$\hat{}$o
$(X_n)_{n\geq 0}$
d
$\sigma^2 I_d$
d
$\sigma^2 >$
0
$X_1 =$
$(X_1^1, \dots, X_1^d)$
$^i_1 =$
$0, X_1^i X_1^j =$
$\sigma^2 \delta_{ij}, 1 \leq$
$i, j \leq$
$d.$
X_n
$n.$
$S_n =$
$X_1 +$
$\dots +$
X_n
$^{(n)}_t =$
$\frac{1}{\sqrt{n}} S_{nt}, t \geq$
$0.$
??
$\vdots$
1
d
$t >$
0
$x \in^d$
$_t(x) =$
$\frac{1}{(2\pi t)^{d/2}} \exp -\frac{x^2}{2t}, whichisthedensityoftheGaussiandistribution$
$^d)$
tI_d
??
$(m, 0)$
m
$0 =$
$t_1 <$
$t_2 <$
$\dots <$
t_k
$B^{(n)}$
$t_1, \dots, t_k$
$n \to$
∞
f
$x_0 =$
0
$t_0 =$
0
$^{(n)}_{t_1}, \dots, B^{(n)}_{t_k} \to$
$\int_{(^d)^k} f(x_1, \dots, x_k) \prod_{1 \leq i \leq k} p_{\sigma^2(t_i - t_{i-1})}(x_i -$
$x_{i-1}) dx_i as n \to$
$\infty.$
$(B^{(n)}_{t_1}, \dots, B^{(n)}_{t_k})$
$(G_1, G_2, \dots, G_k)$
$(G_1, G_2 -$
$G_1, \dots, G_k -$
$G_{k-1}) =$
$(N_1, \dots, N_k)$
$\sigma^2(t_i -$
$t_{i-1}) I_d$
$(B^{(n)}_{t_i} -$
$B^{(n)}_{t_{i-1}}, 1 \leq$
$i \leq$
$k)$
$(N_i)_{1 \leq i \leq k}$
N_i
$N_i \sim$
$(0, \sigma^2(t_i -$
$t_{i-1}) I_d)$
$B^{(n)}_t$
$(B^{(n)}_t) \to$
$(0, \sigma^2 t I_d)$
$B^{(n)}$
B
2
$B^{(n)}_{t_1}, B^{(n)}_{t_2} - B^{(n)}_{t_1}, \dots, B^{(n)}_{t_k} - B^{(n)}_{t_{k-1}}$
$1 \leq$
$i \leq$
k
$^{(n)}_{t_i} -$
$B^{(n)}_{t_{i-1}} =$

$(\xi_j)_{1\leq j\leq k}$
$B^{(n)}_{t_j}-$
$B^{(n)}_{t_{j-1}}$
$\exp i\sum_{j=1}^k \xi_j B^{(n)}_{t_j} - B^{(n)}_{t_{j-1}} =$
$\prod_{j=1}^k \exp i\xi_j B^{(n)}_{t_j} - B^{(n)}_{t_{j-1}} (by Theorem ??)$
$\to$
$\prod_{j=1}^k \exp(i\xi_j(G_j - G_{j-1}))(by Theorem ??)$
$=$
$\exp i\sum_{j=1}^k \xi_j(G_j - G_{j-1})(by Theorem ??). as we assume_j -$
G_{j-1}
??
$B^{(n)}$
t_i
G_i
??
f
$_{t_1}, \ldots, B_{t_k}) \to$
$\int_{(^d)^k} f(x_1,\ldots,x_k)\prod_{i=1}^k p_{\sigma^2(t_i-t_{i-1})}(x_i -$
$x_{i-1})dx_1\ldots dx_k as \to$
∞
d
$(B_t)_{t\geq 0}$
$B_0 =$
0
$0 =$
$t_0 \leq$
$t_1 \leq$
$t_2 \leq$
$\ldots \leq$
t_k
$(B_{t_1} -$
$B_{t_0}, B_{t_2} -$
$B_{t_1}, \ldots, B_{t_k} -$
$B_{t_{k-1}})$
$t, s \geq$
0
$B_{t+s} -$
B_t
sI_d
$B_0 =$
0
d
$(B_t)_{t\geq 0}$
$0 =$
$t_0 \leq$
$t_1 \leq$
$t_2 \leq$
$\ldots \leq$
t_k
$(B_{t_1} -$
$B_{t_0}, B_{t_2} -$
$B_{t_1}, \ldots, B_{t_k} -$
$B_{t_{k-1}})$
$t, s \geq$
0
$B_{t+s} -$
B_t
sI_d
??
??
$d =$
1
$(B_t)_{t\in[0,1]}$
?
$_0 =$
$\{0, 1\}$
$_n =$
$\{k2^{-n}, 0 \leq$
$k <$
$2^n\}$
$n \geq$
1
$\bigcup_{n\geq 0}{}_n$
$[0, 1]$
$(\Omega, ,)$
$(Z_d)_{d\in D}$
$(0, 1)$
$X_1, X_2, \ldots$
$X_1, X_2, \ldots$
$(X_i, X_j) =$
$X_iX_j =$
0
$i \neq$
j^3
d^-
d^+
d

