Ref of Fu's 1993 paper

February 19, 2016

1 The formula of δW_k

The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = i \frac{e}{M} \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) \frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \tag{1}$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\omega_c} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \kappa$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i \hat{\mathbf{b}} \times \nabla F \cdot \nabla}{\Omega \partial F / \partial \epsilon}$, $\epsilon = \frac{1}{2} v^2$, $\delta \mathbf{E}_\perp = i\omega \vec{\xi} \times \mathbf{B}$, $\omega_c = \frac{Be}{M}$ is the particle cyclotron frequency.[Berk, Phys. Fluid B 4 1992]

The term $\mathbf{v}_d \cdot \delta \mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp = \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp = i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \tilde{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{2}$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \vec{\kappa} \cdot \vec{\xi} = -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \left(\nabla\theta \kappa_\theta + \nabla r \kappa_r\right) \cdot \left(\xi_\theta \nabla\theta + \xi_r \nabla r\right)$$

$$= -i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \left(\nabla\theta \cdot \nabla\theta \xi_\theta \kappa_\theta + \nabla r \cdot \nabla r \xi_r \kappa_r + \nabla\theta \cdot \nabla r \xi_r \kappa_\theta + \nabla r \cdot \nabla\theta \xi_\theta \kappa_r\right)$$

$$= -i\omega B \frac{\epsilon}{\omega_c} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r \cdot \nabla r$, $g^{\theta r} = \nabla\theta \cdot \nabla r$, $g^{\theta\theta} = \nabla\theta \cdot \nabla\theta$. $\xi_r \nabla r = \xi_r \mathbf{e}_r$, $\xi_\theta \nabla\theta = \frac{1}{r} \xi_\theta \mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial \theta}$, $\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2 R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0)\cos\theta$, $\epsilon = \frac{1}{2}v^2$, $\delta \mathbf{E}_\perp^\star = -i\omega \vec{\xi}^\star \times \mathbf{B}$. Thus, the complex conjugate term is

$$\mathbf{v}_d \cdot \delta \mathbf{E}_\perp^\star = \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \kappa \cdot \delta \mathbf{E}_\perp^\star = -i\omega \frac{v_\perp^2/2 + v_\parallel^2}{\omega_c} \hat{\mathbf{b}} \times \tilde{\kappa} \cdot \left(\vec{\xi}^\star \times \mathbf{B}\right) \tag{3}$$

$$= i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\vec{\kappa}\cdot\vec{\xi}^\star = i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\left(\nabla\theta\kappa_\theta+\nabla r\kappa_r\right)\cdot\left(\xi_\theta^\star\nabla\theta+\xi_r^\star\nabla r\right)$$

$$= i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\left(\nabla\theta\cdot\nabla\theta\xi_\theta^\star\kappa_\theta+\nabla r\cdot\nabla r\xi_r^\star\kappa_r+\nabla\theta\cdot\nabla r\xi_r^\star\kappa_\theta+\nabla r\cdot\nabla\theta\xi_\theta^\star\kappa_r\right)$$

$$= i\omega B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta^\star+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = i\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{4}$$

$$\frac{d}{dt}g = H\left(r,\theta,\phi,t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint\frac{d\psi}{\dot{\psi}} = \oint\frac{d\theta}{\dot{\theta}} \tag{5}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}\left(r,\theta\right)\exp\left(-i\omega t+in\phi\right) \tag{6}$$

Note that $\hat{X}^{(1)}\left(r,\theta\right)$ is complex, and we take the real part of RHS for any physical variable, i.e. $X^{(1)}$. Thus, for internal kink mode, the displacement $\vec{\xi}$ is $\vec{\xi} = \xi_\theta'\mathbf{e}_\theta+\xi_r'\mathbf{e}_r$ and $\xi_r' = \xi_0\exp\left(i\left(\phi-\theta-\omega t\right)\right)$ within the region $q=1$ rational surface $r=r_s$. With cylindrical approximation, it can apply the relation $\nabla\cdot\vec{\xi} = 0$, and thus obtain $\xi_\theta' = -i\xi_0\exp\left[i\left(\phi-\theta-\omega t\right)\right]$. Thus, the covariant forms of the perturbation are

$$\xi_r = \xi_0\exp\left(i\left(\phi-\theta-\omega t\right)\right), \tag{7}$$

$$\xi_\theta = -i\xi_0 r\exp\left(i\left(\phi-\theta-\omega t\right)\right) \tag{8}$$

within the region $q=1$ surface. Similarly, the covariant forms of the perturbation are

$$\xi_r = \xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r} \right) \exp\left(i\left(\phi - \theta - \omega t\right)\right), \tag{9}$$

$$\xi_\theta = -i\xi_0 r \left(\frac{\Delta r - 2r + (r_s - \Delta r/2)}{\Delta r} \right) \exp\left(i\left(\phi - \theta - \omega t\right)\right) \tag{10}$$

in the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$. And $\xi_r = \xi_\theta = 0$ in the rest region.

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} i \frac{e}{M} \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B \frac{\epsilon}{\omega_c} G\left(\tau\right) d\tau \tag{11}$$

with

$$G = \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r \right)$$

where $G\left(\tau\right) = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-i\omega\tau + in\phi\left(\tau\right)\right)$ and the τ dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}} \cdot \nabla r, \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla \theta, \dot{\phi} = \dot{\mathbf{R}} \cdot \nabla \phi \tag{12}$$

Let us separate $\phi\left(\tau\right)$ into its secular and oscillating parts:

$$\phi\left(\tau\right) = \left\langle \dot{\phi} \right\rangle \tau + \widetilde{\phi}\left(\tau\right) \tag{13}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(in\tilde{\phi}\left(\tau\right)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}\left(\tau\right) = \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(ip\omega_b\tau\right) \tag{14}$$

where,

$$Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{15}$$

with $r\left(\tau\right) = \bar{r} + \rho_d \cos\theta\left(\tau\right)$, ρ_d represents the finite orbit width for passing particles. $\rho_d = \Omega_d/\omega_t$, $\Omega_d = \frac{\left(v_\perp^2/2 + v_\parallel^2\right)}{\omega_c R_0}$, $\omega_t = \frac{v_\parallel}{qR_0}$. Thus,

$$\rho_d = \frac{q}{\omega_c} \sqrt{\frac{\epsilon}{2\left(1 - \Lambda/b\right)}} \left[\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right] \tag{16}$$

Carrying out the time integration, the solution of g is obtained

$$g=\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\omega_c}\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\epsilon,\bar{r};\sigma\right)\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega\right)t\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega}\tag{17}$$

The formula of δW_k is derived as follows.

$$\delta W_k=\int d^3x\vec{\xi}^\star\cdot\nabla\cdot\delta\mathbf{P}_k=e\int d^3x\int d^3v\left(\frac{i}{\omega}\mathbf{v}_d\cdot\delta\mathbf{E}_\perp\right)^\star g$$

$$=e\int d^3x\int d^3vgB\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta^\star+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

$$=e\int d^3x\int d^3vgB\frac{\epsilon}{\omega_c}G^\star\tag{18}$$

where $G^\star=\hat{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(i\omega\tau-in\phi\left(\tau\right)\right)$. Let $\tilde{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right)\right]=\hat{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-in\tilde{\phi}\left(\tau\right)\right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star\left(\tau\right)=\sum_{-\infty}^{\infty}Y_p^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(-ip\omega_b\tau\right)\tag{19}$$

where,

$$Y_p^\star\left(\Lambda,\bar{r};\sigma\right)=\frac{1}{\tau_b}\oint d\tau\tilde{G}^\star\left(\tau\right)\exp\left(ip\omega_b\tau\right)\tag{20}$$

with $r\left(\tau\right)=\bar{r}+\rho_d\cos\theta\left(\tau\right)$.

