Problem 1

Algebraic numbers form a field. Recall that $\alpha \in \mathbb{C}$ is called an algebraic number if there exist $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. Denote the set of algebraic numbers by $\overline{\mathbb{Q}}$. We will show that $(\overline{\mathbb{Q}}, +, \cdot)$ is a field, in various steps.

- Show that, if $\alpha \in \overline{\mathbb{Q}}$, then $-\alpha \in \overline{\mathbb{Q}}$ and, if $\alpha \neq 0$, then $\frac{1}{\alpha} \in \overline{\mathbb{Q}}$.
- Consider the ring $(\mathbb{Q}[\alpha], +, \cdot)$ with $\alpha \in \overline{\mathbb{Q}}$, that is, all elements of the form $\sum_{i=0}^{\infty} a_i \alpha^i$ with $a_i \in \mathbb{Q}$ and the usual addition and multiplication. Show that $\mathbb{Q}[\alpha]$ is finitely generated, i.e. you want to show that α^m can be written as a finite sum $\sum_{i=0}^{s} a_i \alpha^i$ with $s < m$, for all m sufficiently large. **Hint**: use division theorem on the minimal polynomial $p(x) \in \mathbb{Q}[x]$ of α and x^m.
- Show that, if V is a ring in $\mathbb{C}$ that can be viewed as a finite dimensional vector space over $\mathbb{Q}$ [1], and $\beta \in \mathbb{C}$ satisfies $\beta \cdot V \subseteq V$. Show then that $\beta \in \overline{\mathbb{Q}}$. **Hint**: Show that β is the eigenvalue of a $\mathbb{Q}$-valued operator.
- Apply the previous two results to the following example: $\alpha = e^{2\pi i/3}$. First, find a basis for $\mathbb{Q}[\alpha]$. This shows you explicitly that $\mathbb{Q}[\alpha]$ is finite dimensional vector space over $\mathbb{Q}$. Find the action of α in this basis [2], determining the $\mathbb{Q}$-valued operator previously mentioned and the polynomial annihilating α.
- Consider now $\alpha, \beta \in \overline{\mathbb{Q}}$ and the vector space $V[\alpha, \beta]$ defined as the vector space over $\mathbb{Q}$ spanned by elements of the form $\alpha^i \beta^j$ for any i, j positive integers. Show that $V[\alpha, \beta]$ is finite dimensional. Then, show that $(\alpha + \beta)V[\alpha, \beta] \subseteq V[\alpha, \beta]$ and $\alpha\beta V[\alpha, \beta] \subseteq V[\alpha, \beta]$.
- Use the previous results to conclude that $\overline{\mathbb{Q}}$ is a field.

Problem 2

For this problem fix $V = \mathbb{R}^3$ and $v \cdot v' \in \mathbb{R}$ ($v, v' \in \mathbb{R}^3$) denote the usual inner product on $\mathbb{R}^3$. Then, determine the invariant subspaces of the following operators $T \in \mathcal{L}(V)$:

- **Reflection**. Fix a unit vector $u \in V$ (i.e. $u \cdot u = 1$), then define $T(v) = v - 2(u \cdot v)u$.

[1] This means you can think of V as a vector space over $\mathbb{Q}$ but multiplication by $\beta \in \mathbb{C}$ defines a linear operator in V. In other words $\beta \cdot v \in V$, where, you can just keep the operation $\cdot$ as an abstract linear operation for this problem

[2] Now, define the $\cdot$ operation of the previous problem, as ordinary multiplication i.e. $\alpha \cdot \sum_{i=0}^{n} a_i \alpha^i = \sum_{i=0}^{n} a_i \alpha^{i+1}$ for $\sum_{i=0}^{n} a_i \alpha^i \in \mathbb{Q}[\alpha]$.

- **Rotation**. Consider T be the rotation by an angle $\alpha \in (0, \pi)$ about an axis passing through the origin of V. **Hint**: You may want to consider working on a basis where the rotation axis is one of the unit vectors, for example $(0, 0, 1)$. [3]
- **Homothety**. $T(v) = \alpha v$ with $\alpha \in \mathbb{R} \setminus \{0\}$.

Problem 3

- Consider $P \in \mathcal{M}_{n,n}(\mathbb{R})$ orthogonal matrix, i. e. $P^{-1} = P^t$. Show that, if $u, v \in \mathbb{R}^n$ are such that $v = P^t u$, then $\|u\| = \|v\|$ (here, we denote $\|u\| = \sqrt{\sum_{i=1}^n u_i^2}$).
- Consider $H \in \mathcal{M}_{n,n}(\mathbb{R})$ symmetric matrix, i. e. $H = H^t$. Show that,

$$\alpha\|u\|^2 \leq u^t H u \leq \beta\|u\|^2 \qquad \text{for all} u \in \mathbb{R}^n$$

  where α is the smallest eigenvalue of H and β, the largest. **Hint**: you can use the fact that a symmetric matrix can always be diagonalized in the form $H = P^{-1}DP$ where P is orthogonal, D is diagonal and its eigenvalues are all real.

Problem 4

- Consider $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$. If A is invertible, show that AB and BA have the same characteristic polynomial.
- Show that, if $A \in \mathcal{M}_{n,n}(\mathbb{R})$ is diagonalizable and has a single eigenvalue, then $A = \lambda\mathbf{1}$, $\lambda \in \mathbb{R}$.
- Show that, if $A \in \mathcal{M}_{n,n}(\mathbb{R})$ is diagonalizable and such that $A^k = 0$ for some $k \in \mathbb{Z}_{>0}$, then $A = 0$.
- Show that, if $A \in \mathcal{M}_{n,n}(\mathbb{R})$ is diagonalizable, then A^t is also diagonalizable.

Problem 5

Consider $A \in \mathcal{M}_{n,n}(\mathbb{R})$ diagonalizable and such that $1, -1 \notin \sigma(A)$, then show that

$$B = \mathbf{1} + \sum_{i=1}^{k} A^i$$

[3] Recall the rotation matrix by an angle α, on $\mathbb{R}^2$ is given by $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

is invertible and write B^{-1} explicitly. **Hint**: May be useful for you to consider the geometric sum $\sum_{i=0}^{k} a^i = \frac{1-a^{k+1}}{1-a}$.