Homework 8

Problem 1

For this problem, fix $T \in \mathcal{L}(V)$ with V a vector space over $\mathbb{F}$ (you can think of $\mathbb{F}$ to be $\mathbb{C}$ or $\mathbb{R}$).

- Consider $p(x) \in \mathbb{F}[x]$ and show that $p(T) \in \mathcal{L}(V)$. Also show that, under a change of basis $p(\mathcal{M}(T))$ transforms as expected (here $\mathcal{M}(T)$ denotes the matrix of T, in some basis).
- Consider $p(x), g(x) \in \mathbb{F}[x]$ and define $v(x) = g(x)p(x) \in \mathbb{F}[x]$ and $f(x) = p(x) + g(x) \in \mathbb{F}[x]$ by the usual rules of multiplication and addition of polynomials. Then, show, $p(T) + g(T) = f(T)$, $v(T) = g(T)p(T)$ and $g(T)p(T) = p(T)g(T)$.
- Consider now $p(x) \in \mathbb{F}[x]$ being the annihilator polynomial of a nonzero vector $v \in V$ (i.e. $p(T)v = 0$). Moreover, suppose that $p(x)$ is of minimal degree between the annihilator polynomials of v (this is called the minimal polynomial of the vector v). Show that every annihilator polynomial of v is divided by the minimal polynomial. Conclude then, that the minimal polynomial is unique up to a constant factor.
- Consider $A = \begin{pmatrix} 37 & 180 \\ -4 & -17 \end{pmatrix}$ Use Cayley-Hamilton theorem to compute A^{79}. Also, compute A^{79} by diagonalizing A and compare. **Hint**: Consider division theorem on x^{79} and the characteristic polynomial $p(x)$ of A. Then use appropriate identities to find the remainder $r(x)$.
- In the previous case, A was diagonalizable, but, consider for instance

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

  which is not diagonalizable and has eigenvalues of (algebraic) multiplicity higher than 1. Generalize the method you use to compute A^{79}, to compute B^{79}. **Hint**: All works analogous you just need more identities to find the remainder $r(x)$.

Problem 2

Fibonacci Sequence. The famous Fibonacci sequence is given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n > 0$, so it looks like $1, 1, 2, 3, 5, 8, \ldots$.

- Find $T \in \mathcal{M}_{2,2}(\mathbb{R})$ such that

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = T \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

  Hint: just use Fibonacci equation for F_{n+2}.

- Note then

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = T^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

  use this to find an expression for F_n.

- Use the previous results to find the limit $\lim_{k\to\infty} F_{k+1}/F_k$ (you can use $\lim_{k\to\infty} a^k = 0$ for a satisfying $|a| < 1$). If you do this correctly, you should find this limit is the golden ratio.

Problem 3

Consider $M \in \mathcal{M}_{n,n}(\mathbb{F})$ (here $\mathbb{F}$ can be $\mathbb{C}$ or $\mathbb{R}$). Show that the following statements are equivalent:

- There is a proper invariant subspace $W \subsetneq \mathbb{F}^n$.
- M is similar to a matrix of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

  for some matrices A, B, C of appropriate dimensions (specify them).

- The characteristic polynomial of M is reducible in $\mathbb{F}$.