Example: Suppose two identical particles, each with mass m and kinetic energy T, collide head-on. Question: What is their relative kinetic energy T' (i.e., the kinetic energy of one in the rest frame of the other)?

Solution:

In the CM frame, $p^{\mu}_{tot} = (2E, \vec{0})$

In the rest frame of one of the particles, $p'^{\mu}_{tot} = (E'+m, \vec{p}\,')$

using $p^2_{tot} = p'^{\mu}_{tot} p'_{\mu\, tot} = p^{\mu}_{tot} p_{\mu\, tot}$

$$\Rightarrow \quad (2E)^2 - 0^2 = (E'+m)^2 - |\vec{p}\,'|^2$$

using $E'^2 - |\vec{p}\,'|^2 = m^2$

$$\Rightarrow \quad (2E)^2 = E'^2 + m^2 + 2E'm + m^2 - E'^2$$

$$\Rightarrow \quad 2E = [2m(E'+m)]^{\frac{1}{2}}$$

using $T = E - m, \quad T' = E' - m$

$$\Rightarrow \quad 2(T+m) = [2m(T'+2m)]^{\frac{1}{2}}$$

$$\Rightarrow \quad T' = \frac{[2(T+m)]^2 - 4m^2}{2m}$$

$$= \frac{4T^2 + 8Tm}{2m}$$

$$= 4T\left(1 + \frac{T}{2m}\right)$$

For the LHC, $T \simeq E = 7\,\text{TeV}, \quad m = 0.938\,\text{MeV}$

$$\Rightarrow \quad T' \simeq 1.05 \times 10^5\,\text{TeV}$$

Note that for $T \gg m$ & $T' \gg m$, then $E \simeq T$ & $E' \simeq T'$,

$$\Rightarrow \quad 2E \simeq \sqrt{2mE'} \;\&\; 2T \simeq \sqrt{2mT'} \;\&\; T' \gg T \;\&\; E' \gg E$$

That's why a collider is prefered compared to fix target experiment.

Symmetries

Why study symmetries in particle physics?

Because

① symmetries are closely related to conservation laws

② we can make some progress (e.g., do some calculations to compare with experimental data, build models) when a complete dynamical theory is not yet available

An example of the power of symmetry

Given an odd function $f(x) = -f(-x)$, then you immediately deduce that, e.g. $\int_{-a}^{+a} f(x)\,dx = 0$, and the Taylor series of it only contains odd powers of x. To know these properties, you do not need to know the functional form of $f(x)$.

Note that symmetries are manifest in the equations of motion rather than in particular solutions of these equations.

e.g., Newton's law of gravitation has spherical symmetry, but the orbits of the planets are elliptical. (This is due to the initial condition which does not have spherical symmetry)

$\boxed{\text{Noether's theorem relates symmetries and conservation laws}}$

e.g.

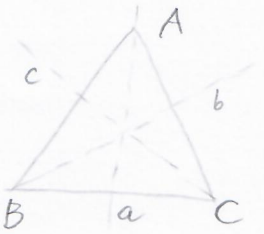
if a system is invariant under
- translation in time $\longleftrightarrow$ energy conservation
- ——— space $\longleftrightarrow$ momentum ———
- rotation $\longleftrightarrow$ angular momentum
- internal transformation $\longleftrightarrow$ charge conservation (electric charge, baryon number, etc.)

What is a symmetry? (a practical definition)

It is an operation you can perform (at least conceptually) on a system that leaves it invariant — that carries it into a configuration indistinguishable from the original one.

e.g., for the odd function example, the operation to leave it invariant is $f(x) \to -f(-x)$.

for an equilateral triangle, the operations to leave it invariant include, e.g., a clockwise rotation through $120°$, flipping it about the axis a, etc., and note that do nothing is also an operation (though a trivial one) that leaves it invariant.



In fact, the set of all symmetry operations on a particular system forms a group, satisfying

① closure: if R_i and R_j are in the set, then R_iR_j is also in the set;

② identity: there is an element I such that for all R_i, $IR_i = R_iI = R_i$;

③ inverse: for every R_i, there is an inverse R_i^{-1} in the set, such that $R_iR_i^{-1} = R_i^{-1}R_i = I$

④ associativity: $R_i(R_jR_k) = (R_iR_j)R_k$

group representations

Every group G can be represented by a group of matrices: for every group element "a" there is a corresponding matrix "M_a", and the correspondance respects group multiplication, in the sense that if $ab = c$, then $M_aM_b = M_c$.

angular momentum

(1) orbital angular momentum

in quantum mechanics, we can simultaneously measure $L^2 = \vec{L}\cdot\vec{L}$ and one component (say, L_z), but not simultaneously measure two components (say, L_x and L_z).

The eigenvalues of L^2 is $l(l+1)\hbar^2$, where $l = 0, 1, 2, \cdots$, and the eigenvalues of L_z is $m_l \hbar$, where $m_l = -l, -l+1, \cdots, -1, 0, 1, \cdots, l$

(2) spin angular momentum

Similar to the orbital angular momentum, we can simultaneously measure $S^2 = \vec{S}\cdot\vec{S}$ and one component (say, S_z), and the values are

$s(s+1)\hbar^2$, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$

and $m_s \hbar$, where $m_s = -s, -s+1, \cdots, s-1, s$.

(3) addition of angular momentum.

If we combine states $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$, we get a state $|j\, m\rangle$, where $m = m_1 + m_2$, and $j = |j_1 - j_2|, |j_1 - j_2| + 1, \cdots, j_1 + j_2 - 1, j_1 + j_2$

For example, a quark and an antiquark are bound together, in a state of zero orbital angular momentum, to form a meson, and the possible meson's spin are $\frac{1}{2} + \frac{1}{2} = 1$ and $\frac{1}{2} - \frac{1}{2} = 0$.

To add three angular momentum, we just combine two of them first, and then add the third.

For example, three quarks are bound together, in a state of zero orbital angular momentum, then first $\frac{1}{2} + \frac{1}{2} = 1$, $\frac{1}{2} - \frac{1}{2} = 0$; then $1 + \frac{1}{2} = \frac{3}{2}$, $1 - \frac{1}{2} = \frac{1}{2}$, $0 + \frac{1}{2} = \frac{1}{2}$. Therefore, there're two ways to get total spin $\frac{1}{2}$, and one way to get $\frac{3}{2}$.

(4) How to decompose $|j_1\ m_1\rangle\,|j_2\ m_2\rangle$ into $|j\,m\rangle$:

$$|j_1\ m_1\rangle\,|j_2\ m_2\rangle = \sum_{j=|j_1-j_2|}^{(j_1+j_2)} C^{j\,j_1\,j_2}_{m\,m_1\,m_2}\,|j\,m\rangle, \text{ with } m=m_1+m_2.,$$

$C^{j\,j_1\,j_2}_{m\,m_1\,m_2}$ are Clebsch-Gordan coefficients, which tell you the probability of getting $j(j+1)\hbar^2$ for any particular allowed j, if we measure J^2 on a system consisting of two angular momentum states $|j_1\ m_1\rangle$ and $|j_2\ m_2\rangle$. Note that the probability is the square of the corresponding C-G coefficient.

You can find the commonly used C-G coefficients in PDG.

For example

$1/2 \times 1/2$

m_1	m_2	$J=1$, $M=+1$	$J=1$, $M=0$	$J=0$, $M=0$	$J=1$, $M=-1$
$+1/2$	$+1/2$	1			
$+1/2$	$-1/2$		$1/2$	$1/2$	
$-1/2$	$+1/2$		$1/2$	$-1/2$	
$-1/2$	$-1/2$				1

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
$\vdots$	$\vdots$			

Note that a square-root sign is to be understood over every coefficients, e.g., for $-1/2$ read $-\sqrt{1/2}$

Therefore.

$$|\tfrac12\ \tfrac12\rangle|\tfrac12\ \tfrac12\rangle = |1\ 1\rangle$$

$$|\tfrac12\ \tfrac12\rangle|\tfrac12\ -\tfrac12\rangle = \frac{1}{\sqrt2}|1\ 0\rangle + \frac{1}{\sqrt2}|0\ 0\rangle$$

$$|\tfrac12\ -\tfrac12\rangle|\tfrac12\ \tfrac12\rangle = \frac{1}{\sqrt2}|1\ 0\rangle - \frac{1}{\sqrt2}|0\ 0\rangle$$

$$|\tfrac12\ -\tfrac12\rangle|\tfrac12\ -\tfrac12\rangle = |1\ -1\rangle.$$

$\Rightarrow$ the spin 1 triplet are

$$\begin{cases} |1\ 1\rangle = |\tfrac12\ \tfrac12\rangle|\tfrac12\ \tfrac12\rangle \\ |1\ 0\rangle = \frac{1}{\sqrt2}\left(|\tfrac12\ \tfrac12\rangle|\tfrac12\ -\tfrac12\rangle + |\tfrac12\ -\tfrac12\rangle|\tfrac12\ \tfrac12\rangle\right) \\ |1\ -1\rangle = |\tfrac12\ -\tfrac12\rangle|\tfrac12\ -\tfrac12\rangle \end{cases}$$

the spin 0 singlet is $|0\ 0\rangle = \frac{1}{\sqrt2}\left(|\tfrac12\ \tfrac12\rangle|\tfrac12\ -\tfrac12\rangle - |\tfrac12\ -\tfrac12\rangle|\tfrac12\ \tfrac12\rangle\right)$

Also note that $|j\ m\rangle = \sum_{m_1, m_2} C^{j\,j_1\,j_2}_{m\,m_1\,m_2} |j_1\ m_1\rangle |j_2\ m_2\rangle$, we can actually directly get, by reading the table along the column rather than along the row, e.g.

$$|1\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle + |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle\right)$$

Note that the triplet is symmetric under interchange of the particles $1 \leftrightarrow 2$, whereas the singlet is antisymmetric.

In the singlet, the spins are oppositely aligned (i.e., antiparallel).

In the triplet, the spins are parallel for $|1\ 1\rangle$ and $|1\ -1\rangle$, but antiparallel for $|1\ 0\rangle$.

(5) spin $\frac{1}{2}$

denote the $m_{s_z} = +\frac{1}{2}$ spin state as $|\frac{1}{2}\ \frac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, i.e., spin up ↑

---------- $-\frac{1}{2}$ ---------- $|\frac{1}{2}\ -\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, ------ down ↓.

An arbitrary spin state is the linear combination

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix},$$ where α and β are complex numbers.

$|\alpha|^2$ is the probability that a measurement of S_z would yield the value $+\frac{1}{2}\hbar$, and $|\beta|^2$ is the probability of getting $-\frac{1}{2}\hbar$.

$$|\alpha|^2 + |\beta|^2 = 1.$$

Note that the $\hat{S}_z$ operator is $\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, and hence

$$\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}, \quad \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

How about $\hat{S}_x$ and $\hat{S}_y$?

We can use the raising and lowering operators, $\hat{S}_\pm \equiv \hat{S}_x \pm i\hat{S}_y$, to construct them :

In general, the angular momentum (not just the spin angular momentum) lowering and raising operators

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

satisfy, from

$$[J_x, J_y] = i\hbar J_z$$
$$[J_y, J_z] = i\hbar J_x$$
$$[J_z, J_x] = i\hbar J_y.$$

(the hat are omitted here and after)

$$\Rightarrow [J_z, J_\pm] = [J_z, J_x \pm iJ_y] = \hbar[iJ_y \pm i(-iJ_x)] = \pm(J_x \pm iJ_y)\hbar$$
$$= \pm\hbar J_\pm$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = \hbar[-iiJ_z + i(-i)J_z]$$
$$= 2\hbar J_z$$

using $J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$

$J_z|jm\rangle = m\hbar|jm\rangle$, $\langle jm|jm\rangle = 1$

$$\Rightarrow J_z J_\pm |jm\rangle = (J_\pm J_z + [J_z, J_\pm])|jm\rangle$$
$$= (J_\pm J_z \pm \hbar J_\pm)|jm\rangle$$
$$= \hbar(m \pm 1) J_\pm |jm\rangle$$

Since $J_z|j(m\pm1)\rangle = \hbar(m\pm1)|j(m\pm1)\rangle$,

then $J_+|jm\rangle = a|j(m+1)\rangle$

$J_-|jm\rangle = b|j(m-1)\rangle$

where a and b are complex numbers.

Since $J_+^\dagger = (J_x + iJ_y)^\dagger = J_x - iJ_y = J_-$,

$J_-^\dagger = J_+$

then $\langle jm | J_+^\dagger J_+ | jm \rangle = \langle jm | J_- J_+ | jm \rangle = \langle j\,(m+1) | a^* a | j(m+1) \rangle$
$= |a|^2$

$\langle jm | J_-^\dagger J_- | jm \rangle = \langle jm | J_+ J_- | jm \rangle = \langle j\,(m-1) | b^* b | j\,(m-1) \rangle$
$= |b|^2$.

Also, since $J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y]$
$= J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z$

$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y]$
$= J^2 - J_z^2 + \hbar J_z$

$\Rightarrow |a|^2 = j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2 = \hbar^2 (j-m)(j+m+1)$

$|b|^2 = j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2 = \hbar^2 (j+m)(j-m+1)$

we can choose a and b to be real and positive, then

$a = \hbar\sqrt{(j-m)(j+m+1)}$

$b = \hbar\sqrt{(j+m)(j-m+1)}$

$$\Rightarrow \boxed{\begin{aligned} J_+ | jm \rangle &= \hbar\sqrt{(j-m)(j+m+1)}\, | j \; m+1 \rangle \\ J_- | jm \rangle &= \hbar\sqrt{(j+m)(j-m+1)}\, | j \; m-1 \rangle \end{aligned}}$$

Note that $J_+ | j\, j \rangle = 0$, $J_- | j\, -j \rangle = 0$ from the above formula, as they should be (since $-j \le m \le j$)

Therefore,

$$\hat{S}_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 , \quad \hat{S}_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \sqrt{\left(\frac{1}{2}-\left(-\frac{1}{2}\right)\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \hat{S}_{+i1} = 0 , \quad \hat{S}_{+12} = \hbar , \quad \hat{S}_{+22} = 0$$

i → row, 1 → column

$(i = 1, 2)$

$$\Rightarrow \quad \hat{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\hat{S}_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad \hat{S}_{-i2} = 0, \quad \hat{S}_{-11} = 0, \quad \hat{S}_{-21} = \hbar$$

$$\Rightarrow \quad \hat{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Since $\begin{cases} \hat{S}_x + i \hat{S}_y = \hat{S}_+ \\ \hat{S}_x - i \hat{S}_y = \hat{S}_- \end{cases}$

then

$$\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right) = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right) = \frac{\hbar}{2i}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Then, how about the eigenfunctions for $\hat{S}_x$ and $\hat{S}_y$?

