Chapter 1 质点运动学

1. 参考系　2. 位矢

为了描述一个物体的运动需要先确定一个$\boxed{\text{参考物}}$(注：有时需明确指出，有时隐含约定(如地面参考系)). 进一步，

为了定量描述一个质点相对于此参考物的$\boxed{\text{空间位置}}$需要建立固定在参考物上的$\boxed{\text{坐标系}}$(比如以参考物上某一固定点为原点，以此原点沿3个相互垂直方向建立笛卡尔坐标系). 进一步，

为了描述质点的运动需要在坐标系的每一点配置$\boxed{\text{同步了的钟}}$。这样就知道了物体的运动情况——即在何时处于何处。

$\boxed{\text{位置}}$：质点的空间坐标值 (x, y, z)，或者写为

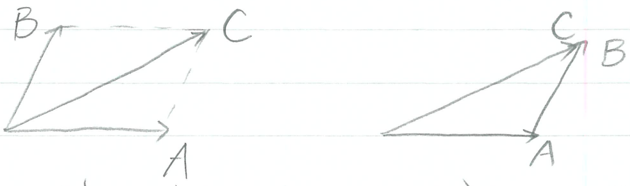
$$\vec r = x\vec i + y\vec j + z\vec k$$

即位置可用矢量表示出来，所以也叫$\boxed{\text{位置矢量}}$，简称$\boxed{\text{位矢}}$。

所以描述质点的运动就是想知道 $\vec r(t)$，即 $\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$

注：位矢是从原点向要描述的点引的一条有向线段。

$\boxed{\text{USEFUL}}$ 矢量求和(差)的平行四边形定则和三角定则：



$$\vec A + \vec B = \vec C \text{，即 } \vec B = \vec C - \vec A$$

3. 位移、速度、加速度

$$\begin{aligned} \Delta \vec r &= \vec r(t+\Delta t) - \vec r(t) \\ &= \left(x(t+\Delta t)\,\vec i + y(t+\Delta t)\,\vec j + z(t+\Delta t)\,\vec k\right) - \left(x(t)\,\vec i + y(t)\,\vec j + z(t)\,\vec k\right) \\ &= \Delta x\,\vec i + \Delta y\,\vec j + \Delta z\,\vec k \end{aligned}$$

$|\Delta\vec{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$

$\boxed{\text{NOTE:}}$ $\Delta r = r(t+\Delta t) - r(t) \neq |\Delta\vec{r}|$.

e.g., 以原点为圆心的圆周运动 $\Delta r = 0$，但 $|\Delta\vec{r}| \neq 0$ (unless go back to the same point).

$\boxed{\text{平均速度}}$ $\bar{v} = \frac{\Delta\vec{r}}{\Delta t}$

平均速度在 $\Delta t \to 0$ 时称为$\boxed{\text{瞬时速度}}$，简称$\boxed{\text{速度}}$：

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$v = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right| = \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t}$$

$\boxed{\text{速率}}$

质点运动所经过的路线叫$\boxed{\text{轨道}}$，在一段时间内沿轨道经过的距离叫$\boxed{\text{路程}}$

用 Δs 表示在 Δt 时间内质点沿轨道所经过的路程，则当 $\Delta t \to 0$ 时

$|\Delta\vec{r}| = \Delta s$.

$$\Rightarrow \quad v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

在笛卡尔坐标系下，

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$
$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

同理，平均加速度为

$$\overline{a} = \frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}$$

瞬时加速度即加速度为

$$\vec{a} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

在笛卡尔坐标下，

$$\begin{aligned}\vec{a} &= \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k} \\ &= a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \\ &= \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}\end{aligned}$$

$$a = |\vec{a}| = \sqrt{a_x^2+a_y^2+a_z^2}$$

↑
加速度的大小.

例 1.1 P_{21}	See my mathematica code "example 1.1.nb".
	这是一道经典的一维运动位置 速度 加速度的题目

注：在学了冲量后，此题可从反过来做：

对火箭进行受力分析.

$$F_{\uparrow} = \frac{dP}{dt} = \alpha u$$ （注：运用冲量定理和牛顿第三定律）.

