# INTRODUCTION

# TO

# LINEAR

# ALGEBRA

# Fifth Edition

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MANUAL FOR INSTRUCTORS

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## Problem Set 2.1, page 41

**1** The row picture for $A = I$ has 3 perpendicular planes $x = 2$ and $y = 3$ and $z = 4$. Those are perpendicular to the $x$ and $y$ and $z$ axes: $z = 4$ is a horizontal plane at height $4$.

The column vectors are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$. Then $\boldsymbol{b} = (2, 3, 4)$ is the linear combination $2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

**2** The planes in a row picture are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The three column vectors are changed; but the same combination (coefficients $z$, produces $\boldsymbol{b} = 34$), $(4, 9, 16)$.

**3** The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

**4** If $z = 2$ then $x + y = 0$ and $x - y = 2$ give the point $(x, y, z) = (1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

**5** If $x, y, z$ satisfy the first two equations they also satisfy the third equation $=$ sum of the first two. The line $\mathbf{L}$ of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$. (Notice that requirement $c + d = 1$. If you allow all $c$ and $d$, you get a plane.)

**6** Equation $1 +$ equation $2 -$ equation $3$ is now $0 = -4$. The intersection line $L$ of planes $1$ and $2$ misses plane $3$: *no solution*.

**7** Column $3 =$ Column $1$ makes the matrix singular. For $\boldsymbol{b} = (2, 3, 5)$ the solutions are $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$. $\boldsymbol{b} = (4, 6, c)$ needs $c = 10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns and the three equations add to $0 = 0$).

**8** Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if $A$ has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$. Solve them in reverse order!

**9** (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

**10** Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x} = (18, 5, 0)$ and $(3, 4, 5, 5)$. By rows or by columns: **9** separate multiplications when $A$ is 3 by 3.

**11** $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

**12** $A\boldsymbol{x}$ equals $(z, y, x)$ and $(0, 0, 0)$ and $(3, 3, 6)$.

**13** (a) $\boldsymbol{x}$ has $n$ components and $A\boldsymbol{x}$ has $m$ components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in $n$-dimensional space. The columns of $A$ are in $m$-dimensional space.

**14** $2x+3y+z+5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [\,2 \;\; 3 \;\; 1 \;\; 5\,]$ : *one row*. The solutions $(x, y, z, t)$ fill a 3D "plane" in 4 dimensions. It could be called a ***hyperplane***.

**15** (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = "identity" (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ = "permutation"

**16** $90^\circ$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180^\circ$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

**17** $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces $\begin{bmatrix} y \\ z \\ x \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. $Q$ is the *inverse* of $P$. Later we write $QP = I$ and $Q = P^{-1}$.

**18** $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

**19** $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$ and $E^{-1}E\boldsymbol{v}$ recovers $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

**20** $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the $x$-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the $y$-axis. The vector $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ projects to $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

**21** $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by 45° . The columns of $R$ are the results from rotating $(1,0)$ and $(0,1)$!

**22** The dot product $A\boldsymbol{x} = [\,1 \;\; 4 \;\; 5\,]\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (1 by 3)(3 by 1) is zero for points $(x,y,z)$ on a plane in three dimensions. The 3 columns of $A$ are one-dimensional vectors.

**23** $A = [\,1 \;\; 2 \;\; \textbf{;} \;\; 3 \;\; 4\,]$ and $\boldsymbol{x} = [\,5 \;\; -2\,]'$ or $[\,5 \;\; \textbf{;} \;\; -2\,]$ and $\boldsymbol{b} = [\,1 \;\; 7\,]'$ or $[\,1 \;\; \textbf{;} \;\; 7\,]$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as two zeros.

**24** $A * \boldsymbol{v} = [\,3 \;\; 4 \;\; 5\,]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$. But $\boldsymbol{v} * A$ gives an error message from 3 by 1 times 3 by 3.

**25** **ones**$(4,4) *$ **ones**$(4,1) =$ column vector $[\,4 \;\; 4 \;\; 4 \;\; 4\,]'$; $B * \boldsymbol{w} = [\,10 \;\; 10 \;\; 10 \;\; 10\,]'$.

**26** The row picture has two lines meeting at the solution $(4,2)$. The column picture will have $4(1,1) + 2(-2,1) =$ **4**(column 1) $+$ **2**(column 2) $=$ right side $(0,6)$.

**27** The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally fill a ***line in*** $3$***-dimensional space***.

**28** The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

**29** $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive. Their components still add to 1.

**30** $\boldsymbol{u}_7$ and $\boldsymbol{v}_7$ have components adding to 1; they are close to $\boldsymbol{s} = (.6, .4)$. $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ ***steady state*** $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

**31** $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15,15,15)$; $M_4(1,1,1,1) = (34,34,34,34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

**32** $A$ is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

**33** $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$. **Linearity** means: When $w$ is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

**34** $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

**35** $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$ are in Section 2.7. The same 6 permutations of ***blocks*** of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

## Problem Set 2.2, page 53

**1** Multiply equation 1 by $\ell_{21} = \frac{10}{2} = \mathbf{5}$ and subtract from equation 2 to find $2x + 3y = 1$ (unchanged) and $-6y = 6$. The pivots to circle are 2 and $-6$.

**2** $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

**3** Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $\boldsymbol{3y = 3}$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

**4** Subtract $\ell = \frac{c}{a}$ times equation 1 from equation 2. The new second pivot multiplying $y$ is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$. Notice the "determinant of $A$" $= ad - bc$. It must be nonzero for this division.

**5** $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x+2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. The two lines in the row picture are the same line, containing all solutions.

**6** Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines $2x + 4y = 16$ and $4x + 8y = 32$ become the ***same***: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

**7** If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

**8** If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

**9** On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line in the row picture). The column picture has both columns along the same line.