For $\hat{S}_x$, from $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$, we get.

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} \equiv 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}.$$

for $\lambda = +\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \qquad y = x$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $+\frac{\hbar}{2}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

for $\lambda = -\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \qquad x = -y$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $-\frac{\hbar}{2}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

For $\hat{S}_y$, from $\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$, we get.

$$\begin{vmatrix} -\lambda & -\frac{\hbar}{2}i \\ i\frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{\hbar}{2}\right)^2 \equiv 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

for $\lambda = +\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \qquad x = -iy$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $+\frac{\hbar}{2}$ is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$

for $\lambda = -\frac{\hbar}{2}$,

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \qquad x = iy$$

$\Rightarrow$ the normalized eigenfunction for the eigenvalue $-\frac{\hbar}{2}$ is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Therefore, for an arbitrary spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, if we measure S_x, then from

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv a\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + b\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Rightarrow \qquad \begin{cases} \alpha = \frac{1}{\sqrt{2}}(a+b) \\ \beta = \frac{1}{\sqrt{2}}(a-b) \end{cases}$$

$$\Rightarrow \qquad a = \frac{1}{\sqrt{2}}(\alpha+\beta)$$

$$b = \frac{1}{\sqrt{2}}(\alpha-\beta)$$

So, the probability that a measurement of S_x will yield the value $\frac{1}{2}\hbar$ is $|a|^2 = \frac{1}{2}|\alpha+\beta|^2$, the probability of getting $-\frac{1}{2}\hbar$ is $|b|^2 = \frac{1}{2}|\alpha-\beta|^2$;

Evidently, $|a|^2+|b|^2 = \frac{1}{2}|\alpha+\beta|^2 + \frac{1}{2}|\alpha-\beta|^2 = |\alpha|^2+|\beta|^2 = 1$

How about if we measure $(S_x)^2$?

Again, first let's find the matrix form of the operator,

$$\hat{S}_x^2 = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

then let's find the eigenvalues and eigenfunctions,

$$\frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \lambda\begin{pmatrix}x\\y\end{pmatrix}$$

$$\Rightarrow \begin{vmatrix}\frac{\hbar^2}{4}-\lambda & 0\\ 0 & \frac{\hbar^2}{4}-\lambda\end{vmatrix} \equiv 0 = \left(\frac{\hbar^2}{4}-\lambda\right)^2$$

$$\Rightarrow \quad \lambda = \frac{\hbar^2}{4}$$

Then from $\frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}x\\y\end{pmatrix}$

$$\Rightarrow \quad x = x, \quad y = y$$

So the normalized eigenfunction is just $\begin{pmatrix}x\\y\end{pmatrix}$ with $|x|^2+|y|^2=1$

That is, any 2×1 matrix is an eigenfunction of $\hat{S}_x^2$, with eigenvalue $\frac{\hbar^2}{4}$.

Therefore, a measurement of S_x^2 certainly yields the value $\frac{\hbar^2}{4}$.

The same goes for S_y^2 and S_z^2, and therefore $S^2 = S_x^2+S_y^2+S_z^2$.

That is, any 2×1 matrix (i.e., spinor) is an eigenfunction of $\hat{S}_x^2$, $\hat{S}_y^2$ and $\hat{S}_z^2$, with eigenvalue $\frac{\hbar^2}{4}$, as well as an eigenfunction of S^2, with eigenvalue $\frac{3\hbar^2}{4} = \frac{\hbar^2}{4}+\frac{\hbar^2}{4}+\frac{\hbar^2}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2$.

Often we prefer to use Pauli matrices, so that $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$.

Flavor Symmetries

The proton and neutron have almost the same mass, $m_p \simeq 938.28 MeV$, $m_n = 939.57 MeV$; the strong force experienced by protons and neutrons are identical.

These are the motivations to consider proton and neutron being two states of a single particle, the nucleon.

To implement this idea (proposed by Heisenberg), we write the nucleon as a two component column matrix $N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, with

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By direct analogy with spin, we are led to introduce isospin, $\vec{I}$. (note that nuclear physicists use the word isobaric spin).

However, note that $\vec{I}$ is not a vector in ordinary space, it is in the isospin space, with components I_1, I_2 & I_3. The nucleon carries isospin $\frac{1}{2}$, and the third component, I_3, has the eigenvalues $+\frac{1}{2}$ for proton and $-\frac{1}{2}$ for neutron. Note that there is no $\hbar$, that is, isospin is dimensionless.

$$p = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle, \quad n = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

If the strong interactions are invariant under rotations in isospin space, then, by Noether's theorem, isospin is conserved in all strong interactions.

The isospin symmetry is an internal symmetry, because it has nothing to do with space and time, but rather with the relations between different particles. For example, a rotation through 180° about axis number 1 (or, number 2, or actually any direction in the 1 & 2 plane) converts protons into neutrons, and vice versa (Note that you need to interchange ALL protons and neutrons).

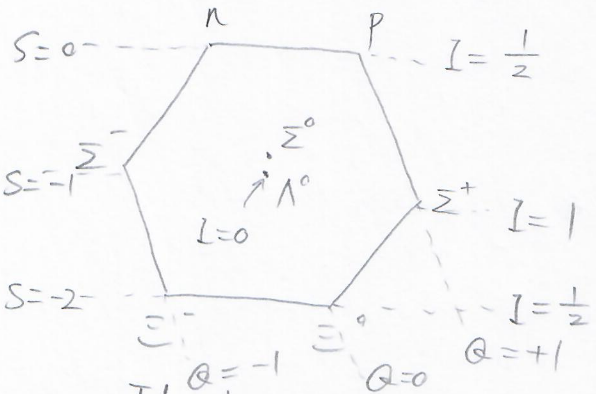
Each row of the Eightfold Way diagrams has the same isospin. (not including the isospin singlet in the middle of the diagrams)

e.g. $p = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$, $n = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

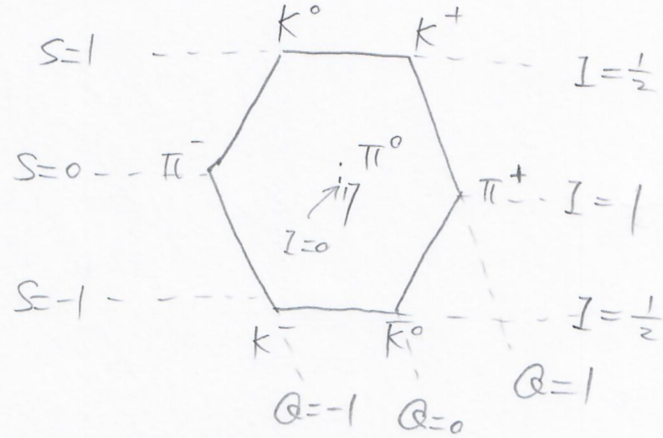
$\pi^+ = |1\ 1\rangle$, $\pi^0 = |1\ 0\rangle$, $\pi^- = |1\ -1\rangle$

$\Delta^{++} = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle$, $\Delta^+ = |\tfrac{3}{2}\ \tfrac{1}{2}\rangle$, $\Delta^0 = |\tfrac{3}{2}, -\tfrac{1}{2}\rangle$, $\Delta^- = |\tfrac{3}{2}\ -\tfrac{3}{2}\rangle$

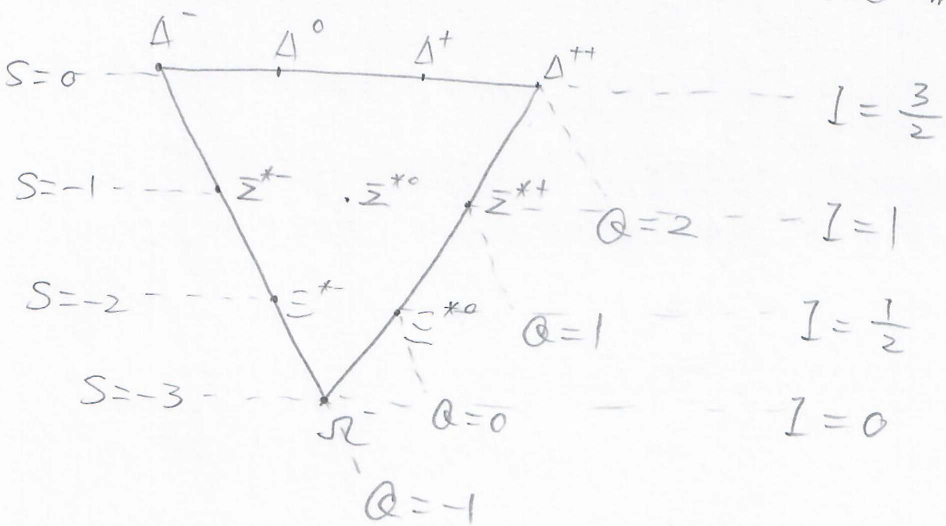
The number of particles in the multiplet is $(2I+1)$.



The baryon octet



The meson octet



For hadrons composed of u, d, and s quarks only, there is the Gell-Mann-Nishijima formula:

$$Q = I_3 + \tfrac{1}{2}(B+S),$$

where B is the baryon number and S is the strangeness.

In the content of the quark model, the Gell-Mann-Nishijima formula follows simply from the isospin assignments for quarks: u and d form a doublet $u = |\frac{1}{2}\ \frac{1}{2}\rangle$ & $d = |\frac{1}{2}\ -\frac{1}{2}\rangle$, and all the other quarks carry isospin zero.

Since Q, I_3, B & S are all additive quantum numbers, then as long as each quark (and antiquark) flavor satisfies the Gell-Mann-Nishijima formula, the hadrons made by quarks (and antiquarks) satisfy it.

check: for u quark, $Q = \frac{2}{3}$, $I_3 = \frac{1}{2}$, $B = \frac{1}{3}$, $S = 0$

$$\Rightarrow \frac{2}{3} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + 0\right) \checkmark$$

Since the quantum numbers Q, I_3, B & S are opposite for quarks & antiquarks, the Gell-Mann-Nishijima formula works for antiquarks as long as it works quarks.

for d quark, $Q = -\frac{1}{3}$, $I_3 = -\frac{1}{2}$, $B = \frac{1}{3}$, $S = 0$

$$\Rightarrow -\frac{1}{3} = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + 0\right) \checkmark$$

for s quark, $Q = -\frac{1}{3}$, $I_3 = 0$, $B = \frac{1}{3}$, $S = -1$

$$\Rightarrow -\frac{1}{3} = 0 + \frac{1}{2}\left(\frac{1}{3} - 1\right) \checkmark$$

To accommodate the c, b & t quarks and their antiquarks, the Gell-Mann-Nishijima formula is extended as

$$Q = I_3 + \frac{1}{2}(B + S + C + B' + T)$$

where the C, B' & T are the charmness, bottomness & topness numbers, with the assignment $C = +1$ for c quark, $B' = -1$ for b quark, and $T = +1$ for t quark.

Furthermore, since $I_3 = \frac{1}{2}(U+D)$, where U is the upness, with $U=+1$ for u quark; D is the downness, with $D=-1$ for d quark, and $B = \frac{1}{3}(U-D+C-S+T-B')$

$$\Rightarrow Q = \frac{1}{2}(U+D) + \frac{1}{2}\frac{1}{3}(U-D+C-S+T-B') + \frac{1}{2}(S+C+B'+T)$$

$$= \frac{2}{3}(U+C+T) + \frac{1}{3}(D+S+B')$$

Isospin symmetry has important dynamical implications:

1. deuteron is in the isospin state $|0\ 0\rangle$.

reason:

two nucleons combination gives a total isospin of 1 or 0:

$$\left.\begin{aligned} |1\ 1\rangle &= pp \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}(pn + np\rangle \\ |1\ -1\rangle &= nn \end{aligned}\right\} \text{isotriplet}$$

$$|0\ 0\rangle = \frac{1}{\sqrt{2}}(pn - np) \quad \text{—— isosinglet}$$

Experimentally, there is no bound states of two protons or of two neutrons. Therefore, the deuteron must be an isosinglet. If it were a triplet, all three states would have to occur.

2. cross section ratio of nucleon-nucleon scattering

For the processes

(a) $p + p \rightarrow d + \pi^+$

(b) $p + n \rightarrow d + \pi^0$

the cross section ratio $\sigma_a : \sigma_b$ calculated is consistent with experimental result.

explanation:

Since $d = |0\ 0\rangle$, $\pi^+ = |1\ 1\rangle$, $\pi^0 = |1\ 0\rangle$, then for (a), the isospin state on the right is $|1\ 1\rangle$, while it is $|1\ 0\rangle$ for (b).

For (a), the LHS isospin is $pp = |1\ 1\rangle$; for (b), the LHS isospin is

$$pn = \frac{1}{\sqrt{2}}\left(|1\ 0\rangle + |0\ 0\rangle\right)$$

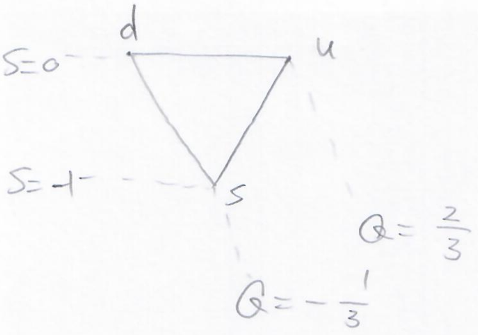
From isospin conservation, for the pn in (b), only the $|1\ 0\rangle$ part contributes in the cross section, therefore the scattering amplitudes are in the ratio $M_a : M_b = 1 : \frac{1}{\sqrt{2}}$

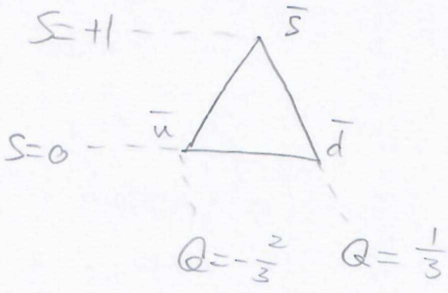
$$\Rightarrow \quad \sigma_a : \sigma_b = 2 : 1$$

this prediction is consistent with experiment.

In the late 1950s, it was found that the nucleons, the Λ, the Σ's, and the Ξ's, all carry spin $\frac{1}{2}$, and their masses are similar (although range from 940 MeV for nucleons to 1320 MeV for Ξ's), so it was temping to regard these eight baryons as supermultiplet, and this presumably meant that they belonged in the same representation of some enlarged symmetry group, in which the SU(2) of isospin would be incorporated as a subgroup. The symmetry group is SU(3), and the above mentioned eight baryons constitute an eight-dimensional representation of SU(3) (i.e., the baryon octet).

