The Mathematical Exploration of Lagrange Points

Word Count: ?

Julian Joaquin
Candidate Number: JYZ314
Mathematics SL
Westwood Collegiate
May 2022

Table of Contents

1 Introduction

With the deployment of the James Webb Space Telescope, many people are excited to see what discoveries it will uncover about the Universe. Its development has involved bleeding edge technology and engineering innovations to make its mission objectives possible,[1] many of which are not easily understood by the general public. One aspect of the JWST that is not well known is the Lagrange point of which the telescope will be orbiting. High school physics teaches us that satellites can orbit around stars and planets with gravity. Yet, when the JWST will orbit around L2, it appears as if the telescope is orbiting around nothing! How is this possible?

This investigation aims to locate the collinear Lagrange point, L2, and attempt to model the orbit of a satellite around said point using IB level mathematics as a base and expanding concepts as necessary. Identifying the locations of the Lagrange points will involve an understanding of algebra and Newtonian mechanics. The derivation of the halo orbits around the Lagrange points will be more complex, requiring vectors, calculus, and a series of concepts that must be addressed before approaching the solutions to halo orbits.

With that being said, key theorems must be developed from other fields of mathematics that have not yet been covered in IB SL Mathematics. These theorems will be explored in the preparation section of this paper, as well as a few assertions throughout this investigation. In this way, this paper will be much more of an adventure to collect the concepts we need before we actually tackle the orbit around L2.

2 Finding the Lagrange Points

We will be determining the collinear Lagrange points from a rotating frame of reference relative to a circular orbit of Earth. This simplifies the solution to a single dimension which bisects the Sun and Earth.

R will be the distance between the Sun and Earth, while r will be the distance from the main body to the Lagrange point. The masses of the Sun, Earth, and a point located at the Lagrange point are denoted by m_1, m_2, and m_3, respectively. For the sake of maintaining

1. "New Technology Innovations Webb Telescope," NASA, accessed January 11, 2022, https://jwst.nasa.gov/content/about/innovations/index.html.

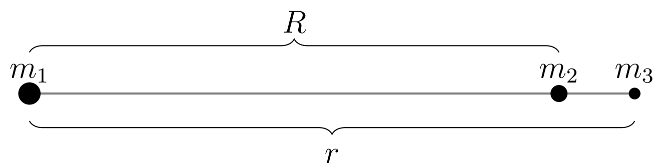


Figure 1: Simple diagram of the Sun-Earth system.

direction, bodies positioned to the right will have a positive distance, and vice versa.

It is important to understand what factors are at play when dealing with Lagrange points. What will be most important is Newton's Law of Gravitation,

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

where F is the force exerted between two bodies of mass, G is the gravitational constant, m is the mass of a body, and r is the distance between the two bodies. We must also consider the centrifugal and Coriolis forces acted on the object at the Lagrange point. We can rule out the Coriolis force, since the collinear Lagrange points will not be moving in our rotating frame of reference. As for centrifugal force, it will be proportional to the centripetal force,

$$F = m\omega^2 r$$

where ω is the angular velocity of the object. Understanding that angular velocity can be defined as

$$\omega = \frac{2\pi}{T}$$

and the period of a circular orbit from Kepler's third law as

$$T = 2\pi\sqrt{\frac{R^3}{G(m_1+m_2)}},$$

the equation for angular velocity simplifies to

$$\omega^2 = \frac{G(m_1+m_2)}{R^3}.$$

Therefore, the sum of forces acting on an object at a Lagrange point is

$$F = m_3a = -\frac{Gm_1m_3}{r^2} - \frac{Gm_2m_3}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3}rm_3.$$