$\lim_{n\to\infty} {}_A f B^*_{t_1}, \ldots, B^*_{t_m}$
??
${}_A f B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_m} =$
$\lim_{n\to\infty} {}_A f B^*_{t_1}, \ldots, B^*_{t_m}$
$=$
$A \lim_{n\to\infty} f B^*_{t_1}, \ldots, B^*_{t_m} =$
$A f B^{(T)}_{t_1}, \ldots, B^{(T)}_{t_m}.$
B
T_+
??
$(T =$
$\infty) >$
0
A
$A \cap$
$T < \infty$
$T \leq \infty$
$\tau =$
$\inf t \geq 0 : B_t = \max_{0\leq s\leq 1} B_s$
τ
$\tau <$
1
$B_{t+\tau} -$
B_τ
??
T
$(B_t)_{t\geq 0}$
$(B_t)_{t\geq 0}$
$t =$
$B_{tt\leq T} +$
$(2B_T -$
$B_t)_{t>T} is also a standard Brownian motion and we call it Brownian motion reflected at.$
??
$B^{(T)} =$
$(B_{T+t} -$
$B_T)_{t\geq 0}$
$(B_t)_{0\leq t\leq T}$
$-B^{(T)} =$
$(B_T -$
$B_{t+T})_{t\geq 0}$
$(B_t)_{0\leq t\leq T}$
$((B_t)_{0\leq t\leq T}, B^{(T)})$
$((B_t)_{0\leq t\leq T}, -B^{(T)})$
T
X
Y
$\Psi(X,Y)(t) =$
$X_{tt\leq T} +$
$(X_T +$
$Y_{t-T})_{t>T}.$
Ψ_T
B
$B^{(T)}$
B
B
$-B^{(T)}$
B
σ
$C[0,\infty)$
$[0,\infty)$
Ψ_T
$(C[0,\infty) \times$
$C[0,\infty), \otimes)$
$(C[0,\infty),)$
T
B
B
$(B_t)_{t\geq 0}$
$a >$
0
$b \leq$
a
$t \geq$
0
$_t >$
$a, B_t \leq$
$b) =$
$(B_t \geq$
$2a -$
$b) where_t =$
$\sup_{0\leq s\leq t} B_s$
$T_a =$
$\inf\{t \geq$
$0 :$
$B_t \geq$
$a\}$
9
B_t
$[a,\infty)$
$a >$
0
T

$\frac{4}{\sqrt{2\pi\sigma^2 t}} \sum_{k=1}^{\infty} (-1)^{k+1} (2k-1) \exp - \frac{(2k-1)^2 a^2}{2\sigma^2 t}, W_t = \sigma \sqrt{\frac{\pi t}{2}}.$

??

