$dwk++$ User Manual

Feng Wang (fwang@pppl.gov)

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PPPL, Princeton, NJ, 08540,USA.

Contents

1 Introduction

dwk++ is a small code to calculate δW_k and solver fishbone dispersion relation for toakmak plasmas. The δW_k is calculate by 3D integration in phase space (r, Λ, E). A slowing down distribution function and a kink like mode structure is used. With a small growth rate (imag part of Ω) and tokamak parameters as a input, this code find out the fishbone mode frequency and fast ion $\beta_{h,0}$.

2 The defintion of δW_k

2.1 The normalized δW_k in the code:

$$\delta W_k = \sum^p \int_0^1 \frac{J}{q} dr \int d\Lambda \int E^3 dE \tau_b (\omega - \omega_\star) \frac{\partial F}{\partial E} \frac{|Y_p|^2}{n\omega_\phi + p\omega_b - \omega} \tag{1}$$

In current version, we only keep signle n. $\sigma = \pm 1$ is the direction of $v_\parallel$, m and n is poloidal and toroidal mode number, $\Lambda = \frac{\mu}{E}$, E is the fast ion energy, τ_b is the particle bounce time. The slowing down distribution function of fast ions is:

$$F = \frac{2^{3/2}}{C_f} \hat{F}(r, \epsilon, \Lambda) = \frac{2^{3/2}}{C_f} \frac{1}{E^{3/2} + E_c^{3/2}} erfc\left(\frac{E - E_0}{\Delta E}\right) e^{-(\frac{r - r_0}{\Delta r})^2} e^{-(\frac{\Lambda - \Lambda_0}{\Delta \Lambda})^2} \tag{2}$$

$$C_f = \int d^3\mathbf{v} \frac{1}{E^{3/2} + E_c^{3/2}} erfc\left(\frac{E - E_0}{\Delta E}\right) e^{-(\frac{\Lambda - \Lambda_0}{\Delta \Lambda})^2} \tag{3}$$

And $d^3\mathbf{v} = \frac{\sqrt{2}\pi}{b\sqrt{1 - \frac{\Lambda}{b}}} d\Lambda E^{1/2} dE$.

$$C_f = \int \frac{\sqrt{2}\pi}{b\sqrt{1-\frac{\Lambda}{b}}} \frac{1}{E^{3/2}+E_c^{3/2}} erfc\left(\frac{E-E_0}{\Delta E}\right) e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} d\Lambda E^{1/2} dE \tag{4}$$

$$\frac{\partial F}{\partial E} = \frac{-2^{3/2}}{C_f}\left[\frac{2\exp(-(\frac{E-E_0}{\Delta E})^2)}{\sqrt{\pi}\Delta E(E^{3/2}+E_c^{3/2})} + \frac{3\sqrt{E}erfc(\frac{E-E_0}{\Delta E})}{2(E^{3/2}+E_c^{3/2})^2} - \frac{2\Lambda(\Lambda-\Lambda_0)erfc\left(\frac{E-E_0}{\Delta E}\right)}{E\Delta\Lambda^2(E^{3/2}+E_c^{3/2})}\right] e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} \tag{5}$$

$$\frac{\partial F}{\partial r} = \frac{2^{3/2}}{C_f} \frac{erfc\left(\frac{E-E_0}{\Delta E}\right)}{E^{3/2}+E_c^{3/2}} \frac{2(r_0-r)}{\Delta r^2} e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} \tag{6}$$

The diamagnetic frequency:

$$\omega_* = \frac{nq}{2r}\frac{\rho_0}{\varepsilon_0}\frac{\partial F/\partial r}{\partial F/\partial E} \tag{7}$$

where ρ_0 is the gyro radius with injection energy. ε_0 is the inverse aspect-ratio, and $\varepsilon = \frac{r}{R_0}$. The transit frequency for passing particle is given below:

$$\omega_b = \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{E} \tag{8}$$

where $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, and K denotes the complete elliptic integral of the first kind. The transit frequency in toroidal:$\omega_\phi = q\omega_b$. Particle bounce time: $\tau_b = \frac{2\pi}{\omega_b}$. The integral along the particle orbits:

$$Y(r,\Lambda,E) = \frac{1}{2\pi}\int_0^{2\pi} \chi d\theta B_\Lambda\left(\Lambda_b + 2\left(1-\Lambda_b\right)\right) G(r,\theta,E) e^{-i\chi p\Theta} \tag{9}$$

where $\chi(r,\Lambda) = \frac{\sigma\pi\sqrt{\kappa}\sqrt{\varepsilon\Lambda/2}}{K(\kappa^{-1})}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$. K denotes the complete elliptic integral of the first kind. $\Lambda_b = \frac{\Lambda}{b}$, $B_\Lambda(r,\Lambda,\theta) = \frac{1}{b\sqrt{(1-\Lambda_b)}}$, $b = B/B_0 = 1 + (r/R_0)cos\theta$.

