Ref of Fu's 1993 paper

January 24, 2016

1 The formula of δW_k

The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = i\frac{e}{M}\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right)\frac{i}{\omega}\mathbf{v}_d \cdot \delta\mathbf{E}_\perp \tag{1}$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\omega_c} \times \left(\mu\nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \kappa$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i\hat{\mathbf{b}}\times\nabla F\cdot\nabla}{\Omega\partial F/\partial\epsilon}$, $\epsilon = \frac{1}{2}v^2$, $\delta\mathbf{E}_\perp = i\omega\vec{\xi} \times \mathbf{B}$, $\omega_c = \frac{Be}{M}$ is the particle cyclotron frequency.[Berk, Phys. Fluid B 4 1992]

The term $\mathbf{v}_d \cdot \delta\mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta\mathbf{E}_\perp = \frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \kappa \cdot \delta\mathbf{E}_\perp = i\omega\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \tilde{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{2}$$

$$= -i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\vec{\kappa}\cdot\vec{\xi} = -i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\left(\nabla\theta\kappa_\theta + \nabla r\kappa_r\right)\cdot\left(\xi_\theta\nabla\theta + \xi_r\nabla r\right)$$

$$= -i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\left(\nabla\theta\cdot\nabla\theta\xi_\theta\kappa_\theta + \nabla r\cdot\nabla r\xi_r\kappa_r + \nabla\theta\cdot\nabla r\xi_r\kappa_\theta + \nabla r\cdot\nabla\theta\xi_\theta\kappa_r\right)$$

$$= -i\omega B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r\cdot\nabla r$, $g^{\theta r} = \nabla\theta\cdot\nabla r$, $g^{\theta\theta} = \nabla\theta\cdot\nabla\theta$. $\xi_r\nabla r = \xi_r\mathbf{e}_r$, $\xi_\theta\nabla\theta = \frac{1}{r}\xi_\theta\mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial\theta}$, $\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0)\cos\theta$, $\epsilon = \frac{1}{2}v^2$, $\delta\mathbf{E}_\perp^\star = -i\omega\vec{\xi}^\star \times \mathbf{B}$. Thus, the complex conjugate term is

$$\mathbf{v}_d \cdot \delta\mathbf{E}_\perp^\star = \frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \kappa \cdot \delta\mathbf{E}_\perp^\star = -i\omega\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\hat{\mathbf{b}} \times \tilde{\kappa} \cdot \left(\vec{\xi}^\star \times \mathbf{B}\right) \tag{3}$$

$$= i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\vec{\kappa}\cdot\vec{\xi}^\star = i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\left(\nabla\theta\kappa_\theta+\nabla r\kappa_r\right)\cdot\left(\xi_\theta^\star\nabla\theta+\xi_r^\star\nabla r\right)$$

$$= i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\omega_c}\left(\nabla\theta\cdot\nabla\theta\xi_\theta^\star\kappa_\theta+\nabla r\cdot\nabla r\xi_r^\star\kappa_r+\nabla\theta\cdot\nabla r\xi_r^\star\kappa_\theta+\nabla r\cdot\nabla\theta\xi_\theta^\star\kappa_r\right)$$

$$= i\omega B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta^\star+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt}g = i\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r\right) \tag{4}$$

$$\frac{d}{dt}g = H\left(r,\theta,\phi,t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint\frac{d\psi}{\dot{\psi}} = \oint\frac{d\theta}{\dot{\theta}} \tag{5}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}\left(r,\theta\right)\exp\left(-i\omega t+in\phi\right) \tag{6}$$

Note that $\hat{X}^{(1)}\left(r,\theta\right)$ is complex, and we take the real part of RHS for any physical variable, i.e. $X^{(1)}$. Thus, for internal kink mode, the displacement $\vec{\xi}$ is $\vec{\xi}=\xi_\theta'\mathbf{e}_\theta+\xi_r'\mathbf{e}_r$ and $\xi_r'=\xi_0\exp\left(i\left(\phi-\theta-\omega t\right)\right)$ within the region $q=1$ rational surface $r=r_s$. With cylindrical approximation, it can apply the relation $\nabla\cdot\vec{\xi}=0$, and thus obtain $\xi_\theta'=-i\xi_0\exp\left[i\left(\phi-\theta-\omega t\right)\right]$. Thus, the covariant forms of the perturbation are

$$\xi_r = \xi_0\exp\left(i\left(\phi-\theta-\omega t\right)\right), \tag{7}$$

$$\xi_\theta = -i\xi_0 r\exp\left(i\left(\phi-\theta-\omega t\right)\right) \tag{8}$$

within the region $q=1$ surface. Similarly, the covariant forms of the perturbation are

$$\xi_r=\xi_0\left(\frac{\Delta r-r+(r_s-\Delta r/2)}{\Delta r}\right)\exp\left(i\left(\phi-\theta-\omega t\right)\right), \tag{9}$$

$$\xi_\theta=-i\xi_0 r\left(\frac{\Delta r-2r+(r_s-\Delta r/2)}{\Delta r}\right)\exp\left(i\left(\phi-\theta-\omega t\right)\right) \tag{10}$$

in the inertial region $r_s-\frac{\Delta r}{2}\le r\le r_s+\frac{\Delta r}{2}$. And $\xi_r=\xi_\theta=0$ in the rest region.

The formal solution of the nonadibatic distribution g is

$$g=\int_{-\infty}^{t} i\frac{e}{M}\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\omega_c}G\left(\tau\right)d\tau \tag{11}$$

with

$$G=\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

where $G\left(\tau\right)=\hat{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-i\omega\tau+in\phi\left(\tau\right)\right)$ and the τ dependence is through the following equations:

$$\dot{r}=\dot{\mathbf{R}}\cdot\nabla r,\dot{\theta}=\dot{\mathbf{R}}\cdot\nabla\theta,\dot{\phi}=\dot{\mathbf{R}}\cdot\nabla\phi \tag{12}$$

Let us separate $\phi\left(\tau\right)$ into its secular and oscillating parts:

$$\phi\left(\tau\right)=\left\langle\dot{\phi}\right\rangle\tau+\widetilde{\phi}\left(\tau\right) \tag{13}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]=\hat{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(in\tilde{\phi}\left(\tau\right)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}\left(\tau\right)=\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(ip\omega_b\tau\right) \tag{14}$$

where,

$$Y_p\left(\Lambda,\epsilon,\bar{r};\sigma\right)=\frac{1}{\tau_b}\oint d\tau\tilde{G}\left(\tau\right)\exp\left(-ip\omega_b\tau\right) \tag{15}$$

with $r\left(\tau\right)=\bar{r}+\rho_d\cos\theta\left(\tau\right)$, ρ_d represents the finite orbit width for passing particles. $\rho_d=\Omega_d/\omega_t$, $\Omega_d=\frac{\left(v_\perp^2/2+v_\parallel^2\right)}{\omega_c R_0}$, $\omega_t=\frac{v_\parallel}{qR_0}$. Thus,

$$\rho_d=\frac{q}{\omega_c}\sqrt{\frac{\epsilon}{2\left(1-\Lambda/b\right)}}\left[\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right] \tag{16}$$

