

## Ref of Fu's 1993 paper

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**The linearized drift kinetic equation is given by**

$$\left( \partial_t + \mathbf{v}_d \cdot \nabla + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla \right) g = ie \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) \frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_{\perp}$$

where,  $\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\Omega} \times \left( \mu \nabla B + \kappa v_{\parallel}^2 \right) \approx \frac{v_{\perp}^2/2 + v_{\parallel}^2}{\Omega} \hat{\mathbf{b}} \times \kappa$ ,  $\mu = \frac{v_{\perp}^2}{2B}$ ,  $\omega_* = \frac{i \hat{\mathbf{b}} \times \nabla F \cdot \nabla}{\Omega \partial F / \partial \epsilon}$ ,  
 $\delta \mathbf{E}_{\perp} = i \omega \vec{\xi} \times \mathbf{B}$ .

**The term  $\mathbf{v}_d \cdot \delta \mathbf{E}_{\perp}$  can be expressed by the following form.**

$$\begin{aligned} \mathbf{v}_d \cdot \delta \mathbf{E}_{\perp} &= \frac{v_{\perp}^2/2 + v_{\parallel}^2}{\Omega} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_{\perp} = i \omega \frac{v_{\perp}^2/2 + v_{\parallel}^2}{\Omega} \hat{\mathbf{b}} \times \vec{\kappa} \cdot \left( \vec{\xi} \times \mathbf{B} \right) \quad (1) \\ &= -i \omega B \frac{v_{\perp}^2/2 + v_{\parallel}^2}{\Omega} \vec{\kappa} \cdot \vec{\xi} = -i \omega B \frac{v_{\perp}^2/2 + v_{\parallel}^2}{\Omega} (\nabla \theta \kappa_{\theta} + \nabla r \kappa_r) \cdot (\xi_{\theta} \nabla \theta + \xi_r \nabla r) \\ &= -i \omega B \frac{v_{\perp}^2/2 + v_{\parallel}^2}{\Omega} (\nabla \theta \cdot \nabla \theta \xi_{\theta} \kappa_{\theta} + \nabla r \cdot \nabla r \xi_r \kappa_r + \nabla \theta \cdot \nabla r \xi_r \kappa_{\theta} + \nabla r \cdot \nabla \theta \xi_{\theta} \kappa_r) \\ &= -i \omega B \frac{\epsilon}{\Omega} \left( \frac{\Lambda}{b} + 2 \left( 1 - \frac{\Lambda}{b} \right) \right) ((g^{\theta\theta} \kappa_{\theta} + g^{r\theta} \kappa_r) \xi_{\theta} + (g^{rr} \kappa_r + g^{r\theta} \kappa_{\theta}) \xi_r) \\ &\quad \text{with } g^{rr} = \nabla r \cdot \nabla r, g^{\theta r} = \nabla \theta \cdot \nabla r, g^{\theta\theta} = \nabla \theta \cdot \nabla \theta. \xi_r \nabla r = \xi_r \mathbf{e}_r, \xi_{\theta} \nabla \theta = \frac{1}{r} \xi_{\theta} \mathbf{e}_{\theta}, \\ &\quad \kappa_{\theta} = -\frac{1}{R} \frac{\partial R}{\partial \theta}, \kappa_r \approx -\frac{1}{R} \frac{\partial R}{\partial r} - \frac{r}{q^2 R^2} \text{ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]}. \\ &\quad \Lambda = \frac{\mu B_0}{\epsilon}, b = B_0/B \approx 1 + (r/R_0) \cos \theta, \epsilon = \frac{1}{2} v^2. \end{aligned}$$