$$G(\text{r,}\Lambda\text{,E,}\theta) = (g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r)\xi_\theta(\hat{r}(\bar{r},\rho_d,\theta),\theta) + (g^{rr}\kappa_r + g^{r\theta}\kappa_\theta)\xi_r(\hat{r}(\bar{r},\rho_d,\theta),\theta) \tag{10}$$

where $\hat{r} = \bar{r} + \rho_d cos\theta$.

$$\rho_d(r,\Lambda,E) = \frac{q}{2}\rho_0\sqrt{\frac{E}{1-\Lambda/b}}\left[\frac{\Lambda}{b} + 2(1-\frac{\Lambda}{b})\right] = \frac{q}{2}\rho_0\sqrt{\frac{E}{1-\Lambda_b}}\left[\Lambda_b + 2(1-\Lambda_b)\right] \tag{11}$$

To simplify the code, we use Λ_0 instead of Λ in ρ_d.

$$\rho_d(r,E) = \frac{q}{2}\rho_0\sqrt{\frac{E}{1-\Lambda_{0,b}}}\left[\Lambda_{0b} + 2(1-\Lambda_{o}b)\right] \tag{12}$$

For $\Lambda_0 = 0$

$$\rho_d(r,E) = q\rho_0\sqrt{E}$$

$$G(\text{r,E,}\theta) = (g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r)\xi_\theta(\hat{r}(\bar{r},\rho_d,\theta),\theta) + (g^{rr}\kappa_r + g^{r\theta}\kappa_\theta)\xi_r(\hat{r}(\bar{r},\rho_d,\theta),\theta) \tag{13}$$

$g^{rr} = 1 + \frac{\varepsilon cos\theta}{2}$, $g^{\theta\theta} = \frac{1}{r^2}\left(1 - \frac{5}{2}\varepsilon cos\theta\right)$, $g^{r\theta} = -\frac{3}{2r}\varepsilon sin\theta$.

$$\Theta(\theta,r,\Lambda) = \int_0^\theta d\theta' \frac{1}{b\sqrt{(1-\Lambda_b)}} = \int_0^\theta B_\Lambda \tag{14}$$

2.2 The mode structure

In the current version, the mode structure is source code, and it is a kink structure with a fixed boundary at $r = 1$, and with a finite resonance layer width Δr_s.

$\xi_r(r,\theta) = \xi_{r0}(r)exp(i(\phi-\theta-\omega t))$, $\xi_\theta(r,\theta) = -i\xi_{\theta 0}(r)rexp(i(\phi-\theta-\omega t))$

$$\xi_{r0}(r) = \begin{cases} \xi_0 & r \le r_s - \Delta r_s/2 \\ \xi_0\frac{\Delta r_s - r + r_s - \Delta r_s/2}{\Delta r_s} & r_s - \frac{\Delta r_s}{2} < r < r_s + \frac{\Delta r_s}{2} \\ 0 & r \ge r_s + \frac{\Delta r_s}{2} \end{cases}$$

$$\xi_{\theta 0}(r)=\begin{cases}\xi_0 & r\le r_s-\Delta r/2\\ \xi_0\frac{\Delta r_s-2r+r_s-\Delta r_s/2}{\Delta r_s} & r_s-\frac{\Delta r_s}{2}<r<r_s+\frac{\Delta r_s}{2}\\ 0 & r\ge r_s+\frac{\Delta r_s}{2}\end{cases}$$

If the flag in dwk.cfg input file input_i=1, then the code will read the structure in file with name defined in input_filename.

The file format is define below:

the first line is a integer for the grid number in r.

from the second line to the last is formated:

r xi_r xi_theta

2.3 The normalized quantities used for δW_k:

$v_0=\sqrt{2T_0/M}$, T_0 is the fast ions injection energy, M is the fast ion's mass. $\omega_0=\frac{v_0}{R_0}$. $F_0=\frac{n_0}{v_0^3}$, n_0 is the fast ion density at axis. $r_0=a$ is the minor radius, $\varepsilon=a/R_0$, $E_0=T_0/M$, B_0 is the torodial magnetic field at magnetic axis. $\delta W_{k,0}=\pi^2a^2R_0n_0T_0$.

