Problem 1

Denote $\mathbb{F}_0[x]$ the set of monic polynomials. Show that $\mathbb{F}_0[x]$ plus divisibility form a partially ordered set (poset, for short). That is, it has the following properties for $f, g, h \in \mathbb{F}_0[x]$

- Reflexivity: $f|f$.
- Antisymmetry: If $f|g$ and $g|f$ then $f = g$.
- Transitivity: If $f|h$ and $h|g$ then $f|g$.

If we consider $\mathbb{F}[x]$ instead of $\mathbb{F}_0[x]$, does divisibility still defines a poset?

Problem 2

- Show that if $p = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ has a rational root $\frac{r}{s} \in \mathbb{Q}$ (with $gcd(r, s) = 1$), then $r|a_0$ and $s|a_n$.
- If $f \in \mathbb{Z}[x]$ is monic and $\deg(f) \geq 1$, then show that if f has a rational root then the root is necessarily an integer.
- Let us apply these results. Consider $p = x^3 + 6x^2 - 3x - 4$, find all its rational roots (if any).

Problem 3

Poor man's Nullstellensatz. Hilbert's Nullstellensatz is a theorem that goes back to the 1900's and it is one of the fundamental theorems in commutative algebra and algebraic geometry. The theorem states that if $f_1 = \ldots = f_r = 0$ is a system of multivariate polynomials (over $\mathbb{C}$), then it has no solution if and only if the exist polynomials $\alpha_1, \ldots, \alpha_r$ such that $\sum_{i=1}^{r} \alpha_i f_i = 1$. We will prove it for the case of univariate polynomials (over $\mathbb{C}$), i.e., when all polynomials belong to $\mathbb{C}[x]$. Start by proving the following statements:

- If $q, r, f \in \mathbb{C}[x]$ satisfy $f = qg + r$ then $gcd(f, g) = gcd(g, f - qg) = gcd(g, r)$
- $gcd(f_1, \ldots, f_s) = gcd(f_1, gcd(f_2, \ldots, f_s))$.
- If g_1, g_2 are $gcd(f_1, \ldots, f_s)$ then $g_1 = cg_2$, with $c \in \mathbb{C}$.
- Show that the Euclidean algorithm works (i.e., that indeed returns $gcd(f, g)$).

Note that these results then gives you an algorithm to compute $gcd(f_1, \ldots, f_s)$ for any s. Now, we are almost ready, first show the following

- If h is $gcd(f_1, \dots, f_s)$, there exist polynomials $\alpha_1, \dots, \alpha_s \in \mathbb{C}[x]$ such that

$$h(x) = \sum_{i=1}^{s} \alpha_i(x) f_i(x) \tag{1}$$

  Hint: you can try induction in s.

then, using the previous results show Hilbert's Nullstellensatz for univariate polynomials:

- The polynomials $f_1, \dots, f_s \in \mathbb{C}[x]$ have no common root if and only if there exist $\alpha_1, \dots, \alpha_s \in \mathbb{C}[x]$ such that

$$1 = \sum_{i=1}^{s} \alpha_i(x) f_i(x) \tag{2}$$

Problem 4

Three proofs of Lagrange interpolation. We will prove Lagrange interpolation in three different ways. Let $x_0, \dots, x_n \in \mathbb{C}$ be $n+1$ distinct numbers and consider the numbers $w_0, \dots, w_n \in \mathbb{C}$ (not necessarily distinct). Our objective is to prove that there exist a unique polynomial $f(x) \in \mathbb{C}[x]$ of degree at most n satisfying $f(x_i) = w_i$ for all $i = 0, \dots, n$ (i.e. there exist a unique polynomial function of degree less or equal than n whose graph in $\mathbb{R}^2$ passes through all the points (x_i, w_i), $i = 0, \dots, n$).

- **Via Vandermonde matrix:** Show that solving the linear system for finding $f(x)$ is equivalent to invert a Vandermonde matrix[1] and has a unique solution if and only if all $x_i's$ are distinct (here you can use what your results from Homework 2).
- **Via rank-nullity theorem:** Define $T : \mathbb{C}_n[x] \to \mathbb{C}^{n+1}$ (here $\mathbb{C}_n[x] = \{p \in \mathbb{C}[x] | \deg(p) \leq n\}$) by $T(f) = (f(x_0), \dots, f(x_n))$.
  1. Prove that T is linear (you can use that $\mathbb{C}_n[x]$ and $\mathbb{C}^{n+1}$ are vector spaces).
  2. Show that $\mathbb{C}_n[x]$ is finite dimensional by finding a basis of $\mathbb{C}_n[x]$[2]
  3. Use rank-nullity theorem to show that T is bijective and conclude the statement of Lagrange interpolation[3].

[1] See Homework 2 for the definition of Vandermonde matrix.

[2] Remember that a basis of a finite dimensional vector space V (of dimension s), over $\mathbb{C}$ is a list of vectors $v_1, \dots, v_s$ such that every element in V can be written as $\sum_{i=1}^{s} a_i v_i$ with the a_i's in $\mathbb{C}$.

[3] Remember from Linear Algebra 1 that the rank-nullity theorem for a linear function $T : V \to W$ states that $\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Image}(T)) = \dim(V)$. Naturally, this only works for finite dimensional vector spaces

- **Via explicit solution:** Consider the following collection of functions:

$$p_k := \prod_{j \neq k}(x - x_k) = (x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n) \qquad k = 0, \ldots, n \quad (3)$$

  using the functions p_k's construct a explicit solution for the interpolation problem i.e. write a explicit solution for $f(x) \in \mathbb{C}[x]$ of degree at most n satisfying the required properties (**Hint**: what are the values of $p_k(x_j)$ for $j = 0, \ldots, n$?)

Problem 5

Consider $\mathbb{R}_n[x]$ defined as in Problem 4 and define the function $T : \mathbb{R}_5[x] \to \mathbb{R}_{10}[x]$ by $T(p(x)) = p(x^2)$.

- Prove that T is linear.
- Find a basis for Image(T) (**Hint**: you can consider taking the image of a basis of the domain).
- Show that T is injective.