Carrying out the time integration, the solution of g is obtained

$$g=\frac{e}{M}\frac{\partial F}{\partial \epsilon}\left(\omega-\omega_\star\right)B\frac{\epsilon}{\omega_c}\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\epsilon,\bar{r};\sigma\right)\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega\right)t\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega} \tag{17}$$

The formula of δW_k is derived as follows.

$$\delta W_k=\int d^3x\vec{\xi}^\star\cdot\nabla\cdot\delta\mathbf{P}_k=e\int d^3x\int d^3v\left(\frac{i}{\omega}\mathbf{v}_d\cdot\delta\mathbf{E}_\perp\right)^\star g$$

$$=e\int d^3x\int d^3vgB\frac{\epsilon}{\omega_c}\left(\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta+g^{r\theta}\kappa_r\right)\xi_\theta^\star+\left(g^{rr}\kappa_r+g^{r\theta}\kappa_\theta\right)\xi_r^\star\right)$$

$$=e\int d^3x\int d^3vgB\frac{\epsilon}{\omega_c}G^\star \tag{18}$$

where $G^\star=\hat{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(i\omega\tau-in\phi\left(\tau\right)\right)$. Let $\tilde{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right)\right]=\hat{G}^\star\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-in\tilde{\phi}\left(\tau\right)\right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star\left(\tau\right)=\sum_{-\infty}^{\infty}Y_p^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(-ip\omega_b\tau\right) \tag{19}$$

where,

$$Y_p^\star\left(\Lambda,\bar{r};\sigma\right)=\frac{1}{\tau_b}\oint d\tau\tilde{G}^\star\left(\tau\right)\exp\left(ip\omega_b\tau\right) \tag{20}$$

with $r\left(\tau\right)=\bar{r}+\rho_d\cos\theta\left(\tau\right)$.

$$\delta W_k=\frac{e^2}{M}\int d^3x\int d^3v\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B^2\frac{\epsilon^2}{\omega_c^2}\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\bar{r};\sigma\right)$$

$$\cdot\frac{\exp\left[i\left(n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega\right)\tau\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega}\sum_{-\infty}^{\infty}Y_{p'}^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(i\omega\tau-in\left\langle\dot{\phi}\right\rangle\tau-ip'\omega_b\tau\right) \tag{21}$$

$$\delta W_k=\frac{e^2}{M}\int d^3x\int d^3v\frac{\partial F}{\partial\epsilon}\left(\omega-\omega_\star\right)B^2\frac{\epsilon^2}{\omega_c^2}$$

$$\cdot\sum_{-\infty}^{\infty}Y_p\left(\Lambda,\epsilon,\bar{r};\sigma\right)\frac{\exp\left[ip\omega_b\tau\right]}{n\left\langle\dot{\phi}\right\rangle+p\omega_b-\omega}\sum_{-\infty}^{\infty}Y_{p'}^\star\left(\Lambda,\epsilon,\bar{r};\sigma\right)\exp\left(-ip'\omega_b\tau\right) \tag{22}$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = \frac{e^2}{M} \int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\omega_c^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p (\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp(ip\omega_b \tau)}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^{\star} (\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b \tau) \tag{23}$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon} b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_\parallel$, one finally obtains

$$\delta W_k = \frac{4\pi^2}{M} \frac{e^2 B^2}{\omega_c^2} \frac{1}{R_0} \int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b (\omega - \omega_\star)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{|Y_p|^2}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega}, \tag{24}$$

which is similar to Eq.(35) of Fu's 1993 paper with replacing ϵ and J by $\epsilon \equiv \frac{1}{2} M v^2$ and $B = qR/J$. Note that $\tilde{\phi} \cong 0$, $\left\langle \dot{\phi} \right\rangle \cong \omega_D^0 + q\omega_b$,$\omega_D^0 \approx 0$ for passing particles.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{25}$$

$$J_t = \frac{1}{2\pi} \int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1} \sin^2 \frac{\theta}{2}} \frac{d\theta}{\pi} \tag{26}$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E} \right)^{-1} = \frac{\pi \omega_\parallel}{2K(\kappa)}, \kappa < 1 \tag{27}$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E} \right)^{-1} = \frac{\pi \sqrt{\kappa} \omega_\parallel}{K(\kappa^{-1})}, \kappa > 1 \tag{28}$$

where $\omega_\parallel = \frac{1}{qR} \sqrt{\varepsilon \mu B_0} = \frac{\sqrt{\epsilon}}{qR} \sqrt{\varepsilon \Lambda}$, $\kappa = \frac{1 - \Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $F = \frac{n_0}{v_h^3}\bar{F}$**,** $v_h = \sqrt{\frac{2T_h}{M}}$**,** $\epsilon = \frac{T_h}{M}\bar{\epsilon}$**,** $r = a\bar{x}$**,** $J = aR_0\bar{J}$**,** $R = R_0\bar{R}$**,** $\omega_t = \frac{v_h}{R_0}\bar{\omega}_t$, $\frac{1}{\tau_t} = \frac{v_h}{2\pi R_0}\bar{\omega}_t = \frac{v_h}{R_0}\frac{\bar{\omega}_t}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_t}$, $\omega = \frac{v_h}{R_0}\bar{\omega}$, $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$, $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$ **.**

$$\delta W_k = \pi^2 a^2 R_0 n_0 T_h \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b \left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\left\langle \dot{\phi} \right\rangle + p\bar{\omega}_b - \bar{\omega}} \tag{29}$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q} \sqrt{\bar{\epsilon}} \tag{30}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\, b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{31}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{32}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(r,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\hat{\xi}_\theta\left(\theta, \bar{r} + \rho_d\cos\theta\right) + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\hat{\xi}_r\left(\theta, \bar{r} + \rho_d\cos\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta+\bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta,\bar{x}+\frac{\rho_d}{a}\cos\theta\right)+\left(\bar{g}^{rr}\bar{\kappa}_r+\bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta,\bar{x}+\frac{\rho_d}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m}=\hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm}=\hat{\xi}_{rm}/a$.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F\left(x,\bar{\epsilon},\Lambda\right)=\frac{n_0}{v_h^3}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{33}$$

where n_0 is determined by

$$n\left(x\right)=\frac{n_0}{v_h^3}\int d^3\mathbf{v}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right]$$

at $x=x_0$.

