The Mathematical Exploration of Lagrange Points

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1 Introduction

In this day and age, the aerospace industry has become a vital component of the global economy. In a publication by the Organisation for Economic Co-operation and Development, the space economy plays a key role in the globalization and digitization of the modern world, with space activities making crucial contributions to the social, economic, and scientific aspects of society.[1] With an increasing investment for spaceflight, there is future need to develop infrastructure that can facilitate the growing space sector, which includes the necessity to understand the nature of orbital mechanics.

Since the longest time, I've always been interested in the development of spaceflight. There's a sort of cosmic beauty with how objects behave in space, subject predominantly and almost exclusively to the fundamental force that is gravity. And it is also because of this exclusivity to a single force that makes it ripe for mathematical exploration. In particular, the launch of the James Webb Space Telescope is a unique mission where the space telescope will sit *behind* the Earth relative to the Sun, in orbit around the Lagrange point L2. This stands against high school intuition that objects in space orbit around bodies with mass; somehow, the JWST is able to exist in a (somewhat) stable orbit around seemingly nothing! **It is for this reason that this investigation aims to explore the mathematics behind Lagrange points.** More specifically, I seek to learn about the nature of Lagrange points, specifically L2, from a basic physics understanding; an understanding through my mathematical knowledge from school; and from a higher level of mathematics such that we can model what the orbit around a collinear Lagrange point would look like.

It is important to note that this paper does not seek out to give a method which can accurately predict the motion of an object in space with clear solutions; there exists a good reason why orbital mechanics is not taught in detail in high school. Instead, this paper investigates Lagrange points in such a way that allows for a conceptual understanding that can still facilitate for numerical computations of the motion of objects near a collinear Lagrange point. It is also due to the complexity of this investigation that more advanced utilities will be used. Primarily, Python libraries will be used for numerical computations of large numbers, numerical approximation of equations with no solutions, and graphing three-dimensional space.

1. OECD, *The Space Economy in Figures* (2019), 200, https://doi.org/https://doi.org/10.1787/c5996201-en, https://www.oecd-ilibrary.org/content/publication/c5996201-en.

2 Locations of Lagrange Points

From a basic understanding of classical mechanics, we can determine the location of Lagrange points. Before we do any math on Lagrange points, its important to know what they are exactly. According to NASA, Lagrange points are special solutions in what is know as the "three-body problem." At these points in space, the gravitational and rotational forces of the other two bodies effectively cancel each other out, allowing small objects to seemingly stay in place.[2]

With this in mind, lets consider a simple system of two static masses, m_1 and m_2, where $m_1 > m_2$.

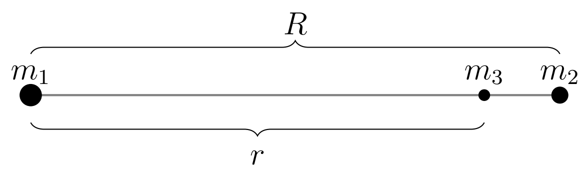


Figure 1: System of two stationary bodies.

In Figure 1, R represents the distance between the two bodies, and r represents the distance from the first body to the equilibrium point, point mass m_3, where the net gravitational force is 0. Using Newton's Law of Gravitation,

$$F = \frac{Gm_1m_2}{r^2}$$

where F is the force exerted between two bodies of mass and G is the gravitational constant, we can write the net force of the system as:

$$\frac{Gm_1m_3}{r^2} = \frac{Gm_2m_3}{(R-r)^2}$$

If we let $m_1 = 4m_2$ and $R = 1$, than we can directly solve for r:

$$\frac{m_1}{r^2} = \frac{m_2}{(R-r)^2}$$
$$\left(\frac{R-r}{r}\right)^2 = \frac{m_2}{m_1}$$

2. "Orbit - Webb/NASA," NASA, accessed January 11, 2022, https://jwst.nasa.gov/content/about/orbit.html.

$$\begin{aligned}\left(\frac{1-r}{r}\right)^2 &= \frac{m_2}{4m_2}\\ \frac{1-r}{r} &= \frac{1}{2}\\ 2-2r &= r\\ \frac{2}{3} &= r\end{aligned}$$

