第四周作业

① Strang书 9.3节 (Page 449-450)

4, 6, 8, 11, 13, 14, 16

② Linear alg done right. (Ex 5.A)

(1) 设 $T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ 2x_2 \\ \vdots \\ n x_n \end{pmatrix}$$

求T的全部不变子空间

Problem Set 9.3

1 Multiply the three matrices in equation (3) and compare with F. In which six entries do you need to know that $i^2 = -1$?

2 Invert the three factors in equation (3) to find a fast factorization of F^{-1}.

3 F is symmetric. So transpose equation (3) to find a new Fast Fourier Transform!

4 All entries in the factorization of F_6 involve powers of $w_6 =$ sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}.$$

Write down these matrices with $1, w_6, w_6^2$ in D and $w_3 = w_6^2$ in F_3. Multiply!

5 If $\boldsymbol{v} = (1, 0, 0, 0)$ and $\boldsymbol{w} = (1, 1, 1, 1)$, show that $F\boldsymbol{v} = \boldsymbol{w}$ and $F\boldsymbol{w} = 4\boldsymbol{v}$. Therefore $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} =$ _____.

6 What is F^2 and what is F^4 for the 4 by 4 Fourier matrix?

7 Put the vector $\boldsymbol{c} = (1, 0, 1, 0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0, 1, 0, 1)$.

8 Compute $\boldsymbol{y} = F_8\boldsymbol{c}$ by the three FFT steps for $\boldsymbol{c} = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $\boldsymbol{c} = (0, 1, 0, 1, 0, 1, 0, 1)$.

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

10 (a) Draw all the sixth roots of 1 on the unit circle. Prove they add to zero.

(b) What are the three cube roots of 1? Do they also add to zero?

11 The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P (see Section 8.3). Multiply PF to find the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}.$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$. The eigenvector matrix (usually X) is F.

12 The equation $\det(P - \lambda I) = 0$ is $\lambda^4 = 1$. This shows again that the eigenvalues are $\lambda =$ _____. Which permutation P has eigenvalues $=$ cube roots of 1?

13 (a) Two eigenvectors of C are $(1,1,1,1)$ and $(1,i,i^2,i^3)$. Find the eigenvalues e.

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_2 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}.$$

(b) $P = F\Lambda F^{-1}$ immediately gives $P^2 = F\Lambda^2 F^{-1}$ and $P^3 = F\Lambda^3 F^{-1}$. Then $\boldsymbol{C} = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 = F(c_0 I + c_1 \Lambda + c_2 \Lambda^2 + c_3 \Lambda^3) F^{-1} = \boldsymbol{FEF^{-1}}$. That matrix E in parentheses is diagonal. It contains the _____ of C.

14 Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix from $E = 2I - \Lambda - \Lambda^3$, with the eigenvalues of P in Λ. The -1's in the corners make this matrix periodic:

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } c_0 = 2, c_1 = -1, c_2 = 0, c_3 = -1.$$

15 ***Fast convolution* $=$ *Fast multiplication by* $\boldsymbol{C}$*:*** To multiply C times a vector $\boldsymbol{x}$, we can multiply $F(E(F^{-1}\boldsymbol{x}))$ instead. The direct way uses n^2 separate multiplications. Knowing E and F, the second way uses only $n \log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

16 **Notice.** Why is row i of $\overline{F}$ the same as row $N - i$ of F (numbered 0 to $N-1$)?

17 What is the *bit-reversed order* of the numbers $0, 1, \ldots, 7$? Write them all in binary (base 2) as $000, 001, \ldots, 111$ and reverse each order. The 8 numbers are now _____.