Homework 12

Problem 1

Consider a matrix $A \in \mathcal{M}_{n,n}(\mathbb{F})$, and a function $f : \mathbb{F} \to \mathbb{F}$, in some cases, we can define a matrix valued function (i.e., we can make sense of $f(A)$), for example by the series expansion of $f(x)$ around a point x_0, as we did to define e^A. For the case of $f(x) \in \mathbb{F}[x]$, $\deg f = m$, we have the finite Taylor expansion:

$$f(x+y) = \sum_{k=0}^{m} f^{(k)}(x)\frac{y^k}{k!} \qquad f^{(k)}(x) = \frac{d^k f(x)}{dx^k} \qquad f^{(0)}(x) = f(x) \tag{1}$$

We will write an expression for $f(A)$ using the formula above. Assume that, if A is block diagonal, i.e. $A = \mathrm{diag}(A_1, \dots, A_k)$, then $f(A) = \mathrm{diag}(f(A_1), \dots, f(A_k))$. Consider A given in Jordan normal form and write it as $A = D + N$ where D is diagonal and N is nilpotent.

- Use this decomposition, the Taylor expansion (1), and the property of f on block diagonal matrices to write an expression for $f(A)$. **Hint**: consider the Taylor expansion of $f(D+N)$ with $x = D$ and $y = N$.
- The expression found in the previous item, based on the expansion (1), can actually be generalized for other functions f (not necessarily polynomial). Consider again the decomposition $A = D + N$ and use it along with the Taylor expansion:

$$f(x+y) = \sum_{k=0}^{\infty} f^{(k)}(x)\frac{y^k}{k!} \tag{2}$$

  to write a matrix expression for $f(A) = e^{As}$, where s is a scalar constant. **Hint**: Since $N^k = 0$ for k large enough, $f(D+N)$ will have a finite expansion.

Problem 2

Consider a vector of functions $x(t) = (x_1(t), \dots, x_n(t))$, so each $x_i(t)$ is a function $x_i : \mathbb{F} \to \mathbb{F}$, $i = 1, \dots, n$. We can also consider matrices of functions, i.e., $A(t)$ is a $n \times m$ matrix such that each entry $A_{ij}(t)$ is a function. Define the matrix $\frac{dA}{dt}$ as the matrix whose entries are $\frac{dA_{ij}(t)}{dt}$.

- Show that Leibniz rule hold for $\frac{d}{dt}$ on matrices i.e. $\frac{d(AB)}{dt} = \frac{dA}{dt}B + A\frac{dB}{dt}$, where $A(t)$ is a $n \times m$ matrix and $B(t)$ is a $m \times r$ matrix. **Hint**: just write the entries of $\frac{d(AB)}{dt}$.
- Use the derivative of e^{At} from homework 11, and the Leibniz rule to show that $x(t) = e^{At}c$, where c is constant vector and A is a constant $n \times n$ matrix (i.e. they are independent of t), satisfies

$$\frac{dx}{dt} = Ax \tag{3}$$

this means $x(t) = e^{At}c$ is the solution of a first order system of linear homogeneous differential equations.

- Consider now a constant, invertible, $n \times n$ matrix C. Show then that $y(t) = C^{-1}x(t)$ satisfies the equation
$$\frac{dy}{dt} = A'y \qquad A' = C^{-1}AC$$

- Because of the previous result, if we want to study the equations (3) we can restrict our attention to the cases in which A is in Jordan canonical form. Consider (3) for the case $n = 2$ and $\mathbb{F} = \mathbb{R}$ and write explicitly the solutions $x(t) = e^{At}c$ for two of the three possible cases of Jordan forms that can occur [1]. For this, use the expressions for e^{At} found in problem 1.

- In the previous problem, there is one case where we cannot write the matrix A directly as a sum $D + N$. Using the series expansion of e^A, show that
$$e^{C^{-1}AC} = C^{-1}e^{A}C$$
then use this identity to compute $x(t) = e^{At}c$ in the case where A cannot be written as $D + N$ [2].

Problem 3

Suppose $u, v, w \in V$, then prove
$$\|w - \frac{1}{2}(u + v)\|^2 = \frac{\|w - u\|^2 + \|w - v\|^2}{2} - \frac{\|u - v\|^2}{4}$$

[1] Do it just for the cases where A can be written as $D + N$

[2] Even though the transformation matrices C you will use may be complex valued, your final answer for $x(t)$ must be real.