Ref of Fu's 1993 paper

October 13, 2015

The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = ie \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) \frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp$$

where, $\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \kappa$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i\hat{\mathbf{b}} \times \nabla F \cdot \nabla}{\Omega \partial F / \partial \epsilon}$, $\delta \mathbf{E}_\perp = i\omega \vec{\xi} \times \mathbf{B}$.

The term $\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp = \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp = i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \qquad (1)$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \vec{\kappa} \cdot \vec{\xi} = -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \kappa_\theta + \nabla r \kappa_r\right) \cdot \left(\xi_\theta \nabla\theta + \xi_r \nabla r\right)$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \cdot \nabla\theta \xi_\theta \kappa_\theta + \nabla r \cdot \nabla r \xi_r \kappa_r + \nabla\theta \cdot \nabla r \xi_r \kappa_\theta + \nabla r \cdot \nabla\theta \xi_\theta \kappa_r\right)$$

$$= -i\omega B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r \cdot \nabla r$, $g^{\theta r} = \nabla\theta \cdot \nabla r$, $g^{\theta\theta} = \nabla\theta \cdot \nabla\theta$. $\xi_r \nabla r = \xi_r \mathbf{e}_r$, $\xi_\theta \nabla\theta = \frac{1}{r}\xi_\theta \mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial \theta}$, $\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2 R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0)\cos\theta$, $\epsilon = \frac{1}{2}v^2$.

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = ie \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{2}$$

$$\frac{d}{dt}g = H\left(r, \theta, \phi, t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint \frac{d\psi}{\dot{\psi}} = \oint \frac{d\theta}{\dot{\theta}} \tag{3}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}(r, \theta) \exp(-i\omega t + in\phi) \tag{4}$$

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} ie \frac{\partial F}{\partial \epsilon}(\omega - \omega_\star) B \frac{\epsilon}{\Omega} G(\tau) \, d\tau \tag{5}$$

with

$$G = \left(\frac{\Lambda}{b} + 2 \left(1 - \frac{\Lambda}{b} \right) \right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r \right) \xi_\theta + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta \right) \xi_r \right)$$

where $G(\tau) = \hat{G}[r(\tau), \theta(\tau)] \exp(-i\omega\tau + in\phi(\tau))$ and the τ dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}} \cdot \nabla r, \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla \theta, \dot{\phi} = \dot{\mathbf{R}} \cdot \nabla \phi \tag{6}$$

Let us separate $\phi(\tau)$ into its secular and oscillating parts:

$$\phi(\tau) = \left\langle \dot{\phi} \right\rangle \tau + \widetilde{\phi}(\tau) \tag{7}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}[r(\tau), \theta(\tau)] = \hat{G}[r(\tau), \theta(\tau)] \exp\left(in\tilde{\phi}(\tau) \right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}(\tau) = \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \exp(ip\omega_b\tau) \tag{8}$$

where,

$$Y_p(\Lambda, \epsilon, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}(\tau) \exp(-ip\omega_b\tau) \tag{9}$$

with $r(\tau) = \bar{r} + \rho \cos\theta(\tau)$, ρ represents the finite orbit width.
When $\tilde{G}$ is expressed by $\cos k\theta$, $\sin k\theta$ instead of $\exp ik\theta$, for $p = 0$,

$$Y_p(\Lambda, \epsilon, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}(\tau) \exp(-ip\omega_b\tau) \tag{10}$$

for $p \neq 0$,

$$Y_p \left(\Lambda, \epsilon, \bar{r}; \sigma \right) = \frac{2}{\tau_b} \oint d\tau \tilde{G} \left(\tau \right) \exp \left(-ip\omega_b \tau \right) \tag{11}$$

Carrying out the time integration, the solution of g is obtained

$$g = e \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star \right) B \frac{\epsilon}{\Omega} \sum_{-\infty}^{\infty} Y_p \left(\Lambda, \epsilon, \bar{r}; \sigma \right) \frac{\exp \left[i \left(n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega \right) t \right]}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \tag{12}$$

The formula of δW_k is derived as follows.

