Problem Set 9.1

Questions 1–8 are about operations on complex numbers.

1 Add and multiply each pair of complex numbers:

(a) $2+i, 2-i$ (b) $-1+i, -1+i$ (c) $\cos\theta + i\sin\theta, \cos\theta - i\sin\theta$

2 Locate these points on the complex plane. Simplify them if necessary:

(a) $2+i$ (b) $(2+i)^2$ (c) $\frac{1}{2+i}$ (d) $|2+i|$

3 Find the absolute value $r = |z|$ of these four numbers. If θ is the angle for $6-8i$, what are the angles for the other three numbers?

(a) $6-8i$ (b) $(6-8i)^2$ (c) $\frac{1}{6-8i}$ (d) $(6+8i)^2$

4 If $|z| = 2$ and $|w| = 3$ then $|z \times w| =$ _____ and $|z+w| \le$ _____ and $|z/w| =$ _____ and $|z-w| \le$ _____.

5 Find $a+ib$ for the numbers at angles $30°, 60°, 90°, 120°$ on the unit circle. If w is the number at $30°$, check that w^2 is at $60°$. What power of w equals 1?

6 If $z = r\cos\theta + ir\sin\theta$ then $1/z$ has absolute value _____ and angle _____. Its polar form is _____. Multiply $z \times 1/z$ to get 1.

7 The complex multiplication $M = (a+bi)(c+di)$ is a 2 by 2 real multiplication

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

The right side contains the real and imaginary parts of M. Test $M = (1+3i)(1-3i)$.

8 $A = A_1 + iA_2$ is a complex n by n matrix and $\boldsymbol{b} = \boldsymbol{b}_1 + i\boldsymbol{b}_2$ is a complex vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x}_1 + i\boldsymbol{x}_2$. Write $A\boldsymbol{x} = \boldsymbol{b}$ as a real system of size $2n$:

Complex n by n
Real $2n$ by $2n$
$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}.$$

Questions 9–16 are about the conjugate $\overline{z} = a - ib = re^{-i\theta} = z^*$.

9 Write down the complex conjugate of each number by changing i to $-i$:

(a) $2-i$ (b) $(2-i)(1-i)$ (c) $e^{i\pi/2}$ (which is i)

(d) $e^{i\pi} = -1$ (e) $\frac{1+i}{1-i}$ (which is also i) (f) $i^{103} =$ _____.

10 The sum $z + \overline{z}$ is always _____. The difference $z - \overline{z}$ is always _____. Assume $z \neq 0$. The product $z \times \overline{z}$ is always _____. The ratio $z/\overline{z}$ has absolute value _____.

11 For a real matrix, the conjugate of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is $A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$. This proves two things: $\overline{\lambda}$ is another eigenvalue and $\overline{\boldsymbol{x}}$ is its eigenvector. Find the eigenvalues $\lambda, \overline{\lambda}$ and eigenvectors $\boldsymbol{x}, \overline{\boldsymbol{x}}$ of $A = [\, a \quad b; \quad -b \quad a \,]$.

12 The eigenvalues of a real 2 by 2 matrix come from the quadratic formula:

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

gives the two eigenvalues $\lambda = \left[a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)} \right] /2$.

(a) If $a = b = d = 1$, the eigenvalues are complex when c is ____ .

(b) What are the eigenvalues when $ad = bc$?

13 In Problem 12 the eigenvalues are not real when $(\text{trace})^2 = (a+d)^2$ is smaller than ____ . Show that the λ's *are* real when $bc > 0$.

14 A real skew-symmetric matrix ($A^{\mathrm{T}} = -A$) has pure imaginary eigenvalues. First proof: If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then block multiplication gives

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ i\boldsymbol{x} \end{bmatrix} = i\lambda \begin{bmatrix} \boldsymbol{x} \\ i\boldsymbol{x} \end{bmatrix}.$$

This block matrix is symmetric. Its eigenvalues must be ____ ! So λ is ____ .

Questions 15–22 are about the form $re^{i\theta}$ of the complex number $r\cos\theta + ir\sin\theta$.

15 Write these numbers in Euler's form $re^{i\theta}$. Then square each number:

(a) $1 + \sqrt{3}i$ (b) $\cos 2\theta + i \sin 2\theta$ (c) $-7i$ (d) $5 - 5i$.

16 (A favorite) Find the absolute value and the angle for $z = \sin\theta + i\cos\theta$ (careful). Locate this z in the complex plane. Multiply z by $\cos\theta + i\sin\theta$ to get ____ .

17 Draw all eight solutions of $z^8 = 1$ in the complex plane. What is the rectangular form $a + ib$ of the root $z = \overline{w} = \exp(-2\pi i/8)$?

18 Locate the cube roots of 1 in the complex plane. Locate the cube roots of -1. Together these are the sixth roots of ____ .

19 By comparing $e^{3i\theta} = \cos 3\theta + i \sin 3\theta$ with $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$, find the "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

20 Suppose the conjugate $\overline{z}$ is equal to the reciprocal $1/z$. What are all possible z's?

21 (a) Why do e^i and i^e both have absolute value 1?

(b) In the complex plane put stars near the points e^i and i^e.

(c) The number i^e could be $(e^{i\pi/2})^e$ or $(e^{5i\pi/2})^e$. Are those equal?

22 Draw the paths of these numbers from $t = 0$ to $t = 2\pi$ in the complex plane:

(a) e^{it} (b) $e^{(-1+i)t} = e^{-t}e^{it}$ (c) $(-1)^t = e^{t\pi i}$.

