The Mathematical Exploration of Lagrange Points

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1 Introduction

In this day and age, the aerospace industry has become a vital component of the global economy. In a publication by the Organisation for Economic Co-operation and Development, the space economy plays a key role in the globalization and digitization of the modern world, with space activities making crucial contributions to the social, economic, and scientific aspects of society.[1] With an increasing investment for spaceflight, there is future need to develop infrastructure that can facilitate the growing space sector, which includes the necessity to understand the nature of orbital mechanics.

Since the longest time, I've always been interested in the development of spaceflight. There's a sort of cosmic beauty with how objects behave in space, subject predominantly and almost exclusively to the fundamental force that is gravity. And it is also because of this exclusivity to a single force that makes it ripe for mathematical exploration. In particular, the launch of the James Webb Space Telescope (JWST) is a unique mission where the space telescope will sit *behind* the Earth relative to the Sun, in orbit around the Lagrange point L2. This stands against high school intuition that objects in space orbit around bodies with mass; somehow, the JWST is able to exist in a (somewhat) stable orbit around seemingly nothing! **It is for this reason that this investigation aims to explore the mathematics behind Lagrange points.** More specifically, I seek to learn about the nature of Lagrange points, mainly L2, from a basic physics understanding; an understanding through my mathematical knowledge from school; and through large numerical computations such that we can model what an orbit around a collinear Lagrange point would look like.

It is important to note that this paper does not seek out to give a method which can accurately predict the motion of an object in space with clear solutions; there exists a good reason why orbital mechanics is not taught in detail in high school. Instead, this paper investigates Lagrange points in such a way that allows for a conceptual understanding that can still facilitate for numerical computations of the motion of objects near a collinear Lagrange point. It is also due to the complexity of this investigation that more advanced utilities will be used. Primarily, Python libraries will be used for numerical computations of large numbers, numerical approximation of equations with no solutions, and graphing three-dimensional space.

1. OECD, *The Space Economy in Figures* (2019), 200, https://doi.org/https://doi.org/10.1787/c5996201-en, https://www.oecd-ilibrary.org/content/publication/c5996201-en.

2 Locations of Lagrange Points

From a basic understanding of classical mechanics, we can determine the location of Lagrange points. But before we do any math on Lagrange points, its important to know what they are exactly. According to NASA, Lagrange points are special solutions in what is known as the "three-body problem." At these points in space, the gravitational and rotational forces of the other two bodies effectively cancel each other out, allowing small objects to seemingly stay in place.[2]

With this in mind, lets consider a simple system of two static masses, m_1 and m_2, where $m_1 > m_2$.

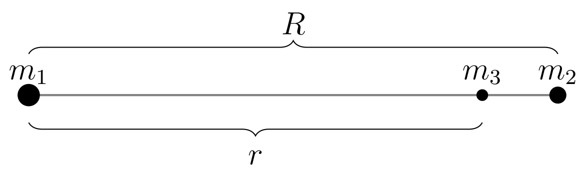


Figure 1: System of two stationary bodies.

In Figure 1, R represents the distance between the two bodies, and r represents the distance from the first body to the equilibrium point, point mass m_3, where the net gravitational force is 0. Using Newton's Law of Gravitation,

$$F = \frac{Gm_1m_2}{r^2}$$

where F is the force exerted between two bodies of mass and G is the gravitational constant, we can write the net force of the system as:

$$\frac{Gm_1m_3}{r^2} = \frac{Gm_2m_3}{(R-r)^2}$$

If we let $m_1 = 4m_2$ and $R = 1$, than we can directly solve for r:

$$\frac{m_1}{r^2} = \frac{m_2}{(R-r)^2}$$
$$\left(\frac{R-r}{r}\right)^2 = \frac{m_2}{m_1}$$

2. "Orbit - Webb/NASA," NASA, accessed January 11, 2022, https://jwst.nasa.gov/content/about/orbit.html.

$$\left(\frac{1-r}{r}\right)^2 = \frac{m_2}{4m_2}$$
$$\frac{1-r}{r} = \frac{1}{2}$$
$$2 - 2r = r$$
$$\frac{2}{3} = r$$