$$(M_0-\alpha t)g$$

$$\Rightarrow (M_0-\alpha t)a = \alpha u - (M_0-\alpha t)g$$

$$\Rightarrow a = \frac{\alpha u}{M_0-\alpha t} - g$$

From $\int_0^v dv' = \int_0^t a\,dt' \Rightarrow v - 0 = \int_0^t \left(\frac{\alpha u}{M_0-\alpha t'} - g\right)dt'$

$$= u\ln\left(\frac{M_0}{M_0-\alpha t}\right) - gt$$

$$\Rightarrow v = u\ln\left(\frac{M_0}{M_0-\alpha t}\right) - gt$$

From $\int_0^z dz' = \int_0^t v\,dt' \Rightarrow z - 0 = \int_0^t \left[u \ln\left(\frac{M_0}{M_0-\alpha t'}\right) - g t'\right] dt'$

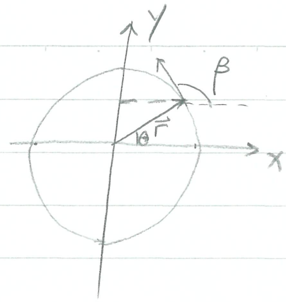
$$\Rightarrow z = ut\ln M_0 - u\underbrace{\int_0^t \ln(M_0-\alpha t')\,dt'}_{\underbrace{\frac{1}{\alpha}\int_0^t \ln(M_0-\alpha t')\,d(\alpha t')}_{\underbrace{\frac{1}{\alpha}\int_t^0 \ln(M_0-\alpha t')\,d(-\alpha t')}_{\underbrace{\frac{1}{\alpha}\int_{M_0-\alpha t}^{M_0} \ln x\,dx}_{\underbrace{\frac{1}{\alpha}\left(-x + x\ln x\right)\Big|_{M_0-\alpha t}^{M_0}}_{\underbrace{\frac{1}{\alpha}\left(-M_0 + M_0-\alpha t + M_0\ln M_0 - (M_0-\alpha t)\ln(M_0-\alpha t)\right)}_{\frac{1}{\alpha}\left(-\alpha t + M_0\ln\frac{M_0}{M_0-\alpha t} + \alpha t\ln(M_0-\alpha t)\right)}}}}}} - \frac{1}{2}gt^2$$

$$= ut\ln M_0 + ut - \frac{uM_0}{\alpha}\left(\ln\frac{M_0}{M_0-\alpha t} - ut\ln(M_0-\alpha\alpha)\right)$$
$$-\frac{1}{2}gt^2$$

$$= ut\ln\frac{M_0}{M_0-\alpha t} + ut\left(1 - \frac{M_0}{\alpha t}\ln\frac{M_0}{M_0-\alpha t}\right) - \frac{1}{2}gt^2$$

$$= ut\left[1 + \left(1 - \frac{M_0}{\alpha t}\right)\ln\left(\frac{M_0}{M_0-\alpha t}\right)\right] - \frac{1}{2}gt^2.$$

例 1.2 P_{22}



直接用位矢、速度和加速度的关系公式.

$$\vec{r} = x\vec{i} + y\vec{j} = R\cos\omega t\,\vec{i} + R\sin\omega t\,\vec{j}$$

$$\Rightarrow |\vec{r}| = [(R\cos\omega t)^2 + (R\sin\omega t)^2]^{\frac{1}{2}} = R$$

注意矢量方向的表达

✓ $\tan\theta = \frac{y}{x} = \tan\omega t. \Rightarrow \theta = \omega t.$ 这是位矢和 x 轴的夹角.

$$\vec{v} = \frac{d\vec{r}}{dt} = -R\omega\sin\omega t\,\vec{i} + R\omega\cos\omega t\,\vec{j} = v_x\vec{i} + v_y\vec{j}$$

$|\vec{v}| = R\omega,$

✓ $\tan\beta = \frac{v_y}{v_x} = -\cot\omega t = \tan(\frac{\pi}{2} + \omega t) = \tan(\frac{\pi}{2}+\theta) \Rightarrow \beta = \frac{\pi}{2}+\theta$

这是速度矢量和 x 轴的夹角.

$$\vec{a} = \frac{d\vec{v}}{dt} = -R\omega^2\cos\omega t\,\vec{i} - R\omega^2\sin\omega t\,\vec{j} = a_x\vec{i} + a_y\vec{j}$$

$$= -\omega^2\vec{r}$$

所以加速度方向和位矢方向相反.

