# Sols Hw 9



## P1

a) $\Lambda : V \longrightarrow U$ is defined by $(\Lambda v)(\varphi) = \varphi(v)$, since $\Lambda v$ must be a map $V^* \to \mathbb{F}$ (i.e. $U = \mathcal{L}(V^*, \mathbb{F})$)

Consider $\alpha v + \beta u \in V$ with $u, v \in V$ and $\alpha, \beta \in \mathbb{F}$, then, we want to show

$$\Lambda(\alpha v + \beta u) = \alpha \Lambda v + \beta \cdot \Lambda u$$

in order to do this, we check how $\Lambda(\alpha v + \beta u)$ acts on an arbitrary vector $\varphi \in V^*$

$$(\Lambda(\alpha v + \beta u))(\varphi) = \varphi(\alpha v + \beta u) = \alpha \varphi(v) + \beta \varphi(u) = (\alpha(\Lambda v) + \beta \cdot (\Lambda u))(\varphi)$$

use linearity of $\varphi \in V^* = \mathcal{L}(V, \mathbb{F})$

b) $T \in \mathcal{L}(V)$, $T^* \in \mathcal{L}(V^*)$ then $(T^*)^* \in \mathcal{L}((V^*)^*)$ and

$(V^*)^* = \mathcal{L}(V^*, \mathbb{F}) = U$ so $(T^*)^* \in \mathcal{L}(U)$ and is defined by

$$(*) \quad (T^*)^*(\psi) = \psi \circ T^* \quad \text{for } \psi \in U$$

Now we want to check how the map $(T^*)^* \circ \Lambda \in \mathcal{L}(V, U)$ behaves, so

$$(T^*)^* \circ \Lambda v \in U \quad \text{for } v \in V,$$

hence we again take an arbitrary $\varphi \in V^*$ and check how $(T^*)^* \circ \Lambda v$ acts on it

$$\Big((T^*)^* \circ \underbrace{\Lambda v}_{\in U}\Big)(\varphi) = \cancel{\Lambda v}\big(T^*(\varphi)\big) = \Lambda v \circ (\overbrace{\varphi \circ T}^{\in \mathcal{L}(V^*, \mathbb{F})})$$

use $(*)$, i.e. definition of $(T^*)^*$

(or more precisely, just definition of dual map)

$$= (\varphi \circ T)(v) \quad (1)$$

definition of $\Lambda$

on the other hand $(\Lambda \circ \overbrace{Tv}^{\in V})(\varphi) = \varphi(Tv) = \varphi \circ Tv = (1)$

as desired!

c) Since $\dim V = \dim V^* = \underset{\overset{\parallel}{V}}{\dim(V^*)^*}$, then we only need to show $\Lambda$ is injective (we already showed $\Lambda$ is linear), then, consider

$$\Lambda v = 0 \iff (\Lambda v)\varphi = \varphi(v) = 0 \quad \text{for } \underline{\text{any}} \ \varphi \in V^*$$

So in a particular basis of $V$, this means $M(\varphi)\cdot v = 0$ for <u>any</u> matrix $M(\varphi)$, in particular invertible matrices. The only solution is $v = 0$, hence $\Lambda$ is injective.

d) Consider $V$, a real vector space, of dimension $m < \infty$, then $V^* = \mathcal{L}(V, \mathbb{F})$ and given a basis $\{v_1, \dots, v_m\}$ of $V$, we define the dual basis $\{\varphi_1, \dots, \varphi_m\}$ of $V^*$ by the functions $\varphi_i$ satisfying

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

So, for example, in our case $\{1, x, x^2, x^3, \dots, x^m\}$ is a basis of $\mathbb{R}_m[x]$ and so, the dual basis of it is given by $\{\varphi_0, \dots, \varphi_m\}$ that satisfies

$$\varphi_j(x^k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

then, we know $\{\varphi_0, \dots, \varphi_m\}$ is a basis of $(\mathbb{R}_m[x])^*$ so, if we write

$$f_j = \sum_{i=0}^{m} L_{ji}\,\varphi_i \quad \text{, we just need to show that}$$

$L \in M_{m+1,m+1}(\mathbb{R})$ is invertible. So, let's find the coefficients of $L$:

$$f_j(x^k) = \int_0^{a_j} x^k \, dx = \frac{a_j^{k+1}}{k+1} = \sum_{i=0}^{m} L_{ji}\,\varphi_i(x^k) = L_{jk}$$

