Problem 1. Assume there exists $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{C}[x]$ s.t. $f(x_i) = w_i$ $i=1\ldots n$

therefore
$$\begin{cases} x_1^n + a_1 x_1^{n-1} + \cdots + a_n = w_1 \\ x_2^n + a_1 x_2^{n-1} + \cdots + a_n = w_2 \\ \quad \vdots \\ x_n^n + a_1 x_n^{n-1} + \cdots + a_n = w_n \end{cases} \Rightarrow.$$

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ & & \vdots & \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \end{pmatrix} = \begin{pmatrix} w_1 - x_1^n \\ \vdots \\ w_n - x_n^n \end{pmatrix}$$

Since $x_1 \ldots x_n$ distinct, $\det \begin{vmatrix} 1 & \cdots & x_1^{n-1} \\ \vdots & & \\ 1 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_j - x_i) \neq 0.$

Then the equation above has unique solution.

Problem 2: $\sum_{j=0}^{n-1} e^{i(\theta+j\varphi)}$

$$= \sum_{j=0}^{n-1} \left(\cos(\theta+j\varphi) + i \sin(\theta+j\varphi)\right)$$

$$= \sum_{j=0}^{n-1} \cos(\theta+j\varphi) + i \sum_{j=0}^{n-1} \sin(\theta+j\varphi).$$

$$\sum_{j=0}^{n-1} e^{i(\theta+j\varphi)}$$

$$= \sum_{j=0}^{n-1} e^{i\theta} \cdot e^{ij\varphi}$$

$$= e^{i\theta} \left(\sum_{j=0}^{n-1} (e^{i\varphi})^j\right)$$

$$= e^{i\theta} \cdot \frac{e^{i\varphi n} - 1}{e^{i\varphi} - 1}$$

$$= (\cos\theta + i \sin\theta) \cdot \frac{\cos n\varphi + i \sin n\varphi - 1}{\cos\varphi + i\sin\varphi - 1}$$

$$= \left[\cos[(n-1)\varphi + \theta] + \cos\theta - \cos(\theta - \varphi) - \cos(\theta + n\varphi)\right] / 2 - 2\cos\theta +$$

$$i\left[\sin[(n-1)\varphi + \theta] + \sin\theta - \sin(\theta-\varphi) - \sin(\theta + n\varphi)\right] / 2 - 2\cos\theta.$$

Problem 3.

(a) "$\Rightarrow$" Assume $p=(x-\alpha)^k h(x)$, then $\frac{dp}{dx}\Big|_\alpha = k(x-\alpha)^{k-1}h(x) + (x-\alpha)^k h'(x)\Big|_\alpha$
$= 0$.

"$\Leftarrow$" Assume multiplicity of α is not >1. If it is 1, $p(x)=(x-\alpha)h(x)$.
where $h(x)$ does not contain $(x-\alpha)$ factor, then $p'(x)|_\alpha = h(x)+(x-\alpha)h'(x)$
$\neq 0$; If it is 0, obviously, $p'(x)|_\alpha \neq 0$.

(b). By (a), it suffices to show that the roots of $\frac{dp}{dx}$ is not that of $p(x)$'s.
Eg. $p=t^4-t$. $p'(x)=4t^3-1$. $t_k=\sqrt[3]{\frac{1}{4}}\left(\cos\frac{2\pi k}{3}+i\sin\frac{2\pi k}{3}\right)$ $k=0,1,2$.
$p(t_k)\neq 0$ for $k=0,1,2$, then p has no multiple roots.

Problem 4. By hw 3. problem 2. to determine the rational roots.

$p=t^7-1 \Rightarrow t=1$.

$p=t^8-1 \Rightarrow t=\pm 1$

$p=2t^2-3t+4 \Rightarrow$ no rational roots

$p=3t^3+t-5 \Rightarrow$ - - - - - -

$p=2t^4-4t+3 \Rightarrow$ - - - - - -

Problem 5.