Of course, this scenario is not exemplary of Lagrange points, let alone the interaction between celestial bodies. Let us increase the complexity of our initial system by placing m_2 in a circular orbit around m_1. With consideration for rotational force, centripetal force is expressed as:

$$F = \frac{4\pi^2 m_3 r}{T^2}$$

where T expresses the orbital period of m_2. Given that, from Kepler's Third Law,

$$T = 2\pi\sqrt{\frac{R^3}{G(m_1+m_2)}}$$

centripetal force can be rewritten as:

$$F = \frac{G(m_1+m_2)}{R^3} m_3 r$$

Considering how the sign of each term of force indicates direction, the net force between m_1 and m_2 is written as:

$$F = m_3 a = -\frac{Gm_1m_3}{r^2} - \frac{Gm_2m_3}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3} m_3 r$$

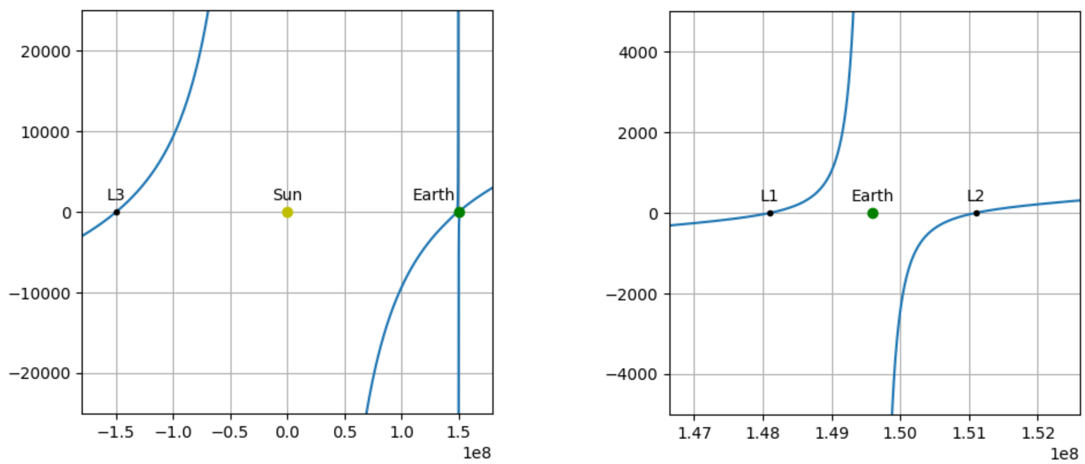
By cancelling off m_3 to find centripetal acceleration, we get the equation:

$$a = -\frac{Gm_1}{r^2} - \frac{Gm_2}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3} r$$

Most of the variables are known constants, with r being the only unknown value which represents the distance of the Lagrange point from m_1, given that the net radial acceleration is 0. It is worth noting that, with respect to direction, r^2 and $(r-R)^2$ will only reflect acceleration due to gravity in the negative direction and will not be sufficient to tell us where L1 and L3 are. Knowing that, cautiously, $n^2 = n \times |n|$, the formula is rewritten as such to preserve the sign of r:

$$a = -\frac{Gm_1}{r|r|} - \frac{Gm_2}{(r-R)|r-R|} + \frac{G(m_1+m_2)}{R^3}r \tag{1}$$

To save ourselves the agony of whether it is possible to algebraically solve for r, we will use Python to plot the centripetal acceleration with respect to distance. The Lagrange points are located where the acceleration is 0.



(a) Radial acceleration between the Sun and Earth.

(b) Radial acceleration around the Earth.

Figure 2: Net radial acceleration vs distance of the Sun-Earth system. Graphs are created using the matplotlib Python library.

Given that we are only concerned with L2, the distance of L2 from the Sun is computed as 1.511×10^8km.

Note how Equation (1) for acceleration is dependent on the distance an object is from either body. Should we try to predict the motion of an object, there would be no way to express distance as a function of time as it is dependant on acceleration. This is a major hurdle that occurs in physics and, while there can be specific situations where this can be overcome, most of the time there is no solution. We should not let this caveat daunt us, however, as there is still a lot to learn about how we describe the motion of objects in such a relation.

3 Deriving Equations of Motion

Using IB calculus, plus some vectors, we will be able to represent the motion of an object near a Lagrange point using equations. It is important, however, that we first establish the system that we will be working with as well as restrictions that will make the mathematics slightly easier.