Of course, this scenario is not exemplary of Lagrange points, let alone the interaction between celestial bodies. Let us increase the complexity of our initial system by placing m_2 in a circular orbit around m_1. With consideration for rotational force, centripetal force is expressed as:

$$F = \frac{4\pi^2 m_3 r}{T^2}$$

where T expresses the orbital period of m_3. Given that, from Kepler's Third Law,

$$T = 2\pi\sqrt{\frac{R^3}{G(m_1+m_2)}}$$

centripetal force can be rewritten as:

$$F = \frac{G(m_1+m_2)}{R^3} r m_3$$

Therefore, the net force between m_1 and m_2 is written as:

$$F = m_3 a = -\frac{Gm_1m_3}{r^2} - \frac{Gm_2m_3}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3} r m_3$$

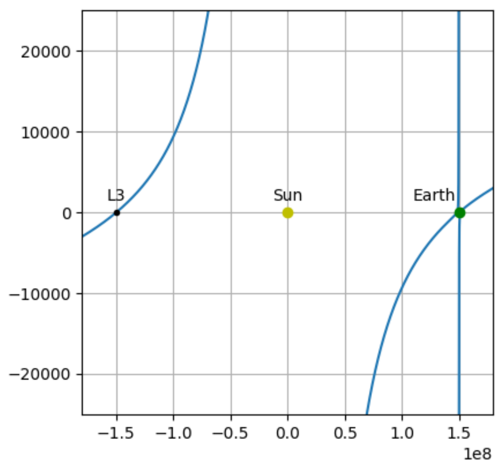
Solving for the centripetal acceleration, we get the formula:

$$a = -\frac{Gm_1}{r^2} - \frac{Gm_2}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3} r$$

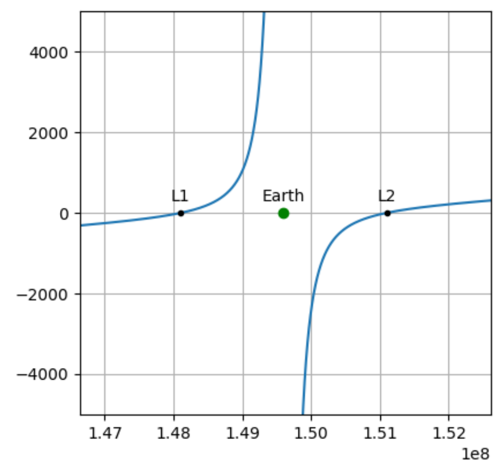
Most of the variables are known constants, with r being the only unknown value which represents the distance of the Lagrange point from m_1, given that the net radial acceleration is 0. It is worth noting that, with respect to direction, r^2 and $(r-R)^2$ will only reflect acceleration due to gravity in the negative direction and will not be sufficient to tell us where L1 and L3 are. Knowing that, cautiously, $n^2 = n \times |n|$, the formula is rewritten as such to preserve the sign of r:

$$a = -\frac{Gm_1}{r|r|} - \frac{Gm_2}{(r-R)|r-R|} + \frac{G(m_1+m_2)}{R^3}r \tag{1}$$

To save ourselves the agony of whether it is possible to isolate for r, we will use Python to plot the centripetal acceleration with respect to distance. The Lagrange points are located where the acceleration is 0.



(a) Radial acceleration between the Sun and Earth.



(b) Radial acceleration around the Earth.

Figure 2: Net radial acceleration vs distance of the Sun-Earth system. Graphs are created using the matplotlib Python library.

Given that we are only concerned with L2, the distance of L2 from the Sun is computed as 1.511×10^8km.

Note how Equation (1) for acceleration is dependent on the distance an object is from either body. Should we try to predict the motion of an object, there would be no way to express distance as a function of time. This is a major hurdle that occurs in physics and, while there can be specific situations where this can be overcome, most of the time there is no solution. Instead, we would have to numerically integrate the distance travelled over time. Not to be confused with integration, this means that we would need to calculate the acceleration at a certain distance, let our object travel for an interval of time at said acceleration, than recalculate acceleration for our new distance. This is tedious work that, for the sake of both my sanity and time, we will delegate to computers. We should not let this caveat daunt us, however, as there is still a lot to learn about how we describe the motion of objects in such a relation.

3 Deriving Equations of Motion

Using IB calculus, plus some vectors, we will be able to represent the motion of an object near a Lagrange point using equations. It is important, however, that we first establish the system that we will be working with as well as restrictions that will make the mathematics slightly easier.

