Ref of Fu's 1993 paper

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1 The formula of δW_k

The linearized drift kinetic equation is given by

$$\left(\partial_t + \mathbf{v}_d \cdot \nabla + v_\parallel \hat{\mathbf{b}} \cdot \nabla\right) g = i\frac{e}{M}\frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right)\frac{i}{\omega}\mathbf{v}_d \cdot \delta\mathbf{E}_\perp \tag{1}$$

where,$\mathbf{v}_d = \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\mu \nabla B + \kappa v_\parallel^2\right) \approx \frac{v_\perp^2/2+v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa$, $\mu = \frac{v_\perp^2}{2B}$, $\omega_\star = \frac{i\hat{\mathbf{b}}\times\nabla F\cdot\nabla}{\Omega\partial F/\partial\epsilon}$, $\epsilon = \frac{1}{2}v^2$, $\delta\mathbf{E}_\perp = i\omega\vec{\xi} \times \mathbf{B}$, $\Omega = \frac{Be}{M}$ is the particle cyclotron frequency.[Berk, Phys. Fluid B 4 1992]

The term $\mathbf{v}_d \cdot \delta\mathbf{E}_\perp$ can be expressed by the following form.

$$\mathbf{v}_d \cdot \delta\mathbf{E}_\perp = \frac{v_\perp^2/2+v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa \cdot \delta\mathbf{E}_\perp = i\omega\frac{v_\perp^2/2+v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \tilde{\kappa} \cdot \left(\vec{\xi} \times \mathbf{B}\right) \tag{2}$$

$$= -i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\Omega}\vec{\kappa}\cdot\vec{\xi} = -i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\Omega}\left(\nabla\theta\kappa_\theta + \nabla r\kappa_r\right)\cdot\left(\xi_\theta\nabla\theta + \xi_r\nabla r\right)$$

$$= -i\omega B\frac{v_\perp^2/2+v_\parallel^2}{\Omega}\left(\nabla\theta\cdot\nabla\theta\xi_\theta\kappa_\theta + \nabla r\cdot\nabla r\xi_r\kappa_r + \nabla\theta\cdot\nabla r\xi_r\kappa_\theta + \nabla r\cdot\nabla\theta\xi_\theta\kappa_r\right)$$

$$= -i\omega B\frac{\epsilon}{\Omega}\left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right)\left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\xi_r\right)$$

with $g^{rr} = \nabla r\cdot\nabla r$, $g^{\theta r} = \nabla\theta\cdot\nabla r$, $g^{\theta\theta} = \nabla\theta\cdot\nabla\theta$. $\xi_r\nabla r = \xi_r\mathbf{e}_r$, $\xi_\theta\nabla\theta = \frac{1}{r}\xi_\theta\mathbf{e}_\theta$, $\kappa_\theta = -\frac{1}{R}\frac{\partial R}{\partial\theta}$, $\kappa_r \approx -\frac{1}{R}\frac{\partial R}{\partial r} - \frac{r}{q^2R^2}$ [G. Y. Fu, PHYSICS OF PLASMAS 13 2006]. $\Lambda = \frac{\mu B_0}{\epsilon}$, $b = B_0/B \approx 1 + (r/R_0)\cos\theta$, $\epsilon = \frac{1}{2}v^2$, $\delta\mathbf{E}_\perp^\star = -i\omega\vec{\xi}^\star \times \mathbf{B}$. Thus, the complex conjugate term is

$$\mathbf{v}_d \cdot \delta\mathbf{E}_\perp^\star = \frac{v_\perp^2/2+v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \kappa \cdot \delta\mathbf{E}_\perp^\star = -i\omega\frac{v_\perp^2/2+v_\parallel^2}{\Omega}\hat{\mathbf{b}} \times \tilde{\kappa} \cdot \left(\vec{\xi}^\star \times \mathbf{B}\right) \tag{3}$$

$$= i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \vec{\kappa} \cdot \vec{\xi}^\star = i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \kappa_\theta + \nabla r \kappa_r \right) \cdot \left(\xi_\theta^\star \nabla\theta + \xi_r^\star \nabla r \right)$$

$$= i\omega B \frac{v_\perp^2/2 + v_\parallel^2}{\Omega} \left(\nabla\theta \cdot \nabla\theta \xi_\theta^\star \kappa_\theta + \nabla r \cdot \nabla r \xi_r^\star \kappa_r + \nabla\theta \cdot \nabla r \xi_r^\star \kappa_\theta + \nabla r \cdot \nabla\theta \xi_\theta^\star \kappa_r \right)$$

$$= i\omega B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r \right) \xi_\theta^\star + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta \right) \xi_r^\star \right)$$

The linearized drift kinetic equation is rewritten as

$$\frac{d}{dt} g = i \frac{e}{M} \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star \right) B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r \right) \xi_\theta + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta \right) \xi_r \right) \tag{4}$$

$$\frac{d}{dt} g = H\left(r, \theta, \phi, t\right)$$

The solution of perturbed distribution function g is obtained in the followings. At equilibrium, the projection of the orbit on the poloidal cross section is a closed curve. For either mirror-trapped or passing orbit, we define the bounce time [F. Porcelli, R. Stankiewicz, and W. Kerner, Phys. Plasmas 1 1994]

$$\tau_b = \oint d\tau = \oint \frac{d\psi}{\dot{\psi}} = \oint \frac{d\theta}{\dot{\theta}} \tag{5}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}\left(r, \theta\right) \exp\left(-i\omega t + in\phi\right) \tag{6}$$

Note that $\hat{X}^{(1)}\left(r, \theta\right)$ is complex, and we take the real part of RHS for any physical variable, i.e. $X^{(1)}$. Thus, for internal kink mode, the displacement $\vec{\xi}$ is $\vec{\xi} = \xi_\theta' \mathbf{e}_\theta + \xi_r' \mathbf{e}_r$ and $\xi_r' = \xi_0 \exp\left(i\left(\phi - \theta - \omega t\right)\right)$ within the region $q = 1$ rational surface $r = r_s$. With cylindrical approximation, it can apply the relation $\nabla \cdot \vec{\xi} = 0$, and thus obtain $\xi_\theta' = -i\xi_0 \exp\left[i\left(\phi - \theta - \omega t\right)\right]$. Thus, the covariant forms of the perturbation are

$$\xi_r = \xi_0 \exp\left(i\left(\phi - \theta - \omega t\right)\right), \tag{7}$$

$$\xi_\theta = -i\xi_0 r \exp\left(i\left(\phi - \theta - \omega t\right)\right) \tag{8}$$

within the region $q=1$ surface. Similarly, the covariant forms of the perturbation are

$$\xi_r = \xi_0 \left(\frac{\Delta r - r + (r_s - \Delta r/2)}{\Delta r} \right) \exp\left(i\left(\phi - \theta - \omega t\right)\right), \tag{9}$$

$$\xi_\theta = -i\xi_0 r \left(\frac{\Delta r - 2r + (r_s - \Delta r/2)}{\Delta r} \right) \exp\left(i\left(\phi - \theta - \omega t\right)\right) \tag{10}$$

in the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$. And $\xi_r = \xi_\theta = 0$ in the rest region.