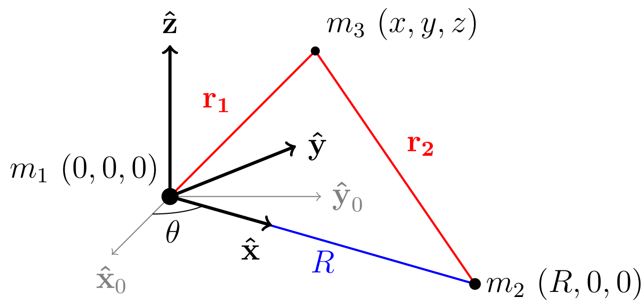


Figure 3: three-dimensional diagram of the Sun-Earth system with basis vectors relative to the Earth's orbit. Positions are exaggerated and not to scale.

Firstly, we must assert that $m_1 > m_2$, allowing the movement of the Sun due to the gravity of Earth is negligible. This allows the Sun to be placed at the center of the coordinate system, avoiding the need to make considerations for the center of mass of the system. It is also asserted that $m_2 > m_3$, so that, similarly to the previous assertion, the satellite has a negligible effect on Earth. Secondly, continuing with the conditions from the previous calculations, it is assumed that the Earth is in a circular orbit, making the system a little easier to comprehend. Thirdly, we assume that the orbit of the Earth has no inclination, meaning that it is co-planar to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Lastly, and as shown in Figure 3, we will have basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ drawn relative to the orbit of the Earth, with initial vectors $\hat{\mathbf{x}}_0$, $\hat{\mathbf{y}}_0$, and $\hat{\mathbf{z}}_0$. This is necessary to keep track of three dimensional space around the Lagrange points and also so that we can compose our equations of motion in terms of each dimension. The satellite around L2, m_3, will have the coordinates (x, y, z) represented by the vector $\mathbf{r}$.

It is also important to acknowledge that we will need to take the derivative of vectors. Hence, let us assert that, for some vector $\mathbf{v}$ with the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ scalars a and b:

$$\frac{d}{dt}\mathbf{v} = \frac{da}{dt}\hat{\mathbf{i}} + \frac{db}{dt}\hat{\mathbf{j}}.$$

In other words, the derivative of a vector is equivalent to the derivative of its components. This will allow us to analyze the movement of an object through kinematics in all three dimensions x, y, and z. Keep in mind that $\mathbf{r}$—hence x, y, z, and θ—are functions of time t.

One last thing: because we are dealing with vectors, Newton's law of gravitation can be rewritten in vector form:

$$\mathbf{F} = m\mathbf{a} = \frac{Gm_1m_2}{\|r\|^3}\mathbf{r},$$

where r is the magnitude of $\mathbf{r}$.

To start off, let us analyze our basis vectors. We will use the notation x' as the derivative of x with respect to time t. Because they are not actually static in the system ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ rotate about the origin), their movement must be taken into account:

$$\begin{aligned}\hat{\mathbf{x}} &= (\cos\theta)\hat{\mathbf{x}}_0 + (\sin\theta)\hat{\mathbf{y}}_0\\ \hat{\mathbf{y}} &= (-\sin\theta)\hat{\mathbf{x}}_0 + (\cos\theta)\hat{\mathbf{y}}_0\\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}_0.\end{aligned}$$

near Taking their first derivatives, we get:

$$\begin{aligned}\hat{\mathbf{x}}' &= (-\sin\theta)\theta'\hat{\mathbf{x}}_0 + (\cos\theta)\theta'\hat{\mathbf{y}}_0 = \theta'\hat{\mathbf{y}}\\ \hat{\mathbf{y}}' &= -(\cos\theta)\theta'\hat{\mathbf{x}}_0 + (-\sin\theta)\theta'\hat{\mathbf{y}}_0 = -\theta'\hat{\mathbf{x}}\\ \hat{\mathbf{z}}' &= 0.\end{aligned}$$

