Problem 1

Sylvester matrix. Let $\mathbb{F}$ be a field and $f, g \in \mathbb{F}[x]$ nonconstant polynomials such that $\deg(f) = n$, $\deg(g) = m$.

- Show that f and g have a nonconstant common factor if and only if there exist two nonzero polynomials $s, t \in \mathbb{F}[x]$ such that $\deg(s) < n$, $\deg(t) < m$ and $sg + tf = 0$.
- Write $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{i=0}^{m} b_i x^i$, $s = \sum_{i=0}^{n-1} c_i x^i$ and $t = \sum_{i=0}^{m-1} d_i x^i$. Then, $p = sg + tf$ is a polynomial of degree $m + n - 1$, write its coefficients explicitly.
- If we want to find s and t satisfying $sg + tf = 0$, this means we have to determine their coefficients. So setting $p = 0$ gives a system of equations for them. Define $v = (c_0, c_1, \ldots, c_{n-1}, d_0, \ldots, d_{m-1})$ and write the equations for the coefficients in a matrix form: $A \cdot v = 0$. Write A explicitly. This matrix is known as the Sylvester matrix and denoted $\mathrm{Syl}(f, g)$.
- We now can apply this. A nontrivial solution to $A \cdot v = 0$ exists if and only if $\det(A) = 0$. This determinant is called the resultant of f and g ($\mathrm{Res}(f, g) = \det(A)$). Compute the resultant of $f = 3x^2 + 7x + 6$ and $g = x^2 + 1$. Do they have a common factor in $\mathbb{Q}[x]$?.

Problem 2

Multivariate polynomials. In this problem we play a bit with multivariate polynomials. The set of multivariate polynomials is denoted by $\mathbb{C}[x_1, \ldots, x_s]$ and $f \in \mathbb{C}[x_1, \ldots, x_s]$ can be defined inductively, by $f = \sum_{i=0}^{n} a_i x_s^i$ with $a_i \in \mathbb{C}[x_1, \ldots, x_{s-1}]$, and so on. For example $f = y^3 + (x+1)y^2 + (x^2 + 2x)y + x \in \mathbb{C}[x, y]$. There are a few things we can show with only our knowledge about univariate polynomials.

- Consider $f(x_1, \ldots, x_s) \in \mathbb{C}[x_1, \ldots, x_s]$ such that $f \neq 0$. Show that there exist $z_1, \ldots, z_s \in \mathbb{C}$ such that $f(z_1, \ldots, z_s) \neq 0$. **Hint**: use induction in s.
- Each summand in $f(x_1, \ldots, x_s) \neq 0$ takes the form

$$c \prod_{i=1}^{s} x_i^{d_i} \qquad d_1, \ldots, d_s \in \mathbb{Z}_{\geq 0}, c \in \mathbb{C} \tag{1}$$

  we define the total degree of a monomial $m = c \prod_{i=1}^{s} x_i^{d_i}$ as $\deg(m) = \sum_{i=1}^{s} d_i$. Consider then f such that all its monomials have total degree less or equal than $d > 0$ (such a f is called homogeneous of degree d). Show that there exist a change of coordinates of the form $x_i = y_i + \lambda_i y_s$ for $i = 1, \ldots, s-1$ and $x_s = y_s$ and a choice of constant α such that

$$\alpha f(y_1 + \lambda_1 y_s, \ldots, y_{s-1} + \lambda_{s-1} y_s, y_s) = y_s^d + \sum_{j=0}^{d-1} p_j y_s^j \qquad p_0, \ldots, p_{d-1} \in \mathbb{C}[y_1, \ldots, y_{s-1}] \quad (2)$$

i.e., we can make αf monic with respect to y_s.

- **Parametric curves**. A circle in $\mathbb{R}^2$ can be described by all the points (x, y) that satisfy the equation $x^2 + y^2 = 1$. Analogously, I can describe it by all the points of the form $x = \cos t$, $y = \sin t$ with $t \in (0, 2\pi]$. The latter is called the parametric form of the circle and the former, an implicit equation for the circle. Consider what is called a rational curve, that is, a curve described in parametric form as $x = \frac{f_1(t)}{g_1(t)}$, $y = \frac{f_1(t)}{g_1(t)}$ for $t \in \mathbb{R}$, where $f_i, g_i \in \mathbb{R}[t]$ [1]. We can compute the implicit form of parametric curves by using the resultant of problem 1. Consider $g = -f_1(t) + xg_1(t)$ and $h = -f_2(t) + yg_2(t)$ as polynomials in $R[t]$ with $R = \mathbb{R}[x, y]$ [2]. Then, compute $\text{Res}(f, h)$ of the following parametric curves: $(x = t^2, y = t^2(t+1))$ and $(x = \frac{t-1}{t^2}, y = t - 1)$ and check that $\text{Res}(f, h)$ indeed gives the implicit equation, by direct substitution.

