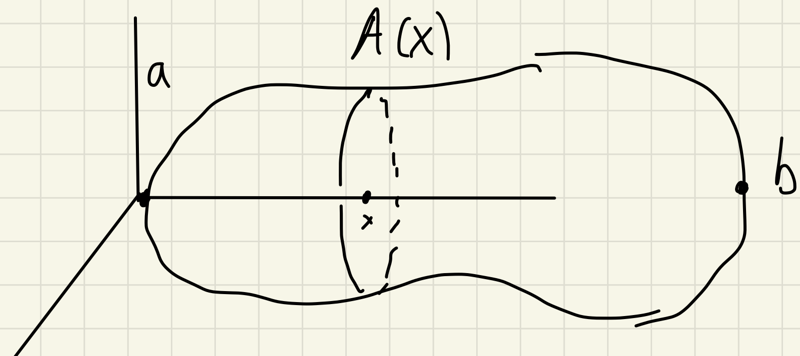
Clase 23

Temario:

- Volúmenes
- Técnicas de integración
- Longitud de arco

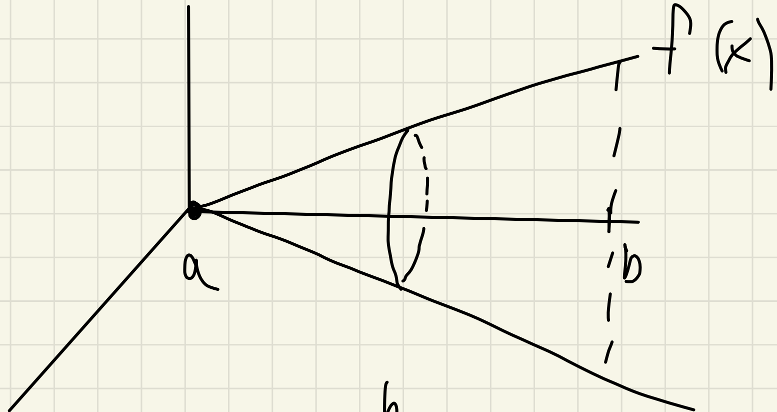
Volumenes:

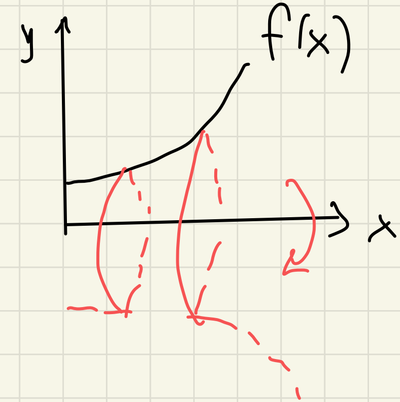
Por área
secciones
transv:



$$Vol = \int_a^b A(x)\,dx$$

Método de discos:

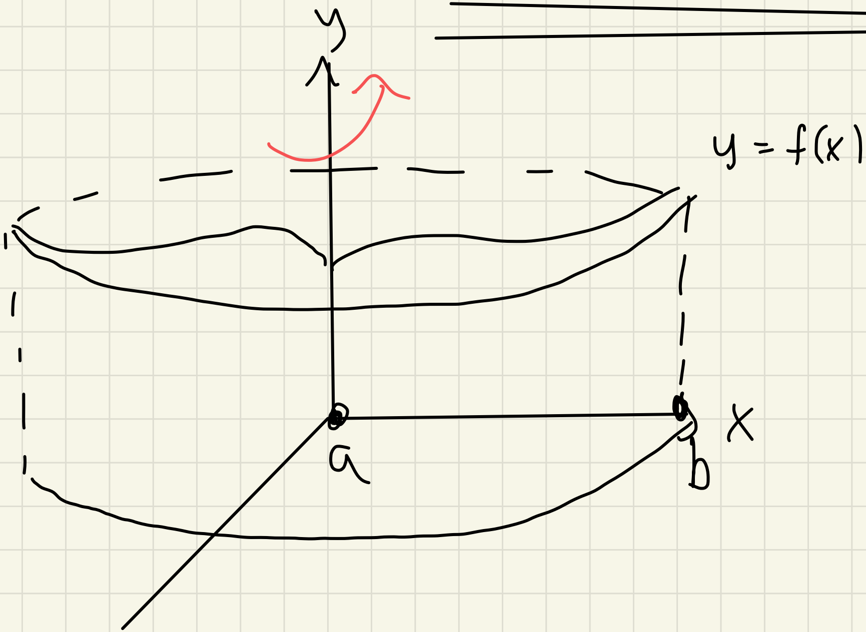




$$Vol = \int_a^b \pi (f(x))^2 dx$$

$$Vol = \int_a^b \pi (f(x))^2 dx - \int_a^b \pi (g(x))^2$$

$$= \int_a^b \pi \left(f(x)^2 - g(x)^2\right) dx$$



$$\text{Vol} = \int_a^b 2\pi x f(x)\, dx$$

Ej: Calcule el volumen generado al rotar la función $y = x^{3/2}$ en torno al eje x, entre $x=1$ y $x=3$

$$\text{Vol} = \int_1^3 \pi \left(x^{3/2}\right)^2 dx$$

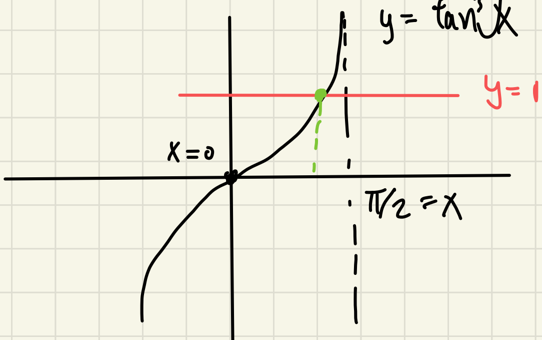
$$= \int_1^3 \pi x^3 dx$$

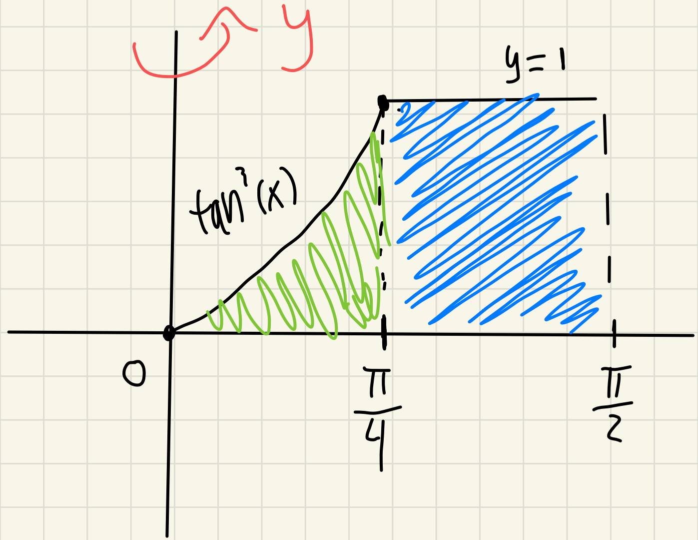
$$= \pi \frac{x^4}{4}\Big|_1^3$$

$$= \pi\left(\frac{81}{4} - \frac{1}{4}\right) = \pi \frac{80}{4} = 20\pi$$

Ej: Escriba una integral para calcular el volumen del sólido generado al rotar la función

a) $y = \tan^3 x$, entre $y = 1$, $x = 0$, $x = \pi/2$, alrededor del eje y.