**10** The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = 20 - 4 = c = 16$.

**11** (a) Another solution is $\frac{1}{2}(x+X, y+Y, z+Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

**12** Elimination leads to this upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + \phantom{3}z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

If a zero is at the start of row 2 or row 3, that avoids a row operation.

**13**
$$\begin{aligned} 2x - 3y \phantom{+ z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y + 3z &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$$

Here are steps $1, 2, 3$: Subtract 2 $\times$ row 1 from row 2, subtract 1 $\times$ row 1 from row 3, subtract 2 $\times$ row 2 from row 3

**14** Subtract 2 times row 1 from row 2 to reach $(d-10)y-z=2$. Equation (3) is $y-z=3$. If $d=10$ exchange rows 2 and 3. If $d=11$ the system becomes singular.

**15** The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. But equation $(3)$ is the same so there is a *line of solutions* $(x,y,z)=(1,1,-1)$.

**16** (a) **Example of 2 exchanges**
$$\begin{aligned} 0x+0y+2z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$
(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then breakdown**
$$\begin{aligned} 0x+3y+4z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$
(rows 1 and 3 are not consistent)

**17** If row $1=$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and row 3 has no pivot. If column $2=$ column $1$, then column 2 has no pivot.

**18** *Example* $x+2y+3z=0$, $4x+8y+12z=0$, $5x+10y+15z=0$ has 9 different coefficients but rows 2 and 3 become $0=0$: infinitely many solutions to $A\boldsymbol{x}=\mathbf{0}$ but almost surely no solution to $A\boldsymbol{x}=\boldsymbol{b}$ for a random $\boldsymbol{b}$.

**19** Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular—no third pivot. Then if $t=5$ the third equation is $0=0$ which allows infinitely many solutions. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

**20** Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1{+}2{=}$row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution. The three planes in the row picture form a triangular tunnel.

**21** (a) Pivots $2,\frac{3}{2},\frac{4}{3},\frac{5}{4}$ in the equations $2x+y=0$, $\frac{3}{2}y+z=0$, $\frac{4}{3}z+t=0$, $\frac{5}{4}t=5$ after elimination. Back substitution gives $t=4, z=-3, y=2, x=-1$. (b) If the off-diagonal entries change from $+1$ to $-1$, the pivots are the same. The solution is $(1,2,3,4)$ instead of $(-1,2,-3,4)$.

**22** The fifth pivot is $\frac{6}{5}$ for both matrices (1's or $-1$'s off the diagonal). The $n$th pivot is $\frac{n+1}{n}$.

**23** If ordinary elimination leads to $x + y = 1$ and $2y = 3$, the original second equation could be $2y + \ell(x+y) = 3 + \ell$ for any $\ell$. Then $\ell$ will be the multiplier to reach $2y = 3$, by subtracting $\ell$ times equation 1 from equation 2.

**24** Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$. (You could notice that the determinant $a^2 - 2a$ is zero for $a = 2$ and $a = 0$.)

**25** $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

**26** Solvable for $s = 10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2 + s$ on the right sides). So 12 must agree with $2 + s$, which makes $s = 10$. The four equations for $a, b, c, d$ are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

**27** Elimination leaves the diagonal matrix diag$(3, 2, 1)$ in $3x = 3, 2y = 2, z = 2$. Then $x = 1, y = 1, z = 2$.

**28** $A(2, :) = A(2, :) - 3 * A(1, :)$ subtracts 3 times row 1 from row 2.

**29** The average pivots for rand(3) ***without*** row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! ***With row exchanges*** in MATLAB's **lu** code, the averages $.75$ and $.50$ and $.365$ are much more stable (and should be predictable, also for **randn** with normal instead of uniform probability distribution for the numbers in $A$).

**30** If $A(5, 5)$ is 7 not 11, then the last pivot will be $0$ not $4$.

**31** Row $j$ of $U$ is a combination of rows $1, \ldots, j$ of $A$ (when there are no row exchanges). If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). $U$ just keeps the diagonal of $A$ when $A$ is ***lower triangular.***

**32** The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when $A$ is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A = \textbf{ones}(100)$. A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through** $\mathbf{0}$. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

## Problem Set 2.3, page 66

**1** $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

**2** $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, 0)$. When $E_{32}$ comes first, row 3 feels no effect from row 1.

**3** $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

Those $E$'s are in the right order to give $MA = U$.

**4** Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $\boldsymbol{z} = -5, y = \frac{1}{2}, x = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

**5** Changing $a_{33}$ from 7 to 11 will change the third pivot from 5 to 9. Changing $a_{33}$ from 7 to 2 will change the pivot from 5 to *no pivot*.

**6** Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

**7** To reverse $E_{31}$, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mathbf{7} & 0 & 1 \end{bmatrix}$ is $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{7} & 0 & 1 \end{bmatrix}$. Multiplication confirms $EE^{-1} = I$.

**8** $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$. $\det M* = a(d - \ell b) - b(c - \ell a)$ reduces to $ad - bc$! Subtracting row 1 from row 2 doesn't change $\det M$.

**9** $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need $E_{31}$ (not $E_{21}$) to act on the new row 3.

**10** $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

**11** An example with two negative pivots is $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The diagonal entries can change sign during elimination.

**12** The first product is $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

**13** (a) $E$ times the third column of $B$ is the third column of $EB$. A column that starts at zero will stay at zero. (b) $E$ could add row 2 to row 3 to change a zero row to a nonzero row.

**14** $E_{21}$ has $-\ell_{21}=\frac{1}{2}$, $E_{32}$ has $-\ell_{32}=\frac{2}{3}$, $E_{43}$ has $-\ell_{43}=\frac{3}{4}$. Otherwise the $E$'s match $I$.

**15** $a_{ij} = 2i - 3j$: $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & \mathbf{0} & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & \mathbf{-12} & -24 \end{bmatrix}$. The zero became $-12$, an example of ***fill-in***. To remove that $-12$, choose $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

Every 3 by 3 matrix with entries $a_{ij} = ci + dj$ is **singular**!