However, there is no naturally occuring particles fall into the fundamental (three-dimensional) representation of SU(3) (while for the SU(2), the nucleons, the K's etc., fall into the fundamental two-dimensional representation). It turns out this role was "reserved" for the u, d and s quarks: they together form a three-dimensional representation of SU(3), which breaks down into an isodoublet (u, d) and an isosinglet (s) under SU(2).





Why is the SU(2) isospin a good symmetry (e.g., the members of an isospin multiplet differ in mass by at most 2 or 3%.), the SU(3) flavor a fair symmetry (e.g. the mass splittings within the baryon octet are around 40%.), and flavor SU(4) (i.e., put the charm quark the same as the u, d & s quarks), SU(5) (i.e., put the bottom quark the same as the u, d, s & c quarks) and SU(6) (i.e., put the top quark the same as the u, d, s, c & b quarks). so poor symmetries?

Answer: it is due to the quarks' masses structure: The u & d quarks have very small bare masses (several MeV) so that they have very similar effective masses (about 350 MeV); The s quark has a bare mass of about 100 MeV, and has an effective mass of about 500 MeV; However, the c quark, b quark and t quark have much larger masses (both the bare masses and the effective masses) than the u, d & s quarks; Actually, the t quark is too heavy to form bound states (i.e., hadrons), since it decays before it can form bound states.

The difference of the bare masses and effective masses, as well as the difference of effective masses of quarks in baryons and mesons, could be analogue with the inertia of a spoon, that is, you feel different when you swing the spoon in the air, stir honey or stir water. (The bare quark mass is also called the current quark mass, the effective quark mass is also called the constituent quark mass. The latter is the former surrounded by a cloud of virtual quarks and gluons)

Discrete Symmetries

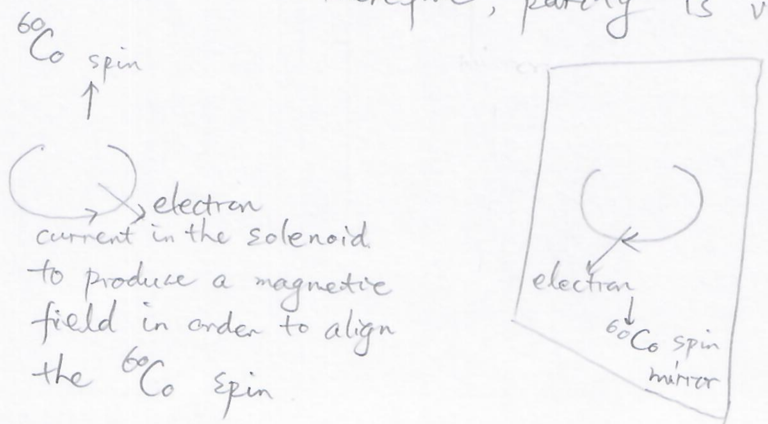
1. Parity

Parity invariance indicates that the mirror image of any physical process also represents a perfectly possible physical process in the real world.

In 1956, Lee and Yang searched the literature and noticed that although there was ample evidence for parity invariance in strong and electromagnetic processes, there was no confirmation in the case of weak interactions.

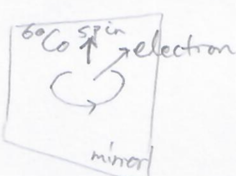
Later that year, Wu used ${}^{60}_{27}Co$ beta decay ${}^{60}Co \to {}^{60}Ni + e + \bar{\nu}_e$ to confirm that parity is not an invariance of the weak interactions.

What Wu found was that the electrons emitted had a preferred direction relative to the ${}^{60}Co$ nuclear spin direction. This indicated that the mirror image of this experiment results is not achieved in the real world. Therefore, parity is violated.



☆ Note that it is the change of the current direction in the solenoid that causes the change of the direction of the spin in the mirror.

If the mirror is putting downside, then

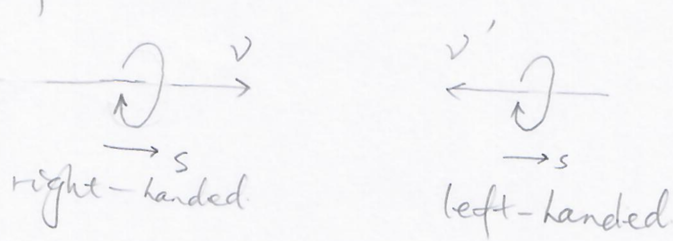


Again, in the mirror the relative orientation of the ^{60}Co spin and electron direction is not achieved in the real world.

Helicity: if we choose the direction of motion of a particle as the z-axis, then the value of m_{sz}/s for this axis is called the helicity of the particle.

Thus a particle of spin $\frac{1}{2}$ can have a helicity of $+1$ ($m_{sz} = \frac{1}{2}$) or -1 ($m_{sz} = -\frac{1}{2}$); we call the former 'right-handed' and the latter 'left-handed'.

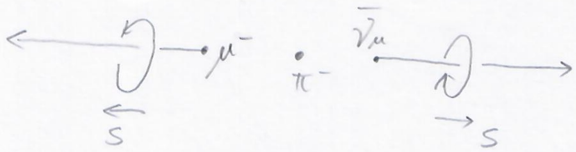
Helicity is not Lorentz-invariant for a massive particle, since we can always find a reference frame which moves faster than the speed of the particle so that the velocity of the particle changes to opposite direction in the new reference frame, and therefore the helicity changes direction (left-handed $\Leftrightarrow$ right-handed), since the spin direction does not change in the new reference frame.



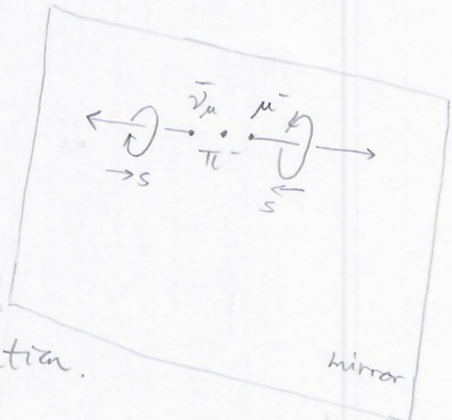
Helicity is Lorentz invariant for a massless particle, since it is impossible to reverse the direction of motion of a massless particle by getting into a faster-moving reference frame.

Through the analysis of $\pi^{\pm}$ decay, $\pi^+ \to \mu^+ + \nu_\mu$, $\pi^- \to \mu^- + \bar{\nu}_\mu$, the handness of the neutrino and anti-neutrino can be inferred: In the π^- rest frame, the μ^- and $\bar{\nu}_\mu$ come back to back; since the π^- has spin 0, the μ^- and $\bar{\nu}_\mu$ spins must be oppositely aligned, i.e., $m_{sz}(\mu^-) = -m_{sz}(\bar{\nu}_\mu)$, and therefore they have the same helicity (note that the orbital angular momentum, if there is any, points perpendicular to the outgoing velocities, so it does not affect the argument). In experiments, the μ^- helicity is measured to be right-handed always, and μ^+ is measured to be left-handed always, and therefore, $\bar{\nu}_\mu$ is right-handed and ν_μ is left-handed. The key point for this experiment is that by conservation of angular momentum, the helicity of ν_μ and $\bar{\nu}_\mu$ can be inferred from the helicity of μ^+ and μ^-, which are easier to be measured in experiment than to directly measure the ν_μ and $\bar{\nu}_\mu$ helicities.

By contrast, for $\pi^0 \longrightarrow \gamma + \gamma$, again the two photons have the same helicity, but this is an electromagnetic process, which respects parity, and thus, on average, we get just as many right-handed photon pairs as left-handed pairs.

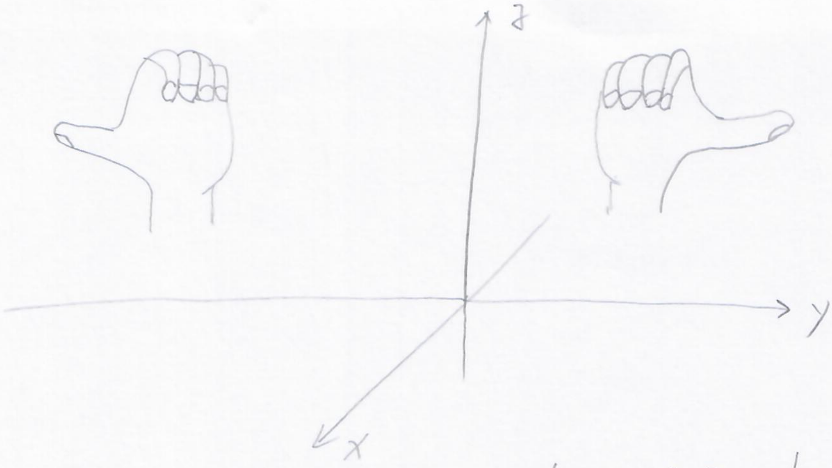


the mirror process is not achieved in the real world — parity violation.



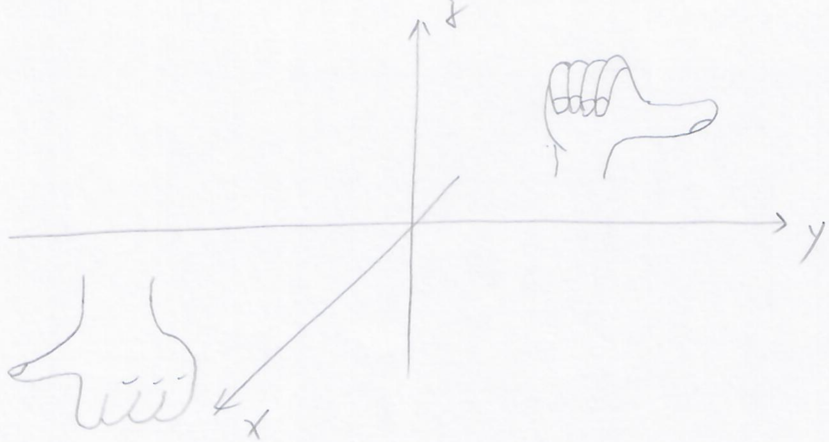
Since both reflection and inversion turn a right hand into a left hand, it does not matter which one to use for parity transformation, as long as the system also possess rotational symmetry. This avoids the arbitrariness in choosing the plane of the mirror if using reflection for parity transformation.

Inversion carries every point through the origin to the diametrically opposite location; it is nothing but a reflection followed by a rotation.



reflection in the x-z plane.

$(x, y, z) \rightarrow (x, -y, z)$



inversion $(x, y, z) \rightarrow (-x, -y, -z)$

We use P to denote inversion, and call it the 'parity operator'.

Act on a vector $\vec{a}$, $P(\vec{a}) = -\vec{a}$.

Act on a pseudo vector $\vec{c}$, $P(\vec{c}) = \vec{c}$. (An example is if $\vec{c} = \vec{a} \times \vec{b}$, where $\vec{a}$ and $\vec{b}$ are vectors)

Examples of vector: electric field $\vec{E}$, velocity $\vec{v}$;

Examples of pseudo vector: magnetic field $\vec{B}$, angular momentum $\vec{J}$.

Note that in a theory with parity invariance, you can not add a vector to a pseudovector. For example, $(\vec{E} + \vec{v} \times \vec{B})$ is OK, but $(\vec{E} + \vec{B})$ is not.

Note that a vector is also called a polar vector, a pseudovector is also called an axial vector.

Also, act on a scalar s, $P(s) = s$; act on a pseudoscalar s, $P(s) = -s$

e.g., a dot product of two polar vectors. e.g., a dot product of a polar and a pseudo vectors.

Since apply parity operator twice get back to the original, $P^2 = I$, then the eigenvalues of P are ± 1.

$$P|\varphi\rangle = x|\varphi\rangle$$

$$|\varphi\rangle = I|\varphi\rangle = P\cdot P|\varphi\rangle = xP|\varphi\rangle = x^2|\varphi\rangle$$

$$\Rightarrow \quad x^2 = 1 \Rightarrow \quad x = \pm 1.$$

Scalars and pseudovectors have eigenvalue $+1$, whereas vectors and pseudoscalars have eigenvalue -1.

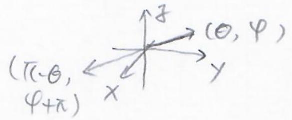
Note that hadrons are eigenstates of P and can be classified according to their eigenvalue just as they can be classified by spin, charge, isospin, strangeness, and so on.

Rules:

The parity of a fermion must be opposite to that of the corresponding antiparticle, while the parity of a boson is the same as its antiparticle. By convention, the intrinsic parity of the quark is taken to be positive, and the antiquarks negative.

Parity of a composite system is given by the product of the parity of the constituents, with an additional contribution of $(-1)^l$ according to the orbital angular momentum l. Therefore, mesons carry parity $(-1)^{l+1}$, baryons carry parity $(+1)^3 \cdot (-1)^l = (-1)^l$. This $(-1)^l$ factor comes from the angular part of the spatial wave function $Y_l^m(\theta, \varphi)$, which becomes, under a P transformation, $Y_l^m(\theta,\varphi) \xrightarrow{P} Y_l^m(\pi-\theta, \varphi+\pi) = (-1)^l Y_l^m(\theta,\varphi)$

For the photon, its intrinsic parity is -1.



$$\begin{cases} x = r\sin\theta\sin\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases}$$

Motivation for Lee and Yang's proposal that weak interaction may not respect parity conservation: the 'tau-theta puzzle' — two strange mesons, called at the time τ and θ, appeared to be identical in every respect (mass, spin, charge, etc.), except that one of them decays into two pions and the other decays into three pions, $\theta^+ \to \pi^+ + \pi^0$

$$\tau^+ \to \begin{cases} \pi^+ + \pi^0 + \pi^0 \\ \pi^+ + \pi^- + \pi^- \end{cases}$$

Since both θ^+ and τ^+ have spin 0, and the $\pi^\pm$ and π^0 have spin 0, then both the two pion and three pion final states cannot have orbital angular momentum (by conservation of total angular momentum).

Since the $\pi^\pm$ and π^0 have odd parity (i.e., $P = -1$), then the two pion state has parity $(-1)^2 = +1$, whereas the three pion states have parity $(-1)^3 = -1$.

It seemed strange that two otherwise identical particles θ^+ and τ^+ should carry opposite parity, if parity is conserved. The alternative, suggested by Lee and Yang was that τ^+ and θ^+ are really the same particle (now known as the K^+, with spin 0 and parity -1), and parity is simply not conserved in one of the decays (i.e., the two pion decay)

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terminology for 4-vector:

$a^\mu = (a^0, \vec{a})$ is called a vector if $P(\vec{a}) = -\vec{a}$, while it is called pseudovector if $P(\vec{a}) = \vec{a}$.