$$\tan^3(x) = 1$$

$$\tan(x) = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

método de anillos: $\int 2\pi x \cdot f(x)\, dx$

Volumen total al girar el área es el vol al girar area verde + vol al girar área azul

$$\text{vol verde} = \int_0^{\pi/4} 2\pi x \tan^3(x)\, dx$$

$$\text{vol azul} = \int_{\frac{\pi}{4}}^{\pi/2} 2\pi x \cdot 1\, dx$$

$$\underset{total}{Vol} = \int_0^{\pi/2} 2\pi x \tan^3 x \, dx + \int_{\frac{\pi}{4}}^{\pi/2} 2\pi x \, dx$$

Métodos de integración:

1. Sustitución

$$u = g(x)$$
$$du = g'(x)\,dx$$

Ej: $\int \sqrt{x+7}\,dx = \int (x+7)^{1/2}\,dx$

$u = x+7$

$du = dx$ $\qquad = \int u^{1/2}\,du = \dfrac{u^{3/2}}{3/2} + C$

$= \dfrac{2}{3}(x+7)^{3/2} + C$

Ej: $\int \operatorname{sen}(x^3+3)\,x^2\,dx = \int \operatorname{sen}(u)\,\dfrac{du}{3}$

$u = x^3+3$

$$du = 3x^2\,dx$$
$$\frac{du}{3} = x^2\,dx$$

$$= -\frac{\cos(u)}{3} + C$$

$$= -\frac{\cos(x^3+3)}{3} + C$$

Ej: $\int e^{\tan(x)} \sec^2(x)\,dx = \int e^u\,du$

$$u = \tan(x)$$
$$du = \sec^2(x)\,dx$$

$$= e^u + C$$
$$= e^{\tan(x)} + C$$

2. Integración por partes

$$\int u\,dv = uv - \int v\,du$$

$u =$ ILATE

Ej: $\int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$

$u = x \quad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C$$

Ej: $\int x^5 \ln x \, dx$ $\qquad u =$ ILATE

$u = \ln x \qquad dv = x^5 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^6}{6}$

$$\int x^5 \ln x = \overbrace{(\ln x)\cdot \frac{x^6}{6}}^{u\cdot v} - \overbrace{\int \frac{x^6}{6}\cdot \frac{1}{x}\,dx}^{-\int v\,du}$$

$$= (\ln x)\cdot \frac{x^6}{6} - \int \frac{x^5}{6}\,dx$$

$$= (\ln x)\cdot \frac{x^6}{6} - \frac{x^6}{36} + C$$

Ej: $\int e^{\sqrt{x}}\,dx$

$$\int e^x dx = e^x + C$$

$$t = \sqrt{x} = x^{1/2} \iff t^2 = x \qquad 2t\,dt = dx$$

$$dt = \frac{1}{2} x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx$$

$$dx = 2\sqrt{x}\, dt = 2t\,dt$$

$$\int e^{\sqrt{x}} dx = \int e^t 2t\,dt$$

$$= 2\int e^t t\,dt$$

$u = t \qquad dv = e^t dt$

$du = dt \qquad v = e^t$

$$= 2\left(te^t - \int e^t dt\right)$$

$$= 2(te^t - e^t) + C \qquad t = \sqrt{x}$$

$$= 2(\sqrt{x}\, e^{\sqrt{x}} - e^{\sqrt{x}}) + C$$

$sen^2 + cos^2 = 1$

$x = a\,sen(t)$

3. Fracciones parciales:

$\sqrt{a^2 - x^2}$

Ej: $\int \frac{x-9}{x^2+3x-10}\,dx$

$$x^2+3x-10 = (x+5)(x-2)$$

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \frac{A}{x+5} + \frac{B}{x-2}$$

$$= \frac{A(x-2)+B(x+5)}{(x+5)(x-2)}$$

$$\frac{x-9}{(x+5)(x-2)} = \frac{x(A+B) + (-2A + 5B)}{(x+5)(x-2)}$$

$$A + B = 1$$
$$-2A + 5B = -9$$

$$2A + 2B = 2$$
$$-2A + 5B = -9$$
$$7B = -7$$
$$B = -7/7 = -1$$
$$A = 1 - B = 1 + 1 = 2$$
$$= 2$$

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int 2 \cdot \frac{1}{x+5}\,dx - \int 1 \cdot \frac{1}{x-2}\,dx$$

$$= 2\ln|x+5| - \ln|x-2| + C$$

$$\int \frac{x-7}{x(x+2)^2}\,dx = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x+2)x + Cx}{x(x+2)^2}$$