$$\delta W_k = \int d^3x \vec{\xi}^\star \cdot \nabla \cdot \delta \mathbf{P}_k = -e \int d^3x \int d^3v \left(\frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \right)^\star g$$

$$= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2 \left(1 - \frac{\Lambda}{b} \right) \right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r \right) \xi_\theta^\star + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta \right) \xi_r^\star \right)$$

$$= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} G^\star$$

where $G^\star = \hat{G}^\star \left[r \left(\tau \right), \theta \left(\tau \right) \right] \exp \left(i\omega\tau - in\phi \left(\tau \right) \right)$. Let $\tilde{G}^\star \left[r \left(\tau \right), \theta \left(\tau \right) \right] = \hat{G}^\star \left[r \left(\tau \right), \theta \left(\tau \right) \right] \exp \left(-in\tilde{\phi} \left(\tau \right) \right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star \left(\tau \right) = \sum_{-\infty}^{\infty} Y_p^\star \left(\Lambda, \epsilon, \bar{r}; \sigma \right) \exp \left(-ip\omega_b \tau \right) \tag{13}$$

where,

$$Y_p^\star \left(\Lambda, \bar{r}; \sigma \right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}^\star \left(\tau \right) \exp \left(ip\omega_b \tau \right) \tag{14}$$

with $r \left(\tau \right) = \bar{r} + \rho \cos \theta \left(\tau \right)$.

$$\delta W_k = -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star \right) B^2 \frac{\epsilon^2}{\Omega^2} \sum_{-\infty}^{\infty} Y_p \left(\Lambda, \bar{r}; \sigma \right)$$

$$\cdot \frac{\exp \left[i \left(n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega \right) \tau \right]}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star \left(\Lambda, \epsilon, \bar{r}; \sigma \right) \exp \left(i\omega\tau - in \left\langle \dot{\phi} \right\rangle \tau - ip'\omega_b \tau \right) \tag{15}$$

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$$\delta W_k = -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \frac{\exp\left[ip\omega_b\tau\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(-ip'\omega_b\tau\right) \quad (16)$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = -e^2 \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \frac{\exp\left(ip\omega_b\tau\right)}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(-ip'\omega_b\tau\right) \quad (17)$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon}b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_\parallel$, one finally obtains

$$\delta W_k = -4\pi^2 \frac{e^2 B^2}{\Omega^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b \left(\omega - \omega_\star\right)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{|Y_p|^2}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \quad (18)$$

Note that $\tilde{\phi} \cong 0$, $\left\langle\dot{\phi}\right\rangle \cong \omega_D^0 + q\omega_b, \omega_D^0 \approx 0$ for passing particles. For internal kink mode $\nabla \cdot \vec{\xi} = 0$, the forms of the perturbation are $\xi_r = \xi_0 \cos\theta$, $\xi_\theta = -\xi_0 r \sin\theta$ within the region $q = 1$ rational surface $r_s = 1$, and $\xi_r = \xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right) \cos\theta$, $\xi_\theta = -\xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right) r \sin\theta + \xi_0 \left(\frac{r}{\Delta r}\right) r \sin\theta$ in the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1}\sin^2\frac{\theta}{2}} \frac{d\theta}{\pi} \quad (19)$$

$$J_t = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1}\sin^2\frac{\theta}{2}} \frac{d\theta}{\pi} \quad (20)$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

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$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E}\right)^{-1} = \frac{\pi \omega_\parallel}{2K(\kappa)}, \kappa < 1 \tag{21}$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E}\right)^{-1} = \frac{\pi \sqrt{\kappa} \omega_\parallel}{K(\kappa^{-1})}, \kappa > 1 \tag{22}$$

where $\omega_\parallel = \frac{1}{qR}\sqrt{\varepsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR}\sqrt{\varepsilon \Lambda}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $F = \frac{n}{v_h^3}\bar{F}_0$, $v_h = \sqrt{\frac{2T_h}{M}}$, $\epsilon = \frac{T_h}{M}\bar{\epsilon}$, $r = a\bar{x}$, $J = aR_0\bar{J}$, $R = R_0\bar{R}$, $\omega_t = \frac{v_h}{R_0}\bar{\omega}_t$, $\frac{1}{\tau_t} = \frac{v_h}{2\pi R_0}\bar{\omega}_t = \frac{v_h}{R_0}\frac{\bar{\omega}_t}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_t}$, $\omega = \frac{v_h}{R_0}\bar{\omega}$, $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$, $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$.