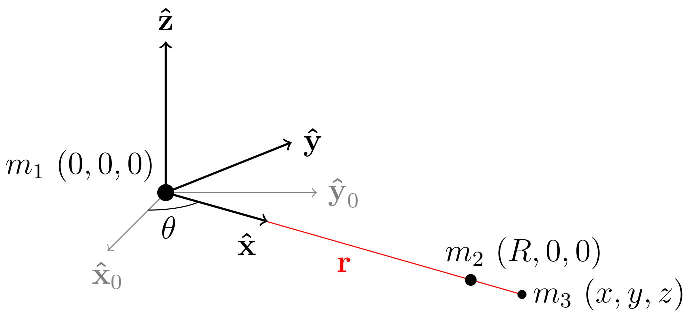


Figure 3: three-dimensional diagram of the Sun-Earth system with unit vectors relative to the Earth's orbit. Not drawn to scale.

Firstly, we must assert that $m_1 > m_2$, allowing the movement of the Sun due to the gravity of Earth is negligible. This allows the Sun to be placed at the center of the coordinate system, avoiding the need to make considerations for the center of mass of the system. It is also asserted that $m_2 > m_3$, so that, similarly to the previous assertion, the satellite has a negligible effect on Earth. Secondly, continuing with the conditions from the previous calculations, it is assumed that the Earth is in a circular orbit, making the system a little easier to comprehend. Thirdly, we assume that the orbit of the Earth has no inclination, meaning that it is on the same plane as $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Lastly, and as shown in Figure 3, we will have basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ drawn relative to the orbit of the Earth, with initial vectors $\hat{\mathbf{x}}_0$, $\hat{\mathbf{y}}_0$, and $\hat{\mathbf{z}}_0$. This is necessary to keep track of three dimensional space around the Lagrange points and also so that we can compose our vectors in terms of each dimension. The satellite around L2, m_3, will have the coordinates (x, y, z) represented by the vector $\mathbf{r}$.

It is also important to acknowledge that we will need to take the derivative of vectors. Hence, let us assert that, for some vector $\mathbf{v}$ with the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ scalars a and b:

$$\frac{d}{dt}\mathbf{v} = \frac{da}{dt}\hat{\mathbf{i}} + \frac{db}{dt}\hat{\mathbf{j}}.$$

In other words, the derivative of a vector is equivalent to the derivative of its components. This will allow us to analyze the movement of an object through kinematics in all three dimensions x, y, and z.

One last thing: because we are dealing with vectors, Newton's law of gravitation can be rewritten in vector form:

$$\mathbf{F} = m\mathbf{a} = \frac{Gm_1m_2}{r^3}\mathbf{r},$$

where r is the magnitude of $\mathbf{r}$.

To start off, let us analyze our basis vectors. We will use the notation x' as the derivative of x with respect to time t. Because they are not actually static in the system ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ rotate about the origin), their movement must be taken into account:

$$\begin{aligned}
\hat{\mathbf{x}} &= (\cos\theta)\hat{\mathbf{x}}_0 + (\sin\theta)\hat{\mathbf{y}}_0 \\
\hat{\mathbf{y}} &= (-\sin\theta)\hat{\mathbf{x}}_0 + (\cos\theta)\hat{\mathbf{y}}_0 \\
\hat{\mathbf{z}} &= \hat{\mathbf{z}}_0.
\end{aligned}$$

Taking their first derivatives, we get:

$$\begin{aligned}
\hat{\mathbf{x}}' &= (-\sin\theta)\theta'\hat{\mathbf{x}}_0 + (\cos\theta)\theta'\hat{\mathbf{y}}_0 = \theta'\hat{\mathbf{y}} \\
\hat{\mathbf{y}}' &= -(\cos\theta)\theta'\hat{\mathbf{x}}_0 + (-\sin\theta)\theta'\hat{\mathbf{y}}_0 = -\theta'\hat{\mathbf{x}} \\
\hat{\mathbf{z}}' &= 0.
\end{aligned}$$

