Discussion of the Diffraction Limit

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1. Introduction to my question

When we discuss the simplist case of imaging system: A lens with a circular pupil of diameter w, the angular and line diffraction limit is always given by:

$$\begin{aligned} \delta\theta &= 1.22\frac{\lambda}{w} \\ \delta x &= 1.22\frac{\lambda z_i}{w} \end{aligned} \tag{1}$$

I wonder: if a lens with an infinitly large pupil is used, i.e. $w \to \infty$, will $\delta x \to 0$?

2. Derivation of the above formula

The above formula Equation 1 can be easily accessed by a conclusion from Fourier Optics [1]

> **Theorem 2.1** : For a diffraction-limited system, the impulse response of the imaging system is the Fourier Transform of the pupil function

So for a circular pupil, its FT is the J_1, an Airy disk, with first zero point at $1.22\frac{\lambda}{w}$.

But the derivation of this conclusion uses the Fresnel diffraction formula rather than the accurate angular spectrum. The transmmission function used in the derivation is

$$H(f_x, f_y) = \exp(jkz)\exp\left[-j\pi\lambda z\left(f_x^2 + f_y^2\right)\right] \tag{2}$$

which is a first-order approximation of the angular spectrum method, whose formula should be

$$H(f_x, f_y) = \exp\left(jkz\sqrt{1 - (\lambda f_x)^2 - \left(\lambda f_y\right)^2}\right) \tag{3}$$

This approximation usually holds **ONLY for $|\boldsymbol{\lambda f_x}|, |\boldsymbol{\lambda f_y}| \ll \mathbf{1}$**, which is small angle approximation. The exact applicable could be derived from the Taylor expansion of the sqrt function. Suppose θ is the $\vec{k}$'s angle with the z axis. The Fresnel formula holds when:

$$\frac{\theta^4 z}{4\lambda} \ll 1 \tag{4}$$

If we want $w \to \infty$, of course the θ will be larger than the above criterion, so the Equation 1 is no longer valid. We **CANNOT** make $w \to \infty$ and claim the diffraction limit is broken.

3. Diffraction Limit in the general case

Next question is: If the above Equation 1 only holds for paraxial approximation, what formula should we use to describe the diffraction limit in the general case?

The diffraction limit can be rewritten as

$$\delta x = 1.22\frac{\lambda}{2\mathrm{NA}} = \frac{\lambda}{2n\sin\alpha} \geq \frac{\lambda}{2n} \tag{5}$$

But Equation 5 seems contradict with Equation 1, as Equation 1 indicates a $\tan$ relationship, rather than sin. This can be explained by the **Abbe sine condition**.

Theorem 3.1 : the sine of the object-space angle α_o should be proportional to the sine of the image space angle α_i.

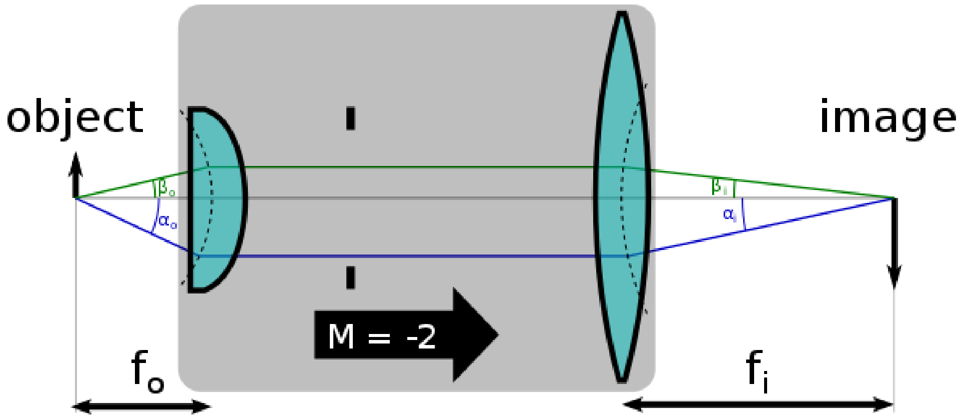


Figure 1: Abbe sine condition from Wikipedia

A collary of the Abbe sine condition [2] establishes a linear correspondence between transverse wavevector components in the object (or image) plane, and spatial coordinates in the pupil plane.