2. Charge Conjugation

Charge conjugation, C, converts each particle into its antiparticle:

$$C|p\rangle = |\bar{p}\rangle.$$

This operation changes the sign of all the internal quantum numbers — charge, baryon number, lepton number, strangeness, etc., while leaving mass, energy, momentum, and spin untouched.

As with parity, application of C twice brings us back to the original state, so $C^2 = I$, and the eigenvalue of C are ± 1.

However, note that only those particles that are their own antiparticle can be eigenstates of C. For if $|p\rangle$ is an eigenstate of C, then

$$C|p\rangle = \pm|p\rangle = |\bar{p}\rangle,$$

so that $|p\rangle$ and $|\bar{p}\rangle$ at most differ by a sign, which means that they represent the same physical state.

The photon is its own antiparticle. Because the photon is the quanta of the electromagnetic field ($\vec{E}$ & $\vec{B}$), which changes sign under C (since under a change in the sign of all electric charges, $\vec{E}$ & $\vec{B}$ change sign), the 'charge conjugation number' for photon is -1.

Also, a system consisting of a spin $\frac{1}{2}$ particle and its antiparticle, in a configuration with orbital angular momentum l and total spin s, constitutes an eigenstate of C with eigenvalue $(-1)^{l+s}$. Therefore, for π^0, which has $l=0$ and $s=0$, it is $C=(-1)^{0+0}=+1$; while for ρ^0, which has $l=0$ and $s=1$, it is $C=(-1)^{0+1}=-1$.

Bilinear combinations of fermion fields under P, C, and T

By imposing that the free fermion Lagrangian is invariant under seperate transformations of P, C, and T, we derive how the bilinear combinations of fermion fields transform under P, C, and T.

1. Parity

$x \to \tilde{x}$: means $t \to t$, $\vec{x} \to -\vec{x}$.

$$\mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x)$$

From $\psi(x) \longrightarrow \eta A \psi(\tilde{x})$, where η is a phase ($|\eta|^2 = 1$), and A is a 4x4 matrix independent of space-time coordinate.

$$\Rightarrow \psi^+(x) \longrightarrow \eta^* \psi^+(\tilde{x}) A^+$$

$$\Rightarrow i\,\eta^* \psi^+(\tilde{x}) A^+ \gamma^0 \gamma^\mu \partial_\mu (\eta A \psi(\tilde{x})) - m\,\eta^* \psi^+(\tilde{x}) A^+ \gamma^0 \eta A \psi(\tilde{x})$$

$$\overset{\text{require}}{\equiv} i\bar{\psi}(\tilde{x})\gamma^\mu \tilde{\partial}_\mu \psi(\tilde{x}) - m\bar{\psi}(\tilde{x})\psi(\tilde{x})$$

note that since the action integrates over all $\vec{x}$, we can change $\vec{x} \to -\vec{x}$ and the action will be invariant.

$$\Rightarrow \begin{cases} \gamma^0 A^+ \gamma^0 \gamma^\mu A = \tilde{\gamma}^\mu = \begin{cases} \gamma^0, & \mu = 0 \\ -\gamma^i, & \mu = 1,2,3 \end{cases} & ① \\ \gamma^0 A^+ \gamma^0 A = 1 \end{cases}$$

From ①, $\begin{cases} \gamma^0 A^+ \gamma^0 \gamma^0 A = \gamma^0 & ② \\ \gamma^0 A^+ \gamma^0 \gamma^i A = -\gamma^i & ③ \end{cases}$ $\Rightarrow A^+A = 1 \Rightarrow$ (right times A^+) $A^+AA^+ = A^+$

$\Downarrow$ left times $(A^+)^{-1}$

$\underbrace{(A^+)^{-1}A^+}_{=1} AA^+ = (A^+)^{-1}A^+$

$\Downarrow$

$AA^+ = 1$

So, A is a unitary matrix.

From ② $\overset{\text{left times } \gamma^0}{\Rightarrow}$ $A^+\gamma^0 A = \gamma^0$.

From ③ $\Rightarrow$ $\gamma^0 A^+ \gamma^0 \underbrace{AA^+}_{\text{use } A \text{ is unitary}} \gamma^i A = -\gamma^i \Rightarrow A^+\gamma^i A = -\gamma^i$

$$\Rightarrow \boxed{A^+\gamma^\mu A = \tilde{\gamma}^\mu = \gamma_\mu}$$

① (checked)

Therefore, $\bar{\psi}_1(x)\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\gamma^\mu\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \underbrace{\gamma^0 \gamma^\mu}_{AA^\dagger} A\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x})\gamma^0\gamma_\mu\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}(\tilde{x})\gamma_\mu\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)i\gamma_5\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \underbrace{\gamma^0 i}_{AA^\dagger}\gamma_5 A\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) A^\dagger i\gamma_5 A\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) A^\dagger i\, i\underbrace{\gamma^0\gamma^1}_{AA^\dagger}\underbrace{\gamma^2}_{AA^\dagger}\underbrace{\gamma^3}_{AA^\dagger} A\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\, i^2\gamma^0\gamma_1\gamma_2\gamma_3\psi_2(\tilde{x})$$

$$= -\eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\, i\gamma_5\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\gamma^\mu\gamma_5\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \underbrace{\gamma^0\overbrace{\gamma^\mu}^{AA^\dagger}}_{AA^\dagger}\gamma_5 A\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\gamma_\mu(-\gamma_5)\psi_2(\tilde{x})$$

$$= -\eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\gamma_\mu\gamma_5\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\sigma^{\mu\nu}\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 \overbrace{\frac{i}{2}}^{AA^\dagger}(\gamma^\mu\overbrace{\gamma^\nu-\gamma^\nu}^{AA^\dagger}\overbrace{\gamma^\mu}^{AA^\dagger}) A\psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\sigma_{\mu\nu}\psi_2(\tilde{x})$$

② (checked)

2. Charge conjugation

since $\mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x)$

$= \partial_\mu[i\bar{\psi}(x)\gamma^\mu\psi(x)] - i(\partial_\mu\bar{\psi}(x))\gamma^\mu\psi(x) - m\bar{\psi}(x)\psi(x)$

then it is OK for $\mathcal{L}'(x)$ takes the form $-i(\partial_\mu\bar{\psi}(x))\gamma^\mu\psi(x) - m\bar{\psi}(x)\psi(x)$ to fulfill the invariance of charge conjugation

Also, since charge conjugation change particle to antiparticle, ψ should transform to ψ^*; considering that $\bar{\psi}$ is more frequently used, the transformation rule is

$\psi(x) \longrightarrow \varepsilon B\bar{\psi}^T(x)$, where ε is a phase, $|\varepsilon|^2 = 1$,
B is a 4x4 matrix independent of spacetime

$\Rightarrow \psi^+(x) \longrightarrow \varepsilon^*(B\bar{\psi}^T(x))^+ = \varepsilon^*\bar{\psi}^*(x)B^+ = \varepsilon^*(\psi^+(x)\gamma^0)^*B^+$
$= \varepsilon^*\psi^T(x)\gamma^{0*}B^+$
$= \varepsilon^*\psi^T(x)\gamma^{0T}B^+$.

$\Rightarrow i\,\varepsilon^*\psi^T(x)\gamma^{0T}B^+\gamma^0\gamma^\mu\partial_\mu(\varepsilon B\bar{\psi}^T(x)) - m\varepsilon^*\psi^T(x)\gamma^{0T}B^+\gamma^0\varepsilon B\bar{\psi}^T(x)$

$= i\psi^T(x)\gamma^{0T}B^+\gamma^0\gamma^\mu B\,\partial_\mu\bar{\psi}^T(x) - m\,\psi^T(x)\gamma^{0T}B^+\gamma^0 B\bar{\psi}^T(x)$

$= i\left(\psi^T(x)\gamma^{0T}B^+\gamma^0\gamma^\mu B\,\partial_\mu\bar{\psi}^T(x)\right)^T - m\left(\psi^T(x)\gamma^{0T}B^+\gamma^0 B\bar{\psi}^T(x)\right)^T$

$= -\,i(\partial_\mu\bar{\psi}(x))B^T\gamma^{\mu T}\gamma^{T0}B^*\gamma^0\psi(x) + m\bar{\psi}(x)B^T\gamma^{0}B^*\gamma^0\psi(x)$

due to exchange the position of two fermion operator

$\overset{\text{require}}{\equiv} -i(\partial_\mu\bar{\psi}(x))\gamma^\mu\psi(x) - m\bar{\psi}(x)\psi(x)$

right times γ^0 on both sides

$\Rightarrow \begin{cases} B^T\gamma^{0T}B^*\gamma^0 = -1 \Rightarrow B^T\gamma^{0T}B^* = -\gamma^0 \Rightarrow B^+\gamma^{0+}B = -\gamma^{0*} & \cdots ① \\ B^T\gamma^{\mu T}\gamma^{T0}B^*\gamma^0 = \gamma^\mu & \cdots ② \end{cases}$

do complex conjugate

use $\begin{cases}\gamma^0\gamma^\mu\gamma^0 = \gamma^{+\mu} \\ \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\end{cases} \Rightarrow \gamma^0 = \gamma^{+0},\ \gamma^{+i} = -\gamma^i \Rightarrow \gamma^{*0} = \gamma^{T0},\ \gamma^{*i} = -\gamma^{Ti}$

from ① $\Rightarrow B^+\gamma^0 B = -\gamma^{T0}$, from ② $\Rightarrow \begin{cases} B^T\gamma^{T0}\gamma^{T0}B^*\gamma^0 = \gamma^0 \Rightarrow B^TB^* = 1 \Rightarrow B^+B = 1, \text{ so } B \text{ is a unitary matrix} \\ B^T\gamma^{Ti}\gamma^{T0}B^*\gamma^0 = \gamma^i \end{cases}$

③ (checked)

From $B^T \gamma^{iT} \gamma^{0T} B^* \gamma^0 = \gamma^i$

$$\Rightarrow B^\dagger \gamma^{i\dagger} \gamma^{0\dagger} B \gamma^{*0} = \gamma^{*i} \Rightarrow -B^\dagger \underbrace{\gamma^i \gamma^0}_{BB^\dagger} B \gamma^{T0} = -\gamma^{Ti} \Rightarrow B^\dagger \gamma^i B(-\gamma^{T0})\gamma^{T0} = \gamma^{Ti}$$

do complex conjugate

$$\Rightarrow B^\dagger \gamma^i B = -\gamma^{Ti}$$

$$\Rightarrow \boxed{B^\dagger \gamma^\mu B = -\gamma^{T\mu}}$$

Therefore, $\bar\psi_1(x)\psi_2(x)$

$$\to \varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{0T} B^\dagger \gamma^0 B \bar\psi_2^T(x) = \varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{0T}(-\gamma^{0T})\bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 \psi_1^T(x)\bar\psi_2^T(x) = -\varepsilon_1^*\varepsilon_2(\psi_1^T(x)\,\bar\psi_2^T(x))^T$$

$$= \varepsilon_1^*\varepsilon_2 \bar\psi_2(x)\psi_1(x)$$

$\bar\psi_1(x)\gamma^\mu\psi_2(x)$

$$\to \varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{0T} B^\dagger \underbrace{\gamma^0 \gamma^\mu}_{BB^\dagger} B \bar\psi_2^T(x) = \varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{0T}(-\gamma^{0T})(-\gamma^{T\mu})\bar\psi_2^T(x)$$

$$= \varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{T\mu}\bar\psi_2^T(x) = \varepsilon_1^*\varepsilon_2(\psi_1^T(x)\gamma^{T\mu}\bar\psi_2^T(x))^T$$

$$= -\varepsilon_1^*\varepsilon_2 \bar\psi_2(x)\gamma^\mu\psi_1(x)$$

$\bar\psi_1(x)i\gamma_5\psi_2(x)$

$$\to \varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{0T} B^\dagger \underbrace{\gamma^0 i\gamma_5}_{BB^\dagger} B \bar\psi_2^T(x) = -\varepsilon_1^*\varepsilon_2 \psi_1^T(x)\gamma^{0T}\gamma^{0T} iB^\dagger\gamma_5 B\bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 \psi_1^T(x)\, iB^\dagger i \underbrace{\gamma^0 \gamma^1}_{BB^\dagger} \underbrace{\gamma^1 \gamma^2}_{BB^\dagger} \underbrace{\gamma^2\gamma^3}_{BB^\dagger} B \bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 \psi_1^T(x)\, i^2\gamma^{0T}\gamma^{1T}\gamma^{2T}\gamma^{3T}\bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 (\psi_1^T(x)\, i^2\gamma^{0T}\gamma^{1T}\gamma^{2T}\gamma^{3T}\bar\psi_2^T(x))^T$$

$$= \varepsilon_1^*\varepsilon_2 \bar\psi_2(x)\, i^2\gamma^3\gamma^2\gamma^1\gamma^0\psi_1(x)$$

$$= \varepsilon_1^*\varepsilon_2 \bar\psi_2\, i\gamma_5\psi_1(x)$$

$$\bar{\psi}_1(x)\gamma^\mu\gamma_5\psi_2(x)$$

$$\rightarrow \epsilon_1^*\epsilon_2\,\psi_1^T(x)\gamma^{T0}B^+\gamma^0\underset{BB^+}{\wedge}\gamma^\mu\overset{BB^+}{\vee}\gamma_5 B\bar{\psi}_2^T(x)$$

$$= \epsilon_1^*\epsilon_2\,\psi_1^T(x)\underline{\gamma^{T0}\gamma^{T0}}\gamma^{T\mu}B^+\gamma_5 B\bar{\psi}_2^T(x)$$

$$= \epsilon_1^*\epsilon_2\,\psi_1^T(x)\gamma^{T\mu}i\gamma^{T0}\gamma^{T1}\gamma^{T2}\gamma^{T3}\bar{\psi}_2^T(x)$$

$$= \epsilon_1^*\epsilon_2\left(\psi_1^T(x)\gamma^{T\mu}i\gamma^{T0}\gamma^{T1}\gamma^{T2}\gamma^{T3}\bar{\psi}_2^T(x)\right)^T$$

$$= -\epsilon_1^*\epsilon_2\,\bar{\psi}_2(x)i\gamma^3\gamma^2\gamma^1\gamma^0\gamma^\mu\psi_1(x)$$

$$= -\epsilon_1^*\epsilon_2\,\bar{\psi}_2(x)\gamma_5\gamma^\mu\psi_1(x)$$

$$= \epsilon_1^*\epsilon_2\,\bar{\psi}_2(x)\gamma^\mu\gamma_5\psi_1(x)$$

$$\bar{\psi}_1(x)\sigma^{\mu\nu}\psi_2(x)$$

$$\rightarrow \epsilon_1^*\epsilon_2\,\psi_1^T(x)\gamma^{T0}B^+\gamma^0\sigma^{\mu\nu}B\bar{\psi}_2^T(x)$$

$$= \epsilon_1^*\epsilon_2\,\psi_1^T(x)\gamma^{T0}B^+\gamma^0\underset{BB^+}{\wedge}\frac{i}{2}(\gamma^\mu\underset{BB^+}{\wedge}\gamma^\nu-\gamma^\nu\underset{BB^+}{\wedge}\gamma^\mu)B\bar{\psi}_2^T(x)$$

$$= -\epsilon_1^*\epsilon_2\,\psi_1^T(x)\underline{\gamma^{T0}\gamma^{T0}}\frac{i}{2}(\gamma^{T\mu}\gamma^{T\nu}-\gamma^{T\nu}\gamma^{T\mu})\bar{\psi}_2^T(x)$$

$$= -\epsilon_1^*\epsilon_2\left(\psi_1^T(x)\frac{i}{2}(\gamma^{T\mu}\gamma^{T\nu}-\gamma^{T\nu}\gamma^{T\mu})\bar{\psi}_2^T(x)\right)^T$$

$$= \epsilon_1^*\epsilon_2\,\bar{\psi}_2(x)\frac{i}{2}(\gamma^\nu\gamma^\mu-\gamma^\mu\gamma^\nu)\psi_1(x)$$

$$= -\epsilon_1^*\epsilon_2\,\bar{\psi}_2(x)\sigma^{\mu\nu}\psi_1(x)$$

3. time inversion. $\gamma^\mu \to \tilde{\gamma}^\mu$ $\begin{cases} t \to -t \\ \vec{x} \to \vec{x} \end{cases}$

since time inversion exchange initial and final state, that is, it changes a 'bra' to a 'ket', then it involves complex conjugate