**16** (a) The ages of $X$ and $Y$ are $x$ and $y$: $x - 2y = 0$ and $x + y = 33$; $x = 22$ and $y = 11$ (b) The line $y = mx + c$ contains $x = 2$, $y = 5$ and $x = 3$, $y = 7$ when $2m + c = 5$ and $3m + c = 7$. Then $m = 2$ is the slope.

**17** The parabola $y=a+bx+cx^2$ goes through the 3 given points when $\begin{matrix} a+ & b+ & c= & 4 \\ a+ & 2b+ & 4c= & 8 \\ a+ & 3b+ & 9c= & 14 \end{matrix}$.

Then $a = 2$, $b = 1$, and $c = 1$. This matrix with columns $(1, 1, 1)$, $(1, 2, 3)$, $(1, 4, 9)$ is a "Vandermonde matrix."

**18** $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

**19** $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. In the opposite order, two row exchanges give $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = I$. If $M$ exchanges rows 2 and 3 then $M^2 = I$ (also $(-M)^2 = I$). There are many square roots of $I$: Any matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ has $M^2 = I$ if $a^2 + bc = 1$.

**20** (a) Each column of $EB$ is $E$ times a column of $B$ (b) $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\ \mathbf{1} & \mathbf{2} & \mathbf{4}\end{bmatrix} =$ $\begin{bmatrix}1 & 2 & 4\\ \mathbf{2} & \mathbf{4} & \mathbf{8}\end{bmatrix}$. All rows of $EB$ are ***multiples*** of $\begin{bmatrix}1 & 2 & 4\end{bmatrix}$.

**21** **No.** $E = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $F = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ give $EF = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$ but $FE = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$.

**22** (a) $\sum a_{3j}x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(EA\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j}x_j$.

**23** $E(EA)$ subtracts $4$ times row $1$ from row $2$ ($EEA$ does the row operation twice). $AE$ subtracts $2$ times column $2$ of $A$ from column $1$ (multiplication by $E$ on the right side acts on **columns** instead of rows).

**24** $\begin{bmatrix}A & \boldsymbol{b}\end{bmatrix} = \begin{bmatrix}2 & 3 & \mathbf{1}\\ 4 & 1 & \mathbf{17}\end{bmatrix} \to \begin{bmatrix}2 & 3 & \mathbf{1}\\ 0 & -5 & \mathbf{15}\end{bmatrix}$. The triangular system is $\begin{aligned}2x_1 + 3x_2 &= 1\\ -5x_2 &= 15\end{aligned}$
Back substitution gives $x_1 = 5$ and $x_2 = -3$.

**25** The last equation becomes $0 = 3$. If the original 6 is 3, then row $1 +$ row $2 =$ row 3. Then the last equation is $0 = 0$ and the system has infinitely many solutions.

**26** (a) Add two columns $\boldsymbol{b}$ and $\boldsymbol{b}^*$ to get $[A\ \ \boldsymbol{b}\ \ \boldsymbol{b}^*]$. The example has
$\begin{bmatrix}1 & 4 & 1 & 0\\ 2 & 7 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 4 & 1 & 0\\ 0 & -1 & -2 & 1\end{bmatrix} \to \boldsymbol{x} = \begin{bmatrix}-7\\ 2\end{bmatrix}$ and $\boldsymbol{x}^* = \begin{bmatrix}4\\ -1\end{bmatrix}$.

**27** (a) No solution if $d=0$ and $c \neq 0$ (b) Many solutions if $d=0=c$. No effect from $a, b$.

**28** $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

**29** $E = \begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$ subtracts each row from the next row. The result $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 2 & 1\end{bmatrix}$
still has multipliers $= 1$ in a $3$ by $3$ Pascal matrix. The product $M$ of all elimination

matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$. This "alternating sign Pascal matrix" is on page $91$.

**30** (a) $E = A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ will reduce row 2 of $EM$ to $[2 \;\; 3]$.

(b) Then $F = B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ will reduce row 1 of $FEM$ to $[1 \;\; 1]$.

(c) Then $E = A^{-1}$ twice will reduce row 2 of $EEFEM$ to $[0 \;\; 1]$

(d) Now $EEFEM = B$. Move $E$'s and $F$'s to get $\boldsymbol{M} = \boldsymbol{ABAAB}$. This question focuses on positive integer matrices $M$ with $ad - bc = 1$. The same steps make the entries smaller and smaller until $M$ is a product of $A$'s and $B$'s.

**31** $E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & b & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & c & 1 \end{bmatrix}$,

$$E_{43}\, E_{32}\, E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ ab & b & 1 & \\ abc & bc & c & 1 \end{bmatrix}$$

## Problem Set 2.4, page 77

**1** If all entries of $A, B, C, D$ are 1, then $BA = 3\,\textbf{ones}(5)$ is 5 by 5; $AB = 5\,\textbf{ones}(3)$ is 3 by 3; $ABD = 15\,\textbf{ones}(3,1)$ is 3 by 1. $DC$ and $A(B + C)$ are not defined.

**2** (a) $A$ (column 2 of $B$) (b) (Row 1 of $A$) $B$ (c) (Row 3 of $A$)(column 5 of $B$)
(d) (Row 1 of $C$)$D$(column 1 of $E$). (Part (c) assumed 5 columns in $B$)

**3** $AB + AC$ is the same as $A(B+C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

**4** $A(BC) = (AB)C$ by the ***associative law***. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Column 1 of $AB$ and row 2 of $C$ are zero (then multiply columns times rows).

**5** (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

**6** $(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

**7** (a) True (b) False (c) True (d) False: usually $(AB)^2 = ABAB \neq A^2B^2$.

**8** The rows of $DA$ are 3 (row 1 of $A$) and 5 (row 2 of $A$). Both rows of $EA$ are row 2 of $A$. The columns of $AD$ are 3 (column 1 of $A$) and 5 (column 2 of $A$). The first column of $AE$ is zero, the second is column 1 of $A$ + column 2 of $A$.

