Universidade Federal do Ceará

Optimum power for a set of m lamps illuminating a set of n flat patches to best approach a target illumination

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Master's Course in Teleinformatics Engineering
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TIP8300 - Nonlinear System Optimization

Contents

1 Problem Definition

Consider m lamps illuminating n (small flat) patches. The illumination intensity I_k at the k-th patch depends linearly on the lamp power p_j as:

$$I_k = \sum_{j=1}^{m} a_{k,j} p_j, \quad \text{with } a_{k,j} = r_{k,j}^{-2} \max\{\cos(\theta_{k,j}), 0\},$$

where $r_{k,j}$ is the length of the vector $r_{k,j}$ connecting the center of the k-th patch to the position of the m-th lamp and $\theta_{k,j}$ is the angle between the patch normal vector $\mathbf{n}_k$ and $\mathbf{r}_{k,j}$. See the Convex Optimization book slides for more details.

The proposed problem is to achieve a desired illumination I_{des} with bounded lamp powers (p_{max}), i.e.,

$$\begin{aligned} \min \quad & \max_{k=1,2,\ldots,n} |\log(I_k) - \log(I_{des})| \\ \text{s.t.} \quad & 0 \le p_j \le p_{max}, j = 1, 2, \ldots, m. \end{aligned} \tag{1}$$

Suboptimally solve the problem using, e.g., Python or Octave, according to the following approaches:

1. Using uniform power, i.e., $p_j = p, 0 \le p \le p_{max}$.
2. Using least-squares, i.e., $\min . \sum_{k=1}^{n} (I_k - I_{des})^2$, and rounding p_j as $p_j = \max\{0, \min\{p_j, p_{max}\}\}$.
3. Using weighted least-squares, i.e., $\min . \sum_{k=1}^{n} (I_k - I_{des})^2 + w_j \sum_{j=1}^{n} (p_j - p_{max})^2$ and iteratively adjusting the weights w_j until $0 \le p \le p_{max}, \forall j$.
4. Using linear programming, i.e.,

$$\begin{aligned} \min \quad & \max_{k=1,2,\ldots,n} |I_k - I_{des}| \\ \text{s.t.} \quad & 0 \le p_j \le p_{max}, \quad j = 1, 2, \ldots, m. \end{aligned}$$

Solve the problem optimally using convex optimization.

For this goal, consider the equivalent convex problem

$$\begin{aligned} \min \quad & \max_{k=1,2,\ldots,n} h\left(\frac{I_k}{I_{des}}\right) \\ \text{s.t.} \quad & 0 \le p_j \le p_{max}, \quad j = 1, 2, \ldots, m, \end{aligned}$$

where $h(u) = \max\left\{u, \frac{1}{u}\right\}$.

For this work, $n = 4$ lamps illuminating $m = 5$ patches are considered. Matrices P_l and P_p contain the x and y coordinates of the lamps and the patches' ends, respectively (1). Figure 1 shows how the lamps and the patches are displayed.

$$P_l = \begin{bmatrix} 0.7609 & 6.9497 \\ 1.9382 & 8.0610 \\ 4.4674 & 6.7316 \\ 7.4095 & 7.3987 \end{bmatrix} \quad P_p = \begin{bmatrix} 0.1329 & 3.9041 \\ 1.5435 & 2.6782 \\ 3.3102 & 3.3981 \\ 4.7796 & 2.2716 \\ 5.8071 & 2.4345 \\ 7.0621 & 2.1015 \end{bmatrix}$$