The formal solution of the nonadibatic distribution g is

$$g = \int_{-\infty}^{t} i \frac{e}{M} \frac{\partial F}{\partial \epsilon} \left(\omega - \omega_\star\right) B \frac{\epsilon}{\Omega} G\left(\tau\right) d\tau \tag{11}$$

with

$$G = \left(\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right) \right) \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r \right) \xi_\theta + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta \right) \xi_r \right)$$

where $G\left(\tau\right) = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-i\omega\tau + in\phi\left(\tau\right)\right)$ and the τ dependence is through the following equations:

$$\dot{r} = \dot{\mathbf{R}} \cdot \nabla r, \dot{\theta} = \dot{\mathbf{R}} \cdot \nabla\theta, \dot{\phi} = \dot{\mathbf{R}} \cdot \nabla\phi \tag{12}$$

Let us separate $\phi\left(\tau\right)$ into its secular and oscillating parts:

$$\phi\left(\tau\right) = \left\langle \dot{\phi} \right\rangle \tau + \widetilde{\phi}\left(\tau\right) \tag{13}$$

where the brackets indicate bounce averaging.

The quantity $\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \hat{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(in\tilde{\phi}\left(\tau\right)\right)$ is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}\left(\tau\right) = \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(ip\omega_b\tau\right) \tag{14}$$

where,

$$Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{15}$$

with $r\left(\tau\right) = \bar{r} + \rho_d \cos\theta\left(\tau\right)$, ρ_d represents the finite orbit width for passing particles. $\rho_d = \Omega_d/\omega_t$, $\Omega_d = \frac{\left(v_\perp^2/2 + v_\parallel^2\right)}{\Omega R_0}$, $\omega_t = \frac{v_\parallel}{qR_0}$. Thus,

$$\rho_d = \frac{q}{\Omega} \sqrt{\frac{\epsilon}{2\left(1 - \Lambda/b\right)}} \left[\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right) \right] \tag{16}$$

Carrying out the time integration, the solution of g is obtained

$$g = \frac{e}{M} \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B \frac{\epsilon}{\Omega} \sum_{-\infty}^{\infty} Y_p (\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp \left[i \left(n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega \right) t \right]}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \tag{17}$$

The formula of δW_k is derived as follows.

$$\delta W_k = \int d^3x \vec{\xi}^\star \cdot \nabla \cdot \delta \mathbf{P}_k = e \int d^3x \int d^3v \left(\frac{i}{\omega} \mathbf{v}_d \cdot \delta \mathbf{E}_\perp \right)^\star g$$

$$= e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} \left(\frac{\Lambda}{b} + 2 \left(1 - \frac{\Lambda}{b} \right) \right) \left(\left(g^{\theta\theta} \kappa_\theta + g^{r\theta} \kappa_r \right) \xi_\theta^\star + \left(g^{rr} \kappa_r + g^{r\theta} \kappa_\theta \right) \xi_r^\star \right)$$

$$= e \int d^3x \int d^3v g B \frac{\epsilon}{\Omega} G^\star \tag{18}$$

where $G^\star = \hat{G}^\star [r(\tau), \theta(\tau)] \exp(i\omega\tau - in\phi(\tau))$. Let $\tilde{G}^\star [r(\tau), \theta(\tau)] = \hat{G}^\star [r(\tau), \theta(\tau)] \exp \left(-in\tilde{\phi}(\tau) \right)$, which is a periodic function of τ, which can be expanded in Fourier series,

$$\tilde{G}^\star (\tau) = \sum_{-\infty}^{\infty} Y_p^\star (\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip\omega_b \tau) \tag{19}$$

where,

$$Y_p^\star (\Lambda, \bar{r}; \sigma) = \frac{1}{\tau_b} \oint d\tau \tilde{G}^\star (\tau) \exp(ip\omega_b \tau) \tag{20}$$

with $r(\tau) = \bar{r} + \rho_d \cos \theta(\tau)$.

$$\delta W_k = \frac{e^2}{M} \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2} \sum_{-\infty}^{\infty} Y_p (\Lambda, \bar{r}; \sigma)$$

$$\cdot \frac{\exp \left[i \left(n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega \right) \tau \right]}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star (\Lambda, \epsilon, \bar{r}; \sigma) \exp \left(i\omega\tau - in \left\langle \dot{\phi} \right\rangle \tau - ip'\omega_b \tau \right) \tag{21}$$

$$\delta W_k = \frac{e^2}{M} \int d^3x \int d^3v \frac{\partial F}{\partial \epsilon} (\omega - \omega_\star) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p (\Lambda, \epsilon, \bar{r}; \sigma) \frac{\exp[ip\omega_b \tau]}{n \left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star (\Lambda, \epsilon, \bar{r}; \sigma) \exp(-ip'\omega_b \tau) \tag{22}$$

Using $d^3v = \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon$, $d^3x = 2\pi J dr d\theta$, yields

$$\delta W_k = \frac{e^2}{M}\int 2\pi J dr d\theta \int \sqrt{2}\pi \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda \epsilon^{1/2} d\epsilon \frac{\partial F}{\partial \epsilon}\left(\omega - \omega_\star\right) B^2 \frac{\epsilon^2}{\Omega^2}$$

$$\cdot \sum_{-\infty}^{\infty} Y_p\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \frac{\exp\left(ip\omega_b \tau\right)}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega} \sum_{-\infty}^{\infty} Y_{p'}^\star\left(\Lambda, \epsilon, \bar{r}; \sigma\right) \exp\left(-ip'\omega_b \tau\right) \quad (23)$$

