Example: Suppose two identical particles, each with mass m and kinetic energy T, collide head-on. Question: What is their relative kinetic energy, T' (i.e., the kinetic energy of one in the rest frame of the other)?

Solution:

In the CM frame, $p^\mu_{tot} = (2E, \vec{0})$

In the rest frame of one of the particles, $p'^\mu_{tot} = (E'+m, \vec{p}')$

using $p^2_{tot} = p'^\mu_{tot} p'_{\mu\,tot} = p^\mu_{tot} p_{\mu\,tot}$

$$\Rightarrow (2E)^2 - 0^2 = (E'+m)^2 - |\vec{p}'|^2$$

using $E'^2 - |\vec{p}'|^2 = m^2$

$$\Rightarrow (2E)^2 = E'^2 + m^2 + 2E'm + m^2 - E'^2$$

$$\Rightarrow 2E = [2m(E'+m)]^{\frac{1}{2}}$$

using $T = E - m$, $T' = E' - m$

$$\Rightarrow 2(T+m) = [2m(T'+2m)]^{\frac{1}{2}}$$

$$\Rightarrow T' = \frac{[2(T+m)]^2 - 4m^2}{2m}$$

$$= \frac{4T^2 + 8Tm}{2m}$$

$$= 4T\left(1 + \frac{T}{2m}\right)$$

For the LHC, $T \simeq E = 7\ \text{TeV}$, $m = 0.938\ \text{MeV}$

$$\Rightarrow T' \simeq 1.05 \times 10^5\ \text{TeV}$$

Note that for $T \gg m$ & $T' \gg m$, then $E \simeq T$ & $E' \simeq T'$.

$\Rightarrow 2E \simeq \sqrt{2mE'}$ & $2T \simeq \sqrt{2mT'}$ & $T' \gg T$ & $E' \gg E$

That's why a collider is prefered compared to fix target experiment.

Symmetries

Why study symmetries in particle physics ?

Because

① symmetries are closely related to conservation laws

② we can make some progress (e.g., do some calculations to compare with experimental data, build models) when a complete dynamical theory is not yet available

An example of the power of symmetry

Given an odd function $f(x) = -f(-x)$, then you immediately deduce that, eg. $\int_{-a}^{+a} f(x)\,dx = 0$, and the Taylor series of it only contains odd powers of x. To know these properties, you do not need to know the functional form of $f(x)$.

<u>Note</u> that symmetries are manifest in the equations of motion rather than in particular solutions of these equations.

e.g., Newton's law of gravitation has spherical symmetry, but the orbits of the planets are elliptical. (This is due to the initial condition which does not have spherical symmetry)

$\boxed{\text{Noether's theorem relates symmetries and conservation laws}}$

e.g.

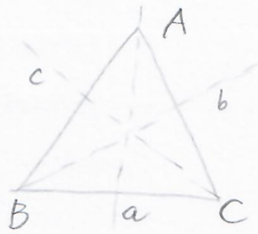
if a system is invariant under
- translation in time $\longleftrightarrow$ energy conservation
- — — space $\longleftrightarrow$ momentum — —
- rotation $\longleftrightarrow$ angular momentum.
- internal transformation $\longleftrightarrow$ charge conservation (electric charge, baryon number, etc.)

What is a symmetry ? (a practical definition)

It is an operation you can perform (at least conceptually) on a system that leaves it invariant — that carries it into a configuration indistinguishable from the original one.

e.g., for the odd function example, the operation to leave it invariant is $f(x) \to -f(-x)$.

for an equilateral triangle, the operations to leave it invariant include, e.g., a clockwise rotation through 120°, flipping it about the axis a, etc., and note that do nothing is also an operation (though a trivial one) that leaves it invariant.



In fact, the set of all symmetry operations on a particular system forms a group, satisfying

① closure: if R_i and R_j are in the set, then R_iR_j is also in the set;

② identity: there is an element I such that for all R_i,
$$IR_i = R_i I = R_i;$$

③ inverse: for every R_i, there is an inverse R_i^{-1} in the set, such that $R_iR_i^{-1} = R_i^{-1}R_i = I$.

④ associativity: $R_i(R_jR_k) = (R_iR_j)R_k$

group representations

Every group G can be represented by a group of matrices: for every group element "a" there is a corresponding matrix "M_a", and the correspondance respects group multiplication, in the sense that if $ab = c$, then $M_aM_b = M_c$.

angular momentum

(1.) orbital angular momentum

in quantum mechanics, we can simultaneously measure $L^2 = \vec{L}\cdot\vec{L}$ and one component (say, L_z), but not simultaneously measure two components (say, L_x and L_z).

