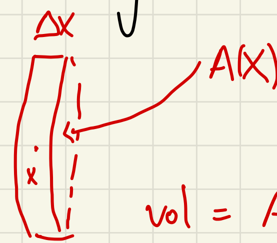
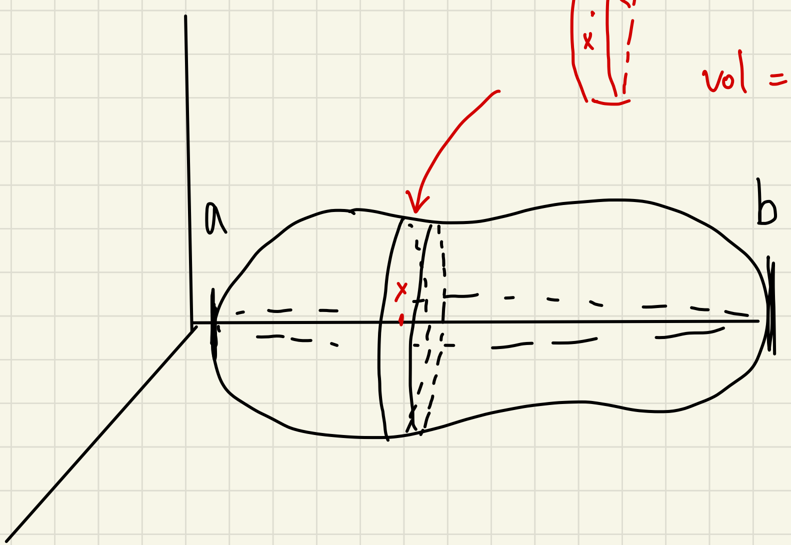
Clase 29: repaso 2

Aplicaciones de la integral

1. Volúmenes:



$$vol = A(x)\Delta x$$



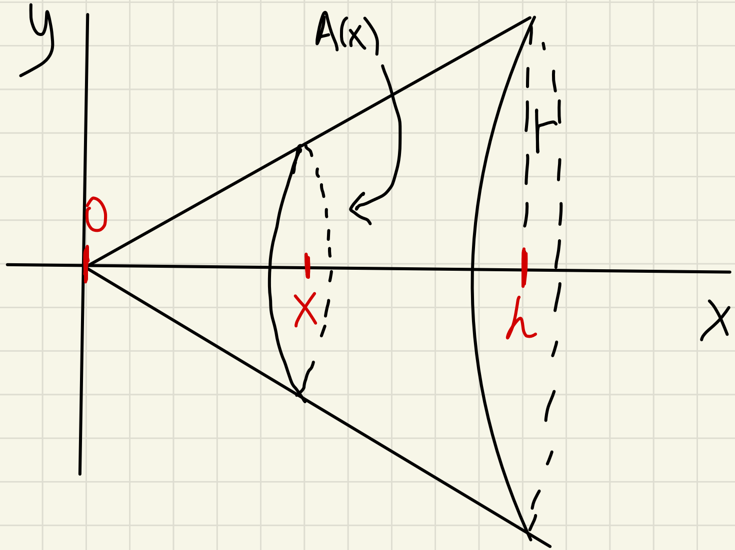
$$vol = \sum A(x)\Delta x$$

$$\downarrow$$

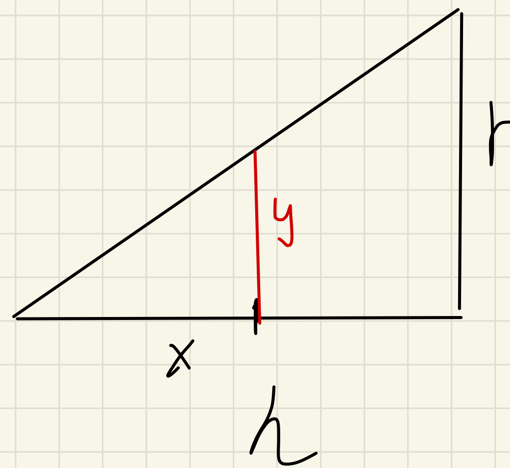
$$\int_a^b A(x)\,dx$$

Método de áreas transversales.