$$\delta W_k=\frac{e^2}{M}\int d^3x\int d^3v\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B^2\frac{\epsilon^2}{\omega_c^2}\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\bar{r};\sigma\right)$$

$$\cdot\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega\right)\tau\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega}\sum_{-\infty}^{\infty}Y_{p'}^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(i\omega\tau-in\left\langle\dot{\phi}\right\rangle\tau-ip'\omega_b\tau\right)\tag{21}$$

$$\delta W_k=\frac{e^2}{M}\int d^3x\int d^3v\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B^2\frac{\epsilon^2}{\omega_c^2}$$

$$\cdot\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\epsilon,\bar{r};\sigma\right)\frac{\exp\left[ip\omega_b\tau\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega}\sum_{-\infty}^{\infty}Y_{p'}^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(-ip'\omega_b\tau\right)\tag{22}$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = \frac{e^2}{M} \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\omega_c^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p \left(\Lambda, \epsilon, \bar{r}; \sigma\right) \frac{\exp\left(ip\omega_b \tau\right)}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star} \left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(-ip'\omega_b \tau\right) \tag{23}$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon} b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_\parallel$, one finally obtains

$$\delta W_k = \frac{4\pi^2}{M} \frac{e^2 B^2}{\omega_c^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b \left(\omega - \omega_\star\right)$$
$$\cdot \sum_{-\infty}^{\infty} \frac{\left|Y_p\right|^2}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega}, \tag{24}$$

which is similar to Eq.(35) of Fu's 1993 paper with replacing ϵ and J by $\epsilon \equiv \frac{1}{2}Mv^2$ and $B = qR/J$. Note that $\tilde{\phi} \cong 0$, $\left\langle \dot{\phi} \right\rangle \cong \omega_D^0 + q\omega_b$,$\omega_D^0 \approx 0$ for passing particles.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{25}$$

$$J_t = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{26}$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E}\right)^{-1} = \frac{\pi \omega_\parallel}{2K\left(\kappa\right)}, \kappa < 1 \tag{27}$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E}\right)^{-1} = \frac{\pi\sqrt{\kappa}\omega_\parallel}{K\left(\kappa^{-1}\right)}, \kappa > 1 \tag{28}$$

where $\omega_\parallel = \frac{1}{qR}\sqrt{\varepsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR}\sqrt{\varepsilon\Lambda}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $F = \frac{n_0}{v_h^3}\bar{F}$**,** $v_h = \sqrt{\frac{2T_h}{M}}$**,** $\epsilon = \frac{T_h}{M}\bar{\epsilon}$**,** $r = ax$**,** $J = aR_0\bar{J}$**,** $R = R_0\bar{R}$**,** $\omega_b = \frac{v_h}{R_0}\bar{\omega}_b$, $\frac{1}{\tau_b} = \frac{v_h}{2\pi R_0}\bar{\omega}_b = \frac{v_h}{R_0}\frac{\bar{\omega}_b}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_b}$, $\omega = \frac{v_h}{R_0}\bar{\omega}$, $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$, $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$ **.** T_h **is arbitrary temperature/energy which can be charastic quantity, i.e. birth energy of fast ions.**

$$\delta W_k = \pi^2 a^2 R_0 n_0 T_h \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b \left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\overline{\left\langle \dot{\phi} \right\rangle} + p\bar{\omega}_b - \bar{\omega}} \tag{29}$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{30}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}} b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{31}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{32}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(r,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\hat{\xi}_\theta\left(\theta, \bar{r} + \rho_d\cos\theta\right) + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\hat{\xi}_r\left(\theta, \bar{r} + \rho_d\cos\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(x,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_{\theta}+\bar{g}^{r\theta}\bar{\kappa}_{r}\right)\bar{\hat{\xi}}_{\theta}\left(\theta,\bar{x}+\frac{\rho_{d}}{a}\cos\theta\right)+\left(\bar{g}^{rr}\bar{\kappa}_{r}+\bar{g}^{r\theta}\bar{\kappa}_{\theta}\right)\bar{\hat{\xi}}_{r}\left(\theta,\bar{x}+\frac{\rho_{d}}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m}=\hat{\xi}_{\theta m}/a^{2}$, $\bar{\hat{\xi}}_{rm}=\hat{\xi}_{rm}/a$.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F\left(x,\bar{\epsilon},\Lambda\right)=\frac{n_{0}}{v_{h}^{3}}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_{c}^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_{0}}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_{0}}{\Delta x}\right)^{2}\right]\exp\left[-\left(\frac{\Lambda-\Lambda_{0}}{\Delta\Lambda}\right)^{2}\right] \tag{33}$$

where n_0 is determined by

$$n\left(x\right)=\frac{n_{0}}{v_{h}^{3}}\int d^{3}\mathbf{v}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_{c}^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_{0}}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_{0}}{\Delta x}\right)^{2}\right]\exp\left[-\left(\frac{\Lambda-\Lambda_{0}}{\Delta\Lambda}\right)^{2}\right]$$

at $x=x_0$.

The normalized metric tensors are

$$\bar{g}^{rr}=1+2\Delta'\cos\theta \tag{34}$$

with $\Delta'=\left(\varepsilon+\alpha\right)/4$, $\varepsilon=\frac{r}{R_0}$, $\alpha=-R_0q^2d\beta/dr$, $\beta=\frac{2\mu_0P}{B^2}$ set $\alpha=0$ if $\beta=0$, or assume $\bar{g}^{rr}=1$ without toroidal effect, θ independent. Specially, in low beta limit,

$$\bar{g}^{rr}=1+\frac{1}{2}\varepsilon\cos\theta \tag{35}$$

$$\bar{g}^{\theta\theta}=\frac{1}{x^{2}}\left[1-2\left(\varepsilon+\Delta'\right)\cos\theta\right] \tag{36}$$

assume $\bar{g}^{\theta\theta}=\frac{1}{x^2}$ without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta}=\frac{1}{x^{2}}\left[1-\frac{5}{2}\varepsilon\cos\theta\right] \tag{37}$$

$$\bar{g}^{r\theta}=-\frac{1}{x}\left[\varepsilon+\left(r\Delta'\right)'\right]\sin\theta \tag{38}$$

specially,

$$\bar{g}^{r\theta}=-\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{39}$$

for low beta limit. and $\bar{g}^{r\theta}=0$ without toroidal effect.

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta + \frac{a}{R}\frac{\varepsilon}{4} - \frac{a}{R}\frac{5}{4}\varepsilon\left(\cos 2\theta - 1\right) - \left(\frac{a}{R}\right)^2\frac{x}{q} \tag{40}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta + \frac{5}{4}\varepsilon^2\sin 2\theta \tag{41}$$

with $R = R_0 + r\cos\theta - \Delta(r) + r\eta(r)(\cos 2\theta - 1).\eta(r) = (\varepsilon + \Delta')/2$. The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}} \tag{42}$$

where, m is poloidal mode number, $\rho_h = v_h/\omega_c$, $v_h = \sqrt{2T_h/M}, \omega_c = Be/M$. The normalized ρ_d is

$$\bar{\rho}_d = \frac{\rho_d}{a} = \frac{q}{2}\frac{\rho_h}{a}\sqrt{\frac{\bar{\epsilon}}{(1-\Lambda/b)}}\left[\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right] \tag{43}$$

The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\exp(-i\theta)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The normalized $\delta\bar{W}_k$ is given by

$$\delta\bar{W}_k = \int\frac{\bar{J}}{q}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\bar{\tau}_b\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\sum_{-\infty}^{\infty}\frac{\left|\bar{Y}_p\right|^2}{n\left\langle\overline{\dot{\phi}}\right\rangle + p\bar{\omega}_b - \bar{\omega}}, \tag{44}$$

where $\bar{J} = x$.

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{45}$$

where, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. and

$$\overline{\left\langle \dot{\phi} \right\rangle} \cong q\bar{\omega}_b \tag{46}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\, b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{47}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{48}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[x\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$ and normalized drift orbit width is $\bar{\rho}_d = \frac{\rho_d}{a}$.

