Ref of Fu's 1993 paper

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The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = i \frac{e}{M} \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) \frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \tag{1}$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \kappa$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i\hat{\mathbf{b}} \times \nabla F \cdot \nabla}{\Omega \partial F / \partial \epsilon}$, $\epsilon = \frac{1}{2} v^2$, $\delta \mathbf{E}_\perp = i\omega \vec{\xi} \times \mathbf{B}$, $\Omega = \frac{Be}{M}$ is the particle cyclotron frequency.[Berk, Phys. Fluid B 4 1992]

The term $\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp = \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp = i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{2}$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \vec{\kappa} \cdot \vec{\xi} = -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \kappa_\theta + \nabla r \kappa_r\right) \cdot \left(\xi_\theta \nabla\theta + \xi_r \nabla r\right)$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \cdot \nabla\theta \xi_\theta \kappa_\theta + \nabla r \cdot \nabla r \xi_r \kappa_r + \nabla\theta \cdot \nabla r \xi_r \kappa_\theta + \nabla r \cdot \nabla\theta \xi_\theta \kappa_r\right)$$

$$= -i\omega B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r \cdot \nabla r$, $g^{\theta r} = \nabla\theta \cdot \nabla r$, $g^{\theta\theta} = \nabla\theta \cdot \nabla\theta$. $\xi_r \nabla r = \xi_r \mathbf{e}_r$, $\xi_\theta \nabla\theta = \frac{1}{r}\xi_\theta \mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial\theta}$, $\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2 R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0)\cos\theta$, $\epsilon = \frac{1}{2}v^2$, $\delta \mathbf{E}_\perp^\star = -i\omega \vec{\xi}^\star \times \mathbf{B}$. Thus, the complex conjugate term is

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp^\star = \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp^\star = -i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi}^\star \times \mathbf{B}\right) \tag{3}$$

$$= i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \vec{\kappa} \cdot \vec{\xi}^\star = i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \kappa_\theta + \nabla r \kappa_r\right) \cdot \left(\xi_\theta^\star \nabla\theta + \xi_r^\star \nabla r\right)$$

$$= i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\Omega}\left(\nabla\theta\cdot\nabla\theta\xi_\theta^\star\kappa_\theta+\nabla r\cdot\nabla r\xi_r^\star\kappa_r+\nabla\theta\cdot\nabla r\xi_r^\star\kappa_\theta+\nabla r\cdot\nabla\theta\xi_\theta^\star\kappa_r\right)$$

$$= i\omega B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta^\star+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = i\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{4}$$

$$\frac{d}{dt}g = H\left(r,\theta,\phi,t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint\frac{d\psi}{\dot\psi} = \oint\frac{d\theta}{\dot\theta} \tag{5}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}\left(r,\theta\right)\exp\left(-i\omega t+in\phi\right) \tag{6}$$

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} i\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\Omega}G\left(\tau\right)d\tau \tag{7}$$

with

$$G = \left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

where $G\left(\tau\right) = \hat{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-i\omega\tau+in\phi\left(\tau\right)\right)$ and the τ dependence is through the following equations:

$$\dot r = \dot{\mathbf{R}}\cdot\nabla r, \dot\theta = \dot{\mathbf{R}}\cdot\nabla\theta, \dot\phi = \dot{\mathbf{R}}\cdot\nabla\phi \tag{8}$$

Let us separate $\phi\left(\tau\right)$ into its secular and oscillating parts:

$$\phi\left(\tau\right) = \left\langle\dot\phi\right\rangle\tau+\widetilde{\phi}\left(\tau\right) \tag{9}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}[r(\tau),\theta(\tau)] = \hat{G}[r(\tau),\theta(\tau)]\exp\left(in\tilde{\phi}(\tau)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}(\tau) = \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma)\exp(ip\omega_b\tau) \tag{10}$$

where,

$$Y_p(\Lambda, \epsilon, \bar{r}; \sigma) = \frac{1}{\tau_b}\oint d\tau\tilde{G}(\tau)\exp(-ip\omega_b\tau) \tag{11}$$

with $r(\tau) = \bar{r} + \rho_d\cos\theta(\tau)$, ρ_d represents the finite orbit width for passing particles. $\rho_d = \Omega_d/\omega_t$, $\Omega_d = \frac{\left(v_\perp^2/2+v_\parallel^2\right)}{\Omega R_0}$, $\omega_t = \frac{v_\parallel}{qR_0}$. Thus,

$$\rho_d = \frac{q}{\Omega}\sqrt{\frac{\epsilon}{2(1-\Lambda/b)}}\left[\frac{\Lambda}{b} + 2\left(1-\frac{\Lambda}{b}\right)\right] \tag{12}$$

