Local gauge invariance

The Dirac Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$ is invariant under a global phase transformation for ψ, $\psi \to \psi' = e^{-ic}\psi$, where c is a real number, independent of x^μ.

Since $\bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{ic}$, then $\bar{\psi}'\gamma^\mu\partial_\mu\psi' = \bar{\psi}e^{ic}\gamma^\mu\partial_\mu e^{-ic}\psi$

$= \bar{\psi}\gamma^\mu\partial_\mu\psi$, and $\bar{\psi}'\psi' = \bar{\psi}e^{ic}e^{-ic}\psi = \bar{\psi}\psi$.

To have the Lagrangian be invariant under a local phase transformation for ψ, we are "forced" to introduce a massless vector field A^μ, so that the new Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - k\bar{\psi}\gamma^\mu\psi A_\mu$$ (where k is a real constant)

is invariant under the simultaneous transformation

$$\begin{cases} A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \\ \psi(x) \longrightarrow \psi'(x) = e^{-ik\Lambda(x)}\psi(x) \end{cases}$$ (where $\Lambda(x)$ is a real function)

check: $\bar{\psi}'(x) = \bar{\psi}(x)e^{ik\Lambda(x)}$

$$\begin{aligned} i\bar{\psi}'(x)\gamma^\mu\partial_\mu\psi'(x) &= i\bar{\psi}(x)e^{ik\Lambda(x)}\gamma^\mu\partial_\mu\left(e^{-ik\Lambda(x)}\psi(x)\right) \\ &= i\bar{\psi}(x)e^{ik\Lambda(x)}\gamma^\mu(-ik)e^{-ik\Lambda(x)}(\partial_\mu\Lambda(x))\psi(x) \\ &\quad + i\bar{\psi}(x)e^{ik\Lambda(x)}\gamma^\mu e^{-ik\Lambda(x)}\partial_\mu\psi(x) \\ &= k(\partial_\mu\Lambda(x))\bar{\psi}(x)\gamma^\mu\psi(x) + i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \end{aligned}$$

$$-m\bar{\psi}'(x)\psi'(x) = -m\bar{\psi}(x)\psi(x)$$

$$-k\bar{\psi}'(x)\gamma^\mu\psi'(x)A'_\mu(x) = -k\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) - k\bar{\psi}(x)\gamma^\mu\psi(x)(\partial_\mu\Lambda(x))$$

$$\Rightarrow i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi' - k\bar{\psi}'\gamma^\mu\psi'A'_\mu = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - k\bar{\psi}\gamma^\mu\psi A_\mu$$

Also, we need to introduce the free part of A_μ, which is

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

Since $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda)$

$= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$

then $-\tfrac{1}{4}F'_{\mu\nu}F'^{\mu\nu} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

However, the mass term for A_μ, i.e., $\tfrac{1}{2}m_A^2 A_\mu A^\mu$ changes to

$$\tfrac{1}{2}m_A^2 A'_\mu A'^\mu = \tfrac{1}{2}m_A^2(A_\mu + \partial_\mu\Lambda)(A^\mu + \partial^\mu\Lambda) = \tfrac{1}{2}m_A^2 A_\mu A^\mu + \tfrac{1}{2}m_A^2(\partial_\mu\Lambda)(\partial^\mu\Lambda) + m_A^2 A_\mu\partial^\mu\Lambda \neq \tfrac{1}{2}m_A^2 A_\mu A^\mu.$$

This mass term should not be there, otherwise the local (phase) gauge invariance is violated.

We can introduce covariant derivative

$$D_\mu \equiv \partial_\mu + i k A_\mu \text{, then}$$

$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is local phase invariant under the simultaneous transformation

$$\begin{cases} A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \\ \psi(x) \longrightarrow \psi'(x) = e^{-ik\Lambda(x)}\psi(x) \end{cases}$$

, where k is a real constant, $\Lambda(x)$ is a real function.

The above is $U(1)$ transformation (since $k\Lambda(x)$ is a 1×1 real function), and we can write $\psi'(x) = U\psi(x)$, where $U = e^{-ik\Lambda(x)}$.

In particular, if A_μ is the EM field, then the equation of motion of A_μ should be just the Maxwell equations

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -k\bar{\psi}\gamma^\nu\psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = k\bar{\psi}\gamma^\nu\psi$$

Since $\int d^3\vec{x} :\bar{\psi}\gamma^0\psi: = \int d^3\vec{p}\,[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}]\sum_s (b^+_{\vec{p},s}b_{\vec{p},s} - d^+_{\vec{p},s}d_{\vec{p},s})$

then k should be $q/|e|$ in order to make $k\bar{\psi}\gamma^\nu\psi$ the EM current

Yang-Mills theory

Suppose now we have two Dirac fields, ψ_1 and ψ_2, then the free Lagrangian is

$$\mathcal{L} = i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 - m_1\bar{\psi}_1\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m_2\bar{\psi}_2\psi_2.$$

Let's write it more compactly by combining ψ_1 and ψ_2 into a two component column

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

so, $\bar{\psi} = (\bar{\psi}_1 \;\; \bar{\psi}_2)$.

(note that the γ^μ matrices act on ψ_1 and ψ_2 individually)

$\Rightarrow \mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\psi}M\psi$, where $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

If $m_1 = m_2 = m$ then $M = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that

$$\bar{\psi}M\psi = m\bar{\psi}\psi$$

The two component ψ admits a $U(2)$ global phase transformation invariance, $\psi \to \psi' = U\psi$, where U is a 2×2 unitary matrix, $U^\dagger U = \mathbb{I}_{2\times2}$,

$$\bar{\psi}'\gamma^\mu\partial_\mu\psi' = \bar{\psi}U^\dagger\gamma^\mu\partial_\mu U\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi$$

$$m\bar{\psi}'\psi' = m\bar{\psi}\psi$$

The 2×2 unitary matrix can be written in the form $U = e^{iH}$, where H is Hermitian, so that $U^\dagger = e^{-iH^\dagger} = e^{-iH}$, and H can be written as $H = \theta\mathbb{I}_{2\times2} + \vec{\sigma}\cdot\vec{h}$, where θ and h^1, h^2, h^3 are four real numbers, and $\sigma^1, \sigma^2, \sigma^3$ are Pauli matrices.

