Sols. Hw 11

P1 $\Rightarrow$ obvious, since then $A = C^{-1}BC$ and then they have same characteristic polynomial $\Rightarrow \sigma(A) = \sigma(B)$ and $(A-\lambda\cdot\mathbb{1})^i = C^{-1}(B-\lambda\cdot\mathbb{1})^i C$

$\Leftarrow$ Suppose $\sigma(A) = \sigma(B)$, so they have the same eigenvalues, but we don't know about their multiplicities, for example. However, we also have $rk(A-\lambda\cdot\mathbb{1})^i = r_i^A$ and $r_i^B = rk(B-\lambda\cdot\mathbb{1})^i$ coincide for all i: $r_i^A = r_i^B$ and so, by uniqueness of Jordan form (See P1 HW 10), we conclude that A and B have the same Jordan form, hence they must be ~~equal~~ similar (i.e. equal up to a change of basis).

P2 A way t show this is using the result from HW 10, P1, where we expressed $r_i = rk(T-\lambda\cdot\mathbb{1})^i$ in terms of l_s, the number of blocks of dimension s associated to the eigenvalue λ.

If T is diagonalizable, then for all $\lambda \in \sigma(T)$ we have $l_1 \neq 0$ and $l_2 = l_3 = \cdots = l_m = 0$ (i.e., for all eigenvalues, only blocks of dimension 1 exist), then the eqs. expressing r_i's in terms of l_s's become

$$\text{for a fixed } \lambda \Rightarrow \left.\begin{aligned} l_1 &= m - 2r_1 + r_2 \\ 0 &= r_1 - 2r_2 + r_3 \\ &\vdots \\ 0 &= r_{p-2} - 2r_{p-1} + r_p \\ 0 &= r_{p-1} - r_p \end{aligned}\right\} (1)$$

$$\Rightarrow r_p = r_{p-1} = r_{p-2} = \cdots = r_2 = r_1, \quad l_1 = m - r_1$$

so, in particular T diagonalizable $\Rightarrow r_1 = r_2$

If we rewrite the eqs. (1) as

$$\begin{aligned} l_1 + \dots + l_p &= n - r_1 \\ l_2 + \dots + l_p &= r_1 - r_2 \\ &\vdots \\ l_p &= r_{p-1} - r_p \end{aligned}$$

then $r_1 = r_2 \Rightarrow l_2 + \dots + l_p = 0 \Rightarrow l_2 = \dots = l_p = 0$ hence only $l_1 \neq 0$ and if this holds for all $\lambda \in \sigma(T)$ then T is diagonalizable

<u>Alternative way</u>: In the Lecture, we showed that if we denote

S_k = # of Jordan blocks of size <u>at least</u> k (assoc. to λ)

then we showed

$$S_2 = \dim \operatorname{Ker}(T - \lambda \cdot 1)^2 - \dim \operatorname{Ker}(T - \lambda \cdot 1)$$

$$\underset{\substack{\text{rank-nullity} \\ \text{theorem}}}{=} rk(T - \lambda \cdot 1) - rk(T - \lambda 1)^2 = r_1 - r_2$$

so if $r_1 - r_2 = 0 \Rightarrow S_2 = 0 \Rightarrow$ there are no Jordan blocks of size 2 or more

$\Rightarrow T$ is diagonalizable (the other ~~implic~~ direction is indeed quite easy)

P31

a) we have 3 options

any of these can occur

$$\underbrace{\text{some basis}}_{\text{Jordan}} \downarrow \quad M(T, \tilde{\mathcal{V}}) = \begin{cases} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} & \text{(Two blocks)} \quad \alpha_1, \alpha_2 \in \mathbb{R} \\ \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} & \text{(one block)} \quad \alpha \in \mathbb{R} \\ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} & \text{(Two blocks)} \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

why? if we write all the possible Jordan forms of a 2×2 complex matrix, we have two options $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, so in a certain basis $\tilde{\mathcal{V}}$ of V, $M(T, \tilde{\mathcal{V}})$ takes the form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, but we also are told that there exist a basis $\mathcal{V}$, where $M(T, \mathcal{V}) \in M_{2,2}(\mathbb{R})$, this means that the characteristic polynomial of T has real coefficients, hence T has eigenvalues λ and $\bar{\lambda}$ i.e. mutually conjugate. Then, we have the posibilities ②