(a) $T: \mathbb{K}\to\mathbb{K}$. $T(a+b\sqrt5)=a-b\sqrt5$. To show T is an isomorphism.

- surjectivity. $\forall\ a+b\sqrt5\in\mathbb{K}$. $\exists\ a-b\sqrt5$ as an element in $\mathbb{K}$, s.t. $T(a-b\sqrt5)=a+b\sqrt5$.
$a,b\in\mathbb{Q}$.

- injectivity. $\forall$. $a_1+b_1\sqrt5 \neq a_2+b_2\sqrt5 \in\mathbb{K}$. $\Rightarrow (a_1-a_2)+(b_1-b_2)\sqrt5\neq 0$.
Then $T(a_1+b_1\sqrt5)-T(a_2+b_2\sqrt5)=(a_1-a_2)-(b_1-b_2)\sqrt5\neq 0$.

- homomorphism. $\forall\ a_1+b_1\sqrt5,\ a_2+b_2\sqrt5\in\mathbb{K}$.
$+$: $T((a_1+b_1\sqrt5)+(a_2+b_2\sqrt5)) = T((a_1+a_2)+(b_1+b_2)\sqrt5)=(a_1+a_2)-(b_1+b_2)\sqrt5$.
$=(a_1-b_1\sqrt5)+(a_2-b_2\sqrt5) = T(a_1+b_1\sqrt5)+T(a_2+b_2\sqrt5)$.
$\cdot$: $T((a_1+b_1\sqrt5)\cdot(a_2+b_2\sqrt5))=T((a_1a_2+5b_1b_2)+(b_1a_2+a_1b_2)\sqrt5)$
$=(a_1a_2+5b_1b_2)-(b_1a_2+a_1b_2)\sqrt5 = (a_1-\sqrt5 b_1)(a_2-\sqrt5 b_2)$.
$=T(a_1+b_1\sqrt5)\cdot T(a_2+b_2\sqrt5)$.

To sum up. $\mathbb{K}$ is an isomorphism $\mathbb{K}\to\mathbb{K}$.

(b). If $a+\sqrt{5}b$ is root of P, then $P(a+\sqrt{5}b)=0$

$$P(a-\sqrt{5}b) = P(T(a+\sqrt{5}b)) = T(P(a+\sqrt{5}b)) = T(0) = 0.$$

(c). $\deg(p)=3$. By (b) $2+\sqrt{5}$ is a root of P, so does $2-\sqrt{5}$. Assume r is the last root, $P=(x-(2+\sqrt{5}))(x-(2-\sqrt{5}))(x-r)$.

$$= (x^2-4x-1)(x-r)$$

$$= x^3+(-4-r)x^2+(4r-1)x+r$$

Since $P\in\mathbb{Q}[x]$, $-4-r, 4r-1, r\in\mathbb{Q}$. □.

Problem 6.

By contradiction, $f(x)=h(x)g(x)$. consider $f(x_i)$, $i=1,\dots,n$.

$$f(x_i) = \prod_{j=1}^{n}(x_i-x_j)+1 = 1. = h(x_i)g(x_i). \Rightarrow h(x_i)=g(x_i)=\pm 1.$$

$\Rightarrow$ There are at least $\frac{m+1}{2}$ 1 or -1 for $i=1\dots n$. Say there are at least $\frac{m+1}{2}$ ± 1.

then $g(x)-1$ has at least $\frac{m+1}{2}$ roots $\Rightarrow \deg(g(x)-1)\geq\frac{m+1}{2} \Rightarrow \deg g(x)\geq\frac{m+1}{2}$

similarly $\deg h(x)\geq\frac{m+1}{2}$

$\deg(gh)\geq m+1$. contradiction. □

Problem 7.

we do induction on n.

For $n=1$, we have $\Phi_1(x)=x-1$, by definition.