**9** $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

**10** $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not the same as $F(EA)$ because multiplication is not commutative: $EF \neq FE$.

**11** Suppose $EA$ does the row operation and then $(EA)F$ does the column operation (because $F$ is multiplying from the right). The associative law says that $(EA)F = E(AF)$ so the column operation can be done first !

**12** (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of $B$ is $1, 0, 0$.

**13** $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $\boldsymbol{b} = \boldsymbol{c} = \mathbf{0}$. Then $AC = CA$ gives $\boldsymbol{a} = \boldsymbol{d}$. The only matrices that commute with $B$ and $C$ (and all other matrices) are multiples of $I$: $A = aI$.

**14** $(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$. In a typical case (when $AB \neq BA$) the matrix $A^2 - 2AB + B^2$ is different from $(A - B)^2$.

**15** (a) True ($A^2$ is only defined when $A$ is square).

(b) False (if $A$ is $m$ by $n$ and $B$ is $n$ by $m$, then $AB$ is $m$ by $m$ and $BA$ is $n$ by $n$).

(c) True by part (b).

(d) False (take $B = 0$).

**16** (a) $mn$ (use every entry of $A$) (b) $mnp = p\times$part (a) (c) $n^3$ ($n^2$ dot products).

**17** (a) Use only column 2 of $B$ (b) Use only row 2 of $A$ (c)–(d) Use row 2 of first $A$.

Column 2 of $AB = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Row 2 of $AB = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ Row 2 of $A^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Row 2 of $A^3 = \begin{bmatrix} 3 & -2 \end{bmatrix}$

**18** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ has $a_{ij} = \min(i, j)$. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has $a_{ij} = (-1)^{i+j} =$ "alternating sign matrix". $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$ has $a_{ij} = i/j$. This will be an example of a *rank one matrix*: 1 column $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^{\mathrm{T}}$ multiplies 1 row $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$.

**19** Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

**20** (a) $a_{11}$ (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - \left(\dfrac{a_{31}}{a_{11}}\right) a_{12}$ (d) $a_{22} - \left(\dfrac{a_{21}}{a_{11}}\right) a_{12}$.

**21** $A^2 = \begin{bmatrix} 0&0&4&0 \\ 0&0&0&4 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0&0&0&8 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A^4 =$ zero matrix for ***strictly triangular*** $A$.

Then $A\boldsymbol{v} = A\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$, $A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$, $A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A^4\boldsymbol{v} = \mathbf{0}$.

**22** $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

**23** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. Note: Any matrix $A =$ column times row $= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ will have $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = 0$ if $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$. $A = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$

but $A^3 = 0$; strictly triangular as in Problem 21.

**24** $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

**25** $\begin{bmatrix} a&b&c \\ d&e&f \\ g&h&i \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix}\begin{bmatrix} 1&0&0 \end{bmatrix} + \begin{bmatrix} d \\ e \\ h \end{bmatrix}\begin{bmatrix} 0&1&0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix}\begin{bmatrix} 0&0&1 \end{bmatrix}$.

**26** **Columns of $A$ times rows of $B$** $\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}3 & 3 & 0\end{bmatrix} + \begin{bmatrix}0\\4\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\end{bmatrix} = \begin{bmatrix}3 & 3 & 0\\6 & 6 & 0\\6 & 6 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\4 & 8 & 4\\1 & 2 & 1\end{bmatrix} =$
$\begin{bmatrix}3 & 3 & 0\\10 & 14 & 4\\7 & 8 & 1\end{bmatrix} = AB.$

**27** (a) (row 3 of $A$) $\cdot$ (column 1 or 2 of $B$) and (row 3 of $A$) $\cdot$ (column 2 of $B$) are all zero.

(b) $\begin{bmatrix}x\\x\\0\end{bmatrix}\begin{bmatrix}0 & x & x\end{bmatrix} = \begin{bmatrix}0 & x & x\\0 & x & x\\0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}x\\x\\x\end{bmatrix}\begin{bmatrix}0 & 0 & x\end{bmatrix} = \begin{bmatrix}0 & 0 & x\\0 & 0 & x\\0 & 0 & x\end{bmatrix}$ : **both upper**.

**28** $A$ times $B$ with cuts $\quad \underbrace{A\Big[\;|\;|\;|\;\Big]}_{\text{4 cols}}, \quad \underbrace{\begin{bmatrix}\text{——}\end{bmatrix}B}_{\text{2 rows}}, \quad \underbrace{\begin{bmatrix}\text{——}\end{bmatrix}\Big[\;|\;|\;|\;\Big]}_{\text{2 rows – 4 cols}}, \quad \underbrace{\Big[\;|\;|\;\Big]\begin{bmatrix}\text{——}\\ \text{——}\end{bmatrix}}_{\text{3 cols – 3 rows}}$

**29** $E_{21} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $E_{31} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-4 & 0 & 1\end{bmatrix}$ produce zeros in the $2,1$ and $3,1$ entries.

Multiply $E$'s to get $E = E_{31}E_{21} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\-4 & 0 & 1\end{bmatrix}$. Then $EA = \begin{bmatrix}2 & 1 & 0\\0 & 1 & 1\\0 & 1 & 3\end{bmatrix}$ is the result of both $E$'s since $(E_{31}E_{21})A = E_{31}(E_{21}A)$.

**30** In **29**, $\boldsymbol{c} = \begin{bmatrix}-2\\8\end{bmatrix}$, $D = \begin{bmatrix}0 & 1\\5 & 3\end{bmatrix}$, $D - \boldsymbol{cb}/a = \begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}$ in the lower corner of $EA$.

