Ref of Fu's 1993 paper

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1 The formula of δW_k

The linearized drift kinetic equation assuming $\delta E_\parallel = 0$ is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = i \frac{e}{M} \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) \frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \tag{1}$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\omega_c} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2+v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \vec{\kappa}$, $\vec{\kappa} = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i\hat{\mathbf{b}} \times \nabla F \cdot \nabla}{\omega_c \partial F/\partial \epsilon}$, $\epsilon = v^2/2 = v_\parallel^2/2 + \mu B$, $\delta \mathbf{E}_\perp = i\omega \vec{\xi} \times \mathbf{B}$, $\omega_c = \frac{Be}{M}$ is the particle cyclotron frequency.[Berk et al, Phys. Fluid B 4 1992]. Note that the alter expression for $\omega_\star$ is $\omega_\star = \frac{\partial F/\partial P_\phi}{M \partial F/\partial \epsilon} i \frac{\partial}{\partial \phi} = \frac{nq \partial_r F}{\omega_c r \partial_\epsilon F}$.

The term $\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp = \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp = i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{2}$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \vec{\kappa} \cdot \vec{\xi} = -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} (\nabla \theta \kappa_\theta + \nabla r \kappa_r) \cdot (\xi_\theta \nabla \theta + \xi_r \nabla r)$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} (\nabla \theta \cdot \nabla \theta \xi_\theta \kappa_\theta + \nabla r \cdot \nabla r \xi_r \kappa_r + \nabla \theta \cdot \nabla r \xi_r \kappa_\theta + \nabla r \cdot \nabla \theta \xi_\theta \kappa_r)$$

$$= -i\omega B \frac{\epsilon}{\omega_c} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r\right) \xi_\theta + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta\right) \xi_r\right)$$

with $g^{rr} = \nabla r \cdot \nabla r$, $g^{\theta r} = \nabla \theta \cdot \nabla r$, $g^{\theta\theta} = \nabla \theta \cdot \nabla \theta$. $\xi_r \nabla r = \xi_r \mathbf{e}_r$, $\xi_\theta \nabla \theta = \frac{1}{r} \xi_\theta \mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R} \frac{\partial R}{\partial \theta}$, $\kappa_r \approx -\frac{1}{R} \frac{\partial R}{\partial r} - \frac{r}{q^2 R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0) \cos \theta$. Using $\delta \mathbf{E}_\perp^\star = -i\omega \vec{\xi}^\star \times \mathbf{B}$, the complex conjugate term thus is

$$\mathbf{v}_d \cdot \delta\mathbf{E}_\perp^\star = \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \kappa \cdot \delta\mathbf{E}_\perp^\star = -i\omega\frac{v_\perp^2/2 + v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \vec{\kappa} \cdot \left(\vec{\xi}^\star \times \mathbf{B}\right) \tag{3}$$

$$= i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\omega_c}\vec{\kappa} \cdot \vec{\xi}^\star = i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\omega_c}\left(\nabla\theta\kappa_\theta + \nabla r\kappa_r\right) \cdot \left(\xi_\theta^\star \nabla\theta + \xi_r^\star \nabla r\right)$$

$$= i\omega B\frac{v_\perp^2/2 + v_\parallel^2}{\omega_c}\left(\nabla\theta \cdot \nabla\theta\xi_\theta^\star\kappa_\theta + \nabla r \cdot \nabla r\xi_r^\star\kappa_r + \nabla\theta \cdot \nabla r\xi_r^\star\kappa_\theta + \nabla r \cdot \nabla\theta\xi_\theta^\star\kappa_r\right)$$

$$= i\omega B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta^\star + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = i\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega - \omega_\star\right)B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{4}$$

$$\frac{d}{dt}g = H\left(r, \theta, \phi, t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint \frac{d\psi}{\dot{\psi}} = \oint \frac{d\theta}{\dot{\theta}} \tag{5}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}\left(r, \theta\right)\exp\left(-i\omega t + in\phi\right) \tag{6}$$

Note that $\hat{X}^{(1)}\left(r, \theta\right)$ is complex, and we take the real part of RHS for any physical variable, i.e. $X^{(1)}$. Thus, for internal kink mode, the displacement $\vec{\xi}$ is $\vec{\xi} = \xi_\theta'\mathbf{e}_\theta + \xi_r'\mathbf{e}_r$ and $\xi_r' = \xi_0\exp\left(i\left(\phi - \theta - \omega t\right)\right)$ within the region $q = 1$ rational surface $r = r_s$. With cylindrical approximation, it can apply the relation $\nabla \cdot \vec{\xi} = 0$, and thus obtain $\xi_\theta' = -i\xi_0\exp\left[i\left(\phi - \theta - \omega t\right)\right]$. Thus, the covariant forms of the perturbation are

$$\xi_r = \xi_0\exp\left(i\left(\phi - \theta - \omega t\right)\right), \tag{7}$$

$$\xi_\theta = -i\xi_0 r \exp\left(i\left(\phi - \theta - \omega t\right)\right) \tag{8}$$

within the region $q = 1$ surface. Similarly, the covariant forms of the perturbation are

$$\xi_r = \xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r}\right) \exp\left(i\left(\phi - \theta - \omega t\right)\right), \tag{9}$$

$$\xi_\theta = -i\xi_0 r \left(\frac{\Delta r - 2r + (r_s - \Delta r/2)}{\Delta r}\right) \exp\left(i\left(\phi - \theta - \omega t\right)\right) \tag{10}$$

in the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$. And $\xi_r = \xi_\theta = 0$ in the rest region.

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} i \frac{e}{M} \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B \frac{\epsilon}{\omega_c} G\left(\tau\right) d\tau \tag{11}$$

with

$$G = \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

where $G\left(\tau\right) = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right), \Lambda\right] \exp\left(-i\omega\tau + in\phi\left(\tau\right)\right)$ and the τ dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}} \cdot \nabla r, \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla \theta, \dot{\phi} = \dot{\mathbf{R}} \cdot \nabla \phi \tag{12}$$

Let us separate $\phi\left(\tau\right)$ into its secular and oscillating parts:

$$\phi\left(\tau\right) = \left\langle \dot{\phi} \right\rangle \tau + \tilde{\phi}\left(\tau\right) \tag{13}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right), \Lambda\right] = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right), \Lambda\right] \exp\left(in\tilde{\phi}\left(\tau\right)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}\left(\tau\right) = \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \bar{r}; \sigma\right) \exp\left(ip\omega_b\tau\right) \tag{14}$$

where,

$$Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{15}$$

with $r\left(\tau\right) = \bar{r} + \rho_d \cos\theta\left(\tau\right)$, ρ_d represents the finite orbit width for passing particles. $\rho_d = \Omega_d/\omega_t$, $\Omega_d = \frac{\left(v_\perp^2/2 + v_\parallel^2\right)}{\omega_c R_0}$, $\omega_t = \frac{v_\parallel}{qR_0}$. Thus,

$$\rho_d = \frac{q}{\omega_c}\sqrt{\frac{\epsilon}{2\left(1-\Lambda/b\right)}}\left[\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right] \tag{16}$$

Carrying out the time integration, the solution of g is obtained

$$g = \frac{e}{M}\frac{\partial F}{\partial \epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\omega_c}\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\bar{r};\sigma\right)\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega\right)t\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega} \tag{17}$$

The formula of δW_k is derived as follows.