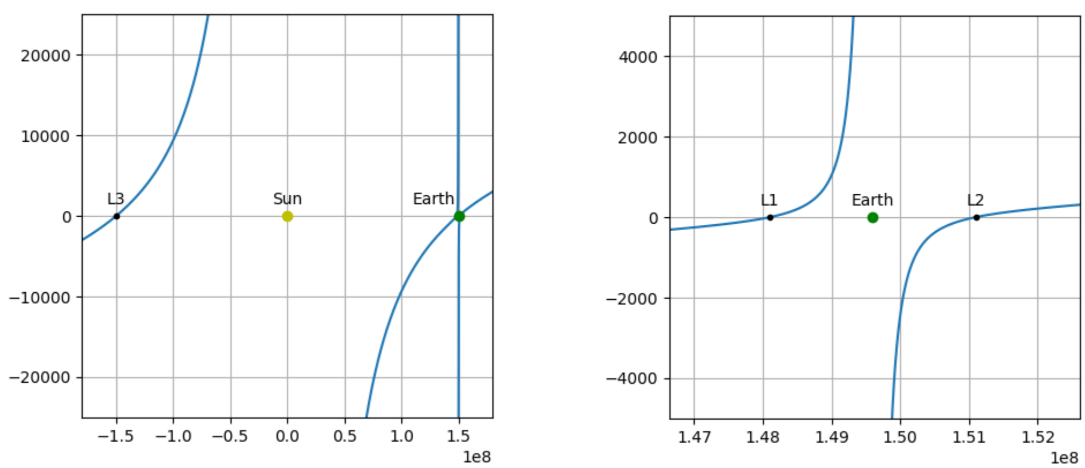
Solving for a radial acceleration of 0, we get the formula:

$$0 = -\frac{Gm_1}{r^2} - \frac{Gm_2}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3}r.$$

Most of the variables are known constants, with r being the only unknown value which represents the distance of the Lagrange point from m_1. It is worth noting that, with respect to direction, r^2 and $(r-R)^2$ will only reflect acceleration in the negative direction and will not be sufficient to tell us where L1 and L3 are. To remedy this, the formula is rewritten as such to preserve the sign of r:

$$0 = -\frac{Gm_1}{r|r|} - \frac{Gm_2}{(r-R)|r-R|} + \frac{G(m_1+m_2)}{R^3}r.$$

Further simplification of the formula results in a rational function that is almost impossible to solve by hand, in part because the formula would involve significantly large values due to the constants used. Instead, the roots of the formula are solved for computationally.



(a) Radial Acceleration between the Sun and Earth.

(b) Radial acceleration around the Earth.

Figure 2: Net acceleration of the Sun-Earth system. Graphs are created using `matplotlib`.

Relative to the distance from the sun, the location of the Lagrange point L2 is 1.511×10^8km.

3 Preparation for Equations of Motion

As stated in the introduction, finding out the motion of a satellite around a Lagrange point will be incredibly difficult. To prepare for this, let us take a step back and tackle these two scenarios.

3.1 Simple Linear Differential Equations

Let us imagine that, for some function f with respect to t,

$$\frac{d}{dt}f(t) = 4f(t). \tag{2}$$

In words, the rate of change of the function f is equal to itself multiplied by four. Let us say we wanted to find what function f is. While it is not as simple to just integrate this equation, we know that e^x is its own derivative. We also know that, because of the chain rule, $[e^{kt}]' = ke^{kt}$. If we substitute $f(t)$ for e^{kt} and let $k = 4$, then Equation (2) is true. Therefore, we can state the following theorem; for any equation

$$\frac{d}{dt}f(t) = kf(t) \ : \ f(t) = ce^{kt} \tag{3}$$

where c is some initial constant that cannot be expressed from integrating the derivative. We can verify that this theorem is correct by employing separation of variables (where $y = f(x)$):

$$\begin{aligned}
\frac{dy}{dx} &= ky \\
\frac{1}{y}\,dy &= k\,dx \\
\int \frac{1}{y}\,dy &= \int k\,dx \\
\ln y + C &= kx + D \\
\ln y &= kx + (D - C) \\
y &= e^{kt} \cdot e^{D-C}, \ \text{let } c = e^{D-C} \\
y &= ce^{kt}.
\end{aligned}$$

Equation (2) is an example of a differential equation, where a function is defined by its derivatives. Differential equations are very common in physics and will also appear in our equations of motion.