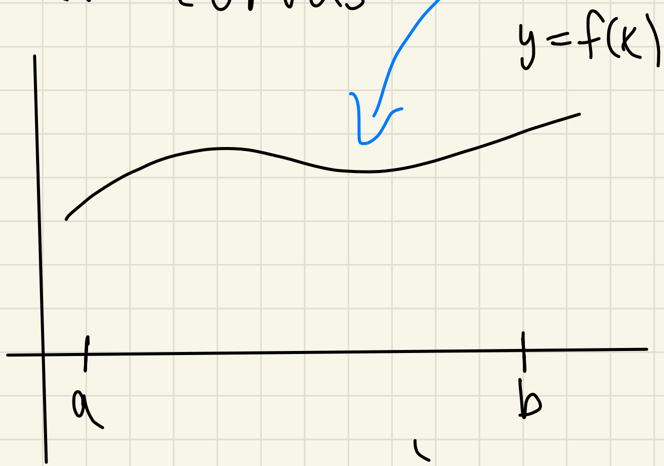
$\vdots$

$$\int \frac{x^3+7x^2+2x+1}{x^2-1}\,dx = \int x + \frac{11}{2(x-1)} - \frac{5}{2(x+1)} + 7\,dx$$

$$= \int x\,dx + \frac{11}{2}\int \frac{1}{x-1}\,dx - \frac{5}{2}\int \frac{1}{x+1}\,dx + \int 7\,dx$$

$$= \frac{x^2}{2} + \frac{11}{2}\ln|x-1| - \frac{5}{2}\ln|x+1| + 7x + C$$

Largo de curvas:



largo de esa cuerda

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Ej: calcule el largo de la curva

a) $y = 1 + 6x^{3/2}$ entre $x = 0$ y $x = 1$

$$f(x) = 1 + 6x^{3/2}$$

$$f'(x) = 6 \cdot \frac{3}{2} \cdot x^{1/2}$$

$$= 9x^{1/2}$$

$$(f'(x))^2 = 81x$$

$$1 + (f'(x))^2 = 1 + 81x$$

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + 81x}$$

$u = 1 + 81x \quad du = 81dx$

$$\int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + 81x}\, dx = \int \sqrt{u}\, du / 81$$

$$= \int u^{1/2} \frac{du}{81} = \frac{u^{3/2}}{3/2 \; 81}$$

$$= \frac{2(1+81x)^{3/2}}{3 \cdot 81} \Big|_0^1 = \frac{2\sqrt[3]{82^2}}{3 \cdot 81} - \frac{2}{3 \cdot 81}$$

Ej: calcule la longitud de la curva

$y = \ln(\cos(x))$ $\quad x = 0 \; \ldots \; x = \pi/3$

$f(x) = \ln(\cos x)$

$$f'(x) = \frac{1}{\cos(x)} \cdot -\operatorname{sen}(x) = -\tan(x)$$

$$(f'(x))^2 = \tan^2(x)$$

$$(f'(x))^2 + 1 = 1 + \tan^2(x) = \sec^2(x)$$

$$\sqrt{(f'(x))^2 + 1} = \sqrt{\sec^2(x)} = |\sec(x)|$$

$$= \sec(x)$$ pues $\cos(x) > 0$ para $x \in [0, \pi/3]$

$$\int_0^{\pi/3} \sqrt{1 + (f'(x))^2}\, dx = \int_0^{\pi/3} \sec(x)\, dx$$

$$= \log(\tan(x) + \sec(x))\Big|_0^{\pi/3} \approx 1.370$$

$$=$$