Problem 1

A very important equation in mathematical physics is the Laplace equation. It governs multiple phenomena such as propagation of heat, incompressible fluids or the potential of an electric field, among others. Consider a positive function $\phi : \mathbb{R}^2 \to \mathbb{R}$, representing, for example the temperature on a flat surface (at the point $(x, y) \in \mathbb{R}^2$ in the surface, the temperature is given by $\phi(x, y) \geq 0$). Then, the laplace equation, satisfied by the function ϕ is given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1}$$

the solution ϕ, to this differential equation is uniquely determined by its boundary conditions. Namely, given the value of ϕ in the boundary $\mathcal{C}$ of a closed region in $\mathbb{R}^2$, its values in the interior Ω (see Fig.1) are uniquely determined by requiring ϕ to satisfy (1).

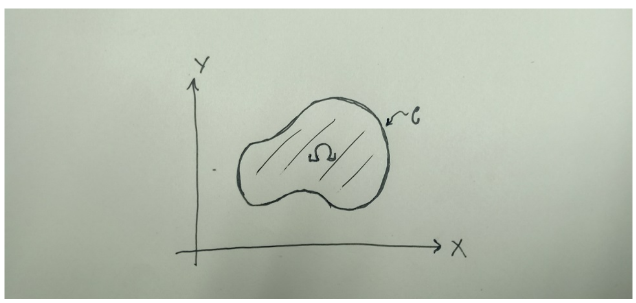

Figure 1: Regions $\mathcal{C}$ and Ω in $\mathbb{R}^2$.

In order to apply linear algebra to this problem we will consider an approximation by reducing $\mathbb{R}^2$ to a lattice. Consider a discrete set of points (x_i, y_j) (for example, you can imagine them given by $(i, j) \in \mathbb{Z}^2 \subset \mathbb{R}^2$) and the denote the value of ϕ at these points by $\phi_{i,j} = \phi(x_i, y_j)$. Also, we need to consider an approximation of the region $\mathcal{C}$, see Fig.2.

We distinguish between two sets of values of $\phi_{i,j}$, the values in the points at the interior Ω and the values at points in $\mathcal{C}$. Call the latter boundary values, and denote them by $\phi^b_{i,j}$ to distinguish them from the interior ones. Our unknowns are then the $\phi_{i,j}$ in Ω and consider the $\phi^b_{i,j} \geq 0$ as given. Then, the discrete laplace equation for ϕ in Ω is given by a system of linear equations in $\phi_{i,j}$:

$$\phi_{i,j} = \frac{1}{4}(\phi_{i+1,j} + \phi_{i,j-1} + \phi_{i-1,j} + \phi_{i,j+1}) \qquad \text{for all} \quad \phi_{i,j} \in \Omega \tag{2}$$

note that in the right hand side of (2) some of the $\phi_{i+1,j}$, etc. can be boundary values.

- Show that given any positive set of boundary values, there exist always a unique solution for the unknowns $\phi_{i,j}$.

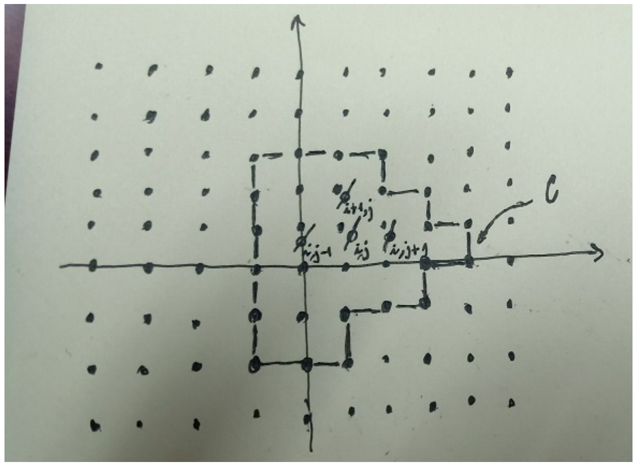

Figure 2: Regions $\mathcal{C}$ and Ω approximated to discrete set of points in $\mathbb{R}^2$, with the values of ϕ over each point indicated.

Problem 2

Suppose $b, c \in \mathbb{R}$ and consider the map $T : \mathbb{R}[x] \to \mathbb{R}^2$ given by

$$T(p) = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c\sin p(0)\right) \tag{3}$$

show that T is linear if and only if $b = c = 0$ (here $p'(x) = \frac{dp(x)}{dx}$).

Problem 3

Consider

$$A = \begin{pmatrix} -\alpha & 2 & 0 & 1 \\ \alpha & -3 & 2 & -1 \\ \alpha & -2 & -1 & 1 \\ 2\alpha & -2 & -4 & \beta \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ -2 \\ -1 \\ \alpha + \beta + 2 \end{pmatrix} \tag{4}$$

determine the values of α and β such that the system $Ax = b$ has:

1. A unique solution.
2. Infinite solutions.
3. No solution.

For $\alpha = 1$ and $\beta = -1$ find A^{-1} and the solution of $Ax = b$.

