Linear regression

Model hypothesis:

$$y = A \cdot x + B.$$

Given a sequence of pairs $((x_i, y_i))_{i\in[1..N]}$, we can estimate A and B by minimizing the square error

$$\mathcal{L}(x_1, \ldots, x_N, y_1, \ldots, y_N, A, B) = \sum_{i=1}^{N} (y_i - A \cdot x_i - B)^{\mathrm{T}} (y_i - A \cdot x_i - B). \tag{1}$$

Solving for B:

$$\begin{aligned}
\nabla_B \mathcal{L}(x_1, \ldots, x_N, y_1, \ldots, y_N, A, B) = -2 \sum_{i=1}^{N} (y_i - A \cdot x_i - B) \equiv 0, \\
\sum_{i=1}^{N} (y_i - A \cdot x_i) - NB = 0, \\
B = \frac{1}{N} \sum_{i=1}^{N} y_i - A \cdot \frac{1}{N} \sum_{i=1}^{N} x_i, \\
B = \bar{y}_N - A \cdot \bar{x}_N.
\end{aligned} \tag{2}$$

Solving for A:

$$\begin{aligned}
&\partial_A \mathcal{L}(x_1, \ldots, x_N, y_1, \ldots, y_N, A, B) = -2 \sum_{i=1} (y_i - A \cdot x_i - B) x_i^{\mathrm{T}} \equiv 0, \\
&\sum_{i=1} (y_i - A \cdot x_i - \bar{y}_N + A \cdot \bar{x}_N) x_i^{\mathrm{T}} = 0, \\
&\sum_{i=1} (y_i - \bar{y}_N) x_i^{\mathrm{T}} - A \sum_{i=1} (x_i - \bar{x}_N) x_i^{\mathrm{T}} = 0, \\
&A \left[\sum_{i=1} (x_i - \bar{x}_N) x_i^{\mathrm{T}} \right] = \sum_{i=1} (y_i - \bar{y}_N) x_i^{\mathrm{T}}, \\
A &= \left[\sum_{i=1} (y_i - \bar{y}_N) x_i^{\mathrm{T}} \right] \left[\sum_{i=1} (x_i - \bar{x}_N) x_i^{\mathrm{T}} \right]^{-1} \\
&= \left[\sum_{i=1} (y_i - \bar{y}_N) x_i^{\mathrm{T}} - \left(\sum_{i=1} y_i \bar{x}_N^{\mathrm{T}} - N \bar{y}_N \bar{x}_N^{\mathrm{T}} \right) \right] \left[\sum_{i=1} (x_i - \bar{x}_N) x_i^{\mathrm{T}} - \left(\sum_{i=1} x_i \bar{x}_N^{\mathrm{T}} - N \bar{x}_N \bar{x}_N^{\mathrm{T}} \right) \right]^{-1} \\
&= \left[\sum_{i=1} (y_i - \bar{y}_N) x_i^{\mathrm{T}} - \sum_{i=1} (y_i - \bar{y}_N) \bar{x}_N^{\mathrm{T}} \right] \left[\sum_{i=1} (x_i - \bar{x}_N) x_i^{\mathrm{T}} - \sum_{i=1} (x_i - \bar{x}_N) \bar{x}_N^{\mathrm{T}} \right]^{-1} \\
&= \left[\sum_{i=1} (y_i - \bar{y}_N)(x_i - \bar{x}_N)^{\mathrm{T}} \right] \left[\sum_{i=1} (x_i - \bar{x}_N)(x_i - \bar{x}_N)^{\mathrm{T}} \right]^{-1} \\
&= \left[\frac{1}{N-1} \sum_{i=1} (y_i - \bar{y}_N)(x_i - \bar{x}_N)^{\mathrm{T}} \right] \left[\frac{1}{N-1} \sum_{i=1} (x_i - \bar{x}_N)(x_i - \bar{x}_N)^{\mathrm{T}} \right]^{-1} \\
&= P_{XY,N} P_{X,N}^{-1}.
\end{aligned} \tag{3}$$