So

$$L = \begin{bmatrix} a_0 & a_0^2/2 & \cdots & a_0^{m+1}/(m+1) \\ a_1 & a_1^2/2 & \cdots & a_1^{m+1}/(m+1) \\ \vdots & & & \vdots \\ a_m & \cdots & \cdots & a_m^{m+1}/(m+1) \end{bmatrix}$$

If we see the ~~charac~~ determinant of $L$ as a polynomial in $R[a_i]$ (with $R=[a_0,\dots,a_{i-1},a_{i+1},\dots,a_m]$) is clear that it vanishes for the values $a_i=0$ or $a_i=a_j$ for any $j\in\{0,\dots,m\}\setminus\{i\}$. Moreover, the polynomial has degree $m+1$ in $a_i$, so, there are all the roots. We can repeat this for any $i=0,\dots,m$, hence we conclude that $\det L$ vanishes, whenever any of the $a_i$'s is zero and whenever $a_i=a_j$ for any pair $i,j$ with $i\neq j$.

Therefore $f_0,\dots,f_m$ form a basis for all values of $a_0,\dots,a_m\in\mathbb{R}\setminus\{0\}$ s.t. they are all distinct.

P2

a) If $v\in\mathcal{L}(V^*,\mathbb{C})$ then, given any $\varphi\in V^*$ $v(\varphi)\in\mathbb{C}$, $\therefore$ $z\cdot v(\varphi)\in\mathbb{C}$ is just ordinary complex multiplication for any $\varphi\in V^*$, hence makes sense to define $z\cdot v\in\mathcal{L}(V^*,\mathbb{C})$

b) $\phi\in\mathcal{L}(V^*,\mathbb{R})$, then $\left(T^{\mathbb{C}}(\phi)\right)(f)=\phi\left(T^*(f)\right)=\phi(f\circ T)$

here $f\circ T\in V^*$ (since $f\circ T:V\to\mathbb{R}$)

by problem 1, we have an isomorphism $\Lambda:V\to\mathcal{L}(V^*,\mathbb{R})$, this means, there exist $v\in V$ s.t. $\Lambda(v)=\phi$, hence we write

$$\left(T^{\mathbb{C}}(\phi)\right)(f)=\Lambda(v)(f\circ T)=f\circ Tv=f(Tv)$$

definition of $\Lambda$

c) $s=v+iu\in V^{\mathbb{C}}$, so $v,u\in V$ and $T^{\mathbb{C}}(s)=T(v)+iT(u)$

$$\therefore\quad \overline{T^{\mathbb{C}}(s)}=\overline{(T(v)+iT(u))}=T(v)-iT(u)$$

$$T^{\mathbb{C}}(\bar{s})=T^{\mathbb{C}}(v-iu)=T(v)+iT(-u)=T(v)-iT(u)=\overline{T^{\mathbb{C}}(s)}$$

[Forgot to include d), sorry!, at the end]

e) Consider a ~~.~~ basis $v_1, \dots, v_m$ of $V$, then we can see them as vectors in $V^{\mathbb{C}}$, just by writing $v_i = v_i + i \cdot 0 \in V^{\mathbb{C}}$

Now consider an arbitrary vector $s \in V^{\mathbb{C}}$, then

$$s = u + it \underset{\underset{u,t \in V}{\nearrow}}{=} \sum_{j=1}^{m} a_j v_j + i \sum_{j=1}^{m} b_j v_j = \sum_{j=1}^{m} (a_j + i b_j) v_j \quad \text{here } a_j, b_j \in \mathbb{R}$$

$$= \sum_{j=1}^{m} c_j v_j$$ where $c_j \in \mathbb{C}$ and $v_j$ is a vector on $V^{\mathbb{C}}$, by the previous reasoning. Then $v_1, \dots, v_m$ (seen as $v_j + i \cdot 0$) form a basis of $V^{\mathbb{C}}$

f) Let's compute the coefficients of $\mathcal{M}(T^{\mathbb{C}})$ in the basis defined above:

$$T^{\mathbb{C}}(v_i) \underset{\substack{\text{definition of } \mathcal{M}(T^{\mathbb{C}}) \\ \text{in a basis } v_1, \dots, v_m \\ \text{of } V^{\mathbb{C}}}}{=} \sum_{j=1}^{m} (T^{\mathbb{C}})_{ji} v_j \underset{\substack{\text{definition of} \\ T^{\mathbb{C}}(v_i)}}{=} T(v_i) = \sum_{j=1}^{m} T_{ji} v_j$$

so $\mathcal{M}(T^{\mathbb{C}}) = \mathcal{M}(T)$, therefore the characteristic polynomial of $\mathcal{M}(T^{\mathbb{C}})$ has real coefficients $\Rightarrow$ its roots come in conjugate pairs

g) Suppose $T^{\mathbb{C}}$ is diagonalizable, i.e. $\exists$ a basis $s_1, \dots, s_m$ of $V^{\mathbb{C}}$ s.t. $T^{\mathbb{C}} s_i = \lambda_i s_i$, $\lambda_i \in \mathbb{C}$