$$\delta W_k = \int d^3x\vec{\xi}^\star\cdot\nabla\cdot\delta\mathbf{P}_k = e\int d^3x\int d^3v\left(\frac{i}{\omega}\mathbf{v}_d\cdot\delta\mathbf{E}_\perp\right)^\star g$$

$$= e\int d^3x\int d^3vgB\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta^\star+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

$$= e\int d^3x\int d^3vgB\frac{\epsilon}{\omega_c}G^\star \tag{18}$$

where $G^\star = \hat{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right),\Lambda\right]\exp\left(i\omega\tau-in\phi\left(\tau\right)\right)$. Let $\tilde{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right),\Lambda\right] = \hat{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right),\Lambda\right]\exp\left(-in\tilde{\phi}\left(\tau\right)\right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star\left(\tau\right) = \sum_{-\infty}^{\infty}Y_p^\star\left(\Lambda,\bar{r};\sigma\right)\exp\left(-ip\omega_b\tau\right) \tag{19}$$

where,

$$Y_p^\star\left(\Lambda,\bar{r};\sigma\right) = \frac{1}{\tau_b}\oint d\tau\tilde{G}^\star\left(\tau\right)\exp\left(ip\omega_b\tau\right) \tag{20}$$

with $r\left(\tau\right) = \bar{r}+\rho_d\cos\theta\left(\tau\right)$.

$$\delta W_k = \frac{e^2}{M}\int d^3x\int d^3v\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B^2\frac{\epsilon^2}{\omega_c^2}\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\bar{r};\sigma\right)$$

$$\cdot\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega\right)\tau\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega}\sum_{-\infty}^{\infty}Y_{p'}^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(i\omega\tau-in\left\langle\dot{\phi}\right\rangle\tau-ip'\omega_b\tau\right) \tag{21}$$

$$\delta W_k = \frac{e^2}{M}\int d^3x\int d^3v\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B^2\frac{\epsilon^2}{\omega_c^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp[ip\omega_b \tau]}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b \tau) \tag{22}$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = \frac{e^2}{M} \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\omega_c^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p(\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp(ip\omega_b \tau)}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star}(\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b \tau) \tag{23}$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon} b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_\parallel$, one finally obtains

$$\delta W_k = \frac{4\pi^2}{M} \frac{e^2 B^2}{\omega_c^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b (\omega - \omega_\star)$$
$$\cdot \sum_{-\infty}^{\infty} \frac{|Y_p|^2}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega}, \tag{24}$$

which is similar to Eq.(35) of Fu's 1993 paper with replacing ϵ and J by $\epsilon \equiv \frac{1}{2} M v^2$ and $B = qR/J$. Note that $\tilde{\phi} \cong 0$, $\left\langle \dot{\phi} \right\rangle \cong \omega_D^0 + q\omega_b, \omega_D^0 \approx 0$ for passing particles.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{25}$$

$$J_t = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{26}$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E}\right)^{-1} = \frac{\pi \omega_\parallel}{2K(\kappa)}, \kappa < 1 \tag{27}$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E}\right)^{-1} = \frac{\pi \sqrt{\kappa} \omega_\parallel}{K(\kappa^{-1})}, \kappa > 1 \tag{28}$$

where $\omega_\parallel = \frac{1}{qR}\sqrt{\varepsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR}\sqrt{\varepsilon \Lambda}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $v_h = \sqrt{\frac{2T_h}{M}}$, $\epsilon = \frac{T_h}{M}\bar{\epsilon}$, $r = ax$, $J = aR_0\bar{J}$, $R = R_0\bar{R}$, $\omega_b = \frac{v_h}{R_0}\bar{\omega}_b$, $\frac{1}{\tau_b} = \frac{v_h}{2\pi R_0}\bar{\omega}_b = \frac{v_h}{R_0}\frac{\bar{\omega}_b}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_b}$, $\omega = \frac{v_h}{R_0}\bar{\omega}$, $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$, $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$. T_h is arbitrary temperature/energy which can be characteristic quantity, i.e. thermal energy or birth energy of fast ions et al.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F(r,\epsilon,\Lambda) = \frac{n_0}{C}\frac{1}{\epsilon^{3/2}+\epsilon_c^{3/2}}Erfc\left(\frac{\epsilon-\epsilon_0}{\Delta\epsilon}\right)\exp\left[-\left(\frac{r-r_0}{\Delta r}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{29}$$

where

$$C = \int d^3\mathbf{v}\frac{1}{\epsilon^{3/2}+\epsilon_c^{3/2}}Erfc\left(\frac{\epsilon-\epsilon_0}{\Delta\epsilon}\right)\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{30}$$

n_0 is the density at $r = r_0$.

Using $d^3v = \sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\epsilon^{1/2}d\epsilon$,

$$C = \int\sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\epsilon^{1/2}d\epsilon\frac{1}{\epsilon^{3/2}+\epsilon_c^{3/2}}Erfc\left(\frac{\epsilon-\epsilon_0}{\Delta\epsilon}\right)\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{31}$$

$$= \int\sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\bar{\epsilon}^{1/2}d\bar{\epsilon}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right]$$

Note C is a normalized quantity and is profile dependent parameter. F can be written as

$$F(r,\epsilon,\Lambda) = \frac{n_0}{C}\frac{1}{(T_0/M)^{3/2}}\hat{F}(x,\bar{\epsilon},\Lambda) \tag{32}$$

where

$$\hat{F}(x,\bar{\epsilon},\Lambda) = \frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{33}$$

Using the above normalized relations including F, δW_k of Eq. (24) then reads

$$\delta W_k = \frac{2^{3/2}}{C}\pi^2 a^2 R_0 n_0 T_0 \delta\bar{W}_k' \tag{34}$$

where

$$\delta \bar{W}_k' = \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \hat{F}}{\partial \bar{\epsilon}} \bar{\tau}_b \left(\bar{\omega} - \bar{\omega}_\star\right) \cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\overline{\left\langle \dot{\phi} \right\rangle} + p\bar{\omega}_b - \bar{\omega}} \tag{35}$$

Note that the normalized coefficient $\frac{2^{3/2}}{C}\pi^2 a^2 R_0 n_0 T_0$ is profile dependent parameter which is not convient for different F especially in the analytical derivation of δW_k. To avoid it, we can use the factor $\frac{n_0}{v_0^3}$ to normalize F's, i.e. $F = \frac{n_0}{v_0^3}\bar{F}$. Thus, $\bar{F} = \left(2^{3/2}/C\right)\hat{F}$ here. And δW_k becomes

$$\delta W_k = \pi^2 a^2 R_0 n_0 T_0 \delta \bar{W}_k \tag{36}$$

where

$$\delta \bar{W}_k = \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b \left(\bar{\omega} - \bar{\omega}_\star\right) \cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\overline{\left\langle \dot{\phi} \right\rangle} + p\bar{\omega}_b - \bar{\omega}} \tag{37}$$

$$\bar{F} = \frac{2^{3/2}}{C} \frac{1}{\bar{\epsilon}^{3/2} + \bar{\epsilon}_c^{3/2}} Erfc\left(\frac{\bar{\epsilon} - \bar{\epsilon}_0}{\Delta \bar{\epsilon}}\right) \exp\left[-\left(\frac{x - x_0}{\Delta x}\right)^2\right] \exp\left[-\left(\frac{\Lambda - \Lambda_0}{\Delta \Lambda}\right)^2\right] \tag{38}$$

Here the normalized coefficient $\pi^2 a^2 R_0 n_0 T_0$ is profile independent.