Thus,

$$A = P_{XY,N} P_{X,N}^{-1}, \tag{4}$$

$$B = \bar{y}_N - P_{XY,N} P_{X,N}^{-1} \cdot \bar{x}_N. \tag{5}$$

Incremental updates to a linear regressor

Computing the sample mean incrementally:

$$\begin{aligned}\bar{x}_N &= \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{N}\sum_{i=1}^{N-L} x_i + \frac{1}{N}\sum_{j=1}^{L} x_{N-L+j} \\ &= \frac{N-L}{N}\frac{1}{N-L}\sum_{i=1}^{N-L} x_i + \sum_{j=1}^{L} x_{N-L+j} \\ &= \frac{N-L}{N}\bar{x}_{N-L} + \frac{1}{N}\sum_{j=1}^{L} x_{N-L+j}. \end{aligned} \tag{6}$$

Equally,

$$\bar{y}_N = \frac{N-L}{N}\bar{y}_{N-L} + \frac{1}{N}\sum_{j=1}^{L} y_{N-L+j}. \tag{7}$$

Let us define

$$S_x = \sum_{j=1}^{L} x_{N-L+j}. \tag{8}$$

Doing the same for the sample covariance:

$$
\begin{aligned}
P_{X,N} &= \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x}_N)(x_i-\bar{x}_N)^{\mathrm{T}} \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N-L}(x_i-\bar{x}_N)(x_i-\bar{x}_N)^{\mathrm{T}}+\sum_{j=1}^{L}(x_{N-L+j}-\bar{x}_N)(x_{N-L+j}-\bar{x}_N)^{\mathrm{T}}\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N-L}\left(x_i-\frac{N-L}{N}\bar{x}_{N-L}-\frac{1}{N}S_x\right)\left(x_i-\frac{N-L}{N}\bar{x}_{N-L}-\frac{1}{N}S_x\right)^{\mathrm{T}} \\
&+ \frac{1}{N-1}\sum_{j=1}^{L}\left(x_{N-L+j}-\frac{N-L}{N}\bar{x}_{N-L}-\frac{1}{N}S_x\right)\left(x_{N-L+j}-\frac{N-L}{N}\bar{x}_{N-L}-\frac{1}{N}S_x\right)^{\mathrm{T}} \\
&= \frac{1}{N-1}\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}+\frac{1}{N}\left(L\bar{x}_{N-L}-S_x\right)\right)\left(x_i-\bar{x}_{N-L}+\frac{1}{N}\left(L\bar{x}_{N-L}-S_x\right)\right)^{\mathrm{T}} \\
&+ \frac{1}{N-1}\sum_{j=1}^{L}\left(\frac{1}{N}\left(Nx_{N-L+j}-S_x\right)-\frac{N-L}{N}\bar{x}_{N-L}\right)\left(\frac{1}{N}\left(Nx_{N-L+j}-S_x\right)-\frac{N-L}{N}\bar{x}_{N-L}\right)^{\mathrm{T}} \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}\right)\left(x_i-\bar{x}_{N-L}\right)^{\mathrm{T}}+\frac{2}{N}\left(L\bar{x}_{N-L}-S_x\right)\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}\right)^{\mathrm{T}}\right] \\
&\frac{1}{N-1}\left[\left(\frac{1}{N}\right)^2\sum_{i=1}^{N-L}\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{x}_{N-L}-S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Nx_{N-L+j}-S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[-2(N-L)\bar{x}_{N-L}\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)^{\mathrm{T}}+\sum_{j=1}^{L}\left(N-L\right)^2\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right)\frac{1}{N-L-1}\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}\right)\left(x_i-\bar{x}_{N-L}\right)^{\mathrm{T}}+\frac{1}{N-1}\left[\frac{2}{N}\left(L\bar{x}_{N-L}-S_x\right)\cancel{\left(\sum_{i=1}^{N-L}x_i-\sum_{i=1}^{N-L}\bar{x}_{N-L}\right)^{\mathrm{T}}}\right] \\
&+ \frac{1}{N-1}\left[\left(\frac{1}{N}\right)^2\left(N-L\right)\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{x}_{N-L}-S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Nx_{N-L+j}-S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[-2(N-L)\bar{x}_{N-L}\left(N\sum_{j=1}^{L}x_{N-L+j}-LS_x\right)^{\mathrm{T}}+L\left(N-L\right)^2\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right)P_{X,N-L}+\frac{1}{N-1}\left[\left(\frac{1}{N}\right)^2\left(N-L\right)\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{x}_{N-L}-S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Nx_{N-L+j}-S_x\right)^{\mathrm{T}}-2(N-L)^2\bar{x}_{N-L}S_x^{\mathrm{T}}+L\left(N-L\right)^2\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}}\right],
\end{aligned}
$$

