1. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a,b,c,d\in\mathbb{R}$

A实正规 $\Longleftrightarrow AA^T=A^TA$，即 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$=\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow b^2=c^2$，若 $b=c$，A实对称（当然正规）

若 $b=-c\neq 0$ $\underset{b(d-a)}{\underset{\shortparallel}{ac+bd}}=\underset{b(a-d)}{\underset{\shortparallel}{ab+cd}}$ $\Rightarrow a-d=0$

即 $a=d$

因此 A实正规 $\Longleftrightarrow$ $b=c$

或 $b=-c$，$a=d$.

2. $\langle u+v,u-v\rangle=\langle u,u\rangle-\langle u,v\rangle+\langle v,u\rangle$

$-\langle v,v\rangle$ $=\langle v,u\rangle-\langle u,v\rangle$

若 $u,v\in\mathbb{R}^n$，则上式 $=0\Rightarrow u+v\perp u-v$

若 $u,v\in\mathbb{C}^n$，则上式 $=2\cdot\mathrm{Im}(\langle v,u\rangle)\cdot i$

例如 $u=\begin{pmatrix}1\\1\end{pmatrix}$，$v=\begin{pmatrix}i\\i\end{pmatrix}$ $\langle v,u\rangle=-2i$

$\Rightarrow\langle u+v,u-v\rangle=-4i\neq 0$

3, 4 略（课程讲义笔记）

5. 设 $A=\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}=(\vec{\alpha_1},\vec{\alpha_2})$

$\vec{\beta_1}=\vec{\alpha_1}$, $\vec{e_1}=\dfrac{\vec{\beta_1}}{\|\vec{\beta_1}\|}=\dfrac{\vec{\alpha_1}}{\|\vec{\alpha_1}\|}=(\vec{\alpha_1},\vec{\alpha_2})\begin{pmatrix} \dfrac{1}{\sqrt{|z_1|^2+|z_3|^2}} \\ 0 \end{pmatrix}$

$\vec{\beta_2}=\vec{\alpha_2}-\langle\vec{e_1},\vec{\alpha_2}\rangle\vec{e_1}$

$=\vec{\alpha_2}-\dfrac{\bar{z}_1z_2+\bar{z}_3z_4}{|z_1|^2+|z_3|^2}\vec{\alpha_1}=(\vec{\alpha_1}\ \vec{\alpha_2})\begin{pmatrix} -\dfrac{\bar{z}_1z_2+\bar{z}_3z_4}{|z_1|^2+|z_3|^2} \\ 1 \end{pmatrix}$

$\vec{e_2}=\dfrac{\vec{\beta_2}}{\|\vec{\beta_2}\|}$ 令 $t=\dfrac{1}{\|\vec{\beta_2}\|}$

$$\underbrace{(\vec{e_1},\ \vec{e_2})}_{U}=\underbrace{(\vec{\alpha_1}\ \vec{\alpha_2})}_{A}\underbrace{\begin{pmatrix} \dfrac{1}{\sqrt{|z_1|^2+|z_3|^2}} & -\dfrac{\bar{z}_1z_2+\bar{z}_3z_4}{|z_1|^2+|z_3|^2}\cdot t \\ 0 & t \end{pmatrix}}_{T}$$

两边取行列式 $|\det U|=1$ $|\det(A)|=|z_1z_4-z_2z_3|$

$|\det T|=t\cdot\dfrac{1}{\sqrt{|z_1|^2+|z_3|^2}}$

$\Rightarrow t=\sqrt{|z_1|^2+|z_3|^2}\Big/|z_1z_4-z_2z_3|$

6. 设 $D=\begin{pmatrix} A & B \\ O & C \end{pmatrix}$ $\quad D^H=\begin{pmatrix} A^H & O \\ B^H & C^H \end{pmatrix}$

D是酉阵 $\Longleftrightarrow D^H D = I_{m+n}$ (D是方阵)

$$\Updownarrow$$

$$\begin{pmatrix} A^H & O \\ B^H & C^H \end{pmatrix} \cdot \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \begin{pmatrix} I_m & O_{m\times n} \\ O_{n\times m} & I_n \end{pmatrix}$$

$$\Updownarrow$$

$$A^H A = I_m,\ A^H B = O_{m\times n} \Rightarrow B=0$$

$$B^H B + C^H C = I_n.$$

7. ① $A^{\in M_n(\mathbb{C})}$ Hermite $\Longleftrightarrow$ $i\cdot A$ 是 skew-Hermite

证明：$A$ Hermite $\Longleftrightarrow A^H = A \Longleftrightarrow iA^H = iA$

$$\Updownarrow$$

$$(iA)^H = -iA$$

② $\lambda$ 是 $A$ 的特征值 $\Longleftrightarrow$ $i\lambda$ 是 $iA$ 的特征值.

③ 应用 Hermite 阵特征值是实数的证法

设 $A$ skew-Hermite，$A^H = -A$

设 $A\vec{\alpha}=\lambda\vec{\alpha}$　$\vec{\alpha}\neq 0$

$\Rightarrow \vec{\alpha}^H A\vec{\alpha}=\lambda\vec{\alpha}^H\vec{\alpha}=\lambda\cdot\|\vec{\alpha}\|^2$

注意 $\vec{\alpha}^H A\vec{\alpha}$ 是一个数，记作 $z$

$\bar{z}=z^H=(\vec{\alpha}^H A\vec{\alpha})^H=-z$　即 $z$ 是纯虚数或 0.

因此　$\lambda=\dfrac{\vec{\alpha}^H A\vec{\alpha}}{\|\vec{\alpha}\|^2}$ 是 0 或纯虚数.