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q} \sqrt{\bar{\epsilon}} \tag{39}$$

$$\begin{aligned} \omega_b t &= \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\, b\sqrt{1-\frac{\Lambda}{b}}} d\theta \\ &= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta \end{aligned} \tag{40}$$

$$\begin{aligned} Y_p\left(\Lambda, \bar{r}; \sigma\right) &= \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \\ &= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right) \\ &= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right) \end{aligned} \tag{41}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(r,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot\left(\left(g^{\theta\theta}\kappa_{\theta}+g^{r\theta}\kappa_{r}\right)\hat{\xi}_{\theta}\left(\theta,\bar{r}+\rho_{d}\cos\theta\right)+\left(g^{rr}\kappa_{r}+g^{r\theta}\kappa_{\theta}\right)\hat{\xi}_{r}\left(\theta,\bar{r}+\rho_{d}\cos\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(x,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_{\theta}+\bar{g}^{r\theta}\bar{\kappa}_{r}\right)\bar{\hat{\xi}}_{\theta}\left(\theta,\bar{x}+\frac{\rho_{d}}{a}\cos\theta\right)+\left(\bar{g}^{rr}\bar{\kappa}_{r}+\bar{g}^{r\theta}\bar{\kappa}_{\theta}\right)\bar{\hat{\xi}}_{r}\left(\theta,\bar{x}+\frac{\rho_{d}}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m}=\hat{\xi}_{\theta m}/a^{2}$, $\bar{\hat{\xi}}_{rm}=\hat{\xi}_{rm}/a$.
The normalized metric tensors are

$$\bar{g}^{rr}=1+2\Delta'\cos\theta \tag{42}$$

with $\Delta'=\left(\varepsilon+\alpha\right)/4$, $\varepsilon=\frac{r}{R_0}$, $\alpha=-R_0q^2d\beta/dr$, $\beta=\frac{2\mu_0P}{B^2}$ set $\alpha=0$ if $\beta=0$, or assume $\bar{g}^{rr}=1$ without toroidal effect, θ independent. Specially, in low beta limit,

$$\bar{g}^{rr}=1+\frac{1}{2}\varepsilon\cos\theta \tag{43}$$

$$\bar{g}^{\theta\theta}=\frac{1}{x^2}\left[1-2\left(\varepsilon+\Delta'\right)\cos\theta\right] \tag{44}$$

assume $\bar{g}^{\theta\theta}=\frac{1}{x^2}$ without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta}=\frac{1}{x^2}\left[1-\frac{5}{2}\varepsilon\cos\theta\right] \tag{45}$$

$$\bar{g}^{r\theta}=-\frac{1}{x}\left[\varepsilon+\left(r\Delta'\right)'\right]\sin\theta \tag{46}$$

specially,

$$\bar{g}^{r\theta}=-\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{47}$$

for low beta limit. and $\bar{g}^{r\theta}=0$ without toroidal effect.
The normalized curvature are in low beta limit

$$\bar{\kappa}_{r}=-\frac{a}{R}\cos\theta+\frac{a}{R}\frac{\varepsilon}{4}-\frac{a}{R}\frac{5}{4}\varepsilon\left(\cos2\theta-1\right)-\left(\frac{a}{R}\right)^{2}\frac{x}{q} \tag{48}$$

$$\bar{\kappa}_{\theta}=\varepsilon\sin\theta+\frac{5}{4}\varepsilon^{2}\sin2\theta \tag{49}$$

with $R = R_0 + r\cos\theta - \Delta(r) + r\eta(r)(\cos 2\theta - 1).\eta(r) = (\varepsilon + \Delta')/2$.
The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{nq}{x}\frac{R_0}{a}\frac{\rho_h}{a}\frac{\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}} \tag{50}$$

or

$$\bar{\omega}_\star = \frac{1}{2}\frac{m}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}} \tag{51}$$

where, n is toroidal mode number, m is poloidal mode number, $\rho_h = v_h/\omega_c$, $v_h = \sqrt{2T_h/M}, \omega_c = Be/M$.
The normalized ρ_d is

$$\bar{\rho}_d = \frac{\rho_d}{a} = \frac{q}{2}\frac{\rho_h}{a}\sqrt{\frac{\bar{\epsilon}}{(1-\Lambda/b)}}\left[\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right] \tag{52}$$

The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0 \exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x \exp(-i\theta)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The normalized $\delta\bar{W}_k$ is given by

$$\delta\bar{W}_k = \int\frac{\bar{J}}{q}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\bar{\tau}_b(\bar{\omega} - \bar{\omega}_\star)$$

$$\cdot\sum_{-\infty}^{\infty}\frac{\left|\bar{Y}_p\right|^2}{n\overline{\left\langle\dot{\phi}\right\rangle} + p\bar{\omega}_b - \bar{\omega}}, \tag{53}$$

with

$$\bar{F}=\frac{2^{3/2}}{C}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{54}$$

and $\bar{J}=x$.

For passing particles,

$$\bar{\omega}_b=\frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{55}$$

where, $\kappa=\frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon=\frac{r}{R_0}$. and

$$\overline{\left\langle\dot{\phi}\right\rangle}\cong q\bar{\omega}_b \tag{56}$$

$$\omega_b t=\bar{\omega}_b\frac{v_h}{R}\int_0^{\theta}\frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}b\sqrt{1-\frac{\Lambda}{b}}}d\theta \tag{57}$$

$$=\int_0^{\theta}\frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\sqrt{\varepsilon\Lambda/2}\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\theta$$

$$Y_p\left(\Lambda,\bar{r};\sigma\right)=\frac{1}{\tau_b}\oint d\tau\tilde{G}\left(\tau\right)\exp\left(-ip\omega_b\tau\right) \tag{58}$$

$$=\frac{\omega_b}{2\pi}\oint d\tau\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\omega_b\tau\right)$$

$$=\frac{1}{2\pi}\oint d\left(\omega_b\tau\right)\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\omega_b\tau\right)$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\int_0^{\theta}d\theta'\frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(x,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta+\bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)+\left(\bar{g}^{rr}\bar{\kappa}_r+\bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m}=\hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm}=\hat{\xi}_{rm}/a$ and normalized drift orbit width is $\bar{\rho}_d=\frac{\rho_d}{a}$.

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon\cos\theta \tag{59}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - \frac{5}{2}\varepsilon\cos\theta\right] \tag{60}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{61}$$

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta \tag{62}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta \tag{63}$$

The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{R_0}{a}\frac{\rho_h}{a}\frac{nq}{x}\frac{\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\epsilon} \tag{64}$$

with n being toroidal mode number.
The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\exp(-i\theta)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

2 The fishbone dispersion relation

The quadratic form is

$$\delta W_{MHD} + \delta W_k + \delta I = 0 \tag{65}$$

where

$$\delta I = \gamma^2 \int \rho_m \left|\vec{\xi}\right|^2 d\vec{r} \tag{66}$$

$$\delta W_{MHD} = \int \vec{\xi}^{\star} \cdot \left(\nabla \cdot \delta \mathbf{P}_f + \delta \mathbf{B} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \delta \mathbf{B}\right) d\vec{r} \tag{67}$$

$$\delta W_k = \int \vec{\xi}^{\star} \cdot \nabla \delta p_h d\vec{r} \tag{68}$$

For fishbone instability, the dispersion relation can be written as

$$-\frac{i\omega}{\omega_A} + \delta \hat{W}_T + \delta \hat{W}_k = 0, \tag{69}$$

where

$$\delta \hat{W}_T = \delta W_{MHD} / \left[2\pi R \xi_0^2 \left(r_s B/2R\right)^2\right] \tag{70}$$

$$\delta \hat{W}_k = \delta W_k / \left[2\pi R \xi_0^2 \left(r_s B/2R\right)^2\right] \tag{71}$$

with $\gamma = i\omega$, $\omega_A = v_A/3^{1/2} R_0 \hat{s}$, $v_A = B/\left(\mu_0 \rho_m\right)^{1/2}$,$\rho_m = m_i n_i$, $\hat{s} = r_s \frac{dq}{dr}_{r=r_s}$. Specially, the MHD potential energy δW_{MHD} of toroidal plasma with circular cross-section for $m=1, n=1$ mode is given by

$$\delta \hat{W}_T = \pi \left(\frac{r_s}{R}\right)^2 3\left(1-q_0\right)\left(\frac{13}{144} - \beta_{ps}^2\right) \tag{72}$$

and with $\delta W_k = \pi^2 a^2 R_0 n_0 T_h \delta \bar{W}_k$, the kinetic potential energy δW_k is given by

$$\delta \hat{W}_k = \frac{1}{4} \frac{1}{\left(r_s/R\right)^2} \frac{1}{\left|\xi_0/a\right|^2} \beta_{h0} \delta \bar{W}_k \tag{73}$$

where $\beta_{h0} = 8\pi n_0 T_h/B^2$ and $\delta \bar{W}_k$ is given by Eq.(53). By normalizing the frequencies $\bar{\omega} = \omega/\left(v_h/R_0\right)$, $\bar{\omega}_A = \omega_A/\left(v_h/R_0\right)$, the dispersion relation Eq. (69) is rewritten by

$$-\frac{i\bar{\omega}}{\bar{\omega}_A} + \delta \hat{W}_T + \delta \hat{W}_k = 0. \tag{74}$$

