Note on Polynomial Approximation

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1 Averaged Taylor Polynomial

Definition 1.1. The Taylor polynomial of order m evaulated at y is given by

$$T_y^m u(x) = \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \tag{1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of non-negative integers, $x \in \mathbb{R}^n$ and

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i},\ \alpha! = \prod_{i=1}^n \alpha_i!,\ |\alpha| = \sum_{i=1}^n \alpha_i$$

Howerer, if u is in a Sobolev space, $D^\alpha u$ may not exist in the usual pointwise sense. We accomplish this by taking an "average" over y on a ball.

Definition 1.2. Suppose u has weak derivatives of order strictly less than m in a region Ω such than $B \subset\subset \Omega$. The corresponding Taylor polynomial of order m of u averaged over B is defined as

$$Q^m u(x) = \int_B T_y^m u(x) \phi(y)\, dy \tag{2}$$

where $T_y^m u(x)$ is defined as (1), B is a ball centerred at x_0 with radius ρ and ϕ is the cut-off function supported in $\overline{B}$.

Remark 1.1. Such a polynomial is not unique, due to the choice od cut-off function ϕ.

Proposition 1.1. *$Q^m u$ is a polynomial of degree less than m in x.*

Proof. The definition of $Q^m u$ does make sence because $u \in W_p^{m-1}(\Omega)$. If we write

$$(x-y)^\alpha = \prod_{i=1}^n (x_i - y_i)^{\alpha_i} = \sum_{\gamma+\beta=\alpha} a_{[\gamma,\beta]} x^\gamma y^\beta$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $a_{[\gamma,\beta]}$ are constants, $Q^m u(x)$ can be written as

$$\begin{aligned} Q^m u(x) &= \int_B T_y^m u(x)\phi(y)\, dy = \sum_{|\alpha|<m} \int_B \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \phi(y)\, dy \\ &= \sum_{|\alpha|<m} \sum_{\gamma+\beta=\alpha} \left(\frac{1}{\alpha!} a_{[\gamma,\beta]} \int_B D^\alpha u(y) y^\beta \phi(y)\, dy \right) x^\gamma \end{aligned}$$

□

Note 1.1. The degree of $Q^m u$ is at most $m-1$.

Though we use the assumption that u is in a Sobolev space $W_p^{m-1}(\Omega)$, we can extend the definition of $Q^m u$ to $L^1(B)$ by integrating by parts:

$$Q^m u(x) = \sum_{|\alpha|<m} \sum_{\gamma+\beta=\alpha} \left(\frac{(-1)^{|\alpha|}}{\alpha!} a_{[\gamma,\beta]} \int_B u(y) D^\alpha (y^\beta \phi(y)) \, dy \right) x^\gamma. \tag{3}$$

It is equivalent to (2) if $u \in W_p^{m-1}(\Omega)$.

Proposition 1.2.

$$Q^m u(x) = \sum_{|\lambda|<m} \left(\int_B \psi_\lambda(y) u(y) \, dy \right) x^\lambda \tag{4}$$

where $\psi_\lambda \in C_0^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp}(\phi_\lambda) \in \overline{B}$.

Proof. This follows from (3) if we define

$$\psi_\lambda(y) = \sum_{\alpha \geq \lambda, |\alpha|<m} \frac{(-1)^{|\alpha|}}{\alpha!} a_{[\lambda,\alpha-\lambda]} D^\alpha (y^{\alpha-\lambda} \phi(y)). \tag{5}$$

□

Corollary 1.3. *If* Ω *is a bounded domain in* $\mathbb{R}^n$*, then for any* k*,* Q^m *is a bounded map of* $L^1(B)$ *into* $W_\infty^k(\Omega)$*. There exist a constant* $C = C(m, n, \rho, \Omega)$*, such that for all* $u \in L^1(B)$*,*

$$\|Q^m u\|_{W_\infty^k(\Omega)} \leq C \|u\|_{L^1(B)}. \tag{6}$$

Proof. This follows from (4) and the fact that both $\sup_{y \in B} |\psi_\lambda(y)|$ and $\sup_{x \in \Omega} |D^\alpha x^\lambda|$ are bounded. □

Proposition 1.4. *For any* α *such than* $|\alpha| \leq m-1$*,*

$$D^\alpha Q^m u = Q^{m-|\alpha|} D^\alpha u, \forall u \in W_1^{|\alpha|}(B). \tag{7}$$

Proof. It's easy to verify this proposition for all $u \in C^\infty(\Omega)$. The proof is completed via a density argument. □

2 Error Representation

Definition 2.1. Ω is star-shaped with respect to the ball B if , for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω.

