more Example of Noether theorem

1 Lorentz transformation of a complex scalar field.

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^*$$

use $\delta x^\mu = \varepsilon^\mu{}_\nu x^\nu = \varepsilon^{\mu\nu} x_\nu$, $\quad \delta_0 \phi(x) = -\frac{i}{2} \varepsilon^{\mu\nu} L_{\mu\nu} \phi(x)$

$$\delta_0 \phi^*(x) = -\frac{i}{2} \varepsilon^{\mu\nu} L_{\mu\nu} \phi^*(x)$$

where $L_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

let $\delta \omega \equiv \varepsilon^{\rho\sigma}$

$\Rightarrow$

$$\frac{\delta x^\mu}{\delta \omega} = \delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho$$

$$\frac{\delta_0 \phi(x)}{\delta \omega} = (x_\rho \partial_\sigma - x_\sigma \partial_\rho) \phi(x)$$

$$\frac{\delta_0 \phi^*(x)}{\delta \omega} = (x_\rho \partial_\sigma - x_\sigma \partial_\rho) \phi^*(x)$$

$$\Rightarrow \quad j^\mu{}_{\rho\sigma} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (x_\rho \partial_\sigma - x_\sigma \partial_\rho) \phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} (x_\rho \partial_\sigma - x_\sigma \partial_\rho) \phi^*(x) + \mathcal{L} (\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho)$$

Since the energy momentum tensor $T^\mu{}_\nu$ introduced by Lorentz translation is $T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \partial_\nu \phi^* - \delta^\mu{}_\nu \mathcal{L}$,

then $j^\mu{}_{\rho\sigma} = x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho$

and $Q_{\rho\sigma} = \int d^3\vec{x} \, j^0{}_{\rho\sigma} = \int d^3\vec{x} \, (x_\rho T^0{}_\sigma - x_\sigma T^0{}_\rho)$

where $T^0{}_\sigma = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_\sigma \phi + \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} \partial_\sigma \phi^* - \delta^0{}_\sigma \mathcal{L}$

The spatial components are

$Q_{ij} = \int d^3\vec{x} \, (x_i P_j - x_j P_i)$, $\left(\frac{dQ_{ij}}{dt} = 0 \right.$ is angular momentum conservation) (orbital only)

where $P_i = T^0{}_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_i \phi + \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} \partial_i \phi^*$.

2 Lorentz transformation of a Dirac field.

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi$$

use $\delta x^\mu = \varepsilon^{\mu\nu} x_\nu$,

$$\delta_0 \psi(x) = -\frac{i}{2}\varepsilon^{\mu\nu}\left(L_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}\right)\psi(x)$$

let $\delta\omega \equiv \varepsilon^{\rho\sigma}$

$$\Rightarrow \quad \frac{\delta x^\mu}{\delta\omega} = \delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho$$

$$\frac{\delta_0\psi(x)}{\delta\omega} = -i\left(L_{\rho\sigma} + \frac{1}{2}\sigma_{\rho\sigma}\right)\psi(x)$$

$$= (x_\rho\partial_\sigma - x_\sigma\partial_\rho)\psi(x) - \frac{i}{2}\sigma_{\rho\sigma}\psi(x)$$

$$\Rightarrow j^\mu{}_{\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\left(x_\rho\partial_\sigma - x_\sigma\partial_\rho - \frac{i}{2}\sigma_{\rho\sigma}\right)\psi(x)$$

$$+ \underbrace{\cancel{\mathcal{L}}}_{\to 0}\left(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho\right)$$

since the ψ in $\mathcal{L}$ satisfies the Dirac equation (i.e., on-shell) so that the second term vanishes.