Applying $d\tau = \frac{qR_0}{\sigma\sqrt{2\epsilon}b\sqrt{1-\frac{\Lambda}{b}}} d\theta$, $\sigma = \pm 1$ for the direction of $v_\parallel$, one finally obtains

$$\delta W_k = \frac{4\pi^2}{M}\frac{e^2 B^2}{\Omega^2}\frac{1}{R_0}\int \frac{J}{q} dr \int d\Lambda \epsilon^3 d\epsilon \frac{\partial F}{\partial \epsilon} \tau_b \left(\omega - \omega_\star\right)$$
$$\cdot \sum_{-\infty}^{\infty} \frac{\left|Y_p\right|^2}{n\left\langle \dot{\phi} \right\rangle + p\omega_b - \omega}, \quad (24)$$

which is similar to Eq.(35) of Fu's 1993 paper with replacing ϵ and J by $\epsilon \equiv \frac{1}{2}Mv^2$ and $B = qR/J$. Note that $\tilde{\phi} \cong 0$, $\left\langle \dot{\phi} \right\rangle \cong \omega_D^0 + q\omega_b$,$\omega_D^0 \approx 0$ for passing particles.

In angle-action coordinate,

$$J_b = \frac{1}{2\pi}\int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\theta_b}^{\theta_b} \sqrt{1 - \kappa^{-1}\sin^2\frac{\theta}{2}} \frac{d\theta}{\pi} \quad (25)$$

$$J_t = \frac{1}{2\pi}\int p_\parallel ds \cong p_{\parallel e} R_\parallel \int_{-\pi}^{\pi} \sqrt{1 - \kappa^{-1}\sin^2\frac{\theta}{2}} \frac{d\theta}{\pi} \quad (26)$$

The formulas of bounce/transit frequency is given by [Alain J. Brizard, PHYSICS OF PLASMAS 18 2011]

$$\omega_b = \frac{\partial H}{\partial J_b} = \left(\frac{\partial J_b}{\partial E}\right)^{-1} = \frac{\pi\omega_\parallel}{2K\left(\kappa\right)}, \kappa < 1 \quad (27)$$

$$\omega_t = \frac{\partial H}{\partial J_t} = \left(\frac{\partial J_t}{\partial E}\right)^{-1} = \frac{\pi\sqrt{\kappa}\omega_\parallel}{K\left(\kappa^{-1}\right)}, \kappa > 1 \quad (28)$$

where $\omega_\parallel = \frac{1}{qR}\sqrt{\varepsilon\mu B_0} = \frac{\sqrt{\epsilon}}{qR}\sqrt{\varepsilon\Lambda}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. K denotes the complete elliptic integral of the first kind.

The normalized relations of the quantities are $F = \frac{n_0}{v_h^3}\bar{F}$**,** $v_h = \sqrt{\frac{2T_h}{M}}$**,** $\epsilon = \frac{T_h}{M}\bar{\epsilon}$**,** $r = a\bar{x}$**,** $J = aR_0\bar{J}$**,** $R = R_0\bar{R}$**,** $\omega_t = \frac{v_h}{R_0}\bar{\omega}_t$, $\frac{1}{\tau_t} = \frac{v_h}{2\pi R_0}\bar{\omega}_t = \frac{v_h}{R_0}\frac{\bar{\omega}_t}{2\pi} = \frac{v_h}{R_0}\frac{1}{\bar{\tau}_t}$**,** $\omega = \frac{v_h}{R_0}\bar{\omega}$, $\omega_\phi = \frac{v_h}{R_0}\bar{\omega}_\phi$, $\omega_\star = \frac{v_h}{R_0}\bar{\omega}_\star$ **.**

$$\delta W_k = \pi^2 a^2 R_0 n_0 T_h \int \frac{\bar{J}}{q} dx \int d\Lambda \bar{\epsilon}^3 d\bar{\epsilon} \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \bar{\tau}_b \left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot \sum_{-\infty}^{\infty} \frac{\left|\bar{Y}_p\right|^2}{n\left\langle \dot{\bar{\phi}} \right\rangle + p\bar{\omega}_b - \bar{\omega}} \tag{29}$$

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{q} \sqrt{\bar{\epsilon}} \tag{30}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R} \int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\, b\sqrt{1-\frac{\Lambda}{b}}} d\theta \tag{31}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \sqrt{\varepsilon\Lambda/2} \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} d\theta$$

$$Y_p\left(\Lambda, \bar{r}; \sigma\right) = \frac{1}{\tau_b} \oint d\tau \tilde{G}\left(\tau\right) \exp\left(-ip\omega_b\tau\right) \tag{32}$$

$$= \frac{\omega_b}{2\pi} \oint d\tau \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \oint d\left(\omega_b\tau\right) \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] \exp\left(-ip\int_0^\theta d\theta' \frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)} \frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[r\left(\tau\right), \theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(r,\theta\right)} + 2\left(1 - \frac{\Lambda}{b\left(r,\theta\right)}\right)\right)$$

$$\cdot \left(\left(g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r\right)\hat{\xi}_\theta\left(\theta, \bar{r} + \rho_d\cos\theta\right) + \left(g^{rr}\kappa_r + g^{r\theta}\kappa_\theta\right)\hat{\xi}_r\left(\theta, \bar{r} + \rho_d\cos\theta\right)\right)$$

Note that the effect of finite orbit width is induced radially in the perturbed $\xi\left(r\right)$ for simply. Furthermore, $\tilde{G}$ is a normalized quantity, so is Y_p,

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right]=\left(\frac{\Lambda}{b\left(x,\theta\right)}+2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_{\theta}+\bar{g}^{r\theta}\bar{\kappa}_{r}\right)\bar{\hat{\xi}}_{\theta}\left(\theta,\bar{x}+\frac{\rho_{d}}{a}\cos\theta\right)+\left(\bar{g}^{rr}\bar{\kappa}_{r}+\bar{g}^{r\theta}\bar{\kappa}_{\theta}\right)\bar{\hat{\xi}}_{r}\left(\theta,\bar{x}+\frac{\rho_{d}}{a}\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m}=\hat{\xi}_{\theta m}/a^{2}$, $\bar{\hat{\xi}}_{rm}=\hat{\xi}_{rm}/a$.

The slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$F\left(x,\bar{\epsilon},\Lambda\right)=\frac{n_{0}}{v_{h}^{3}}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_{c}^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_{0}}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_{0}}{\Delta x}\right)^{2}\right]\exp\left[-\left(\frac{\Lambda-\Lambda_{0}}{\Delta\Lambda}\right)^{2}\right] \tag{33}$$

where n_0 is determined by

$$n\left(x\right)=\frac{n_{0}}{v_{h}^{3}}\int d^{3}\mathbf{v}\frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_{c}^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_{0}}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_{0}}{\Delta x}\right)^{2}\right]\exp\left[-\left(\frac{\Lambda-\Lambda_{0}}{\Delta\Lambda}\right)^{2}\right]$$

at $x=x_0$.

The normalized metric tensors are

$$\bar{g}^{rr}=1+2\Delta'\cos\theta \tag{34}$$

with $\Delta'=\left(\varepsilon+\alpha\right)/4$, $\varepsilon=\frac{r}{R_0}$, $\alpha=-R_0q^2d\beta/dr$, $\beta=\frac{2\mu_0P}{B^2}$ set $\alpha=0$ if $\beta=0$, or assume $\bar{g}^{rr}=1$ without toroidal effect, θ independent. Specially, in low beta limit,

$$\bar{g}^{rr}=1+\frac{1}{2}\varepsilon\cos\theta \tag{35}$$

$$\bar{g}^{\theta\theta}=\frac{1}{x^{2}}\left[1-2\left(\varepsilon+\Delta'\right)\cos\theta\right] \tag{36}$$

assume $\bar{g}^{\theta\theta}=\frac{1}{x^2}$ without toroidal effect. Specially, in low beta limit,

$$\bar{g}^{\theta\theta}=\frac{1}{x^{2}}\left[1-\frac{5}{2}\varepsilon\cos\theta\right] \tag{37}$$

$$\bar{g}^{r\theta}=-\frac{1}{x}\left[\varepsilon+\left(r\Delta'\right)'\right]\sin\theta \tag{38}$$

specially,

$$\bar{g}^{r\theta}=-\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{39}$$

for low beta limit. and $\bar{g}^{r\theta}=0$ without toroidal effect.

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta + \frac{a}{R}\frac{\varepsilon}{4} - \frac{a}{R}\frac{5}{4}\varepsilon\left(\cos 2\theta - 1\right) - \left(\frac{a}{R}\right)^2 \frac{x}{q} \tag{40}$$

$$\bar{\kappa}_\theta = \varepsilon \sin\theta + \frac{5}{4}\varepsilon^2 \sin 2\theta \tag{41}$$

with $R = R_0 + r\cos\theta - \Delta\left(r\right) + r\eta\left(r\right)\left(\cos 2\theta - 1\right).\eta\left(r\right) = \left(\varepsilon + \Delta'\right)/2$.
The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m}{x}\frac{R}{a}\frac{\rho_h}{a}\frac{\partial \bar{F}/\partial x}{\partial \bar{F}/\partial \bar{\epsilon}} \tag{42}$$

where, m is poloidal mode number, $\rho_h = v_h/\Omega$, $v_h = \sqrt{2T_h/M}, \Omega = Be/M$.
The normalized ρ_d is

$$\bar{\rho}_d = \frac{\rho_d}{a} = \frac{q}{2}\frac{\rho_h}{a}\sqrt{\frac{\bar{\epsilon}}{\left(1 - \Lambda/b\right)}}\left[\frac{\Lambda}{b} + 2\left(1 - \frac{\Lambda}{b}\right)\right] \tag{43}$$

The normalized ξ are

$$\bar{\xi}_r\left(\theta, x\right) = \bar{\xi}_0 \exp\left(-i\theta\right)$$

$$\bar{\xi}_\theta\left(\theta, x\right) = -i\bar{\xi}_0 x \exp\left(-i\theta\right)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \leq r \leq r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r\left(\theta, x\right) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp\left(-i\theta\right)$$

$$\bar{\xi}_\theta\left(\theta, x\right) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp\left(-i\theta\right)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

The normalized $\delta\bar{W}_k$ is given by

$$\delta\bar{W}_k = \int \frac{\bar{J}}{q}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial \bar{F}}{\partial \bar{\epsilon}}\bar{\tau}_b\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\sum_{-\infty}^{\infty}\frac{\left|\bar{Y}_p\right|^2}{n\left\langle \dot{\bar{\phi}}\right\rangle + p\bar{\omega}_b - \bar{\omega}}, \tag{44}$$

where $\bar{J} = x$.

For passing particles,

$$\bar{\omega}_b = \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{\bar{\epsilon}} \tag{45}$$

where, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, $\varepsilon = \frac{r}{R_0}$. and

$$\left\langle \dot{\bar{\phi}} \right\rangle \cong q\bar{\omega}_b \tag{46}$$

$$\omega_b t = \bar{\omega}_b \frac{v_h}{R}\int_0^\theta \frac{qR_0}{\sqrt{2\left(T/M\right)\bar{\epsilon}}\,b\sqrt{1-\frac{\Lambda}{b}}}d\theta \tag{47}$$

$$= \int_0^\theta \frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\sqrt{\varepsilon\Lambda/2}\frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}d\theta$$

$$Y_p\left(\Lambda,\bar{r};\sigma\right) = \frac{1}{\tau_b}\oint d\tau\tilde{G}\left(\tau\right)\exp\left(-ip\omega_b\tau\right) \tag{48}$$

$$= \frac{\omega_b}{2\pi}\oint d\tau\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi}\oint d\left(\omega_b\tau\right)\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\omega_b\tau\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\tilde{G}\left[r\left(\tau\right),\theta\left(\tau\right)\right]\exp\left(-ip\int_0^\theta d\theta'\frac{\sigma\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$

where

$$\tilde{G}\left[x\left(\tau\right),\theta\left(\tau\right)\right] = \left(\frac{\Lambda}{b\left(x,\theta\right)} + 2\left(1-\frac{\Lambda}{b\left(x,\theta\right)}\right)\right)$$

$$\cdot\left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\hat{\xi}}_\theta\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\hat{\xi}}_r\left(\theta,\bar{x}+\bar{\rho}_d\cos\theta\right)\right)$$

where, the normalized displacements are $\bar{\hat{\xi}}_{\theta m} = \hat{\xi}_{\theta m}/a^2$, $\bar{\hat{\xi}}_{rm} = \hat{\xi}_{rm}/a$ and normalized drift orbit width is $\bar{\rho}_d = \frac{\rho_d}{a}$.

