Sols. HW 6

P1

a) $P_m(x) \in \mathbb{R}[x]$, $m \in \mathbb{Z}_{\geq -1}$, $P_{-1} = 0$, $P_0 = 1$

First note that $P_{m+1} - x P_m \in \mathbb{R}_m[x]$ (Since P_ℓ's are monic)

So, we can write

$$P_{m+1} - x P_m = \sum_{\ell=0}^{m} a_\ell P_\ell \,, \quad a_\ell \in \mathbb{R}$$

take the product $\langle \cdot , P_s \rangle$ with $s \in \{0, \dots, m-2\}$, so we get

$$\underbrace{\langle P_{m+1}, P_s \rangle}_{= 0} - \langle x P_m, P_s \rangle = \underbrace{\sum_{\ell=0}^{m} a_\ell \langle P_\ell, P_s \rangle}_{= a_s \langle P_s, P_s \rangle}$$

and $\langle x P_m, P_s \rangle = \langle P_m, \underbrace{x P_s}_{\in \mathbb{F}_{s+1}[x]} \rangle = 0 \quad \Rightarrow \quad a_s = 0$ for $s = 0, \dots, m-2$

Then, we take the product with P_{m-1}:

$$\underbrace{\langle P_{m+1}, P_{m-1} \rangle}_{= 0} - \langle P_m, x P_{m-1} \rangle = a_{m-1} \langle P_{m-1}, P_{m-1} \rangle$$

here, use that $x \cdot P_{m-1} = P_m + q$, $q \in \mathbb{F}_{m-1}[x]$

$$\Rightarrow \quad a_{m-1} = - \frac{\langle P_m, x P_{m-1} \rangle}{\langle P_{m-1}, P_{m-1} \rangle} = - \frac{\langle P_m, P_m \rangle}{\langle P_{m-1}, P_{m-1} \rangle}$$

Finally we take the product with P_m and we get

$$- \langle P_m, x P_m \rangle = a_m \langle P_m, P_m \rangle \quad \Rightarrow \quad a_m = - \frac{\langle P_m, x P_m \rangle}{\langle P_m, P_m \rangle}$$

So

$$P_{m+1} = \left(x - \frac{\langle P_m, x P_m \rangle}{\langle P_m, P_m \rangle} \right) P_m - \frac{\langle P_m, P_m \rangle}{\langle P_{m-1}, P_{m-1} \rangle} P_{m-1}$$

which is what we wanted

Note: alternatively, we could have considered $f = P_{m+1} - (x-\alpha_m)P_m + \beta_m P_{m-1}$ and show that, ~~adjusting~~ choosing α_m and β_m appropriately $\langle f, P_\ell \rangle = 0$ for $\ell = 0, \dots, m$ and $\deg(f) = m$ hence $f = 0$.

b) we consider $\psi = e^{-x^2/2} \cdot u$, so $\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(-x\, e^{-x^2/2} u + e^{-x^2/2}\frac{du}{dx}\right)$

$$= e^{-x^2/2}\left(x^2 \cdot u - 2x\frac{du}{dx} - u + \frac{d^2u}{dx^2}\right)$$

Replacing back into the Schrödinger eq.:

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + (2\varepsilon - 1)u = 0 \quad (*)$$

c) we try the ansatz $u = \sum_{i=0}^{\infty} c_i x^i$, so $\frac{du}{dx} = \sum_{i=0}^{\infty} i\, c_i x^{i-1}$

$$\frac{d^2u}{dx^2} = \sum_{i=0}^{\infty} i(i-1)c_i x^{i-2}$$

Putting this in $(*)$, we get

$$\sum_{i=0}^{\infty}\left[i(i-1)c_i x^{i-2} - 2ic_i x^i + (2\varepsilon - 1)c_i x^i\right] = 0$$

we look at the coeff. of x^i and get:

$$(i+1)(i+2)C_{i+2} - 2iC_i + (2\varepsilon - 1)C_i = 0 \Rightarrow C_{i+2} = C_i \cdot \frac{(2i+1-2\varepsilon)}{(i+1)(i+2)}$$