The normalized metric tensors are

$$\bar{g}^{rr}=1+2\Delta'\cos\theta \tag{34}$$

with $\Delta'=\left(\varepsilon+\alpha\right)/4$, $\varepsilon=\frac{r}{R_0}$, $\alpha=-R_0q^2d\beta/dr$, $\beta=\frac{2\mu_0P}{B^2}$ set $\alpha=0$ if $\beta=0$, or assume $\bar{g}^{rr}=1$ without toroidal effect, θ independent. Specially, in low beta limit,

$$\bar{g}^{rr}=1+\frac{1}{2}\varepsilon\cos\theta \tag{35}$$

$$\bar{g}^{\theta\theta}=\frac{1}{x^2}\left[1-2\left(\varepsilon+\Delta'\right)\cos\theta\right] \tag{36}$$

assume $\bar{g}^{\theta\theta}=\frac{1}{x^2}$ without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta}=\frac{1}{x^2}\left[1-\frac{5}{2}\varepsilon\cos\theta\right] \tag{37}$$

$$\bar{g}^{r\theta}=-\frac{1}{x}\left[\varepsilon+\left(r\Delta'\right)'\right]\sin\theta \tag{38}$$

specially,

$$\bar{g}^{r\theta}=-\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{39}$$

for low beta limit. and $\bar{g}^{r\theta}=0$ without toroidal effect.

The normalized curvature are in low beta limit

$$\bar{\kappa}_r=-\frac{a}{R}\cos\theta+\frac{a}{R}\frac{\varepsilon}{4}-\frac{a}{R}\frac{5}{4}\varepsilon\left(\cos2\theta-1\right)-\left(\frac{a}{R}\right)^2\frac{x}{q} \tag{40}$$

$$\bar{\kappa}_{\theta}=\varepsilon\sin\theta+\frac{5}{4}\varepsilon^{2}\sin 2\theta \tag{41}$$

with $R=R_0+r\cos\theta-\Delta(r)+r\eta(r)(\cos 2\theta-1).\eta(r)=(\varepsilon+\Delta')/2$.
The normalized $\omega_\star$ is

$$\bar{\omega}_{\star}=\frac{1}{2}\frac{m}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}} \tag{42}$$

where, m is poloidal mode number, $\rho_h=v_h/\omega_c$, $v_h=\sqrt{2T_h/M}$, $\omega_c=Be/M$.
The normalized ρ_d is

$$\bar{\rho}_d=\frac{\rho_d}{a}=\frac{q}{2}\frac{\rho_h}{a}\sqrt{\frac{\bar{\epsilon}}{(1-\Lambda/b)}}\left[\frac{\Lambda}{b}+2\left(1-\frac{\Lambda}{b}\right)\right] \tag{43}$$

The normalized ξ are

$$\bar{\xi}_r(\theta,x)=\bar{\xi}_0\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta,x)=-i\bar{\xi}_0 x\exp(-i\theta)$$

within $q=1$ surface. In the inertial region $r_s-\frac{\Delta r}{2}\le r\le r_s+\frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta,x)=\bar{\xi}_0\left(\frac{\overline{\Delta r}-x+\left(\bar{r}_s-\overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta,x)=-i\bar{\xi}_0 x\left(\frac{\overline{\Delta r}-2x+\left(\bar{r}_s-\overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0=\xi_0/a$, $\bar{r}_s=r_s/a$, $\overline{\Delta r}=\Delta r/a$, $x=r/a$.

The normalized $\delta\bar{W}_k$ is given by

$$\delta\bar{W}_k=\int\frac{\bar{J}}{q}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\bar{\tau}_b(\bar{\omega}-\bar{\omega}_\star)$$
$$\cdot\sum_{-\infty}^{\infty}\frac{\left|\bar{Y}_p\right|^2}{n\left\langle\dot{\phi}\right\rangle+p\bar{\omega}_b-\bar{\omega}}, \tag{44}$$

where $\bar{J}=x$.

For passing particles,

$$\bar{\omega}_b=\frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{45}$$

where, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. and

$$\left\langle \dot{\phi} \right\rangle \cong q\bar{\omega}_b \tag{46}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\,(T/M)\,\bar{\epsilon}} b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{47}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{48}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[x\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$ and normalized drift orbit width is $\bar{\rho}_d = \frac{\rho_d}{a}$.

The normalized slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$\bar{F}\left(x, \bar{\epsilon}, \Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2} + \bar{\epsilon}_c^{3/2}} Erfc\left(\frac{\bar{\epsilon} - \bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right) \exp\left[-\left(\frac{x - x_0}{\Delta x}\right)^2\right] \exp\left[-\left(\frac{\Lambda - \Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{49}$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon\cos\theta \tag{50}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - \frac{5}{2}\varepsilon\cos\theta\right] \tag{51}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{52}$$

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta \tag{53}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta \tag{54}$$

The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\bar{\epsilon}}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \tag{55}$$

The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\exp(-i\theta)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

2 The fishbone dispersion relation

The formula for δW_{MHD}, δI [Miyamoto, "Plasma Physics and Controlled Nuclear Fusion"] The energy principle is

$$\delta W_{MHD} + \delta W_k + \delta I = 0 \tag{56}$$

where

$$\delta I = \frac{\gamma^2}{2}\int\rho_m\left|\vec{\xi}\right|^2 d\vec{r} \tag{57}$$

$$\delta W_k = \frac{1}{2}\int \vec{\xi}\cdot\nabla\delta p_h d\vec{r} \tag{58}$$

Note that δW_k is half of δW_k given by Eq.(15). δW_{MHD} consists of the contribution δW^s_{MHD} from the singular region near the rational surface and the contribution δW^{ext}_{MHD} from the external region.

The MHD potential energy $\delta W^{ext}_{MHDtor}/2\pi R$ of toroidal plasma with circular cross-section is given by

$$\frac{\delta W^{ext}_{MHDtor}}{2\pi R} = \left(1-\frac{1}{n^2}\right)\frac{\delta W^{ext}_{MHDcycl}}{2\pi R} + \frac{\pi B^2_{\theta s}}{2\mu_0}\left|\xi_s\right|^2 \delta\hat{W}_T \tag{59}$$

$$\delta\hat{W}_T = \pi\left(\frac{r_s}{R}\right)^2 3\left(1-q_0\right)\left(\frac{13}{144}-\beta^2_{ps}\right) \tag{60}$$

The term δW^s_{MHD} for the singular region is

$$\frac{\delta W^s_{MHD}}{2\pi R} = \frac{\pi}{2\mu_0}\frac{B^2_{\theta s}}{2\pi} sn\gamma\tau_{A\theta}\left|\xi_s\right|^2$$

where $B_{\theta s} = \frac{r_s B_t}{R q_s}$, $\tau_{A\theta} = \frac{3^{1/2} r_s}{\left(B^2_{\theta s}/\mu_0\rho_m\right)^{1/2}}$, $\rho_m = m_p n_{p_{r=r_s}}$, $s = r_s \frac{dq}{dr}_{r=r_s}$,$\xi_s = \xi_{r_{r=r_s}}$, n is toroidal mode number.