The eigenvalues of L^2 is $l(l+1)\hbar^2$, where $l = 0, 1, 2, \cdots$, and the eigenvalues of L_z is $m_l \hbar$, where $m_l = -l, -l+1, \cdots, -1, 0, 1, \cdots, l$

(2) spin angular momentum

Similar as the orbital angular momentum, we can simultaneously measure $S^2 = \vec{S}\cdot\vec{S}$ and one component (say, L_z), and the values are

$$S(S+1)\hbar^2, \text{ where } S = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \cdots$$

and $$m_s \hbar, \text{ where } m_s = -s, -s+1, \cdots, s-1, s$$

(3) addition of angular momentum

If we combine states $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$, we get a state $|j m\rangle$, where $m = m_1 + m_2$, and $j = |j_1 - j_2|, |j_1 - j_2| + 1, \cdots, j_1 + j_2 - 1, j_1 + j_2$

For example, a quark and an antiquark are bound together, in a state of zero orbital angular momentum, to form a meson, and the possible meson's spin are $\frac{1}{2} + \frac{1}{2} = 1$ and $\frac{1}{2} - \frac{1}{2} = 0$.

To add three angular momentum, we just combine two of them first, and then add the third.

For example, three quarks are bound together, in a state of zero orbital angular momentum, then first $\frac{1}{2} + \frac{1}{2} = 1$, $\frac{1}{2} - \frac{1}{2} = 0$; then $1 + \frac{1}{2} = \frac{3}{2}$, $1 - \frac{1}{2} = \frac{1}{2}$, $0 + \frac{1}{2} = \frac{1}{2}$. Therefore, there're two ways to get total spin $\frac{1}{2}$, and one way to get $\frac{3}{2}$.

(4) How to decompose $|j_1\ m_1\rangle|j_2\ m_2\rangle$ into $|j\,m\rangle$:

$$|j_1\ m_1\rangle|j_2\ m_2\rangle = \sum_{j=|j_1-j_2|}^{(j_1+j_2)} C^{j\,j_1\,j_2}_{m\,m_1\,m_2}\,|j\ m\rangle, \text{ with } m=m_1+m_2,$$

$C^{j\,j_1\,j_2}_{m\,m_1\,m_2}$ are Clebsch-Gordan coefficients, which tell you the probability of getting $j(j+1)\hbar^2$ for any particular allowed j, if we measure J^2 on a system consisting of two angular momentum states $|j_1\ m_1\rangle$ and $|j_2\ m_2\rangle$. Note that the probability is the square of the corresponding C-G coefficient.

You can find the commonly used C-G coefficients in PDG.

<u>For example</u>

$1/2 \times 1/2$

		1	1	0	1
		+1	0	0	−1
+1/2	+1/2	1			
+1/2	−1/2		1/2	1/2	
−1/2	+1/2		1/2	−1/2	
−1/2	−1/2				1

Notation:

		J	J	…
		M	M	…
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

Note that a square-root sign is to be understood over every coefficients, e.g., for $-1/2$ read $-\sqrt{1/2}$

Therefore.

$$|\tfrac12\ \tfrac12\rangle|\tfrac12\ \tfrac12\rangle = |1\ 1\rangle$$

$$|\tfrac12\ \tfrac12\rangle|\tfrac12\ -\tfrac12\rangle = \tfrac{1}{\sqrt2}|1\ 0\rangle + \tfrac{1}{\sqrt2}|0\ 0\rangle$$

$$|\tfrac12\ -\tfrac12\rangle|\tfrac12\ \tfrac12\rangle = \tfrac{1}{\sqrt2}|1\ 0\rangle - \tfrac{1}{\sqrt2}|0\ 0\rangle$$

$$|\tfrac12\ -\tfrac12\rangle|\tfrac12\ -\tfrac12\rangle = |1\ -1\rangle.$$

$\Rightarrow$ the spin 1 triplet are
$$\begin{cases} |1\ 1\rangle = |\tfrac12\ \tfrac12\rangle|\tfrac12\ \tfrac12\rangle \\ |1\ 0\rangle = \tfrac{1}{\sqrt2}\left(|\tfrac12\ \tfrac12\rangle|\tfrac12\ -\tfrac12\rangle + |\tfrac12\ -\tfrac12\rangle|\tfrac12\ \tfrac12\rangle\right) \\ |1\ -1\rangle = |\tfrac12\ -\tfrac12\rangle|\tfrac12\ -\tfrac12\rangle \end{cases}$$

the spin 0 singlet is $|0\ 0\rangle = \tfrac{1}{\sqrt2}\left(|\tfrac12\ \tfrac12\rangle|\tfrac12\ -\tfrac12\rangle - |\tfrac12\ -\tfrac12\rangle|\tfrac12\ \tfrac12\rangle\right)$

Also note that $|j\ m\rangle = \sum_{m_1, m_2} C^{j\, j_1\, j_2}_{m\, m_1\, m_2} |j_1\ m_1\rangle |j_2\ m_2\rangle$, we can actually directly get, by reading the table along the column rather than along the row, e.g.

$$|1\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle\right)$$

Note that the triplet is symmetric under interchange of the particles $1 \leftrightarrow 2$, whereas the singlet is antisymmetric.
In the singlet, the spins are oppositely aligned (i.e., antiparallel).
In the triplet, the spins are parallel for $|1\ 1\rangle$ and $|1\ -1\rangle$, but antiparallel for $|1\ 0\rangle$.