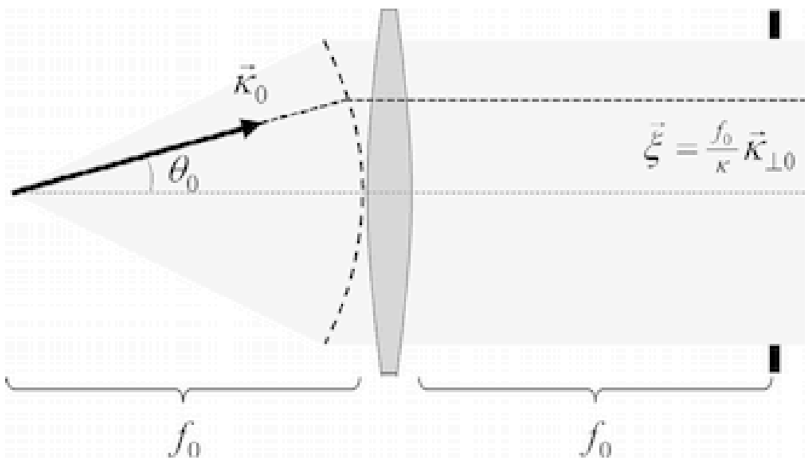


Figure 2: linear correspondence

4. Numerical Aperture

As Rudolf Kingslake explains, “It is a common **error** to suppose that the ratio [D/2f] is actually equal to tan θ, and not sin θ … The tangent would, of course, be correct if the principal planes were really plane. However, the complete theory of the Abbe sine condition shows that if a lens is corrected for coma and spherical aberration, as all good photographic objectives must be, the second principal plane becomes a portion of a sphere of radius f centered about the focal point”

WHYYYY? Why not tan? Why sin?

5. Physics underlying the diffraction limit

5.1. From Fourier Optics’s point of view

According to the idea of angular spectrum method of Equation 3, only the low frequency components ($f_x^2 + f_y^2 < 1/\lambda^2$) can propagate in space, so high frequency components are lost. From this point of view, imaging cannot be perfect, as long as there is propagation in free space.

Calculation: The maximum frequency that can propagate is on order of $f_x = \frac{1}{\lambda}$, so the diffraction limit is with the same order of $1/f_x = \lambda$.

5.2. From Heisenberg Uncertainty Principle’s point of view

With the Heisenberg Uncertainty Principle:

$$\Delta x \Delta p_x > h \tag{6}$$

(right hand side of Equation 6 should be $\frac{\hbar}{2}$. h is adopted here for magnitude estimation)

The photon has a momentum

$$p = \frac{h\nu}{c} \tag{7}$$

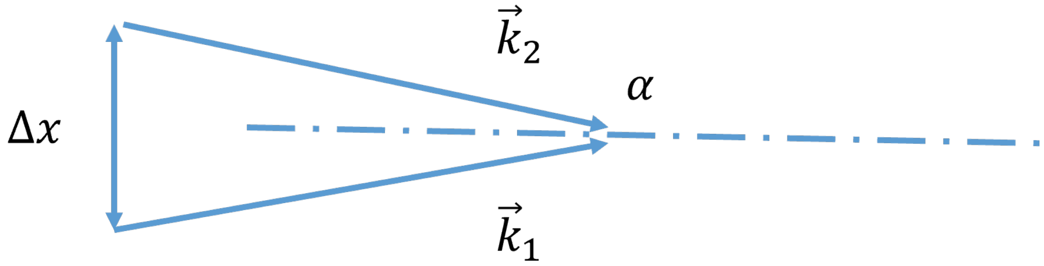
Which is equivalent to

$$p = \frac{h}{\lambda} = \hbar k \tag{8}$$

$$\Delta x \Delta k_x > \pi \tag{9}$$

$$\Delta k_x = 2k \sin\alpha \leq 2k = \frac{2\pi}{\lambda} \tag{10}$$

The equal sign is reached when the lens pupil reaches infinity.



$(\alpha \to 90, \sin\alpha \to 1)$

$$\Delta x > \frac{\pi}{\Delta k_x} = \frac{\lambda}{2} \tag{11}$$

6. Super-Resolution

The Heisenberg Uncertainty Principle offers a great physical insight into the diffraction limit. With the same point of view, we can explore ways of breaking the diffraction limit.

The key seems counter intuitive: should $\sin\alpha \leq 1$ **hold for all scenarios?**

NO! By having a complex $k_z = j\,|k_z|$, have

$$k_x^2 = k^2 - k_z^2 = k^2 + |k_z|^2 > k^2 \tag{12}$$

As the uncertainty principle requires:$\Delta x > \frac{\pi}{\Delta k_x}$, if we use **evanescent waves**: $k_x \to \infty \leftrightarrow \Delta x \to 0$

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Bibliography

[1] J. W. Goodman, 傅里叶光学导论 *(第四版)*. 北京: 科学出版社, 2020, pp. 146–152.

[2] J. Mertz, *Introduction to Optical Microscopy*, 2nd ed. Cambridge University Press, 2019.