$y > 0, z < 0$

$_{X_t}(x, Y_t \leq y, Z_t \geq z) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \exp - \frac{(x+2kz-2ky)^2}{2\sigma^2 t} - \exp - \frac{(x-2kz+2(k-1)y)^2}{2\sigma^2 t}.(*)$

$_{X_t,Z_t}(x, Y_t \leq y, z)$

$= -\frac{1}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \frac{-2k(x-2k(y-z))}{\sigma^2 t} \exp - \frac{(x-2k(y-z))^2}{2\sigma^2 t} - \frac{2k(x-2y+2k(y-z))}{\sigma^2 t} \exp - \frac{(x-2y+2k(y-z))^2}{2\sigma^2 t}$

$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \frac{2k(x-2k(y-z))}{\sigma^2 t} \exp - \frac{(x-2k(y-z))^2}{2\sigma^2 t} + \frac{2k(x-2y+2k(y-z))}{\sigma^2 t} \exp - \frac{(x-2y+2k(y-z))^2}{2\sigma^2 t}$

$_{X_t,Y_t,Z_t}(x, y, z) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \frac{4k^2(x-2k(y-z))^2}{\sigma^4 t^2} - \frac{4k^2}{\sigma^2 t} \exp - \frac{(x-2k(y-z))^2}{2\sigma^2 t}$

$+ \frac{1}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \frac{4k(k-1)}{\sigma^2 t} - \frac{4k(k-1)(x-2y+2k(y-z))^2}{\sigma^4 t^2} \exp - \frac{(x-2y+2k(y-z))^2}{2\sigma^2 t}$

$= 4 \sum_{k=-\infty}^{\infty} k^2 \phi x - 2k(y-z) - k(k-1) \phi(x - 2y + 2k(y-z). where \phi(x) = \frac{x^2 - \sigma^2 t}{\sigma^5 \sqrt{2\pi t^5}} \exp - \frac{x^2}{2\sigma^2 t}$

$*$

$t \leq a = \int_{-a}^{a} \frac{1}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \exp - \frac{(x-4ka)^2}{2\sigma^2 t} - \exp - \frac{(x+2(2k-1)a)^2}{2\sigma^2 t} dx.$

a

$_{W_t}(a) = \frac{4}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} \int_{-a}^{a} \frac{2k(x+4ka)}{2\sigma^2 t} \exp - \frac{(x+4ka)^2}{2\sigma^2 t} + \frac{(2k-1)(x+2(2k-1)a)}{2\sigma^2 t} \exp - \frac{(x+2(2k-1)a)^2}{2\sigma^2 t} dx$

$= \frac{2}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} 2k \exp - \frac{(4k+1)^2 a^2}{2\sigma^2 t} - 2k \exp - \frac{(4k-1)^2 a^2}{2\sigma^2 t}$

$- \frac{2}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} (2k-1) \exp - \frac{(4k-1)^2 a^2}{2\sigma^2 t} + (2k-1) \exp - \frac{(4k-3)^2 a^2}{2\sigma^2 t}$

$= \frac{2}{\sqrt{2\pi\sigma^2 t}} \sum_{k=-\infty}^{\infty} (4k+1) \exp - \frac{(4k+1)^2 a^2}{2\sigma^2 t} - (4k-1) \exp - \frac{(4k-1)^2 a^2}{2\sigma^2 t}$

$= \frac{4}{\sqrt{2\pi\sigma^2 t}} \sum_{k=1}^{\infty} (-1)^{k+1} (2k-1) \exp - \frac{(2k-1)^2 a^2}{2\sigma^2 t}.$

$_t = \int_0^{\infty} \frac{4}{\sqrt{2\pi\sigma^2 t}} \sum_{k=1}^{\infty} (-1)^{k+1} (2k-1) a \exp - \frac{(2k-1)^2 a^2}{2\sigma^2 t} da$

$= \frac{4}{\sqrt{2\pi\sigma^2 t}} \sum_{k=1}^{\infty} (-1)^{k+1} (2k-1) \int_0^{\infty} a \exp - \frac{(2k-1)^2 a^2}{2\sigma^2 t} da$

$= \frac{4}{\sqrt{2\pi\sigma^2 t}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sigma^2 t}{2k-1} = 4\sigma \sqrt{\frac{t}{2\pi}}. \frac{\pi}{4} = \sigma \sqrt{\frac{\pi t}{2}}.$

??

