Ref of Fu's 1993 paper

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The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = ie\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right)\frac{i}{\omega}\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i\hat{\mathbf{b}} \times \nabla F \cdot \nabla}{\Omega \partial F/\partial \epsilon}$, $\delta \mathbf{E}_\perp = i\omega \vec{\xi} \times \mathbf{B}$.

The term $\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp = \frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp = i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{1}$$

$$= -i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\vec{\kappa} \cdot \vec{\xi} = -i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\left(\nabla\theta \kappa_\theta + \nabla r \kappa_r\right) \cdot \left(\xi_\theta \nabla \theta + \xi_r \nabla r\right)$$

$$= -i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\Omega}\left(\nabla\theta \cdot \nabla\theta \xi_\theta \kappa_\theta + \nabla r \cdot \nabla r \xi_r \kappa_r + \nabla\theta \cdot \nabla r \xi_r \kappa_\theta + \nabla r \cdot \nabla \theta \xi_\theta \kappa_r\right)$$

$$= -i\omega B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r \cdot \nabla r$, $g^{\theta r} = \nabla\theta \cdot \nabla r$, $g^{\theta\theta} = \nabla\theta \cdot \nabla\theta$. $\xi_r \nabla r = \xi_r \mathbf{e}_r$, $\xi_\theta \nabla\theta = \frac{1}{r}\xi_\theta \mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial \theta}$, $\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2 R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0)\cos\theta$, $\epsilon = \frac{1}{2}v^2$.

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = ie\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right)B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{2}$$

$$\frac{d}{dt}g = H\left(r, \theta, \phi, t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the polodial cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint \frac{d\psi}{\dot{\psi}} = \oint \frac{d\theta}{\dot{\theta}} \tag{3}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}\left(r, \theta\right) \exp\left(-i\omega t + in\phi\right) \tag{4}$$

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} ie\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right) B\frac{\epsilon}{\Omega} G\left(\tau\right) d\tau \tag{5}$$

with

$$G = \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

where $G\left(\tau\right) = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-i\omega\tau + in\phi\left(\tau\right)\right)$ and the τ dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}} \cdot \nabla r, \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla\theta, \dot{\phi} = \dot{\mathbf{R}} \cdot \nabla\phi \tag{6}$$

Let us separate $\phi\left(\tau\right)$ into its secular and osillating parts:

$$\phi\left(\tau\right) = \left\langle\dot{\phi}\right\rangle\tau + \widetilde{\phi}\left(\tau\right) \tag{7}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(in\tilde{\phi}\left(\tau\right)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}\left(\tau\right) = \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(ip\omega_b\tau\right) \tag{8}$$

where,

$$Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) = \frac{1}{\tau_b}\oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{9}$$

with $r\left(\tau\right) = \bar{r} + \rho\cos\theta\left(\tau\right)$, ρ repesents the finite orbit width.

Carrying out the time integration, the solution of g is obtained

$$g = e\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right) B\frac{\epsilon}{\Omega}\sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right)\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega\right)t\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega} \tag{10}$$

The formula of δW_k is derived as follows.

$$\delta W_k = \int d^3x \vec{\xi}^\star \cdot \nabla \cdot \delta \mathbf{P}_k = -e \int d^3x \int d^3v \left(\frac{i}{\omega}\mathbf{v}_d \cdot \delta \mathbf{E}_\perp\right)^\star g$$

$$= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta^\star + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

$$= -e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} G^\star$$

where $G^\star = \hat{G}^\star\left[r(\tau), \theta(\tau)\right]\exp\left(i\omega\tau - in\phi(\tau)\right)$. Let $\tilde{G}^\star\left[r(\tau), \theta(\tau)\right] = \hat{G}^\star\left[r(\tau), \theta(\tau)\right]\exp\left(-in\tilde{\phi}(\tau)\right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star(\tau) = \sum_{-\infty}^{\infty} Y_p^\star\left(\Lambda, \epsilon, \bar{r}; \sigma\right)\exp\left(-ip\omega_b\tau\right) \tag{11}$$

where,

$$Y_p^\star\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b}\oint d\tau \tilde{G}^\star(\tau)\exp\left(ip\omega_b\tau\right) \tag{12}$$

with $r(\tau) = \bar{r} + \rho\cos\theta(\tau)$.

