Problem 1

Dual vector spaces. Remember that given a vector space V (over $\mathbb{F}$), the dual is defined as $V^* = \mathcal{L}(V, \mathbb{F})$ i.e., the space of linear maps from V to $\mathbb{F}$. Also, given $T \in \mathcal{L}(V)$ we define its dual as well by $T^* \in \mathcal{L}(V^*)$ acting on $\varphi \in V^*$ as $T^*(\varphi) = \varphi \circ T \in V^*$ [1]. Define the map $\Lambda : V \to U$ where $U = (V^*)^* = \mathcal{L}(V^*, \mathbb{F})$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V^*$

- Show that Λ is a linear map (so $\Lambda \in \mathcal{L}(V, U)$). **Hint**: Check how Λv acts on an arbitrary map $\varphi \in V^*$ for an appropriately chosen v.
- Show that if $T \in \mathcal{L}(V)$ then $(T^*)^* \circ \Lambda = \Lambda \circ T$. **Hint**: Use carefully the definition of a dual map, namely $(T^*)^* \in \mathcal{L}(U)$ and $(T^*)^*(\psi) = \psi \circ T^*$ for $\psi \in U$. Then study $((T^*)^* \circ \Lambda v)(\varphi) = ((T^*)^*(\Lambda v))(\varphi)$ for arbitrary v and φ.
- Show that if V is finite dimensional then Λ is an isomorphism. **Hint**: you can use $\dim(V) = \dim(V^*) = \dim(U)$.
- An example: consider the collections of maps $f_i \in (\mathbb{R}_n[x])^*$ defined by

$$f_i(p(x)) = \int_0^{a_i} p dx \qquad i = 0, \dots, n$$

  where $a_i \in \mathbb{R}$. Find, for which values of $a_0, \dots, a_n$ the maps f_i form a basis of $(\mathbb{R}_n[x])^*$. **Hint**: a typical example of a basis of a dual space V^* is the dual basis of a given basis of V, say $\mathcal{V} = \{v_1, \dots, v_n\}$. Then the dual basis, spanned by $\phi_i \in V^*$, $i = 1, \dots, n$ is defined by the equations: $\phi_i(v_j) = 1$ if $i = j$ and 0 otherwise. You can then show that there is an invertible map from the proposed vectors $f_0, \dots, f_n$ to the dual of a basis of $\mathbb{R}_n[x]$ of your choice.

Problem 2

Dual spaces and complexification. In this problem, V is a real, finite dimensional, vector space. Only for the first two parts use the following definition of $V^{\mathbb{C}} = \mathcal{L}(V^*, \mathbb{C})$. For the rest, work with the one we used in the lecture.

- Given $z \in \mathbb{C}$ argue that scalar multiplication $z \cdot v$ with $v \in V^{\mathbb{C}}$ makes sense.

[1] A reference for dual spaces and operators as well as its properties is Ch. 3.F of S. Axler's *Linear Algebra Done Right*. But this problem is actually self contained.

- Define $T^{\mathbb{C}}$ as $(T^{\mathbb{C}}(\phi))(f) = \phi(T^*(f))$ where $\phi \in V^{\mathbb{C}}$ and $f \in V^*$ and $T^* \in \mathcal{L}(V^*)$. Show that, if $\phi \in \mathcal{L}(V^*, \mathbb{R})$ then $(T^{\mathbb{C}}(\phi))(f) = f(Tv)$ for some $v \in V$ (**Hint**: use the isomorphism Λ from problem 1).

- Write $s = v + iu \in V^{\mathbb{C}}$ as we did in class. Then define conjugation in $V^{\mathbb{C}}$ by

$$\bar{s} = v - iu$$

  and show $\overline{T^{\mathbb{C}}(s)} = T^{\mathbb{C}}(\bar{s})$. This means, $T^{\mathbb{C}}$ is what is called a real operator.

- Show, using the definition and properties of $T^{\mathbb{C}}$ we used in the class, that if T is block diagonalizable, with each block has of dimension at most 2 [2] then $T^{\mathbb{C}}$ is diagonalizable.

- Show that a basis of V is a basis of $V^{\mathbb{C}}$. **Hint**: This makes sense since we saw $V \subset V^{\mathbb{C}}$. Remember that these are vector spaces over different fields.

- Show that, in the previously defined basis $\mathcal{M}(T) = \mathcal{M}(T^{\mathbb{C}})$ and that complex (and not real) eigenvalues of $T^{\mathbb{C}}$ come in conjugate pairs.

- Use the previous result to show that, if $T^{\mathbb{C}}$ is diagonalizable, then T is block diagonalizable, in the same sense as above.

Problem 3

Here V is a real vector space and $T \in \mathcal{L}(V)$ and v is a principal vector of T with eigenvalue λ and grade m.

- Consider the basis generated by v of the cyclic subspace $M_\lambda \subseteq V$. Show that in this basis $\mathcal{M}(T)|_{M_\lambda}$ is a matrix of the form $\text{diag}(\lambda, \ldots, \lambda) + N$ where N is a matrix that only has 1's in the upper diagonal and all other entries 0.

- The space spanned by all principal vectors with eigenvalue λ [3] is a cyclic subspace by itself, so we will abuse notation and call it M_λ as well. Using this information show that $M_\lambda = \text{Ker}((T - \lambda\mathbf{1})^n)$ where $n = \dim V$ (**Hint**: use double inclusion) and then conclude that there cannot be an eigenvalue of a principal vector such that $\lambda \notin \sigma(T)$.

- If we want to compute M_λ, is not so hard to compute $\text{Ker}((T - \lambda\mathbf{1})^n)$, but in order to get all the vectors we need to make sure that say, if $(T - \lambda\mathbf{1})^j x = 0$ then $(T - \lambda\mathbf{1})^{j-1} x \neq 0$.

[2] The precise statement was given in class, recall the 2 dimensional blocks are of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$ and $b \neq 0$.

[3] That is, all vectors $u \neq 0$ satisfying $(T - \lambda\mathbf{1})^j u = 0$ for some j.

Compute all the spaces M_λ of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

- Compute the principal vectors of grade $1, 2$ and 3 of A by solving the corresponding linear problem. Show that the principal vector of grade 3 generates a cyclic space (which coincides with M_λ, as promised).

Problem 4

Consider a cyclic subspace $M_\lambda \subset V$ generated by v, principal vector of $T \in \mathcal{L}(V)$.

- Show that any $y \in M_\lambda$ can be written always as $(T - \mu\mathbf{1})z$ for some vector z and $\mu \neq \lambda$.
- Use the theorem we showed in class, that $y \in M_\lambda$ can always be written as $y = f(T)v$ to show that if $(x - \lambda)$ divides $f(x)$, then we can write $y = (T - \lambda\mathbf{1})z'$ for some vector $z' \in M_\lambda$