Homework 11

Problem 1

Show that the matrices $A, B \in \mathcal{M}_{n,n}(\mathbb{C})$ are similar if and only if $\sigma(A) = \sigma(B)$ and for each eigenvalue λ and each $i \leq n$, we have

$$\mathrm{rk}(A - \lambda \mathbf{1})^i = \mathrm{rk}(B - \lambda \mathbf{1})^i$$

Problem 2

Show that $T \in \mathcal{L}(V)$ (V is a finite dimensional complex vector space) is diagonalizable if and only if for all $\lambda \in \sigma(T)$, we have

$$\mathrm{rk}(T - \lambda \mathbf{1}) = \mathrm{rk}(T - \lambda \mathbf{1})^2$$

Problem 3

Consider $T \in \mathcal{L}(V)$ (V is a finite dimensional complex vector space)

- Consider the case $\dim V = 2$ and suppose there exist a basis $\mathcal{V}$ of V where $\mathcal{M}(T, \mathcal{V}) \in \mathcal{M}_{2,2}(\mathbb{R})$. Under these assumptions, write all the possible Jordan normal forms of T (up to ordering of the blocks). **Hint**: there are three.

- Suppose now that V is the complexification of a real vector space U i.e. $V = U^{\mathbb{C}}$ and $\dim V = 2$. Consider $A \in \mathcal{L}(U)$ and then, recall the definition of $A^{\mathbb{C}} \in \mathcal{L}(V)$:

  $$A^{\mathbb{C}}(x) = A(u) + iA(v) \qquad x = u + iv \in V$$

  where $u, v \in U$. Then $A^{\mathbb{C}}$ as an eigenvector $x = x_1 + ix_2$ with eigenvalue λ. Assuming $\lambda \in \mathbb{C} \setminus \mathbb{R}$ write $\mathcal{M}(A)$ in the basis x_1, x_2. What are the eigenvalues of $\mathcal{M}(A)$?

- Use the previous two results to write the three possible Jordan normal forms (up to ordering of the blocks) of an operator on a real vector space of dimension two.

- Use the previous results to write the most general Jordan normal form of an operator A over a real vector space of dimension n. **Hint**: start from the most general canonical form of $A^{\mathbb{C}}$, then use the fact that eigenvalues come in conjugate pairs to reorder the blocks in a way that you can apply the results for the case of $\dim V = 2$.

Problem 4

Find the Jordan normal form (up to ordering) of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Problem 5

Functions of matrices. In homework 10 we defined e^A, where $A \in \mathcal{M}_{n,n}(\mathbb{C})$ as a series (again, do not worry about convergence for this problem). Use that definition of e^A in the following

- Consider $A, B \in \mathcal{M}_{n,n}(\mathbb{C})$. Check, up to order 3 in A and B that, if $[A, B] = AB - BA = 0$ then $e^{A+B} = e^A e^B$. **Hint**: just compare the series, up to order 3 at both sides.

- Assume that if $[A, B] = 0$ then $e^{A+B} = e^A e^B$ holds at all orders. Use this to show

  $$\frac{d}{dt} e^{At} = A e^{At}$$

  where t is a scalar and the derivative is defined by

  $$\frac{d}{dt} e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h}$$

  Hint: using the property $e^{A+B} = e^A e^B$ and the fact that $As = sA$ for a scalar s, should let you manipulate the expression $\frac{e^{A(t+h)} - e^{At}}{h}$ to bring it into a form where the limit can be computed just by evaluating it at $h = 0$.