(Note: an $n\times n$ complex matrix has $2n^2$ degrees of freedom, the unitary condition $U^\dagger U = UU^\dagger = \mathbb{I}_{n\times n}$ gives n^2 restriction: $\sum_k U_{ik}(U^\dagger)_{kj} = \sum_k U_{ik}U^*_{jk} = \delta_{ij} = \sum_k (U^\dagger)_{ik}U_{kj} = \sum_k U^*_{ki}U_{kj}$, so there are $2n^2 - n^2 = n^2$ degrees of freedom left.)

check:

For $U(2)$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the unitary condition means that

$$\begin{cases} a^*c + b^*d = 0 \\ a^*a + b^*b = 1 \\ c^*c + d^*d = 1 \\ a^*b + c^*d = 0 \\ a^*a + c^*c = 1 \\ b^*b + d^*d = 1 \end{cases}$$

let $a = xe^{i\alpha}$, $x \geq 0$ is real, α is real.
$b = ye^{i\beta}$, $y \geq 0$ is real, β is real
$c = ze^{i\gamma}$, $z \geq 0$ is real, γ is real
$d = we^{i\varphi}$, $w \geq 0$ is real, φ is real

$$\Rightarrow \begin{cases} xze^{i(-\alpha+\gamma)} + ywe^{i(-\beta+\varphi)} = 0 \\ x^2 + y^2 = 1 \\ z^2 + w^2 = 1 \\ xye^{i(-\alpha+\beta)} + zwe^{i(-\gamma+\varphi)} = 0 \\ x^2 + z^2 = 1 \\ y^2 + w^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 + y^2 = 1 \\ y = z, \quad x = w, \\ xye^{i(-\alpha+\gamma)} + xye^{i(-\beta+\varphi)} = 0 \Rightarrow xy = 0 \text{ or } e^{i(-\alpha+\gamma)} + e^{i(-\beta+\varphi)} = 0 \\ xye^{i(-\alpha+\beta)} + xye^{i(-\gamma+\varphi)} = 0 \Rightarrow xy = 0 \text{ or } e^{i(-\alpha+\beta)} + e^{i(-\gamma+\varphi)} = 0 \end{cases}$$

① If $x = 0$, then $w = 0$, $y = z = 1 \Rightarrow \begin{pmatrix} 0 & e^{i\beta} \\ e^{i\gamma} & 0 \end{pmatrix}$

check: $\begin{pmatrix} 0 & e^{i\beta} \\ e^{i\gamma} & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & e^{-i\gamma} \\ e^{-i\beta} & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & e^{-i\gamma} \\ e^{-i\beta} & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{i\beta} \\ e^{i\gamma} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$$

$$\begin{pmatrix} 0 & e^{i\beta} \\ e^{i\gamma} & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-i\gamma} \\ e^{-i\beta} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$$

② If $y = 0$, then $z = 0$, $x = w = 1 \Rightarrow \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

check: $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}^+ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$$

⑥ If $e^{i(-\alpha+\gamma)} + e^{i(-\beta+\varphi)} = e^{i(-\alpha+\beta)} + e^{i(-\gamma+\varphi)} = 0$ and $x^2+y^2=1$

$$\Rightarrow \begin{cases} -\alpha+\gamma = -\beta+\varphi + (2k+1)\pi, & k=0,\pm1,\pm2,\cdots \\ -\alpha+\beta = -\gamma+\varphi+(2m+1)\pi, & m=0,\pm1,\pm2,\cdots \end{cases}$$

$$\Rightarrow \begin{pmatrix} (\cos k)e^{i\alpha} & (\sin k)e^{i\beta} \\ (\sin k)e^{i\gamma} & -(\cos k)e^{i(-\alpha+\beta+\gamma)} \end{pmatrix}, \quad k \in (0, \tfrac{\pi}{2})$$

check: $$\begin{pmatrix} (\cos k)e^{i\alpha} & (\sin k)e^{i\beta} \\ (\sin k)e^{i\gamma} & -(\cos k)e^{i(-\alpha+\beta+\gamma)} \end{pmatrix}^{+}$$

$$= \begin{pmatrix} \cos k\, e^{-i\alpha} & (\sin k)e^{-i\gamma} \\ (\sin k)e^{-i\beta} & -(\cos k)e^{i(\alpha-\beta-\gamma)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} (\cos k)e^{-i\alpha} & (\sin k)e^{-i\gamma} \\ (\sin k)e^{-i\beta} & -(\cos k)e^{i(\alpha-\beta-\gamma)} \end{pmatrix} \begin{pmatrix} (\cos k)e^{i\alpha} & (\sin k)e^{i\beta} \\ (\sin k)e^{i\gamma} & -(\cos k)e^{i(-\alpha+\beta+\gamma)} \end{pmatrix}$$

$$= \begin{pmatrix} (\cos k)^2+(\sin k)^2 & (\cos k)(\sin k)e^{i(-\alpha+\beta)} - (\cos k)(\sin k)e^{i(-\alpha+\beta)} \\ (\sin k)(\cos k)e^{i(\alpha-\beta)} - (\sin k)(\cos k)e^{i(\alpha-\beta)} & (\sin k)^2+(\cos k)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} (\cos k)e^{i\alpha} & (\sin k)e^{i\beta} \\ (\sin k)e^{i\gamma} & -(\cos k)e^{i(-\alpha+\beta+\gamma)} \end{pmatrix} \begin{pmatrix} (\cos k)e^{-i\alpha} & (\sin k)e^{-i\gamma} \\ (\sin k)e^{-i\beta} & -(\cos k)e^{i(\alpha-\beta-\gamma)} \end{pmatrix}$$

$$= \begin{pmatrix} (\cos k)^2+(\sin k)^2 & (\cos k)(\sin k)e^{i(\alpha-\gamma)} - (\sin k)(\cos k)e^{i(\alpha-\gamma)} \\ (\sin k)(\cos k)e^{i(-\alpha+\gamma)} - (\sin k)(\cos k)e^{i(-\alpha+\gamma)} & (\sin k)^2+(\cos k)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