So, for arbitrary $\varepsilon \in \mathbb{R}$, these solutions are infinite series. Let's write some of the terms on this recursion relation, to see if we can truncate them for special values of ε:

$$C_2 = C_0 \frac{(1-2\varepsilon)}{2}$$

$$C_3 = C_1 \cdot \frac{(3-2\varepsilon)}{6}$$

$$C_4 = C_2 \frac{(5-2\varepsilon)}{12}$$

$$C_5 = C_3 \frac{(7-2\varepsilon)}{20}$$

$$C_6 = C_4 \frac{(9-2\varepsilon)}{30}$$

$$\vdots$$

and so on. Clearly, C_k with k odd are related between them and same for k even. So, in order for the series to truncate, we need $\varepsilon = \frac{2l+1}{2} = l + \frac{1}{2}$ with $l = 0, 1, 2, \ldots$. This will set $C_k = 0$ for $k \geq \delta$ for certain δ ~~odd~~, for all odd or even k, and we can set $C_s = 0$ for all even or odd s respectively, hence solving all equations for the C_i coefficients ~~and~~ by setting only a finite number of them to be non zero.

~~Let's do examples~~ The eq. looks like then:

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2l \cdot u = 0 \qquad l = 0, 1, 2, \ldots$$

Let's write some of the solutions we found:

$l = 0$, $\varepsilon = \frac{1}{2}$ $\Rightarrow$ $u = 1$ (only $C_0 \neq 0$, we normalize $C_0 = 1$)

$l = 1$, $\varepsilon = \frac{3}{2}$ $\Rightarrow$ $u = x$ (only $C_1 \neq 0$, we set $C_0 = C_2 = C_4 = \ldots = 0$, and normalize to make it monic)

$l = 2$, $\varepsilon = \frac{5}{2}$ $\Rightarrow$ $u = C_0 - 2x^2 \cdot C_0 \xrightarrow[\text{monic}]{\text{make it}} u = -\frac{1}{2} + x^2$

$l = 3$, $\varepsilon = \frac{7}{2}$ $\Rightarrow$ $u = -\frac{3}{2}x + x^3$

~~[illegible]~~

$l=4,\ \mathcal{E}=\frac{9}{2} \Rightarrow u=\frac{3}{4}-3x^2+x^4$

P2|

a) Suppose $\lambda \in \sigma(A)$, then $\exists\ v \neq 0$ s.t. $Av=\lambda v$, also WLOG we can assume k is s.t. $A^k=0$ and $A^{k-1}\neq 0$, so we have

$$\underbrace{A^k \cdot v}_{\overset{\shortparallel}{0}} = \lambda \cdot A^{k-1} v = \lambda^k \cdot v = 0$$

Since $v\neq 0 \Rightarrow \lambda^k=0 \Rightarrow \lambda=0$

)

$$Q=\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}$$ if $Q\in \mathcal{M}_{m,m}(\mathbb{F})$ then ~~[illegible]~~ if the block A has dimension k (i.e. $A\in\mathcal{M}_{k,k}(\mathbb{F})$) then $B\in\mathcal{M}_{m-k,m-k}(\mathbb{F})$ and $D\in\mathcal{M}_{k,m-k}(\mathbb{F})$ (and the block marked as 0, is of dimension $(m-k)\times k$)

then we can write (where $\mathbb{1}_l$ is the identity of dimension l)

$$\det(Q-\lambda\cdot\mathbb{1}_m)=\det\begin{pmatrix} A-\lambda\cdot\mathbb{1}_k & D \\ 0 & B-\lambda\cdot\mathbb{1}_{m-k}\end{pmatrix}=\det\left((A-\lambda\cdot\mathbb{1}_k)\cdot(B-\lambda\cdot\mathbb{1}_{m-k})\right)$$

$$=\det(A-\lambda\cdot\mathbb{1}_k)\det(B-\lambda\cdot\mathbb{1}_{m-k})$$

) We did this one in the lecture. ~~Up to [illegible]~~

If we write $v\in V$ as $v=\sum_{i=1}^m a_i v_i$, then

$$Qv=\sum_{i=1}^m a_i Qv_i=\sum_{i=1}^k a_i\lambda_i v_i+\sum_{l=k+1}^m a_l \sum_{j=1}^m \mathcal{M}(Q)_{j,l}v_j$$