And their second derivatives:

$$\begin{aligned}
\hat{\mathbf{x}}'' &= \theta''\hat{\mathbf{y}} + \theta'\hat{\mathbf{y}}' = \theta''\hat{\mathbf{y}} - \theta'^2\hat{\mathbf{x}} \\
\hat{\mathbf{y}}'' &= \theta''\hat{\mathbf{y}} + \theta'\hat{\mathbf{x}}' = -\theta''\hat{\mathbf{x}} - \theta'^2\hat{\mathbf{y}}.
\end{aligned}$$

Given that this is uniform circular motion, $\theta'' = 0$. The position vector r is expressed as:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

We can take the second derivative of our position vector $\mathbf{r}$ to find the acceleration.

$$\mathbf{a} = \mathbf{r}'' = (x\hat{\mathbf{x}})'' + (y\hat{\mathbf{y}})'' + (z\hat{\mathbf{z}})''$$

Our acceleration vector is then expanded to:

$$\mathbf{a} = x''\hat{\mathbf{x}} + 2x'\hat{\mathbf{x}}' + x\hat{\mathbf{x}}'' + y'\hat{\mathbf{y}} + 2y'\hat{\mathbf{y}}' + x\hat{\mathbf{y}}'' + z''\hat{\mathbf{z}} + 2z'\hat{\mathbf{z}}' + z\hat{\mathbf{z}}''$$

Substituting the derivatives of the basis vectors that we calculated earlier,

$$\mathbf{a} = x''\hat{\mathbf{x}} + 2x'(\theta'\hat{\mathbf{y}}) + x(\theta''\hat{\mathbf{y}} - \theta'^2\hat{\mathbf{x}}) + y''\hat{\mathbf{y}} - 2y'(\theta'\hat{\mathbf{x}}) - x(\theta''\hat{\mathbf{x}} + \theta'\hat{\mathbf{y}}) + z''\hat{\mathbf{z}}$$
$$\mathbf{a} = (x'' - \theta'^2 x - 2y'\theta')\hat{\mathbf{x}} + (y'' + 2x'\theta' - \theta'^2 y)\hat{\mathbf{y}} + z''\hat{\mathbf{z}}. \tag{2}$$

This equation represents the general centripetal acceleration of an object in circular motion in three dimensions. To account for gravity, the acceleration due to gravity is expressed as:

$$\mathbf{F} = m_3\mathbf{a} = -\frac{Gm_1m_3}{r_1^3}\mathbf{r}_1 - \frac{Gm_2m_3}{r_2^3}\mathbf{r}_2$$
$$\mathbf{a} = -\frac{Gm_1}{r_1^3}\mathbf{r}_1 - \frac{Gm_2}{r_2^3}\mathbf{r}_2$$

Where $\mathbf{r}_1$ and $\mathbf{r}_2$ is the distance of our satellite from m_1 and m_2, respectively. $\mathbf{r}_1$ and $\mathbf{r}_2$ and their magnitudes are represented as:

$$\mathbf{r}_1 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad r_1 = \sqrt{x^2 + y^2 + z^2}$$
$$\mathbf{r}_2 = (x - R)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad r_2 = \sqrt{(x - R)^2 + y^2 + z^2}$$

Under this definition, the acceleration vector can be expanded for each dimension:

$$\mathbf{a} = \left(-\frac{Gm_1}{r_1^3}x - \frac{Gm_2}{r_2^3}(x - R)\right)\hat{\mathbf{x}} + \left(-\frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y\right)\hat{\mathbf{y}} + \left(-\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z\right)\hat{\mathbf{z}}$$

Now that it is written in terms of its components, just like Equation (2), we can set the components of each vector equal to each other (note that $\theta' = \omega$, which is radial velocity).

$$\hat{\mathbf{x}}: \qquad x'' - \omega^2 x - 2y'\omega = -\frac{Gm_1}{r_1^3}x - \frac{Gm_2}{r_2^3}(x - R)$$
$$\hat{\mathbf{y}}: \qquad y'' - \omega^2 y + 2x'\omega = -\frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y$$
$$\hat{\mathbf{z}}: \qquad z'' = -\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z$$

The variables can then be rearranged to indicate acceleration:

$$\hat{\mathbf{x}}: \qquad x'' = +\omega^2 x + 2y'\omega - \frac{Gm_1}{r_1^3}x - \frac{Gm_2}{r_2^3}(x - R)$$
$$\hat{\mathbf{y}}: \qquad y'' = +\omega^2 y - 2x'\omega - \frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y$$
$$\hat{\mathbf{z}}: \qquad z'' = -\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z$$

These are our equations of motion that govern the movement of a satellite around a Lagrange point. Technically, we can use them to represent motion anywhere in our system, not just the Lagrange points. In physics, these equations are simplified versions of those found in the "circular restricted three-body problem."