3 Fishbone dispersion relation

Assume $\delta W_{mhd}=0$, the normalized dispersion relation is:

$$\frac{4}{\pi}(\frac{r_s}{R_0})^2|\frac{\xi_0}{a})|^2(-i\frac{\omega}{\omega_A})+\beta_hC_p\delta W_k=0 \tag{15}$$

or:

$$i\omega=C\beta_{h,0}C_p\delta W_k \tag{16}$$

where:

$$C=\frac{\omega_A}{\omega_0}\frac{1}{4(\frac{r_s}{R_0})^2|\frac{\xi_0}{a}|^2} \tag{17}$$

and $C_p=\frac{p_0}{n_0T_0}$. Here $\xi_s/\xi_0=1$, $\omega_A=\frac{v_A}{3^{1/2}R_0s}$, $v_A=\frac{B}{\sqrt{\mu_0\rho_m}}$, $s=r_s\frac{dq}{dr(r=r_s)}$, and$\beta_{h,0}=8\pi n_0T_h/B_t^2$.
Considering MHD contribution from $m=1,n=1$:

$$i\omega=C\beta_{h,0}C_P\delta W_k+\frac{\omega_A}{\omega_0}\delta W_T \tag{18}$$

where $\delta W_T=3\pi\left(\frac{r_s}{R}\right)^2(1-q_0)\left(\frac{13}{144}-\beta_{ps}^2\right)$.

4 How to run dwk++

4.1 Compile the code

dwk++ code is using c++ language as the main language, so it need a c++ compiler to compile the code. Gnu/g++ on max os and Linux and intel compiler on Linux was tested. To compile the code, libconfig with version >1.5 is needed, and set environment variable LIBCONFIG_DIR to the path where libconfig located. Set CXX to the c++ compiler (g++, icpc), and run 'make' at 'dwk-/' directory. If everything correct, a executable file 'dwk++' should be generated in 'dwk-/src/' directory and be copied to current directory. Then you can run dwk++ with dwk.cfg input file. A example of dwk.cfg can be find in 'dwk++/examples'.

4.2 Arguments

- -h print help information.
- -i input file name, dwk.cfg by default.
- -o outfile name, omega_dwk.out by default.
- -s scan dwk(omega), only find Ω_0 and $\beta_{h,0}$.
- -y write Yps_3D to Yps.nc.
- -x find solution for dispersion relation using iteration.
- -g find solution for dispersion relation using newton iteration with initial guess in gomegar and gomegai.
- -B fint the solution by newton for beta_h to beta_hb.

4.3 Input file

Here is an example of input file:

Listing 1: a input file example

```
/*the input file for dwk++, parameters units is in ( ) */
//tokamak parameters
tokamak=
{
        a=0.38;           //minor radius (m).
        R0=1.30;          //major radius (m).
        Bt =0.84;         //Toroidal magnetic field at axis without plasma, (Tesla).
        n0 =1.7e19;       //thermal plasma density, (m^-3).
        mi =2.0;          //ion mass, (protom mass,m_p).
        E_i0 =25.142;     //Fost ion injection energy, (KeV).
        m_ep =2.0;        //fat ion mass (protom mass,m_p).
        //qc[0:7] q profile, q=qc[0] +qc[1]*r +qc[2] *r^2 ..... qc[7]*r^7.
        qc=[0.8, 0.0, 1.384189];
        q_s=1.0;          //q at resonance surface
        beta_h=0.0055;    //Fast ion beta
        beta_hb=0.02;
        nbeta=10;

}
//grid parameters
grid=
{
        nx=100; //grid size should be 3n + 1, n is a positive integer.
        nL=100;
        nE=100;
        ntheta=100;
}
//fast ion distribution
slowing=
{
        rflag =1;         //0 for exp profile, 1 for polynomial profile
        r0=0.1;           //(a)
        rd=0.2;           //(a)
        rc=[1.0, -0.063047119517556, -5.467717976626049, -15.106613290761315, 1.02317698952035
                          //rc[0:8], F(r) =rc[0] +rc[1]*r +rc[2]*r^2 ... rc[8]*r^8;
        L0=0.01;          //Lambda_0
        Ld=0.02;          //Delta_Lambda
        E0=1.0;           //(E_i0)
        Ed=0.01;          //(E_i0)
        Ec=0.01;          //(E_i0)
        sigma=1;          //co(1) ? count (-1)
}
//perturbation and omega range for the soultion
mode=
{
        n=1;     //toroidal mode number.
        m=1;     //poloidal mode number.
        pa=0;    //resounces: sum Yps/(n*omega_phi +p*omega_theta -\omega).
        pb=0;    //sum from pa to pb.
        delta_r=0.001; //step function width for kink. (a)
        input_i=1;        //1: using mode structure from input file
        input_filename="mode.dat"; // mode structure data
        omega_0=0.1;  //find mode frequency and scan dwk between omega_0 and omega_1.
        omega_1=0.95;  //omega unit is (v_0/R_0), v_0 is the fast ion injection speed.
        omega_i=0.005;  //image part of omega, the growth rate.
        omega_n=100;        //scan steps.
        omega_err=1.0e-5; //residual of omega_0.
        max_iter =100;      //maximum iteration number to find the omega_0.
        max_iterg =2;       //
```

```
        dw_f=0.00;            //dw_mhd.
        zero_rhod=0;          //1: with drift orbit width effect , 0 without .
        zero_iner=1;          //1: with iner layer , 0 without .
        xi_0 =0.01;           //(a) displacement at r=0.
        gomegar=0.04;
        gomegai=0.001;
}

dwkopt=
{
        omega_star_off=1;          //1: omega_star term on, 0: omega_star term off
        omega_off=1;               //1: omega term on, 0; omega term off
}
```