Again, since the energy momentum tensor $T^\mu{}_\nu$ introduced by Lorentz translation is $T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi - \delta^\mu{}_\nu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi$

then
$$j^\mu{}_{\rho\sigma} = x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho - \frac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\sigma_{\rho\sigma}\psi(x)$$

$$= x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho - \frac{i}{2}\bar{\psi} i\gamma^\mu\sigma_{\rho\sigma}\psi(x)$$

and $Q_{\rho\sigma} = \int d^3\vec{x}\, j^0{}_{\rho\sigma} = \int d^3\vec{x}\left(x_\rho T^0{}_\sigma - x_\sigma T^0{}_\rho\right) + \int d^3\vec{x}\, \psi^\dagger(x)\frac{\sigma_{\rho\sigma}}{2}\psi(x)$

The spatial components are

$$Q_{ij} = \int d^3\vec{x}\,(x_i P_j - x_j P_i) + \int d^3\vec{x}\,\psi^\dagger(x)\frac{\sigma_{ij}}{2}\psi(x),$$

where $P_j = T^0{}_j = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\partial_j\psi = \bar{\psi} i\gamma^0\partial_j\psi = i\psi^\dagger(x)\partial_j\psi(x)$

2 (checked)

$\frac{dQ_{ij}}{dt}=0$ is total angular momentum conservation, where the first term is the orbital part and the second term is spin part.

Therefore, the spin operator for Dirac field is

$$\int d^3\vec{x}\, \psi^\dagger(x) \frac{\sigma_{ij}}{2} \psi(x)$$

Define $\vec{\sigma}_{4\times4} \equiv (\sigma_{23}, \sigma_{31}, \sigma_{12}) = (\sigma^{23}, \sigma^{31}, \sigma^{12})$

then $\hat{S}^k \equiv \frac{1}{2}\varepsilon^{ijk} \int d^3\vec{x}\, \psi^\dagger(x) \frac{\sigma_{ij}}{2} \psi(x)$

that is

$$\hat{S}^1 = \int d^3\vec{x}\, \psi^\dagger(x) \frac{\sigma_{23}}{2} \psi(x)$$

$$\hat{S}^2 = \int d^3\vec{x}\, \psi^\dagger(x) \frac{\sigma_{31}}{2} \psi(x)$$

$$\hat{S}^3 = \int d^3\vec{x}\, \psi^\dagger(x) \frac{\sigma_{12}}{2} \psi(x)$$

$$\Rightarrow [\hat{S}^i, \hat{S}^j] = \frac{1}{2}\varepsilon^{mni} \frac{1}{2}\varepsilon^{efj} \Big[\int d^3\vec{x}\, \psi^\dagger(x) \frac{\sigma_{mn}}{2} \psi(x), \int d^3\vec{y}\, \psi^\dagger(y) \frac{\sigma_{ef}}{2} \psi(y) \Big]$$

(note that $\psi(x) = \psi(\vec{x}, t)$, $\psi(y) = \psi(\vec{y}, t)$)

$$= \frac{1}{2}\varepsilon^{mni} \frac{1}{2}\varepsilon^{efj} \int d^3\vec{x} \int d^3\vec{y}\, \frac{(\sigma_{mn})_{ab}}{2} \frac{(\sigma_{ef})_{cd}}{2} \Big(\psi_a^\dagger(x)\psi_b(x)\psi_c^\dagger(y)\psi_d(y) - \psi_c^\dagger(y)\psi_d(y)\psi_a^\dagger(x)\psi_b(x) \Big)$$

where $\psi_a^\dagger(x) \underline{\psi_b(x)\psi_c^\dagger(y)} \psi_d(y)$

$$= -\psi_a^\dagger(x)\psi_c^\dagger(y)\psi_b(x)\psi_d(y) + \delta_{bc}\delta^3(\vec{x}-\vec{y})\psi_a^\dagger(x)\psi_d(y)$$

$$= \psi_c^\dagger(y)\psi_a^\dagger(x)\psi_b(x)\psi_d(y) + \delta_{bc}\delta^3(\vec{x}-\vec{y})\psi_a^\dagger(x)\psi_d(y)$$

$$= -\psi_c^\dagger(y)\psi_a^\dagger(x)\psi_d(y)\psi_b(x) + \delta_{bc}\delta^3(\vec{x}-\vec{y})\psi_a^\dagger(x)\psi_d(y)$$

$$= \psi_c^\dagger(y)\psi_d(y)\psi_a^\dagger(x)\psi_b(x) - \delta_{ad}\delta^3(\vec{x}-\vec{y})\psi_c^\dagger(y)\psi_b(x) + \delta_{bc}\delta^3(\vec{x}-\vec{y})\psi_a^\dagger(x)\psi_d(y)$$