Thus, for $m=1, n=1$, the total sum of MHD contributions are

$$\begin{aligned}\delta W_{MHD} + \delta I &= 2\pi R\frac{B^2_{\theta s}}{2\mu_0}\left|\xi_s\right|^2\left(\delta\hat{W}_T + \gamma\tau_{A\theta}\frac{s}{2} + \pi\gamma^2\tau^2_{A\theta}\right) &(61)\\ &\approx 2\pi R\frac{B^2_{\theta s}}{2\mu_0}\left|\xi_s\right|^2\left(\delta\hat{W}_T + \gamma\tau_{A\theta}\frac{s}{2}\right), &(62)\end{aligned}$$

when $\gamma\tau_{A\theta} \ll 1$. With $\gamma = -i\omega$, the dispersion relation is

$$2\pi R\frac{B^2_{\theta s}}{2\mu_0}\left|\xi_s\right|^2\left(\delta\hat{W}_T + \frac{-i\omega}{\omega_A}\right) + \frac{1}{2}\pi^2 a^2 R_0 n_0 T_h \delta\bar{W}_k = 0. \tag{63}$$

where, $\omega_A \equiv \left(\tau_{A\theta} s/2\right)^{-1}$ and $\delta\bar{W}_k$ is given by Eq.(44).
Finally,

$$\frac{4}{\pi}\frac{1}{q_s^2}\left(\frac{r_s}{R_0}\right)^2\left|\frac{\xi_s}{a}\right|^2\frac{1}{\beta_h}\delta\hat{W}_T + \frac{2\sqrt{3}}{\pi}\frac{s}{q_s}\sqrt{\frac{m_p n_p}{M n_0}}\left(\frac{r_s}{R_0}\right)^2\left|\frac{\xi_s}{a}\right|^2\frac{1}{\beta_h^{1/2}}(-i\bar{\omega}) + \delta\bar{W}_k = 0, \tag{64}$$

or

$$-\frac{i\bar{\omega}}{\bar{\omega}_A} + \delta\hat{W}_T + \delta\hat{W}_k = 0, \tag{65}$$

with

$$\delta \hat{W}_k = \frac{1}{4}\pi \frac{1}{\left(r_s/Rq_s\right)^2}\frac{1}{\left|\xi_s/a\right|^2}\beta_{h0}\delta \bar{W}_k$$

where $\beta_{h0} = 2\mu_0 n_0 T_h/B_t^2$, n_p is plasma density at $r = r_s$. n_0 is energetic particle density at $x = x_0$. m_p, M is ion mass, and energetic particle mass respectively. And $\bar{\omega} = \omega/\left(v_h/R_0\right)$, $\bar{\omega}_A = \omega_A/\left(v_h/R_0\right)$.

3 Analytic form of the dispersion relation with passing particles and large aspect ratio approximation

For $\varepsilon \ll 1$, the normalized metric tensors are approximated as

$$\bar{g}^{rr} \approx 1 \tag{66}$$

$$\bar{g}^{\theta\theta} \approx \frac{1}{x^2} \tag{67}$$

$$\bar{g}^{r\theta} \approx -\frac{1}{x}\frac{3}{2}\varepsilon \sin\theta \tag{68}$$

and the normalized curvature are in low beta limit

$$\bar{\kappa}_r \approx -\frac{a}{R}\cos\theta \tag{69}$$

$$\bar{\kappa}_\theta \approx \varepsilon \sin\theta \tag{70}$$

The formula of ξ_θ and ξ_r are given by

$$\bar{\xi}_r\left(\theta, x\right) = \bar{\xi}_0\left(x\right)\exp\left(-i\theta\right), \tag{71}$$

$$\bar{\xi}_\theta\left(\theta, x\right) = -i\bar{\xi}_0\left(x\right)x\exp\left(-i\theta\right) \tag{72}$$

where, $\bar{\xi}_0\left(x\right) = \bar{\xi}_s H\left(x_s - x\right)$. $H\left(x\right)$ is Heaviside step function, $H = 1$ for $x > 0$ and $H = 0$ for $x < 0$, $dH/dx = \delta\left(x\right)$. Together with $\Lambda \ll 1$, one obtains

$$\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(r,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\tilde{\xi}}_\theta\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\tilde{\xi}}_r\left(\theta, \bar{x} + \bar{\rho}_d\cos\theta\right)\right)$$

$$\approx 2\left(\frac{1}{x^2}\varepsilon\sin\theta\bar{\xi}_\theta - \frac{a}{R}\cos\theta\bar{\xi}_r\right)$$

$$\approx 2\left(-\frac{a}{R}i\sin\theta\bar{\xi}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)-\frac{a}{R}\cos\theta\bar{\xi}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\right)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\left(i\sin\theta+\cos\theta\right)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right) \tag{73}$$

For $\kappa \gg 1$, the ellipitic fuction K becomes $K\left(\kappa^{-1}\right)=\pi/2$. Thus,

$$\frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \approx \frac{\pi\sqrt{\frac{1}{2\varepsilon\Lambda}}}{\pi/2}\frac{\sqrt{\frac{\varepsilon\Lambda}{2}}}{b\sqrt{1-\frac{\Lambda}{b}}}$$

$$= \frac{1}{b\sqrt{1-\Lambda/b}} \tag{74}$$

Using Eq.(73) and Eq.(74), Y_p is rewritten as

$$Y_p = -\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}2\frac{a}{R}\bar{\xi}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\int_0^{\theta}d\theta'\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

$$= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right)$$

$$= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-\bar{x}-\bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right) \tag{75}$$

For $\bar{\rho}_d = 0$ and $p = 0$, we have

$$\begin{aligned} Y_0 &= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-x\right)\exp\left(-i\theta\right)\exp\left(i\theta\right) && (76)\\ &= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-x\right) && (77)\\ &= -2\frac{a}{R}\bar{\xi}_s H\left(x_s-x\right) && (78)\end{aligned}$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 1$, $\delta\bar{W}_k$ of Eq.(44) becomes

$$\delta\bar{W}_k = \int_0^{x_s}\bar{J}dx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega}-\bar{\omega}_{\star}\right)$$

$$\cdot\frac{\left|\bar{Y}_0\right|^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}}, \tag{79}$$

here Y_0 is zero in the region $x > x_s$ where $H = 0$. According to Y_0 and $\bar{J}$ as shown above, $\delta \bar{W}_k$ becomes

$$\delta \bar{W}_k = \int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \left(\bar{\omega} - \bar{\omega}_\star \right)$$
$$\cdot \frac{\left(-2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}, \tag{80}$$