$$\delta W_k = -\pi^2 \frac{e^2 B^2}{\Omega^2} a^2 R_0 n_0 \frac{T_h}{M} \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b (\bar{\omega} - \bar{\omega}_\star)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{|\bar{Y}_p|^2}{n\langle \dot{\bar{\phi}} \rangle + p\bar{\omega}_b - \bar{\omega}} \tag{23}$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{q} \sqrt{\bar{\epsilon}} \tag{24}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2(T/M)\bar{\epsilon}}b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{25}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}(\tau) \exp(-ip\omega_b\tau) \tag{26}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}[r(\tau), \theta(\tau)] \exp(-ip\omega_b\tau)$$

$$= \frac{1}{2\pi} \oint d(\omega_b\tau) \tilde{G}[r(\tau), \theta(\tau)] \exp(-ip\omega_b\tau)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}[r(\tau), \theta(\tau)] \exp\left(-ip \int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(r,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\hat{\xi}_\theta\left(\theta,\bar{r}+\rho\cos\theta\right)+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\hat{\xi}_r\left(\theta,\bar{r}+\rho\cos\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(x,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta+\bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta,\bar{x}+\frac{\rho}{a}\cos\theta\right)+\left(\bar{g}^{rr}\bar{\kappa}_r+\bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta,\bar{x}+\frac{\rho}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m}=\hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm}=\hat{\xi}_{rm}/a$.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F\left(x,\bar{\epsilon},\Lambda\right)=\frac{n_0}{v_h^3}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda^{0.2}}\right)^2 \tag{27}$$

The normalized metric tensors are

$$\bar{g}^{rr}=1+2\Delta'\cos\theta \tag{28}$$

with $\Delta'=\left(\varepsilon+\alpha\right)/4$, $\varepsilon=\frac{r}{R_0}$, $\alpha=-R_0q^2d\beta/dr$, $\beta=\frac{2\mu_0P}{B^2}$ set $\alpha=0$ if $\beta=0$, or assume $\bar{g}^{rr}=1$ without toroidal effect, θ independent. Specially, in low beta limit,

$$\bar{g}^{rr}=1+\frac{1}{2}\varepsilon\cos\theta \tag{29}$$

$$\bar{g}^{\theta\theta}=\frac{1}{x^2}\left[1-2\left(\varepsilon+\Delta'\right)\cos\theta\right] \tag{30}$$

assume $\bar{g}^{\theta\theta}=\frac{1}{x^2}$ without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta}=\frac{1}{x^2}\left[1-\frac{5}{2}\varepsilon\cos\theta\right] \tag{31}$$

$$\bar{g}^{r\theta}=-\frac{1}{x}\left[\varepsilon+\left(r\Delta'\right)'\right]\sin\theta \tag{32}$$

specially,

$$\bar{g}^{r\theta}=-\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{33}$$

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for low beta limit. and $\bar{g}^{r\theta} = 0$ without toroidal effect.

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta + \frac{a}{R}\frac{\varepsilon}{4} - \frac{a}{R}\frac{5}{4}\varepsilon\left(\cos 2\theta - 1\right) - \left(\frac{a}{R}\right)^2\frac{x}{q} \qquad (34)$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta + \frac{5}{4}\varepsilon^2\sin 2\theta \qquad (35)$$

with $R = R_0 + r\cos\theta - \Delta\left(r\right) + r\eta\left(r\right)\left(\cos 2\theta - 1\right)$. $\eta\left(r\right) = \left(\varepsilon + \Delta'\right)/2$.

The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \qquad (36)$$

where, m is poloidal mode number, $\rho_h = v_h/\Omega$, $v_h = \sqrt{2T_h/M}$, $\Omega = Be/M$.

The normalized ξ are

$$\bar{\xi}_r\left(\theta, x\right) = \bar{\xi}_0\cos\theta$$

$$\bar{\xi}_\theta\left(\theta, x\right) = -\bar{\xi}_0 x\sin\theta$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r\left(\theta, x\right) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\cos\theta$$

$$\bar{\xi}_\theta\left(\theta, x\right) = -\bar{\xi}_0\left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)x\sin\theta + \bar{\xi}_0\left(\frac{x}{\overline{\Delta r}}\right)x\sin\theta$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The normalized $\delta\bar{W}_k$ is given by

$$\delta\bar{W}_k = \int\frac{\bar{J}}{q}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\bar{\tau}_b\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\sum_{-\infty}^{\infty}\frac{\left|\bar{Y}_p\right|^2}{n\left\langle\dot{\bar{\phi}}\right\rangle + p\bar{\omega}_b - \bar{\omega}}, \qquad (37)$$

where $\bar{J} = x$.

