

Bilinear combinations of fermion fields under P, C, and T

By imposing that the free fermion Lagrangian is invariant under separate transformations of P, C, and T, we derive how the bilinear combinations of fermion fields transform under P, C, and T.

1. Parity

$\chi \rightarrow \tilde{\chi}$: means $t \rightarrow t$, $\vec{x} \rightarrow -\vec{x}$.

$$L(x) = i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - m \bar{\psi}(x) \psi(x)$$

note that this formula means $P \bar{\psi}(t, \vec{x}) P^{-1} = \bar{\psi}(t, -\vec{x})$

From $\psi(x) \rightarrow \eta A \psi(\tilde{x})$, where η is a phase ($|\eta|^2 = 1$), and A is a 4×4 matrix independent of space-time coordinate.

$$\Rightarrow \psi^+(x) \rightarrow \eta^* \psi^+(\tilde{x}) A^+ \leftarrow \text{it means } P \psi^+(t, \vec{x}) P^{-1} = \eta^* \psi^+(t, -\vec{x}) A^+$$

$$\Rightarrow = i \eta^* \psi^+(\tilde{x}) A^+ \gamma^0 \gamma^\mu \partial_\mu (\eta A \psi(\tilde{x})) - m \eta^* \psi^+(\tilde{x}) A^+ \gamma^0 \eta A \psi(\tilde{x})$$

note that
P acts on
Hilbert space
operator
only, so not
on ∂_μ here

require

$$\begin{cases} \gamma^0 A^+ \gamma^0 \gamma^\mu A = \tilde{\gamma}^\mu = \begin{cases} \gamma^0, & \mu=0 \\ -\gamma^i, & \mu=1,2,3 \end{cases} \\ \gamma^0 A^+ \gamma^0 A = 1 \end{cases} \quad \textcircled{1}$$

note that
Since the
action integrates
over all \tilde{x} ,
we can change
 $\tilde{x} \rightarrow -\tilde{x}$ and
the action
will be invariant.

$$\begin{cases} \gamma^0 A^+ \gamma^0 \gamma^0 A = \gamma^0 \Rightarrow A^+ A = 1 \Rightarrow A^+ A A^+ = A^+ \\ \gamma^0 A^+ \gamma^0 \gamma^i A = -\gamma^i \end{cases} \quad \textcircled{2}$$

$$\begin{aligned} &\text{right times } A^+ \\ &(A^+)^{-1} A A^+ = (A^+)^{-1} A^+ \\ &\Downarrow = 1 \\ &A A^+ = 1 \end{aligned}$$

So, A is a unitary matrix.

$$\text{From } \textcircled{2} \Rightarrow A^+ \gamma^0 A = \gamma^0$$

$$\text{From } \textcircled{3} \Rightarrow \gamma^0 A^+ \gamma^0 \underline{A A^+} \gamma^i A = -\gamma^i \Rightarrow A^+ \gamma^i A = -\gamma^i$$

use A is unitary.

$$\Rightarrow \boxed{A^+ \gamma^\mu A = \tilde{\gamma}^\mu = \gamma_\mu}$$

① (checked)

$$\text{Therefore, } \bar{\psi}_1(x) \psi_2(x) \leftarrow \text{it means that } P \bar{\psi}_1(x) \psi_2(x) P^{-1} \\ \rightarrow \eta^* \eta_2 \bar{\psi}_1^+(x) A^+ \gamma^0 A \psi_2(x) = \eta^* \eta_2 \bar{\psi}_1(x) \psi_2(x)$$

$$\bar{\psi}_1(x) \gamma^\mu \psi_2(x) \\ \rightarrow \eta^* \eta_2 \bar{\psi}_1^+(x) A^+ \gamma^0 \underbrace{\gamma^\mu}_{AA^+} A \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1^+(x) \gamma^0 \gamma_\mu \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1(x) \gamma_\mu \psi_2(x)$$

$$\bar{\psi}_1(x) i \gamma_5 \psi_2(x) \\ \rightarrow \eta^* \eta_2 \bar{\psi}_1^+(x) A^+ \underbrace{\gamma^0 i \gamma_5}_{AA^+} A \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1(x) A^+ i \gamma_5 A \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1(x) A^+ i \underbrace{\gamma^0 \gamma^1 \gamma^2 \gamma^3}_{AA^+ AA^+} A \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1(x) i^2 \gamma^0 \gamma_1 \gamma_2 \gamma_3 \psi_2(x) \\ = -\eta^* \eta_2 \bar{\psi}_1(x) i \gamma_5 \psi_2(x)$$

