1. 设 $M_1, M_2 \in M_n(\mathbb{C})$，则它们可写成

$M_1 = A_1 + iB_1$, $\quad M_2 = A_2 + iB_2$

其中 $A_1, A_2, B_1, B_2 \in M_n(\mathbb{R})$

$$T(M_1 \cdot M_2) = T[(A_1A_2 - B_1B_2) + i(A_1B_2 + B_1A_2)]$$

$$= \begin{pmatrix} A_1A_2 - B_1B_2 & -A_1B_2 - B_1A_2 \\ A_1B_2 + B_1A_2 & A_1A_2 - B_1B_2 \end{pmatrix}$$

$$T(M_1) \cdot T(M_2) = \begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & -B_2 \\ B_2 & A_2 \end{pmatrix}$$

$$= T(M_1 \cdot M_2)$$

$$T(M_1^H) = T(A_1^T - iB_1^T) = \begin{pmatrix} A_1^T & B_1^T \\ -B_1^T & A_1^T \end{pmatrix}$$

$$= T(M_1)^T$$

注：① 若 $M, N \in M_n(\mathbb{C})$，$T(M)=T(N)$.

则 $M=N$.

② 由上述结论，易证.

$M=A+iB$ 正规 $\Longleftrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ 正规

Hermite $\Longleftrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ 实对称

酉阵 $\Longleftrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ 正交

2. (1) $\forall x \in \mathbb{C}^n$. 令 $y = A^H x$

则 $x^H A A^H x = y^H y \geq 0$.

特别地，若 x 是 AA^H 的特征向量，$AA^H x = \lambda x$.

则 $\lambda \neq 0$ (A 可逆)，且

$$\lambda x^H x = x^H A A^H x = y^H y > 0$$

$$\Rightarrow \lambda = \frac{y^H y}{x^H x} > 0 \quad (x \neq 0 \Rightarrow \|x\| \neq 0)$$

(2) AA^H 是一个 Hermite 阵，存在酉阵 U

$$U^H (AA^H) U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad \lambda_i > 0 \quad i = 1, \cdots, n$$

则 $AA^H = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^H$

令 $R = U \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} U^H$，则 $R^2 = AA^H$.

R是唯一的，因为若$\lambda_1, \cdots, \lambda_n$全相等，则$R=\sqrt{\lambda_1} I_n$，否则$\lambda_1, \cdots, \lambda_n$有$t$个互异值，不妨设$\lambda_1, \cdots, \lambda_t$互不相同。

令$f(\lambda) = \sum_{i=1}^{t} \sqrt{\lambda_i} \prod_{\substack{k=1 \\ k \neq i}}^{t} \frac{\lambda - \lambda_k}{\lambda_i - \lambda_k}$ (Lagrange 插值多项式)

则$f(\lambda_i) = \sqrt{\lambda_i} \quad i=1, \cdots, t$.

$$f(AA^H) = R$$

(3) 令$V = R^{-1}A$，因为R Hermite.

R^{-1}也是，$V^H = A^H R^{-1}$

$$VV^H = R^{-1}AA^H R^{-1} = R^{-1} R^2 R^{-1}$$

$$= I_n \Rightarrow V \text{ 酉阵，且 } A = RV.$$

(4). 因为 U 酉阵, 存在酉阵 W

$$W^H U W = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

其中 $\lambda_k = e^{i\theta_k}$ $k=1,\cdots,n$ $\underline{\theta_k \in \mathbb{R}}$.

令 $H = W \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{pmatrix} W^H$

则 H 是 Hermite 阵

$iH = W \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} W^H$ 是

skew-Hermite 阵

从而 $\underline{e^{iH} = U}$.

注: 合并 (3) (4). $A = Re^{iH}$.

(极分解).

3. $\|u+v\|^2=\|u\|^2+\|v\|^2+\langle u,v\rangle+\langle v,u\rangle$

$\|u-v\|^2=\|u\|^2+\|v\|^2-\langle u,v\rangle-\langle v,u\rangle$

$i\|u+iv\|^2=i\langle u+iv,\ u+iv\rangle$

$=i\|u\|^2-\langle u,v\rangle+\langle v,u\rangle+i\|v\|^2$

$i\|u-iv\|^2=i\|u\|^2+\langle u,v\rangle-\langle v,u\rangle+i\|v\|^2$

$\Rightarrow \|u+v\|^2-\|u-v\|^2=2\langle u,v\rangle+2\langle v,u\rangle$

$i\|u-iv\|^2-i\|u+iv\|^2$

$=2\langle u,v\rangle-2\langle v,u\rangle$

4. (1) 若 $\langle f(x),f(x)\rangle=0$ 则 $\int_{-1}^{1}|f(x)|^2dx=0$

$\Rightarrow f(x)=0$.

