$dwk++$ User Manual

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Contents

1 Introduction

dwk++ is a small code to calculate δW_k and solver fishbone dispersion relation for toakmak plasmas. The δW_k is calculate by 3D integration in phase space (r, Λ, E). A slowing down distribution function and a kink like mode structure is used. With a small growth rate (imag part of Ω) and tokamak parameters as a input, this code find out the fishbone mode frequency and fast ion $\beta_{h,0}$.

2 The defintion of δW_k

2.1 The normalized δW_k in the code:

$$\delta W_k = \sum^p \int_0^1 \frac{J}{q} dr \int d\Lambda \int E^3 dE \tau_b (\omega - \omega_\star) \frac{\partial F}{\partial E} \frac{|Y_p|^2}{n\omega_\phi + p\omega_b - \omega} \tag{1}$$

In current version, we only keep signle n. $\sigma = \pm 1$ is the direction of $v_\parallel$, m and n is poloidal and toroidal mode number, $\Lambda = \frac{\mu}{E}$, E is the fast ion energy, τ_b is the particle bounce time. The slowing down distribution function of fast ions is:

$$F = \frac{2^{3/2}}{C_f} \hat{F}(r, \epsilon, \Lambda) = \frac{2^{3/2}}{C_f} \frac{1}{E^{3/2} + E_c^{3/2}} erfc\left(\frac{E - E_0}{\Delta E}\right) e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda - \Lambda_0}{\Delta \Lambda})^2} \tag{2}$$

$$C_f = \int d^3\mathbf{v} \frac{1}{E^{3/2} + E_c^{3/2}} erfc\left(\frac{E - E_0}{\Delta E}\right) e^{-(\frac{\Lambda - \Lambda_0}{\Delta \Lambda})^2} \tag{3}$$

And $d^3\mathbf{v} = \frac{\sqrt{2}\pi}{b\sqrt{1-\frac{\Lambda}{b}}} d\Lambda E^{1/2} dE$.

$$C_f = \int \frac{\sqrt{2}\pi}{b\sqrt{1-\frac{\Lambda}{b}}} \frac{1}{E^{3/2}+E_c^{3/2}} erfc\left(\frac{E-E_0}{\Delta E}\right) e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} d\Lambda E^{1/2} dE \tag{4}$$

$$\frac{\partial F}{\partial E} = \frac{-2^{3/2}}{C_f}\left[\frac{2\exp(-(\frac{E-E_0}{\Delta E})^2)}{\sqrt{\pi}\Delta E(E^{3/2}+E_c^{3/2})} + \frac{3\sqrt{E}erfc(\frac{E-E_0}{\Delta E})}{2(E^{3/2}+E_c^{3/2})^2} - \frac{2\Lambda(\Lambda-\Lambda_0)erfc\left(\frac{E-E_0}{\Delta E}\right)}{E\Delta\Lambda^2(E^{3/2}+E_c^{3/2})}\right] e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} \tag{5}$$

$$\frac{\partial F}{\partial r} = \frac{2^{3/2}}{C_f} \frac{erfc\left(\frac{E-E_0}{\Delta E}\right)}{E^{3/2}+E_c^{3/2}} \frac{2(r_0-r)}{\Delta r^2} e^{-(\frac{r-r_0}{\Delta r})^2} e^{-(\frac{\Lambda-\Lambda_0}{\Delta\Lambda})^2} \tag{6}$$

The diamagnetic frequency:

$$\omega_* = \frac{nq}{2r}\frac{\rho_0}{\varepsilon_0}\frac{\partial F/\partial r}{\partial F/\partial E} \tag{7}$$

where ρ_0 is the gyro radius with injection energy. ε_0 is the inverse aspect-ratio, and $\varepsilon = \frac{r}{R_0}$. The transit frequency for passing particle is given below:

$$\omega_b = \frac{\pi\sqrt{\kappa}}{K(\kappa^{-1})}\frac{\sqrt{\varepsilon\Lambda/2}}{q}\sqrt{E} \tag{8}$$

