

Current Law  $A^T \mathbf{y} = \mathbf{0}$ , we get  $A^T C A \mathbf{x} = \mathbf{0}$ . This is *almost* the central equation for network flows. The only thing wrong is the zero on the right side! The network needs power from outside—a voltage source or a current source—to make something happen.

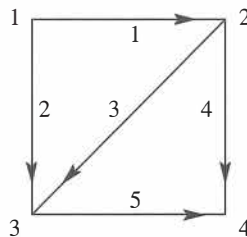
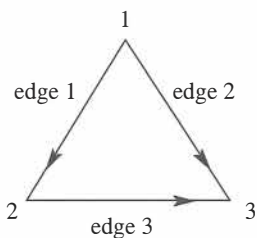
*Note about signs* In circuit theory we change from  $A\mathbf{x}$  to  $-A\mathbf{x}$ . The flow is from higher potential to lower potential. There is (positive) current from node 1 to node 2 when  $x_1 - x_2$  is positive—whereas  $A\mathbf{x}$  was constructed to yield  $x_2 - x_1$ . The minus sign in physics and electrical engineering is a plus sign in mechanical engineering and economics.  $A\mathbf{x}$  versus  $-A\mathbf{x}$  is a general headache but unavoidable.

*Note about applied mathematics* Every new application has its own form of Ohm's Law. For springs it is Hooke's Law. The stress  $\mathbf{y}$  is (elasticity  $C$ ) times (stretching  $A\mathbf{x}$ ). For heat conduction,  $A\mathbf{x}$  is a temperature gradient. For oil flows it is a pressure gradient. For least squares regression in statistics (Chapter 12)  $C^{-1}$  is the covariance matrix.

My textbooks *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) are practically built on  $A^T C A$ . This is the key to equilibrium in matrix equations and also in differential equations. Applied mathematics is more organized than it looks! *In new problems I have learned to watch for  $A^T C A$ .*

## Problem Set 10.1

Problems 1–7 and 8–14 are about the incidence matrices for these graphs.



- 1 Write down the 3 by 3 incidence matrix  $A$  for the triangle graph. The first row has  $-1$  in column 1 and  $+1$  in column 2. What vectors  $(x_1, x_2, x_3)$  are in its nullspace? How do you know that  $(1, 0, 0)$  is not in its row space?
- 2 Write down  $A^T$  for the triangle graph. Find a vector  $\mathbf{y}$  in its nullspace. The components of  $\mathbf{y}$  are currents on the edges—how much current is going around the triangle?
- 3 Eliminate  $x_1$  and  $x_2$  from the third equation to find the echelon matrix  $U$ . What tree corresponds to the two nonzero rows of  $U$ ?

$$-x_1 + x_2 = b_1$$

$$-x_1 + x_3 = b_2$$

$$-x_2 + x_3 = b_3.$$

- 4 Choose a vector  $(b_1, b_2, b_3)$  for which  $Ax = b$  can be solved, and another vector  $b$  that allows no solution. How are those  $b$ 's related to  $y = (1, -1, 1)$ ?
- 5 Choose a vector  $(f_1, f_2, f_3)$  for which  $A^T y = f$  can be solved, and a vector  $f$  that allows no solution. How are those  $f$ 's related to  $x = (1, 1, 1)$ ? The equation  $A^T y = f$  is Kirchhoff's \_\_\_\_\_ law.
- 6 Multiply matrices to find  $A^T A$ . Choose a vector  $f$  for which  $A^T Ax = f$  can be solved, and solve for  $x$ . Put those potentials  $x$  and the currents  $y = -Ax$  and current sources  $f$  onto the triangle graph. Conductances are 1 because  $C = I$ .
- 7 With conductances  $c_1 = 1$  and  $c_2 = c_3 = 2$ , multiply matrices to find  $A^T CA$ . For  $f = (1, 0, -1)$  find a solution to  $A^T CAx = f$ . Write the potentials  $x$  and currents  $y = -CAx$  on the triangle graph, when the current source  $f$  goes into node 1 and out from node 3.
- 8 Write down the 5 by 4 incidence matrix  $A$  for the square graph with two loops. Find one solution to  $Ax = 0$  and two solutions to  $A^T y = 0$ .
- 9 Find two requirements on the  $b$ 's for the five differences  $x_2 - x_1, x_3 - x_1, x_3 - x_2, x_4 - x_2, x_4 - x_3$  to equal  $b_1, b_2, b_3, b_4, b_5$ . You have found Kirchhoff's \_\_\_\_\_ law around the two \_\_\_\_\_ in the graph.
- 10 Reduce  $A$  to its echelon form  $U$ . The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.
- 11 Multiply matrices to find  $A^T A$  and guess how its entries come from the graph:
  - (a) The diagonal of  $A^T A$  tells how many \_\_\_\_\_ into each node.
  - (b) The off-diagonals  $-1$  or  $0$  tell which pairs of nodes are \_\_\_\_\_.
- 12 Why is each statement true about  $A^T A$ ? *Answer for  $A^T A$  not  $A$ .*
  - (a) Its nullspace contains  $(1, 1, 1, 1)$ . Its rank is  $n - 1$ .
  - (b) It is positive semidefinite but not positive definite.
  - (c) Its four eigenvalues are real and their signs are \_\_\_\_\_.
- 13 With conductances  $c_1 = c_2 = 2$  and  $c_3 = c_4 = c_5 = 3$ , multiply the matrices  $A^T CA$ . Find a solution to  $A^T CAx = f = (1, 0, 0, -1)$ . Write these potentials  $x$  and currents  $y = -CAx$  on the nodes and edges of the square graph.
- 14 The matrix  $A^T CA$  is not invertible. What vectors  $x$  are in its nullspace? Why does  $A^T CAx = f$  have a solution if and only if  $f_1 + f_2 + f_3 + f_4 = 0$ ?
- 15 A connected graph with 7 nodes and 7 edges has how many loops?
- 16 For the graph with 4 nodes, 6 edges, and 3 loops, add a new node. If you connect it to one old node, Euler's formula becomes  $( ) - ( ) + ( ) = 1$ . If you connect it to two old nodes, Euler's formula becomes  $( ) - ( ) + ( ) = 1$ .

- 17 Suppose  $A$  is a 12 by 9 incidence matrix from a connected (but unknown) graph.
- (a) How many columns of  $A$  are independent?
  - (b) What condition on  $\mathbf{f}$  makes it possible to solve  $A^T \mathbf{y} = \mathbf{f}$ ?
  - (c) The diagonal entries of  $A^T A$  give the number of edges into each node. What is the sum of those diagonal entries?
- 18 Why does a complete graph with  $n = 6$  nodes have  $m = 15$  edges? A tree connecting 6 nodes has \_\_\_\_\_ edges.

*Note* The *stoichiometric matrix* in chemistry is an important “generalized” incidence matrix. Its entries show how much of each chemical species (each column) goes into each reaction (each row).