The normalized slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$\bar{F}\left(x, \bar{\epsilon}, \Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2} + \bar{\epsilon}_c^{3/2}} Erfc\left(\frac{\bar{\epsilon} - \bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right) \exp\left[-\left(\frac{x - x_0}{\Delta x}\right)^2\right] \exp\left[-\left(\frac{\Lambda - \Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{49}$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon\cos\theta \tag{50}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - \frac{5}{2}\varepsilon\cos\theta\right] \tag{51}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{52}$$

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta \tag{53}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta \tag{54}$$

The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \tag{55}$$

with m being poloidal mode number.
The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\exp(-i\theta)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + (\bar{r}_s - \overline{\Delta r}/2)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

2 The fishbone dispersion relation

The formula for δW_{MHD}, δI [Miyamoto, "Plasma Physics and Controlled Nuclear Fusion"] The energy principle is

$$\delta W_{MHD} + \delta W_k + \delta I = 0 \tag{56}$$

where

$$\delta I = \frac{\gamma^2}{2} \int \rho_m \left| \vec{\xi} \right|^2 d\vec{r} \tag{57}$$

$$\delta W_k = \frac{1}{2} \int \vec{\xi} \cdot \nabla \delta p_h d\vec{r} \tag{58}$$

Note that δW_k is half of δW_k given by Eq.(24). δW_{MHD} consists of the contribution δW^s_{MHD} from the singular region near the rational surface and the contribution δW^{ext}_{MHD} from the external region.

The MHD potential energy $\delta W^{ext}_{MHDtor}/2\pi R$ of toroidal plasma with circular cross-section is given by

$$\frac{\delta W^{ext}_{MHDtor}}{2\pi R} = \left(1 - \frac{1}{n^2}\right) \frac{\delta W^{ext}_{MHDcycl}}{2\pi R} + \frac{\pi B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \delta \hat{W}_T \tag{59}$$

$$\delta \hat{W}_T = \pi \left(\frac{r_s}{R}\right)^2 3\left(1 - q_0\right) \left(\frac{13}{144} - \beta^2_{ps}\right) \tag{60}$$

The term δW^s_{MHD} for the singular region is

$$\frac{\delta W^s_{MHD}}{2\pi R} = \frac{\pi}{2\mu_0} \frac{B^2_{\theta s}}{2\pi} sn\gamma\tau_{A\theta} \left|\xi_s\right|^2$$

where $B_{\theta s} = \frac{r_s B_t}{R q_s}$, $\tau_{A\theta} = \frac{3^{1/2} r_s}{\left(B^2_{\theta s}/\mu_0 \rho_m\right)^{1/2}}$, $\rho_m = m_p n_{p_{r=r_s}}$, $s = r_s \frac{dq}{dr}_{r=r_s}$, $\xi_s = \xi_{r_{r=r_s}}$, n is toroidal mode number.

Thus, for $m = 1, n = 1$, the total sum of MHD contributions are

$$\begin{aligned} \delta W_{MHD} + \delta I &= 2\pi R \frac{B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \left(\delta \hat{W}_T + \gamma\tau_{A\theta} \frac{s}{2} + \pi\gamma^2\tau^2_{A\theta}\right) && (61) \\ &\approx 2\pi R \frac{B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \left(\delta \hat{W}_T + \gamma\tau_{A\theta} \frac{s}{2}\right), && (62) \end{aligned}$$

when $\gamma\tau_{A\theta} \ll 1$. With $\gamma = -i\omega$, the dispersion relation is

$$2\pi R \frac{B^2_{\theta s}}{2\mu_0} \left|\xi_s\right|^2 \left(\delta \hat{W}_T + \frac{-i\omega}{\omega_A}\right) + \frac{1}{2}\pi^2 a^2 R_0 n_0 T_h \delta \bar{W}_k = 0. \tag{63}$$

where, $\omega_A \equiv (\tau_{A\theta} s/2)^{-1}$ and $\delta \bar{W}_k$ is given by Eq.(44).
Finally,

$$-\frac{i\bar{\omega}}{\bar{\omega}_A} + \delta \hat{W}_T + \delta \hat{W}_k = 0, \tag{64}$$

with

$$\delta \hat{W}_k = \frac{1}{4}\pi \frac{1}{(r_s/Rq_s)^2} \frac{1}{|\xi_s/a|^2} \beta_{h0} \delta \bar{W}_k$$

where $\beta_{h0} = 2\mu_0 n_0 T_h / B_t^2$, n_p is plasma density at $r = r_s$. n_0 is energetic particle density at $x = x_0$. m_p, M is ion mass, and energetic particle mass respectively. And $\bar{\omega} = \omega / (v_h/R_0)$, $\bar{\omega}_A = \omega_A / (v_h/R_0)$.

3 Analytic form of the dispersion relation with passing particles and large aspect ratio approximation

For $\varepsilon \ll 1$, the normalized metric tensors are approximated as

$$\bar{g}^{rr} \approx 1 \tag{65}$$

$$\bar{g}^{\theta\theta} \approx \frac{1}{x^2} \tag{66}$$

$$\bar{g}^{r\theta} \approx -\frac{1}{x}\frac{3}{2}\varepsilon \sin\theta \tag{67}$$

and the normalized curvature are in low beta limit

$$\bar{\kappa}_r \approx -\frac{a}{R}\cos\theta \tag{68}$$

$$\bar{\kappa}_\theta \approx \varepsilon \sin\theta \tag{69}$$

The formula of ξ_θ and ξ_r are given by

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0(x) \exp(-i\theta), \tag{70}$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0(x)\, x \exp(-i\theta) \tag{71}$$

where, $\bar{\xi}_0(x) = \bar{\xi}_s H(x_s - x)$. $H(x)$ is Heaviside step function, $H = 1$ for $x > 0$ and $H = 0$ for $x < 0$, $dH/dx = \delta(x)$. Together with $\Lambda \ll 1$, one obtains

$$\tilde{G}[r(\tau), \theta(\tau)] = \left(\frac{\Lambda}{b(r,\theta)} + 2\left(1 - \frac{\Lambda}{b(r,\theta)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\right)$$

$$\approx 2\left(\frac{1}{x^2}\varepsilon\sin\theta\bar{\xi}_\theta - \frac{a}{R}\cos\theta\bar{\xi}_r\right)$$

$$\approx 2\left(-\frac{a}{R}i\sin\theta\bar{\xi}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right) - \frac{a}{R}\cos\theta\bar{\xi}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\right)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\left(i\sin\theta + \cos\theta\right)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right) \tag{72}$$

For $\kappa \gg 1$, the ellipitic fuction K becomes $K\left(\kappa^{-1}\right) = \pi/2$. Thus,

$$\frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \approx \frac{\pi\sqrt{\frac{1}{2\varepsilon\Lambda}}}{\pi/2}\frac{\sqrt{\frac{\varepsilon\Lambda}{2}}}{b\sqrt{1-\frac{\Lambda}{b}}}$$

$$= \frac{1}{b\sqrt{1-\Lambda/b}} \tag{73}$$

Using Eq.(72) and Eq.(73), Y_p is rewritten as

$$Y_p = -\frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}2\frac{a}{R}\bar{\xi}_r\left(\theta, x + \bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\int_0^\theta d\theta'\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

$$= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_r\left(\theta, x + \bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right)$$

$$= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s - x - \bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right) \tag{74}$$

The distribution function of passing particles for analytical purpose is given by

$$\bar{F}\left(x, \bar{\epsilon}, \Lambda\right) = \frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right), \tag{75}$$

and its derivative of $\bar{\epsilon}$ is read as

$$\frac{\partial\bar{F}}{\partial\bar{\epsilon}} = \frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\left[-\frac{3}{2}\bar{\epsilon}^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) - \bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'\left(\Lambda\right)H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) - \bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right)\right].$$

$$\delta\bar{W}_k = I_1 + I_2 \tag{76}$$

where I_1 represents the term with $\bar{\omega}$ and I_2 the term with $\bar{\omega}_\star$. Then I_1 decomposes three parts corresponding to $\partial\bar{F}/\partial\bar{\epsilon}$.