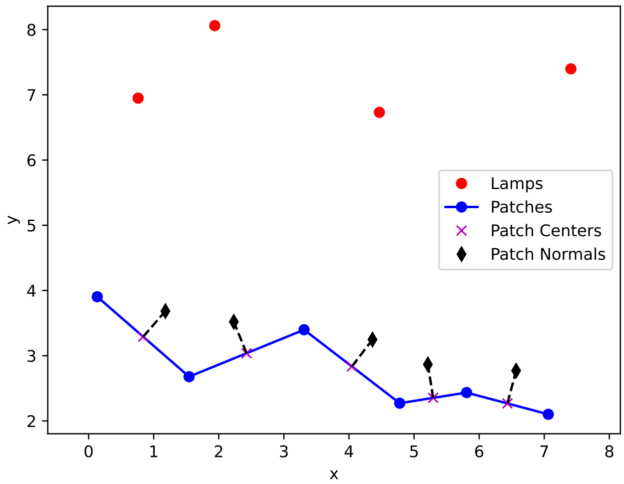


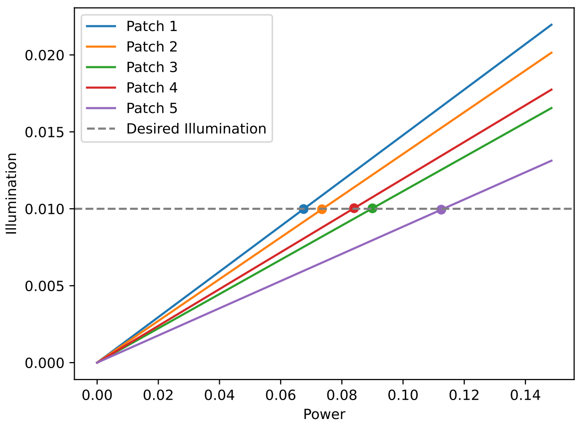
Figure 1: Lamps and patches positions.

2 Suboptimal approaches

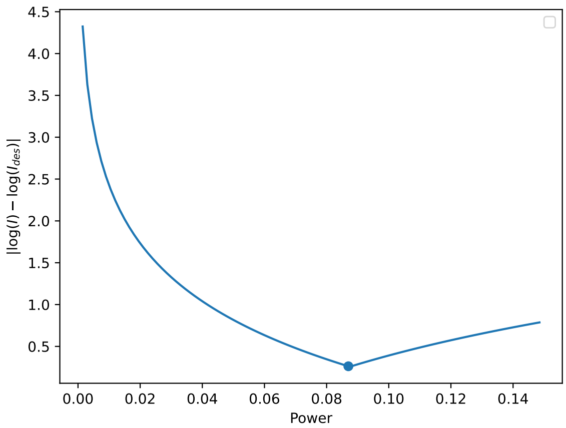
All the code developed for this work is available online[1].

2.1 Uniform power

For this approach, a value of power p is assumed to be the same for all the lamps. By varying the value of p, it is possible to see how the illuminations of each patch behave, as shown in Figure 2a.



(a)



(b)

Figure 2: Patches' illumination when varying a uniform power (a) and its cost function (b).

We see that the desired illumination is met for each patch at different values of p. In this manner, even though it is possible to reach the desired illumination for a patch, the others either have less or more illumination that they should.

The uniform power p is chosen to minimize the original cost function (1), as shown in Figure 2b. The illumination coefficients obtained with p are displayed in Table 1. Although most of the patches have illuminations close to the desired, patches 1 and 5 are quite distant from the objective.

[1] Lamps and Patches - GitHub Repository

Table 1: Illumination results for uniform power.

	Patch 1	Patch 2	Patch 3	Patch 4	Patch 5
Illumination	1.29×10^{-2}	1.18×10^{-2}	0.97×10^{-2}	1.04×10^{-2}	0.77×10^{-2}

2.2 Least-squares

In the least-squares approach, we aim at determining the vector of powers p that minimizes the squared difference between the patche's illuminations and the desired illumination (2).

$$\min \sum_{k=1}^{n} (I_k - I_{des})^2 \tag{2}$$

In this approach, it is not possible to set the constraints for p, i.e., $(0 <= p <= p_{max})$. A possible way to get over this is to set negative elements of p to 0 and values greater than p_{max} to be equal to p_{max}, i.e., $p_j = max\{0, min\{p_j, p_{max}\}\}$.

The matrix form of the problem is $\min(Ap - I_{des})^2$,, where A is the matrix of illumination coefficients a_{kj}, that represent the illumination coefficient of lamp j to the patch k. p is the vector of lamps' powers. Each element of Ap gives the illumination of a patch.

The least-squares problem has a closed form for determining p, which is $p = (A^T A)^{-1} A^T I_{des}$.

From Table 2 it is possible to see that the solution obtained through least-squares produces illuminations quite close to the desired. However, since it does not have constraints, there are lamps with negative powers (Table 3). The solution could also have lamps using more power than it was established. When fixing the negative powers however, the patches' illumination becomes far away from the desired.

Table 2: Illumination results for least-squares.

