第三周作业一解答

1. 设 T_1, T_2 在标准基下矩阵分别是 A_1, A_2，则 $i(T_1T_2-T_2T_1)$ 在上述基下矩阵是 $i(A_1A_2-A_2A_1) \xlongequal{记作} A_3$

$$A_3^H = \bar{i}(A_1A_2-A_2A_1)^H$$
$$= -i(A_2^HA_1^H - A_1^HA_2^H)$$
$$= A_3$$

因此，它是一个 Hermite 变换.

2. 设 $(T(f_1), \cdots, T(f_n)) = (f_1, \cdots, f_n)A$

$(T^*(f_1), \cdots, T^*(f_n)) = (f_1, \cdots, f_n)B$

因为 $T(f_k) = \langle f_1, T(f_k)\rangle f_1 + \cdots + \langle f_n, T(f_k)\rangle f_n$

即 A 的第 k 列 $= \begin{pmatrix} \langle f_1, T(f_k)\rangle \\ \vdots \\ \langle f_n, T(f_k)\rangle \end{pmatrix}$

即 $a_{ij} = \langle f_i, T(f_j) \rangle$

设 $B = (b_{ij})$，类似讨论 $b_{ij} = \langle f_i, T^*(f_j) \rangle$

因为 $\langle f_i, T^*(f_j) \rangle = \langle T(f_i), f_j \rangle$

$= \overline{\langle f_j, T(f_i) \rangle} = \overline{a_{ji}}$

$\Rightarrow B = A^H$.

3. (1) 设 $\vec{v} \in \mathbb{C}^n$，
$\vec{v} = c_1 \vec{v_1} + \cdots + c_n \vec{v_n}$.
$\vec{w} = d_1 \vec{v_1} + \cdots + d_n \vec{v_n}$

$$\langle \vec{v}, \vec{w} \rangle = \langle \sum_{i=1}^{n} c_i v_i , \sum_{j=1}^{n} d_j v_j \rangle$$

$$= \sum_{i,j=1}^{n} \overline{c_i} d_j \langle v_i, v_j \rangle$$

$$= (\overline{c_1}, \cdots, \overline{c_n}) \underbrace{\begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_n \rangle \\ \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_n \rangle \end{pmatrix}}_{G_{\mathcal{B}}} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

(2) 令 $G_B = (g_{ij})$ $g_{ij} = \langle v_i, v_j \rangle$

则 $g_{ij} = \overline{g_{ji}} \Rightarrow G_B^H = G_B$.

(3) $\langle v, w \rangle = [v]_B^H G_B [w]_B$ (由(1))

$= [v]_{B'}^H G_{B'} [w]_{B'}$

因为 $[v]_B = P[v]_{B'}$, $[w]_B = P[w]_{B'}$

代入,得 $[v]_B^H G_B [w]_B$

$= [v]_B^H P^H G_{B'} P [w]_B$

因为 v, w 的任意性,

$G_B = P^H G_{B'} P$.

4. 令 $A=\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ $\quad A^H A = A A^H$ (A为可正交)

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda-1 & -i \\ -i & \lambda-1 \end{vmatrix} = (\lambda-1)^2+1 = \lambda^2-2\lambda+2$$

$\lambda_1 = 1+i,\quad \lambda_2 = 1-i$

$$Ax=\lambda_1 x \Longrightarrow e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Ax=\lambda_2 x \Longrightarrow e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

T 在基 e_1, e_2 下矩阵 $=\begin{pmatrix} 1+i & \\ & 1-i \end{pmatrix}$

5. 设 T 在标准基下矩阵 $=A=(a_{ij})$ 则

$A^H = A$，且 $\forall v \in \mathbb{C}^n$，$v^H A v = 0$

令 $v=e_i$，$e_i^H A e_i = a_{ii} = 0$

令 $v = e_i + e_j \ (i \neq j)$，$(e_i+e_j)^H A (e_i+e_j) = 0$

$= e_i^H A e_j + e_j^H A e_i = a_{ij} + a_{ji} = 0$

但 $a_{ij} = \overline{a_{ji}} \Rightarrow a_{ij} + \overline{a_{ij}} = 0 \Rightarrow a_{ij}$ 实部 $= 0$

令 $v = e_s + ie_t$, $(s \neq t)$ $(e_s + ie_t)^H A (e_s + ie_t) = 0$

$\|$

$e_s^H A (ie_t) + (ie_t)^H A e_s$

$= i(e_s^H A e_t) - i(e_t^H A e_s)$

$= i[a_{st} - a_{ts}]$

即 $a_{st} = a_{ts}$ 但 $a_{st} = \overline{a_{ts}} \Rightarrow a_{st} \in \mathbb{R}$

$\Rightarrow a_{st} = 0 \; (s \neq t) \Rightarrow A = 0$

6. V_2 由 $1, e^{it}, e^{i2t}, e^{-it}, e^{-i2t}$

生成 因为 $e^{it} = \cos t + i \sin t$,

$\cos t = \dfrac{e^{it} + e^{-it}}{2}$, $\sin t = \dfrac{e^{it} - e^{-it}}{2i}$

$\Rightarrow 1, \cos t, \sin t, \cos 2t, \sin 2t \in V_2$

且生成 V_2.

