

Note on Polynomial Approximation

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1 Averaged Taylor Polynomial

Definition 1.1. The Taylor polynomial of order m evaluated at y is given by

$$T_y^m u(x) = \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha u(y) (x - y)^\alpha \quad (1)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of non-negative integers, $x \in \mathbb{R}^n$ and

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^n \alpha_i!, \quad |\alpha| = \sum_{i=1}^n \alpha_i$$

However, if u is in a Sobolev space, $D^\alpha u$ may not exist in the usual pointwise sense. We accomplish this by taking an “average” over y on a ball.

Definition 1.2. Suppose u has weak derivatives of order strictly less than m in a region Ω such that $B \subset \subset \Omega$. The corresponding Taylor polynomial of order m of u averaged over B is defined as

$$Q^m u(x) = \int_B T_y^m u(x) \phi(y) dy \quad (2)$$

where $T_y^m u(x)$ is defined as (1), B is a ball centered at x_0 with radius ρ and ϕ is the cut-off function supported in \bar{B} .

Remark 1.1. Such a polynomial is not unique, due to the choice of cut-off function ϕ .

Proposition 1.1. $Q^m u$ is a polynomial of degree less than m in x .

Proof. The definition of $Q^m u$ does make sense because $u \in W_p^{m-1}(\Omega)$. If we write

$$(x - y)^\alpha = \prod_{i=1}^n (x_i - y_i)^{\alpha_i} = \sum_{\gamma + \beta = \alpha} a_{[\gamma, \beta]} x^\gamma y^\beta$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $a_{[\gamma, \beta]}$ are constants, $Q^m u(x)$ can be written as

$$\begin{aligned} Q^m u(x) &= \int_B T_y^m u(x) \phi(y) dy = \sum_{|\alpha| < m} \int_B \frac{1}{\alpha!} D^\alpha u(y) (x - y)^\alpha \phi(y) dy \\ &= \sum_{|\alpha| < m} \sum_{\gamma + \beta = \alpha} \left(\frac{1}{\alpha!} a_{[\gamma, \beta]} \int_B D^\alpha u(y) y^\beta \phi(y) dy \right) x^\gamma \end{aligned}$$

□

Note 1.1. The degree of $Q^m u$ is at most $m - 1$.

Though we use the assumption that u is in a Sobolev space $W_p^{m-1}(\Omega)$, we can extend the definition of $Q^m u$ to $L^1(B)$ by integrating by parts:

$$Q^m u(x) = \sum_{|\alpha| < m} \sum_{\gamma + \beta = \alpha} \left(\frac{(-1)^{|\alpha|}}{\alpha!} a_{[\gamma, \beta]} \int_B u(y) D^\alpha (y^\beta \phi(y)) dy \right) x^\gamma. \quad (3)$$

It is equivalent to (2) if $u \in W_p^{m-1}(\Omega)$.

Proposition 1.2.

$$Q^m u(x) = \sum_{|\lambda| < m} \left(\int_B \psi_\lambda(y) u(y) dy \right) x^\lambda \quad (4)$$

where $\psi_\lambda \in C_0^\infty(\mathbb{R}^n)$ and $\text{supp}(\phi_\lambda) \in \overline{B}$.

Proof. This follows from (3) if we define

$$\psi_\lambda(y) = \sum_{\alpha \geq \lambda, |\alpha| < m} \frac{(-1)^{|\alpha|}}{\alpha!} a_{[\lambda, \alpha - \lambda]} D^\alpha (y^{\alpha - \lambda} \phi(y)). \quad (5)$$

□

Corollary 1.3. If Ω is a bounded domain in \mathbb{R}^n , then for any k , Q^m is a bounded map of $L^1(B)$ into $W_\infty^k(\Omega)$. There exist a constant $C = C(m, n, \rho, \Omega)$, such that for all $u \in L^1(B)$,

$$\|Q^m u\|_{W_\infty^k(\Omega)} \leq C \|u\|_{L^1(B)}. \quad (6)$$

Proof. This follows from (4) and the fact that both $\sup_{y \in B} |\psi_\lambda(y)|$ and $\sup_{x \in \Omega} |D^\alpha x^\lambda|$ are bounded. □

Proposition 1.4. For any α such that $|\alpha| \leq m - 1$,

$$D^\alpha Q^m u = Q^{m-|\alpha|} D^\alpha u, \forall u \in W_1^{|\alpha|}(B). \quad (7)$$

Proof. It's easy to verify this proposition for all $u \in C^\infty(\Omega)$. The proof is completed via a density argument. □

2 Error Representation

Definition 2.1. Ω is star-shaped with respect to the ball B if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω .

From now on, we assume that Ω is star-shaped with respect to the ball B . Let C_x denote the convex hull of $\{x\} \cup B$.

The integral form of the Taylor remainder for $f \in C^m([0, 1])$ is given by

$$f(s) = \sum_{k=0}^{m-1} \frac{1}{k!} f^{(k)}(0) + \int_0^s \frac{1}{(m-1)!} f^{(m)}(t) (s-t)^{m-1} dt. \quad (8)$$

Let u be a C^m function on Ω . For $x \in \Omega$ and $y \in B$, define $f(s) = u(y + s(x-y))$. Then, we obtain

$$\frac{1}{k!} f^{(k)}(s) = \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(y + s(x-y)) (x-y)^\alpha. \quad (9)$$

Hence,

$$\begin{aligned} f(s) &= \sum_{k=0}^{m-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(y) (x-y)^\alpha \\ &\quad + m \int_0^s \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha u(y + t(x-y)) (x-y)^\alpha \right) (s-t)^{m-1} dt. \end{aligned} \quad (10)$$

Take $s = 1$, we have $f(1) = u(x)$. By using the variable substitution $t \rightarrow 1 - t$ in the integral, we obtain

$$\begin{aligned} u(x) &= \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha u(y) (x - y)^\alpha \\ &\quad + m \int_0^1 \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha u(x + t(y - x)) (x - y)^\alpha \right) t^{m-1} dt. \end{aligned} \quad (11)$$

$$\begin{aligned} &= \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha u(y) (x - y)^\alpha \\ &\quad + m \sum_{|\alpha|=m} (x - y)^\alpha \int_0^1 \frac{1}{\alpha!} D^\alpha u(x + t(y - x)) t^{m-1} dt. \end{aligned} \quad (12)$$

Definition 2.2. The m -th order renaubder term of $Q^m u$ is given by

$$R^m u(x) = u(x) - Q^m u(x)$$

Using the definition of Q^m and cut-off function, we obtain

$$\begin{aligned} R^m u(x) &= u(x) - Q^m u(x) \\ &= \int_B u(x) \phi(y) dy - \int_B T_y^m u(x) \phi(y) dy \\ &= \int_B (u(x) - T_y^m u(x)) \phi(y) dy \\ &= \int_B \left(m \sum_{|\alpha|=m} (x - y)^\alpha \int_0^1 \frac{1}{\alpha!} D^\alpha u(x + t(y - x)) t^{m-1} dt \right) \phi(y) dy. \end{aligned} \quad (13)$$

Proposition 2.1. The remainder $R^m u$ satisfies

$$R^m u(x) = m \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x, z) D^\alpha u(z) dz \quad (14)$$

where $z = x + t(y - x)$, $k_\alpha(x, z) = \frac{1}{\alpha!} (x - z)^\alpha k(x, z)$ and

$$|k(x, z)| \leq C \left(1 + \frac{1}{\rho} |x - x_0| \right)^n |x - z|^{-n} \quad (15)$$

The proof is omitted.

Definition 2.3. Suppose Ω has diameter d and is star-shaped with respect to a ball B . Then the chunkiness parameter of Ω is defined as

$$\gamma = \frac{d}{\rho_{\max}},$$

where ρ_{\max} is the largest radius of all ball B ,

$$\rho_{\max} = \sup\{\rho \mid \Omega \text{ is star-shaped with respect to a ball of radius } \rho\}$$

Corollary 2.2. The ball B can be chosen so that the function $k(x, z)$ in proposition 2.1 satisfies the following estimate:

$$|k(x, z)| \leq C(1 + \gamma)^n |x - z|^{-n}, \forall x \in \Omega, \quad (16)$$

where γ is the chunkiness parameter of Ω .