$$\bar{\psi}_1(x) \gamma^\mu \gamma_5 \psi_2(x) \quad AA^+ \\ \rightarrow \eta^* \eta_2 \bar{\psi}_1^+(x) A^+ \underbrace{\gamma^0 \gamma^\mu \gamma_5}_{AA^+} A \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1(x) \gamma_\mu (-\gamma_5) \psi_2(x) \\ = -\eta^* \eta_2 \bar{\psi}_1(x) \gamma_\mu \gamma_5 \psi_2(x)$$

$$\bar{\psi}_1(x) \sigma^{\mu\nu} \psi_2(x) \quad AA^+ \quad AA^+ \quad AA^+ \\ \rightarrow \eta^* \eta_2 \bar{\psi}_1^+(x) A^+ \gamma^0 \underbrace{i}_{2} \underbrace{(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)}_{\sigma^{\mu\nu}} A \psi_2(x) \\ = \eta^* \eta_2 \bar{\psi}_1(x) \sigma_{\mu\nu} \psi_2(x)$$

2. Charge conjugation.

$$\text{Since } \mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x)$$

$$= \partial_\mu [i\bar{\psi}(x)\gamma^\mu \psi(x)] - i(\partial_\mu \bar{\psi}(x))\gamma^\mu \psi(x) - m\bar{\psi}(x)\psi(x)$$

then it is OK for $\mathcal{L}'(x)$ takes the form $-i(\partial_\mu \bar{\psi}(x))\gamma^\mu \psi(x) - m\bar{\psi}(x)\psi(x)$ to fulfill the invariance of charge conjugation.

Also, since charge conjugation change particle to antiparticle, ψ should transform to ψ^+ ; considering that $\bar{\psi}$ is more frequently used, the transformation rule is \leftarrow it means $C\psi(x)C^{-1} = \Sigma B \bar{\psi}^T(x)$

$$\psi(x) \rightarrow \Sigma B \bar{\psi}^T(x), \text{ where } \Sigma \text{ is a phase, } |\Sigma|^2 = 1,$$

B is a 4×4 matrix independent of spacetime

$$\Rightarrow \psi^+(x) \rightarrow \Sigma^* (B \bar{\psi}^T(x))^+ = \Sigma^* \bar{\psi}^*(x) B^+ = \Sigma^* (\gamma_0 \gamma^i)^* B^+$$

$$= \Sigma^* \bar{\psi}^T(x) \gamma^a \gamma^* B^+$$

$$= \Sigma^* \bar{\psi}^T(x) \gamma^a B^+. \quad \leftarrow \text{it means } C\psi^+(x)C^{-1} = \Sigma^* \bar{\psi}^T(x) \gamma^a B^+$$

$$\begin{aligned} & \Rightarrow i \Sigma^* \bar{\psi}^T(x) \gamma^a B^+ \gamma^a \gamma^\mu \partial_\mu (\Sigma B \bar{\psi}^T(x)) - m \Sigma^* \bar{\psi}^T(x) \gamma^a B^+ \gamma^a \Sigma B \bar{\psi}^T(x) \\ &= i \bar{\psi}^T(x) \gamma^a B^+ \gamma^a \gamma^\mu \partial_\mu \bar{\psi}^T(x) - m \bar{\psi}^T(x) \gamma^a B^+ \gamma^a \bar{\psi}^T(x) \\ &= i (\bar{\psi}^T(x) \gamma^a B^+ \gamma^a \gamma^\mu \partial_\mu \bar{\psi}^T(x))^T - m (\bar{\psi}^T(x) \gamma^a B^+ \gamma^a \bar{\psi}^T(x))^T \\ &= -i (\partial_\mu \bar{\psi}(x)) B^+ \gamma^\mu \gamma^a B^* \gamma^a \psi(x) + m \bar{\psi}(x) B^+ \gamma^a B^* \gamma^a \psi(x) \end{aligned}$$

\uparrow
due to exchange the position of two fermion operator

require
 $\equiv -i(\partial_\mu \bar{\psi}(x)) \gamma^\mu \psi(x) - m \bar{\psi}(x) \psi(x)$
 right times γ^a on both sides

$$\Rightarrow \begin{cases} B^+ \gamma^a B^* \gamma^a = -1 \Rightarrow B^+ \gamma^a B^* = -\gamma^a \Rightarrow B^+ \gamma^a B = -\gamma^a & \dots \textcircled{1} \\ B^+ \gamma^\mu \gamma^a B^* \gamma^a = \gamma^\mu & \text{do complex conjugate} \end{cases} \quad \dots \textcircled{2}$$

use $\begin{cases} \gamma^a \gamma^\mu \gamma^a = \gamma^\mu \\ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \end{cases} \Rightarrow \gamma^a = \gamma^a, \gamma^{+i} = -\gamma^i \Rightarrow \gamma^a = \gamma^a, \gamma^i = -\gamma^i$

from $\textcircled{1} \Rightarrow B^+ \gamma^a B = -\gamma^a$, from $\textcircled{2} \Rightarrow \begin{cases} B^+ \gamma^a B^* \gamma^a = \gamma^a \Rightarrow B^+ B^* = 1 \Rightarrow B^+ B = 1, \text{ so } B \text{ is a} \\ \text{unitary matrix} \end{cases}$