So in general all the three cases can be written as

$$e^{i(\beta+\gamma)} \begin{pmatrix} (\cos k)\, e^{i(\alpha-\beta-\gamma)} & (\sin k)\, e^{-i\gamma} \\ (\sin k)\, e^{-i\beta} & -(\cos k)\, e^{-i\alpha} \end{pmatrix}$$

let $\beta+\gamma = \theta'$ then $\gamma = \theta' - \beta$

$\Rightarrow$ a general $U(2)$ matrix is

$$e^{i\theta'} \begin{pmatrix} (\cos k) e^{i(\alpha-\theta')} & (\sin k) e^{-i(\theta'-\beta)} \\ (\sin k) e^{-i\beta} & -(\cos k) e^{-i\alpha} \end{pmatrix},$$

$$\Rightarrow e^{i(\theta' - \frac{\theta'}{2} - \frac{\pi}{2})} \begin{pmatrix} (\cos k)\, e^{i(\alpha - \frac{\theta'}{2} + \frac{\pi}{2})} & (\sin k)\, e^{-i(\frac{\theta'}{2} - \beta - \frac{\pi}{2})} \\ -(\sin k)\, e^{+i(\frac{\theta'}{2} - \beta - \frac{\pi}{2})} & (\cos k)\, e^{-i(\alpha - \frac{\theta'}{2} + \frac{\pi}{2})} \end{pmatrix}$$

let $\theta = \theta' - \frac{\theta'}{2} - \frac{\pi}{2}$, $a \equiv (\cos k) e^{i(\alpha - \frac{\theta'}{2} + \frac{\pi}{2})}$, $b \equiv (\sin k) e^{-i(\frac{\theta'}{2} - \beta - \frac{\pi}{2}}$

then $$e^{i\theta} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where $|a|^2 + |b|^2 = 1$.

Since $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}^{\dagger} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}^{\dagger} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = |a|^2 + |b|^2 = 1$.

We conclude that we can decompose $U(2)$ as $U(1) \otimes SU(2)$

The general $SU(2)$ matrix $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ can be written as

$e^{i\vec{\sigma}\cdot\vec{h}}$, where h^1, h^2, h^3 are three real numbers.

check: use $e^{i\vec{\sigma}\cdot\vec{h}} = \cos h + i(\hat{h}\cdot\vec{\sigma}) \sin h$, where $h \equiv |\vec{h}|$ and $\hat{h} = \vec{h}/h$

$$\Rightarrow \begin{pmatrix} \cos h + i\hat{h}^3 \sin h & (i\hat{h}^1 + \hat{h}^2)\sin h \\ (i\hat{h}^1 - \hat{h}^2)\sin h & \cos h - i\hat{h}^3 \sin h \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$\Rightarrow \quad 2i\hat{h}^3 \sin h = a - a^* = 2i\,I_m a \Rightarrow \hat{h}^3 \sin h = I_m a$$

$$2i\hat{h}^1 \sin h = b - b^* = 2i\,I_m b \Rightarrow \hat{h}^1 \sin h = I_m b$$

$$2\cos h = a + a^* = 2R_e a \Rightarrow \cos h = R_e a$$

$$2\hat{h}^2 \sin h = b + b^* = 2R_e b \Rightarrow \hat{h}^2 \sin h = R_e b$$

$$(e^{i\vec{\sigma}\cdot\vec{h}})^+ e^{i\vec{\sigma}\cdot\vec{h}} = (\cos h - i(\hat{h}\cdot\vec{\sigma})\sin h)(\cos h + i(\hat{h}\cdot\vec{\sigma})\sin h)$$

$$= (\cos h)^2 + (\sin h)^2\, \hat{h}\cdot\hat{h} = 1$$

$$e^{i\vec{\sigma}\cdot\vec{h}}(e^{i\vec{\sigma}\cdot\vec{h}})^+ = 1$$

$$\det(e^{i\vec{\sigma}\cdot\vec{h}}) = (\cos h)^2 + (\hat{h}^3)^2 \sin^2 h + \sin^2 h\,((\hat{h}^2)^2 + (\hat{h}^1)^2) = 1$$

So, indeed, any $U(2)$ matrix can be written as

$$U = e^{iH} \equiv e^{i(\theta 1_{2\times 2} + \vec{\sigma}\cdot\vec{h})} = e^{i\theta} e^{i\vec{\sigma}\cdot\vec{h}}$$, with θ, h^1, h^2, h^3 real.

We have already explored the implications of the $U(1)$ part, i.e., the phase transformation $e^{i\theta}$, let's concentrate on the $SU(2)$ part $S = e^{i\vec{\sigma}\cdot\vec{h}}$ (i.e., $U = e^{i\theta}S$)

Apparently, the Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$, where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, is invariant under $\underbrace{\overset{\text{global}}{U(2)}}_{\text{(have checked)}}$, $\underbrace{SU(2)}_{\text{(let } \theta=0)}$ and $\underbrace{U(1)}_{\text{(let } \vec{h}=0)}$.