**31** $\begin{bmatrix}A & -B\\B & A\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}A\boldsymbol{x} - B\boldsymbol{y}\\ B\boldsymbol{x} + A\boldsymbol{y}\end{bmatrix}$ real part / imaginary part. Complex matrix times complex vector needs **4** real times real multiplications.

**32** $A$ times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

**33** $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$ ; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $\boldsymbol{x}_1 = (1,1,1), \boldsymbol{x}_2 = (0,1,1), \boldsymbol{x}_3 = (0,0,1)$ as columns of its "inverse" $A^{-1}$.

**34** $A *$ **ones** $= \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$ agrees with **ones** $*A = \begin{bmatrix} a+c & b+b \\ a+c & b+d \end{bmatrix}$ when $b = c$ and $a = d$

Then $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. These are the matrices that commute with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

**35** $S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $S^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$,

| | | |
|---|---|---|
| **aba, ada** | **cba, cda** | These show |
| **bab, bcb** | **dab, dcb** | 16 2-step |
| **abc, adc** | **cbc, cdc** | paths in |
| **bad, bcd** | **dad, dcd** | the graph |

**36** Multiplying $AB =$($m$ by $n$)($n$ by $p$) needs $mnp$ multiplications. Then $(AB)C$ needs $mpq$ more. Multiply $BC =$ ($n$ by $p$)($p$ by $q$) needs $npq$ and then $A(BC)$ needs $mnq$.

(a) If $m, n, p, q$ are $2, 4, 7, 10$ we compare $(2)(4)(7) + (2)(7)(10) = \mathbf{196}$ with the larger number $(2)(4)(10) + (4)(7)(10) = \mathbf{360}$. So $AB$ first is better, we want to multiply that 7 by 10 matrix by as few rows as possible.

(b) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are $N$ by $1$, then $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ needs $2N$ multiplications but $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$ needs $N^2$ to find $\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}$ and $N^2$ more to multiply by the row vector $\boldsymbol{u}^{\mathrm{T}}$. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1} + n^{-1}$ with $p^{-1} + m^{-1}$. This yields a simple important rule. If matrices $A$ and $B$ are multiplying $\boldsymbol{v}$ for $AB\boldsymbol{v}$, **don't multiply the matrices first**. Better to multiply $B\boldsymbol{v}$ and then $A(B\boldsymbol{v})$.

**37** The proof of $(AB)\boldsymbol{c} = A(B\boldsymbol{c})$ used the column rule for matrix multiplication. "The same is true for all other *columns* of $C$."

Even for nonlinear transformations, $A(B(\boldsymbol{c}))$ would be the ***"composition"*** of $A$ with $B$ (applying $B$ then $A$). This composition $A \circ B$ is just written as $AB$ for matrices.

One of many uses for the associative law: The left-inverse $B$ = the right-inverse $C$ because $B = B(AC) = (BA)C = C$.

**38** (a) Multiply the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ by the rows $\boldsymbol{a}_1^{\mathrm{T}}, \ldots, \boldsymbol{a}_m^{\mathrm{T}}$ and add the resulting matrices.

(b) $A^{\mathrm{T}}CA = c_1 \boldsymbol{a}_1 \boldsymbol{a}_1^{\mathrm{T}} + \cdots + c_m \boldsymbol{a}_m \boldsymbol{a}_m^{\mathrm{T}}$. Diagonal $C$ makes it neat.

## Problem Set 2.5, page 92

**1** $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

**2** For the first, a simple row exchange has $P^2 = I$ so $P^{-1} = P$. For the second, $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always $P^{-1}$ = "transpose" of $P$, coming in Section 2.7.

**3** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination. For a different matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, you could find a first column for $A^{-1}$ but not a second column—so $A$ would be singular (*no inverse*).

**4** The equations are $x + 2y = 1$ and $3x + 6y = 0$. No solution because 3 times equation 1 gives $3x + 6y = 3$.

**5** An upper triangular $U$ with $U^2 = I$ is $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ for any $a$. And also $-U$.

**6** (a) Multiply $AB = AC$ by $A^{-1}$ to find $B = C$ (since $A$ is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

**7** (a) In $A\boldsymbol{x} = (1, 0, 0)$, equation $1 +$ equation $2 -$ equation $3$ is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

**8** (a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column $3 =$ column $1 + 2$: no third pivot.

**9** Yes, $B$ is invertible ($A$ was just multiplied by a permutation matrix $P$). If you exchange rows 1 and 2 of $A$ to reach $B$, you exchange **columns** 1 and 2 of $A^{-1}$ to reach $B^{-1}$. In matrix notation, $B = PA$ has $B^{-1} = A^{-1}P^{-1} = A^{-1}P$ for this $P$.

**10** $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block of B)

**11** (a) If $B = -A$ then certainly $A + B$ = zero matrix is not invertible.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular but $A + B = I$ is invertible.

**12** Multiply $C = AB$ on the left by $A^{-1}$ and on the right by $C^{-1}$. Then $A^{-1} = BC^{-1}$.

**13** $M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply on the left by $C$ and the right by $A$ : $B^{-1} = CM^{-1}A$.

**14** $B^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract *column* 2 of $A^{-1}$ from *column* 1.

**15** If $A$ has a column of zeros, so does $BA$. Then $BA = I$ is impossible. There is no $A^{-1}$.

**16** $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad-bc$

**17** $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E.$

Reverse the order and change $-1$ to $+1$ to get inverses $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} =$ $L = E^{-1}$. Notice that the 1's are unchanged by multiplying inverses in this order.

**18** $A^2B = I$ can also be written as $A(AB) = I$. Therefore $A^{-1}$ is $AB$.

**19** The $(1,1)$ entry requires $4a - 3b = 1$; the $(1,2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b = a$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

**20** $A * \text{ones}(4,1) = A$ (column of 1's) is the zero vector so $A$ cannot be invertible.

**21** Six of the sixteen $0-1$ matrices are invertible: $I$ and $P$ and all four with three 1's.

**22** $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix};$

$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}.$

**23** $[\boldsymbol{A}\ \ \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to$

$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] =$$

$[\boldsymbol{I}\ \boldsymbol{A}^{-1}]$.