When $\tilde{G}$ is expressed by $\cos k\theta$, $\sin k\theta$ instead of $\exp ik\theta$, for $p = 0$,

$$Y_p(\Lambda, \epsilon, \bar{r}; \sigma) = \frac{1}{\tau_b}\oint d\tau\tilde{G}(\tau)\exp(-ip\omega_b\tau) \tag{13}$$

for $p \neq 0$,

$$Y_p(\Lambda, \epsilon, \bar{r}; \sigma) = \frac{2}{\tau_b}\oint d\tau\tilde{G}(\tau)\exp(-ip\omega_b\tau) \tag{14}$$

Carrying out the time integration, the solution of g is obtained

$$g = \frac{e}{M}\frac{\partial F}{\partial\epsilon}(\omega-\omega_\star)B\frac{\epsilon}{\Omega}\sum_{-\infty}^{\infty}Y_p(\Lambda, \epsilon, \bar{r}; \sigma)\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega\right)t\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \tag{15}$$

The formula of δW_k is derived as follows.

$$\delta W_k = \int d^3x\vec{\xi}^\star\cdot\nabla\cdot\delta\mathbf{P}_k = e\int d^3x\int d^3v\left(\frac{i}{\omega}\mathbf{v}_d\cdot\delta\mathbf{E}_\perp\right)^\star g$$

$$= e\int d^3x\int d^3vgB\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta^\star + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

$$= e\int d^3x\int d^3vgB\frac{\epsilon}{\Omega}G^\star \tag{16}$$

where $G^\star = \hat{G}^\star[r(\tau),\theta(\tau)]\exp(i\omega\tau - in\phi(\tau))$. Let $\tilde{G}^\star[r(\tau),\theta(\tau)] = \hat{G}^\star[r(\tau),\theta(\tau)]\exp\left(-in\tilde{\phi}(\tau)\right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^{\star}(\tau) = \sum_{-\infty}^{\infty} Y_p^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip\omega_b \tau) \tag{17}$$

where,

$$Y_p^{\star}(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}^{\star}(\tau) \exp(ip\omega_b \tau) \tag{18}$$

with $r(\tau) = \bar{r} + \rho_d \cos\theta(\tau)$.

$$\delta W_k = \frac{e^2}{M} \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2} \sum_{-\infty}^{\infty} Y_p(\Lambda, \bar{r}; \sigma)$$

$$\cdot \frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega\right)\tau\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp\left(i\omega\tau - in\left\langle\dot{\phi}\right\rangle\tau - ip'\omega_b\tau\right) \tag{19}$$

$$\delta W_k = \frac{e^2}{M} \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp[ip\omega_b\tau]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b\tau) \tag{20}$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = \frac{e^2}{M} \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp(ip\omega_b\tau)}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b\tau) \tag{21}$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon}b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_\parallel$, one finally obtains

$$\delta W_k = \frac{4\pi^2}{M} \frac{e^2 B^2}{\Omega^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b (\omega - \omega_\star)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{|Y_p|^2}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \tag{22}$$

Note that $\tilde{\phi} \cong 0$, $\left\langle \dot{\phi} \right\rangle \cong \omega_D^0 + q\omega_b$,$\omega_D^0 \approx 0$ for passing particles. For internal kink mode $\nabla \cdot \vec{\xi} = 0$, the forms of the perturbation are $\xi_r = \xi_0 \cos\theta$, $\xi_\theta = -\xi_0 r \sin\theta$ within the region $q = 1$ rational surface $r_s = 1$, and $\xi_r = \xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right)\cos\theta$, $\xi_\theta = -\xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right) r \sin\theta + \xi_0 \left(\frac{r}{\Delta r}\right) r \sin\theta$ in the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi}\int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1}\sin^2\frac{\theta}{2}}\frac{d\theta}{\pi} \tag{23}$$

$$J_t = \frac{1}{2\pi}\int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1}\sin^2\frac{\theta}{2}}\frac{d\theta}{\pi} \tag{24}$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E}\right)^{-1} = \frac{\pi \omega_\parallel}{2K(\kappa)}, \kappa < 1 \tag{25}$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E}\right)^{-1} = \frac{\pi\sqrt{\kappa}\omega_\parallel}{K(\kappa^{-1})}, \kappa > 1 \tag{26}$$