**The linearized drift kinetic equation is rewritten as**

$$\frac{d}{dt} g = ie \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) B \frac{\epsilon}{\Omega} \left( \frac{\Lambda}{b} + 2 \left( 1 - \frac{\Lambda}{b} \right) \right) ((g^{\theta\theta} \kappa_{\theta} + g^{r\theta} \kappa_r) \xi_{\theta} + (g^{rr} \kappa_r + g^{r\theta} \kappa_{\theta}) \xi_r) \quad (2)$$

$$\frac{d}{dt} g = H(r, \theta, \phi, t)$$

**The solution of perturbed distribution function  $g$  is obtained in the followings.** At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint \frac{d\psi}{\dot{\psi}} = \oint \frac{d\theta}{\dot{\theta}} \quad (3)$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}(r, \theta) \exp(-i\omega t + in\phi) \quad (4)$$

The formal solution of the nonadibatic distribution  $g$  is

$$g = \int_{-\infty}^t ie \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) B \frac{\epsilon}{\Omega} G(\tau) d\tau \quad (5)$$

with

$$G = \left( \frac{\Lambda}{b} + 2 \left( 1 - \frac{\Lambda}{b} \right) \right) ((g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r) \xi_\theta + (g^{rr} \kappa_r + g^{r\theta} \kappa_\theta) \xi_r)$$

where  $G(\tau) = \hat{G}[r(\tau), \theta(\tau)] \exp(-i\omega\tau + in\phi(\tau))$  and the  $\tau$  dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}} \cdot \nabla r, \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla \theta, \dot{\phi} = \dot{\mathbf{R}} \cdot \nabla \phi \quad (6)$$

Let us separate  $\phi(\tau)$  into its secular and osillating parts:

$$\phi(\tau) = \langle \dot{\phi} \rangle \tau + \tilde{\phi}(\tau) \quad (7)$$

where the brackets indicate bounce averaging.

The quantity  $\tilde{G}[r(\tau), \theta(\tau)] = \hat{G}[r(\tau), \theta(\tau)] \exp(in\tilde{\phi}(\tau))$  is a periodic function of  $\tau$ , which can be expanded in Fourier series,

$$\tilde{G}(\tau) = \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \exp(ip\omega_b \tau) \quad (8)$$

where,

$$Y_p(\Lambda, \epsilon, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}(\tau) \exp(-ip\omega_b \tau) \quad (9)$$

with  $r(\tau) = \bar{r} + \rho \cos \theta(\tau)$ ,  $\rho$  represents the finite orbit width.

Carrying out the time integration, the solution of  $g$  is obtained

$$g = e \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) B \frac{\epsilon}{\Omega} \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp \left[ i \left( n \langle \dot{\phi} \rangle + p\omega_b - \omega \right) t \right]}{n \langle \dot{\phi} \rangle + p\omega_b - \omega} \quad (10)$$

The formula of  $\delta W_k$  is derived as follows.

$$\begin{aligned}
\delta W_k &= \int d^3x \vec{\xi}^* \cdot \nabla \cdot \delta \mathbf{P}_k = -e \int d^3x \int d^3v \left( \frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \right)^* g \\
&= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} \left( \frac{\Lambda}{b} + 2 \left( 1 - \frac{\Lambda}{b} \right) \right) ((g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r) \xi_\theta^* + (g^{rr} \kappa_r + g^{r\theta} \kappa_\theta) \xi_r^*) \\
&= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} G^*
\end{aligned}$$

where  $G^* = \hat{G}^*[r(\tau), \theta(\tau)] \exp(i\omega\tau - in\phi(\tau))$ . Let  $\tilde{G}^*[r(\tau), \theta(\tau)] = \hat{G}^*[r(\tau), \theta(\tau)] \exp(-in\tilde{\phi}(\tau))$ , which is a periodic function of  $\tau$ , which can be expanded in Fourier series,

$$\tilde{G}^*(\tau) = \sum_{-\infty}^{\infty} Y_p^*(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip\omega_b\tau) \quad (11)$$

where,

$$Y_p^*(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}^*(\tau) \exp(ip\omega_b\tau) \quad (12)$$

with  $r(\tau) = \bar{r} + \rho \cos \theta(\tau)$ .

