The Mathematical Exploration of Lagrange Points

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1 Introduction

In this day and age, the aerospace industry has become a vital component of the global economy. In a publication by the Organisation for Economic Co-operation and Development, the space economy plays a key role in the globalization and digitization of the modern world, with space activities making crucial contributions to the social, economic, and scientific aspects of society.[1] With an increasing investment for spaceflight, there is future need to develop infrastructure that can facilitate the growing space sector, which includes the necessity to understand the nature of orbital mechanics.

Since the longest time, I've always been interested in the development of spaceflight. There's a sort of cosmic beauty with how objects behave in space, subject predominantly and almost exclusively to the fundamental force that is gravity. And it is also because of this exclusivity to a single force that makes it ripe for mathematical exploration. In particular, the launch of the James Webb Space Telescope is a unique mission where the space telescope will sit *behind* the Earth relative to the Sun, in orbit around the Lagrange point L2. This stands against high school intuition that objects in space orbit around bodies with mass; somehow, the JWST is able to exist in a (somewhat) stable orbit around seemingly nothing! **It is for this reason that this investigation aims to explore the mathematics behind Lagrange points.** More specifically, I seek to learn about the nature of Lagrange points, specifically L2, from a basic physics understanding; an understanding through my mathematical knowledge from school; and from a higher level of mathematics such that we can model what the orbit around a collinear Lagrange point would look like.

It is important to note that this paper does not seek out to give a method which can accurately predict the motion of an object in space with clear solutions; there exists a good reason why orbital mechanics is not taught in detail in high school. Instead, this paper investigates Lagrange points in such a way that allows for a conceptual understanding that can still facilitate for numerical computations of the motion of objects near a collinear Lagrange point. It is also due to the complexity of this investigation that more advanced utilities will be used. Primarily, Python libraries will be used for numerical computations of large numbers, numerical approximation of equations with no solutions, and graphing three-dimensional space.

1. OECD, *The Space Economy in Figures* (2019), 200, https://doi.org/https://doi.org/10.1787/c5996201-en, https://www.oecd-ilibrary.org/content/publication/c5996201-en.

2 What are Lagrange Points?

Before we do any math on Lagrange points, its important to know what they are exactly. According to NASA, Lagrange points are special solutions in what is know as the “three-body problem.” At these points in space, the gravitational and rotational forces of the other two bodies effectively cancel each other out, allowing small objects to seemingly stay in place.[2]

With this in mind, lets consider a simple system of two static masses, m_1 and m_2, where $m_1 > m_2$.

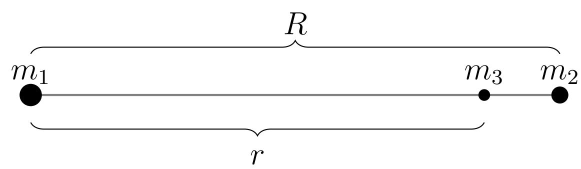


Figure 1: System of two stationary bodies.

In Figure 1, R represents the distance between the two bodies, and r represents the distance from the first body to the equilibrium point, point mass m_3, where the net gravitational force is 0. Using Newton’s Law of Gravitation,

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

where F is the force exerted between two bodies of mass and G is the gravitational constant, we can write the net force of the system as:

$$\frac{Gm_1m_3}{r^2} = \frac{Gm_2m_3}{(R-r)^2}$$

If we let $m_1 = 4m_2$ and $R = 1$, than we can directly solve for r:

$$\frac{m_1}{r^2} = \frac{m_2}{(R-r)^2}$$
$$\left(\frac{R-r}{r}\right)^2 = \frac{m_2}{m_1}$$
$$\left(\frac{1-r}{r}\right)^2 = \frac{m_2}{4m_2}$$

2. “Orbit - Webb/NASA,” NASA, accessed January 11, 2022, https://jwst.nasa.gov/content/about/orbit.html.

$$\begin{aligned}\frac{1-r}{r} &= \frac{1}{2} \\ 2-2r &= r \\ \frac{2}{3} &= r\end{aligned}$$

This non-dimensional look on problems like these will come in handy to simplify more involved equations.