For a local $SU(2)$ gauge transformation $\psi \to \psi' = S\psi$, where $S = e^{i\vec{\sigma}\cdot\vec{h}(x)}$, we have

$$\bar{\psi}'\psi' = \bar{\psi}S^+S\psi = \bar{\psi}\psi$$

$$\bar{\psi}'\gamma^\mu\partial_\mu\psi' = \bar{\psi}S^+\gamma^\mu\partial_\mu(S\psi) = \bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu S^+(\partial_\mu S)\psi$$

To compensate the term $\bar{\psi}\gamma^\mu S^+(\partial_\mu S)\psi$, i.e. to make the Lagrangian invariant under local gauge transformation, we introduce vector fields $\vec{A}_\mu = (A_\mu^1, A_\mu^2, A_\mu^3)$ coupling to ψ, and it transform under local gauge transformation as

$$\vec{\sigma}\cdot\vec{A}_\mu' = S(\vec{\sigma}\cdot\vec{A}_\mu)S^{-1} + i\frac{1}{k}(\partial_\mu S)S^{-1}$$, where k is a constant

so that

$$i\bar{\psi}'\gamma^\mu\partial_\mu\psi' + i\bar{\psi}'\gamma^\mu(ik\vec{\sigma}\cdot\vec{A}_\mu')\psi'$$
$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\bar{\psi}\gamma^\mu S^+(\partial_\mu S)\psi + i\bar{\psi}S^+\gamma^\mu(ik)[S(\vec{\sigma}\cdot\vec{A}_\mu)S^{-1} + i\frac{1}{k}(\partial_\mu S)S^{-1}]S\psi$$
$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\bar{\psi}\gamma^\mu S^+(\partial_\mu S)\psi + i\bar{\psi}\gamma^\mu(ik\vec{\sigma}\cdot\vec{A}_\mu)\psi - i\bar{\psi}S^+\gamma^\mu(\partial_\mu S)\psi$$
$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\bar{\psi}\gamma^\mu(ik\vec{\sigma}\cdot\vec{A}_\mu)\psi$$

So, we can introduce covariant derivative

$$D_\mu \equiv \partial_\mu + ik\vec{\sigma}\cdot\vec{A}_\mu$$

such that $i\bar{\psi}'\gamma^\mu D_\mu'\psi' = i\bar{\psi}\gamma^\mu D_\mu\psi$, i.e., $D_\mu'\psi' = SD_\mu\psi$

(check: $D_\mu'\psi' = \partial_\mu(S\psi) + ik\vec{\sigma}\cdot\vec{A}_\mu'\psi' = (\partial_\mu S)\psi + S\partial_\mu\psi + ikS(\vec{\sigma}\cdot\vec{A}_\mu)\psi + ik\frac{i}{k}(\partial_\mu S)\psi = SD_\mu\psi$.)

We need to introduce the free part of $\vec{A}_\mu$, which is

$$\mathcal{L}_A = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}^{1}F^{\mu\nu 1} - \frac{1}{4}F_{\mu\nu}^{2}F^{\mu\nu 2} - \frac{1}{4}F_{\mu\nu}^{3}F^{\mu\nu 3}$$

Still, the mass term $\frac{1}{2}m_A^2\vec{A}_\mu\cdot\vec{A}^\mu = \frac{1}{2}m_A^2(A_\mu^1 A^{1\mu} + A_\mu^2 A^{2\mu} + A_\mu^3 A^{3}$

is not invariant under local gauge transformation

(check: $(\vec{\sigma}\cdot\vec{A}_\mu)(\vec{\sigma}\cdot\vec{A}^\mu) = \sigma^i\sigma^j A_\mu^i A^{j\mu} = (\delta^{ij} + i\varepsilon^{ijk}\sigma^k)A_\mu^i A^{j\mu}$

$$= \vec{A}_\mu\vec{A}^\mu + i\underbrace{(\vec{A}_\mu\times\vec{A}^\mu)}_{\vec{A}_0\times\vec{A}^0 + \vec{A}_1\times\vec{A}^1 + \vec{A}_2\times\vec{A}^2 + \vec{A}_3\times\vec{A}^3 = \vec{A}_0\times\vec{A}_0 - \sum_{i=1}^{3}\vec{A}_i\times \; = 0 - 0 = 0}\cdot\vec{\sigma} = \vec{A}_\mu\cdot\vec{A}^\mu$$

lorentz index

$$\Rightarrow \frac{1}{2}m_A^2\vec{A}_\mu'\cdot\vec{A}^{\mu\prime} = \frac{1}{2}m_A^2(\vec{\sigma}\cdot\vec{A}_\mu')(\vec{\sigma}\cdot\vec{A}^{\mu\prime})$$

$$= \frac{1}{2}m_A^2\left[S(\vec{\sigma}\cdot\vec{A}_\mu)S^{-1} + \frac{i}{k}(\partial_\mu S)S^{-1}\right]\left[S(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1} + \frac{i}{k}(\partial^\mu S)S^{-1}\right]$$

$$= \frac{1}{2}m_A^2\Big[S(\vec{\sigma}\cdot\vec{A}_\mu)(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1} + \left(\frac{i}{k}\right)^2(\partial_\mu S)S^{-1}(\partial^\mu S)S^{-1}$$

$$+ \frac{i}{k}S(\vec{\sigma}\cdot\vec{A}_\mu)S^{-1}(\partial^\mu S)S^{-1} + \frac{i}{k}(\partial_\mu S)(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1}\Big]$$

where $(\partial^\mu S)S^{-1} = \partial^\mu(SS^{-1}) - S(\partial^\mu S^{-1}) = -S(\partial^\mu S^{-1})$

$$\Rightarrow \left(\frac{i}{k}\right)^2(\partial_\mu S)S^{-1}(\partial^\mu S)S^{-1} = \left(\frac{i}{k}\right)^2(\partial_\mu S)S^{-1}(-S(\partial^\mu S^{-1}))$$

$$= -\left(\frac{i}{k}\right)^2(\partial_\mu S)(\partial^\mu S^{-1})$$

$$\frac{i}{k}S(\vec{\sigma}\cdot\vec{A}_\mu)S^{-1}(\partial^\mu S)S^{-1} = -\frac{i}{k}S(\vec{\sigma}\cdot\vec{A}_\mu)(\partial^\mu S^{-1})$$

$$\Rightarrow \frac{1}{2}m_A^2\vec{A}_\mu'\cdot\vec{A}^{\mu\prime}$$

$$= \frac{1}{2}m_A^2\Big[S(\vec{A}_\mu\cdot\vec{A}^\mu)S^{-1} - \left(\frac{i}{k}\right)^2(\partial_\mu S)(\partial^\mu S^{-1}) - \frac{i}{k}S(\vec{\sigma}\cdot\vec{A}_\mu)(\partial^\mu S^{-1})$$

$$+ \frac{i}{k}(\partial_\mu S)(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1}\Big]$$

$$\neq \frac{1}{2}m_A^2\vec{A}_\mu\cdot\vec{A}^\mu.$$

)