$$\begin{aligned}
\delta W_k &= -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) B^2 \frac{\epsilon^2}{\Omega^2} \sum_{-\infty}^{\infty} Y_p(\Lambda, \bar{r}; \sigma) \\
&\cdot \frac{\exp \left[ i \left( n \langle \dot{\phi} \rangle + p\omega_b - \omega \right) \tau \right]}{n \langle \dot{\phi} \rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^*(\Lambda, \epsilon, \bar{r}; \sigma) \exp \left( i\omega\tau - in \langle \dot{\phi} \rangle \tau - ip'\omega_b\tau \right)
\end{aligned} \quad (13)$$

$$\begin{aligned}
\delta W_k &= -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) B^2 \frac{\epsilon^2}{\Omega^2} \\
&\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp[ip\omega_b\tau]}{n \langle \dot{\phi} \rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^*(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b\tau)
\end{aligned} \quad (14)$$

Using  $d^3v = \sqrt{2\pi} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$ ,  $d^3x = 2\pi J dr d\theta$ , yields

$$\delta W_k = -e^2 \int 2\pi J dr d\theta \int \sqrt{2\pi} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} (\omega - \omega_*) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp(ip\omega_b\tau)}{n\langle\dot{\phi}\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^*(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b\tau) \quad (15)$$

Applying  $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon b}\sqrt{1-\frac{\Lambda}{b}}}d\theta$ ,  $\sigma = \pm 1$  for the direction of  $v_{\parallel}$ , one finally obtains

$$\delta W_k = -4\pi^2 \frac{e^2 B^2}{\Omega^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b (\omega - \omega_*)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{|Y_p|^2}{n\langle\dot{\phi}\rangle + p\omega_b - \omega} \quad (16)$$

Note that  $\tilde{\phi} \cong 0$ ,  $\langle\dot{\phi}\rangle \cong \omega_D^0 + q\omega_b, \omega_D^0 \approx 0$  for passing particles. For internal kink mode  $\nabla \cdot \tilde{\xi} = 0$ , the forms of the perturbation are  $\xi_r = \xi_0 \cos \theta$ ,  $\xi_\theta = -\xi_0 r \sin \theta$  within the region  $q = 1$  rational surface  $r_s = 1$ , and  $\xi_r = \xi_0 \left( \frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r} \right) \cos \theta$ ,  $\xi_\theta = -\xi_0 \left( \frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r} \right) r \sin \theta + \xi_0 \left( \frac{r}{\Delta r} \right) r \sin \theta$  in the inertial region  $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$ .

In angle-action coordinate,

$$J_b = \frac{1}{2\pi} \int p_{\parallel} ds \cong p_{\parallel e} R_{\parallel} \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1} \sin^2 \theta} \frac{\theta}{2} \frac{d\theta}{\pi} \quad (17)$$

$$J_t = \frac{1}{2\pi} \int p_{\parallel} ds \cong p_{\parallel e} R_{\parallel} \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1} \sin^2 \theta} \frac{\theta}{2} \frac{d\theta}{\pi} \quad (18)$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left( \frac{\partial J_b}{\partial E} \right)^{-1} = \frac{\pi \omega_{\parallel}}{2K(\kappa)}, \kappa < 1 \quad (19)$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left( \frac{\partial J_t}{\partial E} \right)^{-1} = \frac{\pi \sqrt{\kappa} \omega_{\parallel}}{K(\kappa^{-1})}, \kappa > 1 \quad (20)$$

where  $\omega_{\parallel} = \frac{1}{qR} \sqrt{\epsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR} \sqrt{\epsilon \Lambda}$ ,  $\kappa = \frac{1 - \Lambda(1 - \epsilon)}{2\epsilon \Lambda}$ ,  $\epsilon = \frac{r}{R_0}$ .  $K$  denotes the complete elliptic integral of the first kind.