3.2 Eigenvalues and Eigenvectors

Imagine that there are two species ants, p and q, whose populations are dependent on each other. That is to say that both species try to kill the other species whilst reproducing their own. We can represent the populations of these two ant species through the following recursive system of equations:

$$\begin{aligned} p_{n+1} &= 4p_n - 2q_n \\ q_{n+1} &= -3p_n + 5q_n, \end{aligned}$$

where each recursive step n represents some interval of time. Say we knew that, for some initial population of both species, their population would double after the given time interval and we wanted to know what that initial population would be. We can determine the initial population by changing our system:

$$\begin{aligned} 2p_k &= 4p_k - 2q_k \\ 2q_k &= -3p_k + 5q_k. \end{aligned}$$

Then, all that is needed is to solve for p_k and q_k:

$$\left.\begin{aligned} 2p_k &= 4p_k - 2q_k \\ 2q_k &= -3p_k + 5q_k \end{aligned}\right\} \implies 0 = p_k - q_k \implies p_k = q_k.$$

Here, $p_k = q_k$ indicates that any amount of ants is a solution so long as both initial populations have a $1:1$ ratio. For example, $p_k = 200$ and $q_k = 200$ is a solution:

$$\begin{aligned} 2(200) &= 4(200) - 2(200) = 400 \\ 2(200) &= -3(200) + 5(200) = 400. \end{aligned}$$

In linear algebra, the idea of a linear system with a set of variables being equivalent to scaling said variables is termed an eigenvalue. More precisely, an eigenvalue is a numerical constant which represents the factor of which an eigenvector (in our case, ratio of initial ant populations) is scaled (which would be 2). Because linear algebra is beyond the scope of this investigation, we will only explore what is necessary to understand orbital motion.

It is, in fact, possible to determine both the eigenvalues and eigenvectors of most linear systems without knowing either to begin with. Looking back to our first scaled system, we can generalize it for some eigenvalue λ as such:

$$\begin{aligned}\lambda p_k &= 4p_k - 2q_k\\ \lambda q_k &= -3p_k + 5q_k.\end{aligned}$$

The eigenvalues can then be found through substitution and cancelling out the other variables.

$$\begin{aligned}4p_k - 2q_k &= \lambda q_k\\ (4-\lambda)p_k &= 2q_k\\ \frac{(4-k)p_k}{2} &= q_k\\ -3p_k + 5q_k &= \lambda q_k\\ -3p_k &= (\lambda-5)q_k\\ -3p_k &= (\lambda-5)\frac{(4-\lambda)p_k}{2}\\ 3\cdot 2 &= (5-\lambda)(4-\lambda)\\ 0 &= (5-\lambda)(4-\lambda) - 3\cdot 2 \qquad (4)\\ 0 &= \lambda^2 - 9\lambda + 14\\ 0 &= (\lambda-2)(\lambda-7)\end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 7$. We can find the respective eigenvector for λ_2 by doing the same operation for λ_1.

$$\left.\begin{aligned}7p_k &= 4p_k - 2q_k\\ 7q_k &= -3p_k + 5q_k\end{aligned}\right\} \implies 0 = -3p_k - 2q_k \implies 3p_k = -2q_k.$$

This gives us an eigenvector of $(-2, 3)$. While a negative amount of ants does not make much sense, these values will be important later for when we want to generalize this system.

From Equation (4), we can generalize the computation of any linear system with 2 variables and 2 equations as:

$$0 = (a-\lambda)(d-\lambda) - bc,$$

where a, b, c, d are the coefficients of the linear system. This computation of solving for eigenvalues can be extrapolated for larger systems involving n variables and n equations. For a linear system with 3 variables and 3 equations:

$$0 = (a - \lambda)((e - \lambda)(i - \lambda) - fh) - b(d(i - \lambda) - fg) + c(dh - eg).$$

And for 4 variables and 4 equations:

$$1 = 1$$

Their solutions are worked through in Appendix A.

3.3 Systems of Linear Differential Equations

Now that we have some understanding of what can be done with a linear system, it is time to combine linear systems with calculus.