Suppose WLOG $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, then if we write $s_j = v_j + i u_j$ for $j = 1, \dots, r$ we have

$$T v_j + i T u_j = \lambda_j v_j + i \lambda_j u_j \Rightarrow \begin{matrix} T v_j = \lambda_j v_j \\ T u_j = \lambda_j u_j \end{matrix} \quad j = 1, \dots, r$$

Since $s_1, \dots, s_r$ are L.I., then, there exist $r$ L.I. vectors in the list $\{u_1, \dots, u_r, v_1, \dots, v_r\}$ (To see this just remember that we can pick the basis of $V$, as in part e) and write it as a linear combination, with complex coefficients, of the basis $s_1, \dots, s_m$)

Then, just pick any $r$ L.I. vectors from this list ~~and~~, say $e_1, \dots, e_r$

then, $Te_i = \lambda_i' e_i$ ~~(depending which ones are)~~ for some $\lambda_i' \in \mathbb{R}$

Now, if $\lambda_j \in \mathbb{C}$, we know $\bar{\lambda}_j$ is also is eigenvalue (by part ~~f~~)

and so $\quad T^{\mathbb{C}} s_j = T^{\mathbb{C}}(\tilde{v}_j + i\tilde{u}_j) = T\tilde{v}_j + iT\tilde{u}_j = \lambda_j \tilde{s}_j \quad j = r+1, \dots, l$ with $2l = m+r$

$$= Re(\lambda_j)\tilde{v}_j - Im(\lambda_j)\tilde{u}_j + i\,Re(\lambda_j)\tilde{u}_j + i\,Im(\lambda_j)\tilde{v}_j$$

and $T^{\mathbb{C}}\bar{s}_j = \bar{\lambda}_j \bar{s}_j$ (from part c)), $\bar{s}_j = \tilde{v}_j - i\tilde{u}_j$

So, from this eqs, we get

$$T\begin{pmatrix} \tilde{v}_j \\ \tilde{u}_j \end{pmatrix} = \begin{pmatrix} Re(\lambda_j) & -Im(\lambda_j) \\ Im(\lambda_j) & Re(\lambda_j) \end{pmatrix}\begin{pmatrix} \tilde{v}_j \\ \tilde{u}_j \end{pmatrix}$$

and so, in the basis $\{e_1, \dots, e_r, \tilde{v}_{r+1}, \tilde{u}_{r+1}, \dots, \tilde{v}_l, \tilde{u}_l\}$

we can write

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1' & & & & \\ & \ddots & & & \\ & & \lambda_r' & & \\ & & & B_{r+1} & \\ & & & & \ddots \\ & & & & & B_l \end{pmatrix}, \quad B_i = \begin{pmatrix} Re(\lambda_i) & -Im(\lambda_i) \\ Im(\lambda_i) & Re(\lambda_i) \end{pmatrix}$$

d) ~~Let~~ Suppose $\exists$ a basis $v_1, \dots, v_m$ of $V$ s.t. $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & B_1 & \\ & & & & \ddots \\ & & & & & B_l \end{pmatrix}$

where ~~the~~ $\lambda_i \in \mathbb{R}$ and the blocks $B_i$ are of the form $B_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$, then

we can write, in the basis $v_i^1, v_i^2$ at each ~~of~~ of these $2\times 2$ blocks:

$$Tv_i^1 = a_i v_i^1 - b_i v_i^2$$
$$Tv_i^2 = a_i v_i^2 + b_i v_i^1$$

then is easy to see, that in the basis $\{v_1, \dots, v_r, (v_1^1 + i v_1^2), \dots, (v_l^1 + i v_l^2),$
$(v_1^1 - i v_1^2), \dots, (v_l^1 - i v_l^2)\}$, $T^{\mathbb{C}}$ is diagonal.