We need to take $\vec{F}^{\mu\nu} \equiv \partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu - 2k(\vec{A}^\mu\times\vec{A}^\nu)$

to make $\mathcal{L}_A$ invariant under local gauge transformation.

check $\quad F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu \vec{A}'^\mu - 2k(\vec{A}'^\mu \times \vec{A}'^\nu)$

$$\vec{\sigma}\cdot\vec{F}'^{\mu\nu} = \partial^\mu(\vec{\sigma}\cdot\vec{A}'^\nu) - \partial^\nu(\vec{\sigma}\cdot\vec{A}'^\mu) - 2k\,\vec{\sigma}\cdot(\vec{A}'^\mu\times\vec{A}'^\nu)$$

using $\quad \vec{\sigma}\cdot\vec{A}'^\mu = S(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1} + \frac{i}{k}(\partial^\mu S)S^{-1}$

and $\quad (\vec{\sigma}\cdot\vec{A}^\mu)(\vec{\sigma}\cdot\vec{A}^\nu) = \sigma^i A^{i\mu}\sigma^j A^{j\nu} = (\delta^{ij} + i\epsilon^{ijk}\sigma^k)A^{i\mu}A^{j\nu}$

$$= \vec{A}^\mu\cdot\vec{A}^\nu + i\vec{\sigma}\cdot(\vec{A}^\mu\times\vec{A}^\nu)$$

$$\Rightarrow \quad (\vec{\sigma}\cdot\vec{A}^\mu)(\vec{\sigma}\cdot\vec{A}^\nu) - (\vec{\sigma}\cdot\vec{A}^\nu)(\vec{\sigma}\cdot\vec{A}^\mu)$$

$$= 2i\vec{\sigma}\cdot(\vec{A}^\mu\times\vec{A}^\nu)$$

$$\begin{aligned}
\Rightarrow \vec{\sigma}\cdot\vec{F}'^{\mu\nu} = {} & \partial^\mu\left[S(\vec{\sigma}\cdot\vec{A}^\nu)S^{-1} + \tfrac{i}{k}(\partial^\nu S)S^{-1}\right] \\
& - \partial^\nu\left[S(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1} + \tfrac{i}{k}(\partial^\mu S)S^{-1}\right] \\
& - 2k\frac{1}{2i}\left[(\vec{\sigma}\cdot\vec{A}'^\mu)(\vec{\sigma}\cdot\vec{A}'^\nu) - (\vec{\sigma}\cdot\vec{A}'^\nu)(\vec{\sigma}\cdot\vec{A}'^\mu)\right] \\
= {} & \partial^\mu\left[S(\vec{\sigma}\cdot\vec{A}^\nu)S^{-1}\right] - \partial^\nu\left[S(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1}\right] \\
& + \tfrac{i}{k}\left[(\partial^\nu S)(\partial^\mu S^{-1}) - (\partial^\mu S)(\partial^\nu S^{-1})\right] \\
& + ik\left\{\left[S(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1} + \tfrac{i}{k}(\partial^\mu S)S^{-1}\right]\left[S(\vec{\sigma}\cdot\vec{A}^\nu)S^{-1} + \tfrac{i}{k}(\partial^\nu S)S^{-1}\right] - \mu\leftrightarrow\nu\right\}
\end{aligned}$$

using $\quad 0 = \partial^\mu(S^{-1}S) = (\partial^\mu S^{-1})S + S^{-1}\partial^\mu S$

$$\Rightarrow \quad \partial^\mu S^{-1} = -S^{-1}(\partial^\mu S)S^{-1}$$

$$\partial^\mu S = -S(\partial^\mu S^{-1})S$$

$$\begin{aligned}
\Rightarrow \vec{\sigma}\cdot\vec{F}^{\mu\nu} = {} & \Big\{(\partial^\mu S)(\vec{\sigma}\cdot\vec{A}^\nu)S^{-1} + S\left(\partial^\mu(\vec{\sigma}\cdot\vec{A}^\nu)\right)S^{-1} + S(\vec{\sigma}\cdot\vec{A}^\nu)(\partial^\mu S^{-1}) \\
& + \tfrac{i}{k}(\partial^\nu S)(\partial^\mu S^{-1}) \\
& + ik\Big[S(\vec{\sigma}\cdot\vec{A}^\mu)(\vec{\sigma}\cdot\vec{A}^\nu)S^{-1} + \left(\tfrac{i}{k}\right)^2(\partial^\mu S)S^{-1}\overbrace{(\partial^\nu S)S^{-1}}^{-S(\partial^\nu S^{-1})} \\
& \quad + \tfrac{i}{k}(\partial^\mu S)(\vec{\sigma}\cdot\vec{A}^\nu)S^{-1} + \tfrac{i}{k}S(\vec{\sigma}\cdot\vec{A}^\mu)S^{-1}\underbrace{(\partial^\nu S)S^{-1}}_{-S(\partial^\nu S^{-1})}\Big\} \\
& - (\mu\leftrightarrow\nu)
\end{aligned}$$

$$\begin{aligned}
= {} & S\Big[\left(\partial^\mu(\vec{\sigma}\cdot\vec{A}^\nu)\right) - \left(\partial^\nu(\vec{\sigma}\cdot\vec{A}^\mu)\right) + ik(\vec{\sigma}\cdot\vec{A}^\mu)(\vec{\sigma}\cdot\vec{A}^\nu) \\
& - ik(\vec{\sigma}\cdot\vec{A}^\nu)(\vec{\sigma}\cdot\vec{A}^\mu)\Big]S^{-1} \\
= {} & S\left[\left(\partial^\mu(\vec{\sigma}\cdot\vec{A}^\nu)\right) - \left(\partial^\nu(\vec{\sigma}\cdot\vec{A}^\mu)\right) - 2k\vec{\sigma}\cdot(\vec{A}^\mu\times\vec{A}^\nu)\right]S^{-1} \\
= {} & S(\vec{\sigma}\cdot\vec{F}^{\mu\nu})S^{-1}
\end{aligned}$$

