Solutions HW 8

P1

a) If $T \in \mathcal{L}(V)$, then $T^m \in \mathcal{L}(V)$ for any $m \in \mathbb{N}$ (one can prove this easily by induction and also, if $T_1, T_2 \in \mathcal{L}(V)$, then $\alpha \cdot T_1 + \beta T_2 \in \mathcal{L}(V)$ (where $\alpha\cdot$ is understood as $\alpha \cdot \mathbb{1}$) hence $p(T) \in \mathcal{L}(V)$

Consider $p(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{F}[x]$, then $p(\mathcal{M}(T)) = \sum_{i=0}^{d} a_i \mathcal{M}(T)^i$

under a change of basis $\mathcal{M}(T) \to C \cdot \mathcal{M}(T) \cdot C^{-1}$, $C \in \mathcal{M}_{m,m}(\mathbb{F})$ (where $m = \dim V$) so $\mathcal{M}(T)^k \longrightarrow \underbrace{C\mathcal{M}(T)C^{-1} \cdot C\mathcal{M}(T)C^{-1} \cdots C\mathcal{M}(T)C^{-1}}_{k\text{-times}}$

$= C\mathcal{M}(T)^k C^{-1}$ $\therefore$ $p(C\mathcal{M}(T)C^{-1}) = C p(\mathcal{M}(T)) \cdot C^{-1}$ as expected

b) Consider $p(x) = \sum_{i=0}^{d} a_i x^i$, $g(x) = \sum_{i=0}^{d} b_i x^i \in \mathbb{F}[x]$ $\quad (d = \max(\deg p, \deg g))$

so $(p+g)(x) = \sum_{i=0}^{d} (a_i + b_i) x^i = f(x)$

use $T^i \in \mathcal{L}(V)$, so $\alpha T^i + \beta T^i = (\alpha+\beta) T^i$

hence $p(T) + g(T) = \sum_{i=0}^{d} a_i T^i + \sum_{i=0}^{d} b_i T^i = \sum_{i=0}^{d} (a_i + b_i) T^i = f(T)$

also $(p \cdot g)(x) = \sum_{i=0}^{2d} \sum_{j=0}^{i} a_i b_{i-j} x^i = v(x)$

So, we want to consider $p(T) g(T) = \left(\sum_{i=0}^{d} a_i T^i\right) \cdot \left(\sum_{j=0}^{d} b_j T^j\right)$

by the fact that $T^j \in \mathcal{L}(V)$, then we can simplify the monomials:

$$a_i \cdot T^i \cdot b_j T^j = a_i \cdot b_j \, T^{i+j}$$

then the equalities $p(T) g(T) = v(T) = g(T) p(T)$ follow, just from usual algebra of polynomials.

c) Consider $f \in \mathbb{F}[x]$ s.t. $f(T)v = 0$ and suppose $p \nmid f$

by minimality of degree of p, $\deg f \geq \deg p$ and so we can apply division theorem

$$f = p \cdot q + r \qquad \deg r < \deg p$$

Now we evaluate these polynomials at T and apply the resulting operator to v:

$$f(T) \cdot v = (p \cdot q + r)(T)\, v = 0$$

using b): $(pq+r)(T) = pq(T) + r(T) = p(T)q(T) + r(T) = q(T)p(T) + r(T)$

so $f(T)v = q(T)\underbrace{p(T)v}_{=0} + r(T)v = r(T)v = 0$ so $r(x) \in \mathbb{F}[x]$

is annihilator polynomial of v and $\deg r < \deg p$ which is a contradiction, $\therefore$

$p \mid f$

So, any annihilator polynomial of minimal degree must be divisible by p hence it is proportional to p. $\Leftrightarrow$ p is unique up to a constant factor.

d) $p(x) = \det(A - x \cdot \mathbb{1}) = x^2 - 20x + 91$

we want to find A^{79} and we know (C-H theorem) $p(A) = 0$. So, the idea is to use the division theorem to write

$$x^{79} = p(x) \cdot q(x) + r(x) \qquad \deg r < \deg p = 2$$

so $r(x) = r_0 + r_1 x$ and

$$A^{79} = \underbrace{p(A)}_{=0}\, q(A) + r(A) = r(A) = r_0 \mathbb{1} + r_1 \cdot A$$

So, we only need to determine r_0 and r_1 !