where $\omega_\parallel = \frac{1}{qR}\sqrt{\varepsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR}\sqrt{\varepsilon\Lambda}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $F = \frac{n}{v_h^3}\bar{F}_0$**,** $v_h = \sqrt{\frac{2T_h}{M}}$**,** $\epsilon = \frac{T_h}{M}\bar{\epsilon}$**,** $r = a\bar{x}$**,** $J = aR_0\bar{J}$**,** $R = R_0\bar{R}$**,** $\omega_t = \frac{v_h}{R_0}\bar{\omega}_t$**,** $\frac{1}{\tau_t} = \frac{v_h}{2\pi R_0}\bar{\omega}_t = \frac{v_h}{R_0}\frac{\bar{\omega}_t}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_t}$**,** $\omega = \frac{v_h}{R_0}\bar{\omega}$**,** $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$**,** $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$ **.**

$$\delta W_k = \frac{\pi^2}{M}\frac{e^2B^2}{\Omega^2}a^2R_0n_0\frac{T_h}{M}\int\frac{\bar{J}}{q}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\bar{\tau}_b\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\sum_{-\infty}^{\infty}\frac{\left|\bar{Y}_p\right|^2}{n\left\langle\bar{\dot{\phi}}\right\rangle + p\bar{\omega}_b - \bar{\omega}} \tag{27}$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{28}$$

$$\omega_b t = \bar{\omega}_b\frac{v_h}{R}\int_0^\theta\frac{qR_0}{\sqrt{2(T/M)\bar{\epsilon}}b\sqrt{1-\frac{\Lambda}{b}}}d\theta \tag{29}$$

$$= \int_0^{\theta} \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{30}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^{\theta} d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(r,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\hat{\xi}_\theta\left(\theta, \bar{r} + \rho_d\cos\theta\right) + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\hat{\xi}_r\left(\theta, \bar{r} + \rho_d\cos\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta, \bar{x} + \frac{\rho_d}{a}\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta, \bar{x} + \frac{\rho_d}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F\left(x, \bar{\epsilon}, \Lambda\right) = \frac{n_0}{v_h^3} \frac{1}{\bar{\epsilon}^{3/2} + \bar{\epsilon}_c^{3/2}} Erfc\left(\frac{\bar{\epsilon} - \bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right) \exp\left[-\left(\frac{x - x_0}{\Delta x}\right)^2\right] \exp\left[-\left(\frac{\Lambda - \Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{31}$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + 2\Delta'\cos\theta \tag{32}$$

with $\Delta' = (\varepsilon + \alpha)/4$, $\varepsilon = \frac{r}{R_0}$, $\alpha = -R_0 q^2 d\beta/dr$, $\beta = \frac{2\mu_0 P}{B^2}$ set $\alpha = 0$ if $\beta = 0$, or assume $\bar{g}^{rr} = 1$ without toroidal effect, θ independent. Specially, in low beta limit,

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon\cos\theta \tag{33}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - 2(\varepsilon + \Delta')\cos\theta\right] \tag{34}$$

assume $\bar{g}^{\theta\theta} = \frac{1}{x^2}$ without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - \frac{5}{2}\varepsilon\cos\theta\right] \tag{35}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\left[\varepsilon + (r\Delta')'\right]\sin\theta \tag{36}$$

specially,

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{37}$$

for low beta limit. and $\bar{g}^{r\theta} = 0$ without toroidal effect.
The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta + \frac{a}{R}\frac{\varepsilon}{4} - \frac{a}{R}\frac{5}{4}\varepsilon(\cos 2\theta - 1) - \left(\frac{a}{R}\right)^2\frac{x}{q} \tag{38}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta + \frac{5}{4}\varepsilon^2\sin 2\theta \tag{39}$$

with $R = R_0 + r\cos\theta - \Delta(r) + r\eta(r)(\cos 2\theta - 1).\eta(r) = (\varepsilon + \Delta')/2$.
The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\epsilon}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \tag{40}$$

where, m is poloidal mode number, $\rho_h = v_h/\Omega$, $v_h = \sqrt{2T_h/M}, \Omega = Be/M$.
The normalized ρ_d is

$$\bar{\rho}_d = \frac{\rho_d}{a} = \frac{q}{2}\frac{\rho_h}{a}\sqrt{\frac{\bar{\epsilon}}{(1 - \Lambda/b)}}\left[\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right] \tag{41}$$

The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\cos\theta$$

$$\bar{\xi}_\theta(\theta, x) = -\bar{\xi}_0 x\sin\theta$$

within $q=1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0 \left(\frac{\bar{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}} \right) \cos\theta$$

$$\bar{\xi}_\theta(\theta, x) = -\bar{\xi}_0 \left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}} \right) x \sin\theta + \bar{\xi}_0 \left(\frac{x}{\overline{\Delta r}} \right) x \sin\theta$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The normalized $\delta\bar{W}_k$ is given by

$$\delta\bar{W}_k = \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b \left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\left\langle \dot{\bar{\phi}} \right\rangle + p\bar{\omega}_b - \bar{\omega}}, \tag{42}$$

where $\bar{J} = x$.