(2) 令 $\alpha_1=1,\ \alpha_2=x,\ \alpha_3=x^2$.

$\beta_1=\alpha_1,\quad e_1=\dfrac{\beta_1}{\|\beta_1\|}$

$$\|\beta_1\| = \sqrt{\int_{-1}^{1} 1^2\,dx} = 2 \Rightarrow e_1 = \frac{1}{\sqrt{2}} \cdot 1.$$

$$\beta_2 = \alpha_2 - \langle e_1, \alpha_2 \rangle e_1$$

$$= x - \left(\int_{-1}^{1} \left(\frac{1}{\sqrt{2}} x\right) dx\right) \frac{1}{\sqrt{2}} = x$$

$$e_2 = \frac{\beta_2}{\|\beta_2\|} \qquad \|\beta_2\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}.$$

$$\Rightarrow e_2 = \sqrt{\frac{3}{2}}\, x$$

$$\beta_3 = \alpha_3 - \langle e_1, \alpha_3 \rangle e_1 - \langle e_2, \alpha_3 \rangle e_2.$$

$$= x^2 - \left(\int_{-1}^{1} x^2 \sqrt{\frac{1}{2}}\, dx\right) \cdot \sqrt{\frac{1}{2}} - \left(\int_{-1}^{1} x^2 \sqrt{\frac{3}{2}}\, x\, dx\right) \cdot \sqrt{\frac{3}{2}}\, x$$

$$= x^2 - \frac{1}{3}$$

$$\|\beta_3\| = \sqrt{\frac{8}{45}} \Rightarrow e_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right).$$

(3) 由提示：求 $\cos\pi x = \vec{v}$ 在 V 上投影 $\vec{v_p} = u(x)$

在(2)已知 V 的一组标准正交基 e_1, e_2, e_3.

则 $\vec{v_p} = \langle e_1, \vec{v} \rangle e_1 + \langle e_2, \vec{v} \rangle e_2 + \langle e_3, \vec{v} \rangle e_3$

$$= \left(\int_{-1}^{1} \sqrt{\frac{1}{2}} \cos\pi x \, dx\right) \cdot \sqrt{\frac{1}{2}}$$

$$+ \left(\int_{-1}^{1} \sqrt{\frac{3}{2}} x \cos\pi x \, dx\right) \cdot \sqrt{\frac{3}{2}} x$$

$$+ \left(\int_{-1}^{1} \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right) \cos\pi x \, dx\right) \cdot \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right)$$

$$= -\frac{45}{2\pi^2} \left(x^2 - \frac{1}{3}\right).$$

5.(1) 例. $\mathbb{C}^4 \xrightarrow{T} \mathbb{C}^4$ 即 $\begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} i & & & \\ & i & & \\ & & i & \\ & & & i \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix}$

$x \longmapsto ix$.

T normal, not Hermite.

(2) 设 T 在标准基 $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ 下矩阵为 A，则 $T\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right] = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

由条件，A 正规，且 $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$A\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0$. 正规阵属于不同特征值的特征向量正交，即

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z_1 + z_2 + z_3 = 0.$$

6. 求V的一组标准正交基.

$$\|\cos x\|^2=\int_{-\pi}^{\pi}\cos^2 x\,dx=\frac{x}{2}+\frac{\sin 2x}{4}\Big|_{-\pi}^{\pi}=\pi$$

$\Rightarrow \|\cos x\|=\sqrt{\pi}$.

令 $e_j=\dfrac{\cos jx}{\sqrt{\pi}}$, $f_j=\dfrac{\sin jx}{\sqrt{\pi}}$

则 $\dfrac{1}{\sqrt{2\pi}}, e_1,\cdots,e_n, f_1,\cdots,f_n$ 是V的一组标准正交基.

(a) $\forall f,g\in V$, $\langle Df,g\rangle=\langle f,D^*g\rangle$

$$\langle Df,g\rangle=\int_{-\pi}^{\pi}f'(x)g(x)dx=\int_{-\pi}^{\pi}[(f(x)g(x))'-f(x)g'(x)]dx$$

$$=f(x)g(x)\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}f(x)g'(x)dx$$

$$=\langle f(x),-g'(x)\rangle$$

即 $\langle f(x), D^* g(x) \rangle = \langle f(x), -g'(x) \rangle$

$\forall f(x), g(x) \in V$

$\Rightarrow \langle f(x), (D^* + D)(g(x)) \rangle = 0$

$\forall f(x), g(x) \in V$

特别地，令 $f(x) = (D^* + D)(g(x))$

$\Rightarrow (D^* + D)[g(x)] = 0 \quad \forall g(x) \in V.$

$\Rightarrow D^* + D = 0.$

注：D is normal but not Hermitian.

