - f(\psi(x)))$. This implies $A = \text{im } f + \ker \psi$. Suppose $x \in \text{im } f$ and $x \in \ker \psi$. Then $x = f(y)$ for some y in A' . It follows

$$x = f(y) = f(\psi(f(y))) = f(\psi(x)) = 0,$$

so in fact $A = \text{im } f \oplus \ker \psi$.

Let $\tau : A = \text{im } f \oplus \ker \psi \rightarrow A' \oplus A''$ be defined by $f(y) + w \mapsto (y, g(w))$, where $w \in \ker \psi$. We must show τ is an isomorphism. For injectivity suppose $x = f(y) + w \in A$ and $x' = f(y') + w' \in A$ are elements of A so that $\tau(x) = \tau(x')$. Since f is an injection, $y = y'$. By exactness, $w - w' \in \text{im } f$, so $w - w' = 0$. For surjectivity, let $(a', a'') \in A' \oplus A''$. Then for any lift $w \in A$ so that $g(w) = a''$, $f(a') + (w - f(\psi(w)))$ is a preimage of (a', a'') .

Clearly, $\tau \circ f$ is the usual inclusion. Next, $(g \circ \tau^{-1})(a', a'') = g(f(a') + (w - f(\psi(w)))) = g(w) = a''$ is the usual projection, where w is any preimage of a'' under g .

((4) \implies (1)) Suppose we have such an isomorphism τ . Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' \longrightarrow 0 \\
 & & \parallel & & \downarrow \tau & & \parallel \\
 0 & \longrightarrow & A' & \xrightleftharpoons[\psi']{i} & A' \oplus A'' & \xrightleftharpoons[\varphi']{p} & A'' \longrightarrow 0
 \end{array}$$

where $i : a' \mapsto (a', 0)$, $p : (a', a'') \mapsto a''$, $\psi' : (a', a'') \mapsto a'$, and $\varphi' : a'' \mapsto (0, a'')$. Let $\psi = \psi' \circ \tau$ and let $\varphi = \tau^{-1} \circ \varphi'$. Then

$$\psi \circ f = \psi' \circ \tau \circ f = \text{id}_{A'}$$

and

$$g \circ \varphi = g \circ \tau^{-1} \circ \varphi' = \text{id}_{A''}$$

by the commutativity of the diagram. This gives us equations (1) and (2) of the definition. The next equation, $g \circ f = 0$, holds by the hypothesis that $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is exact. Next,

$$\psi \circ \psi = \psi' \circ \tau \circ \tau^{-1} \circ \varphi' = \psi' \circ \varphi' = 0,$$

so we have the fourth equation. Finally,

$$\begin{aligned}
 f \circ \psi + \varphi \circ g &= f \circ \psi' \circ \tau + \tau^{-1} \circ \varphi' \circ g \\
 &= \tau^{-1} \circ (i \circ \psi' + \varphi' \circ p) \circ \tau \\
 &= \tau^{-1} \circ \text{id}_{A' \oplus A''} \circ \tau^{-1} \\
 &= \text{id}_A.
 \end{aligned}$$

Therefore $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$ is split exact. \square

Our definition of short exact sequence makes the following theorem easy.

Theorem 4. The image of a short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ under an additive functor to an abelian category is a short exact sequence.

Proof. Let $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$ be a short exact sequence in an abelian category \mathcal{A} , with associated maps $\varphi : A'' \rightarrow A$ and $\psi : A \rightarrow A'$. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Then $0 \rightarrow F(A') \xrightarrow{Ff} F(A) \xrightarrow{Fg} F(A'') \rightarrow 0$ is a short exact sequence since equations (1)-(5) of the definition are preserved by F . \square

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