Problem Set 9.2

1 Find the lengths of $\boldsymbol{u} = (1+i, 1-i, 1+2i)$ and $\boldsymbol{v} = (i, i, i)$. Find $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ and $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$.

2 Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

3 Solve $A\boldsymbol{z} = \boldsymbol{0}$ to find a vector $\boldsymbol{z}$ in the nullspace of A in Problem 2. Show that $\boldsymbol{z}$ is orthogonal to the columns of A^{H}. Show that $\boldsymbol{z}$ is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer*** $\boldsymbol{C}(A^{\mathrm{T}})$. **Now it is** $\boldsymbol{C}(\boldsymbol{A}^{\mathrm{H}})$.

4 Problem 3 indicates that the four fundamental subspaces are $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$ and _____ and _____. Their dimensions are still r and $n-r$ and r and $m-r$. They are still orthogonal subspaces. *The symbol* $^{\mathrm{H}}$ *takes the place of* $^{\mathrm{T}}$.

5 (a) Prove that $A^{\mathrm{H}}A$ is always a Hermitian matrix.

(b) If $A\boldsymbol{z} = \boldsymbol{0}$ then $A^{\mathrm{H}}A\boldsymbol{z} = \boldsymbol{0}$. If $A^{\mathrm{H}}A\boldsymbol{z} = \boldsymbol{0}$, multiply by $\boldsymbol{z}^{\mathrm{H}}$ to prove that $A\boldsymbol{z} = \boldsymbol{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are _____. Therefore $A^{\mathrm{H}}A$ is an invertible Hermitian matrix when the nullspace of A contains only $\boldsymbol{z} = 0$.

6 True or false (give a reason if true or a counterexample if false):

(a) If A is a real matrix then $A + iI$ is invertible.

(b) If S is a Hermitian matrix then $S + iI$ is invertible.

(c) If Q is a unitary matrix then $Q + iI$ is invertible.

7 When you multiply a Hermitian matrix by a real number c, is cS still Hermitian? Show that iS is skew-Hermitian when S is Hermitian. The 3 by 3 Hermitian matrices are a subspace provided the "scalars" are real numbers.

8 Which classes of matrices does P belong to: invertible, Hermitian, unitary?

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}.$$

Compute P^2, P^3, and P^{100}. What are the eigenvalues of P?

9 Find the unit eigenvectors of P in Problem 8, and put them into the columns of a unitary matrix Q. What property of P makes these eigenvectors orthogonal?

10 Write down the 3 by 3 circulant matrix $C = 2I + 5P$. It has the same eigenvectors as P in Problem 8. Find its eigenvalues.

11 If Q and U are unitary matrices, show that Q^{-1} is unitary and also QU is unitary. Start from $Q^{\mathrm{H}}Q = I$ and $U^{\mathrm{H}}U = I$.

12 How do you know that the determinant of every Hermitian matrix is real?

13 The matrix $A^{\mathrm{H}}A$ is not only Hermitian but also positive definite, when the columns of A are independent. Proof: $\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}}A\boldsymbol{z}$ is positive if $\boldsymbol{z}$ is nonzero because _____.

14 Diagonalize these Hermitian matrices to reach $S = Q\Lambda Q^{\mathrm{H}}$:

$$S = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 2 & 1+i \\ i-1 & 3 \end{bmatrix}.$$

15 Diagonalize this skew-Hermitian matrix to reach $K = Q\Lambda Q^{\mathrm{H}}$. All λ's are _____:

$$K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}.$$

16 Diagonalize this orthogonal matrix to reach $U = Q\Lambda Q^{\mathrm{H}}$. Now all λ's are _____:

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

17 Diagonalize this unitary matrix to reach $U = Q\Lambda Q^{\mathrm{H}}$. Again all λ's are _____:

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

18 If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis for $\mathbf{C}^n$, the matrix with those columns is a _____ matrix. Show that any vector $\boldsymbol{z}$ equals $(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_1 + \cdots + (\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_n$.

19 $\boldsymbol{v} = (1, i, 1), \boldsymbol{w} = (i, 1, 0)$ and $\boldsymbol{z} =$ _____ are an orthogonal basis for _____.

20 If $S = A + iB$ is a Hermitian matrix, are its real and imaginary parts symmetric?

21 The (complex) dimension of $\mathbf{C}^n$ is _____. Find a non-real basis for $\mathbf{C}^n$.

22 Describe all 1 by 1 and 2 by 2 Hermitian matrices and unitary matrices.

23 How are the eigenvalues of A^{H} related to the eigenvalues of the square matrix A?

24 If $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u} = 1$ show that $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is Hermitian and also unitary. The rank-one matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is the projection onto what line in $\mathbf{C}^n$?

25 If $A + iB$ is a unitary matrix (A and B are real) show that $Q = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is an orthogonal matrix.

26 If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$ is symmetric.

27 Prove that the inverse of a Hermitian matrix is also Hermitian (transpose $S^{-1}S = I$).

28 A matrix with orthonormal eigenvectors has the form $N = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{H}}$. *Prove that* $NN^{\mathrm{H}} = N^{\mathrm{H}}N$. These N are exactly the **normal matrices**. Examples are Hermitian, skew-Hermitian, and unitary matrices. Construct a 2 by 2 normal matrix from $Q\Lambda Q^{\mathrm{H}}$ by choosing complex eigenvalues in Λ.