**The normalized relations of the quantities are**  $F = \frac{n}{v_h^3} \bar{F}_0$ ,  $v_h = \sqrt{\frac{2T_h}{M}}$ ,  $\epsilon = \frac{T_h}{M} \bar{\epsilon}$ ,  $r = a\bar{x}$ ,  $J = aR_0 \bar{J}$ ,  $R = R_0 \bar{R}$ ,  $\omega_t = \frac{v_h}{R_0} \bar{\omega}_t$ ,  $\frac{1}{\tau_t} = \frac{v_h}{2\pi R_0} \bar{\omega}_t = \frac{v_h}{R_0} \frac{\bar{\omega}_t}{2\pi} = \frac{v_h}{R_0} \frac{1}{\bar{\tau}_t}$ ,  $\omega = \frac{v_h}{R_0} \bar{\omega}$ ,  $\omega_\phi = \frac{v_h}{R_0} \bar{\omega}_\phi$ ,  $\omega_* = \frac{v_h}{R_0} \bar{\omega}_*$ .

$$\delta W_k = -\pi^2 \frac{e^2 B^2}{\Omega^2} a^2 R_0 n_0 \frac{T_h}{M} \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b (\bar{\omega} - \bar{\omega}_*)$$

$$\sum_{-\infty}^{\infty} \frac{|\bar{Y}_p|^2}{n \langle \dot{\phi} \rangle + p\bar{\omega}_b - \bar{\omega}} \quad (21)$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{q} \sqrt{\bar{\epsilon}} \quad (22)$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2(T/M)} \bar{\epsilon} b \sqrt{1 - \frac{\Lambda}{b}}} d\theta \quad (23)$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1 - \frac{\Lambda}{b}}} d\theta$$

$$Y_p(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}(\tau) \exp(-ip\omega_b\tau) \quad (24)$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}[r(\tau), \theta(\tau)] \exp(-ip\omega_b\tau)$$

$$= \frac{1}{2\pi} \oint d(\omega_b\tau) \tilde{G}[r(\tau), \theta(\tau)] \exp(-ip\omega_b\tau)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1 - \frac{\Lambda}{b}}} \tilde{G}[r(\tau), \theta(\tau)] \exp\left(-ip \int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1 - \frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}[r(\tau), \theta(\tau)] = \left( \frac{\Lambda}{b(r, \theta)} + 2 \left( 1 - \frac{\Lambda}{b(r, \theta)} \right) \right)$$

$$\cdot \left( (g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r) \hat{\xi}_{\theta m}(\bar{r} + \rho \cos \theta) \exp(-im\theta) + (g^{rr}\kappa_r + g^{r\theta}\kappa_\theta) \hat{\xi}_{rm}(\bar{r} + \rho \cos \theta) \exp(-im\theta) \right)$$

Note that the effect of finite orbit width is induced radially in the perturbed  $\xi(r)$  for simply. Furthermore,  $\tilde{G}$  is a normalized quantity, so is  $Y_p$ ,

$$\tilde{G}[x(\tau), \theta(\tau)] = \left( \frac{\Lambda}{b(x, \theta)} + 2 \left( 1 - \frac{\Lambda}{b(x, \theta)} \right) \right)$$

$$\cdot \left( (\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r) \bar{\hat{\xi}}_{\theta m}\left(\bar{x} + \frac{\rho}{a} \cos \theta\right) \exp(-im\theta) + (\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta) \bar{\hat{\xi}}_{rm}\left(\bar{x} + \frac{\rho}{a} \cos \theta\right) \exp(-im\theta) \right)$$

where, the normalized displacements are  $\bar{\xi}_{\theta m} = \hat{\xi}_{\theta m}/a^2$ ,  $\bar{\xi}_{rm} = \hat{\xi}_{rm}/a$ .