From $\psi(x) \longrightarrow \xi D \psi(\tilde{x})$, where ξ is a phase, $|\xi|^2 = 1$, D is a 4x4 matrix independent of spacetime

$\Rightarrow \psi^+(x) \longrightarrow \xi^* \psi^+(\tilde{x}) D^+$

$$\Rightarrow \mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x)$$

$$\longrightarrow (-i)\xi^* \psi^+(\tilde{x}) D^+ \gamma^{0*} \gamma^{\mu*} \partial_\mu (\xi D \psi(\tilde{x}))$$
$$- m\xi^* \psi^+(\tilde{x}) D^+ \gamma^{0*} \xi D \psi(\tilde{x})$$
$$= -i\bar{\psi}(\tilde{x}) \gamma^0 D^+ \gamma^{0*} \gamma^{\mu*} D \partial_\mu \psi(\tilde{x})$$
$$- m\bar{\psi}(\tilde{x}) \gamma^0 D^+ \gamma^{0*} D \psi(\tilde{x})$$

note that since the action integrates over t, we can change $t \to -t$ and the action will be invariant.

$$\overset{\text{require.}}{=} i\bar{\psi}(\tilde{x}) \gamma^\mu \tilde{\partial}_\mu \psi(\tilde{x}) - m\bar{\psi}(\tilde{x})\psi(\tilde{x})$$

$$\Rightarrow \begin{cases} -\gamma^0 D^+ \gamma^{0*} \gamma^{\mu*} D = \tilde{\gamma}^\mu = \begin{cases} -\gamma^0, & \mu = 0 \quad \text{--- ①} \\ \gamma^i, & \mu = 1,2,3 \quad \text{--- ②} \end{cases} \\ \gamma^0 D^+ \gamma^{0*} D = 1 \quad \text{--- ③} \end{cases}$$

From ③ $\Rightarrow D^+ \gamma^{0*} D = \gamma^0 \Rightarrow D^+ \gamma^{0T} D = \gamma^0$

From ① $\Rightarrow D^+ \gamma^{0*} \gamma^{0*} D = 1 \Rightarrow D^+ D = 1$, so D is unitary matrix.

From ② $\Rightarrow -\gamma^0 D^+ \underbrace{\gamma^{0*}}_{DD^+} \gamma^{i*} D = \gamma^i \Rightarrow -\gamma^0 \gamma^0 D^+ \gamma^{i*} D = \gamma^i$

$$\Rightarrow D^+ \gamma^{iT} D = \gamma^i$$

$$\Rightarrow \boxed{D^+ \gamma^{T\mu} D = \gamma^\mu}$$

$$\Rightarrow \boxed{D^+ \gamma^{*\mu} D = \gamma_\mu}$$

⑥ (checked)

Therefore, $\bar{\psi}_1(x)\psi_2(x)$

$$\to \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\underbrace{\gamma^{*0}}_{\gamma^0}D\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\psi_1^+(\tilde{x})\gamma^0\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\bar{\psi}_1(\tilde{x})\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\gamma^\mu\psi_2(x)$

$$\to \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\underbrace{\gamma^{*0}\gamma^{*\mu}}_{DD^+}D\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\psi_1^+(\tilde{x})\gamma^0\gamma_\mu\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\bar{\psi}_1(\tilde{x})\gamma_\mu\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)i\gamma_5\psi_2(x)$

$$\to \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\gamma^{*0}(i\gamma_5)^*D\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\underbrace{\gamma^{*0}}_{DD^+}(-i)(-i)\underbrace{\gamma^{*0}}_{DD^+}\underbrace{\gamma^{*1}}_{DD^+}\underbrace{\gamma^{*2}}_{DD^+}\gamma^{*3}D\psi_2(\tilde{x})$$

$$= -\xi_1^*\xi_2\psi_1^+(\tilde{x})\gamma^0\gamma^0\gamma_1\gamma_2\gamma_3\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\bar{\psi}_1(\tilde{x})\gamma^0\gamma^1\gamma^2\gamma^3\psi_2(\tilde{x})$$

$$= -\xi_1^*\xi_2\bar{\psi}_1(\tilde{x})i\gamma_5\psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\gamma^\mu\gamma_5\psi_2(x)$

$$\to \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\underbrace{\gamma^{*0}\gamma^{*\mu}}_{DD^+}\underbrace{\gamma_5^*}_{DD^+}D\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\psi_1^+(\tilde{x})\gamma^0\gamma_\mu(-i)\gamma^0\gamma_1\gamma_2\gamma_3\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\bar{\psi}_1(\tilde{x})\gamma_\mu\gamma_5\psi_2(\tilde{x})$$

$\bar{\psi}_1\sigma^{\mu\nu}\psi_2(x)$

$$\to \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\gamma^{*0}(\sigma^{\mu\nu})^*D\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\psi_1^+(\tilde{x})D^+\underbrace{\gamma^{*0}}_{DD^+}\left(-\frac{i}{2}\right)(\underbrace{\gamma^{*\mu}\gamma^{*\nu}}_{DD^+}-\underbrace{\gamma^{*\nu}\gamma^{*\mu}}_{DD^+})D\psi_2(\tilde{x})$$

$$= \xi_1^*\xi_2\psi_1^+(\tilde{x})\gamma^0\left(-\frac{i}{2}\right)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\psi_2(\tilde{x})$$

$$= -\xi_1^*\xi_2\bar{\psi}_1(\tilde{x})\sigma_{\mu\nu}\psi_2(\tilde{x})$$

(checked)

To summarize, up to the phase factor $\eta_1^*\eta_2$, $\varepsilon_1^*\varepsilon_2$, $\rho_1^*\rho_2$, we have the following transformation properties for bilinear combinations

	P	C	T	CP	CPT
$\bar\psi_1\psi_2$	$\bar\psi_1\psi_2$	$\bar\psi_2\psi_1$	$\bar\psi_1\psi_2$	$\bar\psi_2\psi_1$	$\bar\psi_2\psi_1$
$\bar\psi_1\gamma^\mu\psi_2$	$\bar\psi_1\gamma_\mu\psi_2$	$-\bar\psi_2\gamma^\mu\psi_1$	$\bar\psi_1\gamma_\mu\psi_2$	$-\bar\psi_2\gamma_\mu\psi_1$	$-\bar\psi_2\gamma^\mu\psi_1$
$\bar\psi_1 i\gamma_5\psi_2$	$-\bar\psi_1 i\gamma_5\psi_2$	$\bar\psi_2 i\gamma_5\psi_1$	$-\bar\psi_1 i\gamma_5\psi_2$	$-\bar\psi_2 i\gamma_5\psi_1$	$\bar\psi_2 i\gamma_5\psi_1$
$\bar\psi_1\gamma^\mu\gamma_5\psi_2$	$-\bar\psi_1\gamma_\mu\gamma_5\psi_2$	$\bar\psi_2\gamma^\mu\gamma_5\psi_1$	$\bar\psi_1\gamma_\mu\gamma_5\psi_2$	$-\bar\psi_2\gamma_\mu\gamma_5\psi_1$	$-\bar\psi_2\gamma^\mu\gamma_5\psi_1$
$\bar\psi_1\sigma^{\mu\nu}\psi_2$	$\bar\psi_1\sigma_{\mu\nu}\psi_2$	$-\bar\psi_2\sigma^{\mu\nu}\psi_1$	$-\bar\psi_1\sigma_{\mu\nu}\psi_2$	$-\bar\psi_2\sigma_{\mu\nu}\psi_1$	$\bar\psi_2\sigma^{\mu\nu}\psi_1$

Note that the argument also changes for P: $\vec{x}\to-\vec{x}$, and T: $t\to -t$, CP: $\vec{x}\to-\vec{x}$, CPT: $t\to -t$, $\vec{x}\to-\vec{x}$.

Recall that for proper homogeneous Lorentz transformation (which does not include P, T and (PT)), the transformation properties for bilinear combinations are

$$\bar\psi_1(x)\psi_2(x) \longrightarrow \bar\psi_1(x)\psi_2(x)$$

$$\bar\psi_1(x)\gamma^\mu\psi_2(x) \longrightarrow \Lambda^\mu{}_\nu\bar\psi_1(x)\gamma^\nu\psi_2(x)$$

$$\bar\psi_1(x) i\gamma_5\psi_2(x) \longrightarrow \underbrace{Det(\Lambda)}_{=1}\bar\psi_1(x) i\gamma_5\psi_2(x) = \bar\psi_1(x) i\gamma_5\psi_2(x)$$

$$\bar\psi_1(x)\gamma^\mu\gamma_5\psi_2(x) \longrightarrow \Lambda^\mu{}_\nu \underbrace{Det(\Lambda)}_{=1}\bar\psi_1(x)\gamma^\nu\gamma_5\psi_2(x) = \Lambda^\mu{}_\nu\bar\psi_1(x)\gamma^\nu\gamma_5\psi_2(x)$$

$$\bar\psi_1(x)\sigma^{\mu\nu}\psi_2(x) \longrightarrow \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar\psi_1(x)\sigma^{\alpha\beta}\psi_2(x)$$

Since electromagnetic interaction conserve C, P and T, then from the invariance of interaction Lagrangian,

$\mathcal{L}_{int} = -\varepsilon |e| \bar{\psi} \gamma^\mu \psi A_\mu$ (where $|e| = \sqrt{4\pi\alpha}$, and $\varepsilon = -1$ when ψ describes the electron-positron field, $\varepsilon = +1$ when ψ describes the proton-antiproton field, $\varepsilon = +\frac{2}{3}$ when ψ describes the up-quark anti-up-quark field, ect), we derive

$$A^\mu(t,\vec{x}) \xrightarrow{P} A_\mu(t,-\vec{x}), \text{ i.e., } \begin{cases} A^0(t,\vec{x}) \xrightarrow{P} A^0(t,-\vec{x}) \\ \vec{A}(t,\vec{x}) \xrightarrow{P} -\vec{A}(t,-\vec{x}) \end{cases}$$

$$A^\mu(t,\vec{x}) \xrightarrow{C} -A^\mu(t,\vec{x})$$

$$A^\mu(t,\vec{x}) \xrightarrow{T} A_\mu(-t,\vec{x}), \text{ i.e., } \begin{cases} A^0(t,\vec{x}) \xrightarrow{T} A^0(-t,\vec{x}) \\ \vec{A}(t,\vec{x}) \xrightarrow{T} -\vec{A}(-t,\vec{x}) \end{cases}$$

From the Maxwell's equations (which is invarance under, C, P and T),

$$\vec{\nabla}\cdot\vec{E} = \rho$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

we get

$$\vec{E}(t,\vec{x}) \xrightarrow{P} -\vec{E}(t,-\vec{x})$$

$$\vec{B}(t,\vec{x}) \xrightarrow{P} \vec{B}(t,-\vec{x})$$

$$\rho(t,\vec{x}) \xrightarrow{P} \rho(t,-\vec{x})$$

$$\vec{j}(t,\vec{x}) \xrightarrow{P} -\vec{j}(t,-\vec{x})$$

$$\vec{\nabla} \xrightarrow{P} -\vec{\nabla}$$

$$\frac{\partial}{\partial t} \xrightarrow{P} \frac{\partial}{\partial t}$$

$$\rho(t,\vec{x}) \xrightarrow{C} -\rho(t,\vec{x})$$

$$\vec{E}(t,\vec{x}) \xrightarrow{C} -\vec{E}(t,\vec{x})$$

$$\vec{B}(t,\vec{x}) \xrightarrow{C} -\vec{B}(t,\vec{x})$$

$$\vec{j}(t,\vec{x}) \xrightarrow{C} -\vec{j}(t,\vec{x})$$

P (checked) $\quad \frac{\partial}{\partial t} \xrightarrow{C} \frac{\partial}{\partial t}, \quad \vec{\nabla} \xrightarrow{C} \vec{\nabla}$

$$\frac{\partial}{\partial t} \xrightarrow{T} -\frac{\partial}{\partial t}$$
$$\vec{\nabla} \xrightarrow{T} \vec{\nabla}$$
$$\rho(t,\vec{x}) \xrightarrow{T} \rho(-t,\vec{x})$$
$$\vec{E}(t,\vec{x}) \xrightarrow{T} \vec{E}(-t,\vec{x})$$
$$\vec{B}(t,\vec{x}) \xrightarrow{T} -\vec{B}(-t,\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{T} -\vec{j}(-t,\vec{x})$$

On the other hand, since $F^{i0} = E^i$, $F^{ij} = -\varepsilon^{ijk}B^k$ (where $\varepsilon^{123} = \varepsilon_{123} = 1$, ε^{ijk} is the Levi-Civita symbol)

and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

then $$E^i = \partial^i A^0 - \partial^0 A^i \Rightarrow \vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}$$

$$-\varepsilon^{ijk}B^k = F^{ij} \Rightarrow -\varepsilon_{ijl}\varepsilon^{ijk}B^k = \varepsilon_{ijl}F^{ij}$$
$$\Rightarrow -2B^l = \varepsilon_{ijl}F^{ij}$$
$$\Rightarrow B^1 = -\tfrac{1}{2}(F^{23} - F^{32}) = -F^{23} = -(\partial^2A^3 - \partial^3A^2)$$
$$B^2 = -\tfrac{1}{2}(F^{31} - F^{13}) = -F^{31} = -(\partial^3A^1 - \partial^1A^3)$$
$$B^3 = -\tfrac{1}{2}(F^{12} - F^{21}) = -F^{12} = -(\partial^1A^2 - \partial^2A^1)$$
$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}.$$

Therefore, we can derive the transformation properties of $\vec{E}$ and $\vec{B}$ directly from the ones for A^μ, i.e.,

$$\vec{E}(t,\vec{x}) \xrightarrow{P} -\vec{E}(t,-\vec{x})$$
$$\vec{E}(t,\vec{x}) \xrightarrow{C} -\vec{E}(t,\vec{x})$$
$$\vec{E}(t,\vec{x}) \xrightarrow{T} E(-t,\vec{x})$$
$$\vec{B}(t,\vec{x}) \xrightarrow{P} \vec{B}(t,-\vec{x})$$
$$\vec{B}(t,\vec{x}) \xrightarrow{C} -\vec{B}(t,\vec{x})$$
$$\vec{B}(t,\vec{x}) \xrightarrow{T} -\vec{B}(-t,\vec{x})$$

Consistent with the properties derived from Maxwell's equations.