where $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$, and K denotes the complete elliptic integral of the first kind. The transit frequency in toroidal:$\omega_\phi = q\omega_b$. Particle bounce time: $\tau_b = \frac{2\pi}{\omega_b}$. The integral along the particle orbits:

$$Y(r,\Lambda,E) = \frac{1}{2\pi}\int_0^{2\pi} \chi d\theta B_\Lambda\left(\Lambda_b + 2\left(1-\Lambda_b\right)\right) G(r,\theta,E) e^{-i\chi p\Theta} \tag{9}$$

where $\chi(r,\Lambda) = \frac{\sigma\pi\sqrt{\kappa}\sqrt{\varepsilon\Lambda/2}}{K(\kappa^{-1})}$, $\kappa = \frac{1-\Lambda(1-\varepsilon)}{2\varepsilon\Lambda}$. K denotes the complete elliptic integral of the first kind. $\Lambda_b = \frac{\Lambda}{b}$, $B_\Lambda(r,\Lambda,\theta) = \frac{1}{b\sqrt{(1-\Lambda_b)}}$, $b = B/B_0 = 1 + (r/R_0)cos\theta$.

$$G(\text{r,}\Lambda\text{,E,}\theta) = (g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r)\xi_\theta(\hat{r}(\bar{r},\rho_d,\theta),\theta) + (g^{rr}\kappa_r + g^{r\theta}\kappa_\theta)\xi_r(\hat{r}(\bar{r},\rho_d,\theta),\theta) \tag{10}$$

where $\hat{r} = \bar{r} + \rho_d cos\theta$.

$$\rho_d(r,\Lambda,E) = \frac{q}{2}\rho_0\sqrt{\frac{E}{1-\Lambda/b}}\left[\frac{\Lambda}{b} + 2(1-\frac{\Lambda}{b})\right] = \frac{q}{2}\rho_0\sqrt{\frac{E}{1-\Lambda_b}}\left[\Lambda_b + 2(1-\Lambda_b)\right] \tag{11}$$

To simplify the code, we use Λ_0 instead of Λ in ρ_d.

$$\rho_d(r,E) = \frac{q}{2}\rho_0\sqrt{\frac{E}{1-\Lambda_{0,b}}}\left[\Lambda_{0b} + 2(1-\Lambda_{ob})\right] \tag{12}$$

For $\Lambda_0 = 0$

$$\rho_d(r,E) = q\rho_0\sqrt{E}$$

$$G(\text{r,E,}\theta) = (g^{\theta\theta}\kappa_\theta + g^{r\theta}\kappa_r)\xi_\theta(\hat{r}(\bar{r},\rho_d,\theta),\theta) + (g^{rr}\kappa_r + g^{r\theta}\kappa_\theta)\xi_r(\hat{r}(\bar{r},\rho_d,\theta),\theta) \tag{13}$$

$g^{rr} = 1 + \frac{\varepsilon cos\theta}{2}$, $g^{\theta\theta} = \frac{1}{r^2}\left(1 - \frac{5}{2}\varepsilon cos\theta\right)$, $g^{r\theta} = -\frac{3}{2r}\varepsilon sin\theta$.

$$\Theta(\theta,r,\Lambda) = \int_0^\theta d\theta' \frac{1}{b\sqrt{(1-\Lambda_b)}} = \int_0^\theta B_\Lambda \tag{14}$$

2.2 The mode structure

In the current version, the mode structure is source code, and it is a kink structure with a fixed boundary at $r = 1$, and with a finite resonance layer width Δr.