The normalized slowing down distribution function of fast ions is given by[M. Schneller 2013]

$$\bar{F}\left(x,\bar{\epsilon},\Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2}+\bar{\epsilon}_c^{3/2}}Erfc\left(\frac{\bar{\epsilon}-\bar{\epsilon}_0}{\Delta\bar{\epsilon}}\right)\exp\left[-\left(\frac{x-x_0}{\Delta x}\right)^2\right]\exp\left[-\left(\frac{\Lambda-\Lambda_0}{\Delta\Lambda}\right)^2\right] \tag{49}$$

The normalized metric tensors are

$$\bar{g}^{rr} = 1 + \frac{1}{2}\varepsilon\cos\theta \tag{50}$$

$$\bar{g}^{\theta\theta} = \frac{1}{x^2}\left[1 - \frac{5}{2}\varepsilon\cos\theta\right] \tag{51}$$

$$\bar{g}^{r\theta} = -\frac{1}{x}\frac{3}{2}\varepsilon\sin\theta \tag{52}$$

The normalized curvature are in low beta limit

$$\bar{\kappa}_r = -\frac{a}{R}\cos\theta \tag{53}$$

$$\bar{\kappa}_\theta = \varepsilon\sin\theta \tag{54}$$

The normalized $\omega_\star$ is

$$\bar{\omega}_\star = \frac{1}{2}\frac{m\partial\bar{F}/\partial x}{\partial\bar{F}/\partial\epsilon}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a} \tag{55}$$

The normalized ξ are

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\exp(-i\theta)$$

within $q = 1$ surface. In the inertial region $r_s - \frac{\Delta r}{2} \le r \le r_s + \frac{\Delta r}{2}$,

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0\left(\frac{\overline{\Delta r} - x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0 x\left(\frac{\overline{\Delta r} - 2x + \left(\bar{r}_s - \overline{\Delta r}/2\right)}{\overline{\Delta r}}\right)\exp(-i\theta)$$

with $\bar{\xi}_0 = \xi_0/a$, $\bar{r}_s = r_s/a$, $\overline{\Delta r} = \Delta r/a$, $x = r/a$.

2 The fishbone dispersion relation

The formula for δW_{MHD}, δI [Miyamoto, "Plasma Physics and Controlled Nuclear Fusion"] The energy principle is

$$\delta W_{MHD} + \delta W_k + \delta I = 0 \tag{56}$$

where

$$\delta I=\frac{\gamma^{2}}{2}\int\rho_{m}\left|\vec{\xi}\right|^{2}d\vec{r} \tag{57}$$

$$\delta W_{k}=\frac{1}{2}\int\vec{\xi}\cdot\nabla\delta p_{h}d\vec{r} \tag{58}$$

Note that δW_k is half of δW_k given by Eq.(15). δW_{MHD} consists of the contribution δW^{s}_{MHD} from the singular region near the rational surface and the contribution δW^{ext}_{MHD} from the external region.

The MHD potential energy $\delta W^{ext}_{MHDtor}/2\pi R$ of toroidal plasma with circular cross-section is given by

$$\frac{\delta W^{ext}_{MHDtor}}{2\pi R}=\left(1-\frac{1}{n^{2}}\right)\frac{\delta W^{ext}_{MHDcycl}}{2\pi R}+\frac{\pi B^{2}_{\theta s}}{2\mu_{0}}\left|\xi_{s}\right|^{2}\delta\hat{W}_{T} \tag{59}$$

$$\delta\hat{W}_{T}=\pi\left(\frac{r_{s}}{R}\right)^{2}3\left(1-q_{0}\right)\left(\frac{13}{144}-\beta^{2}_{ps}\right) \tag{60}$$

The term δW^{s}_{MHD} for the singular region is

$$\frac{\delta W^{s}_{MHD}}{2\pi R}=\frac{\pi}{2\mu_{0}}\frac{B^{2}_{\theta s}}{2\pi}sn\gamma\tau_{A\theta}\left|\xi_{s}\right|^{2}$$

where $B_{\theta s}=\frac{r_s B_t}{R q_s}$, $\tau_{A\theta}=\frac{3^{1/2}r_s}{\left(B^2_{\theta s}/\mu_0\rho_m\right)^{1/2}}$, $\rho_m=m_p n_{p_{r=r_s}}$, $s=r_s\frac{dq}{dr}_{r=r_s}$,$\xi_s=\xi_{r_{r=r_s}}$, n is toroidal mode number.

Thus, for $m=1,n=1$, the total sum of MHD contributions are

$$\begin{aligned}\delta W_{MHD}+\delta I &= 2\pi R\frac{B^{2}_{\theta s}}{2\mu_{0}}\left|\xi_{s}\right|^{2}\left(\delta\hat{W}_{T}+\gamma\tau_{A\theta}\frac{s}{2}+\pi\gamma^{2}\tau^{2}_{A\theta}\right) &(61)\\ &\approx 2\pi R\frac{B^{2}_{\theta s}}{2\mu_{0}}\left|\xi_{s}\right|^{2}\left(\delta\hat{W}_{T}+\gamma\tau_{A\theta}\frac{s}{2}\right), &(62)\end{aligned}$$

when $\gamma\tau_{A\theta}\ll 1$. With $\gamma=-i\omega$, the dispersion relation is

$$2\pi R\frac{B^{2}_{\theta s}}{2\mu_{0}}\left|\xi_{s}\right|^{2}\left(\delta\hat{W}_{T}+\frac{-i\omega}{\omega_{A}}\right)+\frac{1}{2}\pi^{2}a^{2}R_{0}n_{0}T_{h}\delta\bar{W}_{k}=0. \tag{63}$$

where, $\omega_A\equiv(\tau_{A\theta}s/2)^{-1}$ and $\delta\bar{W}_k$ is given by Eq.(42).
Finally,

$$\frac{4}{\pi}\frac{1}{q^{2}_{s}}\left(\frac{r_{s}}{R_{0}}\right)^{2}\left|\frac{\xi_{s}}{a}\right|^{2}\frac{1}{\beta_{h}}\delta\hat{W}_{T}+\frac{2\sqrt{3}}{\pi}\frac{s}{q_{s}}\sqrt{\frac{m_{p}n_{p}}{Mn_{0}}}\left(\frac{r_{s}}{R_{0}}\right)^{2}\left|\frac{\xi_{s}}{a}\right|^{2}\frac{1}{\beta^{1/2}_{h}}\left(-i\bar{\omega}\right)+\delta\bar{W}_{k}=0, \tag{64}$$

where $\beta_h=2\mu_0 n_0 T_h/B^2_t$, n_p is plasma density at $r=r_s$. n_0 is energetic particle density at $x=x_0$. m_p, M is ion mass, and energetic particle mass respectively. And $\bar{\omega}=\omega/\left(v_h/R_0\right)$.