Also, the electric current for a fermion field is $j^\mu = q|e|\bar{\psi}\gamma^\mu\psi$, then from $j^\mu = (\rho, \vec{j})$ and the transformation properties for $\bar{\psi}\gamma^\mu\psi$, we get

$$\rho(t,\vec{x}) \xrightarrow{P} \rho(t,-\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{P} -\vec{j}(t,-\vec{x})$$
$$\rho(t,\vec{x}) \xrightarrow{C} -\rho(t,\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{C} -\vec{j}(t,\vec{x})$$
$$\rho(t,\vec{x}) \xrightarrow{T} \rho(-t,\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{T} -\vec{j}(-t,\vec{x})$$

(note that $q|e|$ is a pure number, and it is not change under transformations)

Consistent with the properties derived from Maxwell's equations.

We can check that the transformation properties of A^μ makes $\mathcal{L}_{EM} \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ invariant under C, P and T:

$$F^{\mu\nu}(t,\vec{x}) = \partial^\mu A^\nu(t,\vec{x}) - \partial^\nu A^\mu(t,\vec{x})$$

$$F^{\mu\nu}(t,\vec{x}) \xrightarrow{P} \partial_\mu A_\nu(t,-\vec{x}) - \partial_\nu A_\mu(t,-\vec{x}) = F_{\mu\nu}(t,-\vec{x})$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x}) \xrightarrow{P} -\frac{1}{4}F_{\mu\nu}(t,-\vec{x})F^{\mu\nu}(t,-\vec{x})$$

Since the action integrates over all $\vec{x}$, we can change $\vec{x} \to -\vec{x}$ and the action will be invariant.

$$F^{\mu\nu}(t,\vec{x}) \xrightarrow{C} -F^{\mu\nu}(t,\vec{x})$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x}) \xrightarrow{C} -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x})$$

$$F^{\mu\nu}(t,\vec{x}) \xrightarrow{T} -\partial_\mu A_\nu(-t,\vec{x}) + \partial_\nu A_\mu(-t,\vec{x}) = -F_{\mu\nu}(-t,\vec{x})$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x}) \xrightarrow{T} -\frac{1}{4}F_{\mu\nu}(-t,\vec{x})F^{\mu\nu}(-t,\vec{x})$$

Since the action integrates over all t, we can change $t \to -t$ and the action will be invariant.

For an arbitrary current obtained by an internal phase transformation of the fermion field, $\psi \to \psi' = e^{-i\delta\alpha}\psi$, where $\delta\alpha$ is a constant,

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

$$Q = \int d^3\vec{x}\, j^0$$

then from the transformation property of $\bar{\psi}\gamma^\mu\psi$, we have, for charge conjugation,

$$j^\mu_{(t,\vec{x})} \xrightarrow{C} -j^\mu(t,\vec{x})$$

$$Q(t) \xrightarrow{C} -Q(t)$$

Therefore, any charge given by this global phase transformation, e.g., electric charge, baryon number, reverses sign by charge conjugation.

For complex scalar field,

$$\phi(t,\vec{x}) \xrightarrow{P} \eta_B\phi(t,-\vec{x}), \quad |\eta_B|^2 = 1$$

$$\phi^+(t,\vec{x}) \xrightarrow{P} \eta_B^*\phi^+(t,-\vec{x}),$$

$$\phi(t,\vec{x}) \xrightarrow{C} \varepsilon_B\phi^+(t,\vec{x}), \quad |\varepsilon_B|^2 = 1$$

$$\phi^+(t,\vec{x}) \xrightarrow{C} \varepsilon_B^*\phi(t,\vec{x})$$

$$\phi(t,\vec{x}) \xrightarrow{T} \zeta_B\phi(-t,\vec{x}), \quad |\zeta_B|^2 = 1$$

$$\phi^+(t,\vec{x}) \xrightarrow{T} \zeta_B^*\phi^+(-t,\vec{x})$$

Apparently, the free field Lagrangian (more accurately, the action)

$$\mathcal{L}(t,\vec{x}) = \partial_\mu\phi(t,\vec{x})\,\partial^\mu\phi^+(t,\vec{x}) - m^2\phi(t,\vec{x})\,\phi^+(t,\vec{x})$$

is unchanged under P, C and T.

(12) [illegible]

The scalar QED interaction Lagrangian

$$\mathcal{L}(t,\vec{x}) = -i\,\mathcal{E}|e|\left[\phi^{\dagger}_{(t,\vec{x})}(\partial^{\mu}\phi_{(t,\vec{x})}) - (\partial^{\mu}\phi^{\dagger}_{(t,\vec{x})})\phi_{(t,\vec{x})}\right]A_{\mu\,(t,\vec{x})} + (\mathcal{E}|e|)^2 A_{\mu\,(t,\vec{x})}A^{\mu}_{(t,\vec{x})}\phi^{\dagger}_{(t,\vec{x})}\phi_{(t,\vec{x})},$$

transforms as

$$\mathcal{L}(t,\vec{x}) \xrightarrow{P} -i\,\mathcal{E}|e|\left[\phi^{\dagger}(t,-\vec{x})(\partial_{\mu}\phi(t,-\vec{x})) - (\partial_{\mu}\phi^{\dagger}(t,-\vec{x}))\underbrace{\,}_{\phi(t,-\vec{x})}\right]A^{\mu}(t,-\vec{x})$$
$$+ (\mathcal{E}|e|)^2 A^{\mu}(t,-\vec{x})A_{\mu}(t,-\vec{x})\,\phi^{\dagger}(t,-\vec{x})\,\phi(t,-\vec{x})$$
$$= \mathcal{L}(t,-\vec{x})$$

$$\mathcal{L}(t,\vec{x}) \xrightarrow{C} -i\,\mathcal{E}|e|\left[\phi(t,\vec{x})(\partial^{\mu}\phi^{\dagger}(t,\vec{x})) - (\partial^{\mu}\phi(t,\vec{x}))\phi^{\dagger}(t,\vec{x})\right](-A_{\mu}(t,\vec{x}))$$
$$+ (\mathcal{E}|e|)^2(-A_{\mu}(t,\vec{x}))(-A^{\mu}(t,\vec{x}))\,\phi(t,\vec{x})\,\phi^{\dagger}(t,\vec{x})$$
$$= \mathcal{L}(t,\vec{x})$$

$$\mathcal{L}(t,\vec{x}) \xrightarrow{T} \underbrace{(-i)^{*}}_{\text{note this}}\mathcal{E}|e|\left[\phi^{\dagger}(-t,\vec{x})(-\partial_{\mu}\phi(-t,\vec{x})) - (-\partial_{\mu}\phi^{\dagger}(-t,\vec{x}))\phi(-t,\vec{x})\right]$$
$$\cdot A^{\mu}(-t,\vec{x}) + (\mathcal{E}|e|)^2 A^{\mu}(-t,\vec{x})A_{\mu}(-t,\vec{x})\,\phi^{\dagger}(-t,\vec{x})\,\phi(-t,\vec{x})$$
$$= \mathcal{L}(-t,\vec{x})$$

So the action is unchanged under P, C, and T.

For an arbitrary current obtained by an internal phase transformation of the complex scalar field, $\varphi \to \varphi' = e^{-i\delta\alpha}\varphi$, where $\delta\alpha$ is a constant,

$$j^{\mu} = i\left[\phi^{\dagger}\partial^{\mu}\phi - (\partial^{\mu}\phi^{\dagger})\phi\right]$$
$$Q = \int d^3\vec{x}\; j^0$$

then

$$j^{\mu}(t,\vec{x}) \xrightarrow{C} -j^{\mu}(t,\vec{x})$$
$$Q(t) \xrightarrow{C} -Q(t)$$

as expected, and the same interpretation as the fermion field current and charge under C.

13 (checked)

For a real scaler field,

$$\phi(t,\vec{x}) \xrightarrow{P} \eta_B \phi(t,-\vec{x}), \quad \eta_B = \pm 1.$$

$$\phi(t,\vec{x}) \xrightarrow{C} \varepsilon_B \phi(t,\vec{x}), \quad \varepsilon_B = \pm 1$$

$$\phi(t,\vec{x}) \xrightarrow{T} \varphi_B \phi(-t,\vec{x}), \quad \varphi_B = \pm 1.$$

The free field Lagrangian (more accurately, the action).

$$\mathcal{L}(t,\vec{x}) = \frac{1}{2}\partial_\mu \phi(t,\vec{x})\partial^\mu \phi(t,\vec{x}) - \frac{1}{2}m^2\phi^2(t,\vec{x})$$

is unchanged under P, C and T.

Now let's look at the quantized fields (i.e., the particle creation and annihilation operators) behavior under P, C, and T.

To aid analysis, we choose the standard representation of gamma matrices,

$$\gamma^0 = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and in this representation the solutions of Dirac equation are

$$u(\vec{p},s) = (E+m)^{\frac{1}{2}}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix}, \quad v(\vec{p},s) = (E+m)^{\frac{1}{2}}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s \\ \eta_s \end{pmatrix},$$

where $s=\pm\frac{1}{2}$, χ_s and η_s are 2×1 columns and satisfy $\chi_s^+\chi_{s'} = \delta_{ss'} = \eta_s^+\eta_{s'}$ and $\sum_s \chi_s\chi_s^+ = \sum_s \eta_s\eta_s^+ = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

[1]

then for Parity transformation, $A^+\gamma^\mu A = \gamma_\mu$, we see that $A=\gamma^0$ satisfies this relation, then

from $\psi(t,\vec{x}) \xrightarrow{P} \eta\gamma^0\psi(t,-\vec{x})$, and the decomposition

$$\psi(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\sum_s \left(u(\vec{p},s)b_{\vec{p},s}e^{-ip\cdot x} + v(\vec{p},s)d^+_{\vec{p},s}e^{ip\cdot x}\right)$$

we have

$$\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u(\vec p, s) b_{\vec p, s} e^{-i(E_{\vec p} t - \vec p\cdot\vec x)} + v(\vec p, s) d^{+}_{\vec p, s} e^{i(E_{\vec p} t - \vec p\cdot\vec x)} \right]$$

$$\xrightarrow{P} \underbrace{\eta \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[\gamma^0 u(\vec p, s) b_{\vec p, s} e^{-i(E_{\vec p} t + \vec p\cdot\vec x)} + \gamma^0 v(\vec p, s) d^{+}_{\vec p, s} e^{i(E_{\vec p} t + \vec p\cdot\vec x)} \right]}_{= **}$$

using $\gamma^0 u(\vec p, s) = (E+m)^{\frac{1}{2}} \cdot \begin{pmatrix} \chi_s \\ -\frac{\vec\sigma\cdot\vec p}{E+m}\chi_s \end{pmatrix} = u(-\vec p, s)$

and $\gamma^0 v(\vec p, s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec p\cdot\vec\sigma}{E+m}\eta_s \\ -\eta_s \end{pmatrix} = -v(-\vec p, s)$

$$\Rightarrow ** = \eta \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u(-\vec p, s) b_{\vec p, s} e^{-i(E_{\vec p} t + \vec p\cdot\vec x)} - v(-\vec p, s) d^{+}_{\vec p, s} e^{i(E_{\vec p} t + \vec p\cdot\vec x)} \right]$$

note that $E_{\vec p} = E_{-\vec p}$

$$\stackrel{\vec p \to -\vec p}{=} \eta \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u(\vec p, s) b_{-\vec p, s} e^{-i(E_{\vec p} t - \vec p\cdot\vec x)} - v(\vec p, s) d^{+}_{-\vec p, s} e^{i(E_{\vec p} t - \vec p\cdot\vec x)} \right]$$

$$\Rightarrow \quad b_{\vec p, s} \xrightarrow{P} \eta\, b_{-\vec p, s}, \quad d^{+}_{\vec p, s} \xrightarrow{P} \eta(-d^{+}_{-\vec p, s})$$

Similarly, from $\psi^{+}(t, \vec x) = \int_{-\infty}^{+\infty} d^3\vec p\, C(E_{\vec p}) \sum_s \left(u^{+}(\vec p, s) b^{+}_{\vec p, s} e^{ip\cdot x} + v^{+}(\vec p, s) d_{\vec p, s} e^{-ip\cdot x} \right)$

and $\psi^{+}(t, \vec x) \xrightarrow{P} \eta^{*} \psi^{+}(t, -\vec x) A^{+} = \eta^{*} \psi^{+}(t, -\vec x)\gamma^0$

we have

$$\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u^{+}(\vec p, s) b^{+}_{\vec p, s} e^{i(E_{\vec p} t - \vec p\cdot\vec x)} + v^{+}(\vec p, s) d_{\vec p, s} e^{-i(E_{\vec p} t - \vec p\cdot\vec x)} \right]$$

$$\xrightarrow{P} \underbrace{\eta^{*} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u^{+}(\vec p, s)\gamma^0 b^{+}_{\vec p, s} e^{i(E_{\vec p} t + \vec p\cdot\vec x)} + v^{+}(\vec p, s)\gamma^0 d_{\vec p, s} e^{-i(E_{\vec p} t + \vec p\cdot\vec x)} \right]}_{= ***}$$

using $\gamma^0 u(\vec p, s) = u(-\vec p, s) \Rightarrow u^{+}(\vec p, s)\gamma^0 = u^{+}(-\vec p, s)$

$\gamma^0 v(\vec p, s) = -v(-\vec p, s) \Rightarrow v^{+}(\vec p, s)\gamma^0 = -v^{+}(-\vec p, s)$

$$\Rightarrow *** = \eta^{*} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u^{+}(-\vec p, s) b^{+}_{\vec p, s} e^{i(E_{\vec p} t + \vec p\cdot\vec x)} - v^{+}(-\vec p, s) d_{\vec p, s} e^{-i(E_{\vec p} t + \vec p\cdot\vec x)} \right]$$

$$= \eta^{*} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s \left[u^{+}(\vec p, s) b^{+}_{-\vec p, s} e^{i(E_{\vec p} t - \vec p\cdot\vec x)} - v^{+}(\vec p, s) d_{-\vec p, s} e^{-i(E_{\vec p} t - \vec p\cdot\vec x)} \right]$$

$$\Rightarrow \quad b^{+}_{\vec p, s} \xrightarrow{P} \eta^{*} b^{+}_{-\vec p, s}, \quad d_{\vec p, s} \xrightarrow{P} \eta^{*}(-d_{-\vec p, s})$$

Moreover, since

$$\psi(t,\vec{x}) \xrightarrow{P} \eta A \psi(t,-\vec{x}) \xrightarrow{P} \eta^2 A^2 \psi(t,\vec{x}),$$

$$\psi^+(t,\vec{x}) \xrightarrow{P} \eta^* \psi^+(t,-\vec{x}) A^+ \xrightarrow{P} \eta^{*2} \psi^+(t,\vec{x})(A^+)^2,$$

then $\eta^2 A^2 = 1$ is required.