$\xi_r(r,\theta) = \xi_{r0}(r)exp(-i\theta)$, $\xi_\theta(r,\theta) = -i\xi_{\theta 0}(r)rexp(-i\theta)$

$$\xi_{r0}(r) = \begin{cases} \xi_0 & r \le r_s - \Delta r/2 \\ \xi_0\frac{\Delta r - r + r_s - \Delta r/2}{\Delta r} & r_s - \frac{\Delta r}{2} < r < r_s + \frac{\Delta r}{2} \\ 0 & r \ge r_s + \frac{\Delta r}{2} \end{cases}$$

$$\xi_{\theta 0}(r) = \begin{cases} \xi_0 & r \le r_s - \Delta r/2 \\ \xi_0\frac{\Delta r - 2r + r_s - \Delta r/2}{\Delta r} & r_s - \frac{\Delta r}{2} < r < r_s + \frac{\Delta r}{2} \\ 0 & r \ge r_s + \frac{\Delta r}{2} \end{cases}$$

2.3 The normalized quantities used for δW_k:

$v_0 = \sqrt{2T_0/M}$, T_0 is the fast ions injection energy, M is the fast ion's mass. $\omega_0 = \frac{v_0}{R_0}$. $F_0 = \frac{n_0}{v_0^3}$, n_0 is the fast ion density at axis. $r_0 = a$ is the minor radius, $\varepsilon = a/R_0$, $E_0 = T_0/M$, B_0 is the torodial magnetic field at magnetic axis. $\delta W_{k,0} = \pi^2 a^2 R_0 n_0 T_0$.

3 Fishbone dispersion relation

Assume $\delta W_{mhd} = 0$, the normalized dispersion relation is:

$$\frac{4}{\pi}(\frac{r_s}{R_0})^2|\frac{\xi_0}{a})|^2(-i\frac{\omega}{\omega_A}) + \beta_h C_p \delta W_k = 0 \tag{15}$$

or:

$$i\omega = C\beta_{h,0} C_p \delta W_k \tag{16}$$

where:

$$C = \frac{\omega_A}{\omega_0}\frac{1}{4(\frac{r_s}{R_0})^2|\frac{\xi_0}{a}|^2} \tag{17}$$

and $C_p = \frac{p_0}{n_0 T_0}$. Here $\xi_s/\xi_0 = 1$, $\omega_A = \frac{v_A}{3^{1/2} R_0 s}$, $v_A = \frac{B}{\sqrt{\mu_0 \rho_m}}$, $s = r_s \frac{dq}{dr(r=r_s)}$, and$\beta_{h,0} = 8\pi n_0 T_h / B_t^2$.
Considering MHD contribution from $m = 1, n = 1$:

$$i\omega = C\beta_{h,0} C_P \delta W_k + \frac{\omega_A}{\omega_0}\delta W_T \tag{18}$$

where $\delta W_T = 3\pi\left(\frac{r_s}{R}\right)^2(1 - q_0)\left(\frac{13}{144} - \beta_{ps}^2\right)$.

4 How to run dwk++

4.1 Compile the code

dwk++ code is using c++ language as the main language, so it need a c++ compiler to compile the code. Gnu/g++ on max os and Linux and intel compiler on Linux was tested. To compile the code, libconfig with version >1.5 is needed, and set environment variable LIBCONFIG_DIR to the path where libconfig located. Set CXX to the c++ compiler (g++, icpc), and run 'make' at 'dwk-/' directory. If everything correct, a executable file 'dwk++' should be generated in 'dwk-/src/' directory and be copied to current directory. Then you can run dwk++ with dwk.cfg input file. A example of dwk.cfg can be find in 'dwk++/examples'.

4.2 Arguments

- -h print help information.
- -i input file name, dwk.cfg by default.
- -o outfile name, omega_dwk.out by default.
- -s scan dwk(omega), only find Ω_0 and $\beta_{h,0}$.
- -y write Yps_3D to Yps.nc.