3 Analytic form of the dispersion relation with passing particles and large aspect ratio approximation

For $\varepsilon \ll 1$, the normalized metric tensors are approximated as

$$\bar{g}^{rr} \approx 1 \tag{75}$$

$$\bar{g}^{\theta\theta} \approx \frac{1}{x^2} \tag{76}$$

$$\bar{g}^{r\theta} \approx -\frac{1}{x}\frac{3}{2}\varepsilon \sin\theta \tag{77}$$

and the normalized curvature are in low beta limit

$$\bar{\kappa}_r \approx -\frac{a}{R}\cos\theta \tag{78}$$

$$\bar{\kappa}_\theta \approx \varepsilon \sin\theta \tag{79}$$

The formula of ξ_θ and ξ_r are given by

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0(x)\exp(-i\theta), \tag{80}$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0(x)\, x \exp(-i\theta) \tag{81}$$

where, $\bar{\xi}_0(x) = \bar{\xi}_s H(x_s - x)$. $H(x)$ is Heaviside step function, $H = 1$ for $x > 0$ and $H = 0$ for $x < 0$, $dH/dx = \delta(x)$. Together with $\Lambda \ll 1$, one obtains

$$\tilde{G}[r(\tau), \theta(\tau)] = \left(\frac{\Lambda}{b(r,\theta)} + 2\left(1 - \frac{\Lambda}{b(r,\theta)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\bar{\xi}}_\theta(\theta, \bar{x} + \bar{\rho}_d\cos\theta) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\bar{\xi}}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)\right)$$

$$\approx 2\left(\frac{1}{x^2}\varepsilon\sin\theta\bar{\xi}_\theta - \frac{a}{R}\cos\theta\bar{\xi}_r\right)$$

$$\approx 2\left(-\frac{a}{R}i\sin\theta\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta) - \frac{a}{R}\cos\theta\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)\right)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)(i\sin\theta + \cos\theta)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)\exp(i\theta) \tag{82}$$

For $\kappa \gg 1$, the ellipitic fuction K becomes $K\left(\kappa^{-1}\right) = \pi/2$. Thus,

$$\frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\approx\frac{\pi\sqrt{\frac{1}{2\varepsilon\Lambda}}}{\pi/2}\frac{\sqrt{\frac{\varepsilon\Lambda}{2}}}{b\sqrt{1-\frac{\Lambda}{b}}}$$

$$=\frac{1}{b\sqrt{1-\Lambda/b}} \tag{83}$$

Using Eq.(82) and Eq.(83), Y_p is rewritten as

$$Y_p=-\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}2\frac{a}{R}\bar{\xi}_r\left(\theta,x+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\int_0^{\theta}d\theta'\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

$$=-\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi}d\theta\bar{\xi}_r\left(\theta,x+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right)$$

$$=-\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi}d\theta\bar{\xi}_sH\left(x_s-x-\bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right) \tag{84}$$

The distribution function of passing particles for analytical purpose is given by

$$\bar{F}\left(x,\bar{\epsilon},\Lambda\right)=\frac{p_h\left(x\right)}{\pi n_0T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right), \tag{85}$$

and its derivative of $\bar{\epsilon}$ is read as

$$\frac{\partial\bar{F}}{\partial\bar{\epsilon}}=\frac{p_h\left(x\right)}{\pi n_0T_h\bar{\epsilon}_0}\left[-\frac{3}{2}\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)-\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)-\bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\right].$$

$$\delta\bar{W}_k=I_1+I_2 \tag{86}$$

where I_1 represents the term with $\bar{\omega}$ and I_2 the term with $\bar{\omega}_\star$. Then I_1 decomposes three parts corresponding to $\partial\bar{F}/\partial\bar{\epsilon}$.

$$I_1=I_1^{(1)}+I_1^{(2)}+I_1^{(3)} \tag{87}$$

3.1 The dispersion relation for the case of $\rho_d=0$, $p=0$

For $\bar{\rho}_d=0$ and $p=0$, we have

$$Y_0=-\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi}d\theta\bar{\xi}_sH\left(x_s-x\right)\exp\left(-i\theta\right)\exp\left(i\theta\right) \tag{88}$$

$$=-\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi}d\theta\bar{\xi}_sH\left(x_s-x\right) \tag{89}$$

$$=-2\frac{a}{R}\bar{\xi}_sH\left(x_s-x\right) \tag{90}$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 0$, $\delta\bar{W}_k$ of Eq.(53) becomes

$$\delta\bar{W}_k = \int_0^{x_s} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right) \cdot\frac{\left|\bar{Y}_0\right|^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}, \tag{91}$$

here Y_0 is zero in the region $x > x_s$ where $H = 0$. According to Y_0 and $\bar{J}$ as shown above, $\delta\bar{W}_k$ becomes

$$\delta\bar{W}_k = \int_0^{x_s} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right) \cdot\frac{\left(-2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}, \tag{92}$$

For $F_h\left(x,\epsilon,\Lambda\right) = c_0\left(x\right)\frac{1}{\epsilon^{3/2}}\delta\left(\Lambda\right)H\left(\epsilon_0-\epsilon\right)$, we can get

$$\begin{aligned}
p_h\left(x\right) &= \int d^3vM\left(v_\parallel^2+\frac{1}{2}v_\perp^2\right)F_h \\
&\approx \int d^3vMv_\parallel^2F_h \\
&= M\int\sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\epsilon^{1/2}d\epsilon 2\epsilon F_h \\
&= M\int 2^{\frac{3}{2}}\pi\epsilon^{\frac{3}{2}}\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}c_0\left(x\right)\frac{1}{\epsilon^{3/2}}\delta\left(\Lambda\right)H\left(\epsilon_0-\epsilon\right)d\Lambda d\epsilon \\
&= \pi 2^{\frac{3}{2}}Mc_0\left(x\right)\epsilon_0
\end{aligned}$$

Thus, $F_h = \left[p_h\left(x\right)/\left(\pi M2^{\frac{3}{2}}\epsilon_0\right)\right]\frac{1}{\epsilon^{3/2}}\delta\left(\Lambda\right)H\left(\epsilon_0-\epsilon\right)$ with $c_0 = p_h\left(x\right)/\left(\pi M2^{\frac{3}{2}}\epsilon_0\right)$. Furthermore, using $F_h = \left(n_0/v_0^3\right)\bar{F}_h$, $\bar{F}_h = \frac{p_h(x)}{\pi n_0T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{\frac{3}{2}}}H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\delta\left(\Lambda\right)$.