The distribution function of passing particles for analytical purpose is given by

$$\bar{F}\left(x, \bar{\epsilon}, \Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right], \tag{81}$$

and its derivative of $\bar{\epsilon}$ is read as

$$\frac{\partial \bar{F}}{\partial \bar{\epsilon}} = \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right] \left[-\frac{3}{2} \bar{\epsilon}^{-\frac{5}{2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) - \bar{\epsilon}^{-\frac{5}{2}} \Lambda \delta'\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) - \bar{\epsilon}^{-\frac{3}{2}} \delta\left(\Lambda\right) \delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \right].$$

$$\delta \bar{W}_k = I_1 + I_2 \tag{82}$$

where I_1 represents the term with $\bar{\omega}$ and I_2 the term with $\bar{\omega}_\star$. Then I_1 decomposes three parts corresponding to $\partial \bar{F} / \partial \bar{\epsilon}$.

$$I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)} \tag{83}$$

3.1 The dispersion relation for the case of $\rho_d = 0$, $p = 0$

$$I_1^{(1)} = \int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \left(-\frac{3}{2}\right) \bar{\epsilon}^{-\frac{5}{2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_c - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right] \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}$$
$$\cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$
$$= \left(2\frac{a}{R}\bar{\xi}_s\right)^2 2\pi \left(-\frac{3}{2}\right) \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right] dx \int_0^{\bar{\epsilon}_0} d\bar{\epsilon} \frac{\bar{\omega}}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$
$$= \left(2\frac{a}{R}\bar{\xi}_s\right)^2 2\pi \left(-\frac{3}{2}\right) \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right] dx \int_0^{\bar{\epsilon}_0} 2\sqrt{\bar{\epsilon}} d\sqrt{\bar{\epsilon}} \frac{\bar{\omega}}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$
$$= -6\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\omega} \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right] dx \int_0^{\bar{\epsilon}_0} \frac{\sqrt{\bar{\epsilon}} - \bar{\omega} + \bar{\omega}}{\sqrt{\bar{\epsilon}} - \bar{\omega}} d\sqrt{\bar{\epsilon}}$$
$$= -6\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\omega} \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2 \right] dx \int_0^{\bar{\epsilon}_0} \left(1 + \frac{\bar{\omega}}{\sqrt{\bar{\epsilon}} - \bar{\omega}}\right) d\sqrt{\bar{\epsilon}}$$

$$= -6\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\omega} \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx \left(\sqrt{\bar{\epsilon}_c} + \bar{\omega} \ln\left(\frac{\bar{\omega} - \sqrt{\bar{\epsilon}_c}}{\bar{\omega}}\right)\right)$$

Defining $\Omega = \bar{\omega}/\sqrt{\bar{\epsilon}_0}$,

$$I_1^{(1)} = -6\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\epsilon}_c \left(\Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right) \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx$$

$$I_1^{(2)} = \int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \left(-1\right) \bar{\epsilon}^{-\frac{5}{2}} \Lambda \delta'\left(\Lambda\right) H\left(\bar{\epsilon}_c - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= \int_0^{x_s} x dx \int d\Lambda d\bar{\epsilon} \left(-1\right) \Lambda \delta'\left(\Lambda\right) H\left(\bar{\epsilon}_c - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] 2\pi \bar{\omega} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= 4\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx \int_0^{\bar{\epsilon}_0} \sqrt{\bar{\epsilon}} d\sqrt{\bar{\epsilon}} \bar{\omega} \cdot \frac{1}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= 4\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\epsilon}_c \left(\Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right) \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx$$

$$I_1^{(3)} = \int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \left(-1\right) \bar{\epsilon}^{-\frac{3}{2}} \delta\left(\Lambda\right) \delta\left(\bar{\epsilon}_c - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}$$

$$\cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= -2\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\epsilon}_c \frac{\Omega}{1 - \Omega} \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx$$

Thus,

$$I_1 = -2\pi \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \bar{\epsilon}_c \left[\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right] \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx$$

$$\frac{\partial \bar{F}}{\partial x} = \frac{1}{\bar{\epsilon}^{3/2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_c - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \left(-\frac{2x}{\Delta x^2}\right)$$

$$I_2 = -\int_0^{x_s} \bar{J} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \bar{\omega}_\star \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= -\int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} m \frac{\partial \bar{F}}{\partial x} \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= -\int_0^{x_s} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} m \frac{1}{x} \frac{R}{a} \frac{\rho_h}{a} \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$\cdot \frac{1}{\bar{\epsilon}^{3/2}} \delta(\Lambda) H(\bar{\epsilon}_c - \bar{\epsilon}) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \left(-\frac{2x}{\Delta x^2}\right)$$

For the case of poloidal mode number $m = 1$, gives

$$I_2 = -\int_0^{x_s} \left(\frac{-2x}{\Delta x^2}\right) \exp\left(-\frac{x^2}{\Delta x^2}\right) dx \int_0^{\bar{\epsilon}_0} \bar{\epsilon}^3 d\bar{\epsilon} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \frac{1}{2} \frac{R}{a} \frac{\rho_h}{a} \frac{1}{\bar{\epsilon}^{3/2}} \cdot \frac{\left(2\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= -\frac{\pi}{2} \frac{R}{a} \frac{\rho_h}{a} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \left(\frac{-2x}{\Delta x^2}\right) \exp\left(-\frac{x^2}{\Delta x^2}\right) dx \int_0^{\bar{\epsilon}_0} \bar{\epsilon} d\bar{\epsilon} \frac{1}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= -\pi \frac{R}{a} \frac{\rho_h}{a} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \left(\frac{-2x}{\Delta x^2}\right) \exp\left(-\frac{x^2}{\Delta x^2}\right) dx \int_0^{\bar{\epsilon}_0} \left(\sqrt{\bar{\epsilon}}\right)^3 d\sqrt{\bar{\epsilon}} \frac{1}{\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

The intergal identity is given by

$$\int dy \frac{y^3}{y-a} = \int dy \frac{y^3 - a^3 + a^3}{y-a} = \int dy \left(y^2 + ay + a^2\right) + \int dy \frac{a^3}{y-a} \tag{84}$$

$$= \frac{1}{3} y^3 + \frac{1}{2} a y^2 + a^2 y + a^3 \ln(y-a) \tag{85}$$

Therefore,

$$\begin{aligned} I_2 &= -\pi \frac{R}{a} \frac{\rho_h}{a} \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} \left(\frac{-2x}{\Delta x^2}\right) \exp\left(-\frac{x^2}{\Delta x^2}\right) dx \\ &\quad \cdot \left(\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right)\right) \left(\sqrt{\bar{\epsilon}_0}\right)^3 \end{aligned}$$