From now on, we assume that Ω is star-shaped with respect to the ball B. Let C_x denote the convex hull of $\{x\} \cup B$.

The integral form of the Taylor remainder for $f \in C^m([0,1])$ is given by

$$f(s) = \sum_{k=0}^{m-1} \frac{1}{k!} f^{(k)}(0) + \int_0^s \frac{1}{(m-1)!} f^{(m)}(t)(s-t)^{m-1} \, dt. \tag{8}$$

Let u be a C^m function on Ω. For $x \in \Omega$ and $y \in B$, define $f(s) = u(y + s(x-y))$. Then, we obtain

$$\frac{1}{k!} f^{(k)}(s) = \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(y + s(x-y))(x-y)^\alpha. \tag{9}$$

Hence,

$$
\begin{aligned}
f(s) = & \sum_{k=0}^{m-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \\
& + m \int_0^s \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha u(y + t(x-y))(x-y)^\alpha \right) (s-t)^{m-1}\, dt. \qquad (10)
\end{aligned}
$$

Take $s = 1$, we have $f(1) = u(x)$. By using the variable substitution $t \to 1-t$ in the integral, we obtain

$$
\begin{aligned}
u(x) = & \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \\
& + m \int_0^1 \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha u(x + t(y-x))(x-y)^\alpha \right) t^{m-1}\, dt. \qquad (11) \\
= & \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \\
& + m \sum_{|\alpha|=m} (x-y)^\alpha \int_0^1 \frac{1}{\alpha!} D^\alpha u(x + t(y-x)) t^{m-1}\, dt. \qquad (12)
\end{aligned}
$$

Definition 2.2. The m-th order renaubder term of $Q^m u$ is given by

$$
R^m u(x) = u(x) - Q^m u(x)
$$

Using the definition of Q^m and cut-off function, we obtain

$$
\begin{aligned}
R^m u(x) &= u(x) - Q^m u(x) \\
&= \int_B u(x)\phi(y)\, dy - \int_B T_y^m u(x)\phi(y)\, dy \\
&= \int_B \left(u(x) - T_y^m u(x)\right) \phi(y)\, dy \\
&= \int_B \left(m \sum_{|\alpha|=m} (x-y)^\alpha \int_0^1 \frac{1}{\alpha!} D^\alpha u(x + t(y-x)) t^{m-1}\, dt \right) \phi(y)\, dy. \qquad (13)
\end{aligned}
$$

Proposition 2.1. *The remainder $R^m u$ satisfies*

$$
R^m u(x) = m \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x,z) D^\alpha u(z)\, dz \qquad (14)
$$

where $z = x + t(y-z)$, $k_\alpha(x,z) = \frac{1}{\alpha!}(x-z)^\alpha k(x,z)$ *and*

$$
|k(x,z)| \le C \left(1 + \frac{1}{\rho}|x - x_0|\right)^n |x-z|^{-n} \qquad (15)
$$

The proof is omitted.

Definition 2.3. Suppose Ω has diameter d and is star-shaped with respect to a ball B. Then the chunkiness parameter of Ω is defined as

$$
\gamma = \frac{d}{\rho_{\max}},
$$

where $\rho_{\max}$ is the largest radius of all ball B,

$$\rho_{\max} = \sup\{\rho \mid \Omega \text{ is star-shaped with respect to a ball of radius } \rho\}$$

Corollary 2.2. *The ball B can be chosen so that the function $k(x, z)$ in proposition 2.1 satisfies the following estimate:*

$$|k(x, z)| \leq C(1 + \gamma)^n |x - z|^{-n}, \forall x \in \Omega, \tag{16}$$

where γ is the chunkiness parameter of Ω.