Then from $A = \gamma^0$, $\Rightarrow A^2 = 1 \Rightarrow \eta^2 = 1 \Rightarrow \eta = \pm 1$

Therefore,

$$b_{\vec{p},s} \xrightarrow{P} \eta b_{-\vec{p},s} \xrightarrow{P} \eta^2 b_{\vec{p},s} = b_{\vec{p},s}$$

$$d^+_{\vec{p},s} \xrightarrow{P} \eta(-d^+_{-\vec{p},s}) \xrightarrow{P} \eta^2 d^+_{\vec{p},s} = d^+_{\vec{p},s}$$

$$b^+_{\vec{p},s} \xrightarrow{P} \eta^* b^+_{-\vec{p},s} = \eta b^+_{-\vec{p},s} \xrightarrow{P} \eta^2 b^+_{\vec{p},s} = b^+_{\vec{p},s}$$

$$d_{\vec{p},s} \xrightarrow{P} \eta^*(-d_{-\vec{p},s}) = \eta(-d_{-\vec{p},s}) = \eta^2 d_{\vec{p},s} = d_{\vec{p},s},$$

as they should be.

In sum,

$$b_{\vec{p},s} \xrightarrow{P} \eta b_{-\vec{p},s}, \quad b^+_{\vec{p},s} \xrightarrow{P} \eta b^+_{-\vec{p},s},$$

$$d_{\vec{p},s} \xrightarrow{P} -\eta d_{-\vec{p},s}, \quad d^+_{\vec{p},s} \xrightarrow{P} -\eta d^+_{\vec{p},s}$$

where $\eta = \pm 1$.

No matter whether $\eta = +1$ or $\eta = -1$, we have the conclusion that the parity of a Dirac fermion must be opposite to that of the corresponding antiparticle.

For real scalar, the decomposition is

$$\phi(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + a^+_{\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

then $\phi(t,\vec{x}) \xrightarrow{P} \eta_B \phi(t,-\vec{x})$, $\eta_B = \pm 1$

gives

$$\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + a^+_{\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\xrightarrow{P} \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + a^+_{\vec{p}} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})}]$$

$$= \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + a^+_{-\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{P} \eta_B a_{-\vec{p}}, \quad a^+_{\vec{p}} \xrightarrow{P} \eta_B a^+_{-\vec{p}}$$

For complex scalar, we have

$$\phi(t,\vec{x}) \xrightarrow{P} \eta_B \phi(t,-\vec{x}) \xrightarrow{P} \eta_B^2 \phi(t,\vec{x})$$

$$\phi^+(t,\vec{x}) \xrightarrow{P} \eta_B^* \phi^+(t,-\vec{x}) \xrightarrow{P} (\eta_B^*)^2 \phi^+(t,\vec{x})$$

$$\Rightarrow \quad \eta_B^2 = 1 \Rightarrow \eta_B = \pm 1$$

then the transformation of the decomposition is

$$\phi(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b^+_{\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\xrightarrow{P} \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{\vec{p}} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + b^+_{\vec{p}} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} \right]$$

$$= \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b^+_{-\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{P} \eta_B a_{-\vec{p}}, \quad b^+_{\vec{p}} \xrightarrow{P} \eta_B b^+_{-\vec{p}}$$

$$\phi^+(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\xrightarrow{P} \eta_B^* \phi^+(t,-\vec{x}) = \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + b_{\vec{p}} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} \right]$$

$$= \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{-\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\Rightarrow \quad a^+_{\vec{p}} \xrightarrow{P} \eta_B a^+_{-\vec{p}}, \quad b_{\vec{p}} \xrightarrow{P} \eta_B b_{-\vec{p}}$$

Therefore, no matter whether $\eta_B = +1$ or $\eta_B = -1$, we have the conclusion that the parity of a complex scalar particle is the same as its antiparticle.

For the photon field, the decomposition in the Coulomb gauge is

$$A^\mu(t,\vec{x}) = (0, \vec{A}(t,\vec{x}))$$

where $\vec{A}(t,\vec{x}) = \int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda \left[\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^*(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$

To aid the analysis, let's use circular polorization,

$$\vec{e}(\vec{k},+) = \frac{1}{\sqrt{2}}(\hat{g} - i\hat{h}), \quad \vec{e}(\vec{k},-) = -\frac{1}{\sqrt{2}}(\hat{g} + i\hat{h}),$$

where $\hat{g}$, $\hat{h}$, $\hat{k}$ are unit vector perpendicular to each other, and $\hat{g}$, $\hat{h}$ and $\hat{k}$ form a right-handed axes system (like $\hat{x}$, $\hat{y}$ and $\hat{z}$).

Therefore, $\vec{e}^{\,*}(\vec{k},+) = \frac{1}{\sqrt{2}}(\hat{g} + i\hat{h}) = -\vec{e}(\vec{k},-)$

check:

$$\begin{cases} \vec{e}^{\,*}(\vec{k},+)\cdot\vec{e}(\vec{k},+) = 1 \\ \vec{e}^{\,*}(\vec{k},-)\cdot\vec{e}(\vec{k},-) = 1 \\ \vec{e}^{\,*}(\vec{k},+)\cdot\vec{e}(\vec{k},-) = -\vec{e}(\vec{k},-)\cdot\vec{e}(\vec{k},-) = 0 \\ \vec{e}^{\,*}(\vec{k},-)\cdot\vec{e}(\vec{k},+) = -\vec{e}(\vec{k},+)\cdot\vec{e}(\vec{k},+) = 0 \end{cases}$$

as required by the orthergonal and completeness requirement for the basis

$$\vec{e}(\vec{k},+) \times \vec{e}(\vec{k},-) = -\frac{1}{2}(\hat{g} - i\hat{h}) \times (\hat{g} + i\hat{h}) = -\frac{1}{2}(i\hat{k} + i\hat{k}) = -i\hat{k}.$$

Using $\left.\begin{aligned} \vec{e}(-\vec{k},+) &= \vec{e}(\vec{k},-) \\ \vec{e}(-\vec{k},-) &= \vec{e}(\vec{k},+) \end{aligned}\right\} \Rightarrow \vec{e}(-\vec{k},-\lambda) = \vec{e}(\vec{k},\lambda)$

$$\Rightarrow \vec{e}(-\vec{k},+) \times \vec{e}(-\vec{k},-) = \vec{e}(\vec{k},-) \times \vec{e}(\vec{k},+) = +i\hat{k} = -i(-\hat{k})$$

as they should be.

(that is, if choose $\hat{g}$, $\hat{h}$ and $\hat{k}$ as x, y and z direction, respectively, then for the light propogates in the $+z$ direction, if the electric field $\vec{E}$ is $\vec{E} = (E_0 e^{i(kz-\omega t)}, \underbrace{E_0 e^{i(kz-\omega t + \frac{\pi}{2})}}_{= iE_0 e^{i(kz-\omega t)}}, 0)$, then $Re(\vec{E}) = (E_0 \cos(kz-\omega t), -E_0 \sin(kz-\omega t), 0)$

it is called left-handed polarized, since at a fixed t, for example $t=0$, we have

$z=0 \Rightarrow Re(\vec{E}) = (\underbrace{E_0, 0, 0}_{\text{i.e., } +x \text{ direction}})$, $z = \frac{\pi}{2k} \Rightarrow Re(\vec{E}) = (\underbrace{0, -E_0, 0}_{\text{i.e., } -y \text{ direction}})$

$z = \frac{\pi}{k} \Rightarrow Re(\vec{E}) = (\underbrace{-E_0, 0, 0}_{\text{i.e., } -x \text{ direction}})$, $z = \frac{3\pi}{2k} \Rightarrow Re(\vec{E}) = (\underbrace{0, +E_0, 0}_{\text{i.e., } +y \text{ direction}})$;

On the other hand, $\vec{E} = (E_0 e^{i(kz-\omega t)}, \underbrace{E_0 e^{i(kz-\omega t - \frac{\pi}{2})}}_{-iE_0 e^{i(kz-\omega t)}}, 0)$ is right-handed polarized.

While for the light propogates in the $-z$ direction, if $\vec{E}=(E_0 e^{i(-kz-\omega t)}, \underbrace{E_0 e^{i(-kz-\omega t+\frac{\pi}{2})}}_{iE_0 e^{i(-kz-\omega t)}}, 0)$, then $Re(\vec{E})=(E_0 \cos(-kz-\omega t), -E_0 \sin(-kz-\omega t), 0)$,

then for $t=0$, we have $Re(\vec{E})=(E_0\cos(kz), E_0 \sin(kz), 0)$, therefore

$z=0 \Rightarrow Re(\vec{E})=(E_0, 0, 0)$; $z=-\frac{\pi}{2k} \Rightarrow Re(\vec{E})=(0, -E_0, 0)$;

$z=-\frac{\pi}{k} \Rightarrow Re(\vec{E})=(-E_0, 0, 0)$; $z=-\frac{3\pi}{2k} \Rightarrow Re(\vec{E})=(0, +E_0, 0)$;

so it is right-handed polarized;

on the other hand, $\vec{E}=(E_0 e^{i(-kz-\omega t)}, \underbrace{E_0 e^{i(-kz-\omega t-\frac{\pi}{2})}}_{-iE_0 e^{i(-kz-\omega t)}}, 0)$, it is left-handed polarized.

So, $\vec{A}(t,\vec{x}) \xrightarrow{P} -\vec{A}(t,-\vec{x}) = -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-i(E_{\vec{k}}t+\vec{k}\cdot\vec{x})} + \vec{e}^*(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{i(E_{\vec{k}}t+\vec{k}\cdot\vec{x})}]$

$$= -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\vec{e}(-\vec{k},\lambda) a_{-\vec{k},\lambda} e^{-i(E_{\vec{k}}t-\vec{k}\cdot\vec{x})} + \vec{e}^*(-\vec{k},\lambda) a^+_{-\vec{k},\lambda} e^{i(E_{\vec{k}}t-\vec{k}\cdot\vec{x})}]$$

$$= -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\vec{e}(-\vec{k},-\lambda) a_{-\vec{k},-\lambda} e^{-ik\cdot x} + \vec{e}^*(-\vec{k},-\lambda) a^+_{-\vec{k},-\lambda} e^{ik\cdot x}]$$

$$= -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\vec{e}(\vec{k},\lambda) a_{-\vec{k},-\lambda} e^{-ik\cdot x} + \vec{e}^*(\vec{k},\lambda) a^+_{-\vec{k},-\lambda} e^{ik\cdot x}]$$

$$\Rightarrow \quad a_{\vec{k},\lambda} \xrightarrow{P} -a_{-\vec{k},-\lambda}, \quad a^+_{\vec{k},\lambda} \xrightarrow{P} -a^+_{-\vec{k},-\lambda}$$

Therefore, the photon has a negative parity; both the momentum and helicity of the photon change signs under the parity operation.

[2] For charge conjugation,

for Dirac fermion field, in the Standard representation,

$B = i\gamma^2\gamma^0$ satisfies.

$$B^\dagger \gamma^\mu B = -\gamma^{T\mu}$$

(check: $(\gamma^2\gamma^0)^\dagger \gamma^\mu (\gamma^2\gamma^0) = -\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 = \begin{cases} -\gamma^0 = -\gamma^{T}_0, & \mu=0 \\ \gamma^1 = \gamma^{*}_1 = -\gamma^{T}_1, & \mu=1 \\ -\gamma^2 = -\gamma^{T}_2, & \mu=2 \\ \gamma^3 = \gamma^{*3} = -\gamma^{T3}, & \mu=3 \end{cases}$)

from $\psi(x) \xrightarrow{C} \varepsilon B \bar{\psi}^T_{(x)}$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s [u(\vec p,s) b_{\vec p,s} e^{-ip\cdot x} + v(\vec p,s) d^\dagger_{\vec p,s} e^{ip\cdot x}]$$

$$\xrightarrow{C} \varepsilon \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p}) \sum_s [i\gamma^2\gamma^0 \bar{u}^T(\vec p,s) b^\dagger_{\vec p,s} e^{ip\cdot x} + i\gamma^2\gamma^0 \bar{v}^T(\vec p,s) d_{\vec p,s} e^{-ip\cdot x}]$$

where

$$i\gamma^2\gamma^0 \bar{u}^T(\vec p,s) = i\gamma^2\gamma^0 (u^\dagger\gamma^0)^T = i\gamma^2\gamma^0\gamma^{T}_0 u^* = i\gamma^2\gamma^0\gamma^0 u^* = i\gamma^2 u^*$$

$$= i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} (E+m)^{\frac12} \begin{pmatrix} \chi_s^* \\ \frac{\vec\sigma^*\cdot\vec p}{E+m}\chi_s^* \end{pmatrix}$$

$$= (E+m)^{\frac12} \begin{pmatrix} \frac{\vec\sigma\cdot\vec p}{E+m}(-i\sigma^2)\chi_s^* \\ -i\sigma^2\chi_s^* \end{pmatrix}$$

using $\chi_s^\dagger \chi_{s'} = \delta_{ss'}$, we have $[(-i\sigma^2)\chi_s^*]^\dagger(-i\sigma^2\chi_{s'})$

$$= \chi_s^T (i\sigma^2)(-i\sigma^2)\chi_{s'}$$

$$= \chi_s^T \chi_{s'}$$

also, $i\gamma^2\gamma^0 \bar{v}^T(\vec p,s) = i\gamma^2 v^* = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} (E+m)^{\frac12} \begin{pmatrix} \frac{\vec p\cdot\vec\sigma^*}{E+m}\eta_s^* \\ \eta_s^* \end{pmatrix}$

$$= (E+m)^{\frac12} \begin{pmatrix} i\sigma^2\eta_s^* \\ \frac{\vec p\cdot\vec\sigma}{E+m}\, i\sigma^2\eta_s^* \end{pmatrix}$$

we can choose χ_s be real ($s=\pm\frac12$), and $\overbrace{-i\sigma^2\chi_s \equiv \eta_s}^{\text{choose } \eta_s \text{ such that}}$, then we have