$$\frac{\partial\bar{F}}{\partial\bar{\epsilon}} = \frac{p_h\left(x\right)}{\pi n_0T_h\bar{\epsilon}_0}\left[-\frac{3}{2}\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)-\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)-\bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\right].$$

$$\begin{aligned}
I_1^{(1)} &= \int_0^{x_s} xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\frac{p_h\left(x\right)}{\pi n_0T_h\bar{\epsilon}_0}\left(-\frac{3}{2}\right)\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega} \cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}} \\
&= -6\pi\frac{\bar{\omega}}{\pi n_0T_h\bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s} xp_h\left(x\right)dx\int_0^{\bar{\epsilon}_0}\frac{\sqrt{\bar{\epsilon}}d\sqrt{\bar{\epsilon}}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}
\end{aligned}$$

$$= -6\pi \frac{\bar{\omega}}{\pi n_0 T_h \bar{\epsilon}_0} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} x p_h\left(x\right) dx \left(1 + \Omega \ln\left(1 - \frac{1}{\Omega}\right)\right) \sqrt{\bar{\epsilon}_0}$$

$$= -6\pi \frac{1}{\pi n_0 T_h} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \left(\Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right) \int_0^{x_s} x p_h\left(x\right) dx$$

$$I_1^{(2)} = \int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{p_h\left(x\right)}{\pi n_0 T_h \bar{\epsilon}_0} \left(-1\right) \bar{\epsilon}^{-\frac{5}{2}} \Lambda \delta'\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= \int_0^{x_s} x dx \int d\Lambda d\bar{\epsilon} \frac{p_h\left(x\right)}{\pi n_0 T_h \bar{\epsilon}_0} \left(-1\right) \Lambda \delta'\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) 2\pi \bar{\omega} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= 4\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \bar{\omega} \cdot \int_0^{x_s} x p_h\left(x\right) dx \int_0^{\bar{\epsilon}_0} \frac{\sqrt{\bar{\epsilon}}}{\sqrt{\bar{\epsilon}} - \bar{\omega}} d\sqrt{\bar{\epsilon}}$$

$$= 4\pi \frac{1}{\pi n_0 T_h} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \left(\Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right) \int_0^{x_s} x p_h\left(x\right) dx$$

$$I_1^{(3)} = \int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{p_h\left(x\right)}{\pi n_0 T_h \bar{\epsilon}_0} \left(-1\right) \bar{\epsilon}^{-\frac{3}{2}} \delta\left(\Lambda\right) \delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}$$

$$\cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$I_1^{(3)} = -2\pi \frac{1}{\pi n_0 T_h} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \frac{\Omega}{1 - \Omega} \int_0^{x_s} x p_h\left(x\right) dx$$

Thus,

$$I_1 = -2\pi \frac{1}{\pi n_0 T_h} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right] \int_0^{x_s} x p_h\left(x\right) dx.$$

Using

$$\frac{\partial \bar{F}_h}{\partial x} = \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \frac{d p_h\left(x\right)}{dx} \frac{1}{\bar{\epsilon}^{3/2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right)$$

and

$$\bar{\omega}_\star = \frac{1}{2} \frac{m}{x} \frac{R}{a} \frac{\rho_h}{a} \frac{\partial \bar{F} / \partial x}{\partial \bar{F} / \partial \bar{\epsilon}}$$

we obtain

$$I_2 = -\int_0^{x_s} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}_\star \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_0^{x_s} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} \frac{\partial \bar{F}}{\partial x} \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_0^{x_s} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a} \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$\cdot \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \frac{dp_h(x)}{dx} \frac{1}{\bar{\epsilon}^{3/2}} \delta(\Lambda) H(\bar{\epsilon}_0 - \bar{\epsilon})$$

$$= -\frac{1}{\pi n_0 T_h \bar{\epsilon}_0} 2\pi \frac{1}{2} \frac{R}{a} \frac{\rho_h}{a} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \frac{dp_h(x)}{dx} dx \int_0^{\bar{\epsilon}_0} \frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}} d\bar{\epsilon}$$

$$= -\frac{1}{\pi n_0 T_h \bar{\epsilon}_0} 2\pi \frac{R}{a} \frac{\rho_h}{a} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \frac{dp_h(x)}{dx} dx \int_0^{\bar{\epsilon}_0} \frac{\left(\sqrt{\bar{\epsilon}}\right)^3}{\sqrt{\bar{\epsilon}}-\bar{\omega}} d\sqrt{\bar{\epsilon}}$$

$$\begin{aligned} &= -2\pi \frac{R}{a} \frac{\rho_h}{a} \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \frac{dp_h(x)}{dx} dx \\ &\quad \cdot \left(\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right)\right) \left(\sqrt{\bar{\epsilon}_0}\right)^3 \end{aligned}$$

$$\begin{aligned} \delta\bar{W}_k &= I_1 + I_2 \\ &= -2\pi \frac{1}{\pi n_0 T_h} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \left[\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right] \int_0^{x_s} x p_h(x)\, dx \\ &\quad -2\pi \frac{R}{a} \frac{\rho_h}{a} \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \frac{dp_h(x)}{dx} dx \\ &\quad \cdot \left(\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right)\right) \left(\sqrt{\bar{\epsilon}_0}\right)^3 \end{aligned} \tag{93}$$

$$\begin{aligned} \delta\hat{W}_k &= \frac{1}{4} \frac{1}{(r_s/R)^2} \frac{1}{|\xi_0/a|^2} \beta_{h0} \delta\bar{W}_k \\ &= \begin{array}{l} -2\frac{1}{(r_s/a)^2} \frac{8\pi \int_0^{x_s} x p_h(x)\, dx}{B^2} \left[\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right] \\ -2\frac{1}{\Omega_c (r_s/R)^2} \frac{8\pi \int_0^{x_s} \frac{dp_h(x)}{dx} dx}{B_t^2} \left[\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right)\right] \end{array} \end{aligned} \tag{94}$$

where $\Omega_c = \omega_c / \left[(v_h/R_0)\sqrt{\bar{\epsilon}_0}\right]$.
Moreover,

$$\delta\hat{W}_k = \begin{array}{r} -\frac{1}{(r_s/a)^2}\frac{8\pi\langle p_h\rangle}{B_t^2}\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]+2\frac{1}{\Omega_c\,(r_s/Rq_s)^2} \\ \cdot\frac{8\pi\,(p_h\,(0)-p_h\,(x_s))}{B_t^2}\left[\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array} \tag{95}$$

where $\langle p\rangle = \frac{2}{x_s^2}\int_0^{x_s} xp_h\,(x)\,dx$ is volume averaged pressure for $q_s=1$, Then

$$\delta\hat{W}_k = \begin{array}{r} -\frac{1}{(r_s/a)^2}\langle\beta_h\rangle\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]+\frac{1}{\Omega_c\,(r_s/R)^2} \\ \cdot 2\,(\beta_h\,(0)-\beta_h\,(x_s))\left[\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array} \tag{96}$$

where $\beta_h = 8\pi p_h/B^2$.

The dispersion relation for $p=0$ is

$$-\frac{i\Omega}{\Omega_A}+\delta\hat{W}_T+\delta\hat{W}_k=0, \tag{97}$$

where $\Omega_A = \bar{\omega}_A/\sqrt{\bar{\epsilon}_0}$.

3.2 The dispersion relation for the case of $\rho_d \neq 0$, $p=0$

For $\bar{\rho}_d \neq 0$, $p=0$ and using Eq.(84), we have

$$\begin{aligned} x < x_s-\bar{\rho}_d \qquad & Y_0 = -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s\int_0^{2\pi}d\theta = -\frac{2a}{R}\bar{\xi}_s \triangleq Y_0' \\ x_s-\bar{\rho}_d < x < x_s+\bar{\rho}_d \qquad & Y_0 = -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s\int_{\theta^\star}^{2\pi-\theta^\star}d\theta = -\frac{2\pi-2\theta^\star}{\pi}\frac{a}{R}\bar{\xi}_s \triangleq Y_0'' \\ x > x_s+\bar{\rho}_d \qquad & Y_0 = 0 \end{aligned}$$

where $\cos\theta^\star = \frac{x_s-x}{\bar{\rho}_d} \in [0,\pi]$.