Ejemplo:



Preg: Vol cono de radio r y altura h.



$$A(x) = \pi (\text{radio})^2$$
$$= \pi y^2$$

$$\frac{x}{y} = \frac{h}{r} \qquad y = \frac{xr}{h}$$

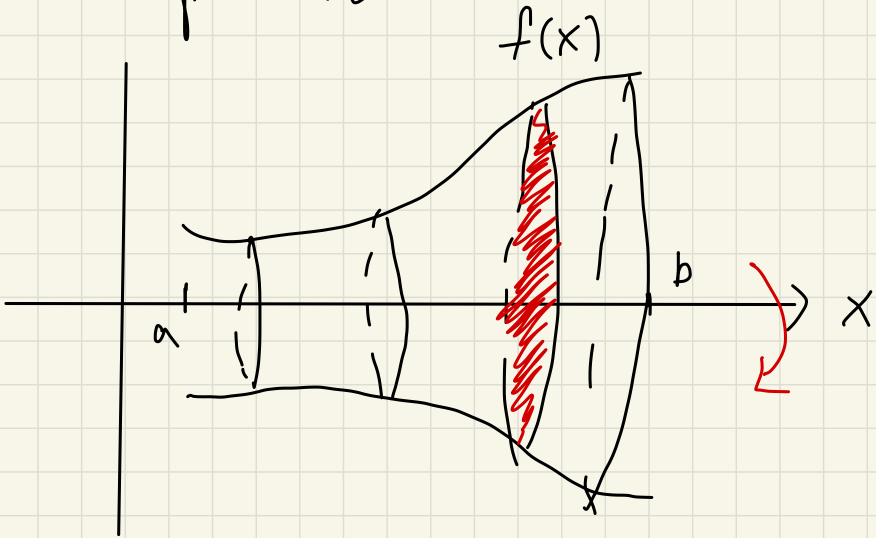
$$A(x) = \pi\left(\frac{xr}{h}\right)^2 = \pi x^2 \frac{r^2}{h^2}$$

$$\text{vol} = \int_0^h A(x)\,dx = \int_0^h \pi \frac{r^2}{h^2} x^2\,dx$$

$$= \pi \frac{r^2}{h^2} \cdot \int_0^h x^2\,dx = \pi \frac{r^2}{h^2} \cdot \frac{x^3}{3} \Big|_0^h$$

$$= \pi \frac{r^2}{\cancel{h^2}} \cdot \frac{h^{\cancel{3}}}{3} = \frac{\pi r^2 h}{3}$$