**24** $$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

**25** $$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-\mathbf{1}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix};\ B\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $B^{-1}$ does not exist.

**26** $E_{21}A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. $E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to reach $DE_{12}E_{21}A = I$. Then $A^{-1} = DE_{12}E_{21} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$.

**27** $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the sign changes); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

**28** $$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}.$$

This is $\begin{bmatrix} I & A^{-1} \end{bmatrix}$: row exchanges are certainly allowed in Gauss-Jordan.

**29** (a) True (If $A$ has a row of zeros, then every $AB$ has too, and $AB = I$ is impossible).

(b) False (the matrix of all ones is singular even with diagonal 1's.

(c) True (the inverse of $A^{-1}$ is $A$ and the inverse of $A^2$ is $(A^{-1})^2$).

**30** Elimination produces the pivots $a$ and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

The matrix $C$ is not invertible if $c = 0$ or $c = 7$ or $c = 2$.

**31** $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$. When the triangular $A$ alternates 1 and $-1$ on its diagonals, $A^{-1}$ has 1's on the diagonal and first superdiagonal.

**32** $\boldsymbol{x} = (1, 1, \ldots, 1)$ has $\boldsymbol{x} = P\boldsymbol{x} = Q\boldsymbol{x}$ so $(P - Q)\boldsymbol{x} = \mathbf{0}$. Permutations do not change this all-ones vector.

**33** $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

**34** $A$ can be invertible with diagonal zeros (example to find). $B$ is singular because each row adds to zero. The all-ones vector $\boldsymbol{x}$ has $B\boldsymbol{x} = \mathbf{0}$.

**35** The equation $LDLD = I$ says that $LD =$ pascal $(4, 1)$ is its own inverse.

**36** hilb(6) is not the exact Hilbert matrix because fractions are rounded off. So inv(hilb(6)) is not the exact inverse either.

**37** The three Pascal matrices have $P = LU = LL^{\mathrm{T}}$ and then inv$(P) =$ inv$(L^{\mathrm{T}})*$inv$(L)$.

**38** $A\boldsymbol{x} = \boldsymbol{b}$ has many solutions when $A =$ ones $(4, 4) =$ ***singular*** and $\boldsymbol{b} =$ ones $(4, 1)$. $A\backslash\boldsymbol{b}$ in MATLAB will pick the shortest solution $\boldsymbol{x} = (1, 1, 1, 1)/4$. This is the only solution that is a combination of the rows of $A$ (later it comes from the "pseudoinverse" $A^+ =$ pinv$(A)$ which replaces $A^{-1}$ when $A$ is singular). Any vector that solves $A\boldsymbol{x} = \mathbf{0}$ could be added to this particular solution $\boldsymbol{x}$.

**39** The inverse of $A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (This would be a good example for the cofactor formula $A^{-1} = C^{\mathrm{T}}/\det A$ in Section 5.3)

**40** $$\begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & d & 1 & \\ 0 & e & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & f & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & d & 1 & \\ c & e & f & 1 \end{bmatrix}$$

In this order the multipliers $a, b, c, d, e, f$ are unchanged in the product (**important for** $A = LU$ **in Section 2.6**).

**41** 4 by 4 still with $T_{11} = 1$ has pivots $1, 1, 1, 1$; reversing to $T^* = UL$ makes $T^*_{44} = 1$.

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

**42** Add the equations $C\boldsymbol{x} = \boldsymbol{b}$ to find $0 = b_1 + b_2 + b_3 + b_4$. So $C$ is singular. Same for $F\boldsymbol{x} = \boldsymbol{b}$.

**43** The block pivots are $A$ and $S = D - CA^{-1}B$ (and $d - cb/a$ is the correct second pivot of an ordinary 2 by 2 matrix). The example problem has Schur complement $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -6 & -5 \end{bmatrix}$.

**44** Inverting the identity $A(I + BA) = (I + AB)A$ gives $(I + BA)^{-1}A^{-1} = A^{-1}(I + AB)^{-1}$. So $I+BA$ and $I+AB$ are both invertible or both singular when $A$ is invertible. (This remains true also when $A$ is singular : Chapter 6 will show that $AB$ and $BA$ have the same nonzero eigenvalues, and we are looking here at the eigenvalue $-1$.)

# Problem Set 2.6, page 104

**1** $\ell_{21} = \mathbf{1}$ multiplied row 1; $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ times $U\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \boldsymbol{c}$ is
$A\boldsymbol{x} = \boldsymbol{b} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. In letters, **$\boldsymbol{L}$ multiplies $\boldsymbol{U\!x = c}$ to give $\boldsymbol{A\!x = b}$.**

**2** $L\boldsymbol{c} = \boldsymbol{b}$ is $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, solved by $\boldsymbol{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ as elimination goes forward.
$U\boldsymbol{x} = \boldsymbol{c}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, solved by $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in back substitution.

**3** $\ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse steps to get $A\boldsymbol{u} = \boldsymbol{b}$ from $U\boldsymbol{x} = \boldsymbol{c}$: 1 times $(x+y+z=5)+2$ times $(y+2z=2)+1$ times $(z=2)$ gives $x+3y+6z=11$.

**4** $L\boldsymbol{c} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$; $\quad U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix}\begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

**5** $EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U.$

With $E^{-1}$ as $L$, $A = LU = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$.

**6** $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$. Then $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} U$ is the same as $E_{21}^{-1}E_{32}^{-1}U = LU$. The multipliers $\ell_{21} = \ell_{32} = 2$ fall into place in $L$.