$$I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)} \tag{77}$$

3.1 The dispersion relation for the case of $\rho_d = 0$, $p = 0$

For $\bar{\rho}_d = 0$ and $p = 0$, we have

$$\begin{aligned}
Y_0 &= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s - x\right)\exp\left(-i\theta\right)\exp\left(i\theta\right) && (78)\\
&= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s - x\right) && (79)\\
&= -2\frac{a}{R}\bar{\xi}_s H\left(x_s - x\right) && (80)
\end{aligned}$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 0$, $\delta\bar{W}_k$ of Eq.(44) becomes

$$\delta\bar{W}_k = \int_0^{x_s} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right) \cdot\frac{\left|\bar{Y}_0\right|^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}, \tag{81}$$

here Y_0 is zero in the region $x > x_s$ where $H = 0$. According to Y_0 and $\bar{J}$ as shown above, $\delta\bar{W}_k$ becomes

$$\delta\bar{W}_k = \int_0^{x_s} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right) \cdot\frac{\left(-2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}, \tag{82}$$

For $F_h\left(x, \epsilon, \Lambda\right) = c_0\left(x\right)\frac{1}{\epsilon^{3/2}}\delta\left(\Lambda\right)H\left(\epsilon_0 - \epsilon\right)$, we can get

$$
\begin{aligned}
p_h(x) &= \int d^3vM\left(v_\parallel^2+\frac{1}{2}v_\perp^2\right)F_h \\
&\approx \int d^3vMv_\parallel^2F_h \\
&= M\int\sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\Lambda\epsilon^{1/2}d\epsilon 2\epsilon F_h \\
&= M\int 2^{\frac{3}{2}}\pi\epsilon^{\frac{3}{2}}\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}c_0(x)\frac{1}{\epsilon^{3/2}}\delta(\Lambda)H(\epsilon_0-\epsilon)\,d\Lambda d\epsilon \\
&= \pi 2^{\frac{3}{2}}Mc_0(x)\,\epsilon_0
\end{aligned}
$$

Thus, $F_h=\left[p_h(x)/\left(\pi M2^{\frac{3}{2}}\epsilon_0\right)\right]\frac{1}{\epsilon^{3/2}}\delta(\Lambda)H(\epsilon_0-\epsilon)$ with $c_0=p_h(x)/\left(\pi M2^{\frac{3}{2}}\epsilon_0\right)$. Furthermore, $\bar{F}_h=\frac{p_h(x)}{\pi n_0T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{\frac{3}{2}}}H(\bar{\epsilon}_0-\bar{\epsilon})\,\delta(\Lambda)$.

$$
\frac{\partial\bar{F}}{\partial\bar{\epsilon}}=\frac{p_h(x)}{\pi n_0T_h\bar{\epsilon}_0}\left[-\frac{3}{2}\bar{\epsilon}^{-\frac{5}{2}}\delta(\Lambda)\,H(\bar{\epsilon}_0-\bar{\epsilon})-\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'(\Lambda)\,H(\bar{\epsilon}_0-\bar{\epsilon})-\bar{\epsilon}^{-\frac{3}{2}}\delta(\Lambda)\,\delta(\bar{\epsilon}_0-\bar{\epsilon})\right].
$$

$$
I_1^{(1)}=\int_0^{x_s}xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\frac{p_h(x)}{\pi n_0T_h\bar{\epsilon}_0}\left(-\frac{3}{2}\right)\bar{\epsilon}^{-\frac{5}{2}}\delta(\Lambda)\,H(\bar{\epsilon}_0-\bar{\epsilon})\,\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}
$$

$$
\cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}
$$

$$
=-6\pi\frac{\bar{\omega}}{\pi n_0T_h\bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s}xp_h(x)\,dx\int_0^{\bar{\epsilon}_0}\frac{\sqrt{\bar{\epsilon}}d\sqrt{\bar{\epsilon}}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}
$$

$$
=-6\pi\frac{\bar{\omega}}{\pi n_0T_h\bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s}xp_h(x)\,dx\left(1+\Omega\ln\left(1-\frac{1}{\Omega}\right)\right)\sqrt{\bar{\epsilon}_0}
$$

$$
=-6\pi\frac{1}{\pi n_0T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\left(\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right)\int_0^{x_s}xp_h(x)\,dx
$$

$$
I_1^{(2)}=\int_0^{x_s}xdx\int d\Lambda\bar{\epsilon}^3d\bar{\epsilon}\frac{p_h(x)}{\pi n_0T_h\bar{\epsilon}_0}(-1)\,\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'(\Lambda)\,H(\bar{\epsilon}_0-\bar{\epsilon})\,\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}
$$

$$
=\int_0^{x_s}xdx\int d\Lambda d\bar{\epsilon}\frac{p_h(x)}{\pi n_0T_h\bar{\epsilon}_0}(-1)\,\Lambda\delta'(\Lambda)\,H(\bar{\epsilon}_0-\bar{\epsilon})\,2\pi\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}
$$

$$= 4\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\bar{\omega}\cdot\int_0^{x_s} x p_h\left(x\right)dx\int_0^{\bar{\epsilon}_0}\frac{\sqrt{\bar{\epsilon}}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}d\sqrt{\bar{\epsilon}}$$

$$= 4\pi\frac{1}{\pi n_0 T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\left(\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right)\int_0^{x_s} x p_h\left(x\right)dx$$

$$I_1^{(3)} = \int_0^{x_s} x dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\left(-1\right)\bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}$$

$$\cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$I_1^{(3)} = -2\pi\frac{1}{\pi n_0 T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{\Omega}{1-\Omega}\int_0^{x_s} x p_h\left(x\right)dx$$

Thus,

$$I_1 = -2\pi\frac{1}{\pi n_0 T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]\int_0^{x_s} x p_h\left(x\right)dx$$

$$\frac{\partial\bar{F}_h}{\partial x} = \frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{dp_h\left(x\right)}{dx}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)$$

$$I_2 = -\int_0^{x_s}\bar{J}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star\cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_0^{x_s} x dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial\bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\cdot\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_0^{x_s} x dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$\cdot\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{dp_h\left(x\right)}{dx}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)$$

$$= -\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}2\pi\frac{1}{2}\frac{R}{a}\frac{\rho_h}{a}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s}\frac{dp_h\left(x\right)}{dx}dx\int_0^{\bar{\epsilon}_0}\frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}d\bar{\epsilon}$$

$$= -\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}2\pi\frac{R}{a}\frac{\rho_h}{a}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s}\frac{dp_h\left(x\right)}{dx}dx\int_0^{\bar{\epsilon}_0}\frac{\left(\sqrt{\bar{\epsilon}}\right)^3}{\sqrt{\bar{\epsilon}}-\bar{\omega}}d\sqrt{\bar{\epsilon}}$$

$$\begin{aligned} = \quad & -2\pi\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s}\frac{dp_h\left(x\right)}{dx}dx \\ & \cdot\left(\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right)\left(\sqrt{\bar{\epsilon}_0}\right)^3 \end{aligned}$$

$$
\begin{aligned}
\delta\bar{W}_k &= I_1 + I_2 \\
&= -2\pi\frac{1}{\pi n_0 T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]\int_0^{x_s} x p_h(x)\,dx \\
&\quad -2\pi\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_0^{x_s}\frac{dp_h(x)}{dx}dx \\
&\quad \cdot\left(\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right)\left(\sqrt{\bar{\epsilon}_0}\right)^3
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\delta\hat{W}_k &= \frac{1}{4}\pi\frac{1}{\left(r_s/Rq_s\right)^2}\frac{1}{\left|\xi_s/a\right|^2}\beta_{h0}\delta\bar{W}_k \\
&= \begin{array}{c}
-2\pi\frac{1}{\left(r_s/aq_s\right)^2}\frac{2\mu_0\int_0^{x_s}xp_h(x)\,dx}{B_t^2}\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right] \\
-2\pi\frac{1}{\Omega_c\left(r_s/Rq_s\right)^2}\frac{2\mu_0\int_0^{x_s}\frac{dp_h(x)}{dx}dx}{B_t^2}\left[\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]
\end{array}
\end{aligned}
\tag{84}
$$

where $\Omega_c = \omega_c/\left[\left(v_h/R_0\right)\sqrt{\bar{\epsilon}_0}\right]$.
Moreover,

$$
\delta\hat{W}_k = \begin{array}{c}
-\pi\frac{2\mu_0\langle p_h\rangle}{B_t^2}\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]+2\pi\frac{1}{\Omega_c\left(r_s/Rq_s\right)^2} \\
\cdot\frac{2\mu_0\left(p_h(0)-p_h(x_s)\right)}{B_t^2}\left[\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]
\end{array}
\tag{85}
$$

where $\langle p\rangle = \frac{2}{x_s^2}\int_0^{x_s}xp_h(x)\,dx$ is volume averaged pressure for $q_s = 1$, Then

$$
\delta\hat{W}_k = \begin{array}{c}
-\pi\left\langle\beta_h\right\rangle\left[\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right]+\pi\frac{1}{\Omega_c\left(r_s/Rq_s\right)^2} \\
\cdot 2\left(\beta_h(0)-\beta_h(x_s)\right)\left[\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]
\end{array}
\tag{86}
$$

The dispersion relation for $p = 0$ is

$$
-\frac{i\Omega}{\Omega_A}+\delta\hat{W}_T+\delta\hat{W}_k = 0,
\tag{87}
$$

where $\Omega_A = \bar{\omega}_A/\sqrt{\bar{\epsilon}_0}$.