Caso importante: volumenes de revolución



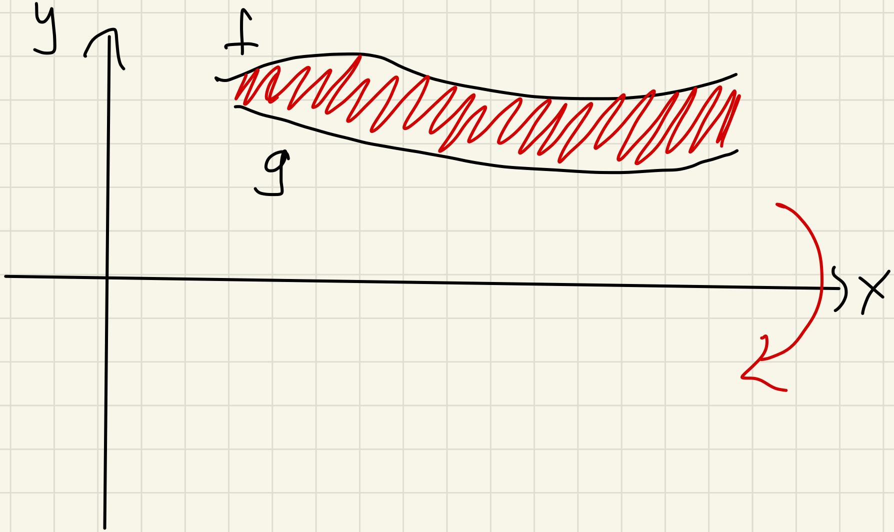
$$\text{vol} = \int_a^b A(x)\,dx$$

$$= \int_a^b f(x)^2 \pi \, dx$$

método de discos

$$A(x) = \pi(\text{radio})^2 = \pi(f(x))^2$$

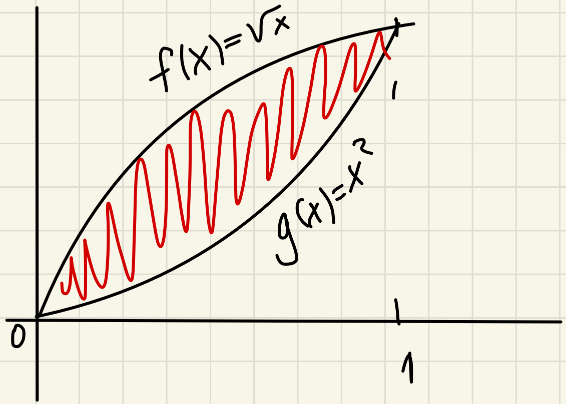
Situación:



En este caso, el volumen generado es

$$\text{vol} = \int_a^b \pi (f(x))^2 \, dx - \int_a^b \pi (g(x))^2 \, dx$$

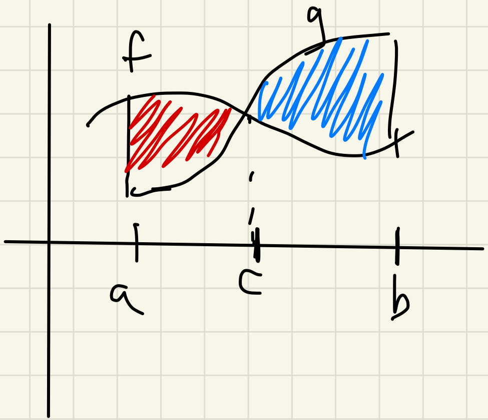
Ej:



$$\text{Vol} = \int_0^1 \pi (\sqrt{x})^2 dx - \int_0^1 \pi (x^2)^2 dx$$

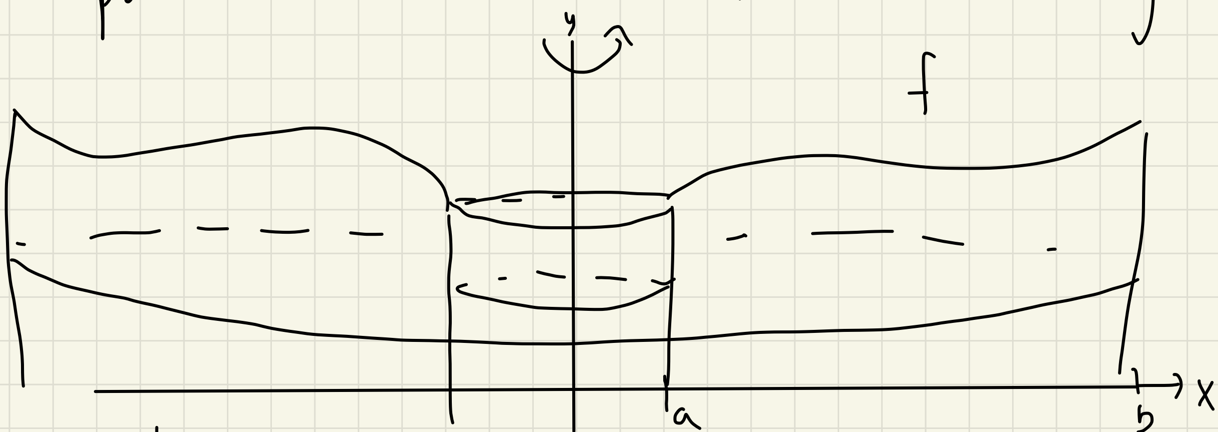
$$= \int_0^1 \pi x^2 - \pi x^4 \, dx = \pi \int_0^1 x^2 dx - \pi \int_0^1 x^4 dx$$