4.3 Input file

Here is an example of input file:

Listing 1: a input file example

```
/*the input file for dwk++, parameters units is in ( ) */
//tokamak parameters
tokamak=
{
        a=0.38;            //minor radius (m).
        R0=1.30;           //major radius (m).
        Bt =0.84;          //Toroidal magnetic field at axis without plasma, (Tesla).
        n0 =1.7e19;        //thermal plasma density, (m^-3).
        mi =2.0;           //ion mass, (protom mass,m_p).
        E_i0 =25.142;      //Fost ion injection energy, (KeV).
```

```
        m_ep =2.0;          //fat ion mass (protom mass,m_p).
        //qc[0:7] q profile, q=qc[0] +qc[1]*r +qc[2] *r^2 ..... qc[7]*r^7.
        qc=[0.8, 0.0, 1.384189];
        q_s=1.0;            //q at resonance surface

}
//grid parameters
grid=
{
        nx=202; //grid size should be 3n + 1, n is a positive integer.
        nL=202;
        nE=202;
        ntheta=202;
}
//fast ion distribution
slowing=
{
        r0=0.0;             //(a)
        rd=0.2;             //(a)
        L0=0.01;            //Lambda_0
        Ld=0.02;            //Delta_Lambda
        E0=1.0;             //(E_i0)
        Ed=0.01;            //(E_i0)
        Ec=0.01;            //(E_i0)
        sigma=1;            //co(1) ? count (-1)
}
//perturbation and omega range for the soultion
mode=
{
        n=1;      //toroidal mode number.
        m=1;      //poloidal mode number.
        pa=0;     //resounces: sum Yps/(n*omega_phi +p*omega_theta -\omega).
        pb=0;     //sum from pa to pb.
        delta_r=0.001; //step function width for kink. (a)
        omega_0=0.1;  //find mode frequency and scan dwk between omega_0 and omega_1.
        omega_1=0.95;  //omega unit is (v_0/R_0), v_0 is the fast ion injection speed.
        omega_i=0.005;  //image part of omega, the growth rate.
        omega_n=100;       //scan steps.
        omega_err=1.0e-5; //residual of omega_0.
        max_iter =100;     //maximum iteration number to find the omega_0.
        max_iterg =2;      //
        dw_f=0.00;         //dw_mhd.
        zero_rhod=0;       //1: with drift orbit width effect, 0 without.
        xi_0 =0.01;        //(a) displacement at r=0.
}

dwkopt=
{
        omega_star_off=1;         //1: omega_star term on, 0: omega_star term off
        omega_off=1;              //1: omega term on, 0; omega term off
}
```

4.4 Output file

- omega_dwk.out,
- Yps.nc.

4.5 Utilities to plot results

There are some matlab & python scripts in 'dwk++/utilities' directory.

5 Benchmark

5.1 Compare with (WANG, Destabilization of internal kink modes at high frequency by energetic circulating ions. Physical Review Letters, 2001, 86) case ('dwk-/runp0_wsj').

Considering deep passing particles ($\Lambda = 0$), and using distribution function:

$$F = \frac{p_h}{\pi n_0 T_0 C_p} \frac{1}{E^{3/2}} \delta(\Lambda) H(E_0 - E) \tag{19}$$

The δW_k analytical results:

$$\delta W_k = (\delta W_{k,d} + \delta W_{k,s})$$

$$\delta W_{k,d} = -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \int_0^{r_s} dr \frac{dp_h}{dr} q \tag{20}$$

$$\delta W_{k,s} = \frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [-\frac{\Omega}{\Omega - 1} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] \int_0^{r_s} r p_h dr \tag{21}$$

Note that the second term $\delta W_{k,s}$ in Eq.21 is different with Eq(13) in Wang's PRL paper. Using PBX parameters: $B = 0.84$T, $\omega_{\zeta,0}/2\pi = 190kHz$, $R_0 = 1.3m$, $a = 0.38m$, the injection energy $T_0 = 25.142keV$. $n_i = 1.7 \times 10^{19} m^{-3}$, $\varepsilon_s = 1/9$, r_s =0.1444 m, $s = 0.4$.