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p=0$, $\delta\bar{W}_k$ of Eq.(53) becomes

$$\delta\bar{W}_k = \delta\bar{W}_k' + \delta\bar{W}_k'' \tag{98}$$

with

$$\delta\bar{W}_k' = \int_0^{x_s-\bar{\rho}_d}\bar{J}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)\frac{|Y_0'|^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

and

$$\delta\bar{W}_k'' = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)\frac{\left|Y_0''\right|^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

Due to $\int_0^{x_s} = \int_0^{x_s-\bar{\rho}_d} + \int_{x_s-\bar{\rho}_d}^{x_s}$, one gets

$$\delta\bar{W}_k' = \delta\bar{W}_k^0 - \delta\bar{W}_k'^s \tag{99}$$

where $\delta\bar{W}_k^0$ is given by the above Eq. (93) , and

$$\delta\bar{W}_k'^s = \int_{x_s-\bar{\rho}_d}^{x_s} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)\frac{\left|Y_0'\right|^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}} \tag{100}$$

$$\begin{aligned} = \quad & -2\pi\frac{1}{\pi n_0 T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]\int_{x_s-\bar{\rho}_d}^{x_s} xp_h\left(x\right)dx \\ & -2\pi\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_{x_s-\bar{\rho}_d}^{x_s} q\frac{dp_h\left(x\right)}{dx}dx \\ & \cdot\left(\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right)\left(\sqrt{\bar{\epsilon}_0}\right)^3 \end{aligned} \tag{101}$$

For simplicity, $\delta\bar{W}_k'$ may be approximate to $\delta\bar{W}_k^0$ as $\bar{\rho}_d \ll x_s$. We let $\delta\bar{W}_k'' = I_1 + I_2$ and $I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)}$. Thus, for $\delta\bar{W}_k''$, one obtains

$$I_1^{(1)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\left(-\frac{3}{2}\right)\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -3\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\int_0^{\bar{\epsilon}_0}\frac{\bar{\omega}d\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xp_h\left(x\right)\left[1-\frac{\theta^\star}{\pi}\right]^2 dx$$

for the normal profile of p_h, the identity $xp_h\left(x\right) = \frac{\delta_x}{2}\frac{p_h}{r_p}$ should be satisfied, where $r_p = -\left[dp_h/pdx\right]^{-1}, \delta_x$ is profile width. and $\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2 dx = \frac{\pi-4}{\pi^2}\bar{\rho}_d$. With $\bar{\rho}_d \simeq q\sqrt{\bar{\epsilon}}\rho_h/a$, it yields

$$I_1^{(1)} \approx -6\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\frac{\rho_h}{a}\bar{\omega}\int_0^{\bar{\epsilon}_0} d\sqrt{\bar{\epsilon}}\frac{\left(\sqrt{\bar{\epsilon}}\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$\begin{aligned} \approx \quad & -6\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\frac{\rho_h}{a}\sqrt{\bar{\epsilon}_0} \\ & \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right] \end{aligned}$$

$$\approx \quad -6\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}(\pi-4)\,\Delta_b$$
$$\cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]$$

where $\Delta_b=\frac{q_s\sqrt{\bar{\epsilon}_0}\rho_h}{a}$ is orbit width.

$$I_1^{(2)}=\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\,(-1)\,\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'(\Lambda)\,H(\bar{\epsilon}_c-\bar{\epsilon})\,\frac{p_h(x)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$=2\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\int_0^{\epsilon_0}\frac{d\bar{\epsilon}\bar{\omega}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xp_h(x)\left[1-\frac{\theta^\star}{\pi}\right]^2dx$$

$$\approx \begin{array}{c} 4\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}(\pi-4)\,\Delta_b \\ \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]\end{array} \tag{102}$$

$$I_1^{(3)}=\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\,(-1)\,\bar{\epsilon}^{-\frac{3}{2}}\delta(\Lambda)\,\delta(\bar{\epsilon}_0-\bar{\epsilon})\,\frac{p_h(x)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$=-2\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\bar{\omega}\int\frac{\bar{\epsilon}d\bar{\epsilon}\delta(\bar{\epsilon}_0-\bar{\epsilon})}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xp_h(x)\left[1-\frac{\theta^\star}{\pi}\right]^2dx$$
$$=-2\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\bar{\omega}\int\frac{\bar{\epsilon}d\bar{\epsilon}\delta(\bar{\epsilon}_0-\bar{\epsilon})}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\bar{\rho}_d$$
$$=-2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}(\pi-4)\,\Delta_b\frac{\Omega}{1-\Omega}$$

Since

$$\frac{\partial\bar{F}(x,\bar{\epsilon},\Lambda)}{\partial x}=\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta(\Lambda)H(\bar{\epsilon}_0-\bar{\epsilon})\,\frac{dp_h(x)}{dx}, \tag{103}$$

and

$$\bar{\omega}_\star=\frac{1}{2}\frac{m}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}} \tag{104}$$

we obtain

$$I_2 = -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star \cdot \frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial\bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \cdot \frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{dp_h\left(x\right)}{dx} \cdot \frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0} \cdot \int_0^{\bar{\epsilon}_0}\frac{\bar{\epsilon}d\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2\frac{dp_h\left(x\right)}{dx}dx$$

Due to $r_p = -\left[dp_h/pdx\right]^{-1}$, the above integral of x yields

$$\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2\frac{dp_h\left(x\right)}{dx}dx$$

$$= -\left(\frac{p_h}{r_p}\right)_{x_s}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2 dx$$

$$= -\left(\frac{p_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\bar{\rho}_d$$

$\Rrightarrow$

$$= \pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0} \cdot \left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\rho_h}{a}\left(\pi-4\right)\int_0^{\bar{\epsilon}_0}\frac{\bar{\epsilon}^{3/2}d\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= \quad 2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h} \cdot \left(\frac{p_h}{r_p}\right)_{x_s}\sqrt{\bar{\epsilon}_0}\left(\pi-4\right)\Delta_b$$

$$\cdot\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right]$$

Therefore,

Figure 1: The integral region

$$
\delta \bar{W}_k'' = \begin{array}{c} -2\pi\left(\pi-4\right)\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}\Delta_b \\ \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)+\frac{\Omega}{1-\Omega}\right] \\ +2\pi\left(\pi-4\right)\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h}\cdot\left(\frac{p_h}{r_p}\right)_{x_s}\sqrt{\bar{\epsilon}_0}\Delta_b \\ \cdot\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array} \tag{105}
$$

Using

$$
\delta \hat{W}_k = \frac{1}{4}\frac{1}{\left(r_s/R\right)^2}\frac{1}{\left|\xi_0/a\right|^2}\beta_{h0}\delta \bar{W}_k \tag{106}
$$

one obtains

$$
\delta \hat{W}_k'' = \begin{array}{c} -\frac{1}{x_s^2}\left(\frac{2}{\pi}-\frac{8}{\pi^2}\right)\frac{\delta_x}{2}\left(\frac{\beta_h}{r_p}\right)_{x_s}\Delta_b\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)+\frac{\Omega}{1-\Omega}\right] \\ +\frac{1}{x_s^2}\left(\frac{2}{\pi}-\frac{8}{\pi^2}\right)\frac{R}{a}\left(\frac{q\beta_h}{r_p}\right)_{x_s}\Delta_b^2\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array}
$$

and the dispersion relation is

$$
-\frac{i\Omega}{\Omega_A}+\delta \hat{W}_T+\delta \hat{W}_k = 0 \tag{107}
$$

where $\Omega_A = \bar{\omega}_A/\sqrt{\bar{\epsilon}_0}$, $\delta \hat{W}_k = \delta \hat{W_k^0} + \delta \hat{W}_k''$.

