1. 设 $A, B \in M_n(\mathbb{C})$, $AB = BA$,

则 $Ae^{Bt} = e^{tB}A$.

2. 设 $A \in M_n(\mathbb{C})$, $A^H = A$,

证明: (1) $(e^A)^H = e^{A^H}$.

(2) e^{iA} 是酉阵,

3. 设 $A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$, 求 e^{At}.

4. 求微分方程

$$\frac{dx(t)}{dt} = Ax(t)$$ 的一般解

(1) $A=\begin{pmatrix} -5 & 1 & 4 \\ -12 & 3 & 8 \\ -6 & 1 & 5 \end{pmatrix}$

(2) $A=\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$

5. 求微分方程

$y''+py'+qy=0$ 的一般解.

其中, $p, q\in\mathbb{C}$ 是常数.

(提示. 令 $u=\begin{pmatrix} y \\ y' \end{pmatrix}$ 则 $u'=\begin{pmatrix} y' \\ y'' \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}u$).

答案

1. $$e^{Bt} = I_n + Bt + \frac{1}{2!}B^2t^2 + \cdots + \frac{1}{k!}B^k t^k$$

$AB = BA \Rightarrow \forall k \in \mathbb{N},$

$$AB^k = AB \cdot B^{k-1} = BAB^{k-1}$$

$$= \cdots = B^k A \quad (*)$$

$$Ae^{Bt} = A + ABt + \cdots + \frac{1}{k!}AB^k t^k + \cdots$$

$$e^{Bt}A = A + BAt + \cdots + \frac{1}{k!}B^k A t^k + \cdots$$

由$(*)$ $\quad Ae^{Bt} = e^{Bt}A$

3. $|\lambda I_2 - A| = \lambda^2 + a^2.$

$\Rightarrow \lambda_1 = ai, \quad \lambda_2 = -ai$

$P = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \quad P^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$

则 $P^{-1}AP = \begin{pmatrix} ai & \\ & -ai \end{pmatrix} = J$

$$e^{At} = P e^{Jt} P^{-1}$$

$$= P \begin{pmatrix} e^{ait} & \\ & e^{-ait} \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}$$

4. (1) $|\lambda I_3 - A| = (\lambda-1)^3$

$$P=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \quad P^{-1}AP=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}=J$$

$$e^{At}P = Pe^{Jt}$$

$$= P\cdot\begin{pmatrix} e^t & & \\ & e^t & te^t \\ & & e^t \end{pmatrix}$$

$$= e^t\begin{pmatrix} 2 & 1 & t \\ 0 & 2 & 2t+1 \\ 3 & 1 & t \end{pmatrix}$$

$\Rightarrow$ 一般解为 $e^{At}P\cdot c \quad c=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

也可以写一般解：$e^{At}\cdot c$

$c=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ 其中

$$e^{At} = e^{(A-I)t}\cdot e^{It}$$

$$= \left[I_3 + (A-I)t\right]\cdot e^t I_3$$

$$= e^t\left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} -6 & 1 & 4 \\ -12 & 2 & 8 \\ -6 & 1 & 4 \end{pmatrix}t\right]$$

$$= e^t\begin{pmatrix} -6t+1 & t & 4t \\ -12t & 2t+1 & 8t \\ -6t & t & 4t+1 \end{pmatrix}.$$

(2) $|\lambda I_3 - A| = (\lambda-1)^3$

$(A-I)^3 = O \qquad (A-I)^2 \neq O$

$$e^{At} = e^{(A-I)t} \cdot e^{It}$$

$$= \left[I_3 + (A-I)t + \frac{1}{2}(A-I)^2 t^2 \right] e^t$$

$$= \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & -3 & 3 \\ -2 & -7 & 13 \\ -1 & -4 & 7 \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 3 & 9 & -18 \\ 1 & 3 & -6 \\ 1 & 3 & -6 \end{pmatrix} t^2 \right] e^t$$

$$= e^t \begin{pmatrix} \frac{3}{2}t^2+1 & \frac{9}{2}t^2-3t & -9t^2+3t \\ \frac{1}{2}t^2-2t & \frac{3}{2}t^2-7t+1 & -3t^2+13t \\ \frac{1}{2}t^2-t & \frac{3}{2}t^2-4t & -3t^2+7t+1 \end{pmatrix}$$

一般解： $e^{At} \cdot C, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$

5. 令 $u=\begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow u'=\begin{pmatrix} y' \\ y'' \end{pmatrix}$

$$u'=\begin{pmatrix} y' \\ -py'-qy \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}u$$

令 $A=\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$

$|\lambda I_2-A|=\lambda^2+p\lambda+q \qquad \Delta=p^2-4q$

① $\Delta\neq 0$, 有两个根 $\lambda_1\neq\lambda_2$,

$$\lambda_1=\frac{-p+\sqrt{p^2-4q}}{2} \qquad \lambda_2=\frac{-p-\sqrt{p^2-4q}}{2}$$

$Ax=\lambda_i x \Rightarrow x_i=\begin{pmatrix} 1 \\ \lambda_i \end{pmatrix} \quad i=1,2$

令 $P=\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$,

则 $P^{-1}AP=\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$

$$\Rightarrow e^{At}=P\begin{pmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{pmatrix}P^{-1}$$

$$\Rightarrow e^{At}P=P\begin{pmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{pmatrix}$$

一般解是 $e^{At}P\begin{pmatrix}c_1\\ c_2\end{pmatrix}$

$$=P\begin{pmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$=c_1e^{\lambda_1 t}x_1+c_2e^{\lambda_2 t}x_2=u(t)$$

注：一般解也可写 $e^{At}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$

形式较繁琐

$$\Rightarrow y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

注：若 $\Delta < 0$ 即 $p^2 - 4q < 0$

此时 $\lambda_1 = a + ib$ $\quad a = -p$

$\lambda_2 = a - ib$ $\quad b = \sqrt{4q - p^2}$

$$e^{\lambda_1 t} = e^{at}(\cos bt + i\sin bt)$$

$$e^{\lambda_2 t} = e^{at}(\cos bt - i\sin bt)$$

$$\Rightarrow y(t) = e^{at} \cdot (c_1 \cos bt + c_2 \sin bt)$$

若 $\Delta = 0$ 即 $p^2 = 4q$.

$$\lambda_1 = \lambda_2 = -\frac{p}{2}$$

$$(A - \lambda_1 I_2)x = 0 \Longrightarrow x = \begin{pmatrix} 2 \\ -p \end{pmatrix}$$

$$(A - \lambda_1 I_2)y = x \Rightarrow y = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{令 } P = \begin{pmatrix} 2 & 0 \\ -p & 2 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{p}{2} & 1 \\ 0 & -\frac{p}{2} \end{pmatrix} = J$$

$$\Rightarrow AP = PJ$$

$$e^{At} \cdot P = P \cdot e^{Jt}$$

一般解是 $e^{At} \cdot P \cdot c \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$= P \cdot e^{Jt} \cdot c$$

$$= \begin{pmatrix} 2 & 0 \\ -p & 2 \end{pmatrix} e^{\lambda_1 t} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= e^{\lambda_1 t} \begin{pmatrix} 2 & 2t \\ -p & -pt+2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= e^{\lambda_1 t} \left[c_1 \begin{pmatrix} 2 \\ -p \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 2-pt \end{pmatrix} \right]$$

$$\Rightarrow y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

$\lambda_1 = -\frac{p}{2}$. $\quad$ c_1, c_2 常数