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \qquad (38)$$

where, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. and

$$\left\langle \dot{\bar{\phi}} \right\rangle \cong q\bar{\omega}_b \tag{39}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2(T/M)\bar{\epsilon}}b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{40}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p(\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}(\tau) \exp(-ip\omega_b\tau) \tag{41}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}[r(\tau), \theta(\tau)] \exp(-ip\omega_b\tau)$$

$$= \frac{1}{2\pi} \oint d(\omega_b\tau) \tilde{G}[r(\tau), \theta(\tau)] \exp(-ip\omega_b\tau)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}[r(\tau), \theta(\tau)] \exp\left(-ip \int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}[x(\tau), \theta(\tau)] = \left(\frac{\Lambda}{b(x,\theta)} + 2\left(1 - \frac{\Lambda}{b(x,\theta)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right) \bar{\hat{\xi}}_\theta\left(\theta, \bar{x} + \frac{\rho}{a}\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right) \bar{\hat{\xi}}_r\left(\theta, \bar{x} + \frac{\rho}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$.

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$$\bar{F}(x, \bar{\epsilon}, \Lambda) = \frac{1}{\bar{\epsilon}^{3/2} + \bar{\epsilon}_c^{3/2}} Erfc\left(\frac{\bar{\epsilon} - \bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right) \exp\left[-\left(\frac{x - x_0}{\Delta x}\right)^2\right] \exp\left(\frac{\Lambda - \Lambda_0}{\Delta\Lambda^{0.2}}\right)^2 \tag{42}$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon\cos\theta \tag{43}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - \frac{5}{2}\varepsilon\cos\theta\right] \tag{44}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{45}$$

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta \tag{46}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta \tag{47}$$

The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \tag{48}$$

The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\cos\theta$$

$$\bar{\xi}_\theta(\theta, x) = -\bar{\xi}_0 x\sin\theta$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\cos\theta$$

$$\bar{\xi}_\theta(\theta, x) = -\bar{\xi}_0\left(\frac{\overline{\Delta r} - x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)x\sin\theta + \bar{\xi}_0\left(\frac{x}{\overline{\Delta r}}\right)x\sin\theta$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The formula for δW_{MHD}, δI [Miyamoto, "Plasma Physics and Controlled Nuclear Fusion"]   The energy principle is

$$\delta W_{MHD} + \delta W_k + \delta I = 0 \tag{49}$$

where

$$\delta I = \frac{\gamma^2}{2}\int\rho_m\left|\vec{\xi}\right|^2 d\vec{r} \tag{50}$$

$$\delta W_k = \frac{1}{2}\int\vec{\xi}\cdot\nabla\delta p_h d\vec{r} \tag{51}$$

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δW_{MHD} consists of the contribution δW_{MHD}^s from the singular region near the rational surface and the contribution δW_{MHD}^{ext} from the external region.

The MHD potential energy $\delta W_{MHDtor}^{ext}/2\pi R$ of toroidal plasma with circular cross-section is given by

$$\frac{\delta W_{MHDtor}^{ext}}{2\pi R} = \left(1 - \frac{1}{n^2}\right) \frac{\delta W_{MHDcycl}^{ext}}{2\pi R} + \frac{\pi B_{\theta s}^2}{2\mu_0} |\xi_s|^2 \, \delta \hat{W}_T \tag{52}$$

$$\delta \hat{W}_T = \pi \left(\frac{r_s}{R}\right)^2 3 \left(1 - q_0\right) \left(\frac{13}{144} - \beta_{ps}^2\right) \tag{53}$$

The term δW_{MHD}^s for the singular region is

$$\frac{\delta W_{MHD}^s}{2\pi R} = \frac{\pi}{2\mu_0} \frac{B_{\theta s}^2}{2\pi} sn\gamma\tau_{A\theta} |\xi_s|^2 \tag{54}$$

Thus, for $m = 1, n = 1$,

$$\begin{aligned}
\delta W_{MHD} + \delta I &= 2\pi R \frac{B_{\theta s}^2}{2\mu_0} |\xi_s|^2 \left(\delta \hat{W}_T + \gamma\tau_{A\theta} \frac{s}{2} + \pi\gamma^2\tau_{A\theta}^2\right) \\
&\approx 2\pi R \frac{B_{\theta s}^2}{2\mu_0} |\xi_s|^2 \left(\delta \hat{W}_T + \frac{\gamma}{\omega_A}\right)
\end{aligned}$$

where $\gamma = -i\omega$, $\omega_A \equiv (\tau_A s/2)^{-1}$. The dispersion relation is

$$\frac{2\pi R \frac{B_{\theta s}^2}{2\mu_0} |\xi_s|^2 \left(\delta \hat{W}_T + \frac{-i\omega}{\omega_A}\right)}{\left(-\pi^2 \frac{e^2 B^2}{\Omega^2} a^2 R_0 n_0 \frac{T_h}{M}\right)} + \delta \bar{W}_k = 0. \tag{55}$$