And their second derivatives:

$$\begin{aligned}\hat{\mathbf{x}}'' &= \theta''\hat{\mathbf{y}} + \theta'\hat{\mathbf{y}}' = -\theta'^2\hat{\mathbf{x}}\\ \hat{\mathbf{y}}'' &= \theta''\hat{\mathbf{x}} - \theta'\hat{\mathbf{x}}' = -\theta'^2\hat{\mathbf{y}}.\end{aligned}$$

Given that this is uniform circular motion, $\theta'' = 0$. The position vector r is expressed in unit vector form as:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

We can take the second derivative of our position vector $\mathbf{r}$ to find the acceleration.

$$\mathbf{a} = \mathbf{r}'' = (x\hat{\mathbf{x}})'' + (y\hat{\mathbf{y}})'' + (z\hat{\mathbf{z}})''$$

Our acceleration vector is then expanded to:

$$\mathbf{a} = x''\hat{\mathbf{x}} + 2x'\hat{\mathbf{x}}' + x\hat{\mathbf{x}}'' + y'\hat{\mathbf{y}} + 2y'\hat{\mathbf{y}}' + x\hat{\mathbf{y}}'' + z''\hat{\mathbf{z}} + 2z'\hat{\mathbf{z}}' + z\hat{\mathbf{z}}''.$$

Substituting the derivatives of the basis vectors that we calculated earlier,

$$\mathbf{a} = x''\hat{\mathbf{x}} + 2x'(\theta'\hat{\mathbf{y}}) + x(-\theta'^2\hat{\mathbf{x}}) + y''\hat{\mathbf{y}} - 2y'(\theta'\hat{\mathbf{x}}) - x(\theta'\hat{\mathbf{y}}) + z''\hat{\mathbf{z}}$$

$$\mathbf{a} = \left(x'' - \theta'^2 x - 2y'\theta'\right)\hat{\mathbf{x}} + \left(y'' + 2x'\theta' - \theta'^2 y\right)\hat{\mathbf{y}} + z''\hat{\mathbf{z}}. \tag{2}$$

This equation represents the general centripetal acceleration of an object in circular motion in three dimensions. To account for gravity, the acceleration due to gravity is expressed as:

$$\mathbf{F} = m_3\mathbf{a} = -\frac{Gm_1m_3}{\|r_1\|^3}\mathbf{r}_1 - \frac{Gm_2m_3}{\|r_2\|^3}\mathbf{r}_2$$

$$\mathbf{a} = -\frac{Gm_1}{\|r_1\|^3}\mathbf{r}_1 - \frac{Gm_2}{\|r_2\|^3}\mathbf{r}_2.$$

Where $\mathbf{r}_1$ and $\mathbf{r}_2$ is the distance of our satellite from m_1 and m_2, respectively. $\mathbf{r}_1$ and $\mathbf{r}_2$ and their magnitudes, $\|r_1\|$ and $\|r_2\|$, are represented as:

$$\mathbf{r}_1 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad \|r_1\| = \sqrt{x^2 + y^2 + z^2}$$

$$\mathbf{r}_2 = (x - R)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad \|r_2\| = \sqrt{(x-R)^2 + y^2 + z^2}.$$

Written in unit vector form, the acceleration vector can be expanded for each dimension:

$$\mathbf{a} = \left(-\frac{Gm_1}{\|r_1\|^3}x - \frac{Gm_2}{\|r_2\|^3}(x-R)\right)\hat{\mathbf{x}} + \left(-\frac{Gm_1}{\|r_1\|^3}y - \frac{Gm_2}{\|r_2\|^3}y\right)\hat{\mathbf{y}} + \left(-\frac{Gm_1}{\|r_1\|^3}z - \frac{Gm_2}{\|r_2\|^3}z\right)\hat{\mathbf{z}}.$$

Now that it is written in terms of its components, just like Equation (2), we can set the components of each vector equal to each other.

$$\hat{\mathbf{x}}: \qquad x'' - \theta'^2 x - 2y'\theta' = -\frac{Gm_1}{\|r_1\|^3}x - \frac{Gm_2}{\|r_2\|^3}(x-R)$$

$$\hat{\mathbf{y}}: \qquad y'' - \theta'^2 y + 2x'\theta' = -\frac{Gm_1}{\|r_1\|^3}y - \frac{Gm_2}{\|r_2\|^3}y$$

$$\hat{\mathbf{z}}: \qquad z'' = -\frac{Gm_1}{\|r_1\|^3}z - \frac{Gm_2}{\|r_2\|^3}z$$