大小为 $|\vec{a}| = R\omega^2$

4. 匀加速运动. ($\vec{a} = constant$)

$$\vec{a} = \frac{d\vec{v}}{dt} \xRightarrow{\vec{a} = constant} \vec{v} = \vec{v}_0 + \vec{a}t \Rightarrow \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

在笛卡尔坐标系下, $a_x = constant$, $a_y = constant$, $a_z = constant$.

$$\begin{cases} v_x = v_{x_0} + a_x t \\ v_y = v_{y_0} + a_y t \\ v_z = v_{z_0} + a_z t \end{cases} \qquad \begin{cases} x = x_0 + v_{x_0} t + \frac{1}{2}a_x t^2 \\ y = y_0 + v_{y_0} t + \frac{1}{2}a_y t^2 \\ z = z_0 + v_{z_0} t + \frac{1}{2}a_z t^2 \end{cases}$$

注：不用分别记忆各情况的公式，只需记住匀加速运动的公式就

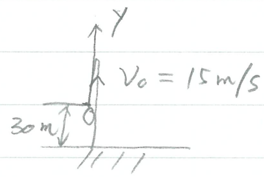
两种常见的匀加速运动：①自由落地运动（竖直上抛、竖直下抛运动）

②抛体运动（斜抛、平抛）

注：①是②的特殊情况

解题方法： [1]通常以抛出点为原点建立坐标系。建议选择x方向为水平方向，y方向为竖直方向。[2]画图 [3]在各方向列出位矢、速度、加速度公式。

例1.3 P26



$$a = -g = -9.80\,m/s^2$$

$$v = v_0 + at = 15m/s - 9.80\,m/s^2 \cdot t$$

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

$$= v_0 t + \frac{1}{2}at^2 = 15m/s\cdot t - \frac{1}{2}\times 9.80m/s^2\, t^2$$

(1) 在最大高度时，$v = 0 \Rightarrow t = \frac{v_0}{g} = \frac{15m/s}{9.80m/s^2}$

$$\Rightarrow y_{max} = 15\,\frac{15}{9.80} - \frac{1}{2}\times 9.80\left(\frac{15}{9.80}\right)^2$$

$$\approx \boxed{11.5\,m}$$

(2) 回落到海面即 $y = -30m$

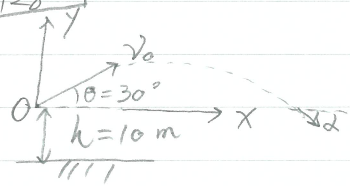
$$\Rightarrow -30 = 15t - \frac{1}{2}\times 9.80t^2$$

$\Rightarrow \boxed{t = 4.44 sec}$ or $t = -1.38$ sec (drop it).

(3) 此时速度为 $v = 15 - 9.80\times 4.44 = \boxed{-28.5\,m/s}$

负号意为速度方向向下.

例1.4. P28



$$\begin{cases} a_x = 0 \\ a_y = -g = -9.8\,m/s^2 \end{cases}$$

$$\begin{cases} v_x = v_0\cos\theta + a_x t = v_0\cos\theta = 20m/s\cos 30° \\ v_y = v_0\sin\theta + a_y t = 20m/s\sin 30° - 9.8m/s^2\cdot t \end{cases}$$

$$\begin{cases} x = x_0 + v_0\cos\theta\cdot t + \frac{1}{2}a_x t^2 = 0 + v_0\cos\theta\cdot t + 0 = 20m/s\cdot\cos 30°\, t \\ y = y_0 + v_0\sin\theta\, t + \frac{1}{2}a_y t^2 = 0 + v_0\sin\theta\, t - \frac{1}{2}gt^2 = 20m/s\,\sin 30°\, t - \frac{1}{2}\times 9.8m/s^2\, t^2 \end{cases}$$

(1) 当球着地时, $y = -10m$

$$\Rightarrow -10m = 20m/s \sin 30^\circ t - \frac{1}{2} \times 9.8 m/s^2 t^2$$

$$\Rightarrow \boxed{t \doteq 2.78 sec} \quad or \quad t \doteq -0.74 sec \text{ (drop it)}$$

(2) 此时 $x = 20 m/s \cdot \cos 30^\circ \times 2.78 sec \doteq \boxed{48.1 m}$

(3) 此时 $v_y = 20m/s \sin 30^\circ - 9.8m/s^2 \times 2.78 sec$

$$\doteq -17.2 m/s$$

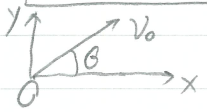
$$\Rightarrow \tan\alpha = \frac{v_y}{v_x} = \frac{-17.2m/s}{20m/s \cos 30^\circ} \simeq -0.99$$

$$\Rightarrow \alpha = \arctan(-0.99) \simeq \boxed{-44.8^\circ}$$

$$v = (v_x^2 + v_y^2)^{\frac{1}{2}} = \boxed{24.4 m/s}$$

即球以 24.4 m/s 的速度落地, 与地面的夹角为 -44.8°.

