Homework 2

Problem 1

Consider a regular n-gon in $\mathbb{C}$ whose vertices $v_a \in \mathbb{C}$, $a = 0, \ldots, n-1$ are given by nth roots of unity. In other words, if we define $\zeta = e^{\frac{2\pi i}{n}}$, then $v_a = \zeta^a$. Find the product of the distances $|v_0 - v_b|$ with $b = 1, \ldots, n-1$, in other words find the value of the following expression

$$|1-\zeta| \cdot |1-\zeta^2| \cdots |1-\zeta^{n-1}| \tag{1}$$

Hint: Use the factorization of $x^n - 1$.

Problem 2

The dth cyclotomic polynomial is defined by[1]

$$\Phi_d(x) = \prod_{\substack{1 \le k \le d \\ \gcd(k,d)=1}} \left(x - e^{\frac{2\pi i k}{d}}\right) \tag{2}$$

this means, the roots of $\Phi_d(x)$ are all the primitive dth roots of unity. Prove that[2]

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \tag{3}$$

Hint: compare the zeroes of both sides of the equation.

Problem 3

Find the value of $\left(\sum_{k=0}^{n} \binom{n}{k}\right)^2$.
Hint: Consider $(1+x)^{2n}$

Problem 4

Show that for a positive integer n and any even positive integer m satisfying $0 \le m \le n$, we have $\binom{n}{0}\binom{n}{m} - \binom{n}{1}\binom{n}{m-1} + \binom{n}{2}\binom{n}{m-2} \pm \ldots \binom{n}{m}\binom{n}{0} = (-1)^{n+\frac{m}{2}}\binom{n}{m/2}$.
Hint: Consider $(1+x)^n(x-1)^n$

[1] $\prod_{\substack{1 \le k \le d \\ \gcd(k,d)=1}}$ means the product over all k that are less or equal than d and relative prime with d.

[2] $\prod_{d|n}$ means the product over all d that divides n. For example, if $n = 4$, the product goes over $d = 1, 2, 4$. If $n = 5$, the product goes over $d = 1, 5$, and so on.

Problem 5

Prove that every polynomial $p \in \mathbb{R}[x]$ of odd degree has a real root.

Problem 6

Consider the angle $|\theta| < \frac{\pi}{2}$ and $x = 1 + i\tan\theta$ (What is $\text{Arg}(x)$?)

1. Consider x^n (What is $\text{Arg}(x^n)$?) and use it to find an expression of $\tan n\theta$ in terms of $\tan\theta$.

Problem 7

Consider a polynomial $p(x)$ of degree $n \geq 1$, such that $p(k) = \frac{k}{k+1}$ for $k = 0, \dots, n$. Find $p(n+1)$.
Hint: Consider $q(x) = (x+1)p(x) - x$. What is the degree of $q(x)$? what are the roots of $q(x)$? then use unique factorization.

Problem 8

Consider the polynomial $p = x^4 - 4x^3 + 10x^2 - 12x + 8$. Using the knowledge that p has only complex roots and one of them has absolute value 2, determine all the roots of p.

Problem 9

1. Show that the roots of $q = x^2 + x + 1$ are roots of unity.
2. Show that q divides $p = x^{3n_1} + x^{3n_2+1} + x^{3n_3+2}$ for any $n_1, n_2, n_3 \in \mathbb{N}$

Problem 10

Consider the Vandermonde matrix

$$A_n = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \in \mathcal{M}_{nn}(\mathbb{C}) \tag{4}$$

we will compute its determinant $\det(A_n)$ using just properties of polynomials.

1. Consider $\det(A_n)$ as a polynomial in $\mathbb{C}[x_1]$ and all the other variables as coefficients. Find all the roots of $\det(A_n)(x_1)$.

2. Show that

$$\det(A_n) = \prod_{1 \le i < j \le n} (x_j - x_i) \tag{5}$$

by induction in n, using your previous result.[3]

[3] The product $\prod_{1 \le i < j \le n}(x_j - x_i)$ means $(x_n - x_{n-1}) \cdots (x_n - x_1)(x_{n-1} - x_{n-2}) \cdots (x_{n-1} - x_1) \cdots (x_2 - x_1)$.