Proof. Choose a ball B such that Ω is star-shaped with respect to B and the radius of B satisfies $\rho \geq \frac{1}{2}\rho_{\max}$. Then

$$\begin{aligned} |k(x, z)| &\leq C \left(1 + \frac{1}{\rho}|x - x_0|\right)^n |x - z|^{-n} \\ &\leq C \left(1 + \frac{2d}{\rho_{\max}}\right)^n |x - z|^{-n} \\ &\leq 2^n C(1 + \gamma)^n |x - z|^{-n}. \end{aligned} \tag{17}$$

□

3 Bounds for Riesz Potentials

Lemma 3.1. *If $f \in L^p(\Omega)$ for $1 < p < \infty$ and $m > n/p$, then*

$$\int_\Omega |x - z|^{-n+m} |f(z)| \, dz \leq C_p d^{m-n/p} \|f\|_{L^p(\Omega)}, \forall x \in \Omega. \tag{18}$$

This inequality also holds for $p = 1$ if $m \geq n$.

Proposition 3.2. *If $1 < p < \infty$ and $m > n/p$, or $p = 1$ and $m \geq n$, there exist a constant $C = C(m, n, \gamma, p)$, such that*

$$\|R^m u\|_{L^\infty(\Omega)} \leq C d^{m-n/p} |u|_{W_p^m(\Omega)} \tag{19}$$

for all $u \in W_p^m(\Omega)$.

Proof. First, we assume that $u \in C^m(\Omega) \cap W_p^m(\Omega)$. We can use the pointwise representation of $R^m u(x)$.

$$\begin{aligned} |R^m u(x)| &= m \left| \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x, z) D^\alpha u(z) \, dz \right| \\ &\leq C \sum_{|\alpha|=m} \int_\Omega |x - z|^{-n+m} |D^\alpha u(z)| \, dz \\ &\leq C' d^{m-n/p} |u|_{W_p^m(\Omega)}. \end{aligned} \tag{20}$$

The proof can be completed via a density argument. □

Lemma 3.3. *Let $f \in L^p(\Omega)$ for $p \geq 1$ and $m \geq 1$ and let*

$$g(x) = \int_\Omega |x - z|^{-n+m} |f(z)| \, dz$$

Then

$$\|g\|_{L^p(\Omega)} \leq C_{m,n} d^m \|f\|_{L^p(\Omega)}. \tag{21}$$

Lemma 3.4 (Bramble-Hilbert)**.** *Let B be a ball in Ω such that Ω is star-shaped with respect to B and such that its radisu $\rho > \frac{1}{2}\rho_{\max}$. Let $Q^m u$ be the Taylor polynomial of order m of u averaged over B where $u \in W_p^m(\Omega)$ and $p \geq 1$. Then*

$$|u - Q^m u|_{W_p^k(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W_p^k(\Omega)}, \ k = 0, 1, \ldots, m, \tag{22}$$

where $d = \mathrm{diam}(\Omega)$.

Note 3.1. Bramble-Hilbert lemma is an important result for the analysis of the approximation properties of finite elements.

Corollary 3.5. *Under the assumption of the Lemma 3.4, the following inequality holds*

$$\inf_{v \in P^{m-1}} \|u - v\|_{W_p^k(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W_p^k(\Omega)}, \ k = 0, 1, \ldots, m, \tag{23}$$

Proof. Lemma 3.4 provides a constructive proof. □

```python
# Python
def hello():
    print("Hello, world!")

hello()
```

Algorithm 1: what

Input: This is some input
Output: This is some output
`/* This is a comment */`

some code here;
$x \leftarrow 0$;
$y \leftarrow 0$;
if $x > 5$ **then**
    x is greater than 5 ; `// This is also a comment`
else
    x is less than or equal to 5;
end
foreach y *in 0..5* **do**
    $y \leftarrow y + 1$;
end
for y *in* 0..5 **do**
    $y \leftarrow y - 1$;
end
while $x > 5$ **do**
    $x \leftarrow x - 1$;
end
return Return something here;

Problem 3.1. Calculate the integral of the function $g(x) = 3x^2$ with respect to x.

Solution 3.1. To calculate the integral of $g(x) = 3x^2$, we use the power rule for integration:

$$\int 3x^2 \, dx = x^3 + C$$

where C is the constant of integration. □

Problem 3.2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

Solution 3.2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$. Since $p > 1$, the series converges by the p-series test. $\square$

Problem 3.3. Solve the equation $2x + 5 = 13$.

Solution 3.3. Subtracting 5 from both sides, we get:

$$2x = 8$$

Dividing both sides by 2, we obtain:

$$x = 4$$

$\square$