$$\mathrm{Tr}\left[(\vec{\sigma}\cdot\vec{F}'^{\mu\nu})(\vec{\sigma}\cdot\vec{F}'_{\mu\nu})\right] = \mathrm{Tr}\left[S(\vec{\sigma}\cdot\vec{F}^{\mu\nu})S^{-1}S(\vec{\sigma}\cdot\vec{F}_{\mu\nu})S^{-1}\right]$$
$$= \mathrm{Tr}\left[(\vec{\sigma}\cdot\vec{F}^{\mu\nu})(\vec{\sigma}\cdot\vec{F}_{\mu\nu})\right]$$

$$\mathrm{Tr}\left[(\vec{\sigma}\cdot\vec{F}^{\mu\nu})(\vec{\sigma}\cdot\vec{F}_{\mu\nu})\right] = F^{i\mu\nu}F^{j}_{\mu\nu}\,\mathrm{Tr}(\sigma^i\sigma^j)$$
$$= F^{i\mu\nu}F^{j}_{\mu\nu}\,\mathrm{Tr}(\delta^{ij}+i\varepsilon^{ijk}\sigma^k) = F^{i\mu\nu}F^{j}_{\mu\nu}\,2\delta^{ij}$$
$$= 2\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}$$

$$\Rightarrow \quad \vec{F}'^{\mu\nu}\cdot\vec{F}'_{\mu\nu} = \vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}.$$

$\Rightarrow$ $\mathcal{L}_A$ is invariant under local gauge transformation.

Therefore, the complete Yang-Mills Lagrangian, which is invariant under local SU(2) transformation, is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu}$$

where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, $D_\mu = \partial_\mu + ik\,\vec{\sigma}\cdot\vec{A}_\mu$,

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - 2k(\vec{A}_\mu\times\vec{A}_\nu)$$

This Lagrangian describes two equal-mass Dirac fields in interaction with three massless vector gauge fields.

<u>QCD</u>

Let's apply Yang-Mills theory to construct the QCD Lagrangian.

$$\psi = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}, \quad \bar{\psi} = (\bar{\psi}_r \ \bar{\psi}_b \ \bar{\psi}_g)$$

Since the three colors of a given flavor are supposed to have the same mass, then the free Lagrangian for a particular flavor is

$$\mathcal{L} = i\bar{\psi}_r \gamma^\mu \partial_\mu \psi_r - m\bar{\psi}_r\psi_r + i\bar{\psi}_b\gamma^\mu\partial_\mu\psi_b - m\bar{\psi}_b\psi_b$$
$$+ i\bar{\psi}_g\gamma^\mu\partial_\mu\psi_g - m\bar{\psi}_g\psi_g$$
$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi.$$

This Lagrangian has a $U(3)$ symmetry, that is, it is invariant under a global $U(3)$ transformation $\psi \to U\psi$ (so that $\bar{\psi} \to \bar{\psi}U^\dagger$), where U is a 3×3 unitary matrix.

Since any unitary matrix can be written as an exponential Hermitian matrix: $U = e^{iH}$, with $H^\dagger = H$, and any 3×3 Hermitian matrix can be expressed in terms of nine real numbers, $a_1, a_2, \cdots a_8$ and θ, as

$H = \theta \mathbb{1}_{3\times3} + \underbrace{\vec{\lambda}\cdot\vec{a}}_{\sum_{i=1}^{8}\lambda_i a_i}$, where $\lambda_1, \cdots, \lambda_8$ are Gell-Mann matrices,

then $U = e^{i\theta} e^{i\vec{\lambda}\cdot\vec{a}}$.

(Check: any 3×3 Hermitian matrix can be written as $H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$ with h_{11}, h_{22}, h_{33} reals, $h_{12}^* = h_{21}$, $h_{13}^* = h_{31}$, $h_{23}^* = h_{32}$, so we can write it using 9 real numbers, as $H = \begin{pmatrix} b & f+ig & h+ik \\ f-ig & c & x+iy \\ h-ik & x-iy & d \end{pmatrix}$

The eight Gell-Mann matrices are $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$, $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix},$$

it is checked that (see my Mathematica code "Gell-Mann matrix property.nb")

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}$$

where f_{abc} are structure constants, they are completely antisymmetric

with $f_{123} = 1$, $f_{147} = f_{246} = f_{257} = f_{345}$, $f_{458} = f_{678} = \frac{\sqrt{3}}{2}$

$= f_{516} = f_{637} = \frac{1}{2}$

(i.e., $f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{acb} = -f_{cba}$)

$$\Rightarrow \theta \mathbb{1} + \vec{\lambda}\cdot\vec{a} = \begin{pmatrix} \theta + a_3 + \frac{1}{\sqrt{3}} a_8 & a_1 - i a_2 & a_4 - i a_5 \\ a_1 + i a_2 & \theta - a_3 + \frac{1}{\sqrt{3}} a_8 & a_6 - i a_7 \\ a_4 + i a_5 & a_6 + i a_7 & \theta - \frac{2}{\sqrt{3}} a_8 \end{pmatrix}$$

So, we obtain $a_1 = f$, $a_2 = -g$, $a_4 = h$, $a_5 = -k$, $a_6 = x$, $a_7 = -$

$$\begin{cases} 3\theta = b+c+d & \Rightarrow \theta = (b+c+d)/3 \\ 2a_3 = b-c & \Rightarrow a_3 = (b-c)/2 \\ \theta - \frac{2}{\sqrt{3}} a_8 = d & \Rightarrow a_8 = \frac{(\theta - d)\sqrt{3}}{2} = \left(\frac{b+c+d}{3} - d\right)\sqrt{3}/2 \\ & = \frac{b+c-2d}{2\sqrt{3}} \end{cases}$$

done the check)

Let's focus on the $e^{i\vec{\lambda}\cdot\vec{a}}$ part of the transformation.