3 Analytic form of the dispersion relation with passing particles and large aspect ratio approximation

For $\varepsilon \ll 1$, the normalized metric tensors are approximated as

$$\bar{g}^{rr} \approx 1 \tag{65}$$

$$\bar{g}^{\theta\theta} \approx \frac{1}{x^2} \tag{66}$$

$$\bar{g}^{r\theta} \approx -\frac{1}{x}\frac{3}{2}\varepsilon \sin\theta \tag{67}$$

and the normalized curvature are in low beta limit

$$\bar{\kappa}_r \approx -\frac{a}{R}\cos\theta \tag{68}$$

$$\bar{\kappa}_\theta \approx \varepsilon \sin\theta \tag{69}$$

The formula of ξ_θ and ξ_r are given by

$$\bar{\xi}_r(\theta, x) = \bar{\xi}_0(x)\exp(-i\theta), \tag{70}$$

$$\bar{\xi}_\theta(\theta, x) = -i\bar{\xi}_0(x)\, x \exp(-i\theta) \tag{71}$$

where, $\bar{\xi}_0(x) = \bar{\xi}_s H(x_s - x)$. $H(x)$ is step function, $H = 1$ for $x > 0$ and $H = 0$ for $x < 0$. Together with $\Lambda \ll 1$, one obtains

$$\tilde{G}[r(\tau), \theta(\tau)] = \left(\frac{\Lambda}{b(r,\theta)} + 2\left(1 - \frac{\Lambda}{b(r,\theta)}\right)\right)$$

$$\cdot \left(\left(\bar{g}^{\theta\theta}\bar{\kappa}_\theta + \bar{g}^{r\theta}\bar{\kappa}_r\right)\bar{\bar{\xi}}_\theta(\theta, \bar{x} + \bar{\rho}_d\cos\theta) + \left(\bar{g}^{rr}\bar{\kappa}_r + \bar{g}^{r\theta}\bar{\kappa}_\theta\right)\bar{\bar{\xi}}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)\right)$$

$$\approx 2\left(\frac{1}{x^2}\varepsilon\sin\theta\bar{\xi}_\theta - \frac{a}{R}\cos\theta\bar{\xi}_r\right)$$

$$\approx 2\left(-\frac{a}{R}i\sin\theta\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta) - \frac{a}{R}\cos\theta\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)\right)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)(i\sin\theta + \cos\theta)$$

$$\approx -2\frac{a}{R}\bar{\xi}_r(\theta, \bar{x} + \bar{\rho}_d\cos\theta)\exp(i\theta) \tag{72}$$

For $\kappa \gg 1$, the ellipitic fuction K becomes $K\left(\kappa^{-1}\right) = \pi/2$. Thus,

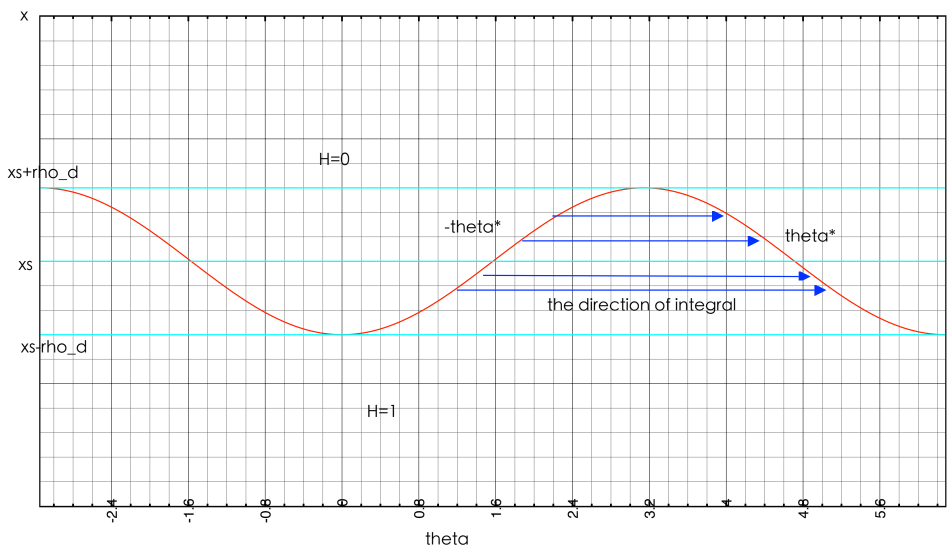

Figure 1: The integral region

$$\frac{\pi\sqrt{\kappa}}{K\left(\kappa^{-1}\right)}\frac{\sqrt{\varepsilon\Lambda/2}}{b\sqrt{1-\frac{\Lambda}{b}}} \approx \frac{\pi\sqrt{\frac{1}{2\varepsilon\Lambda}}}{\pi/2}\frac{\sqrt{\frac{\varepsilon\Lambda}{2}}}{b\sqrt{1-\frac{\Lambda}{b}}}$$
$$= \frac{1}{b\sqrt{1-\Lambda/b}} \tag{73}$$

Using Eq.(73) and Eq.(74), Y_p is rewritten as

$$Y_p = -\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}} 2\frac{a}{R}\bar{\xi}_r\left(\theta, \bar{x}+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\int_0^{\theta} d\theta' \frac{1}{b\sqrt{1-\frac{\Lambda}{b}}}\right)$$
$$= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_r\left(\theta, \bar{x}+\bar{\rho}_d\cos\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right)$$
$$= -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-\bar{x}-\bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right)\exp\left(i\theta\right)\exp\left(-ip\theta\right) \tag{74}$$

For $p=1$, we arrive at

$$Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_0^{2\pi} d\theta\bar{\xi}_s H\left(x_s-\bar{x}-\bar{\rho}_d\cos\theta\right)\exp\left(-i\theta\right) \tag{75}$$