Finally,

$$\delta \bar{W}_k = \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \int_0^{x_s} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx \begin{bmatrix} -2\pi\bar{\epsilon}_0 \left(\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right) \\ + \frac{2\pi}{\Delta x^2} \frac{R}{a} \frac{\rho_h}{a} \left(\sqrt{\bar{\epsilon}_0}\right)^3 \\ \cdot \left(\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right)\right) \end{bmatrix} \tag{86}$$

$$\delta\bar{W}_k = \left(2\frac{a}{R}\bar{\xi}_s\right)^2 \frac{1}{2}\Delta x^2 \left(1 - \exp\left(-\frac{x_s^2}{\Delta x^2}\right)\right) \left[\begin{array}{c} -2\pi\bar{\epsilon}_0\left(\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 \ln\left(1 - \frac{1}{\Omega}\right)\right) \\ +\frac{2\pi}{\Delta x^2}\frac{R}{a}\frac{\rho_h}{a}\left(\sqrt{\bar{\epsilon}_0}\right)^3 \\ \cdot\left(\frac{1}{3} + \frac{1}{2}\Omega + \Omega^2 + \Omega^3 \ln\left(1 - \frac{1}{\Omega}\right)\right) \end{array}\right] \tag{87}$$

$$\begin{aligned} p_h(x) &= \int d^3 v M\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right) F_h \\ &= M \int \begin{array}{c} \sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon\epsilon \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \\ \cdot\frac{n_0}{v_h^3}\frac{1}{\bar{\epsilon}^{3/2}}\delta(\Lambda) H(\bar{\epsilon}_0 - \bar{\epsilon}) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \end{array} \\ &= M\left(\frac{T_h}{M}\right)^{\frac{5}{2}} \int \begin{array}{c} \sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \bar{\epsilon}^{1/2} d\bar{\epsilon}\bar{\epsilon} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \\ \cdot\frac{n_0}{v_h^3}\frac{1}{\bar{\epsilon}^{3/2}}\delta(\Lambda) H(\bar{\epsilon}_0 - \bar{\epsilon}) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \end{array} \end{aligned}$$

for $v_\parallel \gg v_\perp$,

$$\begin{aligned} &= M\left(\frac{T_h}{M}\right)^{\frac{5}{2}} \int \begin{array}{c} 2\sqrt{2}\pi\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \bar{\epsilon}^{1/2} d\bar{\epsilon}\bar{\epsilon} \\ \cdot\frac{n_0}{v_h^3}\frac{1}{\bar{\epsilon}^{3/2}}\delta(\Lambda) H(\bar{\epsilon}_0 - \bar{\epsilon}) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \end{array} \\ &= M\left(\frac{T_h}{M}\right)^{\frac{5}{2}} \frac{n_0}{v_h^3} 2\sqrt{2}\pi \int_0^{\bar{\epsilon}_0} d\bar{\epsilon} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \\ &= M\left(\frac{T_h}{M}\right)^{\frac{5}{2}} \frac{n_0}{(2T_h/M)^{\frac{3}{2}}} 2\sqrt{2}\pi\bar{\epsilon}_0 \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \\ &= T_h n_0 \pi\bar{\epsilon}_0 \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \end{aligned}$$

Thus ,

$$n_0 = \frac{p_h(x)}{\pi T_h \bar{\epsilon}_0} \exp\left[\left(\frac{x}{\Delta x}\right)^2\right] \tag{88}$$

$$\delta\hat{W}_k = \frac{1}{4}\pi\frac{1}{(r_s/Rq_s)^2}\frac{1}{|\xi_s/a|^2}\beta_{h0}\delta\bar{W}_k$$

$$\delta \hat{W}_k = \begin{array}{c} \frac{1}{4}\pi\frac{1}{(r_s/Rq_s)^2}\frac{1}{|\xi_s/a|^2}\beta_{h0}\left(1-\exp\left(-\frac{x_s^2}{\Delta x^2}\right)\right) \\ \cdot\left(2\frac{a}{R}\bar{\xi}_s\right)^2\frac{1}{2}\Delta x^2 \left[\begin{array}{c} -2\pi\bar{\epsilon}_0\left(\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right) \\ +\frac{2\pi}{\Delta x^2}\frac{R}{a}\frac{\rho_h}{a}\left(\sqrt{\bar{\epsilon}_0}\right)^3 \\ \cdot\left(\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right) \end{array}\right] \end{array} \tag{89}$$

$$\delta \hat{W}_k = \begin{array}{c} -\pi\frac{1}{(r_s/aq_s)^2}\frac{2\mu_0\left(p_h(0)-p_h(x_s)\right)\Delta x^2}{B_t^2}\left(\frac{\Omega}{1-\Omega}+\Omega+\Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right) \\ +\frac{\pi}{\Omega_c}\frac{1}{(r_s/Rq_s)^2}\frac{2\mu_0\left(p_h(0)-p_h(x_s)\right)}{B_t^2}\left(\frac{1}{3}+\frac{1}{2}\Omega+\Omega^2+\Omega^3\ln\left(1-\frac{1}{\Omega}\right)\right) \end{array} \tag{90}$$

where $\Omega_c = \omega_c/\left[(v_h/R)\sqrt{\bar{\epsilon}_0}\right]$. The dispersion relation is

$$-\frac{i\Omega}{\Omega_A}+\delta\hat{W}_T+\delta\hat{W}_k=0, \tag{91}$$

where $\Omega_A = \bar{\omega}_A/\sqrt{\bar{\epsilon}_0}$.

3.2 The dispersion relation for the case of $p = 1$, $\rho_d \neq 0$

For $p = 1$, we arrive at

$$Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-\bar{x}-\bar{\rho}_d\cos\theta\right)\exp(-i\theta) \tag{92}$$

From Fig.1, Y_1 is zero in the region $x < x_s - \bar{\rho}_d$ where $H = 1$ for $\theta \in [0, 2\pi]$ since the θ integral of $\sim \exp(-i\theta)$ from 0 to 2π is zero. In the region $x > x_s+\bar{\rho}_d$, Y_1 also is zero since $H = 0$. Obviously, Y_1 is finite in the region $x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d$ where $H = 1$ for $\theta \in [\theta^\star, -\theta^\star + 2\pi]$. $\cos\theta^\star = \frac{x_s-x}{\bar{\rho}_d} \in [0, \pi]$. Thus,

$$Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_{\theta^\star}^{2\pi-\theta^\star} d\theta\bar{\xi}_s\exp(-i\theta) \tag{93}$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(-i\left(2\pi-\theta^\star\right)\right)-\exp\left(-i\theta^\star\right)\right]$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i\left[\exp\left(-i\left(2\pi-\theta^\star\right)\right)-\exp\left(-i\theta^\star\right)\right]$$

$$= \frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\sin\theta^\star$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 1$, $\delta\bar{W}_k$ becomes