This is an SU(3) transformation, i.e., $\det(e^{i\vec{\lambda}\cdot\vec{a}}) = 1$, $(e^{i\vec{\lambda}\cdot\vec{a}})^+ e^{i\vec{\lambda}\cdot\vec{a}}$

$= e^{i\vec{\lambda}\cdot\vec{a}} (e^{i\vec{\lambda}\cdot\vec{a}})^+ =$

Under a global SU(3) transformation, $S = e^{i\vec{\lambda}\cdot\vec{a}}$, the Lagrangian

$$\mathcal{L}' = i\bar{\psi}'\gamma^\mu \partial_\mu \psi' - m\bar{\psi}'\psi' = i\bar{\psi}S^+\gamma^\mu\partial_\mu(S\psi) - m\bar{\psi}S^+S\psi$$

$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = \mathcal{L}, \quad \text{where } \psi = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}$$

Now, let's "promote" it to a local transformation (i.e., $S \equiv S(x) = e^{i\vec{\lambda}\cdot\vec{a}(x)}$)

First, we introduce the covariant derivative

$$D_\mu = \partial_\mu + ik\vec{\lambda}\cdot\vec{A}_\mu.$$

and require that $D'_\mu \psi' = S D_\mu \psi$, and therefore

$$i\bar{\psi}'\gamma^\mu D'_\mu \psi' = i\bar{\psi}\gamma^\mu D_\mu \psi .$$

Since
$$\begin{aligned} D'_\mu \psi' &= (\partial_\mu + ik\vec{\lambda}\cdot\vec{A}'_\mu)(S\psi) \\ &= (\partial_\mu S)\psi + S(\partial_\mu \psi) + ik\vec{\lambda}\cdot\vec{A}'_\mu S\psi \\ &\underset{\text{require}}{\overset{\uparrow}{=}} S D_\mu \psi = S(\partial_\mu \psi + ik\vec{\lambda}\cdot\vec{A}_\mu \psi) \end{aligned}$$

$$\Rightarrow \quad ik\vec{\lambda}\cdot\vec{A}'_\mu S + \partial_\mu S = ikS\,\vec{\lambda}\cdot\vec{A}_\mu$$

$$\Rightarrow \quad \vec{\lambda}\cdot\vec{A}'_\mu = S(\vec{\lambda}\cdot\vec{A}_\mu)S^{-1} + \frac{i}{k}(\partial_\mu S)S^{-1}$$

We also need to introduce the free part Lagrangian for $\vec{A}_\mu$,

$$\mathcal{L}_A = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} = -\frac{1}{4}\sum_{i=1}^{8} F^i_{\mu\nu}F^{i\mu\nu}$$

where $\vec{F}^{\mu\nu} = \partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu - 2k(\vec{A}^\mu\times\vec{A}^\nu)$,

where $(\vec{A}^\mu\times\vec{A}^\nu)^i \equiv f^{ijk}A^{j\mu}A^{k\nu}$

Let's show that $\vec{F}'^{\mu\nu}\cdot\vec{F}'_{\mu\nu} = \vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}$

First of all, we notice that

$$\begin{aligned} &(\vec{\lambda}\cdot\vec{A}^\mu)(\vec{\lambda}\cdot\vec{A}^\nu) - (\vec{\lambda}\cdot\vec{A}^\nu)(\vec{\lambda}\cdot\vec{A}^\mu) \\ &= \lambda^i\lambda^j A^{i\mu}A^{j\nu} - \lambda^j\lambda^i A^{i\mu}A^{\nu j} = [\lambda^i,\lambda^j]A^{i\mu}A^{\nu j} \\ &= 2if^{ijk}\lambda^k A^{i\mu}A^{j\nu} \equiv 2i\vec{\lambda}\cdot(\vec{A}^\mu\times\vec{A}^\nu) \end{aligned}$$

then
$$\begin{aligned} \vec{\lambda}\cdot\vec{F}'^{\mu\nu} &= \partial^\mu(\vec{\lambda}\cdot\vec{A}'^\nu) - \partial^\nu(\vec{\lambda}\cdot\vec{A}'^\mu) - 2k\,\vec{\lambda}\cdot(\vec{A}'^\mu\times\vec{A}'^\nu) \\ &= \left\{\partial^\mu\left[S(\vec{\lambda}\cdot\vec{A}^\nu)S^{-1} + \frac{i}{k}(\partial^\nu S)S^{-1}\right] - (\mu\leftrightarrow\nu)\right\} \\ &\quad - \frac{2k}{2i}\left[(\vec{\lambda}\cdot\vec{A}'^\mu)(\vec{\lambda}\cdot\vec{A}'^\nu) - (\vec{\lambda}\cdot\vec{A}'^\nu)(\vec{\lambda}\cdot\vec{A}'^\mu)\right] \\ &= \Big\{(\partial^\mu S)(\vec{\lambda}\cdot\vec{A}^\nu)S^{-1} + S(\partial^\mu(\vec{\lambda}\cdot\vec{A}^\nu))S^{-1} + S(\vec{\lambda}\cdot\vec{A}^\nu)(\partial^\mu S^{-1}) \\ &\qquad + \frac{i}{k}(\partial^\nu S)(\partial^\mu S^{-1}) - (\mu\leftrightarrow\nu)\Big\} \\ &\quad + ik\Big\{\left[S(\vec{\lambda}\cdot\vec{A}^\mu)S^{-1} + \frac{i}{k}(\partial^\mu S)S^{-1}\right]\left[S(\vec{\lambda}\cdot\vec{A}^\nu)S^{-1} + \frac{i}{k}(\partial^\nu S)S^{-1}\right] \\ &\qquad - (\mu\leftrightarrow\nu)\Big\} \\ &= \Big\{(\partial^\mu S)(\vec{\lambda}\cdot\vec{A}^\nu)S^{-1} + S(\partial^\mu(\vec{\lambda}\cdot\vec{A}^\nu))S^{-1} + S(\vec{\lambda}\cdot\vec{A}^\nu)(\partial^\mu S^{-1}) \\ &\qquad + \frac{i}{k}(\partial^\nu S)(\partial^\mu S^{-1}) \\ &\qquad + ik\Big[S(\vec{\lambda}\cdot\vec{A}^\mu)(\vec{\lambda}\cdot\vec{A}^\nu)S^{-1} - \frac{1}{k^2}(\partial^\mu S)S^{-1}(\partial^\nu S)S^{-1} \\ &\qquad\quad + \frac{i}{k}(\partial^\mu S)(\vec{\lambda}\cdot\vec{A}^\nu)S^{-1} + \frac{i}{k}S(\vec{\lambda}\cdot\vec{A}^\mu)S^{-1}(\partial^\nu S)S^{-1}\Big\} \\ &\qquad - (\mu\leftrightarrow\nu) \end{aligned}$$