Similar to Equation (1), our equations of motion requires numerical integration to determine the actual movement of our satellite.

4 Plotting the Orbit around a Lagrange point

Deriving the equations of motion to predict the movement of a satellite around a Lagrange point will be much more difficult than locating the Lagrange points. We will need to come up with a coordinate system that we can use to define our equations of motion around, as well as specific parameters for the physical system to allow the mathematics to be easier.

5 Conclusion

Bibliography

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"Orbit - Webb/NASA." NASA. Accessed January 11, 2022. https://jwst.nasa.gov/content/about/orbit.html.

A Eigenvalues for Larger Systems

For the 3×3 linear system:

$$\begin{aligned}
ax_1 + bx_2 + cx_3 &= \lambda x_1 \\
dx_1 + ex_2 + fx_3 &= \lambda x_2 \\
gx_1 + hx_2 + ix_3 &= \lambda x_3
\end{aligned}$$

let $\alpha = a - \lambda$, $\epsilon = e - \lambda$, and $\iota = i - \lambda$.

$$\begin{aligned}
\alpha x_1 + bx_2 + cx_3 &= 0 \\
dx_1 + \epsilon x_2 + fx_3 &= 0 \\
gx_1 + hx_2 + \iota x_3 &= 0
\end{aligned}$$

Isolating for x_1 for the equation in the first row,

$$x_1 = -\frac{b}{\alpha}x_2 - \frac{c}{\alpha}x_3$$

Substituting x_1 in the remaining equations gives:

$$\begin{aligned}
\left(\epsilon - \frac{bd}{\alpha}\right)x_2 + \left(f - \frac{cd}{\alpha}\right)x_3 &= 0 \\
\left(h - \frac{bg}{\alpha}\right)x_2 + \left(\iota - \frac{cg}{\alpha}\right)x_3 &= 0
\end{aligned}$$

Isolating for x_2 for the equation in the first row of the remaining equations gives:

$$-\frac{f - \frac{cd}{\alpha}}{\epsilon - \frac{bd}{\alpha}}x_3 = x_2$$

Substituting x_2 for the last equations gives:

$$\frac{\left(h - \frac{bg}{\alpha}\right)\left(f - \frac{cd}{\alpha}\right)}{\epsilon - \frac{bd}{\alpha}}x_3 = \left(\iota - \frac{cg}{\alpha}\right)$$

The x_3 can be cancelled out of the equation, leaving:

$$\left(h - \frac{bg}{\alpha}\right)\left(f - \frac{cd}{\alpha}\right) = \left(\iota - \frac{cg}{\alpha}\right)\left(\epsilon - \frac{bd}{\alpha}\right)$$

$$
\begin{aligned}
\frac{1}{\alpha^2}(h\alpha - bg)(f\alpha - cd) &= \frac{1}{\alpha^2}(\iota\alpha - cg)(\epsilon\alpha - bd) \\
(h\alpha - bg)(f\alpha - cd) &= (\iota\alpha - cg)(\epsilon\alpha - bd) \\
\alpha^2 fh - \alpha cdh - \alpha bfg + bcdg &= \alpha^2\epsilon\iota - \alpha bd\iota - \alpha c\epsilon g + bcdg \\
\alpha^2 fh - \alpha cdh - \alpha bfg &= \alpha^2\epsilon\iota - \alpha bd\iota - \alpha c\epsilon g \\
\alpha fh - \alpha\epsilon\iota + bd\iota - bfg + c\epsilon g - cdh &= 0 \\
\alpha(\epsilon\iota - \alpha f) - b(d\iota - fg) + c(dh - \epsilon g) &= 0 \qquad (3)
\end{aligned}
$$

Then, replacing α, ϵ, and ι:

$$(a-\lambda)[(e-\lambda)(i-\lambda) - (a-\lambda)f] - b[d(i-\lambda) - fg] + c[dh - (e-\lambda)g] = 0$$