So, clearly, for $i=1,\dots,k$ $\quad \lambda_i v_i = Q\cdot v_i \quad (1)$

and for $l=k+1,\dots,m$ $\quad Q\cdot v_l = \sum_{j=1}^{m} \mathcal{M}(Q)_{jl} v_j$

then, from (1):

$$\mathcal{M}(Q)_{ij} = \begin{cases} \lambda_i & i=j \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i,j \in \{1,\dots,k\} \qquad \text{where } \lambda_1 = \dots = \lambda_k = \lambda$$

$$\mathcal{M}(Q)_{lj} = 0 \quad \text{for } \begin{array}{l} l \in \{k+1,\dots,m\} \\ j \in \{1,\dots,k\} \end{array}$$

and for the rest of components of $\mathcal{M}(Q)$ we cannot say anything, but this is enough to write:

$$\mathcal{M}(Q) = \begin{pmatrix} \lambda\cdot\mathbb{1}_k & D \\ 0 & B \end{pmatrix} \qquad \begin{array}{l} D \in \mathcal{M}_{k,m-k}(\mathbb{R}) \\ B \in \mathcal{M}_{m-k,m-k}(\mathbb{R}) \end{array}$$

d) Suppose Q has an eigenvalue $\lambda \in \sigma(Q)$, then denote $\{v_1,\dots,v_k\}$ a basis of the space $V_\lambda \subseteq V$, so $k = \dim V_\lambda = \text{mult}_{\text{geom.}}(\lambda)$ (geometric multiplicity of λ). Then, in this basis we can write (from c)):

$$\mathcal{M}(Q) = \begin{pmatrix} \lambda\cdot\mathbb{1}_k & D \\ 0 & B \end{pmatrix}$$

Now, we consider the characteristic polynomial of Q (that doesn't depend on the ~~basis~~ choice of basis), using b) to write

$$p(\lambda') = \det(Q - \lambda'\cdot\mathbb{1}) = \det(\mathcal{M}(Q) - \lambda'\cdot\mathbb{1}) = (\lambda - \lambda')^k \det(B - \lambda'\cdot\mathbb{1}) \in \mathbb{F}[\lambda']$$

so λ is a root of $p(\lambda')$ of multiplicity (algebraic multiplicity) $\geq k$

An example is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ where $\dim V_{\lambda=1} = 1$ and $\text{mult}_{\text{alg.}}(\lambda=1) = 2$

P3

a) Clearly, $0 \in V$, so we have to show that given $p(x), q(x) \in V$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha p + \beta q \in V$.

Write $p = \sum_{i=0}^{m} a_i x^i$, $a_i = a_{m-i}$

$q = \sum_{i=0}^{m} b_i x^i$, $b_i = b_{m-i}$

so $$\alpha p + \beta q = \sum_{i=0}^{m} (\alpha a_i + \beta b_i) x^i = \sum_{i=0}^{m} c_i x^i$$

and $c_i = \alpha a_i + \beta b_i = \alpha a_{m-i} + \beta b_{m-i} = c_{m-i}$, so $\alpha p + \beta q \in V$

b) Consider $p \in V$, then write $m = 2m$, we can solve the constraints on the coefficients, $a_0 = a_{2m}$, $a_1 = a_{2m-1}$, ..., $a_m = a_m$ and so

$$p = \sum_{i=0}^{2m} a_i x^i = a_0 + a_1 x + \dots + a_{m-1} x^{m-1} + a_m x^m + a_{m-1} x^{m+1} + a_{m-2} x^{m+2} + \dots + a_0 x^{2m}$$

$$= \sum_{i=0}^{m-1} a_i (x^i + x^{m-i}) + a_m x^m$$

So, any $p \in V$ can be written as a linear combination of $\{x^i + x^{m-i}, i = 0, \dots, m-1, x^m\}$. We just have to show this set is L.I., for this we solve the equation for $\alpha_0, \dots, \alpha_m$:

$$\sum_{i=0}^{m-1} \alpha_i (x^i + x^{m-i}) + \alpha_m \cdot x^m = 0$$

$$= \alpha_0 + \alpha_1 x + \dots + \alpha_{m-1} x^{m-1} + \alpha_m x^m + \alpha_{m-1} x^{m+1} + \dots + \alpha_0 x^{\overset{2m}{\overset{\shortparallel}{m}}}$$