From Fig.1, Y_1 is zero in the region $x < x_s - \bar{\rho}_d$ where $H = 1$ for $\theta \in [0, 2\pi]$ since the θ integral of $\sim \exp(-i\theta)$ from 0 to 2π is zero. In in the region $x > x_s + \bar{\rho}_d$, Y_1 also is zero since $H = 0$. Obviously, Y_1 is finite in the region $x_s - \bar{\rho}_d < x < x_s + \bar{\rho}_d$ where $H = 1$ for $\theta \in [\theta^\star, -\theta^\star + 2\pi]$. $\cos\theta^\star = \frac{x_s - x}{\bar{\rho}_d} \in [0, \pi]$. Thus,

$$Y_1 = -\frac{1}{\pi}\frac{a}{R}\int_{\theta^\star}^{2\pi-\theta^\star} d\theta\bar{\xi}_s \exp(-i\theta) \tag{76}$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i \left[\exp\left(-i\left(2\pi - \theta^\star\right)\right) - \exp\left(-i\theta^\star\right)\right]$$

$$= -\frac{1}{\pi}\frac{a}{R}\bar{\xi}_s i \left[\exp\left(-i\left(2\pi - \theta^\star\right)\right) - \exp\left(-i\theta^\star\right)\right]$$

$$= \frac{2}{\pi}\frac{a}{R}\bar{\xi}_s \sin\theta^\star$$

With $\bar{\omega}_b \simeq \sqrt{\bar{\epsilon}}/q$ and $p = 1$, $\delta\bar{W}_k$ becomes

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \bar{J}dx \int d\Lambda\bar{\epsilon}^3 d\bar{\epsilon}\frac{\partial\bar{F}}{\partial\bar{\epsilon}}\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\frac{\left|Y_1\right|^2}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q} - \bar{\omega}}. \tag{77}$$

Using $\bar{J} = x$ and the distribution function of passing particles for analytical purpose

$$\bar{F}\left(x, \bar{\epsilon}, \Lambda\right) = \frac{1}{\bar{\epsilon}^{3/2}}\delta\left(\Lambda\right) H\left(\bar{\epsilon}_c - \bar{\epsilon}\right)\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] \tag{78}$$

we have

$$\delta\bar{W}_k = \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int d\Lambda \int_0^{\bar{\epsilon}_c} \bar{\epsilon}^3 d\bar{\epsilon}\left(-\frac{3}{2}\bar{\epsilon}^{-5/2}\right)\delta\left(\Lambda\right)\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}}. \tag{79}$$

$$= \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} xdx \int \bar{\epsilon}^3 d\bar{\epsilon}\left(-\frac{3}{2}\bar{\epsilon}^{-5/2}\right)\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\frac{2\pi}{\sqrt{\bar{\epsilon}}}\left(\bar{\omega} - \bar{\omega}_\star\right)$$

$$\cdot\frac{\left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar{\epsilon}} + \frac{\sqrt{\bar{\epsilon}}}{q_s} - \bar{\omega}}. \tag{80}$$

$$\delta\bar{W}_k = I_1 + I_2 \tag{81}$$

$$I_1 = -3\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]\left[1-\left(\frac{x_s-x}{\bar{\rho}_d}\right)^2\right] x dx \int d\bar{\epsilon}\bar{\omega}$$

$$\cdot \frac{1}{\sqrt{\bar{\epsilon}}+\frac{\sqrt{\bar{\epsilon}}}{q_s}-\bar{\omega}}$$

$$= -3\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar{\rho}_d}\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx - \frac{1}{2}\frac{\Delta x^4}{\bar{\rho}_d^2} \\ \cdot \left(\exp\left(-\left(\frac{x_s-\bar{\rho}_d}{\Delta x}\right)^2\right) - \exp\left(-\left(\frac{x_s+\bar{\rho}_d}{\Delta x}\right)^2\right)\right) \end{array} \right\}$$

$$\cdot \bar{\omega} \int 2\sqrt{\bar{\epsilon}} d\sqrt{\bar{\epsilon}} \frac{1}{\left(1+\frac{1}{q_s}\right)\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

$$= -3\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar{\rho}_d}\right)^2 \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx - \frac{1}{2}\frac{\Delta x^4}{\bar{\rho}_d^2} \\ \cdot \exp\left(-\left(\frac{x_s-\bar{\rho}_d}{\Delta x}\right)^2\right)\left(1-\exp\left(-\left(\frac{4x_s\bar{\rho}_d}{\Delta x^2}\right)\right)\right) \end{array} \right\}$$

$$\cdot \bar{\omega} \int 2\sqrt{\bar{\epsilon}} d\sqrt{\bar{\epsilon}} \frac{1}{\left(1+\frac{1}{q_s}\right)\sqrt{\bar{\epsilon}}-\bar{\omega}}$$

For the condition of maximum of damping, we only keep the first term in the $\{\cdots\}$ of the above equation and make the approximation, i.e. $\int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} \exp\left[-\left(\frac{x}{\Delta x}\right)^2\right] dx \simeq \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \int_{x_s-\bar{\rho}_d}^{x_s+\bar{\rho}_d} dx = 2\bar{\rho}_d \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$. With $\bar{\rho}_d \simeq q\sqrt{\bar{\epsilon}}\rho_h/a$, we obtain

$$I_1 = -12\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s \Delta x^2 \bar{\omega} \frac{a}{q_s\rho_h} \frac{1}{1+\frac{1}{q_s}} \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \int_0^{\bar{\epsilon}_c} d\sqrt{\bar{\epsilon}}$$

$$\cdot \frac{1}{\sqrt{\bar{\epsilon}}-\bar{\omega}/\left(1+\frac{1}{q_s}\right)}$$

$$= -12\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s \Delta x^2 \bar{\omega} \frac{a}{q_s\rho_h} \frac{1}{1+\frac{1}{q_s}} \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right] \ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}}-\bar{\epsilon}_c^{1/2}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right) \tag{82}$$

$$I_2 = -\int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} \bar J dx \int d\Lambda \bar\epsilon^3 d\bar\epsilon \frac{\partial \bar F}{\partial \epsilon}\frac{2\pi}{\sqrt{\bar\epsilon}}\bar\omega_\star$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}$$

$$= -\int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} x dx \int d\Lambda \bar\epsilon^3 d\bar\epsilon \frac{2\pi}{\sqrt{\bar\epsilon}}\frac{1}{2} m \frac{\partial \bar F}{\partial x}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \sin^2\theta^\star}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}$$