	Patch 1	Patch 2	Patch 3	Patch 4	Patch 5
Not constrained	1.01×10^{-2}	0.98×10^{-2}	0.97×10^{-2}	1.05×10^{-2}	0.99×10^{-2}
Constrained	3.78×10^{-2}	3.69×10^{-2}	2.11×10^{-2}	2.55×10^{-2}	1.77×10^{-2}

Table 3: Lamps power obtained with least-squares.

	Lamp 1	Lamp 2	Lamp 3	Lamp 4
Power	-0.399	0.944	-0.143	0.192

2.3 Weighted least-squares

In the weighted least-squares (WLS) approach, we aim at minimizing a function composed of the least-squares approach function and the squared difference between the vector p and p_{max}, weighted by coefficients w.

$$\min \sum_{k=1}^{n} (I_k - I_{des})^2 + w_j \sum_{j=1}^{n} (p_j - p_{max})^2$$

The purpose of this additional term is to penalise the cost function for values of p_j that deviate from p_{max}. The coefficient w_j determines how strong this penalty is, and it can be adjusted to meet the constraints for p: $0 \leq p \leq p_{max}$.

This problem also has a closed form for p (3), where $W \in \mathbb{R}^{m \times m}$ is the diagonal matrix of weights.

$$p_{wls} = (A^T A + W)^{-1} \cdot (A^T I_{des} + W p_{max}) \tag{3}$$

When all the weights w_j in the WLS approach are all zero, the problem turns out to be a regular least-squares approach. As shown in the previous section, the not constrained least-squares approach has a quite good result in minimizing the problem. Therefore, the value of zero for initializing the weights may be a good choice.

Since it is asked to update the weights until the powers p are in the accepted range, this is a stop condition for the algorithm. Additionally, a maximum number of iterations is also considered.

The vector of weights $\mathbf{w}$ that compose the main diagonal of $\mathbf{W}$ are updated according to Equation 4, where $\mathbf{w}^{(t)}$ and $\mathbf{w}^{(t+1)}$ are the current and the next $\mathbf{w}$, respectively. α is a factor that multiplies the squared difference between the current vector of powers $p_{wls}^{(t)}$ and p_{max}, and the sum of squared errors of the current and desired illuminations. In this way, the weights are individually updated.

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} + \alpha(p_{wls}^{(t)} - p_{max})^2 \sum_{k=1}^{n} (I_k - I_{des})^2 \tag{4}$$

The powers and the illuminations obtained are shown in Tables 4 and 5, for $\alpha = 10^{-2}$.

Table 4: Lamps power obtained with WLS.

	Lamp 1	Lamp 2	Lamp 3	Lamp 4
Power	2.09×10^{-4}	0.195	0.012	0.195

Table 5: Illumination results for WLS.

	Patch 1	Patch 2	Patch 3	Patch 4	Patch 5
Illumination	1.08×10^{-2}	0.922×10^{-2}	0.987×10^{-2}	1.03×10^{-2}	0.973×10^{-2}

2.4 Linear programming

In the Linear Programming (LP) approach, we aim to formulate the problem such that it has a linear objective function and linear constraints. The proposed problem is to minimize the maximum of the absolute difference between the illumination of a patch and the desired illumination.

$$\begin{aligned} \min \quad & \max_{k=1,2,\ldots,n} |I_k - I_{des}| \\ \text{s.t.} \quad & 0 \leq p_j \leq p_{max}, \quad j = 1, 2, \ldots, m. \end{aligned}$$

We can introduce a new variable t, such that $t \leq |I_k - I_{des}|, \quad \forall k = 1, \ldots, n$. However, the absolute value is a nonlinear function. To get around this problem, we can add to it linear inequalities, such as

$$\begin{aligned} t &\geq I_k - I_{des}, \quad \forall k \\ t &\geq -(I_k - I_{des}) \quad \forall k. \end{aligned}$$

This way, the linear problem can be defined as

$$\begin{aligned} \min \quad & t \\ s.t. \quad & t \geq I_k - I_{des} \\ & t \geq -(I_k - I_{des}) \\ & 0 \leq p_j \leq p_{max}, \quad j = 1, \ldots, m. \end{aligned}$$

As shown in Table 6, the LP approach produces illuminations quite close to the desired without consuming much power as the previous approach.