 (checked)

$$\Rightarrow [\hat{S}^i, \hat{S}^j] = \frac{1}{2}\varepsilon^{mni}\frac{1}{2}\varepsilon^{efj}\int d^3\vec{x}\, \frac{(\sigma_{mn})_{ab}}{2}\frac{(\sigma_{ef})_{cd}}{2}\left(\psi_a^+(x)\psi_d(x)\delta_{bc} - \psi_c^+(x)\psi_b(x)\delta_{ad}\right)$$

$$= \frac{1}{2}\varepsilon^{mni}\frac{1}{2}\varepsilon^{efj}\int d^3\vec{x}\,\psi_a^+(x)\left[\frac{\sigma_{mn}}{2}, \frac{\sigma_{ef}}{2}\right]_{ad}\psi_d(x)$$

Since $\frac{1}{2}\varepsilon^{mn1}\frac{1}{2}\varepsilon^{ef2}\left[\frac{\sigma_{mn}}{2}, \frac{\sigma_{ef}}{2}\right]$

$$= \left[\frac{\sigma_{23}}{2}, \frac{\sigma_{31}}{2}\right] = \frac{i}{2}\sigma_{12} = \frac{i}{2}\varepsilon^{ij3}\frac{\sigma_{ij}}{2}$$

$\frac{1}{2}\varepsilon^{mn2}\frac{1}{2}\varepsilon^{ef3}\left[\frac{\sigma_{mn}}{2}, \frac{\sigma_{ef}}{2}\right]$

$$= \left[\frac{\sigma_{31}}{2}, \frac{\sigma_{12}}{2}\right] = \frac{i}{2}\sigma_{23} = \frac{i}{2}\varepsilon^{ij1}\frac{\sigma_{ij}}{2}$$

$\frac{1}{2}\varepsilon^{mn3}\frac{1}{2}\varepsilon^{ef1}\left[\frac{\sigma_{mn}}{2}, \frac{\sigma_{ef}}{2}\right]$

$$= \left[\frac{\sigma_{12}}{2}, \frac{\sigma_{23}}{2}\right] = \frac{i}{2}\sigma_{31} = \frac{i}{2}\varepsilon^{ij2}\frac{\sigma_{ij}}{2}$$

$$\frac{1}{2}\varepsilon^{mni}\frac{1}{2}\varepsilon^{efj}\left[\frac{\sigma_{mn}}{2}, \frac{\sigma_{ef}}{2}\right] = \frac{1}{2}\varepsilon^{efi}\frac{1}{2}\varepsilon^{mnj}\left[\frac{\sigma_{ef}}{2}, \frac{\sigma_{mn}}{2}\right]$$

$$= -\frac{1}{2}\varepsilon^{mnj}\frac{1}{2}\varepsilon^{efi}\left[\frac{\sigma_{mn}}{2}, \frac{\sigma_{ef}}{2}\right]$$

$$\Rightarrow [\hat{S}^i, \hat{S}^j] = i\varepsilon^{ijk}\hat{S}^k$$

Define the longitudinal spin operator

$$\hat{S}_{\vec{p}} \equiv \frac{\hat{\vec{S}}\cdot\vec{p}}{|\vec{p}|} = \int d^3\vec{x}\,\psi^+(x)\frac{\vec{\sigma}_{4\times4}\cdot\vec{p}}{2|\vec{p}|}\psi(x) \equiv \frac{1}{2}\int d^3\vec{x}\,\psi^+(x)\,\sigma_{\vec{p}_{4\times4}}\,\psi(x)$$

where $\sigma_{\vec{p}_{4\times4}} \equiv \frac{\vec{\sigma}_{4\times4}\cdot\vec{p}}{|\vec{p}|}$. $\Rightarrow \sigma_{-\vec{p}_{4\times4}} = -\sigma_{\vec{p}_{4\times4}}$

Furthermore, we choose $u(\vec{p},r)$ and $v(\vec{p},r)$ in the decomposition of $\psi(x)$ such that $\sigma_{\vec{p}_{4\times4}}\,u(\vec{p},r) = (-1)^{r+1}u(\vec{p},r)$ and $\sigma_{\vec{p}_{4\times4}}\,v(\vec{p},r) = (-1)^r v(\vec{p},r)$,

where $r=1,2$.