Common sense: 在平地上以何角度投球能最远 (给定初速度)



$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases} \qquad \begin{cases} v_x = v_0 \cos\theta + a_x t = v_0 \cos\theta \\ v_y = v_0 \sin\theta + a_y t = v_0 \sin\theta - gt \end{cases}$$

$$\begin{cases} x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 = v_0 \cos\theta\, t \\ y = y_0 + v_{y_0} t + \frac{1}{2} a_y t^2 = v_0 \sin\theta\, t - \frac{1}{2} g t^2 \end{cases}$$

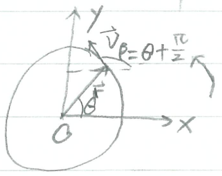
落地时 $y = 0 \Rightarrow t = \frac{2 v_0 \sin\theta}{g}$

$$\Rightarrow x = v_0 \cos\theta \frac{2 v_0 \sin\theta}{g} = \frac{v_0^2}{g} \sin(2\theta)$$

So, when $\theta = 45^\circ$, x is maximize, which is $\frac{v_0^2}{g}$

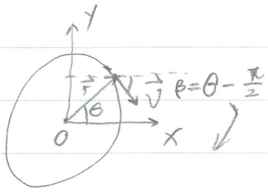
5. 圆周运动.

$$\vec{r} = x\vec{i} + y\vec{j} = R\cos\theta\,\vec{i} + R\sin\theta\,\vec{j}.$$



$$\Rightarrow \frac{y}{x} = \tan\theta, \quad |\vec{r}| = R$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -R\sin\theta\frac{d\theta}{dt}\vec{i} + R\cos\theta\frac{d\theta}{dt}\vec{j} = v_x\vec{i} + v_y\vec{j}$$



$$\Rightarrow \frac{v_y}{v_x} = -\cot\theta \equiv \tan\beta \Rightarrow \begin{cases}\beta=\theta+\frac{\pi}{2}, \text{逆时针转} \\ \beta=\theta-\frac{\pi}{2}, \text{顺时针转}\end{cases}$$

$$v \equiv |\vec{v}| = R\left|\frac{d\theta}{dt}\right| = R|\omega| \qquad \boxed{\text{NOTE}: \omega \equiv \frac{d\theta}{dt}}$$

速度沿切向方向。

$$\Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = \left(-R\cos\theta\left(\frac{d\theta}{dt}\right)^2 - R\sin\theta\frac{d^2\theta}{dt^2}\right)\vec{i} + \left(-R\sin\theta\left(\frac{d\theta}{dt}\right)^2 + R\cos\theta\frac{d^2\theta}{dt^2}\right)\vec{j}$$

$$= a_x\vec{i} + a_y\vec{j}$$

$$= -\left(\frac{d\theta}{dt}\right)^2\vec{r} + \vec{v}\left(\frac{d^2\theta}{dt^2}\right)\Big/\left(\frac{d\theta}{dt}\right)$$

$$\equiv \vec{a}_n + \vec{a}_t$$

$$\Rightarrow a_t = \left|\vec{v}\frac{d^2\theta}{dt^2}\Big/\left(\frac{d\theta}{dt}\right)\right| = R\left|\frac{d^2\theta}{dt^2}\right| = R|\alpha| \qquad \boxed{\text{NOTE}: \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$

$$a_n = \left|-\left(\frac{d\theta}{dt}\right)^2\vec{r}\right| = \omega^2 R = \frac{v^2}{R}$$

切向加速度的方向与速度相同(加速时)或相反(减速时)

法向加速度的方向指向圆心.

$$|\vec{a}| = (a_t^2 + a_n^2)^{\frac{1}{2}} = \left(R^2\alpha^2 + \frac{v^4}{R^2}\right)^{\frac{1}{2}}$$

$$\frac{a_t}{a_n} = \frac{R|\alpha|}{\frac{v^2}{R}} = \frac{R^2|\alpha|}{v^2}.$$

> 对于 $\alpha =$ constant 的情况,
>
> $\omega = \omega_0 + \alpha t$
>
> $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$

例 1.6 P33

$\alpha = \text{constant}$, α is negative.