$\textcircled{3}$ (checked)

$$\text{From } B^T \gamma^i \gamma^0 B^* \gamma^0 = \gamma^i$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{do complex conjugate}}}{B^+} \gamma^i \gamma^0 B \gamma^0 = \gamma^{*i} \Rightarrow -\underset{\substack{\uparrow \\ BB^+}}{B^+} \gamma^i \gamma^0 B \gamma^0 = -\gamma^i \Rightarrow B^+ \gamma^i B (-\gamma^0) \gamma^0 = \gamma^i$$

$$\Rightarrow B^+ \gamma^i B = -\gamma^i$$

$$\Rightarrow \boxed{B^+ \gamma^\mu B = -\gamma^\mu}$$

Therefore, $\bar{f}_1(x) \bar{f}_2(x)$

$$\rightarrow \varepsilon_1^* \varepsilon_2 \bar{f}_1^T(x) \gamma^0 B^+ \gamma^0 B \bar{f}_2^T(x) = \varepsilon_1^* \varepsilon_2 f_1^T(x) \gamma^0 (-\gamma^0) \bar{f}_2^T(x)$$

$$= -\varepsilon_1^* \varepsilon_2 f_1^T(x) \bar{f}_2^T(x) = -\varepsilon_1^* \varepsilon_2 (f_1^T(x), \bar{f}_2^T(x))^T$$

$$= \varepsilon_1^* \varepsilon_2 \bar{f}_2(x) f_1(x) \quad \leftarrow \text{it means that}$$

$$\bar{f}_1(x) \gamma^\mu f_2(x) \quad C \bar{f}_1(x) f_2(x) C^{-1}$$

$$= \varepsilon_1^* \varepsilon_2 \bar{f}_2(x) f_1(x)$$

$$\rightarrow \varepsilon_1^* \varepsilon_2 f_1^T(x) \gamma^0 B^+ \underset{\substack{\uparrow \\ BB^+}}{\gamma^0} \gamma^\mu B \bar{f}_2^T(x) = \varepsilon_1^* \varepsilon_2 f_1^T(x) \gamma^0 (-\gamma^0) (-\gamma^\mu) \bar{f}_2^T(x)$$

$$= \varepsilon_1^* \varepsilon_2 f_1^T(x) \gamma^\mu \bar{f}_2^T(x) = \varepsilon_1^* \varepsilon_2 (f_1^T(x), \bar{f}_2^T(x))^T$$

$$= -\varepsilon_1^* \varepsilon_2 \bar{f}_2(x) \gamma^\mu f_1(x)$$

$\bar{f}_1(x) i \gamma_5 f_2(x)$

$$\rightarrow \varepsilon_1^* \varepsilon_2 f_1^T(x) \gamma^0 B^+ \underset{\substack{\uparrow \\ BB^+}}{\gamma^0} i \gamma_5 B \bar{f}_2^T(x) = -\varepsilon_1^* \varepsilon_2 f_1^T(x) \gamma^0 \gamma^0 i B^+ \gamma_5 B \bar{f}_2^T(x)$$

$$= -\varepsilon_1^* \varepsilon_2 f_1^T(x) i B^+ i \underset{\substack{\uparrow \\ BB^+}}{\gamma^0} \underset{\substack{\uparrow \\ BB^+}}{\gamma^1} \underset{\substack{\uparrow \\ BB^+}}{\gamma^2} \underset{\substack{\uparrow \\ BB^+}}{\gamma^3} B \bar{f}_2^T(x)$$

$$= -\varepsilon_1^* \varepsilon_2 f_1^T(x) i^2 \gamma^0 \gamma^1 \gamma^2 \gamma^3 \bar{f}_2^T(x)$$

$$= -\varepsilon_1^* \varepsilon_2 (f_1^T(x) i^2 \gamma^0 \gamma^1 \gamma^2 \gamma^3 \bar{f}_2^T(x))^T$$

$$= \varepsilon_1^* \varepsilon_2 \bar{f}_2(x) i^2 \gamma^3 \gamma^2 \gamma^1 \gamma^0 f_1(x)$$