**7** $E_{32}E_{31}E_{21}\ A = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\mathbf{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -\mathbf{3} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -\mathbf{2} & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$. This is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$. Put those multipliers $2, 3, 2$ into $L$. Then $A = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{3} & \mathbf{2} & 1 \end{bmatrix} U = LU$.

**8** $E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ ac-b & -c & 1 \end{bmatrix}$ is mixed but $L$ is $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ a & 1 & \\ b & c & 1 \end{bmatrix}$.

**9** 2 by 2: $d = 0$ not allowed; $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}$ $d = 1, e = 1$, then $\ell = 1$; $f = 0$ is not allowed; **no pivot in row 2**

**10** $c = 2$ leads to zero in the second pivot position: exchange rows and not singular. $c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

**11** $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$ has $L = I$ ($A$ is already upper triangular) and $D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}$;

$A = LU$ has $U = A$; $A = LDU$ has $U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ with 1's on the diagonal.

**12** $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$; $\boldsymbol{U}$ **is** $\boldsymbol{L}^{\mathrm{T}}$

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

**13** $\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$. Need $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$. All of the multipliers are $\ell_{ij} = 1$ for this $A$

**14** $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$. Need $a \neq 0$, $b \neq r$, $c \neq s$, $d \neq t$

**15** $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

$A\boldsymbol{x} = \boldsymbol{b}$ is $LU\boldsymbol{x} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Eliminate to $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boldsymbol{c}$.

**16** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Those are forward elimination and back substitution for $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

**17** (a) $L$ goes to $I$ (b) $I$ goes to $L^{-1}$ (c) $LU$ goes to $U$. Elimination multiplies by $L^{-1}$!

**18** (a) Multiply $LDU = L_1 D_1 U_1$ by inverses to get $L_1^{-1} LD = D_1 U_1 U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.

(b) $L, U, L_1, U_1$ have diagonal 1's so $D = D_1$. Then $L_1^{-1} L$ and $U_1 U^{-1}$ are both $I$.

**19** $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} = LIU; \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = L \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} U.$

A tridiagonal matrix $A$ has **bidiagonal factors** $L$ and $U$.

**20** A tridiagonal $T$ has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find $\ell$ and then one for the new pivot!). Only $2n$ operations for elimination on a tridiagonal matrix. $T$=bidiagonal $L$ times bidiagonal $U$.

**21** For the first matrix $A, L$ keeps the 3 zeros at the start of rows. But $U$ may not have the upper zero where $A_{24} = 0$. For the second matrix $B, L$ keeps the bottom left zero at the start of row 4. $U$ keeps the upper right zero at the start of column 4. One zero in $A$ and two zeros in $B$ are filled in.

**22** Eliminating ***upwards***, $\begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$. We reach a ***lower*** triangular $L$, and the multipliers are in an ***upper*** triangular $U$. $A = \boldsymbol{UL}$ with $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

**23** The 2 by 2 upper submatrix $A_2$ has the first two pivots $5, 9$. Reason: Elimination on $A$ starts in the upper left corner with elimination on $A_2$.

**24** The upper left blocks all factor at the same time as $A$: $A_k$ is $L_k U_k$. So $A = LU$ is possible only if all those blocks $A_k$ are invertible.

**25** The $i, j$ entry of $L^{-1}$ is $j/i$ for $i \geq j$. And $L_{i\,i-1}$ is $(1-i)/i$ below the diagonal

**26** $(K^{-1})_{ij} = j(n-i+1)/(n+1)$ for $i \geq j$ (and symmetric): Multiply $K^{-1}$ by $n+1$ (the determinant of $K$) to see all whole numbers.

## Problem Set 2.7, page 117

**1** $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ has $A^{\mathrm{T}} = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$;

$A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ has $A^{\mathrm{T}} = A$ and $A^{-1} = \dfrac{1}{c^2}\begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^{\mathrm{T}}$.

**2** $(AB)^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = B^{\mathrm{T}}A^{\mathrm{T}}$. This answer is different from $A^{\mathrm{T}}B^{\mathrm{T}}$ (except when $AB = BA$ and transposing gives $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$).

**3** (a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$.

(b) If $U$ is upper triangular, so is $U^{-1}$: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

**4** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. But the diagonal of $A^{\mathrm{T}}A$ has dot products of columns of $A$ with *themselves*. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

**5** (a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5$

(b) This is the row $\boldsymbol{x}^{\mathrm{T}}A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ times $\boldsymbol{y}$.

(c) This is also the row $\boldsymbol{x}^{\mathrm{T}}$ times $A\boldsymbol{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

**6** $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ *and* $B^{\mathrm{T}} = C$ *and* $D^{\mathrm{T}} = D$.

**7** (a) False: $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is symmetric only if $A = A^{\mathrm{T}}$.

(b) False: The transpose of $AB$ is $B^{\mathrm{T}}A^{\mathrm{T}} = BA$. So $(AB)^{\mathrm{T}} = AB$ needs $BA = AB$.

(c) True: Invertible symmetric matrices have symmetric inverses! Easiest proof is to transpose $AA^{-1} = I$.

(d) True: $(ABC)^{\mathrm{T}}$ is $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}(= CBA$ for symmetric matrices $A, B$, and $C$).

**8** The 1 in row 1 has $n$ choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

**9** $P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ but $P_2P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
If $P_3$ and $P_4$ exchange ***different*** pairs of rows, $P_3P_4 = P_4P_3 =$ both exchanges.

**10** $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even $P$'s keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ and $(4,3,2,1)$ exchange 2 pairs. $(1,2,3,4)$ makes 12.

