Sols. HW 10

P1

a) Consider $y \in V_i$, then $V_i = \mathrm{Span}(v, (T-\lambda_i 1)v, \dots, (T-\lambda_i 1)^{m_i-1}v)$ for some $v \in V$, hence $y = \sum_{k=0}^{m_i-1} c_k (T-\lambda_i \cdot 1)^k v$, then write

$$(T-\lambda 1)y = (T-\lambda 1 + \lambda_i 1 - \lambda_i 1)y = \underbrace{\sum_{k=0}^{m_i-1} c_k (T-\lambda_i \cdot 1)^{k+1} v}_{\in V_i}$$

$$+ \underbrace{(\lambda_i - \lambda)\cdot y}_{\in V_i} \Rightarrow (T-\lambda\cdot 1)y \in V_i$$

then

$$(T-\lambda\cdot 1)^k y = (T-\lambda\cdot 1)^{k-1} \cdot \underbrace{(T-\lambda\cdot 1)\cdot y}_{\in V_i}$$, so by induction

$$(T-\lambda\cdot 1)^k \cdot y \in V_i$$

b) Now, if we restrict to V_i, we know, that in the basis $\{v, (T-\lambda_i)v, \dots (T-\lambda_i)^{m}v\}$, $\mathcal{M}(T)$ takes the form $\mathcal{M}(T) = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$, hence, is easy to see

$$\det\left((T-\lambda\cdot 1)\big|_{V_i}\right) = (\lambda_i - \lambda)^{m_i}$$, hence if $\lambda \neq \lambda_i$ $(T-\lambda\cdot 1)\big|_{V_i}$ is invertible. Likewise $\det\left((T-\lambda\cdot 1)^k\big|_{V_i}\right) = (\lambda_i-\lambda)^{k\cdot m_i}$, so if $\lambda\neq\lambda_i$ then $(T-\lambda\cdot 1)^k\big|_{V_i}$ is invertible as well, therefore $\dim\left((T-\lambda\cdot 1)^k \cdot V_i\right) = m_i$

If $\lambda = \lambda_i$, then $(T-\lambda_i\cdot 1)^k$ acting on an arbitrary vector $y \in V_i$ gives

$$(T-\lambda_i 1)^k \cdot \left(\sum_{l=0}^{m_i-1} c_l (T-\lambda_i 1)^l v\right) = \sum_{l=k}^{m_i-1} c_l (T-\lambda_i 1)^l v$$

so, any $u \in (T-\lambda_i\cdot 1)^k \cdot V_i$ can be expressed as a linear combination of the vectors

$$(T-\lambda_i 1)^k v, \dots, (T-\lambda_i\cdot 1)^{m_i-1} v$$

$$\therefore \dim (T-\lambda_i 1)^k V_i = m_i - k$$

c) $m' = \sum_{\lambda_i \neq \lambda} \dim V_i$

l_m = number of V_i's of dimension m and eigenvalue λ (i.e. # of blocks of dimension m associated to λ)

p = max dim. of V_i's associated w/ eigenvalue λ

So, in the part b) we learn:

$$\dim (T-\lambda\cdot \mathbb{1})^k V_i = \begin{cases} m_i & \lambda \neq \lambda_i \\ m_i - k & \lambda = \lambda_i \text{ and } k < m_i \\ 0 & \lambda = \lambda_i \text{ and } k \geq m_i \end{cases}$$