$$
\begin{aligned}
P_{X,N} &= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} + \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)\left(L\bar{x}_{N-L} - S_x\right)\left(L\bar{x}_{N-L} - S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[\sum_{j=1}^{L} \left(N x_{N-L+j} - S_x\right)\left(N x_{N-L+j} - S_x\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[-2(N-L)^2 \bar{x}_{N-L} S_x^{\mathrm{T}} + L\left(N-L\right)^2 \bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} + \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)\left(L^2 \bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}} - 2L\bar{x}_{N-L} S_x^{\mathrm{T}} + S_x S_x^{\mathrm{T}}\right)\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[\left(N^2 \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}} - 2N S_x S_x^{\mathrm{T}} + L S_x S_x^{\mathrm{T}}\right) - 2(N-L)^2 \bar{x}_{N-L} S_x^{\mathrm{T}} + L\left(N-L\right)^2 \bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} + \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)(\cancel{L^2} + NL - \cancel{L^2})\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[2(-NL + \cancel{L^2} - N^2 + 2NL - \cancel{L^2})\bar{x}_{N-L} S_x^{\mathrm{T}} + (N - \cancel{L} - 2N + \cancel{L}) S_x S_x^{\mathrm{T}} + N^2 \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)NL\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}} - 2N(N-L)\bar{x}_{N-L} S_x^{\mathrm{T}} - N S_x S_x^{\mathrm{T}} + N^2 \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[(N-L)L\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}} - 2(N-L)\bar{x}_{N-L} S_x^{\mathrm{T}} - S_x S_x^{\mathrm{T}} + N \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[(N-L)\sum_{j=1}^{L} \bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}} - 2(N-L)\bar{x}_{N-L} \sum_{j=1}^{L} x_{N-L+j}^{\mathrm{T}} - S_x S_x^{\mathrm{T}} + N \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} + \frac{1}{N-1}\left(\frac{1}{N}\right) \left[(N-L)\sum_{j=1}^{L} \left(\bar{x}_{N-L}\bar{x}_{N-L}^{\mathrm{T}} - 2\bar{x}_{N-L} x_{N-L+j}^{\mathrm{T}} + x_{N-L+j} x_{N-L+j}^{\mathrm{T}}\right)\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[L \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}} - S_x S_x^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{X,N-L} + \frac{1}{N-1}\frac{1}{N} \left[(N-L)\sum_{j=1}^{L} \left(x_{N-L+j} - \bar{x}_{N-L}\right)\left(x_{N-L+j} - \bar{x}_{N-L}\right)^{\mathrm{T}}\right]. \\
&+ \frac{1}{N-1}\frac{1}{N} \left[L \sum_{j=1}^{L} x_{N-L+j} x_{N-L+j}^{\mathrm{T}} - \left(\sum_{j=1}^{L} x_{N-L+j}\right)\left(\sum_{j=1}^{L} x_{N-L+j}\right)^{\mathrm{T}}\right].
\end{aligned} \tag{9}
$$