Assume $p_h = p_0 exp(-(\frac{r}{\Delta r})^2)$, $q = c_0 + c_2 r^2$.

$$\begin{aligned}
\delta W_{k,d} &= -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \int_0^{r_s} dr \frac{dp_h}{dr} q \\
&= -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] p_0 \int_0^{r_s} dr \frac{de^{-(\frac{r}{\Delta r})^2}}{dr} (c_0 + c_2 r^2) \\
&= \frac{16}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \frac{p_0}{\Delta r^2} \int_0^{r_s} r e^{-(\frac{r}{\Delta r})^2} (c_0 + c_2 r^2) dr \\
&= \frac{16}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \frac{p_0}{\Delta r^2} \{-\frac{1}{2} \Delta r^2 e^{-(\frac{r}{\Delta r})^2} [c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s} \\
&= -\frac{8}{\varepsilon_0} \frac{\rho_h}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] p_0 \{e^{-(\frac{r}{\Delta r})^2} [c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s}
\end{aligned} \tag{22}$$

$$\begin{aligned}
\delta W_{k,s} &= -\frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] \int_0^{r_s} r p_h dr \\
&= -\frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] p_0 \int_0^{r_s} r e^{-(\frac{r}{\Delta r})^2} dr \\
&= - \frac{8}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] p_0 [-\frac{1}{2} \Delta r^2 e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} \\
&= \frac{4}{n_0 T_0 C_p} (\varepsilon_0 \xi_s)^2 [\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 lln(1 - \frac{1}{\Omega})] p_0 \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s}
\end{aligned} \tag{23}$$

So the normalized δW_k is :

$\frac{4(\varepsilon_0 \xi_0)^2 p_0}{n_0 T_0 C_p} \left\{ [\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \{e^{-(\frac{r}{\Delta r})^2} [c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s} \right\}$

Define $W_a = [\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \{e^{-(\frac{r}{\Delta r})^2} [c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s}$.

Consider the $\delta W_{k,0} = \pi^2 a^2 R_0 n_0 T_0$, finally, in physical units:

$$\begin{aligned}
\delta W_{k,0} \delta W_k &= \delta W_{k,0} \frac{4(\varepsilon_0 \xi_0)^2 p_0}{n_0 T_0 C_p} \quad W_a \\
&= \pi^2 a^2 R_0 n_0 T_0 \frac{4(\varepsilon_0 \xi_0)^2 p_0}{n_0 T_0 C_p} \quad W_a \\
&= \frac{4\pi^2 a^2 R_0 (\varepsilon_0 \xi_0)^2 p_0}{C_p} \quad W_a \\
&= \frac{4\pi^2 a^2 R_0 (\varepsilon_0 \xi_0)^2 p_0}{C_p} \quad \Big\{ [\frac{\Omega}{1 - \Omega} + \Omega + \Omega^2 ln(1 - \frac{1}{\Omega})] \Delta r^2 [e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} \\
&\qquad - \frac{2\rho_h}{\varepsilon_0} [\Omega^3 ln(1 - \frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}] \{e^{-(\frac{r}{\Delta r})^2} [c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s} \Big\}
\end{aligned}$$

Then we can get:

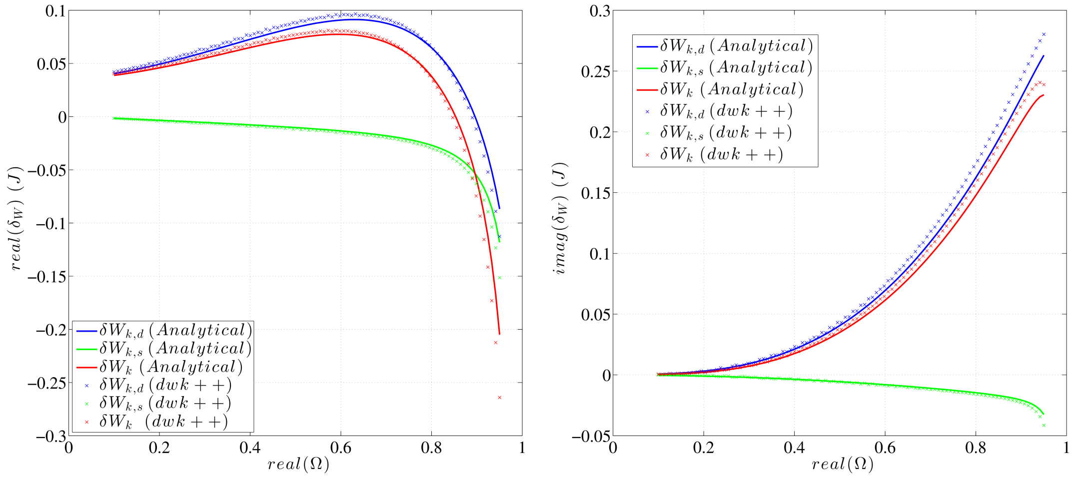

Figure 1: Comparison between $dwk++$ results and analysical's.

$$i\frac{\Omega}{\Omega_A} = \frac{1}{4}\frac{1}{\varepsilon_s^2}\frac{\beta_0}{|\xi_0|^2}\delta W_k \tag{24}$$

$$\begin{aligned} i\frac{\Omega}{\Omega_A} &= \frac{\varepsilon_0^2}{\varepsilon_s^2}\frac{p_0\beta_0}{n_0T_0C_p} \quad \Big\{[\frac{\Omega}{1-\Omega} + \Omega + \Omega^2 ln(1-\frac{1}{\Omega})]\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} \\ &\quad -\frac{2\rho_h}{\varepsilon_0}[\Omega^3 ln(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}]\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s}\Big\} \end{aligned} \tag{25}$$

The image part is $\Omega^3\log(1-\frac{1}{\Omega}) + \Omega^2 + \frac{1}{2}\Omega + \frac{1}{3}$ is $[(\Omega_r^3 - 3\Omega_r\Omega_i^2)\pi + (3\Omega_i\Omega_r^2 - \Omega_i^3)\log\left|1-\frac{1}{\Omega}\right| + 2\Omega_r\Omega_i + \frac{1}{2}\Omega_i]$, and the image part of $-\frac{\Omega}{\Omega-1} + \Omega + \Omega^2 log(1-\frac{1}{\Omega})$ is $\frac{\Omega_i}{(\Omega_r-1)^2+\Omega_i^2} + \Omega_i + (\Omega_r^2 - \Omega_i^2)\pi + 2\Omega_i\Omega_r log\left|1-\frac{1}{\Omega}\right|$. With $\Omega_i = 0$, it becomes $\pi\Omega_r^3$ and $\pi\Omega_r^2$. The dipersion relation is:

$$\frac{\Omega_r}{\Omega_A} = \frac{\varepsilon_0^2}{\varepsilon_s^2}\beta_0\Omega_r^2\pi\{\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0}\Omega_r\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s}\} \tag{26}$$

The critical β_0^{crit} given by

$$\beta_0^{crit} = \frac{\varepsilon_s^2}{\pi\Omega_A\Omega_r\varepsilon_0^2}\frac{1}{\Delta r^2[e^{-(\frac{r}{\Delta r})^2}]|_0^{r_s} - \frac{2\rho_h}{\varepsilon_0}\Omega_r\{e^{-(\frac{r}{\Delta r})^2}[c_2(\Delta r^2 + r^2) + c_0]\}|_0^{r_s}} \tag{27}$$

Using $q = 0.8 + 1.385r^2$, $\Delta r = 0.2$, and $\beta = 0.01$, $p_0 = 2.8083 \times 10^3 pascal$, and $\xi_0/a = 0.01$ compare the analytical results to $dwk++$ results (shown in Fig. 1). For $imag(\Omega) = 0.005$, the real frequency can be found by the code: $\Omega_r = 0.8472$,$(0.29137\omega_A$, $160.965kHz)$, $\beta_{0,crit} = 0.0402$ (without ω term, $\Omega_r = 0.8901$, $\beta_{0,crit} = 0.0333$). And by Eq. 27, $\beta_{0,crit} = 0.04130$ (without ω term, $\beta_{0,crit} = 0.03442$).

6 Appendix

For more details about formula derivation, please read limin_kinetic.pdf(dwk-/doc/limin_kinetic.pdf).