Proof. Choose a ball B such that Ω is star-shaped with respect to B and the radius of B satisfies $\rho \geq \frac{1}{2}\rho_{\max}$. Then

$$\begin{aligned} |k(x, z)| &\leq C \left(1 + \frac{1}{\rho} |x - x_0| \right)^n |x - z|^{-n} \\ &\leq C \left(1 + \frac{2d}{\rho_{\max}} \right)^n |x - z|^{-n} \\ &\leq 2^n C(1 + \gamma)^n |x - z|^{-n}. \end{aligned} \quad (17)$$

□

3 Bounds for Riesz Potentials

Lemma 3.1. *If $f \in L^p(\Omega)$ for $1 < p < \infty$ and $m > n/p$, then*

$$\int_{\Omega} |x - z|^{-n+m} |f(z)| dz \leq C_p d^{m-n/p} \|f\|_{L^p(\Omega)}, \forall x \in \Omega. \quad (18)$$

This inequality also holds for $p = 1$ if $m \geq n$.

Proposition 3.2. *If $1 < p < \infty$ and $m > n/p$, or $p = 1$ and $m \geq n$, there exist a constant $C = C(m, n, \gamma, p)$, such that*

$$\|R^m u\|_{L^\infty(\Omega)} \leq C d^{m-n/p} |u|_{W_p^m(\Omega)} \quad (19)$$

for all $u \in W_p^m(\Omega)$.

Proof. First, we assume that $u \in C^m(\Omega) \cap W_p^m(\Omega)$. We can use the pointwise representation of $R^m u(x)$.

$$\begin{aligned} |R^m u(x)| &= m \left| \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x, z) D^\alpha u(z) dz \right| \\ &\leq C \sum_{|\alpha|=m} \int_{\Omega} |x - z|^{-n+m} |D^\alpha u(z)| dz \\ &\leq C' d^{m-n/p} |u|_{W_p^m(\Omega)}. \end{aligned} \quad (20)$$

The proof can be completed via a density argument. \square

Lemma 3.3. *Let $f \in L^p(\Omega)$ for $p \geq 1$ and $m \geq 1$ and let*

$$g(x) = \int_{\Omega} |x - z|^{-n+m} |f(z)| dz$$

Then

$$\|g\|_{L^p(\Omega)} \leq C_{m,n} d^m \|f\|_{L^p(\Omega)}. \quad (21)$$

Lemma 3.4 (Bramble-Hilbert). *Let B be a ball in Ω such that Ω is star-shaped with respect to B and such that its radius $\rho > \frac{1}{2} \rho_{\max}$. Let $Q^m u$ be the Taylor polynomial of order m of u averaged over B where $u \in W_p^m(\Omega)$ and $p \geq 1$. Then*

$$|u - Q^m u|_{W_p^k(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W_p^k(\Omega)}, \quad k = 0, 1, \dots, m, \quad (22)$$

where $d = \text{diam}(\Omega)$.

Note 3.1. Bramble-Hilbert lemma is an important result for the analysis of the approximation properties of finite elements.

Corollary 3.5. *Under the assumption of the [Lemma 3.4](#), the following inequality holds*

$$\inf_{v \in P^{m-1}} \|u - v\|_{W_p^k(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W_p^k(\Omega)}, \quad k = 0, 1, \dots, m, \quad (23)$$

Proof. [Lemma 3.4](#) provides a constructive proof. \square

```

1  # Python
2  def hello():
3      print("Hello, world!")
4
5  hello()

```

Algorithm 1: what

Input: This is some input

Output: This is some output

/ This is a comment */*

```
1 some code here;
2  $x \leftarrow 0$ ;
3  $y \leftarrow 0$ ;
4 if  $x > 5$  then
5   |  $x$  is greater than 5 ;
6 else
7   |  $x$  is less than or equal to 5;
8 end
9 foreach  $y$  in 0..5 do
10  |  $y \leftarrow y + 1$ ;
11 end
12 for  $y$  in 0..5 do
13  |  $y \leftarrow y - 1$ ;
14 end
15 while  $x > 5$  do
16  |  $x \leftarrow x - 1$ ;
17 end
18 return Return something here;
```

Problem 3.1. Calculate the integral of the function $g(x) = 3x^2$ with respect to x .

Solution 3.1. To calculate the integral of $g(x) = 3x^2$, we use the power rule for integration:

$$\int 3x^2 dx = x^3 + C$$

where C is the constant of integration. □

Problem 3.2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

Solution 3.2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p -series with $p = 2$. Since $p > 1$, the series converges by the p -series test. □

Problem 3.3. Solve the equation $2x + 5 = 13$.

Solution 3.3. Subtracting 5 from both sides, we get:

$$2x = 8$$

Dividing both sides by 2, we obtain:

$$x = 4$$

□