Table 6: Illumination results for linear programming.

Patch 1	Patch 2	Patch 3	Patch 4	Patch 5
1.071×10^{-2}	0.929×10^{-2}	1.06×10^{-2}	1.071×10^{-2}	0.929×10^{-2}

3 Optimal approach: Convex optimization

In this approach, the problem is formulated as to minimize an equivalent convex function

$$\begin{aligned} \min \quad & \max_{k=1,2,\ldots,n} h\left(\frac{I_k}{I_{des}}\right) \\ \text{s.t.} \quad & 0 \leq p_j \leq p_{max}, \quad j = 1, 2, \ldots, m, \end{aligned} \tag{5}$$

where $h(u) = \max\left\{u, \frac{1}{u}\right\}$.

As I_k approaches I_{des}, the fraction $\frac{I_k}{I_{des}}$ gets closer to 1. The function $h(I_k/I_{des})$ produces great values when $I_k < I_{des}$ and linearly crescent values when $I_k > I_{des}$. This way, $h(\cdot)$ has as its minimum the value 1, i.e., when $I_k = I_{des}$. This way, when minimizing $h(\cdot)$, we are aiming at minimizing the difference between I_k and I_{des}. Figure 3 shows the behaviour of $h(\cdot)$.

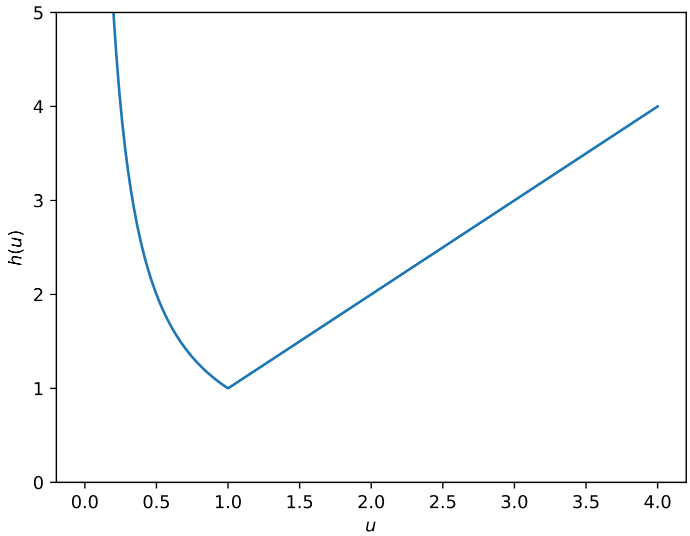


Figure 3: Behaviour of the function $h(u)$.

For this approach, the equivalent problem in Equation 5 is reformulated as

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \begin{cases} \dfrac{Ap}{I_{des}} \leq t \\ \dfrac{I_{des}}{Ap} \leq t \\ 0 \leq p \leq p_{max}. \end{cases} \end{aligned}$$

The illumination results for this approach are shown in Table 7.

Table 7: Illumination results for linear programming.

Patch 1	Patch 2	Patch 3	Patch 4	Patch 5
1.074×10^{-2}	0.931×10^{-2}	1.062×10^{-2}	1.074×10^{-2}	0.931×10^{-2}

4 Results

The objective in the original cost function (1) is to minimize illumination with the worst result considering all the patches in the problem. This way, to compare the results of the different methods applied in this work, we check their worst illumination, i.e.,

$$\max_{k=1,2,\ldots,n} |\log(I_k) - \log(I_{des})|. \tag{6}$$

Table 8 summarizes the results. The uniform power result could vary depending on the cost function used to choose p. With the cost used, its worst result is just not greater than the least-squares approach. When applying the power constraints, the overall performance of the least-squares method is degraded. The same does not happen with the weighted least-squares method, whose worst result is by far lower than the previous methods. However, it must be taken into account that the method strongly depends on how the weights are updated.

Table 8: Worst illumination for each method, according to Equation 6.

Method	**Worst result**
Uniform power	0.263
Least-squares	1.330
Weighted least-squares	0.081
Linear programming	0.074
Convex Optimization	0.071

The LP and the convex optimization methods have the lower worst results when compared with the other approaches. Despite being close, the optimal approach is able to obtain a better result. This may be explained by how similar the original problem (1) and its equivalent (5) are, as shown in Figures 2b and 3.