So, let's act with $(T-\lambda\cdot\mathbb{1})^k$ on $V = \bigoplus_{i=1}^{r} V_i$ and count dimensions:

$$\dim\left((T-\lambda\cdot\mathbb{1})^k V\right) = \sum_{i=1}^{r} \dim (T-\lambda\cdot\mathbb{1})^k V_i = \sum_{i,\, \lambda_i \neq \lambda} \dim (T-\lambda\cdot \mathbb{1})^k V_i$$

$$+ \sum_{i,\, \lambda_i = \lambda} \dim (T-\lambda\cdot\mathbb{1})^k V_i = \underbrace{\sum_{i,\, \lambda_i \neq \lambda} \overset{\dim V_i}{\overset{\shortparallel}{m_i}}}_{\overset{\shortparallel}{m'}} + \underbrace{\sum_{\substack{i,\, \lambda_i = \lambda \\ k < m_i}} (m_i - k)}_{\overset{\shortparallel}{(*)}}$$

Recall that there may be more than one λ_i asso. to V_i w/ value λ

Now, we need to manipulate the second sum. All the m_i's appearing there are the dimensions of V_i's w/ associated eigenvalue $\lambda_i = \lambda$, and, by definition, the possible values of m_i are $k+1, k+2, \dots, p$ (for $m_i > k$ in that sum, so, the possible values of m_i are $k+1, k+2, \dots, p$ (for some maximum value we just call p). So, let's split the sum $(*)$ according to these values:

$$(*) = \sum_{\substack{i,\, \lambda_i = \lambda \\ m_i = k+1}} (k+1-k) + \sum_{\substack{i,\, \lambda_i = \lambda \\ m_i = k+2}} (k+2-k) + \cdots + \sum_{\substack{i,\, \lambda_i = \lambda \\ m_i = p}} (p-k)$$

$$= l_{k+1} + 2\cdot l_{k+2} + \cdots + (p-k)\, l_p$$

Hence we proved

$$\dim\left((T-\lambda\cdot 1)^k V\right) = m' + l_{k+1} + 2\cdot l_{k+2} + \dots + (p-k)\, l_p$$

d) $r_k = rk\left((T-\lambda\cdot 1)^k\right) = \dim (T-\lambda\cdot 1)^k V$

(definition of rank)

Consider $k=0$, then $r_0 = \dim V$ ~~, then k=1 gives r_1~~ and using c):

$$\dim V = l_1 + 2l_2 + \dots + p l_p + m'$$

~~of~~ for

$$r_1 = \dim (T-\lambda\cdot 1)V = l_2 + 2l_3 + \dots + (p-1)\, l_p + m'$$

$$r_2 = \dim (T-\lambda\cdot 1)^2 V = l_3 + 2l_4 + \dots + (p-2)\, l_p + m'$$

$$\vdots$$

$$r_p = \dim (T-\lambda\cdot 1)^p V = m'$$

e) we have the eqs. ($m = \dim V$)

$$l_1 + 2l_2 + \dots + p l_p + m' = m$$

$$l_2 + \dots + (p-1)\, l_p + m' = r_1$$

$$\vdots$$

$$m' = r_p$$

by Gaussian elimination is easy to get

$$l_1 = m - 2r_1 + r_2$$

$$l_2 = r_1 - 2r_2 + r_3$$

$$l_3 = r_2 - 2r_3 + r_4$$

$$\vdots$$

$$l_p = r_{p-1} - r_p$$

hence, the numbers l_i do not depend on the choice of basis (since r_i's and eigenvalue λ do not). l_s is the number of ~~dimension~~ V_i's w/ dimension s, so, knowing the eigenvalues $\lambda_1, \dots, \lambda_m$ of T and the numbers l_s associated to each λ_i

we can completely reconstruct the Jordan form of T. Another way of putting it, is to say, in order to reconstruct the Jordan form of T, I need to know, for each $\lambda \in \sigma(T)$, how many blocks assoc. to λ does T have, and their dimensions. Fixing λ and computing the l_s's gives exactly this information.