For the covariance $P_{XY,N}$,we have

$$
\begin{aligned}
P_{XY,N} &= \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x}_N)(y_i-\bar{y}_N)^{\mathrm{T}} \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N-L}(x_i-\bar{x}_N)(y_i-\bar{y}_N)^{\mathrm{T}}+\sum_{j=1}^{L}(x_{N-L+j}-\bar{x}_N)(y_{N-L+j}-\bar{y}_N)^{\mathrm{T}}\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N-L}\left(x_i-\frac{N-L}{N}\bar{x}_{N-L}-\frac{1}{N}S_x\right)\left(y_i-\frac{N-L}{N}\bar{y}_{N-L}-\frac{1}{N}S_y\right)^{\mathrm{T}} \\
&+ \frac{1}{N-1}\sum_{j=1}^{L}\left(x_{N-L+j}-\frac{N-L}{N}\bar{x}_{N-L}-\frac{1}{N}S_x\right)\left(y_{N-L+j}-\frac{N-L}{N}\bar{y}_{N-L}-\frac{1}{N}S_y\right)^{\mathrm{T}} \\
&= \frac{1}{N-1}\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}+\frac{1}{N}\left(L\bar{x}_{N-L}-S_x\right)\right)\left(y_i-\bar{y}_{N-L}+\frac{1}{N}\left(L\bar{y}_{N-L}-S_y\right)\right)^{\mathrm{T}} \\
&+ \frac{1}{N-1}\sum_{j=1}^{L}\left(\frac{1}{N}\left(Nx_{N-L+j}-S_x\right)-\frac{N-L}{N}\bar{x}_{N-L}\right)\left(\frac{1}{N}\left(Ny_{N-L+j}-S_y\right)-\frac{N-L}{N}\bar{y}_{N-L}\right)^{\mathrm{T}} \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}\right)\left(y_i-\bar{y}_{N-L}\right)^{\mathrm{T}}+\frac{1}{N}\left(L\bar{x}_{N-L}-S_x\right)\sum_{i=1}^{N-L}\left(y_i-\bar{y}_{N-L}\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left[\frac{1}{N}\left(L\bar{y}_{N-L}-S_y\right)\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}\right)^{\mathrm{T}}+\left(\frac{1}{N}\right)^2\sum_{i=1}^{N-L}\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{y}_{N-L}-S_y\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Ny_{N-L+j}-S_y\right)^{\mathrm{T}}-(N-L)\bar{x}_{N-L}\sum_{j=1}^{L}\left(Ny_{N-L+j}-S_y\right)^{\mathrm{T}}\right] \\
&+ +\frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[-(N-L)\bar{y}_{N-L}\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)^{\mathrm{T}}+\sum_{j=1}^{L}\left(N-L\right)^2\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right)\frac{1}{N-L-1}\sum_{i=1}^{N-L}\left(x_i-\bar{x}_{N-L}\right)\left(y_i-\bar{y}_{N-L}\right)^{\mathrm{T}} \\
&+ \frac{1}{N-1}\left[\frac{1}{N}\left(L\bar{x}_{N-L}-S_x\right)\cancel{\left(\sum_{i=1}^{N-L}y_i-\sum_{i=1}^{N-L}\bar{y}_{N-L}\right)^{\mathrm{T}}}+\frac{1}{N}\left(L\bar{y}_{N-L}-S_y\right)\cancel{\left(\sum_{i=1}^{N-L}x_i-\sum_{i=1}^{N-L}\bar{x}_{N-L}\right)^{\mathrm{T}}}\right] \\
&+ \frac{1}{N-1}\left[\left(\frac{1}{N}\right)^2\left(N-L\right)\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{y}_{N-L}-S_y\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Ny_{N-L+j}-S_y\right)^{\mathrm{T}}-(N-L)\bar{x}_{N-L}\left(N\sum_{j=1}^{L}y_{N-L+j}-LS_y\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[-(N-L)\bar{y}_{N-L}\left(N\sum_{j=1}^{L}x_{N-L+j}-LS_x\right)^{\mathrm{T}}+L\left(N-L\right)^2\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right)P_{XY,N-L}+\frac{1}{N-1}\left[\left(\frac{1}{N}\right)^2\left(N-L\right)\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{y}_{N-L}-S_y\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[\sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Ny_{N-L+j}-S_y\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2\left[-(N-L)^2\bar{x}_{N-L}S_y^{\mathrm{T}}-(N-L)^2\bar{y}_{N-L}S_x^{\mathrm{T}}+L\left(N-L\right)^2\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}}\right],
\end{aligned}
$$

