[HW7: Sols.]

P1/

a) If $\alpha \in \overline{\mathbb{Q}}$, then $\exists\ f \in \mathbb{Q}[x]$ s.t. $f(\alpha)=0$, so if $f = \sum_{i=0}^{m} a_i x^i$ then $\tilde{f} = \sum_{i=0}^{m} (-1)^i a_i x^i \in \mathbb{Q}[x]$ satisfies $f(-\alpha)=0$ and if $\alpha \neq 0$, consider

$P = \alpha^m \sum_{i=0}^{m} a_i x^{m-i}$ then $P(\frac{1}{\alpha}) = f(\alpha) = 0$. So if $\alpha \in \overline{\mathbb{Q}}$ so does $-\alpha$ and α^{-1} (if $\alpha \neq 0$).

b) Consider $(\mathbb{Q}[\alpha],+,\cdot)$, we want to show that α^m can be written in terms of a finite number of monomials $\alpha^0 = 1, \alpha, \alpha^2$, etc., for m larger than certain value.

Since $\alpha \in \overline{\mathbb{Q}}$, we know $\exists\ p$ s.t. $p(\alpha)=0$, irreducible, of minimal degree (i.e. the minimal polynomial of α), then denote $l = \deg p$ and so we can take x^m, for any $m \geq l$ and use the division theorem to obtain

$$x^m = p(x)\cdot q(x) + r(x) \quad (*)$$

where $\deg r(x) < l$.

Evaluate $(*)$ at $x=\alpha$, then $\alpha^m = \underbrace{p(\alpha)}_{0}\cdot q(\alpha) + r(\alpha) = r(\alpha) = \sum_{s=0}^{r} b_s \cdot \alpha^s$

where $r < l$, for any $m \geq l$. Therefore, any element of $\mathbb{Q}[\alpha]$ can be written as an element in $\mathrm{Span}_{\mathbb{Q}}(1, \alpha, .., \alpha^l)$ [Note: the subindex $\mathbb{Q}$, is to emphasize that we take the span w/ coeffs. in $\mathbb{Q}$ and, in principle, in certain situations, one may be able to take $\mathrm{Span}_{\mathbb{Q}}(1, \alpha, .., \alpha^{l'})$ with $l' < l$, but the previous result is enough for us, since we only wanted to show that $\mathbb{Q}[\alpha]$ was a finite dim'l vector space over $\mathbb{Q}$]

c) So, here we consider V, a vector space over $\mathbb{Q}$, i.e. if we fix a basis $\{v_1, .., v_m\}$ for V, any element $u \in V$ can be written as

$$u = \sum_{i=1}^{m} u_i v_i \quad , \quad u_i \in \mathbb{Q}, \ i = 1, .., m$$

then we know that $\beta \in \mathbb{C}$ defines an operator in $\mathcal{L}(V)$, i.e. β acts on V as:

$$\beta \cdot u = \beta \cdot \left(\sum_{i=1}^{m} u_i \cdot v_i \right) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \sum_{i=1}^{m} \beta(u_i v_i) \underset{\substack{\uparrow \\ \text{linearity again}}}{=} \sum_{i=1}^{m} u_i \, \beta(v_i)$$

$$= \sum_{i=1}^{m} u_i \sum_{j=1}^{m} M_{ij} v_i \quad \text{where } M_{ij} \in \mathbb{Q} \text{, i.e. } M \text{ is the matrix}$$

of $\beta\cdot$ in the basis $\{v_1, \dots, v_m\}$, so

$$\beta \cdot v_i = M_{ij} v_j \Rightarrow \beta \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = M \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

where $M \in \mathcal{M}_{m,m}(\mathbb{Q})$ so $\underbrace{\beta \cdot \vec{v} - M \cdot \vec{v}}_{\overset{\shortparallel}{(\beta \cdot \mathbb{1} - M)\cdot \vec{v}}} = 0 \Rightarrow \det(\beta \cdot \mathbb{1} - M) = 0$, $\vec{v} \neq 0$

since $M \in \mathcal{M}_{m,m}(\mathbb{Q})$, then $\det(\beta \cdot \mathbb{1} - M) \in \mathbb{Q}[\beta]$, hence $\beta \in \overline{\mathbb{Q}}$.

d) take $\alpha = e^{2\pi i/3}$, α is algebraic, indeed $p(x) = x^2 + x + 1 \in \mathbb{Q}[x]$ satisfies $p(\alpha) = 0$

Clearly a basis for $\mathbb{Q}[\alpha]$ is $\{1, \alpha\}$, since $\alpha^3 = 1$, $\alpha^4 = \alpha$, $\alpha^2 = -\alpha - 1$, etc., so

e.g. $\frac{7}{2} \cdot \alpha^5 + \alpha^{90} + \frac{4}{5}\alpha = \frac{7}{2}\alpha^2 + 1 + \frac{4}{5}\alpha = -\frac{7}{2}(1+\alpha) + 1 + \frac{4}{5}\alpha = -\frac{27}{3}\alpha - \frac{5}{2}$

Take this basis and write $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$, then, say α acts by multiplication

$$\alpha \cdot \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ -1-\alpha \end{pmatrix} = M \cdot \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \qquad M = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and, $p(\lambda) = \det(\lambda \cdot \mathbb{1} - M) = \det \begin{pmatrix} \lambda & -1 \\ 1 & \lambda+1 \end{pmatrix} = \lambda(\lambda+1) + 1 = \lambda^2 + \lambda + 1$

clearly $p(\alpha) = 0$, as expected.