(b) $T = D^2$, $\quad T^* = (D^2)^* = (D^*)^2$

$= (-D) \cdot (-D) = D^2$

$= T$

7. (1) 求证：设 $A \in M_n(\mathbb{C})$，且 $v^H A v = 0$，$\forall v \in \mathbb{C}^n$，则 $A = O$.

证明：正如第3题

$S_1 = \langle u+w, A(u+w)\rangle - \langle u-w, A(u-w)\rangle$

$= \langle u, Au\rangle + \langle u, Aw\rangle + \langle w, Au\rangle + \langle w, Aw\rangle$

$- [\langle u, Au\rangle - \langle u, Aw\rangle - \langle w, Au\rangle + \langle w, Aw\rangle]$

$= 2\langle u, Aw\rangle + 2\langle w, Au\rangle$

$S_2 = \langle u-iw, A(u-iw)\rangle - \langle u+iw, A(u+iw)\rangle$.

$= \langle u, Au\rangle - i\langle u, Aw\rangle + i\langle w, Au\rangle + \langle w, Aw\rangle$

$- [\langle u, Au\rangle + i\langle u, Aw\rangle - i\langle w, Au\rangle + \langle w, Aw\rangle]$

$= 2i\langle w, Au\rangle - 2i\langle u, Aw\rangle$

$\Rightarrow 4\langle w, Au\rangle = -iS_2 + S_1$.

即 $w^H A u = 0 \quad \forall u, w \in \mathbb{C}^n \Rightarrow A = 0$

(取 $w = Au$, $\Rightarrow Au = 0 \ \forall u \in \mathbb{C}^n$, 令 $u = e_1, \cdots, e_n$).

(2) 若 $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, 则 $\forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \in \mathbb{R}^2$.

$\alpha^T A \alpha = 0$ 但 $A \neq 0$

8. 假设 $v_1, v_2, v_3 \in \mathbb{C}^3$ 线性相关.

不妨设 $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$, $a_1 \neq 0$.

$\| a_1(e_1 - v_1) + a_2(e_2 - v_2) + a_3(e_3 - v_3) \|^2$

$= \| a_1 e_1 + a_2 e_2 + a_3 e_3 \|^2 = |a_1|^2 + |a_2|^2 + |a_3|^2$

另一方面, 上式 $= \langle \sum_{i=1}^{3} a_i(e_i - v_i), \sum_{i=1}^{3} a_i(e_i - v_i) \rangle$

$= \left| \sum_{i,j=1}^{3} \overline{a_i} a_j \langle e_i - v_i, e_j - v_j \rangle \right|$

$$\le \sum_{i,j=1}^{3} |a_i|\cdot|a_j|\cdot|\langle e_i - v_i, e_j - v_j\rangle|$$

$$\le \sum_{i,j=1}^{3} |a_i|\cdot|a_j| \|e_i - v_i\|\cdot\|e_j - v_j\|$$

$$< \sum_{i,j=1}^{3} |a_i||a_j| \frac{1}{3} \quad (因为 a_i \ne 0)$$

$$= \frac{1}{3}\left[|a_1|^2 + |a_2|^2 + |a_3|^2 + 2|a_1||a_2| + 2|a_2||a_3| + 2|a_1|\cdot|a_3|\right]$$

$$\le |a_1|^2 + |a_2|^2 + |a_3|^2 \quad 矛盾!$$

9. 由Schur定理，存在酉阵U

$$U^H A U = \begin{pmatrix} \lambda_1 & & b_{ij} \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}_{n\times n} = B$$

则 $B^H = U^H A^H U = \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ \overline{b_{ij}} & & \bar{\lambda}_n \end{pmatrix}$

$$B^H B = C = (c_{ij})$$

$$c_{ii} = \sum_{k=1}^{i-1} |b_{ki}|^2 + |\lambda_i|^2.$$

$$A^H A = D = (d_{ij})$$

$$d_{ii} = \sum_{k=1}^{n} |a_{ki}|^2$$

$$\operatorname{tr}(A^H A) = \sum_{i,j=1}^{n} |a_{ij}|^2.$$

$$\|$$

$$\operatorname{tr}(U B^H U^H U B U^H) = \operatorname{tr}(B^H B)$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i=1}^{n} \sum_{k=1}^{i-1} |b_{ki}|^2$$

$$\Rightarrow \sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2$$

等号成立 $\Leftrightarrow$ $b_{ki} = 0 \ \forall k < i$，从而

B 是对角阵，$\Leftrightarrow$ A 正规.