Now say that we wanted to generalize our system of equations so that we can determine either ant population for some time, t. From what has already been established, this relation would be defined as:

$$\begin{aligned} p'(t) &= 4p(t) - 2q(t) & \qquad\qquad p(0) &= 4 \\ q'(t) &= -3p(t) + 5q(t) & q(0) &= 2 \end{aligned}$$

The approach to solving for $p(t)$ and $p(t)$ is not immediately intuitive. Now is a good time to view eigenvalues and eigenvectors in a more abstract manner. In Figure 3a, where the p and q are viewed as a set of values, the equations representing the ant species are on different lines with an intersection at $(0, 0)$. In comparison, in Figure 3b, the equations have converged onto the same line when $\lambda = 7$. This can be thought of as, for each eigenvalue, $p'(t)$ and $q'(t)$ are transformed onto a function which both share in common. We will call these ‘common’ functions $x(t)$ and $y(t)$. The eigenvalues can define these ‘common’ functions as:

$$\begin{aligned} x'(t) &= \lambda_1 x(t) \\ y'(t) &= \lambda_2 y(t), \end{aligned}$$

and are connected back to the functions $p(t)$ and $q(t)$ by the eigenvectors:

$$\begin{aligned} p(t) &= (1)x(t) + (-2)y(t) \\ q(t) &= (1)x(t) + (3)y(t). \end{aligned}$$

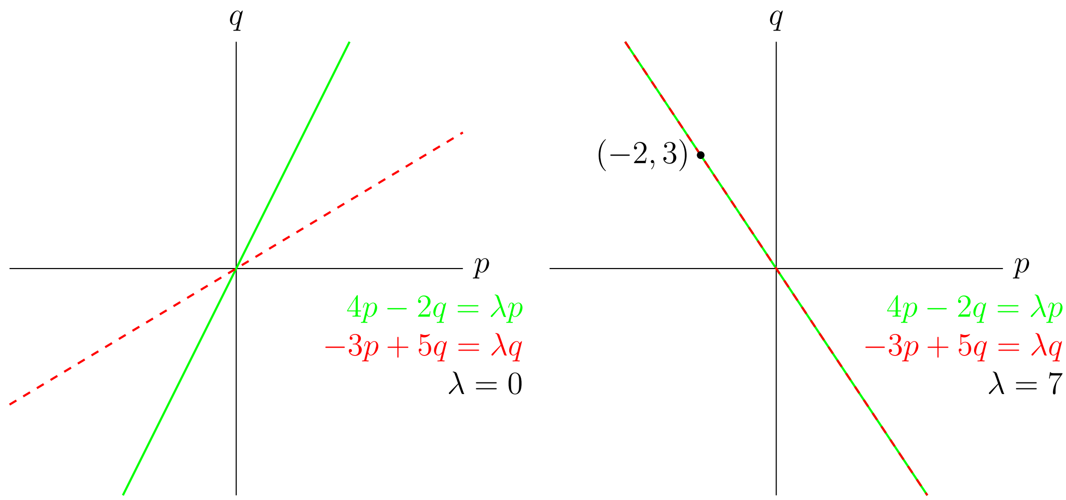


Figure 3: The linear system defined with regard to λ (It is better to think of the graphs as ratios between p and q rather than q as a function of p).

What we have essentially done is factored out a group of functions $x(t)$ and $y(t)$ from the linear system that are much easier to to solve as well as a set of coefficients, the eigenvectors, that relate $x(t)$ and $y(t)$ to $p(t)$ and $q(t)$. Solving for the 'common' functions:

$$\begin{aligned} x'(t) &= 2x(t) & x(t) &= c_1 e^{2t} \\ y'(t) &= 7y(t) & y(t) &= c_2 e^{7t}. \end{aligned}$$

Then we relate the these back to our original functions:

$$\begin{aligned} p(t) &= c_1 e^{2t} - 2c_2 e^{7t} \\ q(t) &= c_1 e^{2t} + 3c_2 e^{7t}. \end{aligned}$$

All that is left is to solve for the given initial conditions:

$$\left.\begin{aligned} p(0) = 4 = c_1 - 2c_2 \\ q(0) = 2 = c_1 + 3c_2 \end{aligned}\right\} \implies 2 = -5c_2 \implies -\frac{2}{5} = c_2$$

$$4 = c_1 - 2\left(-\frac{2}{5}\right) \implies \frac{16}{5} = c_1$$

$$\begin{aligned} p(t) &= \frac{16}{5}e^{2t} + \frac{4}{5}e^{7t} \\ q(t) &= \frac{16}{5}e^{2t} - \frac{6}{5}e^{7t} \end{aligned}$$

This approach to solving equations will be crucial for the next steps involving the equations of motion. More specifically, this will allow us to separate the dimensions from our equations of motion and determine the appropriate initial velocity given initial coordinates to the proximity of the Lagrange points.