Problem 4

Define $\mathcal{H} = \{H = (h_{ij}) \in \mathcal{M}_{nn}(\mathbb{R}) | h_{ij} = 0 \forall i > j+1\}$ (here $\mathcal{M}_{nn}(\mathbb{R})$ denotes the set of $n \times n$ matrices with coefficients in $\mathbb{R}$).

1. Show that $\mathcal{H}$ is a vector subspace of $\mathcal{M}_{nn}(\mathbb{R})$
2. Show that if $T \in \mathcal{M}_{nn}(\mathbb{R})$ is upper triangular and $H \in \mathcal{H}$, then $T \cdot H \in \mathcal{H}$

1. Let $u \in \mathbb{R}^n$ with $\|u\| = \sqrt{u_1^2 + \cdots + u_n^2} = 1$ show that the matrix

$$A = \mathbf{1} - 2uu^t \tag{5}$$

is invertible with $A^{-1} = A$

Problem 5

Determine $a, b, c \in \mathcal{C}$ such that $p(x) = x^5 + ax^2 + b$ is divisible by $d(x) = x^3 + x^2 + cx + 1$.

Problem 6

Consider $J = \{p \in \mathbb{R}[x] | \deg(p) \leq 2, a_0 = 0, a_1 \neq 0\}$ and define the operation $p \Delta q = \sum_{i=1}^{2} c_i x^i$ where $\sum_{i=0}^{4} c_i x^i = p(q(x))$

1. Show that (J, Δ) is a nonabelian group.
2. Let $f : J \to \mathbb{R}$ such that $f(a_1 x + a_2 x^2) = a_1$. Prove that f is a surjective group homomorphism $f : (J, \Delta) \to (\mathbb{R} \setminus \{0\}, \cdot)$.
3. Let $H = \{p \in J | a_2 = 0\}$. Prove that (H, Δ) is an abelian subgroup of (J, Δ).

Note: If you are not familiar with the concepts of groups and group homomorphism, see Appendix.

Problem 7

Consider $p \in \mathbb{R}[x]$ such that p is monic and $\deg(p) = 3$. If we know that $(x-1)$ divides p and the remainders of p divided by $(x-2)$, $(x-3)$ and $(x-4)$ are all equal. Determine p, justifying every step and find all its roots.

Problem 8

Given $p(x) = x^5 - 2x^2 + 1$ divide it by the following polynomials $d(x)$: x^5, $x^2 - 2$, x^3, $x^2 - 3x + 1$ and $x - 1$, In each case obtain explicitly q and r such that $p = qd + r$.

Problem 9

By using the information that $p = 2z^3 - (5 + 6i)z^2 + 9iz - 3i + 1 \in \mathbb{C}[z]$ has one real root a, determine any of the roots of p.

Problem 10

Ruffini's algorithm provides a way to divide a polynomial $p \in \mathbb{R}[x]$ by $(x - c)$ for any $c \in \mathbb{R}$. Set $n = \deg(p)$ and $p = \sum_{i=0}^{n} a_i x^i$. The solution expresses $p = q(x-c)+r$ where $q = \sum_{i=0}^{n-1} b_i x^i$ with the coefficients b_i's defined recursively by

$$b_{n-i} = a_{n-i+1} + b_{n-i+1}c \tag{6}$$

for $i = 1, \ldots, n$ (and $b_n = 0$). What is the value of $r(x)$?. Show that this algorithm gives indeed the correct answer.

Appendix

For problem 6 you will need to recall the definition of a group. A group is a set, G, together with an operation $\bullet$ (called the group law of G) that combines any two elements $a, b \in G$ to form another element, denoted $a \bullet b$. To qualify as a group, the set and operation, $(G, \bullet)$, must satisfy four requirements known as the group axioms:

1. **Closure**
   For all $a, b \in G$, $a \bullet b \in G$.

2. **Associativity**
   For all $a, b, c \in G$, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

3. **Identity element**
   There exists an element $e \in G$ such that, for every element $a \in G$, the equation $e \bullet a = a \bullet e = a$ holds. Such an element is unique, and thus one speaks of the identity element.

4. **Inverse element**
   For each $a \in G$, there exists an element $b \in G$, commonly denoted a^{-1}, such that $a \bullet b = b \bullet a = e$, where e is the identity element.

for a general $(G, \bullet)$, the equation $a \bullet b = b \bullet a$ may not always be true. If it holds for every pair of elements in G, G is called abelian and nonabelian otherwise.
A group homomorphism between $(G, \bullet)$ and $(H, *)$ is a map $f : G \to H$ such that the group operation is preserved: $f(g_1 \bullet g_2) = f(g_1) * f(g_2)$ for all $g_1, g_2 \in G$.