The variables can then be rearranged to give us our equations of motion:

$$x'' = \theta'^2 x + 2y'\theta' - \frac{Gm_1}{\|r_1\|^3}x - \frac{Gm_2}{\|r_2\|^3}(x-R) \tag{3}$$

$$y'' = \theta'^2 y - 2x'\theta' - \frac{Gm_1}{\|r_1\|^3}y - \frac{Gm_2}{\|r_2\|^3}y \tag{4}$$

$$z'' = -\frac{Gm_1}{\|r_1\|^3}z - \frac{Gm_2}{\|r_2\|^3}z. \tag{5}$$

4 Computing the Orbit around a Lagrange point

Equations (3), (4), and (5) are our equations of motion that govern the movement of a satellite around a Lagrange point. Technically, we can use them to represent motion anywhere in our system, not just around the Lagrange points.

Notice how the equations of motion are similar to (1), where the distance, velocity, and acceleration of each dimension can not be represented separately as functions of time. Still, it is possible to make use of these equations through numerical approximation. This means that, given some initial conditions (position and velocity), the output is calculated (acceleration) for a small interval of time. Than, after letting the new output determine the rest of the system for the said time interval, the output is recalculated with the given change from the initial conditions. Recursively, this is generally written as:

$$y_{n+1} = y_n + hf(t_n)$$

where h indicates the length of the time interval and where some value y_0 is known. This particular equation is known as "Euler's method"[3] and is used to approximate equations that are similar to our equations of motion. The method used to approximate the equations of motion will not specifically be Euler's method, but its purpose is practically identical.

G, R, m_1, m_2, and θ' are known constants, with the latter defined as:

$$\theta' = \omega = \sqrt{\frac{G(m_1 + m_2)}{R^3}},$$

meaning the unknown values to the equations of motion are x, y, z, x', y', and z'. By setting our initial x coordinate $(1.511 \times 10^{11} - 10^3)$m, 1000m away from of L2, and the remaining conditions $y, z, x', y', z' = 0$, we can compute our first trajectory for our satellite, m_3. As seen in Figure 4, the satellite unremarkably falls back towards the Earth. This makes sense since there is no initial velocity and the initial position places it in the influence of Earth's gravity.

3. "Euler's Method," Brilliant.org, accessed March 10, 2022, https://brilliant.org/wiki/eulers-method/.

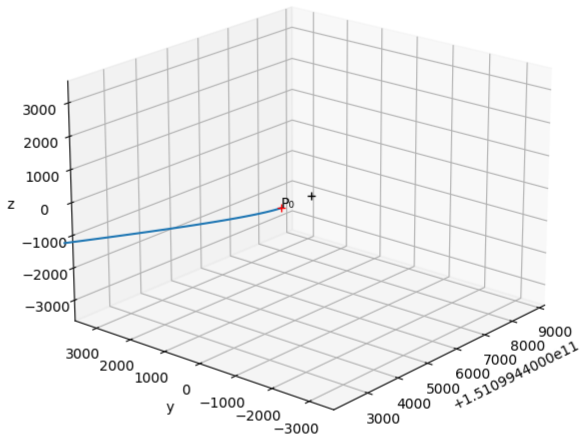


Figure 4: Trajectory plot of an object near L2 fo the Sun-Earth system, x-coordinate is $(1.511 \times 10^{11} - 10^3)$m, all other conditions are 0. The red point, P_0, indicates the starting position of the object. The black point indicates the Lagrange point. Generated through the matplotlib Python library.

In order to create a trajectory that resembles an orbit, we need to apply velocity that is perpendicular to the acceleration to create circular motion. Knowing that $v^2 = ar$, we can write the equation:

$$u_y = a\sqrt{x''r}$$

where u_y is the initial velocity in the y component and a is the proportionality constant, assuming that the orbit being predicted is not circular. After some trial and error, $a = 2.2$ makes for a good approximation.