4 Deriving Equations of Motion

Deriving the equations of motion to predict the movement of a satellite around a Lagrange point will be much more difficult. We will need to come up with a coordinate system that we can use to define our equations of motion around, as well as specific parameters for the physical system to allow the mathematics to be easier.

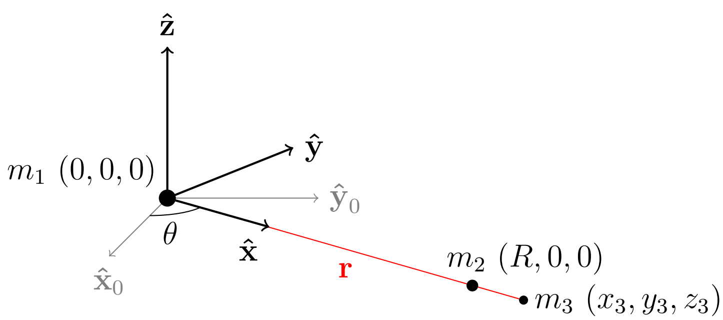


Figure 4: three-dimensional diagram of the Sun-Earth system with unit vectors relative to the Earth's orbit. Not drawn to scale.

Firstly, we must assert that $m_1 \gg m_2$, meaning that the movement of the Sun due to the gravity of Earth is negligible. This allows the Sun to be placed at the center of the coordinate system, avoiding the need to make considerations for the center of mass of the system. It is also asserted that $m_2 \gg m_3$, so that, similarly to the previous assertion, the satellite has a negligible effect on Earth. Secondly, continuing with the conditions from the previous calculations, it is assumed that the Earth is in a circular orbit, making the system a little easier to comprehend. Thirdly, and unlike the previous calculations, we will derive the equations of motion from a inertial reference frame, meaning that we will not be viewing this system from a moving point of reference. This will allow us to completely determine motion without the need to consider fictitious forces.

As shown in Figure 4, we will have unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ drawn relative to the orbit of the Earth. The satellite around L2, m_3, will have the coordinates (x_3, y_3, z_3) represented by the vector $\mathbf{r}$. Taking from my understanding of vectors, the position of a satellite can be expressed as:

$$\mathbf{r} = x_3\hat{\mathbf{x}} + y_3\hat{\mathbf{y}} + z_3\hat{\mathbf{z}}$$

Because the unit vectors are not actually static in the system ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ rotate about the origin), their movement must be taken into account.

$$\hat{\mathbf{x}} = (\cos\theta)\hat{\mathbf{x}}_0 + (\sin\theta)\hat{\mathbf{y}}_0$$
$$\hat{\mathbf{y}} = -(\sin\theta)\hat{\mathbf{x}}_0 + (\cos\theta)\hat{\mathbf{y}}_0$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}_0$$

At this point, it is crucial to notice that these equations are not yet useful to us to describe movement. In order to describe motion, we need to be able to define the velocity and acceleration of the satellite in three-dimensional space. This means being able to take the derivative of a vector. Let us assert the following axiom, for some vector $\mathbf{v}$ with the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components a and b:

$$\frac{d\mathbf{v}}{dt} = \frac{da}{dt}\hat{\mathbf{i}} + \frac{db}{dt}\hat{\mathbf{j}} \tag{5}$$

In other words, the derivative of a vector is the derivative of its components, and would be consistent with the sum rule for derivatives. The notation here can become quite bloated and difficult to read if we continue to use Leibniz's notation. So instead, from here on, we will use Newton's notation where possible.