$$= \frac{\pi}{3} - \frac{\pi}{5}$$



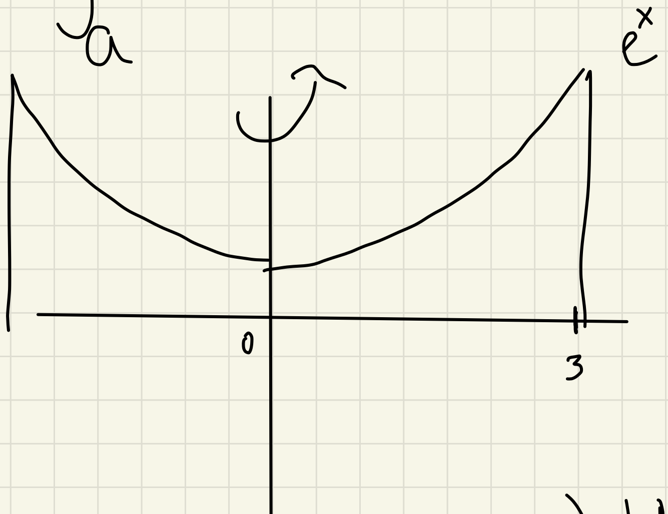
$$Vol = \int_a^c \pi\left((f(x))^2 - (g(x))^2\right) dx + \int_c^b \pi\left(g^2(x) - f^2(x)\right) dx$$

Qué pasa si rotamos en torno al eje y??



$$\mathrm{Vol} = \int_a^b 2\pi x\, f(x)\, dx$$ método cascarones cilíndricos

Ej:



$$\mathrm{vol} = \int_0^1 2\pi x e^x \, dx$$

$$= 2\pi \int_0^1 x e^x \, dx$$

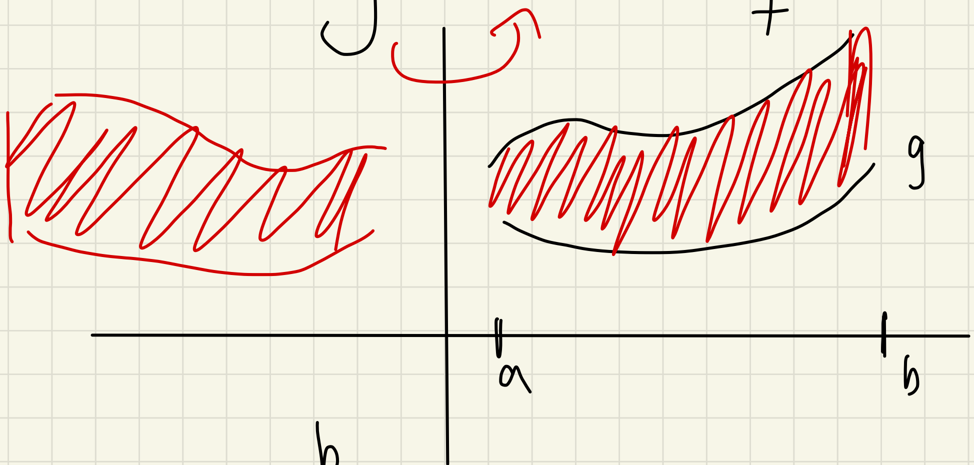
$$= 2\pi \left(x e^x - e^x \right) \Big|_0^1 = 2\pi \left(\cancel{1 e^1} - \cancel{e^1} - \left(\cancel{0 \cdot e^0} - e^0 \right) \right)$$

$$= 2\pi$$

$$\left(x e^x - e^x \right)' = (x)' e^x + x (e^x)' - (e^x)' = e^x + x e^x - e^x$$

$$= x e^x$$

Casos más generales:

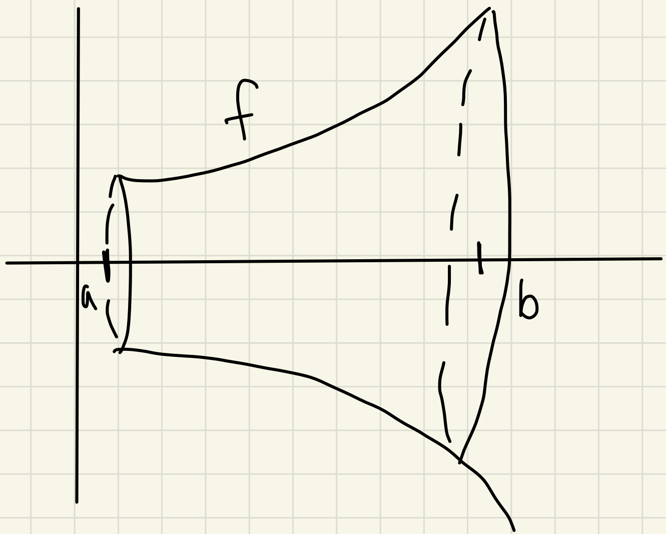


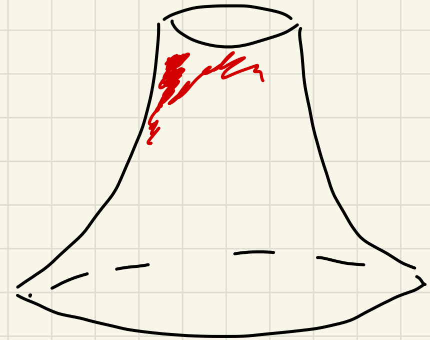
$f \geq g$

$$\mathrm{vol} = \int_a^b 2\pi x\,(f(x) - g(x))\,dx$$

Pregunta: qué pasa con las áreas de los

mantos de estas superficies??





$$\text{Area} = \int_a^b 2\pi\, f(x) \cdot \sqrt{1 + f'(x)^2}\, dx$$

Ej: Calcule el área obtenida al rotar

$$f(x) = \frac{x^3}{6} + \frac{1}{2x} \qquad x = \frac{1}{2}, \ldots, 1$$

$$f'(x) = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2} \qquad (x^{-1})' = -1x^{-2}$$

$$f'(x)^2 = \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$1 + f'(x)^2 = \frac{4}{4} + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$= \frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)$$

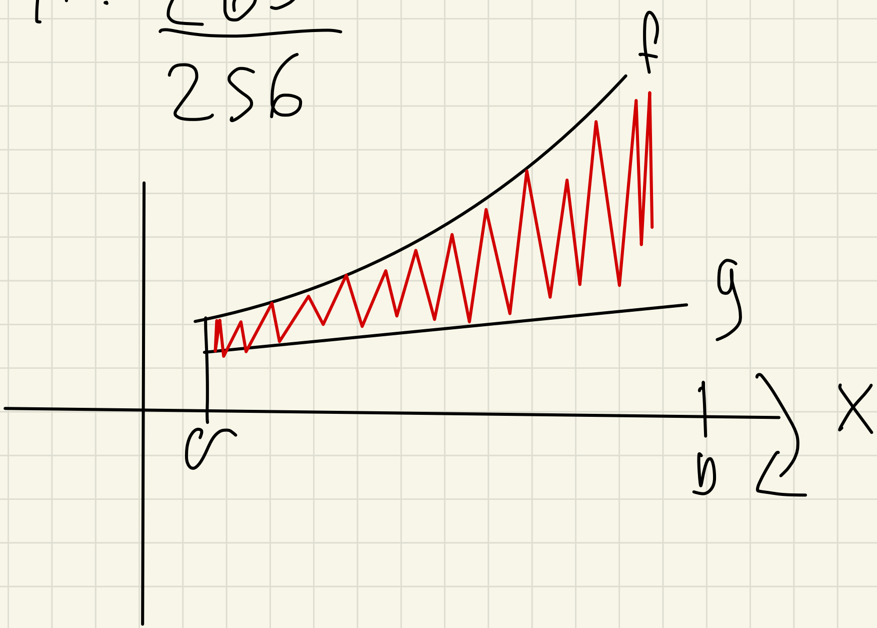
$$= \left(\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\right)^2$$

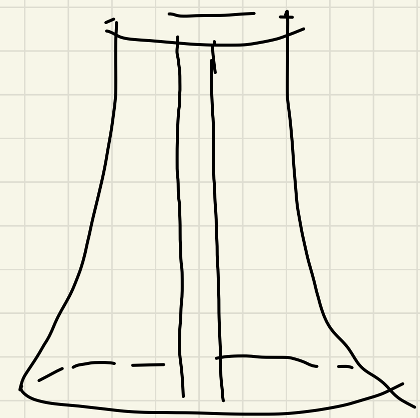
$$\sqrt{1 + f'(x)^2} = \sqrt{\left(\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$$

$$\int_a^b 2\pi f(x) \cdot \sqrt{1 + f'(x)^2}\, dx = \int_{1/2}^{1} 2\pi \cdot \left(\frac{x^3}{6} + \frac{1}{2x}\right) \cdot \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx$$