$$\Rightarrow \quad \hat{S}_{\vec{p}} = \frac{1}{2}\int d^3\vec{x} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left(u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} + v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x} \right)$$

$$(\sigma_{\vec{p}_{4\times4}}) \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_r \left(u(\vec{k},r)\, b_{\vec{k},r}\, e^{-ik\cdot x} + v(\vec{k},r)\, d^+_{\vec{k},r}\, e^{ik\cdot x} \right)$$

$$= \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\, C(E_{\vec{p}}) \sum_{s,r} \Big[u^+(\vec{p},s)\, \sigma_{\vec{p}_{4\times4}}\, u(\vec{p},r)\, b^+_{\vec{p},s}\, b_{\vec{p},r}$$

$$+ v^+(\vec{p},s)\, \sigma_{\vec{p}_{4\times4}}\, v(\vec{p},r)\, d_{\vec{p},s}\, d^+_{\vec{p},r}$$

$$+ u^+(-\vec{p},s)\, \sigma_{\vec{p}_{4\times4}}\, v(-\vec{p},r)\, b^+_{\vec{p},s}\, d^+_{-\vec{p},r}\, e^{2iE_{\vec{p}}t}$$

$$+ v^+(\vec{p},s)\, \sigma_{\vec{p}_{4\times4}}\, u(-\vec{p},r)\, d_{\vec{p},s}\, b_{-\vec{p},r}\, e^{-2iE_{\vec{p}}t} \Big]$$

$$= \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, [C(E_{\vec{p}})]^2 \sum_{s,r} \Big[u^+(\vec{p},s)\, (-1)^{r+1}\, u(\vec{p},r)\, b^+_{\vec{p},s}\, b_{\vec{p},r}$$

$$+ v^+(\vec{p},s)\, (-1)^r\, v(\vec{p},r)\, d_{\vec{p},s}\, d^+_{\vec{p},r}$$

$$+ u^+(-\vec{p},s)\, (-\sigma_{\vec{p}_{4\times4}})\, v(\vec{p},r)\, b^+_{-\vec{p},s}\, d^+_{\vec{p},r}\, e^{2iE_{\vec{p}}t}$$

$$+ v^+(-\vec{p},s)\, (-\sigma_{\vec{p}_{4\times4}})\, u(\vec{p},r)\, d_{-\vec{p},s}\, b_{\vec{p},r}\, e^{-2iE_{\vec{p}}t} \Big]$$

$$= \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, [C(E_{\vec{p}})]^2 \sum_{s,r} \Big[(-1)^{r+1}\, 2E_{\vec{p}}\, \delta_{sr}\, b^+_{\vec{p},s}\, b_{\vec{p},r}$$

$$+ (-1)^r\, 2E_{\vec{p}}\, \delta_{sr}\, d_{\vec{p},s}\, d^+_{\vec{p},r} \Big]$$

$$\Rightarrow\ :\hat{S}_{\vec{p}}: = \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, [C(E_{\vec{p}})]^2\, 2E_{\vec{p}} \sum_s (-1)^{s+1} \left(b^+_{\vec{p},s}\, b_{\vec{p},s} + d^+_{\vec{p},s}\, d_{\vec{p},s} \right)$$

$$\Rightarrow \quad :\hat{S}_{\vec{p}}:\, b^+_{\vec{p},r}|0\rangle = \frac{1}{2}(-1)^{r+1}\, b^+_{\vec{p},r}|0\rangle$$

$$:\hat{S}_{\vec{p}}:\, d^+_{\vec{p},r}|0\rangle = \frac{1}{2}(-1)^{r+1}\, d^+_{\vec{p},r}|0\rangle$$

Therefore, $b^+_{\vec{p},1}|0\rangle$ and $d^+_{\vec{p},1}|0\rangle$ creates spin $+\frac{1}{2}$ states in the $\vec{p}$ direction; $b^+_{\vec{p},2}|0\rangle$ and $d^+_{\vec{p},2}|0\rangle$ - - - - - - - - $-\frac{1}{2}$ - - - - - - - - - - - -. So, $:\hat{S}_{p}:$ is actually the helicity operator of a spin $\frac{1}{2}$ particle with momentum $\vec{p}$.