Suppose it is true for $\forall m<n$. By Problem 2, hw 2, we have

$$x^n-1 = \prod_{d|n}\Phi_d(x)$$

By induction $\prod_{d|n, d\neq n}$ is a monic polynomial with integer coeffecient $\Rightarrow \Phi_n(x)$ has integer coefficients.

Problem 8.

(a). $P_1 P_2\in\mathbb{Q}[x]$ share a root, α, then there exists a minimal polynomial of α, say g. Then $g|P_1$, $g|P_2$. Since P_1 P_2 are irreducible, we must have $P_1=c_1 g$, $c_1 c_2\in\mathbb{Q}$. $c_1, c_2\neq 0$. $\Rightarrow P_1=\frac{1}{c_2}P_2 =: cP_2$.

$P_2 = c_2 g_2$

(b) To prove p^2-q^2 has a rational root.

Since $P(x)$ has root α. $q(x)$ has root β. consider $q(\alpha+\beta-x) \in \mathbb{Q}[x]$ also has root α. and it is irreducible. By (a), we know that

$p(x) = c\, q(\alpha+\beta-x)$

Therefore, $p^2-q^2 = c^2(q^2(\alpha+\beta-x)) - q^2(x)$

$= [c\,q(\alpha+\beta-x) + q(x)][c\,q(\alpha+\beta-x) - q(x)]$.

Since $p(x) = c\,q(\alpha+\beta-x)$ is monic $\Rightarrow c = \pm 1$.

Then $(p^2-q^2)(\frac{\alpha+\beta}{2}) = 0 = c\,q(\frac{\alpha+\beta}{2}) - q(\frac{\alpha+\beta}{2})$ when $c=1$

$= -1\,q(\frac{\alpha+\beta}{2}) + q(\frac{\alpha+\beta}{2})$ when $c=-1$.

And $\frac{\alpha+\beta}{2} \in \mathbb{Q}$. ∎.

Problem 9.

Consider $f(x+1) = (x+1)^{p-1} + \cdots + 1 = x^{p-1} + \binom{p}{1}x^{p-2} + \cdots + \binom{p}{p-1}x + p$.

By Eisenstein's criterion, $f(x+1)$ is irreducible over $\mathbb{Q}$.

$\Rightarrow f(x)$ is irreducible over $\mathbb{Q}$.

Problem 10

(a) For $f = \sum_{i=0}^{m} a_i x^i$ $g = \sum_{j=0}^{n} b_j x^j$. $f g = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j x^{i+j}$

$\frac{df}{dx} = \sum_{i=1}^{m} a_i \cdot i\, x^{i-1}$ $\frac{dg}{dx} = \sum_{j=1}^{n} b_j \cdot j\, x^{j-1}$

$\frac{dfg}{dx} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j (i+j)\, x^{i+j-1} + \sum_{j=1}^{n} b_j \cdot j \cdot a_0\, x^{j-1} + \sum_{i=1}^{m} a_i \cdot i \cdot b_0 \cdot x^{i-1}$ = LHS

$\frac{df}{dx}\cdot g + \frac{dg}{dx}\cdot f = \sum_{i=1}^{m}\sum_{j=0}^{n} a_i \cdot i \cdot x^{i+j} + \sum_{j=1}^{n}\sum_{i=0}^{m} b_j j \cdot x^{j+i}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} (i+j)\, a_i b_j\, x^{i+j-1} + \sum_{j=1}^{n} b_j \cdot j\, a_0\, x^{j-1} + \sum_{i=1}^{m} a_i \cdot i \cdot b_0 \cdot x^{i-1}$ = RHS.

(b). If $f \in \mathbb{Q}[x]$ is irreducible, and has a multiple root α, then $f(\alpha) = 0$ also $f'(\alpha) = 0$. so f and f' have common factor, namely the minimal polynomial $p(x)$ of α. Since p is irreducible, $f = c\cdot p(x)$ and $f'(x) = p(x)\cdot h(x)$. So $\deg p \le \deg f' \le \deg f$. contradicting $f = cp$. ∎.