(5) spin $\frac{1}{2}$

denote the $m_{s_z} = +\frac{1}{2}$ spin state as $|\frac{1}{2}\ \frac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, i.e., spin up $\uparrow$

------ $-\frac{1}{2}$ ------ $|\frac{1}{2}\ -\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, ------ down $\downarrow$.

An arbitrary spin state is the linear combination

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix},$$ where α and β are complex numbers.

$|\alpha|^2$ is the probability that a measurement of S_z would yield the value $+\frac{1}{2}\hbar$, and $|\beta|^2$ is the probability of getting $-\frac{1}{2}\hbar$.

$$|\alpha|^2 + |\beta|^2 = 1.$$

Note that the $\hat{S}_z$ operator is $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, and hence

$$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}, \quad \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

How about $\hat{S}_x$ and $\hat{S}_y$?

We can use the raising and lowering operators, $\hat{S}_\pm \equiv \hat{S}_x \pm i\hat{S}_y$, to construct them:

In general, the angular momentum (not just the spin angular momentum) lowering and raising operators

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

satisfy, from

$$[J_x, J_y] = i\hbar J_z$$
$$[J_y, J_z] = i\hbar J_x$$
$$[J_z, J_x] = i\hbar J_y.$$

(the hat are omitted here and after)

$$\Rightarrow \quad [J_z, J_\pm] = [J_z, J_x \pm iJ_y] = \hbar[iJ_y \pm i(-iJ_x)] = \pm(J_x \pm iJ_y)\hbar$$
$$= \pm\hbar J_\pm$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = \hbar[-i\,i\,J_z + i(-i)J_z]$$
$$= 2\hbar J_z$$

using $J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$

$J_z|jm\rangle = m\hbar|jm\rangle$, $\langle jm|jm\rangle = 1$

$$\Rightarrow \quad J_z J_\pm |jm\rangle = (J_\pm J_z + [J_z, J_\pm])|jm\rangle$$
$$= (J_\pm J_z \pm \hbar J_\pm)|jm\rangle$$
$$= \hbar(m\pm 1) J_\pm |jm\rangle$$

Since $J_z|j(m\pm1)\rangle = \hbar(m\pm1)|j(m\pm1)\rangle$,

then $J_+|jm\rangle = a|j(m+1)\rangle$

$J_-|jm\rangle = b|j(m-1)\rangle$

where a and b are complex numbers.

Since $J_+^\dagger = (J_x + iJ_y)^\dagger = J_x - iJ_y = J_-$,

$J_-^\dagger = J_+$

then $\langle jm | J_+^\dagger J_+ | jm \rangle = \langle jm | J_- J_+ | jm \rangle = \langle j\,(m+1) | a^* a | j\,(m+1) \rangle$
$= |a|^2$

$\langle jm | J_-^\dagger J_- | jm \rangle = \langle jm | J_+ J_- | jm \rangle = \langle j\,(m-1) | b^* b | j\,(m-1) \rangle$
$= |b|^2$.

Also, since $J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y]$
$= J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z$

$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y]$
$= J^2 - J_z^2 + \hbar J_z$

$\Rightarrow |a|^2 = j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2 = \hbar^2 (j-m)(j+m+1)$

$|b|^2 = j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2 = \hbar^2 (j+m)(j-m+1)$

we can choose a and b to be real and positive, then

$a = \hbar\sqrt{(j-m)(j+m+1)}$

$b = \hbar\sqrt{(j+m)(j-m+1)}$

$$\Rightarrow \boxed{\begin{aligned} J_+ |jm\rangle &= \hbar\sqrt{(j-m)(j+m+1)}\; |j \;\; m+1\rangle \\ J_- |jm\rangle &= \hbar\sqrt{(j+m)(j-m+1)}\; |j \;\; m-1\rangle \end{aligned}}$$

Note that $J_+ |j\,j\rangle = 0$, $J_- |j\,-j\rangle = 0$ from the above formula, as they should be (since $-j \leq m \leq j$)