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \longrightarrow \begin{cases} \lambda_1, \lambda_2 \in \mathbb{R} \\ \text{or} \\ \bar{\lambda}_1 = \lambda_2 \end{cases}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \longrightarrow \lambda = \bar{\lambda} \text{ so } \lambda \in \mathbb{R}$$

This gives the 3 possibilities

b) Suppose $V = U^{\mathbb{C}}$, $\dim V = 2$ and $A \in \mathcal{L}(U)$, then $A^{\mathbb{C}}$ has an eigenvector $x = x_1 + i x_2$:

$$A^{\mathbb{C}}(x) = A(x_1) + i A(x_2) = \lambda \cdot x = (\lambda_1 + i\lambda_2)(x_1 + i x_2)$$

Now, we want the matrix of A in the basis x_1, x_2, ~~these~~ denote its entries by A_{ij}, so $A \cdot x_j = \sum_{i=1}^{2} A_{ij}\, x_i$

$$\Rightarrow \quad \mathcal{M}(A) = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$ and is easy to compute its eigenvalues which are λ and $\bar{\lambda}$ as expected.

c) Using the result from parts a), b) we see that there are 3 possible Jordan blocks for $A^{\mathbb{C}}$.

Let start with the real cases: if $\exists$ a basis of V where $\mathcal{M}(A^{\mathbb{C}}) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ with $\alpha_1, \alpha_2 \in \mathbb{R}$, then is easy to ~~see~~ show that there exist a basis of U, where $\mathcal{M}(A) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. Likewise if $\mathcal{M}(A^{\mathbb{C}}) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ with $\alpha \in \mathbb{R}$ is also straightforward to find a basis of U where $\mathcal{M}(A) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$

However, ~~if~~ in the case of $\mathcal{M}(A^{\mathbb{C}}) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$, the best we can do (see part b)) ~~the~~ is to find a basis where $\mathcal{M}(A) = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$ where $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$

There are the 3 blocks.

d) This was sketched in the lecture. The Jordan blocks of $A^{\mathbb{C}}$ takes the form $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$, if $\lambda \in \mathbb{R}$, then we can ~~see that~~ use an ~~blocks as~~ analogous way than the one we used to show $\mathcal{M}(A^{\mathbb{C}}) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ implies $\exists$ a basis of U s.t. $\mathcal{M}(A) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

on the other hand, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then we know there must exist another block of the same dimension of the form $\begin{pmatrix} \bar{\lambda} & 1 & & 0 \\ & \bar{\lambda} & \ddots & \\ & & \ddots & 1 \\ 0 & & & \bar{\lambda} \end{pmatrix}$, we put both blocks together as

$$B = \begin{pmatrix} \lambda & 1 & & & & 0 \\ & \ddots & \ddots & & & \\ & & \lambda & 1 & & \\ & & & \bar{\lambda} & \ddots & \\ & & & & \ddots & 1 \\ 0 & & & & & \bar{\lambda} \end{pmatrix}$$

, using fundamental transformations for permutations of rows and columns, we can show that B is similar to (here, written for $B \in M_{8,8}(\mathbb{C})$ for clarity)

$$\begin{pmatrix} L & \mathbb{1}_2 & 0 & 0 \\ 0 & L & \mathbb{1}_2 & 0 \\ 0 & 0 & L & \mathbb{1}_2 \\ 0 & 0 & 0 & L \end{pmatrix}$$

Where the 2×2 blocks are $L = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$, $\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then, we can use b), c) to show that ~~B is similar~~ if there exist a basis where $\mathcal{M}(A^{\mathbb{C}}) = B$, then we can find a basis where

$$\mathcal{M}(A) = \begin{pmatrix} S & \mathbb{1}_2 & & \\ & S & \mathbb{1}_2 & \\ & & S & \ddots & \mathbb{1}_2 \\ & & & & S \end{pmatrix} = C \quad \text{where } S = \begin{pmatrix} \mathrm{Re}(\lambda) & -\mathrm{Im}(\lambda) \\ \mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix}$$

likewise we could have shown that B is similar to C. This finish the characterization of Jordan blocks of A.