Of course, this scenario is not exemplary of Lagrange points, let alone the interaction between celestial bodies. Let us think increase the complexity of our initial system by having m_2 orbit m_1. With consideration for rotational force, centripetal force is expressed as:

$$F = \frac{4\pi^2 m_3 r}{T^2}$$

where T expresses the orbital period of m_3. Given that, from Kepler's Third Law,

$$T = 2\pi\sqrt{\frac{R^3}{G(m_1+m_2)}}$$

centripetal force can be rewritten as:

$$F = \frac{G(m_1+m_2)}{R^3} r m_3$$

Therefore, the net force between m_1 and m_2 is written as:

$$F = m_3 a = -\frac{G m_1 m_3}{r^2} - \frac{G m_2 m_3}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3} r m_3$$

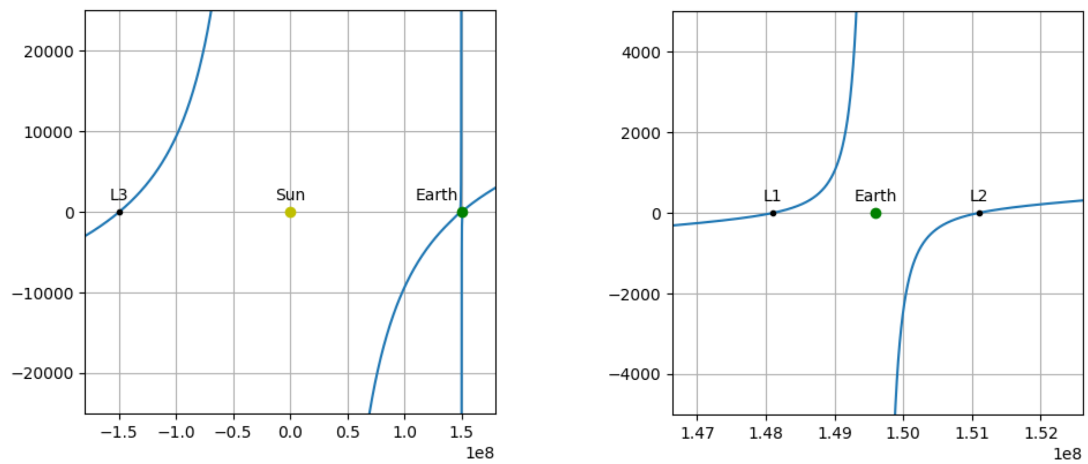
Solving for a radial acceleration, we get the formula:

$$a = -\frac{G m_1}{r^2} - \frac{G m_2}{(r-R)^2} + \frac{G(m_1+m_2)}{R^3} r$$

Most of the variables are known constants, with r being the only unknown value which represents the distance of the Lagrange point from m_1, given that the net radial acceleration is 0. It is worth noting that, with respect to direction, r^2 and $(r-R)^2$ will only reflect acceleration due to gravity in the negative direction and will not be sufficient to tell us where L1 and L3 are. Knowing that, cautiously, $n^2 = n \times |n|$, the formula is rewritten as such to preserve the sign of r:

$$a = -\frac{Gm_1}{r|r|} - \frac{Gm_2}{(r-R)|r-R|} + \frac{G(m_1+m_2)}{R^3}r$$

To save ourselves the agony of whether it is possible to isolate for r, we will use Python to plot the radial acceleration with respect to distance. The Lagrange points are located where the acceleration is 0.



(a) Radial Acceleration between the Sun and Earth.

(b) Radial acceleration around the Earth.

Figure 2: Net acceleration of the Sun-Earth system. Graphs are created using the matplotlib Python library.

Given that we are only concerned with L2, the distance of L2 from the Sun is computed as 1.511×10^8km.

3 Deriving Equations of Motion

As stated in the introduction, finding out the motion of a satellite around a Lagrange point will require us to expand the math that is currently known from IB.

3.1 Simple Linear Differential Equations

Let us imagine that, for some function f with respect to t,

$$\frac{d}{dt}f(t) = 4f(t). \tag{2}$$

In other words, the rate of change of the function f is equal to itself multiplied by four. Let us say we wanted to find what function f is. While it is not as simple as to integrate this equation, we know that e^x is its own derivative. We also know that, because of the chain rule, $[e^{kt}]' = ke^{kt}$. If we substitute $f(t)$ for e^{kt} and let $k = 4$, then Equation (2) is true. Therefore, we can state the following **theorem**; for any equation

$$\frac{d}{dt}f(t) = kf(t) \ : \ f(t) = ce^{kt} \tag{3}$$

where c is some initial constant that cannot be expressed from integrating the derivative. We can verify that this theorem is correct using some algebra and rearranging variables (where $y = f(x)$):

$$\begin{aligned}
\frac{dy}{dx} &= ky \\
\frac{1}{y}\,dy &= k\,dx \\
\int \frac{1}{y}\,dy &= \int k\,dx \\
\ln y + C &= kx + D \\
\ln y &= kx + (D - C) \\
y &= e^{kt} \cdot e^{D-C}, \ \text{let } c = e^{D-C} \\
y &= ce^{kt}.
\end{aligned}$$