(c) (checked)

Now let's find $u(\vec{p},r)$ and $v(\vec{p},r)$ in the Standard representation satisfying $\sigma_{\vec{p}_{4\times4}} u(\vec{p},r) = (-1)^{r+1} u(\vec{p},r)$ and $\sigma_{\vec{p}_{4\times4}} v(\vec{p},r) = (-1)^{r} v(\vec{p},r)$

Since $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$,

then
$$\sigma_{\substack{ij \\ i\neq j}} = \frac{i}{2}[\gamma^i, \gamma^j] = i\gamma^i\gamma^j = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$
$$= i\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\Rightarrow \quad \sigma_{\vec{p}_{4\times4}} = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix} = \begin{pmatrix} \vec{\sigma}\cdot\hat{n} & 0 \\ 0 & \vec{\sigma}\cdot\hat{n} \end{pmatrix}$$

where $\hat{n} = \frac{\vec{p}}{|\vec{p}|} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$, is the direction vector of $\vec{p}$.

$$\Rightarrow \quad \sigma_{\vec{p}_{4\times4}} = \begin{pmatrix} \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi & 0 & 0 \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ 0 & 0 & \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta \end{pmatrix}$$

From $u(\vec{p},r) = (E+m)^{\frac{1}{2}}\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \end{pmatrix}$

and $\sigma_{\vec{p}_{4\times4}} u(\vec{p},r) = (-1)^{r+1} u(\vec{p},r)$

$$\Rightarrow \quad \vec{\sigma}\cdot\hat{n}\,\chi_r = (-1)^{r+1}\chi_r$$

Therefore, we just need to find the eigenfunction χ_r with the eigenvalue $+1$ for χ_1 and -1 for χ_2.

from
$$\begin{pmatrix} \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta \end{pmatrix}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\chi_1} = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\chi_1}$$

and $|a|^2+|b|^2 = 1$ (since $\chi_1^{\dagger}\chi_1 = 1$)

$$\Rightarrow \begin{cases} a\cos\theta + b\sin\theta\, e^{-i\varphi} = a & \Rightarrow\ b = \frac{a(1-\cos\theta)}{\sin\theta} e^{i\varphi} \\ a\sin\theta\, e^{i\varphi} - b\cos\theta = b & \Rightarrow\ b = \frac{a\sin\theta\, e^{i\varphi}}{1+\cos\theta} = \frac{a\sin\theta(1-\cos\theta)}{1-\cos^2\theta} e^{i\varphi} \\ |a|^2 + |b|^2 = 1 & \qquad = \frac{a(1-\cos\theta)}{\sin\theta} e^{i\varphi} \end{cases}$$

$$\Rightarrow |a|^2 \frac{(1-\cos\theta)^2}{\sin^2\theta} + |a|^2 = 1$$

$$\Rightarrow |a|^2 \frac{1+\cos^2\theta - 2\cos\theta + \sin^2\theta}{\sin^2\theta} = 1$$

$$\Rightarrow |a|^2 = \frac{\sin^2\theta}{2-2\cos\theta} = \frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \cos^2\frac{\theta}{2}.$$

Since times a common phase $e^{i\alpha}$ to a and b give another solution, (α is real)

we get

$$\begin{cases} a = \cos\frac{\theta}{2} e^{i\alpha} \\ b = \frac{(1-\cos\theta)}{\sin\theta} e^{i\varphi} \cos\frac{\theta}{2} e^{i\alpha} = \sin\frac{\theta}{2} e^{i\varphi} e^{i\alpha} \end{cases}$$

From

$$\begin{pmatrix} \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta \end{pmatrix} \underbrace{\begin{pmatrix} c \\ d \end{pmatrix}}_{\chi_2} = -\begin{pmatrix} c \\ d \end{pmatrix}$$

and $|c|^2 + |d|^2 = 1$ (since $\chi_2^+ \chi_2 = 1$)