3.2 The dispersion relation for the case of $\rho_d \neq 0$, $p = 0$

For $\bar{\rho}_d \neq 0$, $p = 0$ and using Eq.(74), we have

$$\begin{array}{rll} x < x_s - \bar{\rho}_d & Y_0 = -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s \int_0^{2\pi} d\theta = -\frac{2a}{R}\bar{\xi}_s \triangleq Y_0' \\ x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d & Y_0 = -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s \int_{\theta^\star}^{2\pi-\theta^\star} d\theta = -\frac{2\pi - 2\theta^\star}{\pi}\frac{a}{R}\bar{\xi}_s \triangleq Y_0'' \\ x > x_s + \bar{\rho}_d & Y_0 = 0 \end{array}$$

where $\cos\theta^\star = \frac{x_s - x}{\bar{\rho}_d} \in [0, \pi]$.

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 0$, $\delta\bar{W}_k$ of Eq.(44) becomes

$$\delta\bar{W}_k = \delta\bar{W}_k' + \delta\bar{W}_k'' \tag{88}$$

with

$$\delta\bar{W}_k' = \int_0^{x_s - \bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)\frac{|Y_0'|^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

and

$$\delta\bar{W}_k'' = \int_{x_s - \bar{\rho}_d}^{x_s + \bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)\frac{|Y_0''|^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

Due to $\int_0^{x_s} = \int_0^{x_s - \bar{\rho}_d} + \int_{x_s - \bar{\rho}_d}^{x_s}$, one gets

$$\delta\bar{W}_k' = \delta\bar{W}_k^0 - \delta\bar{W}_k'^s \tag{89}$$

where $\delta\bar{W}_k^0$ is given by the above Eq. (83) , and

$$\delta\bar{W}_k'^s = \int_{x_s - \bar{\rho}_d}^{x_s} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)\frac{|Y_0'|^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}} \tag{90}$$

$$\begin{aligned} = \quad & -2\pi\frac{1}{\pi n_0 T_h}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\left[\frac{\Omega}{1-\Omega} + \Omega + \Omega^2\ln\left(1 - \frac{1}{\Omega}\right)\right]\int_{x_s - \bar{\rho}_d}^{x_s} x p_h(x)\,dx \\ & -2\pi\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\left(2\frac{a}{R}\bar{\xi}_s\right)^2\int_{x_s - \bar{\rho}_d}^{x_s}\frac{dp_h(x)}{dx}dx \\ & \cdot\left(\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3\ln\left(1 - \frac{1}{\Omega}\right)\right)\left(\sqrt{\bar{\epsilon}_0}\right)^3 \end{aligned} \tag{91}$$

For simplicity, $\delta\bar{W}_k'$ may be approximate to $\delta\bar{W}_k^0$ as $\bar{\rho}_d \ll x_s$. We let $\delta\bar{W}_k'' = I_1 + I_2$ and $I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)}$. Thus, for $\delta\bar{W}_k''$, one obtains

$$I_1^{(1)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon} \frac{p_h(x)}{\pi n_0 T_h \bar{\epsilon}_0}\left(-\frac{3}{2}\right)\bar{\epsilon}^{-\frac{5}{2}}\delta(\Lambda) H(\bar{\epsilon}_0-\bar{\epsilon})\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -3\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\int_0^{\bar{\epsilon}_0}\frac{\bar{\omega} d\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xp_h(x)\left[1-\frac{\theta^\star}{\pi}\right]^2 dx$$

for the normal profile of p_h, the identity $xp_h(x) = \frac{\delta_x}{2}\frac{p_h}{r_p}$ should be satisfied, where $r_p = -\left[dp_h/pdx\right]^{-1}, \delta_x$ is profile width. and $\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2 dx = \frac{\pi-4}{\pi^2}\bar{\rho}_d$. With $\bar{\rho}_d \simeq q\sqrt{\bar{\epsilon}}\rho_h/a$, it yields

$$I_1^{(1)} \approx -6\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\frac{\rho_h}{a}\bar{\omega}\int_0^{\bar{\epsilon}_0} d\sqrt{\bar{\epsilon}}\frac{\left(\sqrt{\bar{\epsilon}}\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$\approx -6\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\frac{\rho_h}{a}\sqrt{\bar{\epsilon}_0} \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]$$

$$\approx -6\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}(\pi-4)\Delta_b \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]$$

where $\Delta_b = \frac{q_s\sqrt{\bar{\epsilon}_0}\rho_h}{a}$ is orbit width.

$$I_1^{(2)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}(-1)\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'(\Lambda)H(\bar{\epsilon}_c-\bar{\epsilon})\frac{p_h(x)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= 2\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\int_0^{\epsilon_0}\frac{d\bar{\epsilon}\bar{\omega}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xp_h(x)\left[1-\frac{\theta^\star}{\pi}\right]^2 dx$$

$$\approx \begin{array}{c} 4\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}(\pi-4)\Delta_b \\ \cdot\left[\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right]\end{array} \tag{92}$$

$$I_1^{(3)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\,(-1)\,\bar{\epsilon}^{-\frac{3}{2}}\delta(\Lambda)\,\delta(\bar{\epsilon}_0-\bar{\epsilon})\,\frac{p_h(x)}{\pi n_0 T_h \bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -2\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\bar{\omega}\int\frac{\bar{\epsilon}d\bar{\epsilon}\delta(\bar{\epsilon}_0-\bar{\epsilon})}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xp_h(x)\left[1-\frac{\theta^\star}{\pi}\right]^2 dx$$

$$= -2\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\bar{\omega}\int\frac{\bar{\epsilon}d\bar{\epsilon}\delta(\bar{\epsilon}_0-\bar{\epsilon})}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\bar{\rho}_d$$

$$= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}(\pi-4)\,\Delta_b\frac{\Omega}{1-\Omega}$$

Since

$$\frac{\partial\bar{F}(x,\bar{\epsilon},\Lambda)}{\partial x} = \frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta(\Lambda)H(\bar{\epsilon}_0-\bar{\epsilon})\frac{dp_h(x)}{dx}, \tag{93}$$

$$I_2 = -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\bar{J}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial\bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot\delta(\Lambda)\,H(\bar{\epsilon}_0-\bar{\epsilon})\frac{dp_h(x)}{dx}\cdot\frac{\left(2\frac{a}{R}\frac{\pi-\theta^\star}{\pi}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -\pi\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\int_0^{\bar{\epsilon}_0}\frac{\bar{\epsilon}d\bar{\epsilon}}{\sqrt{\bar{\epsilon}}-\bar{\omega}}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2\frac{dp_h(x)}{dx}dx$$