4.4 Output file

- omega_dwk.out.
- dwk.log.
- Yps.nc.

4.5 Utilities to plot results

There are some matlab & python scripts in 'dwk++/utilities' directory.

5 Benchmark

5.1 Compare with (WANG, Destabilization of internal kink modes at high frequency by energetic circulating ions. Physical Review Letters, 2001, 86) case ('dwk-/runp0_wsj').

Considering deep passing particles ($\Lambda = 0$), and using distribution function:

$$F = \frac{p_h}{\pi n_0 T_0 C_p}\frac{1}{E^{3/2}}\delta(\Lambda)H(E_0 - E) \tag{19}$$

The δW_k analytical results:

$$\delta W_k = (\delta W_{k,d} + \delta W_{k,s})$$

$$\delta W_{k,d} = -\frac{8}{\varepsilon_0}\frac{\rho_h}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]\int_0^{r_s} dr\frac{dp_h}{dr}q \tag{20}$$

$$\delta W_{k,s} = \frac{8}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[-\frac{\Omega}{\Omega-1} + \Omega + \Omega^2 ln(1-\frac{1}{\Omega})]\int_0^{r_s} rp_h dr \tag{21}$$

Note that the second term $\delta W_{k,s}$ in Eq.21 is different with Eq(13) in Wang's PRL paper. Using PBX parameters: $B = 0.84$T, $\omega_{\zeta,0}/2\pi = 190kHz$, $R_0 = 1.3m$, $a = 0.38m$, the injection energy $T_0 = 25.142keV$. $n_i = 1.7\times 10^{19}m^{-3}$, $\varepsilon_s = 1/9$, r_s =0.1444 m, $s = 0.4$.

Assume $p_h = p_0 exp(-(\frac{r}{\Delta r})^2)$, $q = c_0 + c_2 r^2$.

$$\begin{aligned}
\delta W_{k,d} &= -\frac{8}{\varepsilon_0}\frac{\rho_h}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]\int_0^{r_s} dr\frac{dp_h}{dr}q \\
&= -\frac{8}{\varepsilon_0}\frac{\rho_h}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]p_0\int_0^{r_s} dr\frac{de^{-(\frac{r}{\Delta r})^2}}{dr}(c_0 + c_2 r^2) \\
&= \frac{16}{\varepsilon_0}\frac{\rho_h}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]\frac{p_0}{\Delta r^2}\int_0^{r_s} re^{-(\frac{r}{\Delta r})^2}(c_0 + c_2 r^2)dr \\
&= \frac{16}{\varepsilon_0}\frac{\rho_h}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]\frac{p_0}{\Delta r^2}\{-\frac{1}{2}\Delta r^2 e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s} \\
&= -\frac{8}{\varepsilon_0}\frac{\rho_h}{n_0 T_0 C_p}(\varepsilon_0\xi_s)^2[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]p_0\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s}
\end{aligned} \tag{22}$$