We can take the second derivative of our position vector $\mathbf{r}$:

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x_3\hat{\mathbf{x}}}{dt^2} + \frac{d^2y_3\hat{\mathbf{y}}}{dt^2} + \frac{d^2z_3\hat{\mathbf{z}}}{dt^2}$$

Our acceleration vector is then expanded to:

$$\mathbf{a} = \ddot{x}_3\hat{\mathbf{x}} + 2\dot{x}_3\dot{\hat{\mathbf{x}}} + x_3\ddot{\hat{\mathbf{x}}} + \ddot{y}_3\hat{\mathbf{y}} + 2\dot{y}_3\dot{\hat{\mathbf{y}}} + x_3\ddot{\hat{\mathbf{y}}} + \ddot{z}_3\hat{\mathbf{z}} + 2\dot{z}_3\dot{\hat{\mathbf{z}}} + z_3\ddot{\hat{\mathbf{z}}}$$

5 Plotting the Orbit

6 Conclusion

Bibliography

"New Technology Innovations Webb Telescope." NASA. Accessed January 11, 2022. https://jwst.nasa.gov/content/about/innovations/index.html.

A Eigenvalues for Larger Systems

For the 3×3 linear system:

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= \lambda x_1 \\ dx_1 + ex_2 + fx_3 &= \lambda x_2 \\ gx_1 + hx_2 + ix_3 &= \lambda x_3, \end{aligned}$$

let $\alpha = a - \lambda$, $\epsilon = e - \lambda$, and $\iota = i - \lambda$.

$$\begin{aligned} \alpha x_1 + bx_2 + cx_3 &= 0 \\ dx_1 + \epsilon x_2 + fx_3 &= 0 \\ gx_1 + hx_2 + \iota x_3 &= 0. \end{aligned}$$

Isolating for x_1 for the equation in the first row,

$$x_1 = -\frac{b}{\alpha}x_2 - \frac{c}{\alpha}x_3$$

Substituting this in the remaining equations gives:

$$\begin{aligned} \left(\epsilon - \frac{bd}{\alpha}\right)x_2 + \left(f - \frac{cd}{\alpha}\right)x_3 = 0 \\ \left(h - \frac{bg}{\alpha}\right)x_2 + \left(\iota - \frac{cg}{\alpha}\right)x_3 = 0 \end{aligned}$$

Isolating for x_2 for the equation in the first row of the remaining equations gives:

$$-\frac{f - \frac{cd}{\alpha}}{\epsilon - \frac{bd}{\alpha}}x_3 = x_2$$

Substituting x_2 for the last equations gives:

$$\frac{\left(h - \frac{bg}{\alpha}\right)\left(f - \frac{cd}{\alpha}\right)}{\epsilon - \frac{bd}{\alpha}}x_3 = \left(\iota - \frac{cg}{\alpha}\right)$$

The x_3 can be cancelled out of the equation, leaving:

$$\left(h - \frac{bg}{\alpha}\right)\left(f - \frac{cd}{\alpha}\right) = \left(\iota - \frac{cg}{\alpha}\right)\left(\epsilon - \frac{bd}{\alpha}\right)$$

$$\frac{1}{\alpha^2}(h\alpha - bg)(f\alpha - cd) = \frac{1}{\alpha^2}(\iota\alpha - cg)(\epsilon\alpha - bd)$$
$$(h\alpha - bg)(f\alpha - cd) = (\iota\alpha - cg)(\epsilon\alpha - bd)$$
$$\alpha^2 fh - \alpha cdh - \alpha bfg + bcdg = \alpha^2\epsilon\iota - \alpha bd\iota - \alpha c\epsilon g + bcdg$$
$$\alpha^2 fh - \alpha cdh - \alpha bfg = \alpha^2\epsilon\iota - \alpha bd\iota - \alpha c\epsilon g$$
$$\alpha fh - cdh - bfg - \alpha\epsilon\iota + bd\iota + c\epsilon g = 0$$
$$\alpha(\epsilon\iota - \alpha f) - b(d\iota - fg) + c(dh - \epsilon g) = 0$$

Then, replacing α, ϵ, and ι:

$$(a-\lambda)[(e-\lambda)(i-\lambda) - (a-\lambda)f] - b[d(i-\lambda) - fg] + c[dh - (e-\lambda)g] = 0$$