B

$B_t \sim (0, \sigma^2 t)$[13]

$S_t := \sup_{0 \leq s \leq t} B_s$

$I_t := \inf \quad B$

$\infty) =$
1.
??
T_{n^3}
$_0(A_n^c) =_0$
$B_t \leq n for some t \geq S_{n^3} =_0$
$B_{t+S_{n^3}} - B_{S_{n^3}} + B_{S_{n^3}} \leq n for some t \geq 0$
$=$
$\int_3 {}_0 B_t^{(S_{n^3})} + y \leq n for some t \geq 0_0 B_{S_{n^3}} \in dy$
$=$
$\int_3 {}_y B_t \leq n for some t \geq 0_0 B_{S_{n^3}} \in dy =$
$\int_3 {}_{n^3} B_t \leq n for some t \geq 0_0 B_{S_{n^3}} \in dy$
$=_{n^3}$
$B_t \leq n for some t \geq 0 \int_3 {}_0 B_{S_{n^3}} \in dy =_{n^3}$
$B_t \leq n for some t \geq 0$
$=_{n^3}$
$T_n < \infty =$
$\frac{n}{n^3} =$
$\frac{1}{n^2} by (\dagger\dagger). Since the_0(A_n^c)$
??
A_n
0
$B_t \to$
∞
$t \to$
∞
0
B
$t \geq$
0
$n \geq$
1
$\Delta_n :=$
$0 = t_0(n) < t_1(n) < \dots t_{m_n}(n) = t be a subdivision of$
$, such that \eta_n :=$
$\max_{1 \leq i \leq m_n} t_i(n) - t_{i-1}(n)$
$\lim_{n\to\infty} \sum_{i=1}^{m_n} B_{t_i} - B_{t_{i-1}}{}^2 =$
$t in^2(\Omega, ,).$
$\sum_{i=1}^{m_n} B_{t_i} - B_{t_{i-1}}{}^2 \in^2$
$(\Omega, ,)$
$\sum_{i=1}^{m_n} (B_{t_i} -$
$B_{t_{i-1}})^2 -$
$t^2 \to$
$0 as \to$
∞
??
$\sum_{i=1}^{m_n} (B_{t_i} -$
$B_{t_{i-1}})^2 -$
$t^2 =$
$\sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^{2\,2} - 2t \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 + t^2$
$=$
$\sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^4 + \sum_{i\neq j}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 (B_{t_j} - B_{t_{j-1}})^2 - 2t \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 + t^2$
$=$
$3 \sum_{i=1}^{m_n} t_i - t_{i-1}{}^2 +$
$\sum_{i\neq j}^{m_n} (t_i -$
$t_{i-1})(t_j -$
$t_{j-1}) -$
$2t \sum_{i=1}^{m_n} (t_i -$
$t_{i-1}) +$
t^2
$=$
$2 \sum_{i=1}^{m_n} t_i - t_{i-1}{}^2 +$
$\sum_{i=1}^{m_n} t_i - t_{i-1}{}^2 -$
$2t^2 +$
t^2
$=$
$2 \sum_{i=1}^{m_n} (t_i -$
$t_{i-1})^2 \leq$
$2\eta_n \sum_{i=1}^{m_n} (t_i -$
$t_{i-1}) =$
$2t\eta_n \to$
$0 as \to$
∞
B
$t \geq$
0
$n \geq$
1
$^n_t :=$
$\sum_{k=0}^{2^n t - 1} B_{(k+1)2^{-n}} - B_{k2^{-n}}{}^2 for dyadic subdivision.^n_t$
B
n
B
$t \geq$
0
$n \geq$
1
$^n_t \to$
$t a.s..$
$t_{m_n}(n) =$
$2^n t 2^{-n}$