Due to $r_p = -\left[dp_h/pdx\right]^{-1}$, the above intergal of x yields

$$\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2\frac{dp_h(x)}{dx}dx$$

$$= -\left(\frac{p_h}{r_p}\right)_{x_s}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\frac{\theta^\star}{\pi}\right]^2 dx$$

$$= -\left(\frac{p_h}{r_p}\right)_{x_s}\frac{\pi-4}{\pi^2}\bar{\rho}_d$$

$\Rightarrow$

$$= \pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h \bar{\epsilon}_0} \cdot \left(\frac{q p_h}{r_p}\right)_{x_s} \frac{\rho_h}{a}\left(\pi - 4\right) \int_0^{\bar{\epsilon}_0} \frac{\bar{\epsilon}^{3/2} d\bar{\epsilon}}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$\begin{aligned} = \quad & 2\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h} \cdot \left(\frac{p_h}{r_p}\right)_{x_s} \sqrt{\bar{\epsilon}_0}\left(\pi - 4\right)\Delta_b \\ & \cdot \left[\frac{1}{4} + \frac{1}{3}\Omega + \frac{1}{2}\Omega^2 + \Omega^3 + \Omega^4 \ln\left(1 - \frac{1}{\Omega}\right)\right] \end{aligned}$$

Therefore,

$$\delta \bar{W}_k'' = \begin{array}{c} -2\pi\left(\pi - 4\right)\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{\pi n_0 T_h}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s} \Delta_b \\ \cdot \left[\frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right) + \frac{\Omega}{1 - \Omega}\right] \\ +2\pi\left(\pi - 4\right)\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h} \cdot \left(\frac{p_h}{r_p}\right)_{x_s} \sqrt{\bar{\epsilon}_0}\Delta_b \\ \cdot \left[\frac{1}{4} + \frac{1}{3}\Omega + \frac{1}{2}\Omega^2 + \Omega^3 + \Omega^4 \ln\left(1 - \frac{1}{\Omega}\right)\right] \end{array} \tag{94}$$

Using

$$\delta \hat{W}_k = \frac{1}{4}\pi \frac{1}{\left(r_s/Rq_s\right)^2}\frac{1}{\left|\xi_s/a\right|^2}\beta_{h0}\delta \bar{W}_k \tag{95}$$

one obtains

$$\delta \hat{W}_k'' = \begin{array}{l} -\frac{1}{\left(x_s/q_s\right)^2}\left(2 - \frac{8}{\pi}\right)\frac{\delta_x}{2}\left(\frac{\beta_h}{r_p}\right)_{x_s} \Delta_b \left[\frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right) + \frac{\Omega}{1 - \Omega}\right] \\ +\frac{1}{x_s^2}\left(2 - \frac{8}{\pi}\right)\frac{R}{a}\left(\frac{q\beta_h}{r_p}\right)_{x_s} \Delta_b^2 \left[\frac{1}{4} + \frac{1}{3}\Omega + \frac{1}{2}\Omega^2 + \Omega^3 + \Omega^4 \ln\left(1 - \frac{1}{\Omega}\right)\right] \end{array}$$

and the dispersion relation is

$$-\frac{i\Omega}{\Omega_A} + \delta \hat{W}_T + \delta \hat{W}_k = 0 \tag{96}$$

where $\Omega_A = \bar{\omega}_A/\sqrt{\bar{\epsilon}_0}$, $\delta \hat{W}_k = \delta \hat{W_k^0} + \delta \hat{W}_k''$.

3.3 The dispersion relation for the case of $p = 1$, $\rho_d \neq 0$

For $p = 1$, we arrive at

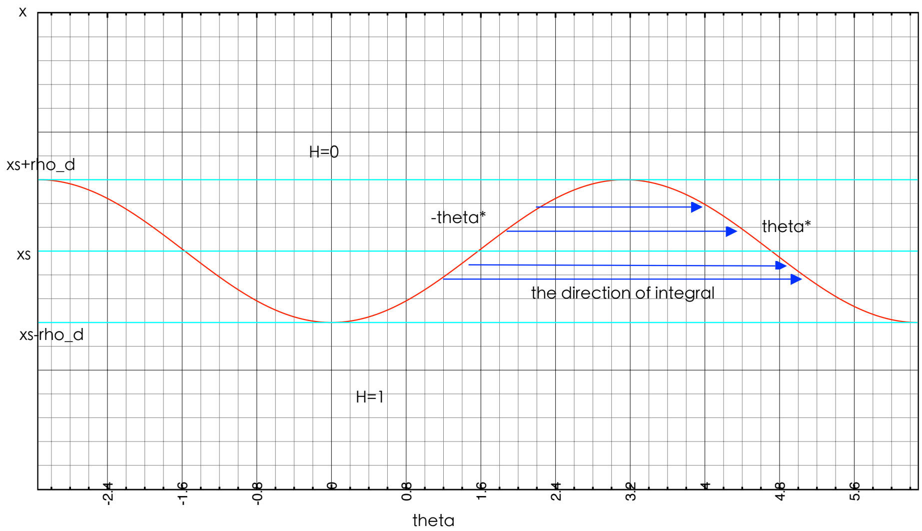

Figure 1: The integral region

$$Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s - \bar{x} - \bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right) \tag{97}$$

From Fig.1, Y_1 is zero in the region $x < x_s - \bar{\rho}_d$ where $H = 1$ for $\theta \in [0, 2\pi]$ since the θ integral of $\sim \exp\left(-i\theta\right)$ from 0 to 2π is zero. In the region $x > x_s + \bar{\rho}_d$, Y_1 also is zero since $H = 0$. Obviously, Y_1 is finite in the region $x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d$ where $H = 1$ for $\theta \in \left[\theta^\star, -\theta^\star + 2\pi\right]$. $\cos\theta^\star = \frac{x_s - x}{\bar{\rho}_d} \in [0, \pi]$. Thus,

$$Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_{\theta^\star}^{2\pi-\theta^\star} d\theta\bar{\xi}_s\exp\left(-i\theta\right) \tag{98}$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(-i\left(2\pi - \theta^\star\right)\right) - \exp\left(-i\theta^\star\right)\right]$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(-i\left(2\pi - \theta^\star\right)\right) - \exp\left(-i\theta^\star\right)\right]$$

$$= \frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\sin\theta^\star$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 1$, $\delta\bar{W}_k$ becomes

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\bar{J}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot \frac{|Y_1|^2}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q}-\bar\omega}. \tag{99}$$

$$\delta \bar W_k = \int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} x dx \int d\Lambda \int_0^{\bar\epsilon_0} \bar\epsilon^3 d\bar\epsilon \frac{\partial \bar F}{\partial \bar\epsilon} \frac{2\pi}{\sqrt{\bar\epsilon}} \left(\bar\omega - \bar\omega_\star\right) \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega} \tag{100}$$

$$\delta \bar W_k = \int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} x dx \int d\Lambda \int_0^{\bar\epsilon_0} \bar\epsilon^3 d\bar\epsilon \frac{\partial \bar F}{\partial \bar\epsilon} \frac{2\pi}{\sqrt{\bar\epsilon}} \left(\bar\omega - \bar\omega_\star\right) \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega} \left(1-\left(\frac{x_s-x}{\bar\rho_d}\right)^2\right) \tag{101}$$

with $\bar J = x$. The distribution function of passing particles for analytical purpose is given by

$$\bar F\left(x,\bar\epsilon,\Lambda\right) = \frac{p_h\left(x\right)}{\pi n_0 T_h \bar\epsilon_0} \frac{1}{\bar\epsilon^{3/2}} \delta\left(\Lambda\right) H\left(\bar\epsilon_0-\bar\epsilon\right), \tag{102}$$

and its derivative of $\bar\epsilon$ is read as

$$\frac{\partial \bar F}{\partial \bar\epsilon} = \frac{p_h\left(x\right)}{\pi n_0 T_h \bar\epsilon_0}\left[-\frac{3}{2}\bar\epsilon^{-\frac{5}{2}}\delta\left(\Lambda\right)H\left(\bar\epsilon_0-\bar\epsilon\right) - \bar\epsilon^{-\frac{5}{2}}\Lambda\delta'\left(\Lambda\right)H\left(\bar\epsilon_0-\bar\epsilon\right) - \bar\epsilon^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar\epsilon_0-\bar\epsilon\right)\right].$$

$$\delta \bar W_k = I_1 + I_2 \tag{103}$$

where I_1 represents the term with $\bar\omega$ and I_2 the term with $\bar\omega_\star$. Then I_1 decomposes three parts corresponding to $\partial \bar F/\partial \bar\epsilon$.

