Sols. Hw 12

P1

a) Suppose $A = D + N$, where D is diagonal, $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}$ and N is nilpotent. If we are considering A a Jordan block, then $D = \lambda \cdot \mathbb{1}_m$ and

$N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ and if the dimension of N is m, then

$N^m = 0$, so

$$f(A) = f(D+N) = \sum_{k=0}^{m-1} f^{(k)}(D) \frac{N^k}{k!}$$

using the property of f on block diagonal matrices: (here $D = \lambda \cdot \mathbb{1}_m$)

$$f^{(k)}(D) = \begin{pmatrix} f^{(k)}(d_1) & & \\ & \ddots & \\ & & f^{(k)}(d_m) \end{pmatrix} = \begin{pmatrix} f^{(k)}(\lambda) & & \\ & \ddots & \\ & & f^{(k)}(\lambda) \end{pmatrix}$$

where $f^{(k)}(\lambda) = \frac{d^k f(\lambda)}{d\lambda^k}$

$f^{(0)}(\lambda) = f(\lambda)$

$$\begin{array}{c} k+1 \text{ column} \\ \downarrow \end{array}$$

So $f^{(k)}(D) . N^k = \begin{pmatrix} 0 & \cdots & f^{(k)}(\lambda) & & \\ & \ddots & & \ddots & \\ & & \ddots & & f^{(k)}(\lambda) \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}$

$$\therefore \quad f(A) = \begin{pmatrix} f(\lambda) & f^{(1)}(\lambda) & \cdots & \frac{f^{(m-1)}(\lambda)}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & f^{(1)}(\lambda) \\ 0 & & & f(\lambda) \end{pmatrix} \quad (*)$$

So, if f is a polynomial of degree $n \leq m-1$, then $f^{(k)}(\lambda)$ for $k \geq n+1$ vanishes, if $n > m-1$, all the entries $f^{(k)}(\lambda)$ of $(*)$ are generically non zero.

b) If we use the ~~sorest~~ Taylor expansion

$$f(D+N) = \sum_{k=0}^{\infty} f^{(k)}(D)\frac{N^k}{k!} = \sum_{k=0}^{m-1} f^{(k)}(D)\frac{N^k}{k!}$$

So, the form of $f(D+N)$ is the same as $(*)$, we just need to use $f(x) = e^{xs}$, hence define $\varphi_k(\lambda) = \frac{e^{\lambda \cdot s}}{k!} \cdot s^k$, then

$$e^{As} = f(A) = \sum_{k=0}^{m-1} f^{(k)}(\overset{\lambda \cdot \mathbb{1}_m}{\overset{\shortparallel}{D}})\frac{N^k}{k!} = \begin{pmatrix} \varphi_0(\lambda) & \varphi_1(\lambda) & \cdots & \varphi_{m-1}(\lambda) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \varphi_1(\lambda) \\ 0 & & & \varphi_0(\lambda) \end{pmatrix}$$

P21

a) $(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$, so $\left(\frac{d(AB)}{dt}\right)_{ij} = \frac{d}{dt}\left(\sum_{k=1}^{m} A_{ik} B_{kj}\right) = \sum_{k=1}^{m}\left(\frac{dA_{ik}}{dt} B_{kj} + A_{ik}\frac{dB_{kj}}{dt}\right)$

$$= \sum_{k=1}^{m}\left(\frac{dA}{dt}\right)_{ik} B_{kj} + \sum_{k=1}^{m} A_{ik}\left(\frac{dB}{dt}\right)_{kj} = \left(\frac{dA}{dt}\cdot B\right)_{ij} + \left(A\cdot\frac{dB}{dt}\right)_{ij}$$

b)

$$\frac{dx}{dt} = \frac{d}{dt}\left(e^{At}\cdot c\right) \underset{\text{Leibniz rule}}{=} \left(\frac{d}{dt}e^{At}\right)\cdot c + e^{At}\overset{0}{\overset{\shortparallel}{\frac{dc}{dt}}} \underset{\text{HW 11}}{=} Ae^{At}c = A\cdot x$$

c)

$$\frac{dy}{dt} = C^{-1}\frac{dx}{dt} = C^{-1}A\cdot x = C^{-1}A\cdot C\cdot C^{-1}x = A'\cdot y$$

d) So we consider two cases for A:

1) $A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, then $x = e^{\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} t} \cdot c = \begin{pmatrix} e^{\alpha_1 t} & 0 \\ 0 & e^{\alpha_2 t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{\alpha_1 t} \\ c_2 e^{\alpha_2 \cdot t} \end{pmatrix}$

Two blocks of 1×1, we can use directly the property of f actiong on diagonal matrices

2) $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, $N^2 = 0$

then $e^{At} = e^{\alpha t} \cdot \mathbb{1} + \alpha e^{\alpha t} \cdot \underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\underset{\parallel}{N}} = \begin{pmatrix} e^{\alpha t} & e^{\alpha t} \cdot t \\ 0 & e^{\alpha t} \end{pmatrix}$

hence $x = \begin{pmatrix} e^{\alpha t} & t \cdot e^{\alpha t} \\ 0 & e^{\alpha t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{\alpha t} + c_2 \cdot t e^{\alpha t} \\ c_2 \cdot e^{\alpha t} \end{pmatrix}$

e) The equality $e^{C^{-1}AC} = C^{-1} e^{A} C$ follows easily from the fact that

$$(C^{-1}AC)^k = C^{-1}A^k C$$

Now, we want to consider the case $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $b \neq 0$, is straightforward to show (just diagonalize A)

$$C^{-1} \cdot A C = \begin{pmatrix} a-ib & 0 \\ 0 & a+ib \end{pmatrix}$$

with $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$

hence $A = C \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix} C^{-1}$ with $\lambda = a+ib$

then $e^{A} = e^{C \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix} C^{-1}} = C e^{\begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix}} C^{-1} = C \begin{pmatrix} e^{\lambda} & \\ & e^{\bar{\lambda}} \end{pmatrix} C^{-1}$

$$= \begin{pmatrix} \cos a & -\sin b \\ \sin b & \cos a \end{pmatrix} \in M_{2,2}(\mathbb{R})$$

as expected

then

$$x = e^{At} c = \begin{pmatrix} c_1 \cos a - c_2 \sin b \\ c_1 \sin b + c_2 \cos a \end{pmatrix}$$

P3) Recall $\|v\| = \sqrt{\langle v, v \rangle}$ and the (easy to prove) parallelogram identity:

$$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$$

then, just use $a = \frac{1}{2}(w-u)$ and $b = \frac{1}{2}(w-v)$