$$= -\int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d} x dx \int d\Lambda \bar\epsilon^3 d\bar\epsilon \frac{2\pi}{\sqrt{\bar\epsilon}}\frac{1}{2}\frac{1}{x}\frac{R}{a}\frac{\rho_h}{a}\left(\frac{-2x}{\Delta x^2}\right)\frac{1}{\bar\epsilon^{3/2}}\delta(\Lambda)\exp\left(-\frac{x^2}{\Delta x^2}\right)$$

$$\cdot \frac{\left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \left[1-\left(\frac{x_s-x}{\bar\rho_d}\right)^2\right]}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}$$

$$= \left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\left(\frac{2\pi}{\Delta x^2}\right)\int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d}\left[1-\left(\frac{x_s-x}{\bar\rho_d}\right)^2\right]\exp\left(-\frac{x^2}{\Delta x^2}\right) x dx \int \bar\epsilon^3 d\bar\epsilon \frac{1}{\sqrt{\bar\epsilon}}\frac{1}{\bar\epsilon^{3/2}}$$

$$\cdot \frac{1}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}$$

$$\begin{aligned} = \quad & \left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \frac{R}{a}\frac{\rho_h}{a}\left(\frac{2\pi}{\Delta x^2}\right)\int_0^{\bar\epsilon_c}\bar\epsilon^3 d\bar\epsilon \frac{1}{\sqrt{\bar\epsilon}}\frac{1}{\bar\epsilon^{3/2}}\frac{1}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega} \\ & \cdot \left\{ \begin{array}{c} x_s\left(\frac{\Delta x}{\bar\rho_d}\right)^2 \int_{x_s-\bar\rho_d}^{x_s+\bar\rho_d}\exp\left[-\left(\frac{x}{\Delta x}\right)^2\right]dx - \frac{1}{2}\frac{\Delta x^4}{\bar\rho_d^2} \\ \cdot\left(\exp\left(-\left(\frac{x_s-\bar\rho_d}{\Delta x}\right)^2\right)-\exp\left(-\left(\frac{x_s+\bar\rho_d}{\Delta x}\right)^2\right)\right) \end{array} \right\} \end{aligned}$$

Simliar to the approximation made for I_1, we obtain

$$I_2 = \left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \frac{R}{a}\frac{4\pi}{q_s} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\int_0^{\bar\epsilon_c} d\bar\epsilon \frac{\sqrt{\bar\epsilon}}{\sqrt{\bar\epsilon}+\frac{\sqrt{\bar\epsilon}}{q_s}-\bar\omega}$$

$$= \left(\frac{2}{\pi}\frac{a}{R}\bar\xi_s\right)^2 \frac{R}{a}\frac{8\pi}{q_s}\frac{1}{1+\frac{1}{q_s}} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\int_0^{\bar\epsilon_c} d\sqrt{\bar\epsilon}\frac{\left(\sqrt{\bar\epsilon}\right)^2}{\sqrt{\bar\epsilon}-\bar\omega/\left(1+\frac{1}{q_s}\right)}$$

The intergal identity is given by

$$\int dy \frac{y^2}{y-a} = \int dy \frac{y^2-a^2+a^2}{y-a} = \int dy\,(y+a) + \int dy \frac{a^2}{y-a} \tag{83}$$

$$= \frac{1}{2}y^2 + ay + a^2 \ln(y-a) \tag{84}$$

Thus

$$I_2 = \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{8\pi}{q_s}\frac{1}{1+\frac{1}{q_s}} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\left[\frac{1}{2}\bar{\epsilon}_c + \frac{\bar{\omega}}{1+\frac{1}{q_s}}\sqrt{\bar{\epsilon}_c} + \left(\frac{\bar{\omega}}{1+\frac{1}{q_s}}\right)^2 \ln\left(\frac{\frac{\bar{\omega}}{1+\frac{1}{q_s}} - \bar{\epsilon}_c^{1/2}}{\frac{\bar{\omega}}{1+\frac{1}{q_s}}}\right)\right] \tag{85}$$

Defining $\Omega = \frac{\bar{\omega}}{(1+1/q_s)\bar{\epsilon}_c^{1/2}}$, I_1 and I_2 can be rewritten by

$$I_1 = -12\pi \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 x_s \Delta x^2 \sqrt{\bar{\epsilon}_c}\frac{a}{q_s\rho_h}\Omega \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\ln\left(1-\frac{1}{\Omega}\right) \tag{86}$$

$$I_2 = \left(\frac{2}{\pi}\frac{a}{R}\bar{\xi}_s\right)^2 \frac{R}{a}\frac{8\pi}{q_s}\frac{\bar{\epsilon}_c}{1+\frac{1}{q_s}} x_s \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]\left[\frac{1}{2} + \Omega + \Omega^2 \ln\left(1-\frac{1}{\Omega}\right)\right] \tag{87}$$

According to Eq. (81), the dispersion relation(64) without MHD contribution thus can be written as

$$-i\Omega + a_1 \Omega \ln\left(1-\frac{1}{\Omega}\right) + b_1\left[\frac{1}{2} + \Omega + \Omega^2 \ln\left(1-\frac{1}{\Omega}\right)\right] = 0 \tag{88}$$

with

$$a_1 = -8\sqrt{3}\frac{x_s}{s}\left(\frac{a}{r_s}\right)^2 \frac{a}{\rho_h}\frac{1}{1+\frac{1}{q_s}}\sqrt{\frac{Mn_0}{m_p n_p}}\Delta x^2 \beta_h^{1/2} \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$$

$$b_1 = \frac{16\sqrt{3}}{3}\left(\frac{a}{R}\right)\left(\frac{R}{r_s}\right)^2 \frac{x_s}{s}\sqrt{\frac{Mn_0}{m_p n_p}}\frac{\bar{\epsilon}_c^{1/2}}{\left(1+\frac{1}{q_s}\right)^2}\beta_h^{1/2} \exp\left[-\left(\frac{x_s}{\Delta x}\right)^2\right]$$

and

$$\left|\frac{a_1}{b_1}\right| = \frac{3}{2}\left(\frac{a}{R}\right)\left(\frac{a}{\rho_h}\right)\Delta x^2 \left(1+\frac{1}{q_s}\right)\bar{\epsilon}_c^{-1/2}.$$