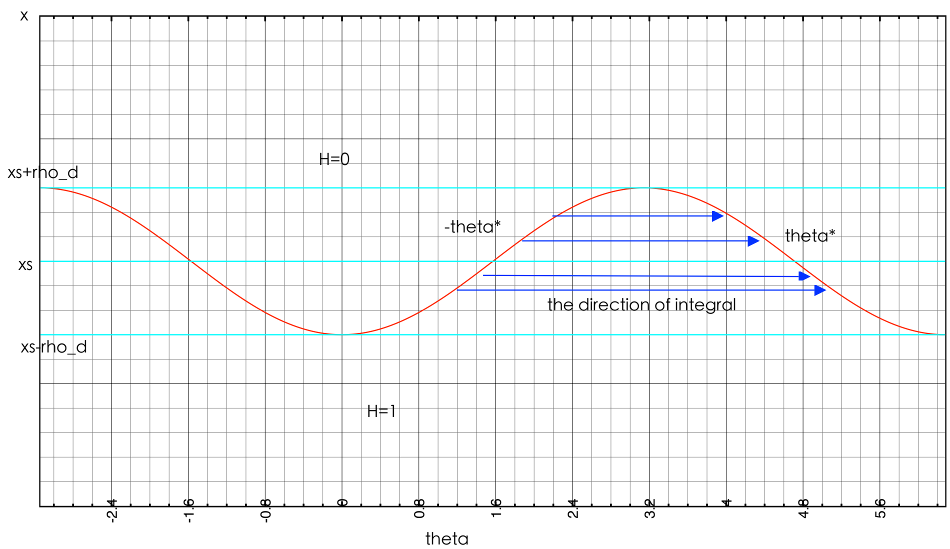

Figure 1: The integral region

$$\delta \bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \left(\bar{\omega} - \bar{\omega}_\star\right) \cdot \frac{\left|Y_1\right|^2}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q} - \bar{\omega}}. \tag{94}$$

$$\delta \bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x dx \int d\Lambda \int_0^{\bar{\epsilon}_0} \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \left(\bar{\omega} - \bar{\omega}_\star\right) \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}} \tag{95}$$

$$\delta \bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x dx \int d\Lambda \int_0^{\bar{\epsilon}_0} \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \frac{2\pi}{\sqrt{\bar{\epsilon}}} \left(\bar{\omega} - \bar{\omega}_\star\right) \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}} \left(1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right) \tag{96}$$

with $\bar{J} = x$. The distribution function of passing particles for analytical purpose is given by

$$\bar{F}\left(x, \bar{\epsilon}, \Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right], \tag{97}$$

and its derivative of $\bar{\epsilon}$ is read as

$$\frac{\partial \bar{F}}{\partial \bar{\epsilon}} = \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \left[-\frac{3}{2}\bar{\epsilon}^{-\frac{5}{2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) - \bar{\epsilon}^{-\frac{5}{2}} \Lambda \delta'\left(\Lambda\right) H\left(\bar{\epsilon}_0 - \bar{\epsilon}\right) - \bar{\epsilon}^{-\frac{3}{2}} \delta\left(\Lambda\right) \delta\left(\bar{\epsilon}_0 - \bar{\epsilon}\right)\right].$$

$$\delta\bar{W}_k = I_1 + I_2 \tag{98}$$

where I_1 represents the term with $\bar{\omega}$ and I_2 the term with $\bar{\omega}_\star$. Then I_1 decomposes three parts corresponding to $\partial\bar{F}/\partial\bar{\epsilon}$.

$$I_1 = I_1^{(1)} + I_1^{(2)} + I_1^{(3)} \tag{99}$$

$$I_1^{(1)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \left(-\frac{3}{2}\right) \bar{\epsilon}^{-\frac{5}{2}} \delta\left(\Lambda\right) H\left(\bar{\epsilon}_c - \bar{\epsilon}\right) \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}} \left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right]$$

$$= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} x \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\left[1 - \left(\frac{x_s - x}{\bar{\rho}_d}\right)^2\right] dx \int_0^{\bar{\epsilon}_0} d\bar{\epsilon}\bar{\omega}$$

$$\cdot \frac{1}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}}$$

$$= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar{\rho}_d}\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx - \frac{1}{2}\frac{\Delta x^4}{\bar{\rho}_d^2} \\ \cdot\left(\exp\left(-\left(\frac{x_s - \bar{\rho}_d}{\Delta x}\right)^2\right) - \exp\left(-\left(\frac{x_s + \bar{\rho}_d}{\Delta x}\right)^2\right)\right) \end{array} \right\}$$

$$\cdot\bar{\omega}\int_0^{\epsilon_0} 2\sqrt{\bar{\epsilon}} d\sqrt{\bar{\epsilon}} \frac{1}{\left(1 + \frac{1}{q_s}\right)\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

$$= -3\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar{\rho}_d}\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx - \frac{1}{2}\frac{\Delta x^4}{\bar{\rho}_d^2} \\ \cdot\exp\left(-\left(\frac{x_s - \bar{\rho}_d}{\Delta x}\right)^2\right)\left(1 - \exp\left(-\left(\frac{4x_s\bar{\rho}_d}{\Delta x^2}\right)\right)\right) \end{array} \right\}$$

$$\cdot\bar{\omega}\int 2\sqrt{\bar{\epsilon}} d\sqrt{\bar{\epsilon}} \frac{1}{\left(1 + \frac{1}{q_s}\right)\sqrt{\bar{\epsilon}} - \bar{\omega}}$$

For the condition of maximum of damping, we only keep the first term in the $\{\cdots\}$ of the above equation and make the approximation, i.e. $\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx \simeq \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} dx = 2\bar{\rho}_d \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$. With $\bar{\rho}_d \simeq q\sqrt{\bar{\epsilon}}\rho_h/a$, we obtain

$$
I_1^{(1)} \approx -12\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s\Delta x^2\bar{\omega}\frac{a}{q_s\rho_h}\frac{1}{1+\frac{1}{q_s}}\exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\int_0^{\bar{\epsilon}_0} d\sqrt{\bar{\epsilon}}
$$

$$
\cdot\frac{1}{\sqrt{\bar{\epsilon}}-\bar{\omega}/\left(1+\frac{1}{q_s}\right)}
$$

$$
= -12\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s\Delta x^2\bar{\omega}\frac{a}{q_s\rho_h}\frac{1}{1+\frac{1}{q_s}}\exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\sqrt{\bar{\epsilon}_0}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right) \tag{100}
$$

$$
I_1^{(2)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\left(-1\right)\bar{\epsilon}^{-\frac{5}{2}}\Lambda\delta'\left(\Lambda\right)H\left(\bar{\epsilon}_c-\bar{\epsilon}\right)\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}
$$