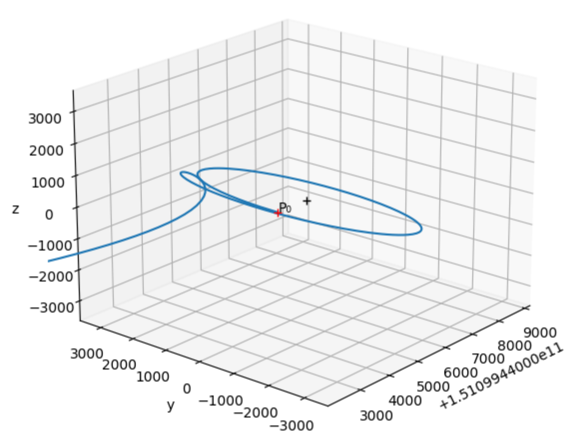
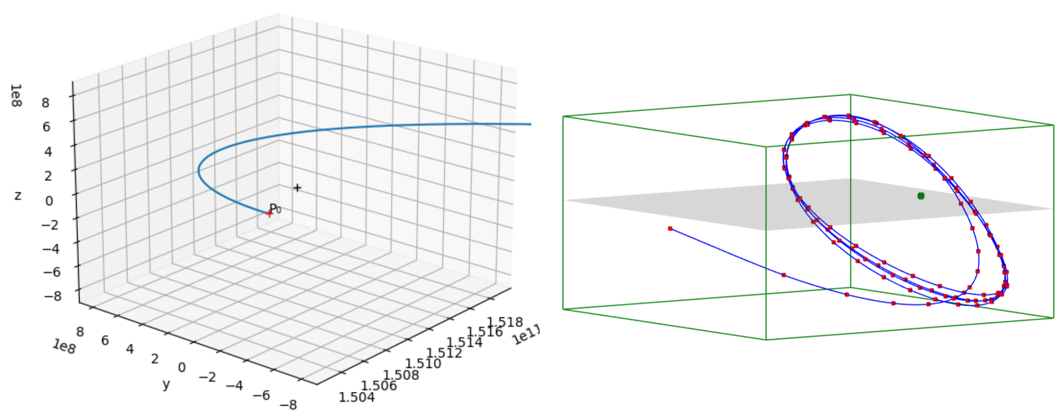


Figure 5: Trajectory plot of an object 1000m in the proximity of L2 in the Sun-Earth system in a semi-stable halo orbit.

And would you look at that; in Figure 5, it seems as if our satellite orbits nothing! Of course, it is also seen that the satellite eventually falls out of its orbit, confirming the instability of collinear Lagrange points.

We can now use this model and compare it to the actual trajectory of objects orbiting Lagrange points. Given that telemetry of the JWST is readily available through JPL Horizons System, we will use its orbit for comparison.



(a) Modelled trajectory plot of the JWST 2.5×10^8m L2. Not modelled with station-keeping.

(b) Trajectory of the JWST according to the JPL Horizons database.

Figure 6: Trajectories of the JWST. Figure (a) has P_0 positioned at $(L_2 - 2.5 \times 10^8, 0, -1.2 \times 10^8)$. Figure (b) generated through SageMath and produced by PM 2Ring on Stack Exchange.

Figure 6a is the trajectory of the JWST given the expected orbital characteristics prior to launch.[4] Figure 6b is the trajectory of the JWST according to the JPL Horizons database from December 26th, 2021 to January 22, 2024.[5] In Figure 6a, it is apparent that station-keeping, the use of onboard engines to stay on a specific orbit, is necessary for larger orbits around Lagrange points. It is possible that a much more precise initial velocity approximations will produce a trajectory that stays in orbit for longer. Still, the model seems to tend towards a similar elliptical, halo-like orbit that is shown in Figure 6b. The difference in trajectories in both figures can be attributed to the degree of sophistication between our model and the model used by JPL. This can include consideration for an elliptical, eccentric orbit of the Earth, the effects of the Moon's gravity, and orbital eccentricity of the JWST during its transfer to L2.

4. "JWST Orbit," JWST User Documentation, accessed January 11, 2022, https://jwst-docs.stsci.edu/jwst-observatory-characteristics/jwst-orbit.

5. "james webb telescope - Is JWST Halo orbit prograde or retrograde and why?," accessed March 1, 2022, https://space.stackexchange.com/questions/57822/is-jwst-halo-orbit-prograde-or-retrograde-and-why.

5 Conclusion

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