1. 设 $A=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$

证明: $e^{A}\cdot e^{B}\neq e^{A+B}$

2. 设 $A\in M_n(\mathbb{C})$.

证明: $\cos A=\frac{1}{2}(e^{iA}+e^{-iA})$

$\sin A=\frac{1}{2i}(e^{iA}-e^{-iA})$

由此证明: $\cos^2 A+\sin^2 A=I_n$.

3. 计算 e^{A}

(1) $A=\begin{pmatrix}1 & 0 & 2\\ 0 & -1 & 0\\ 0 & 4 & 2\end{pmatrix}$ (2) $A=\begin{pmatrix}0 & -a\\ a & 0\end{pmatrix}$

4. 设 $A \in M_n(\mathbb{C})$，求 e^A 的行列式（提示：求 e^A 的特征值和 A 特征值关系）

5. 计算 $\sin(e^{cI_n})$ 和 $\cos(e^{cI_n})$ 其中 $c \neq 0$.

6. 设 $A \in M_n(\mathbb{C})$，A 的特征值模长均 < 1
证明：$\lim\limits_{k \to +\infty} A^k = 0$

7. 设 $A = \begin{pmatrix} -5 & 1 & 4 \\ -12 & 3 & 8 \\ -6 & 1 & 5 \end{pmatrix}$ 求 $\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) \\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{cases}$ 的解.

1. $e^A = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}$, $e^B = \begin{pmatrix} 1 & 0 \\ e-1 & e \end{pmatrix}$

$$e^{A+B} = \begin{pmatrix} \frac{e^2+1}{2} & \frac{e^2-1}{2} \\ \frac{e^2-1}{2} & \frac{e^2+1}{2} \end{pmatrix}$$

$$e^A \cdot e^B = \begin{pmatrix} e^2-e+1 & e^2-e \\ e-1 & e \end{pmatrix}$$

3. (1)

$$e^A = \begin{pmatrix} e & -4e+\frac{4}{3}e^{-1}+\frac{8}{3}e^2 & -2e+2e^2 \\ 0 & e^{-1} & 0 \\ 0 & -\frac{4}{3}e^{-1}+\frac{4}{3}e^2 & e^2 \end{pmatrix}$$

(2) $$e^A=\begin{pmatrix} \frac{1}{2}(e^{ia}+e^{-ia}) & \frac{i}{2}(e^{ia}-e^{-ia}) \\ -\frac{i}{2}(e^{ia}-e^{-ia}) & \frac{1}{2}(e^{ia}+e^{-ia}) \end{pmatrix}$$

4. 设 A 特征值 $\lambda_1, \cdots, \lambda_n$.

则 e^A 特征值 $e^{\lambda_1}, \cdots, e^{\lambda_n}$

$$\det(e^A)=e^{\lambda_1+\cdots+\lambda_n}=e^{tr(A)}.$$

5. $$e^{cI_n}=e^cI_n$$

$$\sin(e^{cI_n})=\sin(e^cI_n)$$

$$=e^cI_n-\frac{1}{3!}(e^cI_n)^3+\frac{1}{5!}(e^cI_n)^5+\cdots$$

$= (\sin e^c) I_n$.

同理 $\cos(e^{cI_n}) = (\cos e^c) I_n$.

6. $P^{-1}AP = J$, $J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_t \end{pmatrix}$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{m_i \times m_i}$$

$$J_i^k = \begin{pmatrix} \lambda_i^k & C_k^1 \lambda_i^{k-1} & C_k^2 \lambda_i^{k-2} & \cdots & C_k^{m_i-1} \lambda_i^{k-m_i+1} \\ & \lambda_i^k & C_k^1 \lambda_i^{k-1} & \cdots & \vdots \\ & & \lambda_i^k & \cdots & \vdots \\ & & & \ddots & \\ & & & & \lambda_i^k \end{pmatrix}$$

$$\lim_{k \to +\infty} \lambda_i^k = 0$$