Equation (2) is an example of a differential equation, where a function is defined by its derivatives. Differential equations are very common in physics and will be how our equations of motion will be defined.

3.2 Eigenvalues and Eigenvectors

Imagine that there are two species of ants, p and q, whose populations are dependent on each other. That is to say that both species try to kill the other species while repopulating their own. We can represent the populations of these two ant species through the following recursive system of equations:

$$\begin{aligned} p_{n+1} &= 4p_n - 2q_n \\ q_{n+1} &= -3p_n + 5q_n, \end{aligned}$$

where each recursive step n represents some interval of time. Say we knew that, for some initial population of both species, their population would double after the given time interval and we want to know what that initial population would be. We can determine the initial population by changing our system:

$$\begin{aligned} 2p_k &= 4p_k - 2q_k \\ 2q_k &= -3p_k + 5q_k. \end{aligned}$$

Then, all that is needed is to solve for p_k and q_k:

$$\left.\begin{aligned} 2p_k &= 4p_k - 2q_k \\ 2q_k &= -3p_k + 5q_k \end{aligned}\right\} \implies 0 = p_k - q_k \implies p_k = q_k.$$

Here, $p_k = q_k$ indicates that any amount of ants is a solution so long as both initial populations have a $1:1$ ratio. For example, $p_k = 200$ and $q_k = 200$ is a solution:

$$\begin{aligned} 2(200) &= 4(200) - 2(200) = 400 \\ 2(200) &= -3(200) + 5(200) = 400. \end{aligned}$$

In linear algebra, the idea of a linear system with a set of variables being equivalent to scaling said variables is termed an eigenvalue. More precisely, an eigenvalue is a numerical constant which represents the factor of which an eigenvector is scaled. In the case of the ants, the eigenvalue is 2, Because linear algebra is beyond the scope of this investigation, we will only explore what is necessary to understand orbital motion.

It is, in fact, possible to determine both the eigenvalues and eigenvectors of most linear systems without knowing either to begin with. Looking back to our first scaled system, we can generalize it for some eigenvalue λ as such:

$$\begin{aligned} \lambda p_k &= 4p_k - 2q_k \\ \lambda q_k &= -3p_k + 5q_k. \end{aligned}$$

The eigenvalues can then be found through substitution and cancelling out the other variables.

$$\begin{aligned} 4p_k - 2q_k &= \lambda q_k \\ (4-\lambda)p_k &= 2q_k \\ \frac{(4-k)p_k}{2} &= q_k \\ -3p_k + 5q_k &= \lambda q_k \\ -3p_k &= (\lambda - 5)q_k \\ -3p_k &= (\lambda - 5)\frac{(4-\lambda)p_k}{2} \\ 3 \cdot 2 &= (5-\lambda)(4-\lambda) \\ 0 &= (5-\lambda)(4-\lambda) - 3 \cdot 2 \qquad (4) \\ 0 &= \lambda^2 - 9\lambda + 14 \\ 0 &= (\lambda - 2)(\lambda - 7) \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 7$. We can find the respective eigenvector for λ_2 by doing the same operation for λ_1.

$$\left.\begin{aligned} 7p_k &= 4p_k - 2q_k \\ 7q_k &= -3p_k + 5q_k \end{aligned}\right\} \implies 0 = -3p_k - 2q_k \implies 3p_k = -2q_k.$$

This gives us an eigenvector of $(-2, 3)$. While a negative amount of ants does not make much sense, these values will be important later for when we want to generalize this system.

From Equation (4), we can generalize the computation of any linear system with 2 variables and 2 equations as:

$$0 = (a - \lambda)(d - \lambda) - bc,$$

where a, b, c, d are the coefficients of the linear system. This computation of solving for eigenvalues can be extrapolated for larger systems involving n variables and n equations.