$$\delta W_k = -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\Omega^2}\sum_{-\infty}^{\infty} Y_p\left(\Lambda, \bar{r}; \sigma\right)$$

$$\cdot \frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega\right)\tau\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega}\sum_{-\infty}^{\infty} Y_{p'}^\star\left(\Lambda, \epsilon, \bar{r}; \sigma\right)\exp\left(i\omega\tau - in\left\langle\dot{\phi}\right\rangle\tau - ip'\omega_b\tau\right) \tag{13}$$

$$\delta W_k = -e^2 \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right)\frac{\exp\left[ip\omega_b\tau\right]}{n\left\langle\dot{\phi}\right\rangle + p\omega_b - \omega}\sum_{-\infty}^{\infty} Y_{p'}^\star\left(\Lambda, \epsilon, \bar{r}; \sigma\right)\exp\left(-ip'\omega_b\tau\right) \tag{14}$$

Using $d^3v = \sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\epsilon^{1/2}d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = -e^2 \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\epsilon^{1/2}d\epsilon\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp(ip\omega_b \tau)}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b \tau) \tag{15}$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon}b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_{\parallel}$, one finially obtains

$$\delta W_k = -4\pi^2 \frac{e^2 B^2}{\Omega^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b (\omega - \omega_\star)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{|Y_p|^2}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \tag{16}$$

Note that $\tilde{\phi} \cong 0$, $\left\langle \dot{\phi} \right\rangle \cong \omega_D^0 + q\omega_b, \omega_D^0 \approx 0$ for passing particles. For internal kink mode $\nabla \cdot \vec{\xi} = 0$, the forms of the perturbation are $\xi_r = \xi_0 \cos\theta$, $\xi_\theta = -\xi_0 r \sin\theta$ within the region $q = 1$ rational surface $r_s = 1$, and $\xi_r = \xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right) \cos\theta$, $\xi_\theta = -\xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right) r \sin\theta + \xi_0 \left(\frac{r}{\Delta r}\right) r \sin\theta$ in the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi} \int p_{\parallel} ds \cong p_{\parallel e} R_{\parallel} \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{17}$$

$$J_t = \frac{1}{2\pi} \int p_{\parallel} ds \cong p_{\parallel e} R_{\parallel} \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{18}$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E}\right)^{-1} = \frac{\pi \omega_{\parallel}}{2K(\kappa)}, \kappa < 1 \tag{19}$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E}\right)^{-1} = \frac{\pi\sqrt{\kappa}\omega_{\parallel}}{K(\kappa^{-1})}, \kappa > 1 \tag{20}$$

where $\omega_{\parallel} = \frac{1}{qR}\sqrt{\varepsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR}\sqrt{\varepsilon\Lambda}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $F = \frac{n}{v_h^3}\bar{F}_0$**,** $v_h = \sqrt{\frac{2T_h}{M}}$**,** $\epsilon = \frac{T_h}{M}\bar{\epsilon}$**,** $r = a\bar{x}$**,** $J = aR_0\bar{J}$**,** $R = R_0\bar{R}$**,** $\omega_t = \frac{v_h}{R_0}\bar{\omega}_t$**,** $\frac{1}{\tau_t} = \frac{v_h}{2\pi R_0}\bar{\omega}_t = \frac{v_h}{R_0}\frac{\bar{\omega}_t}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_t}$**,** $\omega = \frac{v_h}{R_0}\bar{\omega}$**,** $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$**,** $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$ **.**

$$\delta W_k = -\pi^2 \frac{e^2 B^2}{\Omega^2} a^2 R_0 n_0 \frac{T_h}{M} \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b (\bar{\omega} - \bar{\omega}_\star)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\left\langle \dot{\bar{\phi}} \right\rangle + p\bar{\omega}_b - \bar{\omega}} \tag{21}$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{22}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^{\theta} \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\, b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{23}$$

$$= \int_0^{\theta} \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{24}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \cancel{\exp\left(-ip\int_0^{\theta} d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)}$$

where

$$\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(r,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\hat{\xi}_{\theta m}\left(\bar{r} + \rho\cos\theta\right)\exp\left(-im\theta\right) + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\hat{\xi}_{rm}\left(\bar{r} + \rho\cos\theta\right)\exp\left(-im\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_{\theta m}\left(\bar{x} + \frac{\rho}{a}\cos\theta\right)\exp\left(-im\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_{rm}\left(\bar{x} + \frac{\rho}{a}\cos\theta\right)\exp\left(-im\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F\left(x,\bar{\epsilon},\Lambda\right) = \frac{n_0}{v_h^3}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda^{0.2}}\right)^2 \tag{25}$$