Also, since $\chi_1^+ \chi_2 = 0 = \chi_2^+ \chi_1$

we need $a^* c + b^* d = 0$

$$\Rightarrow \begin{cases} c\cos\theta + d\sin\theta\, e^{-i\varphi} = -c & \Rightarrow\ d = -\frac{c(1+\cos\theta)}{\sin\theta} e^{i\varphi} \\ c\sin\theta\, e^{i\varphi} - d\cos\theta = -d & \Rightarrow\ d = \frac{c\sin\theta\, e^{i\varphi}}{\cos\theta - 1} = \frac{c\sin\theta(\cos\theta+1)e^{i\varphi}}{(\cos\theta-1)(\cos\theta+1)} \\ |c|^2 + |d|^2 = 1 & \qquad = -\frac{c(\cos\theta+1)e^{i\varphi}}{\sin\theta} \\ a^* c + b^* d = 0 \end{cases}$$

$$\Rightarrow |c|^2 \frac{(1+\cos\theta)^2}{\sin^2\theta} + |c|^2 = 1$$

$$\Rightarrow |c|^2 \frac{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}{\sin^2\theta} = 1$$

$$\Rightarrow |c|^2 = \frac{\sin^2\theta}{2+2\cos\theta} = \frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{4\cos^2\frac{\theta}{2}} = \sin^2\frac{\theta}{2}$$

 (checked)

$$\Rightarrow \begin{cases} c = \sin\frac{\theta}{2}\, e^{i\beta} & , \ \beta \text{ is real (the common phase)} \\ d = -\dfrac{1+\cos\theta}{\sin\theta}\, c\, e^{i\varphi} = -\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \cdot \sin\frac{\theta}{2}\, e^{i\beta} e^{i\varphi} \\ \quad = -\cos\frac{\theta}{2}\, e^{i\beta} e^{i\varphi} \end{cases}$$

For the requirement $a^* c + b^* d = 0$,

Since $a^*c + b^*d = \cos\frac{\theta}{2}\, e^{-i\alpha} \sin\frac{\theta}{2}\, e^{i\beta} + \sin\frac{\theta}{2}\, e^{-i\varphi} e^{-i\alpha}\left(-\cos\frac{\theta}{2}\, e^{i\beta} e^{i\varphi}\right)$

$$= \cos\frac{\theta}{2}\, e^{-i\alpha} \sin\frac{\theta}{2}\, e^{i\beta}\,(1-1) = 0,$$

then it is automatically satisfied

Therefore $\chi_1 = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\alpha} \\ \sin\frac{\theta}{2}\, e^{i\varphi} e^{i\alpha} \end{pmatrix}$, $\chi_2 = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{i\beta} \\ -\cos\frac{\theta}{2}\, e^{i\varphi} e^{i\beta} \end{pmatrix}$

(check $\sum_s \chi_s \chi_s^\dagger = \mathbb{1}$: $\begin{pmatrix} \cos\frac{\theta}{2} e^{i\alpha} \\ \sin\frac{\theta}{2} e^{i\varphi} e^{i\alpha} \end{pmatrix} \left(\cos\frac{\theta}{2} e^{-i\alpha}, \sin\frac{\theta}{2} e^{-i\varphi} e^{-i\alpha}\right) + \begin{pmatrix} \sin\frac{\theta}{2} e^{i\beta} \\ -\cos\frac{\theta}{2} e^{i\varphi} e^{i\beta} \end{pmatrix} \left(\sin\frac{\theta}{2} e^{-i\beta}, -\cos\frac{\theta}{2} e^{-i\varphi} e^{-i\beta}\right)$
$= \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} e^{-i\varphi} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{i\varphi} & \sin^2\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} \sin^2\frac{\theta}{2} & -\cos\frac{\theta}{2}\sin\frac{\theta}{2} e^{-i\varphi} \\ -\cos\frac{\theta}{2}\sin\frac{\theta}{2} e^{i\varphi} & \cos^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ done the check)

Since $\theta \in [0, \pi)$, and the term such as $\sin\theta$ and $1-\cos\theta$ appear in the denominators in the above calculation, we need to check whether this special case give us any trouble.

check:

When put $\theta = 0$ in the above χ_1 and χ_2, we get

$$\chi_1 = \begin{pmatrix} e^{i\alpha} \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ -e^{i\varphi} e^{i\beta} \end{pmatrix}$$

and $\vec{\sigma}\cdot\hat{n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \vec{\sigma}\cdot\hat{n}\, \chi_1 = \chi_1, \quad \vec{\sigma}\cdot\hat{n}\, \chi_2 = -\chi_2$$

So, the solutions χ_1 and χ_2 are also good for $\theta = 0$.