$$
\begin{aligned}
\delta W_{k,s} &= -\frac{8}{n_0T_0C_p}(\varepsilon_0\xi_s)^2[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]\int_0^{r_s} rp_h dr \\
&= -\frac{8}{n_0T_0C_p}(\varepsilon_0\xi_s)^2[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]p_0\int_0^{r_s} re^{-(\frac{r}{\Delta r})^2} dr \\
&= -\frac{8}{n_0T_0C_p}(\varepsilon_0\xi_s)^2[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]p_0[-\frac{1}{2}\Delta r^2 e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} \\
&= \frac{4}{n_0T_0C_p}(\varepsilon_0\xi_s)^2[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 lln(1-\frac{1}{\Omega})]p_0\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}
\end{aligned} \tag{23}
$$

So the normalized δW_k is :
$\frac{4(\varepsilon_0\xi_0)^2p_0}{n_0T_0C_p}\left\{[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}-\frac{2\rho_h}{\varepsilon_0}[\Omega^3 ln(1-\frac{1}{\Omega})+\Omega^2+\frac{1}{2}\Omega+\frac{1}{3}]\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s}\right\}$
Define $W_a=[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}-\frac{2\rho_h}{\varepsilon_0}[\Omega^3 ln(1-\frac{1}{\Omega})+\Omega^2+\frac{1}{2}\Omega+\frac{1}{3}]\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s}$.
Consider the $\delta W_{k,0}=\pi^2a^2R_0n_0T_0$, finally, in physical units:

$$
\begin{aligned}
\delta W_{k,0}\delta W_k &= \delta W_{k,0}\frac{4(\varepsilon_0\xi_0)^2p_0}{n_0T_0C_p} W_a \\
&= \pi^2a^2R_0n_0T_0\frac{4(\varepsilon_0\xi_0)^2p_0}{n_0T_0C_p} W_a \\
&= \frac{4\pi^2a^2R_0(\varepsilon_0\xi_0)^2p_0}{C_p} W_a \\
&= \frac{4\pi^2a^2R_0(\varepsilon_0\xi_0)^2p_0}{C_p} \Big\{[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} \\
&\quad -\frac{2\rho_h}{\varepsilon_0}[\Omega^3 ln(1-\frac{1}{\Omega})+\Omega^2+\frac{1}{2}\Omega+\frac{1}{3}]\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s}\Big\}
\end{aligned}
$$

Then we can get:

$$
i\frac{\Omega}{\Omega_A}=\frac{1}{4}\frac{1}{\varepsilon_s^2}\frac{\beta_0}{|\xi_0|^2}\delta W_k \tag{24}
$$

$$
\begin{aligned}
i\frac{\Omega}{\Omega_A} &= \frac{\varepsilon_0^2}{\varepsilon_s^2}\frac{p_0\beta_0}{n_0T_0C_p} \Big\{[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2 ln(1-\frac{1}{\Omega})]\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} \\
&\quad -\frac{2\rho_h}{\varepsilon_0}[\Omega^3 ln(1-\frac{1}{\Omega})+\Omega^2+\frac{1}{2}\Omega+\frac{1}{3}]\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s}\Big\}
\end{aligned} \tag{25}
$$

The image part of $\Omega^3\log(1-\frac{1}{\Omega})+\Omega^2+\frac{1}{2}\Omega+\frac{1}{3}$ is $[(\Omega_r^3-3\Omega_r\Omega_i^2)\pi+(3\Omega_i\Omega_r^2-\Omega_i^3)\log\left|1-\frac{1}{\Omega}\right|+2\Omega_r\Omega_i+\frac{1}{2}\Omega_i]$, and the image part of $-\frac{\Omega}{\Omega-1}+\Omega+\Omega^2 log(1-\frac{1}{\Omega})$ is $\frac{\Omega_i}{(\Omega_r-1)^2+\Omega_i^2}+\Omega_i+(\Omega_r^2-\Omega_i^2)\pi+2\Omega_i\Omega_r log\left|1-\frac{1}{\Omega}\right|$. With $\Omega_i=0$, it becomes $\pi\Omega_r^3$ and $\pi\Omega_r^2$. The image part of the dispersion relation is:

$$
\frac{\Omega_r}{\Omega_A}=\frac{\varepsilon_0^2}{\varepsilon_s^2}\beta_0\Omega_r^2\pi\{\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}-\frac{2\rho_h}{\varepsilon_0}\Omega_r\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s}\} \tag{26}
$$