doing the division of x^{79} by $p(x)$ using an algorithm is quite painful, but fortunately can be avoided: we know that if λ is a root of p, then

$$\lambda^{79} = \overbrace{p(\lambda)}^{=0} q(\lambda) + r(\lambda) = r_0 + r_1 \cdot \lambda$$

and we know the roots of $p(x)$ are $x = 13$ and $x = 7$, so

$$7^{79} = r_0 + r_1 \cdot 7$$
$$13^{79} = r_0 + r_1 \cdot 13$$

this determines r_0, r_1 and so $A^{79} = r_0 \mathbb{1} + r_1 \cdot A$

e) $p(x) = \det(B - x \cdot \mathbb{1}) = -(x-2)(x-1)^2$, using the same reasoning:

$(*)$ $x^{79} = p(x)\, q(x) + r(x)$, $r(x) = r_0 + r_1 x + r_2 x^2$ (since $\deg p = 3$)

So

$$B^{79} = r_0 + r_1 \cdot B + r_2 B^2$$

~~So~~ we need r_0, r_1, r_2, but we have only two roots $x = 2$ and $x = 1$, which gives

$$2^{79} = r_0 + r_1 \cdot 2 + r_2 4$$
$$1 = r_0 + r_1 + r_2$$

however 1 is a multiple root, so $\left.\frac{dp}{dx}\right|_{x=1} = 0$ $\therefore$ we can just derive $(*)$ and evaluate it at $x = 1$:

$$79 = r_1 + 2r_2$$

this gives enough eqs. to determine r_0, r_1, r_2 and so B^{79}.

P2 a) $F_{m+2} = F_{m+1} + F_m$, so

$$\begin{pmatrix} F_{m+2} \\ F_{m+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ ? & ? \end{pmatrix} \begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix}$$

but $F_{m+1} = F_{m+1}$, so $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

b) the eigenvalues of T are $\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$

diagonalizing T, gives

$$T = C \cdot D C^{-1} \quad \text{where}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad C = \begin{pmatrix} -\lambda_2^{-1} & -\lambda_1^{-1} \\ 1 & 1 \end{pmatrix}, \quad C^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1^{-1} \\ -1 & -\lambda_2^{-1} \end{pmatrix}$$

so $$T^m = C \cdot \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} C^{-1}$$

then, since $F_0 = 0$, $F_1 = 1$, is straightforward to compute

$$\begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\lambda_2^{-1} \lambda_1^m + \lambda_1^{-1} \lambda_2^m \\ \lambda_1^m - \lambda_2^m \end{pmatrix}$$

So

$$F_m = \frac{1}{\sqrt{5}} \left(\left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right)^m - \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right)^m \right)$$

c) $$\frac{F_{k+1}}{F_k} = \frac{\lambda_1^{-1} \lambda_2^k - \lambda_2^{-1} \lambda_1^k}{\lambda_1^k - \lambda_2^k}$$ and $\lambda_2 = -0,06180\ldots$ so $|\lambda_2| < 1$

hence $$\lim_{k \to \infty} F_{k+1} / F_k = \lim_{k \to \infty} \frac{-\lambda_2^{-1} \cdot \lambda_1^k}{\lambda_1^k} = -\lambda_2^{-1} = \frac{1+\sqrt{5}}{2}$$ which is the golden ratio!

P3/

The proof of a) $\Leftrightarrow$ b) goes completely analogous to the part c) of P2 in HW6, just, instead of assuming $Qv_i = \lambda_i v_i$ for $i=1,\dots,k$, we assume $Qv_i = \sum_{j=1}^{k} A_{ji} v_j$ and the result follows

Once we established a) $\Leftrightarrow$ b) then is obvious that b) $\Rightarrow$ c) just by using the formula for the determinant of a block matrix.

The only trickier part is to show c) $\Rightarrow$ a) or b). For this, suppose $p(x) = \det(M - x\cdot \mathbb{1}) \in \mathbb{F}[x]$ is reducible, then

$$p(x) = f(x)\cdot g(x) \quad , \deg f, \deg g < m \text{ and } f, g \text{ are nonconstant}$$

by C-H theorem, we know $p(M) = 0$ i.e. $p(M)\cdot v = 0 \ \forall v \in \mathbb{F}^m$ so, $\exists v \neq 0$ s.t. $f(M)g(M)\cdot v = 0 \Rightarrow$ either $f(M)v = 0$ or $g(M)v = 0$. Then, WLOG assume $f(M)v = 0$ and $d = \deg f < m$, so we have

$$f(M)v = \sum_{i=0}^{d} a_i M^i \cdot v = 0$$

So, is straightforward to show that (since $d < m$ and $v \neq 0$)

$$\mathrm{Span}(v, Mv, \dots, M^d v) \subsetneq \mathbb{F}^m$$

is a proper invariant subspace.