Similarly, from $v(\vec{p}, r) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \eta_r \\ \eta_r \end{pmatrix}$

and $\sigma_{\hat{p}\,4\times4}\, v(\vec{p}, r) = (-1)^r\, v(\vec{p}, r)$,

$$\Rightarrow \vec{\sigma}\cdot\hat{n}\, \eta_r = (-1)^r \eta_r$$

 (checked)

$$\Rightarrow \quad \eta_1 = \chi_2 e^{i\gamma}, \quad \eta_2 = \chi_1 e^{i\delta}.$$

where γ and δ are real numbers (i.e., phases)

(check $\sum_s \eta_s \eta_s^+ = \mathbb{I}$: $\eta_1\eta_1^+ + \eta_2\eta_2^+ = \chi_1 e^{i\delta}\chi_1^+ e^{-i\delta} + \chi_2 e^{i\gamma}\chi_2^+ e^{-i\gamma} = \sum_s \chi_s \chi_s^+ = \mathbb{I}$, done the check).

To make the analysis of charge conjugation transformation simple, we can choose

$$-i\sigma^2 \chi_s^* = \eta_s, \text{ that is } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi_s^* = \eta_s$$

$$\Rightarrow \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\alpha} \\ \sin\frac{\theta}{2} e^{-i\varphi} e^{-i\alpha} \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2} e^{i\beta} e^{i\gamma} \\ -\cos\frac{\theta}{2} e^{i\varphi} e^{i\beta} e^{i\gamma} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\beta} \\ -\cos\frac{\theta}{2} e^{-i\varphi} e^{-i\beta} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} e^{i\alpha} e^{i\delta} \\ \sin\frac{\theta}{2} e^{i\varphi} e^{i\alpha} e^{i\delta} \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} -\sin\frac{\theta}{2} e^{-i\varphi} e^{-i\alpha} = \sin\frac{\theta}{2} e^{i\beta} e^{i\gamma} \\ \cos\frac{\theta}{2} e^{-i\alpha} = -\cos\frac{\theta}{2} e^{i\varphi} e^{i\beta} e^{i\gamma} \\ \cos\frac{\theta}{2} e^{-i\varphi} e^{-i\beta} = \cos\frac{\theta}{2} e^{i\alpha} e^{i\delta} \\ \sin\frac{\theta}{2} e^{-i\beta} = \sin\frac{\theta}{2} e^{i\varphi} e^{i\alpha} e^{i\delta} \end{cases}$$

$$\Rightarrow \quad \begin{cases} \beta + \gamma + \alpha + \varphi = (2k+1)\pi, & k = 0, \pm 1, \pm 2, \dots \\ \beta + \alpha + \delta + \varphi = 2l\pi, & l = 0, \pm 1, \pm 2, \dots \end{cases}$$

$$\Rightarrow \quad \begin{cases} \gamma - \delta = (2m+1)\pi, & m = 0, \pm 1, \pm 2, \dots \\ \beta + \alpha + \delta + \varphi = 2l\pi, & l = 0, \pm 1, \pm 2, \dots \end{cases}$$

To make the analysis of time inversion transformation simple, we can choose

$$\left\{\begin{matrix} i\sigma^2 \chi_1^* = -\chi_2 \\ i\sigma^2 \chi_2^* = \chi_1 \end{matrix}\right\} \xrightarrow{\text{use } -i\sigma^2\chi_s^* = \eta_s} \begin{cases} -\eta_1 = -\chi_2 \\ -\eta_2 = \chi_1 \end{cases}$$

$$\left\{\begin{matrix} i\sigma^2 \eta_1^* = -\eta_2 \\ i\sigma^2 \eta_2^* = \eta_1 \end{matrix}\right\} \xrightarrow{\text{use } -i\sigma^2\chi_s^* = \eta_s} \begin{cases} i\sigma^2(+i\sigma^{2*}\chi_1) = -\eta_2 \Rightarrow \chi_1 = -\eta_2 \\ i\sigma^2(+i\sigma^{2*}\chi_2) = \eta_1 \Rightarrow \chi_2 = \eta_1 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \chi_1 = -\eta_2 = -\chi_1 e^{i\delta} \\ \chi_2 = \eta_1 = \chi_2 e^{i\gamma} \end{cases}$$

$$\Rightarrow \quad \begin{cases} \delta = (2m+1)\pi, & m = 0, \pm 1, \pm 2, \dots \\ \gamma = 2n\pi, & n = 0, \pm 1, \pm 2 \end{cases}$$