$$I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)} \tag{104}$$

$$I_1^{(1)} = \int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} x dx \int d\Lambda \bar\epsilon^3 d\bar\epsilon \left(-\frac{3}{2}\right) \bar\epsilon^{-\frac{5}{2}} \delta\left(\Lambda\right) H\left(\bar\epsilon_0-\bar\epsilon\right) \frac{p_h\left(x\right)}{\pi n_0 T_h \bar\epsilon_0} \frac{2\pi}{\sqrt{\bar\epsilon}} \bar\omega$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}\left[1-\left(\frac{x_s-x}{\bar\rho_d}\right)^2\right]$$

$$= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \frac{1}{\pi n_0 T_h \bar\epsilon_0} \int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} x p_h\left(x\right)\left[1-\left(\frac{x_s-x}{\bar\rho_d}\right)^2\right] dx \int_0^{\bar\epsilon_0} d\bar\epsilon \bar\omega$$

$$\cdot \frac{1}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}$$

for the normal profile of p_h, the identity $xp_h(x) = \frac{\delta_x}{2}\frac{p_h}{r_p}$ should be satisfied, where $r_p = -[dp_h/pdx]^{-1}$, δ_x is profile width. and $\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx = \frac{4}{3}\bar{\rho}_d$.

$\Rrightarrow$

$$
\begin{aligned}
&= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s} \\
&\cdot\bar{\omega}\int_0^{\bar{\epsilon}_0}2\sqrt{\bar{\epsilon}}d\sqrt{\bar{\epsilon}}\frac{1}{\left(1+\frac{1}{q_s}\right)\sqrt{\bar{\epsilon}}-\bar{\omega}}\frac{4}{3}\bar{\rho}_d
\end{aligned}
$$

With $\bar{\rho}_d \simeq q\sqrt{\bar{\epsilon}}\rho_h/a$, we obtain

$$
\begin{aligned}
I_1^{(1)} &\approx -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{8}{3}\frac{\rho_h}{a}\frac{\bar{\omega}}{1+\frac{1}{q_s}} \\
&\cdot\int_0^{\bar{\epsilon}_0}d\sqrt{\bar{\epsilon}}\frac{\left(\sqrt{\bar{\epsilon}}\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}/\left(1+\frac{1}{q_s}\right)}
\end{aligned}
$$

The intergal identity is given by

$$\int dy\frac{y^2}{y-a} = \int dy\frac{y^2-a^2+a^2}{y-a} = \int dy\,(y+a) + \int dy\frac{a^2}{y-a} \tag{105}$$

$$= \frac{1}{2}y^2 + ay + a^2\ln(y-a) \tag{106}$$

$\Rrightarrow$

$$= -8\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\rho_h}{a}\frac{\bar{\omega}}{1+\frac{1}{q_s}}\left[\frac{1}{2}\left(\sqrt{\bar{\epsilon}_0}\right)^2+\frac{\bar{\omega}}{1+\frac{1}{q_s}}\sqrt{\bar{\epsilon}_0}+\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2\ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\sqrt{\bar{\epsilon}_0}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\right] \tag{107}$$

$$
\begin{aligned}
I_1^{(2)} &= \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\,(-1)\,\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'(\Lambda)\,H(\bar{\epsilon}_c-\bar{\epsilon})\frac{p_h(x)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega} \\
&\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right] \\
&= 2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}xp_h(x)\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx
\end{aligned}
$$

$$\cdot \int_0^{\epsilon_0} d\bar{\epsilon}\bar{\omega}\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$\approx \begin{array}{c} \frac{16}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{\rho_h}{a} \\ \cdot\frac{\bar{\omega}}{1+\frac{1}{q_s}}\left[\frac{1}{2}\left(\sqrt{\bar{\epsilon}_0}\right)^2+\frac{\bar{\omega}}{1+\frac{1}{q_s}}\sqrt{\bar{\epsilon}_0}+\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2\ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\sqrt{\bar{\epsilon}_0}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\right] \end{array} \tag{108}$$

$$I_1^{(3)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\left(-1\right)\bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{p_h\left(x\right)}{\pi n_0 T_h\bar{\epsilon}_0}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]$$

$$= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xp_h\left(x\right)\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx$$

$$\cdot\bar{\omega}\int d\bar{\epsilon}\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx$$

$$\cdot\bar{\omega}\int d\bar{\epsilon}\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{p_h}{r_p}\right)_{x_s}$$

$$\cdot\bar{\omega}\int d\bar{\epsilon}\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{\bar{\epsilon}}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\frac{4}{3}\bar{\rho}_d$$

$$= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{qp_h}{r_p}\right)_{x_s}\frac{4}{3}\frac{\rho_h}{a}\frac{\bar{\omega}\left(\sqrt{\bar{\epsilon}_0}\right)^3}{\sqrt{\bar{\epsilon}_0}+\frac{\sqrt{\bar{\epsilon}_0}}{q_s}-\bar{\omega}}$$

we have

$$\frac{\partial\bar{F}\left(x,\bar{\epsilon},\Lambda\right)}{\partial x} = \frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{dp_h\left(x\right)}{dx}, \tag{109}$$

$$I_2 = -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial \bar{F}}{\partial \bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial \bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot\delta\left(\Lambda\right) H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{dp_h\left(x\right)}{dx}\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h\left(x\right)}{dx}dx$$

$$\cdot\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

Due to $r_p = -\left[dp_h/pdx\right]^{-1}$, the above intergal of x yields

$$\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h\left(x\right)}{dx}dx$$

$$= -\left(\frac{p_h}{r_p}\right)_{x_s}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]dx$$

$$= -\frac{4}{3}\left(\frac{p_h}{r_p}\right)_{x_s}\bar{\rho}_d$$

$\Rightarrow$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\cdot\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left(-\frac{4}{3}\right)\left(\frac{p_h}{r_p}\right)_{x_s}\bar{\rho}_d$$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\left(\frac{\rho_h}{a}\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\left(-\frac{4}{3}\right)\left(\frac{q p_h}{r_p}\right)_{x_s} \cdot \int_0^{\bar{\epsilon}_0} \frac{2\left(\sqrt{\bar{\epsilon}}\right)^4}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}} d\sqrt{\bar{\epsilon}}$$

The intergal identity is given by

$$\int dy \frac{y^4}{y-a} = \int dy \frac{y^4 - a^4 + a^4}{y-a} = \int dy\,(y+a)\left(y^2+a^2\right) + \int dy \frac{a^4}{y-a} \tag{110}$$

$$= \frac{1}{4}y^4 + \frac{1}{3}ay^3 + \frac{1}{2}a^2y^2 + a^3 y + a^4 \ln(y-a) \tag{111}$$

$\Rrightarrow$

$$\begin{aligned}
= \quad & \frac{8}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\left(\frac{\rho_h}{a}\right)^2 \frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\left(\frac{q p_h}{r_p}\right)_{x_s} \frac{1}{1+\frac{1}{q_s}} \\
& \cdot \left[\begin{array}{c} \frac{1}{4}\bar{\epsilon}_0^2 + \frac{1}{3}\frac{\bar{\omega}}{1+\frac{1}{q_s}}\left(\sqrt{\bar{\epsilon}_0}\right)^3 + \frac{1}{2}\left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2\left(\sqrt{\bar{\epsilon}_0}\right)^2 \\ + \left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^3\left(\sqrt{\bar{\epsilon}_0}\right) + \left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^4 \ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}} - \bar{\epsilon}_0^{1/2}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right) \end{array}\right]
\end{aligned}$$

