Homework 6

Remark: In order that you have more time to study for the midterm exam, you have two weeks for doing this homework, i.e. it is due April 15, 2019.

Problem 1

- **Three-term recurrence relation for orthogonal polynomials.** In Homework 5 we saw that Chebyshev polynomials satisfy a recurrence relation. Here we will show that this holds in general. Considering the collection of monic orthogonal polynomials, $p_n(x) \in \mathbb{R}[x]$ (actually, the field does not play an important role), $n = -1, 0, 1, 2, \ldots$ as defined in the lecture (for convenience we add $n = -1$). Set $p_{-1}(x) = 0$ and $p_0(x) = 1$, then show that p_n's satisfy

$$p_{n+1} = (x - \alpha_n)p_n - \beta_n p_{n-1} \qquad n = 0, 1, 2, \ldots \tag{1}$$

  and determine the constants α_n and β_n. **Hint**: From the form of the inner product we defined, one can use that $\langle p(x), g(x)h(x)\rangle = \langle g(x)p(x), h(x)\rangle$, etc. Also it may be convenient to show that if $\deg(p) = n$ and $\langle p_l, p\rangle = 0$ for $l = 0, \ldots, n$, then $p = 0$.

- **Hermite polynomials.** Consider the Schrödinger equation for the quantum harmonic oscillator

$$\frac{d^2\psi(x)}{dx^2} + (2\epsilon - x^2)\psi(x) = 0 \tag{2}$$

  where ϵ is the a dimensionless parameter characterizing the energy of the solution. Consider a solution of the form $\psi(x) = e^{-x^2/2}u(x)$ and write the resulting differential equation for $u(x)$.

- Try a series solution of the form $u(x) = \sum_{i=0}^{\infty} c_i x^i$ and show that the coefficients should satisfy

$$c_{n+2} = c_n \frac{(2n + 1 - 2\epsilon)}{(n + 1)(n + 2)} \tag{3}$$

  there is a special collection for values for ϵ, call them ϵ_n, so that the series truncates i.e. $c_n = 0$ for all $n \geq \delta$ certain δ. What are these values?. This corresponds to the quantization of the energy (the values of the energy are not continuous as in classical mechanics). Write the equation for the resulting equation for $u(x)$ and write the solutions, normalized to be monic, for $n = 0, 1, 2, 3$ and 4. These are the first four Hermite polynomials.

Problem 2

- Show that for a nilpotent operator A (i.e. there exist $k > 0$ such that $A^k = 0$) its spectrum is given by $\sigma(A) = \{0\}$.
- Show that the characteristic polynomial of a block square matrix (note not all blocks are necessarily of the same dimension)
$$Q = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix} \tag{4}$$
takes the form $p(\lambda) = \det(A - \lambda\mathbf{1})\det(B - \lambda\mathbf{1})$ where $\mathbf{1}$ are the identities of the appropriate rank[1]
- Suppose $Q \in \mathcal{L}(V)$ and there is a basis $\{v_1, \ldots, v_n\}$ of V such that $\{v_1, \ldots, v_k\}$ are eigenvectors of Q, with eigenvalue λ. Show that in this basis Q takes the form
$$Q = \begin{pmatrix} \lambda\mathbf{1} & D \\ 0 & B \end{pmatrix} \tag{5}$$
indicating clearly the dimension of all the matrices in the blocks.
- Use the two previous result to show that the geometric multiplicity of an eigenvalue is always less or equal than its algebraic multiplicity. Give an example of a matrix having an eigenvalue whose geometric and algebraic multiplicities differ.

Problem 3

Consider m even and $\mathbb{F}_m[x]$. Define
$$V = \{p = \sum_{i=0}^{m} a_i x^i \in \mathbb{F}_m[x] | i \in \{0, \ldots, m\}, a_i = a_{m-i}\} \tag{6}$$

- Show that V is a vector subspace of $\mathbb{F}_m[x]$.
- Find a basis of V and show its dimension is $n + 1$, where $m = 2n$.
- Show that $\mathbb{F}_m[x] = V \oplus \mathbb{F}_{n-1}[x]$.
- Define
$$V' = \{p = \sum_{i=0}^{m} a_i x^i \in \mathbb{F}_m[x] | i \in \{0, \ldots, m\}, a_i = -a_{m-i}\} \tag{7}$$
Show that $\mathbb{F}_m[x] = V \oplus V'$ (you can assume that V' is a vector subspace of $\mathbb{F}_m[x]$).

[1] You can use the property of determinant of block matrices $\det \begin{pmatrix} A & D \\ C & B \end{pmatrix} = \det(AB - CD)$.

Problem 4

- Diagonalize the matrix (i.e. write in the form $A = P^{-1}DP$ with D diagonal)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{8}$$

- Let $M \in \mathcal{M}_{nn}(\mathbb{R})$ be a symmetric matrix and $\{z_1, \ldots, z_n\}$ and orthogonal basis of $\mathbb{R}^n$ of eigenvectors associated to the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, respectively. Show that $\{z_1, \ldots, z_n\}$ are also eigenvectors of the matrix

$$C = M + z_1 z_1^t \tag{9}$$

  and determine their eigenvalues.

Problem 5

Consider $T \in \mathcal{L}(V)$ and $U \subset V$ a vector subspace of V.

- Show that if $U \subset \text{Ker}T$, then U is an invariant subspace under T.
- Show that if $\text{Image}T \subset U$, then U is an invariant subspace under T.
- Suppose $S \in \mathcal{L}(V)$ and S commutes with T i.e. $ST = TS$. show that $\text{Ker}S$ is an invariant subspace under T.
- For S, T as above, show that $\text{Image}S$ is an invariant subspace under T.