Problem 1

Let $x_1, \ldots, x_n \in \mathbb{C}$ be n distinct numbers and consider the numbers $w_1, \ldots, w_n \in \mathbb{C}$ (not necessarily distinct). Prove that there exist a unique monic polynomial $f(x) \in \mathbb{C}[x]$ of degree n satisfying $f(x_i) = w_i$ for all $i = 1, \ldots, n$.

Problem 2

Find closed formulas (i.e., formulas that does not involve sums) for the following sums $\sum_{j=0}^{n-1} \cos(\theta + j\psi)$ and $\sum_{j=0}^{n-1} \sin(\theta + j\psi)$. **Hint**: you may want to consider $\sum_{j=0}^{n-1} \exp i(\theta + j\psi)$.

Problem 3

- Consider $p \in \mathbb{F}[x]$ such that $\deg(p) \geq 1$ and $\alpha \in \mathbb{F}$ a root of p. Show that the multiplicity of α is > 1 if and only if $\frac{dp}{dx}(\alpha) = 0$ (this means the derivative of the polynomial function $p(x)$ evaluated at α).
- Show that $p = t^4 - t$, $p = t^5 - 5t + 1$ and $p = t^2 + bt + c$, with $b^2 - 4c \neq 0$ have no multiple roots in $\mathbb{C}$.

Problem 4

Determine all the rational roots of $t^7 - 1$, $t^8 - 1$, $2t^2 - 3t + 4$, $3t^3 + t - 5$ and $2t^4 - 4t + 3$. **Hint**: Problem 2, homework 3.

Problem 5

Consider the field $(\mathbb{K}, +, \cdot)$ where $\mathbb{K} = \{a + b\sqrt{5} | a, b \in \mathbb{Q}\}$ (you do not have to prove that $\mathbb{K}$ is a field). Define the function $T : \mathbb{K} \to \mathbb{K}$, $T(a + b\sqrt{5}) = a - b\sqrt{5}$. Show that:

- T is an isomorphism.
- Given $p \in \mathbb{Q}[x]$ then $p(T(x_0)) = T(p(x_0))$ for any $x_0 \in \mathbb{K}$. Conclude that if $a + b\sqrt{5}$ is a root of p, then $a - b\sqrt{5}$ is also a root of p.
- Given $p \in \mathbb{Q}[x]$ such that $\deg(p) = 3$, and $2 + \sqrt{5}$ is a root of p, then p has a root in $\mathbb{Q}$.

Problem 6

Consider the polynomial $f(x) = \prod_{j=1}^{n}(x - x_j) + 1 \in \mathbb{Z}[x]$, with n odd and $x_1, \ldots, x_n$ distinct integers. Show that $f(x)$ is irreducible in $\mathbb{Z}$. **Hint**: you may want to consider proof by contradiction i.e. assume $f = hg$, then consider $f(x_i)$ for all i.

Problem 7

Recall the cyclotomic polynomials from homework 2. Show that $\Phi_d(x) \in \mathbb{Z}[x]$ (set $\Phi_1(x) = x - 1$).

Problem 8

Let $p, q \in \mathbb{Q}[x]$ be monic irreducible polynomials. Suppose p and q have respective roots α and β such that $\alpha + \beta \in \mathbb{Q}$ (note, α and β are not necessarily rational themselves). Prove that $p^2 - q^2$ has a rational root. We will show this as follows:

- Show that if two irreducible polynomials $p_1, p_2 \in \mathbb{Q}[x]$ share a root, then $p_1 = cp_2$ with $c \in \mathbb{Q}$.
- Prove that $p^2 - q^2$ has a rational root.**Hint**: Consider $q(\alpha + \beta - x)$. Note that $q(\alpha + \beta - x) \in \mathbb{Q}[x]$.

Problem 9

Let p be a prime and $f(x) = x^{p-1} + x^{p-2} + \cdots + 1$. Show that $f(x)$ is irreducible over $\mathbb{Q}$. **Hint**: Consider $f(x + a)$ for an appropriately chosen a and Eisenstein criterion.

Problem 10

- We know that Leibniz rule apply to derivatives of product of polynomial functions, namely $\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$ for $f, g \in \mathbb{F}[x]$. Show this explicitly by comparing coefficients.
- Show that an irreducible polynomial over $\mathbb{Q}$ has no multiple roots.