using $0 = \partial^\mu (S S^{-1}) = (\partial^\mu S) S^{-1} + S(\partial^\mu S^{-1})$

$$\Rightarrow \vec{\lambda}\cdot\vec{F}'^{\mu\nu} = S\left[\left(\partial^\mu(\vec{\lambda}\cdot\vec{A}^\nu)\right) - \left(\partial^\nu(\vec{\lambda}\cdot\vec{A}^\mu)\right)\right]S^{-1}$$
$$+ ik\,2i\,S\left[\vec{\lambda}\cdot(\vec{A}^\mu\times\vec{A}^\nu)\right]S^{-1}$$
$$= S(\vec{\lambda}\cdot\vec{F}^{\mu\nu})S^{-1}$$

$$\Rightarrow (\vec{\lambda}\cdot\vec{F}'^{\mu\nu})(\vec{\lambda}\cdot\vec{F}'_{\mu\nu}) = S\left[(\vec{\lambda}\cdot\vec{F}^{\mu\nu})(\vec{\lambda}\cdot\vec{F}_{\mu\nu})\right]S^{-1}$$

$$\Rightarrow \mathrm{Tr}\left[(\vec{\lambda}\cdot\vec{F}'^{\mu\nu})(\vec{\lambda}\cdot\vec{F}'_{\mu\nu})\right] = \mathrm{Tr}\left[S(\vec{\lambda}\cdot\vec{F}^{\mu\nu})(\vec{\lambda}\cdot\vec{F}_{\mu\nu})S^{-1}\right]$$
$$= \mathrm{Tr}\left[(\vec{\lambda}\cdot\vec{F}^{\mu\nu})(\vec{\lambda}\cdot\vec{F}_{\mu\nu})\right]$$

Since $\mathrm{Tr}\left[(\vec{\lambda}\cdot\vec{F}^{\mu\nu})(\vec{\lambda}\cdot\vec{F}_{\mu\nu})\right] = \mathrm{Tr}\left[\lambda^i\lambda^j F^{\mu\nu}_i F_{\mu\nu j}\right]$

$$\underset{\uparrow}{=} \frac{2}{3}\times 3\,\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu} = 2\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}$$

use $\lambda_i\lambda_j = \frac{2}{3}\delta_{ij} + (d_{ijk} + i f_{ijk})\lambda_k$ and $\mathrm{Tr}[\lambda_i] = 0$ for all λ_i

$$\Rightarrow \vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu} = \vec{F}'^{\mu\nu}\cdot\vec{F}'_{\mu\nu}$$

Also, the mass term for $\vec{A}_\mu$, $\frac{1}{2}m_A^2\vec{A}_\mu\cdot\vec{A}^\mu$ is not local gauge invariant, so it cannot be in the full Lagrangian.

check: $\mathrm{Tr}\left[(\vec{\lambda}\cdot\vec{A}^\mu)(\vec{\lambda}\cdot\vec{A}_\mu)\right] = \mathrm{Tr}\left[\lambda^i\lambda^j A^{i\mu}A^j_\mu\right] = 2\vec{A}_\mu\cdot\vec{A}^\mu$

$$\Rightarrow \vec{A}'_\mu\cdot\vec{A}'^\mu = \frac{1}{2}\mathrm{Tr}\left[(\vec{\lambda}\cdot\vec{A}'^\mu)(\vec{\lambda}\cdot\vec{A}'_\mu)\right]$$
$$= \frac{1}{2}\mathrm{Tr}\Big[\left(S(\vec{\lambda}\cdot\vec{A}_\mu)S^{-1} + \frac{i}{k}(\partial_\mu S)S^{-1}\right)$$
$$\left(S(\vec{\lambda}\cdot\vec{A}^\mu)S^{-1} + \frac{i}{k}(\partial^\mu S)S^{-1}\right)\Big]$$
$$= \frac{1}{2}\mathrm{Tr}\left[S(\vec{\lambda}\cdot\vec{A}_\mu)(\vec{\lambda}\cdot\vec{A}^\mu)S^{-1}\right]$$
$$+ \frac{1}{2}\mathrm{Tr}\left[\left(\frac{i}{k}\right)^2(\partial_\mu S)S^{-1}(\partial^\mu S)S^{-1}\right]$$
$$+ \frac{i}{k}\mathrm{Tr}\left[(\partial_\mu S)(\vec{\lambda}\cdot\vec{A}^\mu)S^{-1}\right]$$
$$= \vec{A}_\mu\cdot\vec{A}^\mu + \frac{1}{2}\left(\frac{i}{k}\right)^2(-1)\,\mathrm{Tr}\left[(\partial_\mu S^{-1})(\partial^\mu S)\right]$$
$$+ \frac{i}{k}\mathrm{Tr}\left[(\partial_\mu S)(\vec{\lambda}\cdot\vec{A}^\mu)S^{-1}\right]$$
$$\neq \vec{A}_\mu\cdot\vec{A}^\mu$$

So, the QCD Lagrangian is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu}$$

where $\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - 2k\underbrace{(\vec{A}_\mu\times\vec{A}_\nu)}$

i.e., $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - 2k\, f^{ijk}A^j_\mu A^k_\nu$

$D_\mu = \partial_\mu + ik\vec{\lambda}\cdot\vec{A}_\mu = \partial_\mu + ik\sum_{i=1}^{8}\lambda^i A^i_\mu$

The convension is to take $k = \frac{g_s}{2}$, where $\alpha_s = \frac{g_s^2}{4\pi}$