3.3 The dispersion relation for the case of $p = 1$, $\rho_d \neq 0$

For $p = 1$, we arrive at

$$
Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta \bar{\xi}_s H\left(x_s - \bar{x} - \bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right) \tag{108}
$$

From Fig.1, Y_1 is zero in the region $x < x_s - \bar{\rho}_d$ where $H = 1$ for $\theta \in [0, 2\pi]$ since the θ integral of $\sim \exp\left(-i\theta\right)$ from 0 to 2π is zero. In the region $x > x_s + \bar{\rho}_d$, Y_1 also is zero since $H = 0$. Obviously, Y_1 is finite in the region $x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d$ where $H = 1$ for $\theta \in [\theta^\star, -\theta^\star + 2\pi]$. $\cos\theta^\star = \frac{x_s - x}{\bar{\rho}_d} \in [0, \pi]$. Thus,

$$
Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_{\theta^\star}^{2\pi-\theta^\star} d\theta \bar{\xi}_s \exp\left(-i\theta\right) \tag{109}
$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i \left[\exp\left(-i\left(2\pi-\theta^\star\right)\right)-\exp\left(-i\theta^\star\right)\right]$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i \left[\exp\left(-i\left(2\pi-\theta^\star\right)\right)-\exp\left(-i\theta^\star\right)\right]$$

$$= \frac{2}{\pi}\frac{a}{R}\bar{\xi}_s \sin\theta^\star$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p=1$, $\delta\bar{W}_k$ becomes

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial \bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right) \cdot \frac{\left|Y_1\right|^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q}-\bar{\omega}}. \quad (110)$$

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda \int_0^{\bar{\epsilon}_0} \bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial \bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}} \quad (111)$$

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda \int_0^{\bar{\epsilon}_0} \bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial \bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left(1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right) \quad (112)$$

with $\bar{J}=x$. The distribution function of passing particles for analytical purpose is given by

$$\bar{F}\left(x,\bar{\epsilon},\Lambda\right) = \frac{p_h\left(x\right)}{\pi n_0 T_h \bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right), \quad (113)$$

and its derivative of $\bar{\epsilon}$ is read as

$$\frac{\partial\bar{F}}{\partial\bar{\epsilon}} = \frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\left[-\frac{3}{2}\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)-\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)-\bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\right].$$

$$\delta\bar{W}_k = I_1 + I_2 \quad (114)$$

where I_1 represents the term with $\bar{\omega}$ and I_2 the term with $\bar{\omega}_\star$. Then I_1 decomposes three parts corresponding to $\partial\bar{F}/\partial\bar{\epsilon}$.

$$I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)} \quad (115)$$

$$I_1^{(1)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon}\left(-\frac{3}{2}\right)\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]$$

$$= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x p_h(x)\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx\int_0^{\bar{\epsilon}_0} d\bar{\epsilon}\bar{\omega}$$

$$\cdot\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

for the normal profile of p_h, the identity $xp_h(x)=\frac{\delta_x}{2}\frac{p_h}{r_p}$ should be satisfied, where $r_p=-\left[dp_h/pdx\right]^{-1}$, δ_x is profile width. and $\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx=\frac{4}{3}\bar{\rho}_d$.

$\Rrightarrow$

$$= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}$$

$$\cdot\bar{\omega}\int_0^{\bar{\epsilon}_0}2\sqrt{\bar{\epsilon}}d\sqrt{\bar{\epsilon}}\frac{1}{\left(1+\frac{1}{q_s}\right)\sqrt{\bar{\epsilon}}-\bar{\omega}}\frac{4}{3}\bar{\rho}_d$$

With $\bar{\rho}_d\simeq q\sqrt{\bar{\epsilon}}\rho_h/a$, we obtain

$$I_1^{(1)}\approx -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{8}{3}\frac{\rho_h}{a}\frac{\bar{\omega}}{1+\frac{1}{q_s}}$$

$$\cdot\int_0^{\bar{\epsilon}_0}d\sqrt{\bar{\epsilon}}\frac{\left(\sqrt{\bar{\epsilon}}\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}/\left(1+\frac{1}{q_s}\right)}$$

The integral identity is given by

$$\int dy\frac{y^2}{y-a}=\int dy\frac{y^2-a^2+a^2}{y-a}=\int dy\,(y+a)+\int dy\frac{a^2}{y-a} \tag{116}$$

$$=\frac{1}{2}y^2+ay+a^2\ln(y-a) \tag{117}$$

$\Rrightarrow$

$$= -8\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\rho_h}{a}\frac{\bar{\omega}}{1+\frac{1}{q_s}}\left[\frac{1}{2}\left(\sqrt{\bar{\epsilon}_0}\right)^2+\frac{\bar{\omega}}{1+\frac{1}{q_s}}\sqrt{\bar{\epsilon}_0}+\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2\ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\sqrt{\bar{\epsilon}_0}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\right] \tag{118}$$

$$I_1^{(2)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \left(-1\right) \bar{\epsilon}^{-\frac{5}{2}} \Lambda \delta' \left(\Lambda\right) H \left(\bar{\epsilon}_c - \bar{\epsilon}\right) \frac{p_h\left(x\right)}{\pi n_0 T_h \bar{\epsilon}_0} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}} \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right]$$

$$= 2\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x p_h\left(x\right) \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right] dx$$

$$\cdot \int_0^{\epsilon_0} d\bar{\epsilon} \bar{\omega} \frac{1}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}}$$

$$\approx \begin{array}{c} \frac{16}{3}\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \frac{\delta_x}{2} \left(\frac{q p_h}{r_p}\right)_{x_s} \frac{\rho_h}{a} \\ \cdot \frac{\bar{\omega}}{1+\frac{1}{q_s}} \left[\frac{1}{2}\left(\sqrt{\bar{\epsilon}_0}\right)^2 + \frac{\bar{\omega}}{1+\frac{1}{q_s}}\sqrt{\bar{\epsilon}_0} + \left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2 \ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}} - \sqrt{\bar{\epsilon}_0}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\right] \end{array} \qquad (119)$$

$$I_1^{(3)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \left(-1\right) \bar{\epsilon}^{-\frac{3}{2}} \delta\left(\Lambda\right) \delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \frac{p_h\left(x\right)}{\pi n_0 T_h \bar{\epsilon}_0} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}} \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right]$$

$$= -2\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x p_h\left(x\right) \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right] dx$$

$$\cdot \bar{\omega} \int d\bar{\epsilon} \delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}}$$

$$= -2\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \frac{\delta_x}{2} \left(\frac{p_h}{r_p}\right)_{x_s} \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right] dx$$

$$\cdot \bar{\omega} \int d\bar{\epsilon} \delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}}$$

$$= -2\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \frac{\delta_x}{2} \left(\frac{p_h}{r_p}\right)_{x_s}$$

$$\cdot\bar{\omega}\int d\bar{\epsilon}\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\frac{4}{3}\bar{\rho}_d$$

$$=-2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{4}{3}\frac{\rho_h}{a}\frac{\bar{\omega}\left(\sqrt{\bar{\epsilon}_0}\right)^3}{\sqrt{\bar{\epsilon}_0}+\frac{\sqrt{\bar{\epsilon}_0}}{q_s}-\bar{\omega}}$$

we have

$$\frac{\partial\bar{F}\left(x,\bar{\epsilon},\Lambda\right)}{\partial x}=\frac{1}{\pi n_0T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{dp_h\left(x\right)}{dx}, \tag{120}$$

$$I_2=-\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\bar{J}dx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$=-\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial\bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$=-\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{dp_h\left(x\right)}{dx}\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$=-\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0T_h\bar{\epsilon}_0}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h\left(x\right)}{dx}dx$$

$$\cdot\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

Due to $r_p=-\left[dp_h/pdx\right]^{-1}$, the above integral of x yields

$$\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h\left(x\right)}{dx}dx$$

$$=-\left(\frac{p_h}{r_p}\right)_{x_s}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx$$

$$=-\frac{4}{3}\left(\frac{p_h}{r_p}\right)_{x_s}\bar{\rho}_d$$

$\Rrightarrow$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left(-\frac{4}{3}\right)\left(\frac{p_h}{r_p}\right)_{x_s}\bar{\rho}_d$$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\left(\frac{\rho_h}{a}\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(-\frac{4}{3}\right)\left(\frac{qp_h}{r_p}\right)_{x_s}\cdot\int_0^{\bar{\epsilon}_0}\frac{2\left(\sqrt{\bar{\epsilon}}\right)^4}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}d\sqrt{\bar{\epsilon}}$$

The integral identity is given by

$$\int dy\frac{y^4}{y-a} = \int dy\frac{y^4-a^4+a^4}{y-a} = \int dy\left(y+a\right)\left(y^2+a^2\right)+\int dy\frac{a^4}{y-a} \tag{121}$$

$$= \frac{1}{4}y^4+\frac{1}{3}ay^3+\frac{1}{2}a^2y^2+a^3y+a^4\ln\left(y-a\right) \tag{122}$$