$$\langle 1, \cos t \rangle = \int_0^{2\pi} \cos t \, dt = 0$$

$$\langle 1, \sin t \rangle = \int_0^{2\pi} \sin t \, dt = 0$$

$$\langle \cos t, \sin t \rangle = \int_0^{2\pi} \cos t \sin t \, dt = \int_0^{2\pi} \frac{1}{2} \sin 2t \, dt = 0$$

$$\langle \cos t, \cos 2t \rangle = \int_0^{2\pi} \cos t \cos 2t \, dt$$

$$= \int_0^{2\pi} \frac{e^{i3t} + e^{it} + e^{-it} + e^{-i3t}}{4} dt$$

$$= \frac{1}{4} \cdot \left[\frac{1}{3i} e^{i3t} + \frac{1}{i} e^{it} - \frac{1}{i} e^{-it} - \frac{1}{3i} e^{-i3t} \right]_0^{2\pi}$$

$$= 0$$

同理 $\langle \sin t, \sin 2t \rangle = 0 = \langle \cos 2t, \sin 2t \rangle$

即 $1, \cos t, \sin t, \cos 2t, \sin 2t$ 相互.

正交，是 V_2 的基.

$$\|1\| = \sqrt{\int_0^{2\pi} 1\,dt} = \sqrt{2\pi}.$$

$$\|\cos t\| = \sqrt{\int_0^{2\pi} \cos^2 t\,dt} = \sqrt{\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)\Big|_0^{2\pi}} = \sqrt{\pi}$$

$$\|\sin t\| = \sqrt{\int_0^{2\pi} \sin^2 t\,dt} = \sqrt{\left(\frac{t}{2}-\frac{\sin 2t}{4}\right)\Big|_0^{2\pi}} = \sqrt{\pi}.$$

$$\|\cos 2t\| = \|\sin 2t\| = \sqrt{\pi}.$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}$$ 是

标准正交基.

7. $f(t)=|t|$ (当 $|t|\le 1$)…… $f(t+2)=f(t)$ $T=2$.

令 $e_n(t)=e^{i\frac{2\pi n t}{T}}$

$$\|e_n(t)\|^2=\int_0^T \overline{e_n(t)}\,e_n(t)\,dt$$

$$=\int_0^T 1\,dt = T$$

即 $\left\{\frac{1}{\sqrt{T}}e_n(t)\,\middle|\,n\in\mathbb{Z}\right\}$ 标准正交.

令 $f_n(t)=\frac{1}{\sqrt{T}}e_n(t)$.

$f(t)$ 的 Fourier 级数

$$=\sum_{n=-\infty}^{\infty} c_n f_n(t)$$

其中 $c_n=\int_0^T \overline{f_n(t)}\,f(t)\,dt$ $\langle f_n(t), f(t)\rangle$

$$c_n = \frac{1}{\sqrt{T}} \int_0^2 e^{-2\pi i n t/T} f(t)\,dt$$

$$= \frac{1}{\sqrt{T}} \left[\int_{-1}^{1} e^{-2\pi i n t/T} f(t)\,dt\right]$$

$$= \frac{1}{\sqrt{T}} \left[\int_{-1}^{0} -t e^{-i\pi n t}\,dt + \int_0^1 t e^{-i\pi n t}\,dt\right]$$

$\int_0^1 t e^{-i\pi n t}\,dt$ 令 $a = -i\pi n$.

$$= \frac{1}{a}\left[t e^{at} - \frac{1}{a} e^{at}\right]\Big|_0^1$$

$$= \frac{1}{a} e^{a} - \frac{1}{a^2}(e^{a} - 1)$$

$\int_{-1}^{0} -t e^{-i\pi n t}\,dt$

$$= \frac{-1}{a}\left[t e^{at} - \frac{1}{a} e^{at}\right]\Big|_{-1}^0$$

$$= -\frac{1}{a} e^{-a} - \frac{1}{a^2}(e^{-a} - 1)$$

$$\Rightarrow C_n = \frac{1}{\sqrt{T}}\left[\frac{1}{a}(e^{a}-e^{-a})+\frac{1}{a^2}(2-e^{a}-e^{-a})\right]$$

$$= \frac{1}{\sqrt{T}}\cdot\frac{2}{-n^2\pi^2}(1-\cos n\pi) \quad (n\neq 0)$$

$$= \begin{cases} 0 & n \text{ even } (n\neq 0) \\ \frac{1}{\sqrt{T}}\cdot\frac{4}{-n^2\pi^2} & n \text{ odd } (n\neq 0) \end{cases}$$

$$C_0 = \frac{1}{\sqrt{2}}\int_0^2 |t|\,dt = \frac{1}{\sqrt{2}}\int_{-1}^{1} |t|\,dt$$

$$= \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} C_n f_n(t) = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{n^2\pi^2} e^{i\pi n t}$$

$$= \frac{1}{2} - \sum_{\substack{n=1,\, n \text{ odd} \\ n}}^{+\infty} \frac{4}{n^2\pi^2}\cos \pi n t$$