**11** $PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is upper triangular. Multiplying $A$ ***on the right*** by a permutation matrix $P_2$ exchanges the ***columns*** of $A$. To make this $A$ lower triangular, we also need $P_1$ to exchange rows 2 and 3:

$$P_1AP_2 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

**12** $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ since $P^{\mathrm{T}}P = I$. In general $P\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot P^{\mathrm{T}}\boldsymbol{y} \neq \boldsymbol{x}\cdot P\boldsymbol{y}$:

Non-equality where $P \neq P^{\mathrm{T}}$: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

**13** A cyclic $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose will have $P^3 = I : (1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$. The permutation $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same $P$ has $\widehat{P}^4 = \widehat{P} \neq I$.

**14** The "reverse identity" $P$ takes $(1,\ldots,n)$ into $(n,\ldots,1)$. When rows and also columns are reversed, the $1,1$ and $n,n$ entries of $A$ change places in $PAP$. So do the $1,n$ and $n,1$ entries. In general $(PAP)_{ij}$ is $(A)_{n-i+1,n-j+1}$.

**15** (a) If $P$ sends row 1 to row 4, then $P^{\mathrm{T}}$ sends row 4 to row 1 (b) $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^{\mathrm{T}}$ with $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

**16** $A^2 - B^2$ and also $ABA$ are symmetric if $A$ and $B$ are symmetric. But $(A+B)(A-B)$ and $ABAB$ are generally ***not*** symmetric.

**17** (a) $S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = S^{\mathrm{T}}$ is not invertible (b) $S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ needs row exchange (c) $S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has pivots $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ : no real square root.

**18** (a) $5+4+3+2+1 = 15$ independent entries if $S = S^{\mathrm{T}}$ (b) $L$ has 10 and $D$ has 5; total 15 in $LDL^{\mathrm{T}}$ (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

**19** (a) The transpose of $A^{\mathrm{T}}SA$ is $A^{\mathrm{T}}S^{\mathrm{T}}A^{\mathrm{T}\,\mathrm{T}} = A^{\mathrm{T}}SA = n$ by $n$ when $S^{\mathrm{T}} = S$ (any $m$ by $n$ matrix $A$) (b) $(A^{\mathrm{T}}A)_{jj} =$ (column $j$ of $A$)$\cdot$ (column $j$ of $A$) $=$ (length squared of column $j$) $\geq 0$.

**20**
$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = \boldsymbol{LDL}^{\mathrm{T}}.$$

**21** Elimination on a symmetric 3 by 3 matrix leaves a symmetric lower right 2 by 2 matrix.
$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ lead to } \begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix} \text{ and } \begin{bmatrix} d-b^2 & e-bc \\ e-bc & f-c^2 \end{bmatrix} \text{ : symmetric!}$$

**22** $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

**23** $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P$ and $L = U = I$. Elimination on this $A = P$ exchanges rows 1-2 then rows 2-3 then rows 3-4.

**24** $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and $a_{12}$ is the pivot, $A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

**25** One way to decide even vs. odd is to count all pairs that $P$ has in the wrong order. Then $P$ is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

**26** (a) $E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$ puts 0 in the $2, 1$ entry of $E_{21}A$. Then $E_{21}AE_{21}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ is still symmetric, with zero also in its 1, 2 entry. (b) Now use $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$ to make the 3, 2 entry zero and $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}} = D$ also has zero in its 2, 3 entry. Key point: Elimination from both sides (rows $+$ columns) gives the symmetric $LDL^{\mathrm{T}}$.

**27** $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = A^{\mathrm{T}}$ has $0, 1, 2, 3$ in every row. I don't know any rules for a symmetric construction like this "Hankel matrix" with constant antidiagonals.

**28** Reordering the rows and/or the columns of $\left[\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right]$ will move the entry **a**. So the result cannot be the transpose (which doesn't move **a**).

**29** (a) Total currents are $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC}+y_{BS} \\ -y_{BC}+y_{CS} \\ -y_{CS}-y_{BS} \end{bmatrix}$.

(b) Either way $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$. Six terms.

**30** $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x};\ A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}\begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ $\begin{matrix}\text{1 truck} \\ \text{1 plane}\end{matrix}$

**31** $A\boldsymbol{x}\cdot\boldsymbol{y}$ is the *cost of inputs* while $\boldsymbol{x}\cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value of outputs*.

**32** $P^3 = I$ so three rotations for $360°$; $P$ rotates every $\boldsymbol{v}$ around the $(1,1,1)$ line by $120°$.

**33** $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \boldsymbol{EH} =$ (elementary matrix) times (symmetric matrix).

**34** $L(U^{\mathrm{T}})^{-1}$ is lower triangular times lower triangular, so *lower triangular*. The transpose of $U^{\mathrm{T}}DU$ is $U^{\mathrm{T}}D^{\mathrm{T}}U^{\mathrm{T}\,\mathrm{T}} = U^{\mathrm{T}}DU$ again, so $U^{\mathrm{T}}DU$ is *symmetric*. The factorization multiplies lower triangular by symmetric to get $LDU$ which is $A$.

**35** These are groups: Lower triangular with diagonal 1's, diagonal invertible $D$, permutations $P$, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

**36** Certainly $B^{\mathrm{T}}$ is northwest. $B^2$ is a full matrix! $B^{-1}$ is southeast: $\left[\begin{smallmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{-1}\end{smallmatrix}\right]$. The rows of $B$ are in reverse order from a lower triangular $L$, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the ***columns*** in reverse order from $L^{-1}$. So $B^{-1}$ is ***southeast***. Northwest $B = PL$ times southeast $PU$ is $(PLP)U =$ upper triangular.

**37** There are $n!$ permutation matrices of order $n$. Eventually *two powers of $P$ must be the same permutation*. And if $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^6 = I$.

**38** To split the matrix $M$ into (symmetric $S$) $+$ (anti-symmetric $A$), the only choice is $S = \frac{1}{2}(M + M^{\mathrm{T}})$ and $A = \frac{1}{2}(M - M^{\mathrm{T}})$.

**39** Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$: ***unit vectors***

(b) The off-diagonal entry is $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ (and in general $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$)

(c) The leading example for $Q$ is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.