Therefore,

$$\hat{S}_+\begin{pmatrix}1\\0\end{pmatrix}=0\ ,\quad \hat{S}_+\begin{pmatrix}0\\1\end{pmatrix}=\hbar\sqrt{\left(\frac{1}{2}-\left(-\frac{1}{2}\right)\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\begin{pmatrix}1\\0\end{pmatrix}$$
$$=\hbar\begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow\quad \underbrace{\hat{S}_{+i1}}_{\substack{i:\ \text{row}\\ 1:\ \text{column}\\(i=1,2)}}=0\ ,\quad \hat{S}_{+12}=\hbar\ ,\quad \hat{S}_{+22}=0$$

$$\Rightarrow\quad \hat{S}_+=\hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$\hat{S}_-\begin{pmatrix}1\\0\end{pmatrix}=\hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\begin{pmatrix}0\\1\end{pmatrix}$$
$$=\hbar\begin{pmatrix}0\\1\end{pmatrix}$$

$$\hat{S}_-\begin{pmatrix}0\\1\end{pmatrix}=0$$

$$\Rightarrow\quad \hat{S}_{-i2}=0,\quad \hat{S}_{-11}=0,\quad \hat{S}_{-21}=\hbar$$

$$\Rightarrow\quad \hat{S}_-=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

Since $\begin{cases}\hat{S}_x+i\hat{S}_y=\hat{S}_+\\ \hat{S}_x-i\hat{S}_y=\hat{S}_-\end{cases}$

then

$$\hat{S}_x=\frac{1}{2}(\hat{S}_++\hat{S}_-)=\frac{\hbar}{2}\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]$$
$$=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\hat{S}_y=\frac{1}{2i}(\hat{S}_+-\hat{S}_-)=\frac{\hbar}{2i}\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]$$
$$=\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

Then, how about the eigenfunctions for $\hat{S}_x$ and $\hat{S}_y$?

For $\hat{S}_x$, from $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$, we get.

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}.$$

for $\lambda = +\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad y = x$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $+\frac{\hbar}{2}$ is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

for $\lambda = -\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad x = -y$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $-\frac{\hbar}{2}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

For $\hat{S}_y$, from $\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$, we get.

$$\begin{vmatrix} -\lambda & -\frac{\hbar}{2}i \\ i\frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

for $\lambda = +\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad x = -iy$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $+\frac{\hbar}{2}$ is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

for $\lambda = -\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \qquad x = iy$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $-\frac{\hbar}{2}$ is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Therefore, for an arbitrary spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, if we measure S_x, then from

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv a\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + b\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} \alpha = \frac{1}{\sqrt{2}}(a+b) \\ \beta = \frac{1}{\sqrt{2}}(a-b) \end{cases}$$

$$\Rightarrow \quad a = \frac{1}{\sqrt{2}}(\alpha+\beta)$$
$$b = \frac{1}{\sqrt{2}}(\alpha-\beta)$$

So, the probability that a measurement of S_x will yield the value $\frac{1}{2}\hbar$ is $|a|^2 = \frac{1}{2}|\alpha+\beta|^2$, the probability of getting $-\frac{1}{2}\hbar$ is $|b|^2 = \frac{1}{2}|\alpha-\beta|^2$;

Evidently, $|a|^2+|b|^2 = \frac{1}{2}|\alpha+\beta|^2 + \frac{1}{2}|\alpha-\beta|^2 = |\alpha|^2+|\beta|^2 = 1$

How about if we measure $(S_x)^2$?

Again, 1 first let's find the matrix form of the operator,

$$\hat{S}_x^2 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2 then let's find the eigenvalues and eigenfunctions,

$$\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} \frac{\hbar^2}{4} - \lambda & 0 \\ 0 & \frac{\hbar^2}{4} - \lambda \end{vmatrix} \equiv 0 = \left(\frac{\hbar^2}{4} - \lambda\right)^2$$

$$\Rightarrow \lambda = \frac{\hbar^2}{4}$$

Then from $\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \begin{aligned} x &= x \\ y &= y \end{aligned}$$

So the normalized eigenfunction is just $\begin{pmatrix} x \\ y \end{pmatrix}$ with $|x|^2 + |y|^2 = 1$

That is, any 2×1 matrix is an eigenfunction of $\hat{S}_x^2$, with eigenvalue $\frac{\hbar^2}{4}$.

Therefore, a measurement of S_x^2 certainly yields the value $\frac{\hbar^2}{4}$.

The same goes for S_y^2 and S_z^2, and therefore $S^2 = S_x^2 + S_y^2 + S_z^2$.

That is, any 2×1 matrix (i.e., spinor) is an eigenfunction of $\hat{S}_x^2$, $\hat{S}_y^2$ and $\hat{S}_z^2$, with eigenvalue $\frac{\hbar^2}{4}$, as well as an eigenfunction of S^2, with eigenvalue $\frac{3\hbar^2}{4} = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4} = \frac{1}{2}\left(\frac{1}{2} + 1\right)\hbar^2$.

Often we prefer to use Pauli matrices, so that $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$.