Problem 1

Uniqueness of Jordan normal form. In this problem, V is a complex vector space of dimension n and we consider an arbitrary $T \in \mathcal{L}(V)$. By the decomposition theorem we know that we can write $V = V_1 \oplus \cdots \oplus V_r$, where V_i is a cyclic subspace associated to T and eigenvalue λ_i [1].

- Fix some $\lambda \in \mathbb{C}$. Show that $(T - \lambda\mathbf{1}) \in \mathcal{L}(V_i)$ for all $i = 1, \dots, r$ i.e. you are asked to show that $(T - \lambda\mathbf{1})$ acting on a vector $y \in V_i$ results in another vector also in V_i. Conclude then that we moreover have $(T - \lambda\mathbf{1})^k \in \mathcal{L}(V_i)$ for any $k > 0$.
- Show that, if $\lambda \neq \lambda_i$ then $\dim(T - \lambda\mathbf{1})^k V_i = m_i$ [2] for all k and if $\lambda = \lambda_i$ then $\dim(T - \lambda\mathbf{1})^k V_i = m_i - k$ (and 0 if $k \geq m_i$).
- Define the following numbers: $n' = \sum_{\lambda_i \neq \lambda} \dim V_i$, l_m: the number of V_i's of dimension m and eigenvalue λ. Show that
$$\dim(T - \lambda\mathbf{1})^k V = l_{k+1} + 2l_{k+2} + \cdots + (p-k)l_p + n'$$
where p is the maximal dimension of the V_i's associated with the eigenvalue λ. **Hint**: Use the values of $\dim(T - \lambda\mathbf{1})^k V_i$ computed before. It may be convenient to separate the cases $\lambda \neq \lambda_i$, $\lambda = \lambda_i$ and the latter into $m_i > k$ and $m_i \leq k$.
- Define the rank of $(T - \lambda\mathbf{1})^k$ by $r_k = \mathrm{rk}(T - \lambda\mathbf{1})^k = \dim(T - \lambda\mathbf{1})^k V$. Write expressions for r_k and n' in terms of l_m's for $k = 0, \dots, p$.
- Show that the previous equations allows to determine uniquely l_m's in terms of the r_k's. **Hint**: this is doable just using Gaussian elimination on the explicit equations. Since the rank of an operator, hence the r_k's, is independent of the choice of basis of V, we conclude that the values of l_m's are independent of the basis. Explain why this means that the eigenvalues and the dimensions of the decomposition $V = V_1 \oplus \cdots \oplus V_r$ are independent of the basis and so the Jordan normal form of T is unique, up to permutation of the Jordan blocks.

Problem 2

- Consider a matrix $N \in \mathcal{M}_{n,n}(\mathbb{Z})$ with coefficients 1 in the upper diagonal and 0 everywhere else. Then, show by induction that the coefficients N^r_{ij} of N^r satisfy:
$$N^r_{ij} = 1 \ \text{ if } \ j = i + r \qquad N^r_{ij} = 0 \ \text{ otherwise}$$
conclude $N^n = 0$ proving that N is nilpotent.

[1] This means, there exist $v_i \in V_i$ of grade m_i and eigenvalue λ_i, that generates V_i (hence $\dim V_i = m_i$).

[2] The notation $(T - \lambda\mathbf{1})^k V_i \subseteq V_i$ means Image$(T - \lambda\mathbf{1})^k$ restricted to V_i i.e. image of $(T - \lambda\mathbf{1})^k$ as an operator in V_i, which we showed, makes sense.

- Consider and arbitrary matrix $T \in \mathcal{M}_{n,n}(\mathbb{C})$. Then show that

  $$\det(e^T) = e^{\mathrm{Tr}(T)}$$

  where Tr stands for trace and e^T is defined by the series (do not worry about convergence and $T^0 = \mathbf{1}$):

  $$e^T = \sum_{i=0}^{\infty} \frac{1}{i!} T^i$$

  Hint: you can use the following formula for the rth power of a block diagonal matrix $(\mathrm{diag}(B_1, \ldots, B_s))^r = \mathrm{diag}(B_1^r, \ldots, B_s^r)$