The critical β_0^{crit} given by

$$
\beta_0^{crit}=\frac{\varepsilon_s^2}{\pi\Omega_A\Omega_r\varepsilon_0^2}\frac{1}{\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}-\frac{2\rho_h}{\varepsilon_0}\Omega_r\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s}} \tag{27}
$$

The real part of $\Omega^3\log(1-\frac{1}{\Omega})+\Omega^2+\frac{1}{2}\Omega+\frac{1}{3}$ is the dispersion relation is :$(\Omega_r^3-3\Omega_r\Omega_i^2)ln(\frac{1}{\Omega_r}-1)-\pi(3\Omega_r^2\Omega_i-\Omega_i^3)+\Omega_r^2-\Omega_i^2+\frac{1}{2}\Omega_r+\frac{1}{3}$, and the real part of $-\frac{\Omega}{\Omega-1}+\Omega+\Omega^2 log(1-\frac{1}{\Omega})$ is $\frac{-\Omega_r^2+\Omega_r-\Omega_i^2}{(\Omega_r-1)^2+\Omega_i^2}+\Omega_r+(\Omega_r^2-\Omega_i^2)ln(\frac{1}{\Omega_r}-1)-2\pi\Omega_r\Omega_i$. With $\Omega_i=0$, The read part of dispersion relation is:

$$
0=[\frac{-\Omega_r^2+\Omega_r}{(\Omega_r-1)^2}+\Omega_r+\Omega_r^2 ln(\frac{1}{\Omega_r}-1)]\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}-\frac{2\rho_h}{\varepsilon_0}[\Omega_r^3 ln(\frac{1}{\Omega_r}-1)+\Omega_r^2+\frac{1}{2}\Omega_r+\frac{1}{3}]\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2+r^2)+c_0]\}|_0^{r_s} \tag{28}
$$

Using $q=0.8+1.385r^2$, $\Delta r=0.2$, and $\beta=0.01$, $p_0=2.8083\times10^3 pascal$, and $\xi_0/a=0.01$ compare the analytical results to $dwk++$ results (shown in Fig. 1). For $imag(\Omega)=0.005$, the real frequency can be found by the code: $\Omega_r=0.8472$,$(0.29137\omega_A, 160.965kHz)$, $\beta_{0,crit}=0.0402$ (without ω term, $\Omega_r=0.8901$, $\beta_{0,crit}=0.0333$). And by Eq. 27, $\beta_{0,crit}=0.04130$ (without ω term, $\beta_{0,crit}=0.03442$).

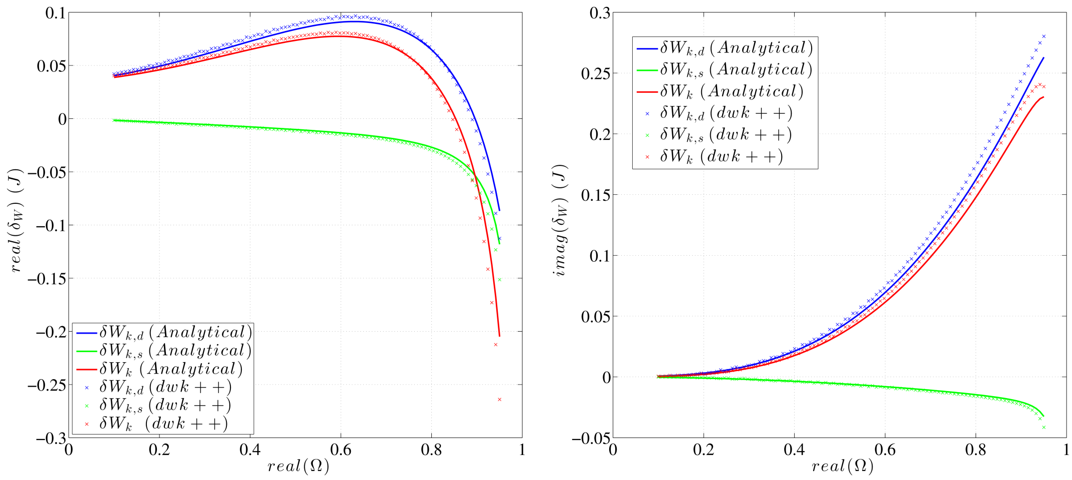

Figure 1: Comparison between $dwk++$ results and analysical's.

5.2 Appendix

For more details about formula derivation, please read limin_kinetic.pdf(dwk-/doc/limin_kinetic.pdf).