$$
\begin{aligned}
P_{XY,N} &= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)\left(L\bar{x}_{N-L}-S_x\right)\left(L\bar{y}_{N-L}-S_y\right)^{\mathrm{T}} + \sum_{j=1}^{L}\left(Nx_{N-L+j}-S_x\right)\left(Ny_{N-L+j}-S_y\right)^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[-(N-L)^2\bar{x}_{N-L}S_y^{\mathrm{T}} - (N-L)^2\bar{y}_{N-L}S_x^{\mathrm{T}} + L\left(N-L\right)^2\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} + \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)\left(L^2\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}} - L\bar{x}_{N-L}S_y^{\mathrm{T}} - L\bar{y}_{N-L}S_x^{\mathrm{T}} + S_xS_y^{\mathrm{T}}\right)\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[\left(N^2\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}} - 2NS_xS_y^{\mathrm{T}} + S_xS_y^{\mathrm{T}}\right)\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[-(N-L)^2\left(\bar{x}_{N-L}S_y^{\mathrm{T}} + \bar{y}_{N-L}S_x^{\mathrm{T}}\right) + L\left(N-L\right)^2\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)(\cancel{L^2} + NL - \cancel{L^2})\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}} + (-NL + \cancel{L^2} - N^2 + 2NL - \cancel{L^2})\left(\bar{x}_{N-L}S_y^{\mathrm{T}} + \bar{y}_{N-L}S_x^{\mathrm{T}}\right)\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N - \cancel{L} - 2N + \cancel{L})S_xS_y^{\mathrm{T}} + N^2\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right)^2 \left[(N-L)NL\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}} - N(N-L)\left(\bar{x}_{N-L}S_y^{\mathrm{T}} + \bar{y}_{N-L}S_x^{\mathrm{T}}\right) - NS_xS_y^{\mathrm{T}} + N^2\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[(N-L)L\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}} - (N-L)\left(\bar{x}_{N-L}S_y^{\mathrm{T}} + \bar{y}_{N-L}S_x^{\mathrm{T}}\right) - S_xS_y^{\mathrm{T}} + N\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} + \frac{1}{N-1}\left(\frac{1}{N}\right) \left[(N-L)\sum_{j=1}^{L}\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}}\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[-(N-L)\left(\bar{x}_{N-L}\sum_{j=1}^{L}y_{N-L+j}^{\mathrm{T}} + \bar{y}_{N-L}\sum_{j=1}^{L}x_{N-L+j}^{\mathrm{T}}\right) - S_xS_y^{\mathrm{T}} + N\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[(N-L)\sum_{j=1}^{L}\left(\bar{x}_{N-L}\bar{y}_{N-L}^{\mathrm{T}} - \bar{x}_{N-L}y_{N-L+j}^{\mathrm{T}} - \bar{y}_{N-L}x_{N-L+j}^{\mathrm{T}} + x_{N-L+j}y_{N-L+j}^{\mathrm{T}}\right)\right] \\
&+ \frac{1}{N-1}\left(\frac{1}{N}\right) \left[L\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}} - S_xS_y^{\mathrm{T}}\right] \\
&= \left(\frac{N-L-1}{N-1}\right) P_{XY,N-L} + \frac{1}{N-1}\frac{1}{N}\left[(N-L)\sum_{j=1}^{L}\left(x_{N-L+j}-\bar{x}_{N-L}\right)\left(y_{N-L+j}-\bar{y}_{N-L}\right)^{\mathrm{T}}\right]. \\
&+ \frac{1}{N-1}\frac{1}{N}\left[L\sum_{j=1}^{L}x_{N-L+j}y_{N-L+j}^{\mathrm{T}} - \left(\sum_{j=1}^{L}x_{N-L+j}\right)\left(\sum_{j=1}^{L}y_{N-L+j}\right)^{\mathrm{T}}\right].
\end{aligned} \tag{10}
$$

