

dwk++ User Manual

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1 Introduction

dwk++ is a small code to calculate δW_k and solve fishbone dispersion relation for tokamak plasmas. The δW_k is calculated by 3D integration in phase space (r, Λ, E) . A slowing down distribution function and a kink like mode structure is used. With a small growth rate (imag part of Ω) and tokamak parameters as input, this code finds out the fishbone mode frequency and fast ion $\beta_{h,0}$.

2 The definition of δW_k

2.1 The normalized δW_k in the code:

$$\delta W_k = \sum_p \int_0^1 \frac{J}{q} dr \int d\Lambda \int E^3 dE \tau_b (\omega - \omega_*) \frac{\partial F}{\partial E} \frac{|Y_p|^2}{n\omega_\phi + p\omega_b - \omega} \quad (1)$$

In current version, we only keep single n . $\sigma = \pm 1$ is the direction of $v_{||}$, m and n is poloidal and toroidal mode number, $\Lambda = \frac{\mu}{E}$, E is the fast ion energy, τ_b is the particle bounce time. The slowing down distribution function of fast ions is:

$$F = \frac{2^{3/2}}{C_f} \hat{F}(r, \epsilon, \Lambda) = \frac{2^{3/2}}{C_f} \frac{1}{E^{3/2} + E_c^{3/2}} \text{erfc} \left(\frac{E - E_0}{\Delta E} \right) e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda - \Lambda_0}{\Delta \Lambda})^2} \quad (2)$$

$$C_f = \int d^3\mathbf{v} \frac{1}{E^{3/2} + E_c^{3/2}} \text{erfc} \left(\frac{E - E_0}{\Delta E} \right) e^{-(\frac{\Lambda - \Lambda_0}{\Delta \Lambda})^2} \quad (3)$$

And $d^3\mathbf{v} = \frac{\sqrt{2}\pi}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda E^{1/2} dE$.

$$C_f = \int \frac{\sqrt{2\pi}}{b\sqrt{1-\frac{\Lambda}{b}}} \frac{1}{E^{3/2} + E_c^{3/2}} \operatorname{erfc}\left(\frac{E-E_0}{\Delta E}\right) e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} d\Lambda E^{1/2} dE \quad (4)$$

$$\frac{\partial F}{\partial E} = \frac{-2^{3/2}}{C_f} \left[\frac{2 \exp(-(\frac{E-E_0}{\Delta E})^2)}{\sqrt{\pi} \Delta E (E^{3/2} + E_c^{3/2})} + \frac{3\sqrt{E} \operatorname{erfc}(\frac{E-E_0}{\Delta E})}{2(E^{3/2} + E_c^{3/2})^2} - \frac{2\Lambda(\Lambda - \Lambda_0) \operatorname{erfc}(\frac{E-E_0}{\Delta E})}{E \Delta \Lambda^2 (E^{3/2} + E_c^{3/2})} \right] e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} \quad (5)$$

$$\frac{\partial F}{\partial r} = \frac{2^{3/2}}{C_f} \frac{\operatorname{erfc}(\frac{E-E_0}{\Delta E})}{E^{3/2} + E_c^{3/2}} \frac{2(r_0 - r)}{\Delta r^2} e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} \quad (6)$$

The diamagnetic frequency:

$$\omega_* = \frac{nq}{2r} \frac{\rho_0}{\varepsilon_0} \frac{\partial F / \partial r}{\partial F / \partial E} \quad (7)$$

where ρ_0 is the gyro radius with injection energy. ε_0 is the inverse aspect-ratio, and $\varepsilon = \frac{r}{R_0}$. The transit frequency for passing particle is given below:

$$\omega_b = \frac{\pi \sqrt{\kappa}}{K(\kappa^{-1})} \frac{\sqrt{\varepsilon \Lambda / 2}}{q} \sqrt{E} \quad (8)$$

where $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, and K denotes the complete elliptic integral of the first kind. The transit frequency in toroidal: $\omega_\phi = q\omega_b$. Particle bounce time: $\tau_b = \frac{2\pi}{\omega_b}$. The integral along the particle orbits:

$$Y(r, \Lambda, E) = \frac{1}{2\pi} \int_0^{2\pi} \chi d\theta B_\Lambda (\Lambda_b + 2(1 - \Lambda_b)) G(r, \theta, E) e^{-ixp\Theta} \quad (9)$$

where $\chi(r, \Lambda) = \frac{\sigma \pi \sqrt{\kappa} \sqrt{\varepsilon \Lambda / 2}}{K(\kappa^{-1})}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$. K denotes the complete elliptic integral of the first kind. $\Lambda_b = \frac{\Lambda}{b}$, $B_\Lambda(r, \Lambda, \theta) = \frac{1}{b\sqrt{(1-\Lambda_b)}}$, $b = B/B_0 = 1 + (r/R_0)\cos\theta$.