$\Rrightarrow$

$$\begin{aligned} = \quad & \frac{8}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\left(\frac{\rho_h}{a}\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{1}{1+\frac{1}{q_s}} \\ & \cdot\left[\begin{array}{c} \frac{1}{4}\bar{\epsilon}_0^2+\frac{1}{3}\frac{\bar{\omega}}{1+\frac{1}{q_s}}\left(\sqrt{\bar{\epsilon}_0}\right)^3+\frac{1}{2}\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2\left(\sqrt{\bar{\epsilon}_0}\right)^2 \\ +\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^3\left(\sqrt{\bar{\epsilon}_0}\right)+\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^4\ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\bar{\epsilon}_0^{1/2}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\end{array}\right]\end{aligned}$$

Defining $\Omega = \frac{\bar{\omega}}{(1+1/q_s)\bar{\epsilon}_0^{1/2}}$, $\delta\bar{W}_k$ can be rewritten by

$$\begin{aligned} \delta\bar{W}_k \quad = \quad & I_1+I_2 \\ & \frac{8}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\left(\frac{\rho_h}{a}\right)\frac{\left(\sqrt{\bar{\epsilon}_0}\right)^3}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s} \\ & \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)+\frac{1}{\Omega-1}\right] \\ = \quad & \\ & +\frac{8}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\left(\frac{\rho_h}{a}\right)^2\frac{\bar{\epsilon}_0^2}{\pi n_0 T_h\bar{\epsilon}_0}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{1}{1+\frac{1}{q_s}} \\ & \cdot\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right]\end{aligned}$$

$$
\delta\hat{W}_k = \frac{1}{4}\frac{1}{(r_s/R)^2}\frac{1}{|\xi_0/a|^2}\beta_{h0}\delta\bar{W}_k
$$

$$
= \begin{array}{l} \frac{8}{3\pi^2}\frac{1}{(r_s/R)^2}\frac{a}{R}\frac{\delta_x}{2}\left(\frac{\beta_h}{r_p}\right)_{x_s}\Delta_b\left[\frac{1}{\Omega-1}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right] \\ +\frac{8}{3\pi^2}\frac{1}{(r_s/R)^2}\frac{1}{q_s+1}\left(\frac{\beta_h}{r_p}\right)_{x_s}\Delta_b^2\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array} \tag{123}
$$

where, $\Delta_b = \frac{q_s\sqrt{\bar{\epsilon}_0}\rho_h}{a}$ is orbit width.

According to Eq. (123), the dispersion relation(69) thus can be written as

$$
-\frac{i\Omega}{\Omega_A}+\delta\hat{W}_T+\delta\hat{W}_k = 0 \tag{124}
$$

where $\Omega_A = \frac{\bar{\omega}_A}{(1+1/q_s)\bar{\epsilon}_0^{1/2}}$.

3.4 The dispersion relation for the case of $p=-1$, $\rho_d \neq 0$

For $p=-1$, we arrive at

$$
Y_{-1} = -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-\bar{x}-\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right) \tag{125}
$$

From Fig.1, Y_{-1} is zero in the region $x < x_s - \bar{\rho}_d$ where $H=1$ for $\theta \in [0, 2\pi]$ since the θ integral of $\sim \exp(-i\theta)$ from 0 to 2π is zero. In the region $x > x_s + \bar{\rho}_d$, Y_1 also is zero since $H=0$. Obviously, Y_{-1} is finite in the region $x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d$ where $H=1$ for $\theta \in [\theta^\star, -\theta^\star + 2\pi]$. $\cos\theta^\star = \frac{x_s - x}{\bar{\rho}_d} \in [0, \pi]$. Thus,

$$
Y_{-1} = -\frac{1}{\pi}\frac{a}{R}\int_{\theta^\star}^{2\pi-\theta^\star} d\theta\bar{\xi}_s\exp\left(i\theta\right) \tag{126}
$$

$$
= \frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(i\left(2\pi-\theta^\star\right)\right)-\exp\left(i\theta^\star\right)\right]
$$

$$
= \frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(i\left(2\pi-\theta^\star\right)\right)-\exp\left(i\theta^\star\right)\right]
$$

$$
= \frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\sin\theta^\star
$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p=-1$, $\delta\bar{W}_k$ becomes

$$
\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\bar{J}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)
$$

$$\cdot \frac{|Y_{-1}|^2}{\sqrt{\bar{\epsilon}} - \frac{\sqrt{\bar{\epsilon}}}{q} - \bar{\omega}}. \tag{127}$$

The fishbone of $\bar{\omega} \sim \omega_{\star i}$ is studied. Thus the term I_2 dominates over the term I_1 and $\left|\Re\left(\delta\hat{W}_k\right)\right| \ll \left|\Im\left(\delta\hat{W}_k\right)\right|$.

$$I_2 = -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}_\star$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}} - \frac{\sqrt{\bar{\epsilon}}}{q} - \bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} m \frac{\partial \bar{F}}{\partial x} \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a}$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}} - \frac{\sqrt{\bar{\epsilon}}}{q} - \bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a} \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot \delta(\Lambda) H(\bar{\epsilon}_0 - \bar{\epsilon}) \frac{dp_h(x)}{dx} \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right]}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}(x - x_s) - \bar{\omega}}$$

$$= -\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a} \frac{\rho_h}{a} \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right] \frac{dp_h(x)}{dx} dx$$

$$\cdot \int_0^{\bar{\epsilon}_0} \bar{\epsilon} d\bar{\epsilon} \frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}(x - x_s) - \bar{\omega}}$$

since $q = q_s + dq/dx\,(x - x_s)$, $s = \frac{x}{q}\frac{dq}{dx}$.
$\Rightarrow$

$$= -\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a} \frac{\rho_h}{a} \frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \int_0^{\bar{\epsilon}_0} \bar{\epsilon} d\bar{\epsilon}$$

$$\cdot \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right] \frac{dp_h(x)}{dx} \frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}(x - x_s) - \bar{\omega}} dx$$

Due to $r_p = -\left[dp_h/pdx\right]^{-1}$, the imaginary part of the above integral of x yields

$$
\begin{aligned}
&\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h(x)}{dx}\frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}(x-x_s)-\bar{\omega}}dx\\
&=-i\pi\left(\frac{p_h}{r_p}\right)_{x_s}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}(x-x_s)-\bar{\omega}}dx\\
&=-i\pi\left(\frac{p_h}{r_p}\right)_{x_s}\left[1-\left(\frac{\bar{\omega}x_s q_s}{\sqrt{\bar{\epsilon}}s\bar{\rho}_d}\right)^2\right]
\end{aligned}
$$

by using $\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}(x-x_s)-\bar{\omega}=0$. The imaginary part of I_2 with maximum drive by ignoring the term $\frac{\bar{\omega}x_s q_s}{\sqrt{\bar{\epsilon}}s\bar{\rho}_d}$ is

$$
\begin{aligned}
I_2^{\Im}&=i\pi^2\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(\frac{p_h}{r_p}\right)_{x_s}\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}\\
&=i\pi^2\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(\frac{p_h}{r_p}\right)_{x_s}\frac{1}{2}\bar{\epsilon}_0^2
\end{aligned}
$$

Thus the imaginary part of $\delta\hat{W}_k$ is

$$
\begin{aligned}
\delta\hat{W}_{k2}^{\Im}&=\frac{1}{4}\frac{1}{(r_s/R)^2}\frac{1}{|\xi_0/a|^2}\beta_{h0}I_2^{\Im}\\
&=i\frac{1}{2\pi}\frac{1}{(r_s/R)^2}\frac{a}{R}\left(\frac{\beta_h}{r_p}\right)_{x_s}\sqrt{\bar{\epsilon}_0}\Delta_b
\end{aligned}
\tag{128}
$$