P4)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \det(A - \lambda \cdot \mathbb{1}) = \lambda^4(\lambda - 1)$$

So, the dimension of the relevant spaces ~~is~~ are (easily computable, just by looking at $\mathrm{rk}(A-\lambda\cdot\mathbb{1})^i$)

$$k_i = \dim \mathrm{Ker}(A - 0\cdot\mathbb{1})^i = \begin{cases} (i=1) & 2 \\ (i=2) & 3 \\ (i=3) & 4 = \text{multiplicity of } \lambda = 0 \end{cases}$$

$\dim \mathrm{Ker}(A - \mathbb{1}) = 1$ = multiplicity of $\lambda = 1$, so we have

- 1 block of size 1, associated to $\lambda = 1$
- $\left.\begin{array}{l} k_1 = 2 \text{ blocks of size at least } 1 \\ k_2 - k_1 = 1 \text{ blocks of size at least } 2 \\ k_3 - k_2 = 1 \text{ blocks of size at least } 3 \end{array}\right\}$ associated to $\lambda = 0$

then $\left.\begin{array}{l} m_1 = 1 \text{ blocks of size } 1 \\ m_2 = 0 \text{ blocks of size } 2 \\ m_3 = 1 \text{ blocks of size } 3 \end{array}\right\}$ asso. to $\lambda = 0$

$\Rightarrow$ Jordan form of A is $\begin{pmatrix} \boxed{1} & & \\ & \boxed{0} & \\ & & \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}} \end{pmatrix}$

P5)

a) $$e^{A+B} = \sum_{i=0}^{\infty} \frac{1}{i!}(A+B)^i = \mathbb{1} + (A+B) + \frac{1}{2}(A^2 + B^2 + \underbrace{AB + BA}_{= 2AB}) + \frac{1}{3!}(A^3 + B^3 + \overbrace{AB^2 + ABA + BA^2 + BAB + B^2A + A^2B}^{= 3AB^2 + 3A^2B}) + \ldots$$

$$e^A e^B = \left(\sum_{i=0}^{\infty} \frac{1}{i!} A^i\right)\left(\sum_{j=0}^{\infty} \frac{1}{j!} B^j\right) = \left(\mathbb{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots\right)\left(\mathbb{1} + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\right)$$

$$= \mathbb{1} + \overbrace{(A+B)}^{\frac{B^2}{2!} + \frac{B^3}{3!} + \ldots} + \frac{A^2}{2!}\left(\mathbb{1} + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\right) + \frac{A^3}{3!}\left(\mathbb{1} + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\right)$$

$$+ A\left(B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\right) + \ldots$$

$$= \mathbb{1} + (A+B) + \left(AB + \frac{A^2}{2} + \frac{B^2}{2}\right) + \left(\frac{A^3}{3!} + \frac{A^2}{2!}B + \frac{AB^2}{2!} + \frac{B^3}{3!}\right) + \ldots$$

at this point is easy to see these first terms are the same.

b) Note first

$$e^{A(t+h)} = e^{At} e^{Ah} = e^{Ah} e^{At}$$

because $[A\cdot t, A\cdot h] = 0$

straightforward, you can use, e.g. $A(t+h) = A(h+t)$

then

$$\frac{e^{A(t+h)} - e^{At}}{h} = \frac{e^{Ah} e^{At} - e^{At}}{h} = \frac{(e^{A\cdot h} - 1)}{h} \cdot e^{At}$$

But

$$\frac{e^{Ah} - 1}{h} = h^{-1}\left(Ah + \frac{(Ah)^2}{2!} + \frac{(Ah)^3}{3!} + \ldots \right)$$

so

$$\lim_{h \to 0} \frac{e^{Ah} - 1}{h} = A$$

$\therefore$

$$\lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h} = A \cdot e^{At}$$

Note: $[A, e^{At}] = 0$, so if one writes (or arrives to) instead $\frac{d}{dt} e^{At} = e^{At} \cdot A$ is still correct.