$$G(r, \Lambda, E, \theta) = (g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r) \xi_\theta(\hat{r}(\bar{r}, \rho_d, \theta), \theta) + (g^{rr} \kappa_r + g^{r\theta} \kappa_\theta) \xi_r(\hat{r}(\bar{r}, \rho_d, \theta), \theta) \quad (10)$$

where $\hat{r} = \bar{r} + \rho_d \cos\theta$.

$$\rho_d(r, \Lambda, E) = \frac{q}{2} \rho_0 \sqrt{\frac{E}{1-\Lambda/b}} \left[\frac{\Lambda}{b} + 2(1 - \frac{\Lambda}{b}) \right] = \frac{q}{2} \rho_0 \sqrt{\frac{E}{1-\Lambda_b}} [\Lambda_b + 2(1 - \Lambda_b)] \quad (11)$$

To simplify the code, we use Λ_0 instead of Λ in ρ_d .

$$\rho_d(r, E) = \frac{q}{2} \rho_0 \sqrt{\frac{E}{1-\Lambda_{0,b}}} [\Lambda_{0,b} + 2(1 - \Lambda_{0,b})] \quad (12)$$

For $\Lambda_0 = 0$

$$\rho_d(r, E) = q\rho_0 \sqrt{E}$$

$$G(r, E, \theta) = (g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r) \xi_\theta(\hat{r}(\bar{r}, \rho_d, \theta), \theta) + (g^{rr} \kappa_r + g^{r\theta} \kappa_\theta) \xi_r(\hat{r}(\bar{r}, \rho_d, \theta), \theta) \quad (13)$$

$$g^{rr} = 1 + \frac{\varepsilon \cos\theta}{2}, g^{\theta\theta} = \frac{1}{r^2} (1 - \frac{5}{2} \varepsilon \cos\theta), g^{r\theta} = -\frac{3}{2r} \varepsilon \sin\theta.$$

$$\Theta(\theta, r, \Lambda) = \int_0^\theta d\theta' \frac{1}{b\sqrt{(1-\Lambda_b)}} = \int_0^\theta B_\Lambda \quad (14)$$

2.2 The mode structure

In the current version, the mode structure is source code, and it is a kink structure with a fixed boundary at $r = 1$, and with a finite resonance layer width Δr .

$$\xi_r(r, \theta) = \xi_{r0}(r) \exp(-i\theta), \xi_\theta(r, \theta) = -i\xi_{\theta0}(r) \exp(-i\theta)$$

$$\xi_{r0}(r) = \begin{cases} \xi_0 & r \leq r_s - \Delta r/2 \\ \xi_0 \frac{\Delta r - r + r_s - \Delta r/2}{\Delta r} & r_s - \frac{\Delta r}{2} < r < r_s + \frac{\Delta r}{2} \\ 0 & r \geq r_s + \frac{\Delta r}{2} \end{cases}$$

$$\xi_{\theta0}(r) = \begin{cases} \xi_0 & r \leq r_s - \Delta r/2 \\ \xi_0 \frac{\Delta r - 2r + r_s - \Delta r/2}{\Delta r} & r_s - \frac{\Delta r}{2} < r < r_s + \frac{\Delta r}{2} \\ 0 & r \geq r_s + \frac{\Delta r}{2} \end{cases}$$

2.3 The normalized quantities used for δW_k :

$v_0 = \sqrt{2T_0/M}$, T_0 is the fast ions injection energy, M is the fast ion's mass. $\omega_0 = \frac{v_0}{R_0}$. $F_0 = \frac{n_0}{v_0^3}$, n_0 is the fast ion density at axis. $r_0 = a$ is the minor radius, $\varepsilon = a/R_0$, $E_0 = T_0/M$, B_0 is the toroidal magnetic field at magnetic axis. $\delta W_{k,0} = \pi^2 a^2 R_0 n_0 T_0$.