For a linear system with 3 variables, 3 equations and 9 constants $a, b, \ldots, i$:

$$0 = (a-\lambda)((e-\lambda)(i-\lambda) - fh) - b(d(i-\lambda) - fg) + c(dh - eg).$$

And for 4 variables, 4 equations,and :

$$1 = 1$$

Their solutions are worked through in Appendix A.

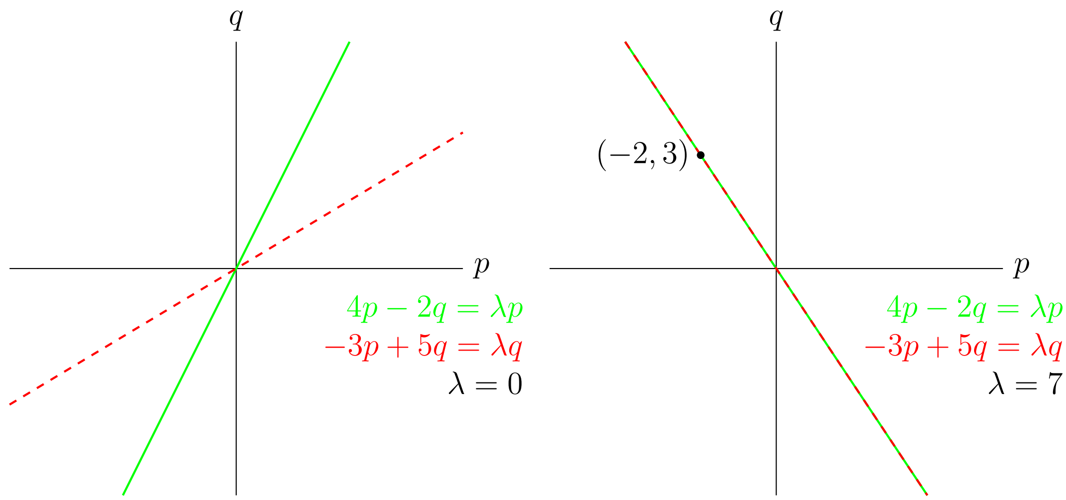
3.3 Systems of Linear Differential Equations

Now that we have some understanding of what can be done with a linear system, it is time to combine linear systems with calculus.

Now say that we wanted to generalize our system of equations so that we can determine either ant population for some time, t. From what has already been established, this relation would be defined as:

$$\begin{aligned} p'(t) &= 4p(t) - 2q(t) \qquad && p(0) = 4 \\ q'(t) &= -3p(t) + 5q(t) \qquad && q(0) = 2 \end{aligned}$$

The approach to solving for $p(t)$ and $p(t)$ is not immediately intuitive. Now is a good time to view eigenvalues and eigenvectors in a more abstract manner.



(a) Linear system with $\lambda = 0$. (b) Linear system with $\lambda = 7$.

Figure 3: The linear system defined with regard to λ (It is better to think of the graphs as ratios between p and q rather than q as a function of p).

In Figure 3a, where the p and q are viewed as a set of values, the equations representing the ant species are on different lines with an intersection at $(0,0)$. In comparison, in Figure 3b, the equations have converged onto the same line when $\lambda = 7$. This can be thought of as, for each eigenvalue, $p'(t)$ and $q'(t)$ are transformed onto a function which both share in common. We will call these 'common' functions $x(t)$ and $y(t)$. The eigenvalues can define these 'common' functions as:

$$x'(t) = \lambda_1 x(t)$$
$$y'(t) = \lambda_2 y(t),$$

and are connected back to the functions $p(t)$ and $q(t)$ by the eigenvectors:

$$p(t) = (1)x(t) + (-2)y(t)$$
$$q(t) = (1)x(t) + (3)y(t)$$

What we have essentially done is factored out a group of functions $x(t)$ and $y(t)$ from the linear system that are much easier to to solve alongside a set of coefficients, the eigenvectors, that relate $x(t)$ and $y(t)$ to $p(t)$ and $q(t)$. Solving for the 'common' functions:

$$x'(t) = 2x(t) \qquad\qquad x(t) = c_1 e^{2t}$$
$$y'(t) = 7y(t) \qquad\qquad y(t) = c_2 e^{7t}$$