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F(x, \bar{\epsilon}, \Lambda) = \frac{n_0}{v_h^3 \bar{\epsilon}^{3/2} + \bar{\epsilon}_c^{3/2}} \text{Erfc} \left( \frac{\bar{\epsilon} - \bar{\epsilon}_0}{\Delta \bar{\epsilon}} \right) \exp \left[ - \left( \frac{x - x_0}{\Delta x} \right)^2 \right] \exp \left( \frac{\Lambda - \Lambda_0}{\Delta \Lambda^{0.2}} \right)^2 \quad (25)$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + 2\Delta' \cos \theta \quad (26)$$

with  $\Delta' = (\varepsilon + \alpha)/4$ ,  $\varepsilon = \frac{r}{R_0}$ ,  $\alpha = -R_0 q^2 d\beta/dr$ ,  $\beta = \frac{2\mu_0 P}{B^2}$  set  $\alpha = 0$  if  $\beta = 0$ , or assume  $\bar{g}^{rr} = 1$  without toroidal effect,  $\theta$  independent. Specially, in low beta limit,

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon \cos \theta \quad (27)$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2} [1 - 2(\varepsilon + \Delta') \cos \theta] \quad (28)$$

assume  $\bar{g}^{\theta\theta} = \frac{1}{x^2}$  without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta} = \frac{1}{x^2} \left[ 1 - \frac{5}{2}\varepsilon \cos \theta \right] \quad (29)$$

$$\bar{g}^{r\theta} = -\frac{1}{x} \left[ \varepsilon + (r\Delta')' \right] \sin \theta \quad (30)$$

specially,

$$\bar{g}^{r\theta} = -\frac{1}{x} \frac{3}{2}\varepsilon \sin \theta \quad (31)$$

for low beta limit. and  $\bar{g}^{r\theta} = 0$  without toroidal effect.

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R} \cos \theta + \frac{a}{R} \frac{\varepsilon}{4} - \frac{a}{R} \frac{5}{4} \varepsilon (\cos 2\theta - 1) - \left( \frac{a}{R} \right)^2 \frac{x}{q} \quad (32)$$

$$\bar{\kappa}_\theta = \varepsilon \sin \theta + \frac{5}{4} \varepsilon^2 \sin 2\theta \quad (33)$$

with  $R = R_0 + r \cos \theta - \Delta(r) + r\eta(r)(\cos 2\theta - 1)$ .  $\eta(r) = (\varepsilon + \Delta')/2$ .

The normalized  $\omega_\star$  is

$$\bar{\omega}_\star = \frac{1}{2} \frac{m \partial \bar{F} / \partial x}{\partial \bar{F} / \partial \bar{\epsilon}} \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a} \quad (34)$$

where,  $m$  is poloidal mode number,  $\rho_h = v_h/\Omega$ ,  $v_h = \sqrt{2T_h/M}$ ,  $\Omega = Be/M$ . The normalized  $\xi$  are

$$\bar{\xi}_r = \bar{\xi}_0 \cos \theta$$

$$\bar{\xi}_\theta = -\bar{\xi}_0 x \sin \theta$$

within  $q = 1$  surface. In the inertial region  $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$ ,

$$\begin{aligned}\bar{\xi}_r &= \bar{\xi}_0 \left( \frac{\bar{\Delta}r - x + (\bar{r}_s - \bar{\Delta}r/2)}{\bar{\Delta}r} \right) \cos \theta \\ \bar{\xi}_\theta &= -\bar{\xi}_0 \left( \frac{\bar{\Delta}r - x + (\bar{r}_s - \bar{\Delta}r/2)}{\bar{\Delta}r} \right) x \sin \theta + \bar{\xi}_0 \left( \frac{x}{\bar{\Delta}r} \right) x \sin \theta\end{aligned}$$

with  $\bar{\xi}_0 = \xi_0/a$ ,  $\bar{r}_s = r_s/a$ ,  $\bar{\Delta}r = \Delta r/a$ ,  $x = r/a$ .