3 Fishbone dispersion relation

Assume $\delta W_{mhd} = 0$, the normalized dispersion relation is:

$$\frac{4}{\pi} \left(\frac{r_s}{R_0} \right)^2 \left| \frac{\xi_0}{a} \right|^2 \left(-i \frac{\omega}{\omega_A} \right) + \beta_h C_p \delta W_k = 0 \quad (15)$$

or:

$$i\omega = C \beta_{h,0} C_p \delta W_k \quad (16)$$

where:

$$C = \frac{\omega_A}{\omega_0} \frac{1}{4 \left(\frac{r_s}{R_0} \right)^2 \left| \frac{\xi_0}{a} \right|^2} \quad (17)$$

and $C_p = \frac{p_0}{n_0 T_0}$. Here $\xi_s/\xi_0 = 1$, $\omega_A = \frac{v_A}{3^{1/2} R_0 s}$, $v_A = \frac{B}{\sqrt{\mu_0 \rho_m}}$, $s = r_s \frac{dq}{dr(r=r_s)}$, and $\beta_{h,0} = 8\pi n_0 T_h / B_t^2$. Considering MHD contribution from $m = 1, n = 1$:

$$i\omega = C \beta_{h,0} C_p \delta W_k + \frac{\omega_A}{\omega_0} \delta W_T \quad (18)$$

where $\delta W_T = 3\pi \left(\frac{r_s}{R} \right)^2 (1 - q_0) \left(\frac{13}{144} - \beta_{ps}^2 \right)$.

4 How to run dwk++

4.1 Compile the code

dwk++ code is using c++ language as the main language, so it need a c++ compiler to compile the code. Gnu/g++ on max os and Linux and intel compiler on Linux was tested. To compile the code, libconfig with version >1.5 is needed, and set environment variable LIBCONFIG_DIR to the path where libconfig located. Set CXX to the c++ compiler (g++, icpc), and run 'make' at 'dwk-/' directory. If everything correct, a executable file 'dwk++' should be generated in 'dwk-/src/' directory and be copied to current directory. Then you can run dwk++ with dwk.cfg input file. A example of dwk.cfg can be find in 'dwk++/examples'.

4.2 Arguments

- -h print help information.
- -i input file name, dwk.cfg by default.
- -o outfile name, omega_dwk.out by default.
- -s scan dwk(omega), only find Ω_0 and $\beta_{h,0}$.
- -y write Yps_3D to Yps.nc.

4.3 Input file

Here is an example of input file:

Listing 1: a input file example

```
/*the input file for dwk++, parameters units is in ( ) */
//tokamak parameters
tokamak=
{
    a=0.38;           //minor radius (m).
    R0=1.30;          //major radius (m).
    Bt =0.84;         //Toroidal magnetic field at axis without plasma, (Tesla).
    n0 =1.7e19;        //thermal plasma density, (m^-3).
    mi =2.0;           //ion mass, (protom mass,m_p).
    E_i0 =25.142;      //Fost ion injection energy, (KeV).
```

```

m_ep =2.0;          //fat ion mass (protom mass,m_p).
//qc[0:7] q profile , q=qc[0] +qc[1]*r +qc[2] *r^2 ..... qc[7]*r^7.
qc=[0.8, 0.0, 1.3850];
q_s=1.0;           //q at resonance surface
}
//grid parameters
grid=
{
    nx=202; //grid size should be 3n + 1, n is a positive integer.
    nL=202;
    nE=400;
    ntheta=202;
}
//fast ion distribution
slowing=
{
    r0=0.0;          //(a)
    rd=0.2;          //(a)
    L0=0.01;         //Lambda_0
    Ld=0.02;         //Delta_Lambda
    E0=1.0;          //(E_i0)
    Ed=0.01;         //(E_i0)
    Ec=0.01;         //(E_i0)
    sigma=1;         //co(1) ? count (-1)
}
//perturbation and omega range for the soultion
mode=
{
    n=1;             //toroidal mode number.
    m=1;             //poloidal mode number.
    pa=0;            //resounces: sum Yps/(n*omega_phi +p*omega_theta -\omega).
    pb=0;            //sum from pa to pb.
    delta_r=0.001;   //step function width for kink. (a)
    omega_0=0.01;    //find mode frequency and scan dwk between omega_0 and omega_1.
    omega_1=0.99;    //omega unit is (v_0/R_0), v_0 is the fast ion injection speed.
    omega_i=0.01;    //image part of omega, the growth rate.
    omega_n=100;     //scan steps.
    omega_err=1.0e-5; //residual of omega_0.
    max_iter =100;   //maximum iteration number to find the omega_0.
    max_iterg =2;    //
    dw_f=0.00;       //dw_mhd.
    zero_rhod=0;     //1: with drift orbit width effect , 0 without.
    xi_0 =0.01;      //(a) displacement at r=0.
}

dwkopt=
{
    omega_star_off=0; //1: omega_star term on, 0: omega_star term off
    omega_off=1;      //1: omega term on, 0; omega term off
}

```

4.4 Output file

- omega_dwk.out,
- Yps.nc.