$$= \pi \int_{1/2}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(x^2 + \frac{1}{x^2}\right) dx$$

$$= \pi \cdot \frac{263}{256}$$





$$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \quad \text{afuera}$$

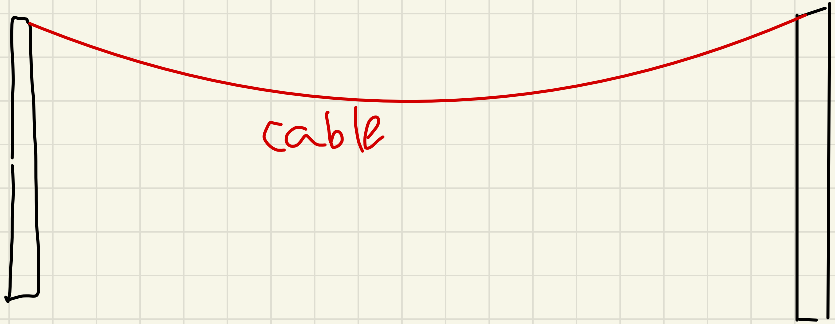
$$\int_a^b 2\pi g(x)\sqrt{1+g'(x)^2}\,dx \quad \text{adentro.}$$

Area total = afuera + adintro

Largo de curvas:

$$\text{largo} = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Ej:



La curva que describe la forma que toma el cable

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Calcule el largo de la curva $f(x) = \cosh x$ entre $x = -1$ y $x = 1$

Sol: $\int_a^b \sqrt{1 + f'(x)^2}\, dx$

$$\begin{cases} f(x) = \cos x \\ f'(x) = -\operatorname{sen} x \\ \operatorname{sen}^2(x) + \cos^2(x) = 1 \end{cases}$$

$\operatorname{senh} = \dfrac{e^x - e^{-x}}{2}$

$f(x) = \cosh(x)$

$\cosh = \dfrac{e^x + e^{-x}}{2}$

$$(f(x))' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2}$$

$$= \operatorname{senh}(x)$$

$$(\operatorname{senh} x)' = \cosh x$$

$$(\cosh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$(\operatorname{senh} x)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$(\cosh x)^2 - (\operatorname{senh} x)^2 = 1$$

$$\cos^2 x + \operatorname{sen}^2 x = 1$$

$$f(x) = \cosh x$$

$$f'(x) = \operatorname{senh} x$$

$$f'(x)^2 = (\operatorname{senh} x)^2$$

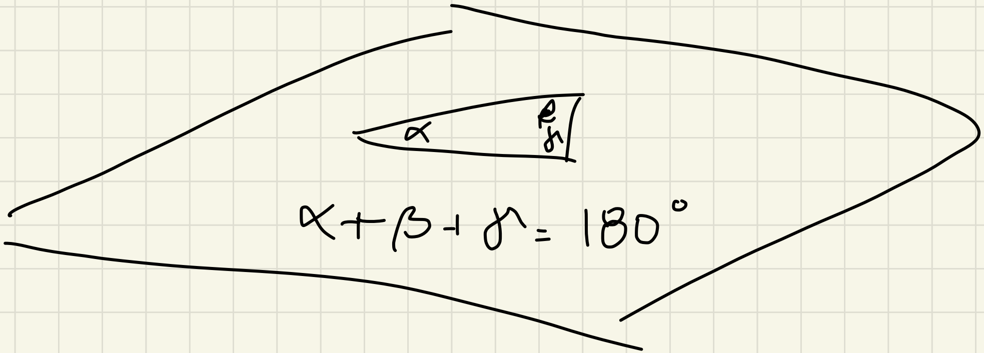
$$1 + f'(x)^2 = 1 + \operatorname{senh}^2 x = \cosh^2 x$$

$$\sqrt{1 + f'(x)^2} = \sqrt{\cosh^2 x} = \cosh x$$

$$\int_{-1}^{1} \sqrt{1 + f'(x)^2}\, dx = \int_{-1}^{1} \cosh x \, dx$$

$$= \sinh x \Big|_{-1}^{1} = \left(\frac{e^x - e^{-x}}{2} \right) \Big|_{-1}^{1}$$

$$= e^1 - e^{-1}$$



$\alpha + \beta + \gamma = 180^\circ$