Ref of Fu's 1993 paper

The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = ie\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right)\frac{i}{\omega}\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2+v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa,\ \mu = \frac{v_\perp^2}{2B},\ \omega_\star = \frac{i\hat{\mathbf{b}}\times\nabla F\cdot\nabla}{\Omega\partial F/\partial\epsilon}$, $\delta\mathbf{E}_\perp = i\omega\xi \times \mathbf{B}$.

The term $\mathbf{v}_d \cdot \delta\mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta\mathbf{E}_\perp = \frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa \cdot \delta\mathbf{E}_\perp = i\omega\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{1}$$

$$= -i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\vec{\kappa} \cdot \vec{\xi} = -i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\left(\nabla\theta\kappa_\theta + \nabla r\kappa_r\right) \cdot \left(\xi_\theta\nabla\theta + \xi_r\nabla r\right)$$

$$= -i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\left(\nabla\theta \cdot \nabla\theta\xi_\theta\kappa_\theta + \nabla r \cdot \nabla r\xi_r\kappa_r + \nabla\theta \cdot \nabla r\xi_r\kappa_\theta + \nabla r \cdot \nabla\theta\xi_\theta\kappa_r\right)$$

$$= -i\omega B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r \cdot \nabla r,\ g^{\theta r} = \nabla\theta \cdot \nabla r,\ g^{\theta\theta} = \nabla\theta \cdot \nabla\theta$. $\xi_r\nabla r = \xi_r\mathbf{e}_r,\ \xi_\theta\nabla\theta = \frac{1}{r}\xi_\theta\mathbf{e}_\theta$. $\Lambda = \frac{\mu B_0}{\epsilon},\ b = B_0/B \approx 1 - (r/R_0)\cos\theta,\ \epsilon = \frac{1}{2}v^2, \kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial\theta},\ \kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2R^2}$.

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = ie\frac{\partial F}{\partial\epsilon}\left(\omega - \omega_\star\right)B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{2}$$

$$\frac{d}{dt}g = H\left(r, \theta, \phi, t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the polodial cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time

$$\tau_b = \oint d\tau = \oint \frac{d\psi}{\dot{\psi}} = \oint \frac{d\theta}{\dot{\theta}} \tag{3}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}(r,\theta)\exp(-i\omega t - in\phi) \tag{4}$$

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} ie\frac{\partial F}{\partial \epsilon}(\omega - \omega_\star) B\frac{\epsilon}{\Omega}G(\tau)\, d\tau \tag{5}$$

with

$$G = \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

where $G(\tau) = \hat{G}[r(\tau), \theta(\tau)]\exp(-i\omega\tau + in\phi(\tau))$ and the τ dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}}\cdot\nabla r, \dot{\theta} = \dot{\mathbf{R}}\cdot\nabla\theta, \dot{\phi} = \dot{\mathbf{R}}\cdot\nabla\phi \tag{6}$$

Let us separate $\phi(\tau)$ into its secular and osillating parts:

$$\phi(\tau) = \left\langle\dot{\phi}\right\rangle\tau + \widetilde{\phi}(\tau) \tag{7}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}[r(\tau), \theta(\tau)] = \hat{G}[r(\tau), \theta(\tau)]\exp\left(in\tilde{\phi}(\tau)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}(\tau) = \sum_{-\infty}^{\infty} Y_p(\Lambda, \bar{r}; \sigma)\exp(ip\omega_b\tau) \tag{8}$$

where,

$$Y_p(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b}\oint d\tau\tilde{G}(\tau)\exp(-ip\omega_b\tau) \tag{9}$$

with $r(\tau) = \bar{r} + \rho\cos\theta(\tau)$.

Carrying out the time integration, the solution of g is obtained

$$g = e\frac{\partial F}{\partial \epsilon}(\omega - \omega_\star) B\frac{\epsilon}{\Omega}\sum_{-\infty}^{\infty} Y_p(\Lambda, \bar{r}; \sigma)\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega\right)t\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \tag{10}$$

The formula of δW_k is derived as follows.

$$\delta W_k = \int d^3x \vec{\xi}^\star \cdot \nabla \cdot \delta \mathbf{P}_k = -e \int d^3x \int d^3v \left(\frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \right)^\star g$$

$$= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2 \left(1 - \frac{\Lambda}{b} \right) \right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r \right) \xi_\theta^\star + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta \right) \xi_r^\star \right)$$

$$= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} G^\star$$

where $G^\star = \hat{G}^\star [r(\tau), \theta(\tau)] \exp(i\omega\tau - in\phi(\tau))$. Let $\tilde{G}^\star [r(\tau), \theta(\tau)] = \hat{G}^\star [r(\tau), \theta(\tau)] \exp\left(-in\tilde{\phi}(\tau)\right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star(\tau) = \sum_{-\infty}^{\infty} Y_p^\star(\Lambda, \bar{r}; \sigma) \exp(-ip\omega_b \tau) \tag{11}$$

where,

$$Y_p^\star(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}^\star(\tau) \exp(ip\omega_b \tau) \tag{12}$$

with $r(\tau) = \bar{r} + \rho \cos\theta(\tau)$.

$$\delta W_k = -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2} \sum_{-\infty}^{\infty} Y_p(\epsilon, \mu, P_\phi; \sigma)$$

$$\cdot \frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega\right)\tau\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star(\Lambda, \bar{r}; \sigma) \exp\left(i\omega\tau - in\left\langle\dot{\phi}\right\rangle\tau - ip'\omega_b\tau\right) \tag{13}$$

$$\delta W_k = -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \bar{r}; \sigma) \frac{\exp[ip\omega_b\tau]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star(\Lambda, \bar{r}; \sigma) \exp(-ip'\omega_b\tau) \tag{14}$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = -e^2 \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \bar{r}; \sigma\right) \frac{\exp\left(ip\omega_b \tau\right)}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}\left(\Lambda, \bar{r}; \sigma\right) \exp\left(-ip'\omega_b \tau\right) \tag{15}$$

Applying $d\tau = \frac{qR_0}{\sqrt{2\epsilon} b \sqrt{1-\frac{\Lambda}{b}}} d\theta$, one finially obtains

$$\delta W_k = -4\pi^2 \frac{e^2 B^2}{\Omega^2} \frac{1}{qR_0} \int J dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b \left(\omega - \omega_\star\right)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{\left|Y_p\right|^2}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \tag{16}$$

The normalized relations of the quantities are $F = \frac{n}{v_{th}^3} \bar{F}_0$, $v_t = \sqrt{\frac{2T}{M}}$, $\epsilon = \frac{T}{M} \bar{\epsilon}$, $r = a\bar{x}$, $J = aR_0 \bar{J}$, $R = R_0 \bar{R}$, $\omega_t = \frac{v_{th}}{R_0} \bar{\omega}_t$, $\frac{1}{\tau_t} = \frac{v_{th}}{2\pi R_0} \bar{\omega}_t = \frac{v_{th}}{R_0} \frac{\bar{\omega}_t}{2\pi} = \frac{v_{th}}{R_0} \frac{1}{\bar{\tau}_t}$.