Summary

Given the previous moments $(\bar{x}_N, \bar{y}_N, P_{X,N}, P_{XY,N})$, and L new pairs $((x_j, y_j))_{j\in[N+1..N+L]}$, the update of the linear regressor is given as follows. Update the moments

$$
\begin{aligned}
\bar{x}_{N+L} &= \frac{N}{N+L}\bar{x}_N + \frac{1}{N+L}\sum_{j=1}^{L} x_{N+j}, \\
\bar{y}_{N+L} &= \frac{N}{N+L}\bar{y}_N + \frac{1}{N+L}\sum_{j=1}^{L} y_{N+j}, \\
P_{X,N+L} &= \left(\frac{N-1}{N+L-1}\right) P_{X,N} \\
&\quad + \frac{1}{N+L-1}\frac{1}{N+L}\left[N\sum_{j=1}^{L}\left(x_{N+j}-\bar{x}_N\right)\left(x_{N+j}-\bar{x}_N\right)^{\mathrm{T}} + L\sum_{j=1}^{L} x_{N+j}x_{N+j}^{\mathrm{T}} - \left(\sum_{j=1}^{L} x_{N+j}\right)\left(\sum_{j=1}^{L} x_{N+j}\right)^{\mathrm{T}}\right] \\
&= \left(\frac{N-1}{N+L-1}\right) P_{X,N} + \frac{1}{N+L-1}\frac{1}{N+L}\left[N\sum_{j=1}^{L}\left(x_{N+j}-\bar{x}_N\right)\left(x_{N+j}-\bar{x}_N\right)^{\mathrm{T}} + L\sum_{j=1}^{L} x_{N+j}(x_{N+j}-\bar{x}_{N+1:N+L})^{\mathrm{T}}\right] \\
P_{XY,N+L} &= \left(\frac{N-1}{N+L-1}\right) P_{XY,N} \\
&\quad + \frac{1}{N+L-1}\frac{1}{N+L}\left[N\sum_{j=1}^{L}\left(x_{N+j}-\bar{x}_N\right)\left(y_{N+j}-\bar{y}_N\right)^{\mathrm{T}} + L\sum_{j=1}^{L} x_{N+j}y_{N+j}^{\mathrm{T}} - \left(\sum_{j=1}^{L} x_{N+j}\right)\left(\sum_{j=1}^{L} y_{N+j}\right)^{\mathrm{T}}\right] \\
&= \left(\frac{N-1}{N+L-1}\right) P_{XY,N} + \frac{1}{N+L-1}\frac{1}{N+L}\left[N\sum_{j=1}^{L}\left(x_{N+j}-\bar{x}_N\right)\left(y_{N+j}-\bar{y}_N\right)^{\mathrm{T}} + L\sum_{j=1}^{L} x_{N+j}(y_{N+j}-\bar{y}_{N+1:N+L})^{\mathrm{T}}\right],
\end{aligned}
$$

and then update the slope and intercept as

$$A_{N+L} = P_{XY,N+L}P_{X,N+L}^{-1}, \tag{11}$$

$$B_{N+L} = \bar{y}_{N+L} - A_{N+L}\cdot\bar{x}_{N+L}. \tag{12}$$