$\Rightarrow i\sigma^2\eta_s^* = i\sigma^2(-i\sigma^2\chi_s) = \chi_s$, and $\chi_s^T\chi_{s'} = \delta_{ss'}$

$$\Rightarrow \quad i\gamma^2\gamma^0\,\bar{u}^T(\vec{p},s) = v(\vec{p},s)$$
$$i\gamma^2\gamma^0\,\bar{v}^T(\vec{p},s) = u(\vec{p},s)$$

$$\Rightarrow \quad b_{\vec{p},s} \xrightarrow{C} \varepsilon\, d_{\vec{p},s}$$
$$d^+_{\vec{p},s} \xrightarrow{C} \varepsilon\, b^+_{\vec{p},s}$$

Also, from $\psi^+(x) \xrightarrow{C} \varepsilon^* \psi^T(x)\,\gamma^{0T} B^+$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \sum_s \left(u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} + v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x} \right)$$

$$\xrightarrow{C} \varepsilon^* \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \sum_s \Big(u^T(\vec{p},s)\, \gamma^{0T}(-i)(\gamma^2\gamma^0)^+\, b_{\vec{p},s}\, e^{-ip\cdot x}$$
$$+ v^T(\vec{p},s)\, \gamma^{0T}(-i)(\gamma^2\gamma^0)^+\, d^+_{\vec{p},s}\, e^{ip\cdot x} \Big)$$

using $\quad u^T(\vec{p},s)\,\gamma^{0T}(-i)(\gamma^2\gamma^0)^+$

$$= u^T(\vec{p},s)\,\gamma^{0T}(+i)\,\gamma^0\gamma^2$$
$$= u^T(\vec{p},s)\, i\gamma^2$$
$$= \left(i\gamma^2 u^*(\vec{p},s)\right)^+ = v^+(\vec{p},s)$$

and $\quad v^T(\vec{p},s)\,\gamma^{0T}(-i)\,(\gamma^2\gamma^0)^+$

$$= v^T(\vec{p},s)\,\gamma^{0T}(+i)\,\gamma^0\gamma^2$$
$$= v^T(\vec{p},s)\; i\gamma^2$$
$$= \left(i\gamma^2 v^*(\vec{p},s)\right)^+$$
$$= u^+(\vec{p},s)$$

$$\Rightarrow \quad b^+_{\vec{p},s} \xrightarrow{C} \varepsilon^* d^+_{\vec{p},s}\,, \quad d_{\vec{p},s} \xrightarrow{C} \varepsilon^* b_{\vec{p},s}$$

Therefore, we conclude that charge conjugation change a Dirac fermion to its antifermion, without changing spin and momentum.

For real scalar field, we can immediately see that (from $\phi(x) \xrightarrow{C} \varepsilon_B \phi(x)$),

$$a_{\vec{p}} \xrightarrow{C} \varepsilon_B a_{\vec{p}}, \quad a^\dagger_{\vec{p}} \xrightarrow{C} \varepsilon_B a^\dagger_{\vec{p}}, \quad \text{where } \varepsilon_B = \pm 1.$$

For a complex scalar field, from $\phi(x) \xrightarrow{C} \varepsilon_B \phi^\dagger(x)$ and $\phi^\dagger(x) \longrightarrow \varepsilon_B^* \phi(x)$, we have

$$\phi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(a_{\vec{p}} e^{-ip\cdot x} + b^\dagger_{\vec{p}} e^{ip\cdot x} \right]$$

$$\xrightarrow{C} \varepsilon_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^\dagger_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x} \right]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{C} \varepsilon_B b_{\vec{p}}$$

$$b^\dagger_{\vec{p}} \xrightarrow{C} \varepsilon_B a^\dagger_{\vec{p}}.$$

$$\phi^\dagger(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^\dagger_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x} \right]$$

$$\xrightarrow{C} \varepsilon_B^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(a_{\vec{p}} e^{-ip\cdot x} + b^\dagger_{\vec{p}} e^{ip\cdot x} \right]$$

$$\Rightarrow \quad a^\dagger_{\vec{p}} \xrightarrow{C} \varepsilon_B^* b^\dagger_{\vec{p}}$$

$$b_{\vec{p}} \xrightarrow{C} \varepsilon_B^* a_{\vec{p}}$$

So, for a complex scalar, charge conjugation transforms to its antiparticle without changing momentum.

For the photon field, we can immediately see that (from $A^\mu(x) \xrightarrow{C} -A^\mu(x)$),

$$a_{\vec{k},\lambda} \xrightarrow{C} -a_{\vec{k},\lambda}, \quad a^\dagger_{\vec{k},\lambda} \longrightarrow -a^\dagger_{\vec{k},\lambda}$$

So, the photon is odd under charge conjugation (i.e., it has charge conjugation number -1).

For a state of scalar-antiscalar particles with orbital angular momentum l:

$$|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\ F_l(\vec{p})\, a^+_{\vec{p}}\, b^+_{-\vec{p}}\, |0\rangle$$, where $F_l(\vec{p})$ is an eigenstate of obital angular momentum l.

Apply charge conjugation,

$$C|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\ F_l(\vec{p})\, b^+_{\vec{p}}\, a^+_{-\vec{p}}\, |0\rangle$$

$$\overset{\text{exchange } b^+ \text{ and } a^+}{=} \int d^3\vec{p}\ F_l(\vec{p})\, a^+_{-\vec{p}}\, b^+_{\vec{p}}\, |0\rangle$$

$$= \int d^3\vec{p}\ F_l(-\vec{p})\, a^+_{\vec{p}}\, b^+_{-\vec{p}}\, |0\rangle$$

$$= \int d^3\vec{p}\ (-1)^l F_l(\vec{p})\, a^+_{\vec{p}}\, b^+_{-\vec{p}}\, |0\rangle$$

$$= (-1)^l |\phi_l(s\bar{s})\rangle .$$

So, this state has charge conjugation number $(-1)^l$.

An example of such system is $\pi^+\pi^-$.

Another example is $\pi^0\pi^0$, where $b^+_{\vec{p}}$ in the above should be substitute by $a^+_{\vec{p}}$; further more, since $C|\phi_l(s\bar{s})\rangle$ exchanges the two identical bozons, then the wavefunction (i.e., $|\phi_l(s\bar{s})\rangle$) has to satisfy $C|\phi_l(s\bar{s})\rangle = |\phi_l(s\bar{s})\rangle$, so that $(-1)^l = 1 \Rightarrow l$ is even.

For a state of fermion-antifermion particles, and neglect the effects of spin-orbit coupling, we have the state being

$$|\psi_{l,s}(f\bar{f})\rangle = \int d^3\vec{p} \sum_{s_1 s_2} F^{s_1 s_2}_{ls}(\vec{p})\, b^+_{\vec{p},s_1}\, d^+_{-\vec{p},s_2}\, |0\rangle ,$$

then

$$C|\psi_{l,s}(f\bar{f})\rangle = \int d^3\vec{p} \sum_{s_1 s_2} F^{s_1 s_2}_{ls}(\vec{p})\, d_{\vec{p},s_1}\, b_{-\vec{p},s_2}\, |0\rangle$$

$$\overset{\text{interchange } b^+ \text{ and } d^+}{=} -\int d^3\vec{p} \sum_{s_1 s_2} F^{s_1 s_2}_{ls}(\vec{p})\, b^+_{-\vec{p},s_2}\, d^+_{\vec{p},s_1}\, |0\rangle$$

$$\overset{\text{interchange } \vec{p} \leftrightarrow -\vec{p},\ s_1 \leftrightarrow s_2}{=} -\int d^3\vec{p} \sum_{s_1 s_2} F^{s_2 s_1}_{ls}(-\vec{p})\, b^+_{\vec{p},s_1}\, d^+_{-\vec{p},s_2}\, |0\rangle$$

using $F^{s_2 s_1}_{ls}(-\vec{p}) = (-1)^{l+s+1} F^{s_1 s_2}_{ls}(\vec{p})$

(note: $s=1$ is spin triplet, which is symmetric under the exchange of the two spin; $s=0$ is spin singlet, which is antisymmetric under the exchange of the two spin.

$F^{s_1 s_2}_{ls}(\vec{p})$ is the product of orbital wave function and total spin wave function).

$$\Rightarrow C|\psi_{l,s}(f\bar{f})\rangle = (-1)^{l+s} |\psi_{l,s}(f\bar{f})\rangle$$

Therefore, the charge conjugation number for fermion-antifermion system is $(-1)^{l+s}$.

So, for a pseudoscalar meson with $l=s=0$, its charge conjugation number is $(-1)^{0+0} = +1$, π^0 is an example; for a vector meson with $l=0$ and $s=1$, its charge conjugation number is $(-1)^{0+1} = -1$, ρ^0 is an example.

[3] For time inversion:

for Dirac fermion field, in the standard representation,

$$D = \gamma^1\gamma^3 \text{ . satisfies}$$

$$D^+ \gamma^{*\mu} D = \gamma_\mu .$$

(check: $(\gamma^1\gamma^3)^+ \gamma^{*0}(\gamma^1\gamma^3) = \gamma^{3+}\gamma^{1+}\gamma^0\gamma^1\gamma^3 = \gamma^3\gamma^1\gamma^0\gamma^1\gamma^3 = \gamma_0$ ✓

$(\gamma^1\gamma^3)^+\gamma^{*1}(\gamma^1\gamma^3) = \gamma^3\gamma^1\gamma^1\gamma^1\gamma^3 = -\gamma^1 = \gamma_1$ ✓

$(\gamma^1\gamma^3)^+\gamma^{*2}(\gamma^1\gamma^3) = -\gamma^3\gamma^1\gamma^2\gamma^1\gamma^3 = -\gamma^2 = \gamma_2$ ✓

$(\gamma^1\gamma^3)^+\gamma^{*3}(\gamma^1\gamma^3) = \gamma^3\gamma^1\gamma^3\gamma^1\gamma^3 = -\gamma^3 = \gamma_3$ ✓).

from $\psi(x) \xrightarrow{T} \varsigma D\psi(\tilde{x})$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [u(\vec{p},s) b_{\vec{p},s} e^{-ip\cdot x} + v(\vec{p},s) d^+_{\vec{p},s} e^{ip\cdot x}]$$

$$\xrightarrow{T} \varsigma \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [\gamma^{*1}\gamma^{*3} u^*(\vec{p},s) b_{\vec{p},s} e^{+i(-E_p t - \vec{p}\cdot\vec{x})} + \gamma^{*1}\gamma^{*3} v^*(\vec{p},s) d^+_{\vec{p},s} e^{-i(-E_p t - \vec{p}\cdot\vec{x})}]$$

$$= \varsigma \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [\gamma^{*1}\gamma^{*3} u^*(\vec{p},s) b_{-\vec{p},s} e^{-ip\cdot x} + \gamma^{*1}\gamma^{*3} v^*(\vec{p},s) d^+_{-\vec{p},s} e^{ip\cdot x}]$$

$= **$

note that complex conjugate is applied.

using

$$\gamma^{*1}\gamma^{*3} = \gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

$$\Rightarrow \gamma^{*1}\gamma^{*3} u^*(\vec{p},s) = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s^* \\ \frac{\vec{\sigma}^*\cdot\vec{p}}{E+m}\chi_s^* \end{pmatrix}$$

$$= (E+m)^{\frac{1}{2}} \cdot \begin{pmatrix} i\sigma^2\chi_s^* \\ -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} i\sigma^2\chi_s^* \end{pmatrix}$$

we can take $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

then $i\sigma^2 \chi_{\frac{1}{2}}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\chi_{-\frac{1}{2}}$.

$i\sigma^2 \chi_{-\frac{1}{2}}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_{\frac{1}{2}}$.

Apparently, $\chi_s^+ \chi_{s'} = \delta_{ss'}$, $\sum_s \chi_s \chi_s^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\Rightarrow \quad i\sigma^2 \chi_s^* = -(-1)^{\frac{1}{2}-s} \chi_{-s}$

$\Rightarrow \quad \gamma^{1*}\gamma^{3*} u^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} \, u(-\vec{p}, -s)$

Also, $\gamma^{1*}\gamma^{3*} v^*(\vec{p}, s) = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}^*}{E+m}\eta_s^* \\ \eta_s^* \end{pmatrix}$

$$= (E+m)^{\frac{1}{2}} \begin{pmatrix} -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} i\sigma^2 \eta_s^* \\ i\sigma^2 \eta_s^* \end{pmatrix}$$

we can take $\eta_{\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\eta_{-\frac{1}{2}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

then $i\sigma^2 \eta_{\frac{1}{2}}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\eta_{-\frac{1}{2}}$.

$i\sigma^2 \eta_{-\frac{1}{2}}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \eta_{\frac{1}{2}}$

Apparently, $\eta_s^+ \eta_{s'} = \delta_{ss'}$, $\sum_s \eta_s \eta_s^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad i\sigma^2 \eta_s^* = -(-1)^{\frac{1}{2}-s} \eta_{-s}$

$\Rightarrow \quad \gamma^{1*}\gamma^{3*} v^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} \, v(-\vec{p}, -s)$

$$\Rightarrow ** = \xi \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_p) \sum_s \Big[\overbrace{u(\vec{p}, -s)}^{-(-1)^{\frac{1}{2}-s}} b_{-\vec{p},s} e^{-ip\cdot x} + \overbrace{v(\vec{p}, -s)}^{-(-1)^{\frac{1}{2}-s}} d^+_{-\vec{p},s} e^{ip\cdot x} \Big]$$

$$= \xi \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_p) \sum_s \Big[\overbrace{u(\vec{p}, s)}^{-(-1)^{\frac{1}{2}+s}} b_{-\vec{p},-s} e^{-ip\cdot x} + \overbrace{v(\vec{p}, s)}^{-(-1)^{\frac{1}{2}+s}} d^+_{-\vec{p},-s} e^{ip\cdot x} \Big]$$

$\Rightarrow$ using $-(-1)^{\frac{1}{2}+s} = (-1)^{\frac{1}{2}-s}$

$$\Rightarrow \quad b_{\vec{p},s} \xrightarrow{T} \varsigma(-1)^{\frac{1}{2}-s}\, b_{-\vec{p},-s}$$

$$d^{+}_{\vec{p},s} \xrightarrow{T} \varsigma(-1)^{\frac{1}{2}-s}\, d^{+}_{-\vec{p},-s}$$