4.5 Utilities to plot results

There are some matlab & python scripts in 'dwk++/utilities' directory.

5 Benchmark

5.1 Compare with (WANG, Destabilization of internal kink modes at high frequency by energetic circulating ions. Physical Review Letters, 2001, 86) case ('dwk-/runp0_ws'j').

Considering deep passing particles ($\Lambda = 0$), and using distribution function:

$$F = \frac{p_h}{\pi n_0 T_0 C_p} \frac{1}{E^{3/2}} \delta(\Lambda) H(E_0 - E) \quad (19)$$

The δW_k analytical results:

$$\delta W_k = (\delta W_{k,d} + \delta W_{k,s})$$

$$\delta W_{k,d} = -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \int_0^{r_s} dr \frac{dp_h}{dr} q \quad (20)$$

$$\delta W_{k,s} = \frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [-\frac{\Omega}{\Omega - 1} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega})] \int_0^{r_s} r p_h dr \quad (21)$$

Note that the second term $\delta W_{k,s}$ in Eq.21 is different with Eq(13) in Wang's PRL paper. Using PBX parameters: $B = 0.84T$, $\omega_{\zeta,0}/2\pi = 190kHz$, $R_0 = 1.3m$, $a = 0.38m$, the injection energy $T_0 = 25.142keV$. $n_i = 1.7 \times 10^{19}m^{-3}$, $\varepsilon_s = 1/9$, $r_s = 0.1444m$, $s = 0.4$.

Assume $p_h = p_0 \exp(-(\frac{r}{\Delta r})^2)$, $q = c_0 + c_2 r^2$.

$$\begin{aligned} \delta W_{k,d} &= -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \int_0^{r_s} dr \frac{dp_h}{dr} q \\ &= -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] p_0 \int_0^{r_s} dr \frac{de^{-(\frac{r}{\Delta r})^2}}{dr} (c_0 + c_2 r^2) \\ &= \frac{16}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \frac{p_0}{\Delta r^2} \int_0^{r_s} r e^{-(\frac{r}{\Delta r})^2} (c_0 + c_2 r^2) dr \\ &= \frac{16}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \frac{p_0}{\Delta r^2} \left\{ -\frac{1}{2} \Delta r^2 e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \right\} \Big|_0^{r_s} \\ &= -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] p_0 \left\{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \right\} \Big|_0^{r_s} \end{aligned} \quad (22)$$

$$\begin{aligned} \delta W_{k,s} &= -\frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] \int_0^{r_s} r p_h dr \\ &= -\frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] p_0 \int_0^{r_s} r e^{-(\frac{r}{\Delta r})^2} dr \\ &= -\frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] p_0 \left[-\frac{1}{2} \Delta r^2 e^{-(\frac{r}{\Delta r})^2} \right] \Big|_0^{r_s} \\ &= \frac{4}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] p_0 \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}] \Big|_0^{r_s} \end{aligned} \quad (23)$$

So the normalized δW_k is :

$$\frac{4(\varepsilon_0 \xi_0)^2 p_0}{n_0 T_0 C_p} \left\{ \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}] \Big|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \} \Big|_0^{r_s} \right\}$$

Define $W_a = \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}] \Big|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \} \Big|_0^{r_s}$.

Consider the $\delta W_{k,0} = \pi^2 a^2 R_0 n_0 T_0$, finally, in physical units:

$$\begin{aligned} \delta W_{k,0} \delta W_k &= \delta W_{k,0} \frac{4(\varepsilon_0 \xi_0)^2 p_0}{n_0 T_0 C_p} W_a \\ &= \pi^2 a^2 R_0 n_0 T_0 \frac{4(\varepsilon_0 \xi_0)^2 p_0}{n_0 T_0 C_p} W_a \\ &= \frac{4\pi^2 a^2 R_0 (\varepsilon_0 \xi_0)^2 p_0}{C_p} W_a \\ &= \frac{4\pi^2 a^2 R_0 (\varepsilon_0 \xi_0)^2 p_0}{C_p} \left\{ \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln(1 - \frac{1}{\Omega}) \right] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}] \Big|_0^{r_s} \right. \\ &\quad \left. - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 \ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \} \Big|_0^{r_s} \right\} \end{aligned}$$

Then we can get:

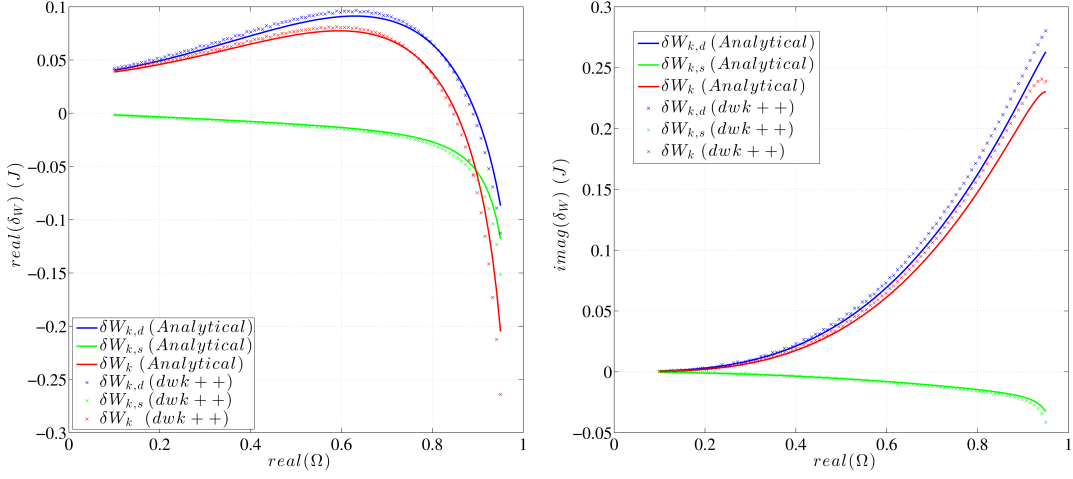


Figure 1: Comparison between $dwk++$ results and analytical's.

$$i \frac{\Omega}{\Omega_A} = \frac{1}{4} \frac{1}{\varepsilon_s^2} \frac{\beta_0}{|\xi_0|^2} \delta W_k \quad (24)$$

$$i \frac{\Omega}{\Omega_A} = \frac{\varepsilon_0^2}{\varepsilon_s^2} \frac{p_0 \beta_0}{n_0 T_0 C_p} \left\{ \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln \left(1 - \frac{1}{\Omega} \right) \right] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 \ln \left(1 - \frac{1}{\Omega} \right) + \Omega^2 + \frac{1}{2} \Omega + \frac{1}{3}] \{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \} |_0^{r_s} \right\} \quad (25)$$

The image part is $\Omega^3 \log(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2} \Omega + \frac{1}{3}$ is $[(\Omega_r^3 - 3\Omega_r \Omega_i^2) \pi + (3\Omega_i \Omega_r^2 - \Omega_i^3) \log |1 - \frac{1}{\Omega}| + 2\Omega_r \Omega_i + \frac{1}{2} \Omega_i]$, and the image part of $-\frac{\Omega}{\Omega - 1} + \Omega + \Omega^2 \log(1 - \frac{1}{\Omega})$ is $\frac{\Omega_i}{(\Omega_r - 1)^2 + \Omega_i^2} + \Omega_i + (\Omega_r^2 - \Omega_i^2) \pi + 2\Omega_i \Omega_r \log |1 - \frac{1}{\Omega}|$. With $\Omega_i = 0$, it becomes $\pi \Omega_r^3$ and $\pi \Omega_r^2$. The dispersion relation is:

$$\frac{\Omega_r}{\Omega_A} = \frac{\varepsilon_0^2}{\varepsilon_s^2} \beta_0 \Omega_r^2 \pi \{ \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} \Omega_r \{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \} |_0^{r_s} \} \quad (26)$$

The critical β_0^{crit} given by

$$\beta_0^{crit} = \frac{\varepsilon_s^2}{\pi \Omega_A \Omega_r \varepsilon_0^2} \frac{1}{\Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} \Omega_r \{ e^{-(\frac{r}{\Delta r})^2} [c_2 (\Delta r^2 + r^2) + c_0] \} |_0^{r_s}} \quad (27)$$

Using $q = 0.8 + 1.385r^2$, $\Delta r = 0.2$, and $\beta = 0.01$, $p_0 = 2.8083 \times 10^3 \text{ pascal}$, and $\xi_0/a = 0.01$ compare the analytical results to $dwk++$ results (shown in Fig. 1). For $\text{imag}(\Omega) = 0.005$, the real frequency can be found by the code: $\Omega_r = 0.8472, (0.29137\omega_A, 160.965 \text{ kHz})$, $\beta_{0,crit} = 0.0402$ (without ω term, $\Omega_r = 0.8901$, $\beta_{0,crit} = 0.0333$). And by Eq. 27, $\beta_{0,crit} = 0.04130$ (without ω term, $\beta_{0,crit} = 0.03442$).

6 Appendix

For more details about formula derivation, please read `limin_kinetic.pdf(dwk-/doc/limin_kinetic.pdf)`.