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q} \sqrt{\bar{\epsilon}} \tag{43}$$

where, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. and

$$\left\langle \dot{\bar{\phi}} \right\rangle \cong q\bar{\omega}_b \tag{44}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\, b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{45}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{46}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\int_0^{\theta} d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_{\theta} + \bar{g}^{r\theta}\bar{\kappa}_{r}\right)\bar{\hat{\xi}}_{\theta}\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_{r} + \bar{g}^{r\theta}\bar{\kappa}_{\theta}\right)\bar{\hat{\xi}}_{r}\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$ and normalized drift orbit width is $\bar{\rho}_d = \frac{\rho_d}{a}$.

The normalized slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$\bar{F}\left(x,\bar{\epsilon},\Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{47}$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1+\frac{1}{2}\varepsilon\cos\theta \tag{48}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1-\frac{5}{2}\varepsilon\cos\theta\right] \tag{49}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{50}$$

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta \tag{51}$$

$$\bar{\kappa}_{\theta} = \varepsilon\sin\theta \tag{52}$$

The normalized $\omega_{\star}$ is

$$\bar{\omega}_{\star} = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \tag{53}$$

The normalized ξ are

$$\bar{\xi}_r\left(\theta,x\right) = \bar{\xi}_0\cos\theta$$

$$\bar{\xi}_\theta(\theta, x) = -\bar{\xi}_0 x \sin\theta$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0 \left(\frac{\bar{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}} \right) \cos\theta$$

$$\bar{\xi}_\theta(\theta, x) = -\bar{\xi}_0 \left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}} \right) x \sin\theta + \bar{\xi}_0 \left(\frac{x}{\overline{\Delta r}} \right) x \sin\theta$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The formula for δW_{MHD}, δI [Miyamoto, "Plasma Physics and Controlled Nuclear Fusion"] The energy principle is

$$\delta W_{MHD} + \delta W_k + \delta I = 0 \tag{54}$$

where

$$\delta I = \frac{\gamma^2}{2} \int \rho_m \left|\vec{\xi}\right|^2 d\vec{r} \tag{55}$$

$$\delta W_k = \frac{1}{2} \int \vec{\xi} \cdot \nabla \delta p_h d\vec{r} \tag{56}$$

δW_{MHD} consists of the contribution δW^s_{MHD} from the singular region near the rational surface and the contribution δW^{ext}_{MHD} from the external region.

The MHD potential energy $\delta W^{ext}_{MHDtor}/2\pi R$ of toroidal plasma with circular cross-section is given by

$$\frac{\delta W^{ext}_{MHDtor}}{2\pi R} = \left(1 - \frac{1}{n^2}\right) \frac{\delta W^{ext}_{MHDcycl}}{2\pi R} + \frac{\pi B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \delta \hat{W}_T \tag{57}$$

$$\delta \hat{W}_T = \pi \left(\frac{r_s}{R}\right)^2 3\left(1 - q_0\right) \left(\frac{13}{144} - \beta^2_{ps}\right) \tag{58}$$

The term δW^s_{MHD} for the singular region is

$$\frac{\delta W^s_{MHD}}{2\pi R} = \frac{\pi}{2\mu_0} \frac{B^2_{\theta s}}{2\pi} sn\gamma\tau_{A\theta} \left|\xi_s\right|^2 \tag{59}$$

Thus, for $m = 1, n = 1$,

$$\begin{aligned} \delta W_{MHD} + \delta I &= 2\pi R \frac{B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \left(\delta \hat{W}_T + \gamma\tau_{A\theta} \frac{s}{2} + \pi\gamma^2\tau^2_{A\theta}\right) \\ &\approx 2\pi R \frac{B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \left(\delta \hat{W}_T + \frac{\gamma}{\omega_A}\right) \end{aligned}$$

where $\gamma = -i\omega$, $\omega_A \equiv (\tau_A s/2)^{-1}$. The dispersion relation is

$$\frac{2\pi R \frac{B_{\theta s}^2}{2\mu_0} \left|\xi_s\right|^2 \left(\delta \hat{W}_T + \frac{-i\omega}{\omega_A}\right)}{\left(\frac{\pi^2}{M} \frac{e^2 B^2}{\Omega^2} a^2 R_0 n_0 \frac{T_h}{M}\right)} + \delta \bar{W}_k = 0. \tag{60}$$