P3|

a) $M_\lambda = \langle v, (T-\lambda \mathbb{1})v, \dots, (T-\lambda\mathbb{1})^{m-1}v \rangle$

then
$$T(T-\lambda\mathbb{1})^j v = (T-\lambda\cdot\mathbb{1}+\lambda\cdot\mathbb{1})(T-\lambda\mathbb{1})^j v = (T-\lambda\cdot\mathbb{1})^{j+1}v + \lambda\cdot(T-\lambda\mathbb{1})^j v$$
in particular $T(T-\lambda\mathbb{1})^{m-1}v = \lambda(T-\lambda\mathbb{1})^{m-1}v$
then, if we define the basis $v_j = (T-\lambda\cdot\mathbb{1})^j v \quad j=0,\dots,m-1$

$$T\begin{pmatrix} v_0 \\ \vdots \\ v_{m-1}\end{pmatrix} = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}\begin{pmatrix} v_0 \\ \vdots \\ v_{m-1}\end{pmatrix}$$

b) Consider $v \in M_\lambda$, $v \neq 0$, then $\exists\, j$ s.t. $(T-\lambda\mathbb{1})^j v = 0$ and $(T-\lambda\cdot\mathbb{1})^{j-1}v \neq 0$, if $j \leq m$, then obviously $v \in Ker(T-\lambda\mathbb{1})^m$. However, the grade of a vector cannot exceed m (otherwise $\langle v, \dots, (T-\lambda\mathbb{1})^{p-1}v\rangle$ will have dimension higher than m) so j cannot be strictly larger than m, hence

$$M_\lambda \subseteq Ker(T-\lambda\mathbb{1})^m$$

If $v \in Ker(T-\lambda\cdot\mathbb{1})^m$, then $(T-\lambda\mathbb{1})^m v = 0 \Rightarrow v \in M_\lambda$, hence

$$M_\lambda = Ker(T-\lambda\cdot\mathbb{1})^m$$

Suppose $\tilde\lambda \notin \sigma(T)$ and $\exists\, v\neq 0$ s.t. $(T-\tilde\lambda\cdot\mathbb{1})^m v = 0$ (since $v \in Ker(T-\lambda\mathbb{1})^m$)
therefore $\det[(T-\tilde\lambda\cdot\mathbb{1})^m] = (\det(T-\tilde\lambda\cdot\mathbb{1}))^m = 0$ i.e. $\tilde\lambda$ is a root of the characteristic polynomial of $T \Rightarrow \tilde\lambda \in \sigma(T)$ $\divideontimes$

c) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, so $\det(A - \lambda \cdot \mathbb{1}) = (1-\lambda)^3$ hence $\sigma(A) = \{1\}$ (4)

since $(A - \mathbb{1})^3 = 0 \Rightarrow M_{\lambda=1} = \mathbb{R}^3$ (M defined as in b))

d) we compute the principal vectors w/ eigenvalue $\lambda = 1$ (normalization is not important for this)

$(A - \mathbb{1}) \cdot v = 0 \Rightarrow v_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ (only sol.)

$(A - \mathbb{1})^2 \cdot v = 0 \Rightarrow \underbrace{v_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{obviously a solution!}}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are sols.

$\underbrace{(A - \mathbb{1})^3}_{= 0} \cdot v = 0 \Rightarrow$ any vector is a sol., in particular $\underbrace{v_1, v_2}_{\text{obvious solutions}}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

note, these vectors satisfy: $\text{gr}\, v_1 = 1$, $\text{gr}\, v_2 = 2$ and $\text{gr}\, v_3 = 3$

we could have chosen another basis, but we always get a vector of grade 3 i.e.

$(A - \mathbb{1})^3 v_3 = 0$ but $(A - \mathbb{1})^2 v_3 \neq 0$, $(A - \mathbb{1}) v_3 \neq 0$

So $\langle (A - \mathbb{1})^2 v_3, (A - \mathbb{1}) v_3, v_3 \rangle = M_{\lambda=1} = \mathbb{R}^3$

P4

a) When restricted to $M_\lambda$, we can always write $T|_{M_\lambda} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{bmatrix}$, so if $\mu \neq \lambda$, very clearly $\det((T - \mu \cdot \mathbb{1})|_{M_\lambda}) \neq 0$ hence $T - \mu \cdot \mathbb{1}$ is invertible, so $z = (T - \mu \cdot \mathbb{1})|_{M_\lambda}^{-1} \cdot y$

b) $y = f(T) v$, $f(x) \in \mathbb{F}[x]$, suppose $(x - \lambda) \mid f(x)$ so $f(x) = (x - \lambda) q(x)$

then $y = (T - \lambda \cdot \mathbb{1}) \cdot \underbrace{q(T) \cdot v}_{= z'}$