Defining $\Omega = \frac{\bar{\omega}}{(1+1/q_s)\bar{\epsilon}_0^{1/2}}$, $\delta\bar{W}_k$ can be rewritten by

$$\begin{aligned}
\delta\bar{W}_k \quad = \quad & I_1 + I_2 \\
& \frac{8}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\left(\frac{\rho_h}{a}\right)\frac{\left(\sqrt{\bar{\epsilon}_0}\right)^3}{\pi n_0 T_h \bar{\epsilon}_0}\frac{\delta_x}{2}\left(\frac{q p_h}{r_p}\right)_{x_s} \\
& \cdot\left[\frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1-\frac{1}{\Omega}\right) + \frac{1}{\Omega - 1}\right] \\
= \quad & \\
& + \frac{8}{3}\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\left(\frac{\rho_h}{a}\right)^2 \frac{\bar{\epsilon}_0^2}{\pi n_0 T_h \bar{\epsilon}_0}\left(\frac{q p_h}{r_p}\right)_{x_s} \frac{1}{1+\frac{1}{q_s}} \\
& \cdot\left[\frac{1}{4} + \frac{1}{3}\Omega + \frac{1}{2}\Omega^2 + \Omega^3 + \Omega^4 \ln\left(1-\frac{1}{\Omega}\right)\right]
\end{aligned}$$

$$\delta\hat{W}_k = \frac{1}{4}\pi\frac{1}{(r_s/Rq_s)^2}\frac{1}{|\xi_s/a|^2}\beta_{h0}\delta\bar{W}_k$$

$$= \begin{array}{c} \frac{8}{3\pi}\frac{1}{(r_s/Rq_s)^2}\left(\frac{q\beta_h}{r_p}\right)_{x_s}\frac{1}{\Omega_c}\frac{\delta_x}{2}\left[\frac{1}{\Omega-1}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right] \\ +\frac{8}{3\pi}\frac{1}{(r_s/Rq_s)^2}\frac{R}{a}\left(\frac{q\beta_h}{r_p}\right)_{x_s}\frac{1}{\Omega_c^2}\frac{1}{1+\frac{1}{q_s}}\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array}$$

$$= \begin{array}{c} \frac{8}{3\pi}\frac{1}{(r_s/Rq_s)^2}\frac{a}{R}\frac{\delta_x}{2}\left(\frac{\beta_h}{r_p}\right)_{x_s}\Delta_b\left[\frac{1}{\Omega-1}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right] \\ +\frac{8}{3\pi}\frac{1}{(r_s/Rq_s)^2}\frac{1}{\Omega_c}\frac{1}{1+\frac{1}{q_s}}\left(\frac{\beta_h}{r_p}\right)_{x_s}\Delta_b\left[\frac{1}{4}+\frac{1}{3}\Omega+\frac{1}{2}\Omega^2+\Omega^3+\Omega^4\ln\left(1-\frac{1}{\Omega}\right)\right] \end{array} \tag{112}$$

where, $\Delta_b = \frac{q_s\sqrt{\bar{\epsilon}_0}\rho_h}{a}$ is orbit width.
According to Eq. (112), the dispersion relation(64) thus can be written as

$$-\frac{i\Omega}{\Omega_A}+\delta\hat{W}_T+\delta\hat{W}_k = 0 \tag{113}$$

where $\Omega_A = \frac{\bar{\omega}_A}{(1+1/q_s)\bar{\epsilon}_0^{1/2}}$.

3.4 The dispersion relation for the case of $p=-1$, $\rho_d \neq 0$

For $p=-1$, we arrive at

$$Y_{-1} = -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-\bar{x}-\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right) \tag{114}$$

From Fig.1, Y_{-1} is zero in the region $x < x_s - \bar{\rho}_d$ where $H=1$ for $\theta \in [0, 2\pi]$ since the θ integral of $\sim \exp(-i\theta)$ from 0 to 2π is zero. In the region $x > x_s + \bar{\rho}_d$, Y_1 also is zero since $H=0$. Obviously, Y_{-1} is finite in the region $x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d$ where $H=1$ for $\theta \in [\theta^\star, -\theta^\star+2\pi]$. $\cos\theta^\star = \frac{x_s - x}{\bar{\rho}_d} \in [0,\pi]$. Thus,

$$Y_{-1} = -\frac{1}{\pi}\frac{a}{R}\int_{\theta^\star}^{2\pi-\theta^\star} d\theta\bar{\xi}_s\exp\left(i\theta\right) \tag{115}$$

$$= \frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(i\left(2\pi-\theta^\star\right)\right)-\exp\left(i\theta^\star\right)\right]$$

$$= \frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(i\left(2\pi-\theta^\star\right)\right)-\exp\left(i\theta^\star\right)\right]$$

$$= \frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\sin\theta^\star$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p=-1$, $\delta\bar{W}_k$ becomes

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_\star\right)$$

$$\cdot\frac{\left|Y_{-1}\right|^2}{\sqrt{\bar{\epsilon}}-\frac{\sqrt{\bar{\epsilon}}}{q}-\bar{\omega}}. \tag{116}$$

The fishbone of $\bar{\omega}\sim\omega_{\star i}$ is studied. Thus the term I_2 dominates over the term I_1 and $\left|\Re\left(\delta\hat{W}_k\right)\right| \ll \left|\Im\left(\delta\hat{W}_k\right)\right|$.

$$I_2 = -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}-\frac{\sqrt{\bar{\epsilon}}}{q}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial\bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}-\frac{\sqrt{\bar{\epsilon}}}{q}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot\delta\left(\Lambda\right)H\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\frac{dp_h\left(x\right)}{dx}\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}\left(x-x_s\right)-\bar{\omega}}$$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h\left(x\right)}{dx}dx$$

$$\cdot\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}\left(x-x_s\right)-\bar{\omega}}$$

since $q = q_s + dq/dx\left(x-x_s\right)$, $s=\frac{x}{q}\frac{dq}{dx}$.
$\Rightarrow$

$$= -\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon}$$

$$\cdot\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\frac{dp_h\left(x\right)}{dx}\frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}\left(x-x_s\right)-\bar{\omega}}dx$$

Due to $r_p = -\left[dp_h/pdx\right]^{-1}$, the imaginary part of the above intergal of x yields

$$\begin{aligned}
&\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right] \frac{dp_h\left(x\right)}{dx} \frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}\left(x-x_s\right)-\bar{\omega}}dx \\
&= -i\pi\left(\frac{p_h}{r_p}\right)_{x_s} \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right] \frac{1}{\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}\left(x-x_s\right)-\bar{\omega}}dx \\
&= -i\pi\left(\frac{p_h}{r_p}\right)_{x_s} \left[1-\left(\frac{\bar{\omega}x_s q_s}{\sqrt{\bar{\epsilon}}s\bar{\rho}_d}\right)^2\right]
\end{aligned}$$

by using $\sqrt{\bar{\epsilon}}\frac{s(x_s)}{q_s x_s}\left(x-x_s\right)-\bar{\omega}=0$. The imaginary part of I_2 with maximum drive by ignoring the term $\frac{\bar{\omega}x_s q_s}{\sqrt{\bar{\epsilon}}s\bar{\rho}_d}$ is

$$\begin{aligned}
I_2^{\Im} &= i\pi^2\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\left(\frac{p_h}{r_p}\right)_{x_s}\int_0^{\bar{\epsilon}_0}\bar{\epsilon}d\bar{\epsilon} \\
&= i\pi^2\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h \bar{\epsilon}_0}\left(\frac{p_h}{r_p}\right)_{x_s}\frac{1}{2}\bar{\epsilon}_0^2
\end{aligned}$$

Thus the imaginary part of $\delta\hat{W}_k$ is

$$\delta\hat{W}_{k2}^{\Im} = \frac{1}{4}\pi\frac{1}{\left(r_s/Rq_s\right)^2}\frac{1}{\left|\xi_s/a\right|^2}\beta_{h0}I_2^{\Im}$$

$$\begin{aligned}
&= \frac{1}{4}\pi\frac{1}{\left(r_s/Rq_s\right)^2}\frac{1}{\left|\xi_s/a\right|^2}\beta_{h0}i\pi^2\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\frac{1}{\pi n_0 T_h\bar{\epsilon}_0}\left(\frac{p_h}{r_p}\right)_{x_s}\frac{1}{2}\bar{\epsilon}_0^2 \\
&= \frac{i}{2}\frac{\bar{\omega}_c}{\left(r_s/a\right)^2}\left(\frac{\beta_h}{r_p}\right)_{x_s}\Delta_b^2 \qquad (117)
\end{aligned}$$