 (checked)

Therefore, to make the analysis of charge conjugation transformation and time inversion transformation simple,

we choose
$$\begin{cases} \beta + \alpha = (2k+1)\pi - \varphi, & k = 0, \pm 1, \pm 2, \dots \\ \gamma = 2n\pi, & n = 0, \pm 1, \pm 2, \dots \\ \delta = (2m+1)\pi, & m = 0, \pm 1, \pm 2, \dots \end{cases}$$

$$\Rightarrow \quad \chi_1 = \begin{pmatrix} \cos\frac{\theta}{2} e^{i\alpha} \\ \sin\frac{\theta}{2} e^{i\varphi} e^{i\alpha} \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\varphi} e^{-i\alpha} \\ +\cos\frac{\theta}{2} e^{-i\alpha} \end{pmatrix}$$
$$= -\eta_2 \qquad\qquad\qquad = \eta_1$$

Furthermore, for $\theta = 0$, we would like to have $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
then we take $e^{i\alpha} = 1$

Therefore $\chi_1 = -\eta_2 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$, $\chi_2 = \eta_1 = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix}$

So, when $\theta = 0$, $\chi_1 = -\eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Since $\hat{S}_p$ is the helicity operator, then we can also use the eigenvalue $\pm\frac{1}{2}$ to label these eigenstates, i.e.,

$$u(\vec{p}, +\tfrac{1}{2}) \equiv u(\vec{p}, 1), \quad u(\vec{p}, -\tfrac{1}{2}) \equiv u(\vec{p}, 2)$$
$$v(\vec{p}, +\tfrac{1}{2}) \equiv v(\vec{p}, 1), \quad v(\vec{p}, -\tfrac{1}{2}) \equiv v(\vec{p}, 2)$$
$$\chi_{\frac{1}{2}} \equiv \chi_1, \quad \chi_{-\frac{1}{2}} \equiv \chi_2, \quad \eta_{\frac{1}{2}} \equiv \eta_1, \quad \eta_{-\frac{1}{2}} \equiv \eta_2.$$

and $-i\sigma^2 \chi_s^* = \eta_s$ where $s = 1, 2$ is equivalent to
$$-i\sigma^2 \chi_s^* = \eta_s \quad \text{where } s = \pm\tfrac{1}{2}.$$

while
$$\begin{cases} i\sigma^2 \chi_1^* = -\chi_2 \\ i\sigma^2 \chi_2^* = \chi_1 \\ i\sigma^2 \eta_1^* = -\eta_2 \\ i\sigma^2 \eta_2^* = \eta_1 \end{cases}$$
is equivalent to

$i\sigma^2 \chi_s^* = (-1)^{\frac{1}{2}+s} \chi_{-s}$ and $i\sigma^2 \eta_s^* = (-1)^{\frac{1}{2}+s} \eta_{-s}$ where $s = \pm\frac{1}{2}$

Finally, for $\vec{P}=0$, we can define $\hat{S}_{\vec{P}=0}$ as $\hat{S}_{\vec{P}=0} \equiv \hat{S}^3$, and we still choose $\sigma_{12} u(0,r)=(-1)^{r+1} u(0,r)$ and $\sigma_{12} v(0,r)=(-1)^{r} v(0,r)$ then we can see that $\hat{S}_p$ (in page ⑧) also catains this case, with $E_{\vec{p}}=m$.

In the standard representation,

$$u(0,1)=(2m)^{\frac{1}{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u(0,2)=(2m)^{\frac{1}{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$v(0,1)=(2m)^{\frac{1}{2}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad v(0,2)=(2m)^{\frac{1}{2}}\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$$

i.e., $\chi_1=-\eta_2=\begin{pmatrix}1\\0\end{pmatrix}$, $\chi_2=\eta_1=\begin{pmatrix}0\\1\end{pmatrix}$

Consistent with the $\theta=0$ case. (in page ⑩)