Problem 3

Hilbert's Nullstellensatz. In homework 3 we proved the univariate version of this famous theorem, now, we go for the full thing. We want to prove that a system of polynomial equations $f_1 = \ldots = f_s = 0$ with $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ has no solution in $\mathbb{C}^n$ if and only if there exist $\alpha_1, \ldots, \alpha_s \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\sum_{i=1}^{s} \alpha_i f_i = 1$. We will prove it in various steps [3]:

- Given $f, g \in R$[x], with R an integral domain, prove that $\text{Res}(f, g) \neq 0$ if and only if $gcd(f, g) = 1$. Also show that there always exist polynomials $s, t \in R[x]$ such that $sf + tg = \text{Res}(f, g)$ for any $f, g \in R[x]$.

- Consider $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ and define $Q(x_1, \ldots, x_n, y) = \sum_{i=2}^{s} y^{i-2} f_i$, where y is some auxiliary variable. Consider f_1 and Q has polynomials in $R[x_n]$ with R an integral domain (which integral domain is this?). Then, the resultant $\text{Res}(f_1, Q; x_n)$ [4],

[1] For this problem, having polynomials in denominators is fine, the equations for x and y are not expected to be polynomials in t, in general

[2] Here R is an integral domain, not a field, but do not worry about this, since it does not affect this problem.

[3] In this problem we work with polynomials in $R[x]$ where R is an integral domain. You can just use all the results from Problem 1, if needed, that you showed for R a field.

[4] The notation $\text{Res}(f_1, Q; x_n)$ is to emphasize we consider a resultant of polynomials in $R[x_n]$.

it takes the form

$$\text{Res}(f_1, Q; x_n) = \sum_{i=0}^{d} p_i(x_1, \ldots, x_{n-1}) y^i \qquad p_0, \ldots, p_d \in \mathbb{C}[x_1, \ldots, x_{n-1}] \tag{3}$$

the exact value of d is irrelevant for us. Show that $f_1 = \ldots = f_s = 0$ has no solution in $\mathbb{C}^n$ if and only if $p_0 = \ldots = p_d = 0$ has no solution in $\mathbb{C}^{n-1}$. **Hint**: you may try by contradiction i.e., assume $p_0 = \ldots = p_d = 0$ has a solution if $f_1 = \ldots = f_s = 0$ has no solution and the other way around.

- Use induction in n to prove Hilbert's Nullstellensatz. For this, note that the case $n = 1$ was proven in homework 3. Then assume that the theorem holds for the system $f_1 = \ldots = f_s = 0$ with $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_{n-1}]$ and prove for $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$. **Hint**: you may want to start considering all the polynomials in $R[x_n]$ with the appropriate choice of integral domain R, introduce the auxiliary variable y and use the previous results.

Problem 4

Chebyshev polynomials. Consider the following form of the Chebyshev polynomials

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1 - x^2}\right)^n + \left(x - i\sqrt{1 - x^2}\right)^n\right] \qquad n = 0, 1, 2, \ldots \tag{4}$$

defined over $x \in \mathbb{R}$

- Show that $T_n \in \mathbb{Z}[x]$.
- Show that T_n satisfy the differential equation

$$(1 - x^2)\frac{d^2 T_n}{dx^2} - x\frac{dT_n}{dx} + n^2 T_n = 0 \tag{5}$$

- Show that $T_n(\cos t) = \cos nt$
- Show that they are indeed orthogonal polynomials by computing

$$\int_{-1}^{1} dx \frac{T_l(x) T_j(x)}{\sqrt{1 - x^2}} \tag{6}$$

Hint: you may find useful the integrals $\int_0^\pi \cos mt \cos nt dt = 0$ if $m \neq n$ and $\int_0^\pi (\cos mt)^2 = \pi/2$ for $m \neq 0$.