Then we relate the these back to our original functions:

$$p(t) = c_1 e^{2t} - 2c_2 e^{7t}$$
$$q(t) = c_1 e^{2t} + 3c_2 e^{7t}$$

All that is left is to solve for the given initial conditions:

$$\left.\begin{aligned} p(0) = 4 = c_1 - 2c_2 \\ q(0) = 2 = c_1 + 3c_2 \end{aligned}\right\} \implies 2 = -5c_2 \implies -\frac{2}{5} = c_2$$
$$4 = c_1 - 2\left(-\frac{2}{5}\right) \implies \frac{16}{5} = c_1$$
$$p(t) = \frac{16}{5}e^{2t} + \frac{4}{5}e^{7t}$$
$$q(t) = \frac{16}{5}e^{2t} - \frac{6}{5}e^{7t}$$

This approach to solving equations will be crucial for the next steps involving the equations of motion. More specifically, this will allow us to separate the dimensions from our equations of motion and determine the appropriate initial velocity given initial coordinates to the proximity of the Lagrange points.

4 Plotting the Orbit around a Lagrange point

Deriving the equations of motion to predict the movement of a satellite around a Lagrange point will be much more difficult than locating the Lagrange points. We will need to come up with a coordinate system that we can use to define our equations of motion around, as well as specific parameters for the physical system to allow the mathematics to be easier.

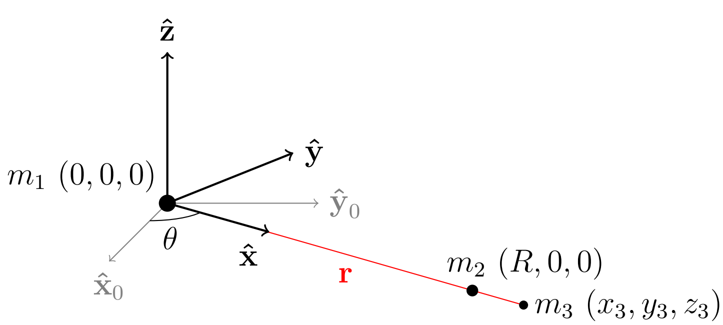


Figure 4: three-dimensional diagram of the Sun-Earth system with unit vectors relative to the Earth's orbit. Not drawn to scale.

Firstly, we must assert that $m_1 \gg m_2$, meaning that the movement of the Sun due to the gravity of Earth is negligible. This allows the Sun to be placed at the center of the coordinate system, avoiding the need to make considerations for the center of mass of the system. It is also asserted that $m_2 \gg m_3$, so that, similarly to the previous assertion, the satellite has a negligible effect on Earth. Secondly, continuing with the conditions from the previous calculations, it is assumed that the Earth is in a circular orbit, making the system a little easier to comprehend. Thirdly, and unlike the previous calculations, we will derive the equations of motion from a inertial reference frame, meaning that we will not be viewing this system from a moving point of reference. This will allow us to completely determine motion without the need to consider fictitious forces.

As shown in Figure 4, we will have unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ drawn relative to the orbit of the Earth. The satellite around L2, m_3, will have the coordinates (x_3, y_3, z_3) represented by the vector $\mathbf{r}$. Taking from my understanding of vectors, the position of a satellite can be expressed as:

$$\mathbf{r} = x_3\hat{\mathbf{x}} + y_3\hat{\mathbf{y}} + z_3\hat{\mathbf{z}}$$

Because the unit vectors are not actually static in the system ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ rotate about the origin), their movement must be taken into account.

$$\begin{aligned} \hat{\mathbf{x}} &= (\cos\theta)\hat{\mathbf{x}}_0 + (\sin\theta)\hat{\mathbf{y}}_0 \\ \hat{\mathbf{y}} &= -(\sin\theta)\hat{\mathbf{x}}_0 + (\cos\theta)\hat{\mathbf{y}}_0 \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}_0 \end{aligned}$$

At this point, it is crucial to notice that these equations are not yet useful to us to describe movement. In order to describe motion, we need to be able to define the velocity and acceleration of the satellite in three-dimensional space. This means being able to take the derivative of a vector. Let us assert the following axiom, for some vector $\mathbf{v}$ with the basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components a and b:

$$\frac{d\mathbf{v}}{dt} = \frac{da}{dt}\hat{\mathbf{i}} + \frac{db}{dt}\hat{\mathbf{j}} \tag{5}$$

In other words, the derivative of a vector is the derivative of its components, and would be consistent with the sum rule for derivatives. The notation here can become quite bloated and difficult to read if we continue to use Leibniz's notation. So instead, from here on, we will use Newton's notation where possible.