e) $V[\alpha,\beta] = \mathrm{Span}_{\mathbb{Q}}\{\alpha^i\beta^j\}_{\substack{i=0,\dots,l_1\\ j=0,\dots,l_2}}$, by the result of part b), I can clearly take l_1, l_2 finite, since α^l w/ $l > l_1$ can be written in terms of $1, \alpha, \alpha^2, \dots, \alpha^{l_1}$ and same for β. So $V[\alpha,\beta]$ is spanned by a finite set of monomials $\alpha^i\beta^j$. So $\alpha\cdot(\alpha^i\beta^j) = \alpha^{i+1}\beta^j \in V[\alpha,\beta]$ for any $i=1,\dots,l_1$ and $j=1,\dots,l_2$ and same holds for β, hence $(\alpha+\beta)V[\alpha,\beta] \subseteq V[\alpha,\beta]$ and $\alpha\beta V[\alpha,\beta] \subseteq V[\alpha,\beta]$ as well.

f) by e) and c) we showed that, if $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}$, then $\alpha\beta \in \overline{\mathbb{Q}}$ and $\alpha+\beta \in \overline{\mathbb{Q}}$ (because $\alpha\beta$ and $\alpha+\beta$ leave a fin. dim'l vector space over $\mathbb{Q}$, in this case $V[\alpha,\beta]$, invariant). Also, from part a), we have that if $\alpha \in \overline{\mathbb{Q}}$ so does $-\alpha$ (additive inverse) and $\frac{1}{\alpha}$, if $\alpha \neq 0$ (multiplicative inverse)

P2

a) Reflection: Any vector v, orthogonal to u satisfies $u\cdot v = 0$ and so $T(v) = v$
So all the planes orthogonal to u are invariant subspaces ($\mathbb{R}^2 \subset \mathbb{R}^3$)

b) Rotation: Was sketched in the lecture. $\exists$ two proper invariant subspaces, $\mathbb{R}^2$, plane orthogonal to the rotation axis and the rotation axis itself ($\mathbb{R} \subset \mathbb{R}^3$)

c) Homothety: Any subspace of $\mathbb{R}^3$ is invariant (just think of the subspaces as $\mathrm{Span}\{v\}$ or $\mathrm{Span}\{v_1, v_2\}$ for any v, v_1, v_2)

P3

a) $v = P^t u$, $v^t = u^t P$ so $\|u\|^2 = u^t u$ and $\|v\|^2 = v^t v = u^t P P^t u = u^t u$ (since $P^t = P^{-1}$) hence $\|u\| = \|v\|$.

b) $H = P^t D P$ where $D = \mathrm{diag}(\alpha_1, \dots, \alpha_m)$, then define $v = Pu$, so

$$u^t H u = v^t D v = \sum_{i=1}^{m} v_i^2 \alpha_i$$

Since $v_i^2 \geq 0$ $\forall i$ and $\alpha_i \in \mathbb{R}$ $\forall i$, then $\sum_{i=1}^{m} v_i^2 \alpha_i \leq \beta \sum_{i=1}^{m} v_i^2 = \beta\, v^t v = \beta \|v\|^2$

$= \beta \|u\|^2$

part a)

Similarly for the smallest eigenvalue $\alpha \|u\|^2 \leq u^t H u$

P4)

a) $\det(AB - \lambda \cdot 1) = \det(A(B - \lambda \cdot A^{-1})) = \det((B - \lambda \cdot A^{-1})A) = \det(BA - \lambda \cdot 1)$

b) $A = P^{-1} D P = P^{-1} \lambda \cdot 1 \cdot P = \lambda \cdot 1$

c) $A = P^{-1} D P \Rightarrow A^k = P^{-1} D^k P = 0 \Rightarrow D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_m^k \end{pmatrix} = 0 \quad , k > 0$

so all λ_i's are zero hence $A = 0$

d) $A = P^{-1} D P \Rightarrow A^t = P^t D^t P^{-t}$ and $D^t = D$

P5) Use $A = P^{-1} D P$

$$B = P^{-1} \cdot P + P^{-1} \underbrace{\left(\sum_{i=1}^{k} D^k \right)}_{= \operatorname{diag}\left(\sum_{i=1}^{k} \lambda_1^i, \dots, \sum_{i=1}^{k} \lambda_m^i\right)} P = P^{-1} \operatorname{diag}\left(\sum_{i=0}^{k} \lambda_1^i, \dots, \sum_{i=0}^{k} \lambda_m^i \right) \cdot P$$

$$= P^{-1} \operatorname{diag}\left(\frac{\lambda_1^{k+1} - 1}{\lambda_1 - 1}, \dots, \frac{\lambda_m^{k+1} - 1}{\lambda_m - 1} \right) \cdot P$$

Note $\dfrac{\lambda_e^{k+1} - 1}{\lambda_e - 1}$ is well defined, since $\lambda_e \notin \{\pm 1\}$, then, is easy to check that

$$B^{-1} = P^{-1} \operatorname{diag}\left(\frac{\lambda_1 - 1}{\lambda_1^{k+1} - 1}, \dots, \frac{\lambda_m - 1}{\lambda_m^{k+1} - 1} \right) P$$

where $\dfrac{\lambda_e - 1}{\lambda_e^{k+1} - 1}$ is well defined for the same reason. So, B is invertible

(we showed it by finding its inverse explicitly)