$\omega = \omega_0 + \alpha t$

(1) $\omega_0 = \frac{180\times 2\pi}{60\,\text{sec}} = 6\pi/\text{sec} \simeq \boxed{18.8/\text{sec}} \Rightarrow V_0 = \omega_0 R = 6\pi \times 0.5 \simeq \boxed{9.4\,m/}$

当 $t = 1.5\,\text{min}$ 时，$\omega = 0 \Rightarrow \alpha = \frac{-\omega_0}{1.5\,\text{min}} = -\frac{6\pi/\text{sec}}{90\,\text{sec}} = -\frac{\pi}{15}/\text{sec}^2$.

(2) $t = 80\,\text{sec}$ 时仍在转，所以

$$\alpha = -\frac{\pi}{15}/\text{sec}^2 \simeq \boxed{-0.209/\text{sec}^2}$$

负号表示减速.

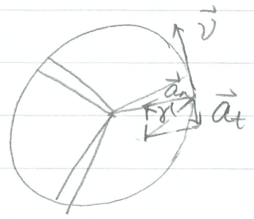
$a_t = |\alpha| R \doteq \boxed{0.105\,m/s^2}$，方向与速度方向相反.

$a_n = \omega^2 R = (\omega_0 + \alpha t)^2 R = \left(\frac{180\times 2\pi}{60\,\text{sec}} - \frac{\pi}{15}\times 80\right)^2 \times 0.5 \simeq \boxed{2.19\,m/s}$

方向指向节筒中心.

$|\vec{a}| = (a_t^2 + a_n^2)^{\frac{1}{2}} \doteq \boxed{2.20\,m/s^2}$,

$\arctan\left(\frac{a_t}{a_n}\right) \simeq \boxed{2.73^\circ = \gamma}$



6. 相对运动.

设有两个相对作平动的参考物 O 和 O′（即以此两个参考物建立坐标系（Oxyz）和（O′x′y′z′），则一物体 A 在这两个参考系中的位矢 $\vec{r}_{OA}$ 和 $\vec{r}_{O'A}$ 有如下关系，$\underbrace{\vec{r}_{OA}}_{\text{A相对于O的位置}} = \vec{r}_{OO'} + \vec{r}_{O'A}$，其中 $\vec{r}_{OO'}$ 为有向线段 OO′ 的位矢.

$$\Rightarrow \frac{d\vec{r}_{OA}}{dt} = \frac{d\vec{r}_{OO'}}{dt} + \frac{d\vec{r}_{O'A}}{dt}$$

即 $\vec{v}_{OA} = \vec{v}_{OO'} + \vec{v}_{O'A}$

$$\frac{d\vec{v}_{OA}}{dt} = \frac{d\vec{v}_{OO'}}{dt} + \frac{d\vec{v}_{O'A}}{dt}$$

即 $\vec{a}_{OA} = \vec{a}_{OO'} + \vec{a}_{O'A}$

NOTE: $\vec{r}_{AB} = -\vec{r}_{BA}$
$\vec{v}_{AB} = -\vec{v}_{BA}$
$\vec{a}_{AB} = -\vec{a}_{BA}$

当建立公共的 x,y,z 方向后，分量关系也成立，所

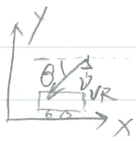
$x_{OA} = x_{OO'} + x_{O'A}$

$v_{x_{OA}} = v_{x_{OO'}} + v_{x_{O'A}}$

$a_{x_{OA}} = a_{x_{OO'}} + a_{x_{O'A}}$

同理 for y and z directions

例 1.7 P_{35}



雨滴相对于车的速度

$$\vec{v}_{VR} = \vec{v}_{VE} + \vec{v}_{ER} = -\vec{v}_{EV} + \vec{v}_{ER}$$

$$\Rightarrow \begin{cases} v_{x_{VR}} = -v_{x_{EV}} + v_{x_{ER}} = -20 + 0 = -20\ m/s \\ v_{y_{VR}} = -v_{y_{EV}} + v_{y_{ER}} = 0 - 10 = -10\ m/s \end{cases}$$

$$\Rightarrow |\vec{v}_{VR}| = (v_{x_{VR}}^2 + v_{y_{VR}}^2)^{\frac{1}{2}} = 10\sqrt{5} \simeq \boxed{22.4\ m/s}$$

$$\frac{v_{y_{VR}}}{v_{x_{VR}}} = \frac{-10}{-20} = \frac{1}{2} \Rightarrow \theta = \arctan\left(\frac{v_{y_{VR}}}{v_{x_{VR}}}\right) = \arctan\left(\frac{1}{2}\right) = \boxed{26.6^\circ}$$

即雨滴以向下偏西 $(180^\circ - \theta) = 63.4^\circ$ 落向车厢。