P2

a) $$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad \text{so} \quad N_{ij} = \begin{cases} 1 & j = i+1, \quad 1 \leq i \leq m-1 \\ 0 & \text{otherwise} \end{cases}$$

so, it holds for $N^{r=1} = N$, assume

$$N^r_{ij} = \begin{cases} 1 & j = i+r \quad 1 \leq i \leq m-r \\ 0 & \text{otherwise} \end{cases}$$

Now compute N^{r+1}:

$$N^{r+1}_{ij} = \sum_{s=1}^{m} N_{is} N^r_{sj} \underset{\text{assume } j > r}{=} N_{i\,j-r} \underbrace{N^r_{j-r\,j}}_{=1} = N_{i\,j-r} = \begin{cases} 1 & j = i+r+1 \\ 0 & \text{otherwise} \end{cases}$$

(Clearly, if $j \leq r$, $N^{r+1}_{ij} = 0$)

hence N^m can only have a nonzero component N^m_{ij} if $j = i+m \Rightarrow N^m = 0$

b) ~~But~~ we know T can be put in Jordan form i.e.

$$T = C \operatorname{diag}(B_1, \dots, B_s) C^{-1} \quad \text{where } B_i = \begin{pmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix} \text{ are}$$

Jordan blocks, hence

$$T^r = C \operatorname{diag}(B_1^r, \dots, B_s^r) C^{-1}$$

So

$$e^T = \sum_{l=0}^{\infty} \frac{1}{l!} C \operatorname{diag}(B_1^l, \dots, B_s^l) C^{-1} = C \left(\sum_{l=0}^{\infty} \frac{1}{l!} \operatorname{diag}(B_1^l, \dots, B_s^l) \right) C^{-1}$$

$$\Rightarrow \quad \det(e^T) = \det\left(\sum_{l=0}^{\infty} \frac{1}{l!} \operatorname{diag}(B_1^l, \dots, B_s^l) \right)$$

easy to see this is strictly upper triangular

$$\text{Now,} \quad B_i^l = (D_i + N)^l = \sum_{k=0}^{l} \underbrace{D_i^{l-k} N^{k}}_{\text{by a), this is upper triangular}} \binom{l}{k} = D_i^l + \sum_{k=1}^{l} D_i^{l-k} N^k \binom{l}{k}$$

$$D_i = \begin{bmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix} = \lambda_i \cdot \mathbb{1}, \qquad N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}$$

binomial and use that $D_i N = N D_i$

$$= \begin{bmatrix} \lambda_i^l & & \\ & \ddots & \\ 0 & & \lambda_i^l \end{bmatrix}$$

Takes this form

$$\text{then} \quad \sum_{l=0}^{\infty} \frac{1}{l!} \operatorname{diag}(B_1^l, \dots, B_s^l) = \sum_{l=0}^{\infty} \operatorname{diag}\Big(\overbrace{\lambda_1^l/l!, \dots, \lambda_1^l/l!}^{\dim B_1 = m_1}, \dots, \overbrace{\lambda_s^l/l!, \dots, \lambda_s^l/l!}^{\dim B_s = m_s} \Big)$$

$+ M$, where M is strictly upper triangular, then

$$\det(e^T) = \det\left(\sum_{l=0}^{\infty} \operatorname{diag}\left(\lambda_1^l/l!, \dots, \lambda_1^l/l!, \dots, \lambda_s^l/l!, \dots, \lambda_s^l/l! \right) \right)$$

$$= \prod_{k=1}^{s} \left(\sum_{l=0}^{\infty} \lambda_k^l / l! \right)^{m_k} = \prod_{k=1}^{s} e^{\lambda_k \cdot m_k}$$

If we don't worry about convergence

On the other hand

$$\mathrm{Tr}(T) = \mathrm{Tr}\left(C \,\mathrm{diag}(B_1, \dots, B_s) C^{-1}\right) = \mathrm{Tr}\left(\mathrm{diag}(B_1, \dots, B_s)\right)$$

$$= m_1\lambda_1 + \dots + m_s\lambda_s$$

then $$e^{\mathrm{Tr}(T)} = e^{\sum_{k=1}^{s} m_k\lambda_k} = \det(e^T)$$