$$
\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]
$$

$$
= 2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int_0^{\epsilon_0} d\bar{\epsilon}\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\bar{\omega}
$$

$$
\cdot\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]
$$

$$
\approx 8\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s\Delta x^2\bar{\omega}\frac{a}{q_s\rho_h}\frac{1}{1+\frac{1}{q_s}}\exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\sqrt{\bar{\epsilon}_0}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right) \tag{101}
$$

$$
I_1^{(3)} = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\left(-1\right)\bar{\epsilon}^{-\frac{3}{2}}\delta\left(\Lambda\right)\delta\left(\bar{\epsilon}_0-\bar{\epsilon}\right)\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}
$$

$$
\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]
$$

$$
= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\bar{\omega}\bar{\epsilon}_c\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]
$$

$$
\cdot\frac{1}{\sqrt{\bar{\epsilon}_0}+\frac{\sqrt{\bar{\epsilon}_0}}{q_s}-\bar{\omega}}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]
$$

$$= -2\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{\bar{\omega}\bar{\epsilon}_0}{\sqrt{\bar{\epsilon}_0}+\frac{\sqrt{\bar{\epsilon}_0}}{q_s}-\bar{\omega}} \left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar{\rho}_d}\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]dx - \frac{1}{2}\frac{\Delta x^4}{\bar{\rho}_d^2} \\ \cdot \exp\left(-\left(\frac{x_s-\bar{\rho}_d}{\Delta x}\right)^2\right)\left(1-\exp\left(-\left(\frac{4x_s\bar{\rho}_d}{\Delta x^2}\right)\right)\right) \end{array} \right\}$$

$$\approx -4\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{\bar{\omega}\sqrt{\bar{\epsilon}_0}}{\sqrt{\bar{\epsilon}_0}+\frac{\sqrt{\bar{\epsilon}_0}}{q_s}-\bar{\omega}} x_s\Delta x^2 \frac{a}{q_s\rho_h}\exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$$

$$I_2 = -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\bar{\omega}_\star$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}m\frac{\partial\bar{F}}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\sin^2\theta^\star}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\left(\frac{-2x}{\Delta x^2}\right)\frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right)\exp\left(-\frac{x^2}{\Delta x^2}\right)$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\left(\frac{2\pi}{\Delta x^2}\right)\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right]\exp\left(-\frac{x^2}{\Delta x^2}\right)xdx\int\bar{\epsilon}^3 d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}}\frac{1}{\bar{\epsilon}^{3/2}}$$

$$\cdot\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2\frac{R}{a}\frac{\rho_h}{a}\left(\frac{2\pi}{\Delta x^2}\right)\int_0^{\bar{\epsilon}_0}\bar{\epsilon}^3 d\bar{\epsilon}\frac{1}{\sqrt{\bar{\epsilon}}}\frac{1}{\bar{\epsilon}^{3/2}}\frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$\cdot\left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar{\rho}_d}\right)^2\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d}\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]dx - \frac{1}{2}\frac{\Delta x^4}{\bar{\rho}_d^2} \\ \cdot\left(\exp\left(-\left(\frac{x_s-\bar{\rho}_d}{\Delta x}\right)^2\right)-\exp\left(-\left(\frac{x_s+\bar{\rho}_d}{\Delta x}\right)^2\right)\right) \end{array} \right\}$$

Simliar to the approximation made for I_1, we obtain

$$I_2 = \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{4\pi}{q_s} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \int_0^{\bar{\epsilon}_0} d\bar{\epsilon} \frac{\sqrt{\bar{\epsilon}}}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{8\pi}{q_s}\frac{1}{1+\frac{1}{q_s}} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \int_0^{\bar{\epsilon}_0} d\sqrt{\bar{\epsilon}} \frac{\left(\sqrt{\bar{\epsilon}}\right)^2}{\sqrt{\bar{\epsilon}}-\bar{\omega}/\left(1+\frac{1}{q_s}\right)}$$

The intergal identity is given by

$$\int dy \frac{y^2}{y-a} = \int dy \frac{y^2-a^2+a^2}{y-a} = \int dy\,(y+a) + \int dy \frac{a^2}{y-a} \tag{102}$$

$$= \frac{1}{2}y^2 + ay + a^2 \ln(y-a) \tag{103}$$

Thus

$$I_2 = \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{8\pi}{q_s}\frac{1}{1+\frac{1}{q_s}} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \left[\frac{1}{2}\bar{\epsilon}_0 + \frac{\bar{\omega}}{1+\frac{1}{q_s}}\sqrt{\bar{\epsilon}_0} + \left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2 \ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\bar{\epsilon}_0^{1/2}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\right] \tag{104}$$

Defining $\Omega = \frac{\bar{\omega}}{(1+1/q_s)\bar{\epsilon}_0^{1/2}}$, I_1 and I_2 can be rewritten by

$$I_1 = -4\pi\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s \Delta x^2 \sqrt{\bar{\epsilon}_c}\frac{a}{q_s\rho_h} \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\left[\frac{\Omega}{1-\Omega} + \Omega\ln\left(1-\frac{1}{\Omega}\right)\right] \tag{105}$$

$$I_2 = \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{8\pi}{q_s}\frac{\bar{\epsilon}_c}{1+\frac{1}{q_s}} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\left[\frac{1}{2} + \Omega + \Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right] \tag{106}$$

According to Eq. (98), the dispersion relation(64) without MHD contribution thus can be written as

$$-i\Omega + a_1\left[\frac{\Omega}{1-\Omega} + \Omega\ln\left(1-\frac{1}{\Omega}\right)\right] + b_1\left[\frac{1}{2} + \Omega + \Omega^2\ln\left(1-\frac{1}{\Omega}\right)\right] = 0 \tag{107}$$

with

$$a_1 \quad = \quad -\frac{8\sqrt{3}}{3}\frac{x_s}{s}\left(\frac{a}{r_s}\right)^2 \frac{a}{\rho_h}\frac{1}{1+\frac{1}{q_s}}\sqrt{\frac{Mn_0}{m_p n_p}}\Delta x^2 \beta_h^{1/2} \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$$

$$b_1 \quad = \quad \frac{16\sqrt{3}}{3}\left(\frac{a}{R}\right)\left(\frac{R}{r_s}\right)^2 \frac{x_s}{s}\sqrt{\frac{Mn_0}{m_p n_p}}\frac{\bar{\epsilon}_0^{1/2}}{\left(1+\frac{1}{q_s}\right)^2}\beta_h^{1/2}\exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$$

and

$$\left|\frac{a_1}{b_1}\right| = \frac{1}{2}\left(\frac{a}{R}\right)\left(\frac{a}{\rho_h}\right)\Delta x^2\left(1+\frac{1}{q_s}\right)\bar{\epsilon}_0^{-1/2}.$$