Since $1, x, \dots, x^m$ is a basis of $\mathbb{F}_m[x]$, the only sol. is $\alpha_i = 0 \ \forall i$

$\therefore \{x^i + x^{m-i}, i = 0, \dots, m-1, x^m\}$ is a basis of $V \Rightarrow \dim V = m+1$

c) we know that $\dim \mathbb{F}_{m-1}[x] = m$ and $\dim V = m+1$ (from b)), so we need to show $V \cap \mathbb{F}_{m-1}[x] = \{0\}$ since

$$\dim(V + \mathbb{F}_{m-1}[x]) = \dim V + \dim \mathbb{F}_{m-1}[x] - \dim(V \cap \mathbb{F}_{m-1}[x])$$

Consider $p = \sum_{i=0}^{m} a_i x^i \in \mathbb{F}_{m-1}[x] \cap V$

Since $p \in \mathbb{F}_{m-1}[x] \Rightarrow p = \sum_{i=0}^{m-1} a_i x^i$ and $a_m = a_{m+1} = \cdots = a_{2m} = 0$

Since also $p \in V$, $a_i = a_{2m-i}$ for $i = 0, \dots, m$ $\Rightarrow$

$$a_0 = a_m = 0$$
$$a_1 = a_{2m-1} = a_{2m-1} = 0$$
$$\vdots$$
$$a_{m-1} = a_{m+1} = 0$$

$\Rightarrow p = 0$ so $V \cap \mathbb{F}_{m-1}[x] = \{0\}$

d) if $p = \sum_{i=0}^{m} a_i x^i \in V' \Rightarrow a_m = a_{2m-m} = a_{m-m} = -a_{m-m} = -a_m \Rightarrow a_m = 0$

hence a general element of V' can be written as

$$p = \sum_{i=0}^{m-1} a_i x^i - \sum_{i=0}^{m-1} a_i x^{2m-i} = \sum_{i=0}^{m-1} a_i (x^i - x^{m-i})$$

by an analogous argument than b), we can show $\dim V' = m$

also is very clear that $V \cap V' = \{0\}$ (Since $p \in V \cap V' \Rightarrow a_i = -a_i \ \forall i$)

$\therefore \ \mathbb{F}_m[x] = V \oplus V'$

P4

a) A is a symmetric matrix, so we can diagonalize A in the form

$$A = P D P^t$$

where P is a matrix made out of an orthonormal basis of eigenvectors of A.

the characteristic polynomial of A is given by

$$\det(A - \lambda \mathbb{1}) = (\lambda - 2)\lambda((1-\lambda)^2 - 1) = \lambda^2 (2-\lambda)^2$$

So we have eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$

V_{λ_1} has a orthogonal basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, we normalize the vectors and

so $V_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \right\}$

Same for λ_2 gives $V_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

so $A = PDP^t$ with $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

b) First let's show z_1 is eigenvector of C:

$$C \cdot z_1 = M \cdot z_1 + z_1 \underbrace{(z_1^t z_1)}_{=1} = (\lambda_1 + 1) z_1$$

So, is eigenvector with eigenvalue $\lambda_1 + 1$. For the rest $(i > 1)$

$$C \cdot z_i = M \cdot z_i + z_1 \underbrace{(z_1^t \cdot z_i)}_{=0} = \lambda_i z_i$$

So z_i, $i > 1$ are eigenvectors w/ eigenvalue λ_i

P5/

a) $u \in U \Rightarrow u \in \operatorname{Ker} T$ so $Tu = 0 \in U$

b) Consider $u \in U$, then $Tu \in \text{Image}\,T \subset U \Rightarrow Tu \in U$

c) Consider $v \in \operatorname{Ker} S$, then $z = Tv$ and $Sz = STv = TSv \underset{\substack{\uparrow \\ v \in \operatorname{Ker} S}}{=} 0$

so $z \in \operatorname{Ker} S$

d) Consider $v \in \text{Image}\,S$, so we can write $v = S \cdot u$, for some $u \in U$, then

$$Tv = TS \cdot u = STu = S(Tu) \in \text{Image}\,S$$