We can take the second derivative of our position vector $\mathbf{r}$:

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x_3\hat{\mathbf{x}}}{dt^2} + \frac{d^2y_3\hat{\mathbf{y}}}{dt^2} + \frac{d^2z_3\hat{\mathbf{z}}}{dt^2}$$

Our acceleration vector is then expanded to:

$$\mathbf{a} = \ddot{x}_3\hat{\mathbf{x}} + 2\dot{x}_3\dot{\hat{\mathbf{x}}} + x_3\ddot{\hat{\mathbf{x}}} + \ddot{y}_3\hat{\mathbf{y}} + 2\dot{y}_3\dot{\hat{\mathbf{y}}} + x_3\ddot{\hat{\mathbf{y}}} + \ddot{z}_3\hat{\mathbf{z}} + 2\dot{z}_3\dot{\hat{\mathbf{z}}} + z_3\ddot{\hat{\mathbf{z}}}$$

5 Conclusion

Bibliography

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"Orbit - Webb/NASA." NASA. Accessed January 11, 2022. https://jwst.nasa.gov/content/about/orbit.html.

A Eigenvalues for Larger Systems

For the 3×3 linear system:

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= \lambda x_1 \\ dx_1 + ex_2 + fx_3 &= \lambda x_2 \\ gx_1 + hx_2 + ix_3 &= \lambda x_3 \end{aligned}$$

let $\alpha = a - \lambda$, $\epsilon = e - \lambda$, and $\iota = i - \lambda$.

$$\begin{aligned} \alpha x_1 + bx_2 + cx_3 &= 0 \\ dx_1 + \epsilon x_2 + fx_3 &= 0 \\ gx_1 + hx_2 + \iota x_3 &= 0 \end{aligned}$$

Isolating for x_1 for the equation in the first row,

$$x_1 = -\frac{b}{\alpha}x_2 - \frac{c}{\alpha}x_3$$

Substituting x_1 in the remaining equations gives:

$$\begin{aligned} \left(\epsilon - \frac{bd}{\alpha}\right)x_2 + \left(f - \frac{cd}{\alpha}\right)x_3 = 0 \\ \left(h - \frac{bg}{\alpha}\right)x_2 + \left(\iota - \frac{cg}{\alpha}\right)x_3 = 0 \end{aligned}$$

Isolating for x_2 for the equation in the first row of the remaining equations gives:

$$-\frac{f - \frac{cd}{\alpha}}{\epsilon - \frac{bd}{\alpha}}x_3 = x_2$$

Substituting x_2 for the last equations gives:

$$\frac{\left(h - \frac{bg}{\alpha}\right)\left(f - \frac{cd}{\alpha}\right)}{\epsilon - \frac{bd}{\alpha}}x_3 = \left(\iota - \frac{cg}{\alpha}\right)$$

The x_3 can be cancelled out of the equation, leaving:

$$\left(h - \frac{bg}{\alpha}\right)\left(f - \frac{cd}{\alpha}\right) = \left(\iota - \frac{cg}{\alpha}\right)\left(\epsilon - \frac{bd}{\alpha}\right)$$

$$
\begin{aligned}
\frac{1}{\alpha^2}(h\alpha - bg)(f\alpha - cd) &= \frac{1}{\alpha^2}(\iota\alpha - cg)(\epsilon\alpha - bd) \\
(h\alpha - bg)(f\alpha - cd) &= (\iota\alpha - cg)(\epsilon\alpha - bd) \\
\alpha^2 fh - \alpha cdh - \alpha bfg + bcdg &= \alpha^2\epsilon\iota - \alpha bd\iota - \alpha c\epsilon g + bcdg \\
\alpha^2 fh - \alpha cdh - \alpha bfg &= \alpha^2\epsilon\iota - \alpha bd\iota - \alpha c\epsilon g \\
\alpha fh - \alpha\epsilon\iota + bd\iota - bfg + c\epsilon g - cdh &= 0 \\
\alpha(\epsilon\iota - \alpha f) - b(d\iota - fg) + c(dh - \epsilon g) &= 0 \qquad (6)
\end{aligned}
$$

Then, replacing α, ϵ, and ι:

$$(a-\lambda)[(e-\lambda)(i-\lambda)-(a-\lambda)f]-b[d(i-\lambda)-fg]+c[dh-(e-\lambda)g]=0$$