$$= \varepsilon_1^* \varepsilon_2 \bar{f}_2(x) i \gamma_5 f_1(x)$$

$$\bar{\psi}_1(x) \gamma^\mu \gamma_5 \psi_2(x)$$

$$\begin{aligned}
& \rightarrow \epsilon_1^* \epsilon_2 \bar{\psi}_1^\top(x) \gamma^0 B^+ \gamma^0 \cancel{\gamma^\mu} \gamma_5 B \bar{\psi}_2^\top(x) \\
& = \epsilon_1^* \epsilon_2 \bar{\psi}_1^\top(x) \cancel{\gamma^0} \gamma^0 \gamma^\mu B^+ \gamma_5 B \bar{\psi}_2^\top(x) \\
& = \epsilon_1^* \epsilon_2 \bar{\psi}_1^\top(x) \gamma^\mu i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \bar{\psi}_2^\top(x) \\
& = \epsilon_1^* \epsilon_2 (\bar{\psi}_1^\top(x) \gamma^\mu i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \bar{\psi}_2^\top(x))^\top \\
& = -\epsilon_1^* \epsilon_2 \bar{\psi}_2^\top(x) i \gamma^3 \gamma^2 \gamma^1 \gamma^0 \gamma^\mu \psi_1(x) \\
& = -\epsilon_1^* \epsilon_2 \bar{\psi}_2^\top(x) \gamma_5 \gamma^\mu \psi_1(x) \\
& = \epsilon_1^* \epsilon_2 \bar{\psi}_2^\top(x) \gamma^\mu \gamma_5 \psi_1(x)
\end{aligned}$$

$$\bar{\psi}_1(x) \sigma^{\mu\nu} \psi_2(x)$$

$$\begin{aligned}
& \rightarrow \epsilon_1^* \epsilon_2 \bar{\psi}_1^\top(x) \gamma^0 B^+ \gamma^0 \sigma^{\mu\nu} B \bar{\psi}_2^\top(x) \\
& = \epsilon_1^* \epsilon_2 \bar{\psi}_1^\top(x) \gamma^0 B^+ \gamma^0 \frac{i}{2} (\underbrace{\gamma^\mu \gamma^\nu}_{BB^+} - \underbrace{\gamma^\nu \gamma^\mu}_{BB^+}) B \bar{\psi}_2^\top(x) \\
& = -\epsilon_1^* \epsilon_2 \bar{\psi}_1^\top(x) \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \bar{\psi}_2^\top(x) \\
& = -\epsilon_1^* \epsilon_2 (\bar{\psi}_1^\top(x) \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \bar{\psi}_2^\top(x))^\top \\
& = \epsilon_1^* \epsilon_2 \bar{\psi}_2^\top(x) \frac{i}{2} (\gamma^\nu \gamma^\mu - \gamma^\mu \gamma^\nu) \psi_1(x) \\
& = -\epsilon_1^* \epsilon_2 \bar{\psi}_2^\top(x) \sigma^{\mu\nu} \psi_1(x)
\end{aligned}$$

⑤ (checked)

$$3. \text{ Time inversion: } x^\mu \rightarrow \tilde{x}^\mu \quad \left\{ \begin{array}{l} t \rightarrow -t \\ \vec{x} \rightarrow \vec{x} \end{array} \right.$$

Since time inversion exchange initial and final state, that is, it changes a 'bra' to a 'ket', then it involves complex conjugate
 \leftarrow it means $T \bar{\psi}(x) T^{-1} = \bar{\psi} D \psi(\bar{x})$

From $\psi(x) \rightarrow \bar{\psi} D \psi(\bar{x})$, where $\bar{\psi}$ is a phase, $|\bar{\psi}|^2 = 1$,

$$\Rightarrow \psi^+(x) \rightarrow \bar{\psi}^* \psi^+(\bar{x}) D^+ \quad \leftarrow \text{it means } T \psi^+(x) T^{-1} = \bar{\psi}^* \psi^+(\bar{x}) D^+$$

$$\Rightarrow \mathcal{L}(x) = i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - m \bar{\psi}(x) \psi(x)$$

$$T \mathcal{L}(t, \bar{x}) T^{-1} = (-i) \bar{\psi}^* \psi^+(\bar{x}) D^+ \gamma^0 \gamma^\mu \partial_\mu (\bar{\psi} D \psi(\bar{x}))$$

$$= T \bar{\psi}(x) \gamma^\mu \psi(x) T^{-1} - m \bar{\psi}^* \psi^+(\bar{x}) D^+ \gamma^0 \bar{\psi} D \psi(\bar{x})$$

$$= T \bar{\psi} T \psi^+(\bar{x}) \gamma^0 \psi(x) T^{-1} = -i \bar{\psi}(\bar{x}) \gamma^0 D^+ \gamma^* \gamma^\mu D \partial_\mu \psi(x)$$

$$= i^* T \psi^+(\bar{x}) \gamma^0 D^+ \gamma^* \gamma^\mu D \bar{\psi}(\bar{x}) - m \bar{\psi}(\bar{x}) \gamma^0 D^+ \gamma^* \gamma^\mu D \psi(x)$$

$$\text{note that since the action integrates over } t, \text{ we can change } t \rightarrow -t \text{ and } \Rightarrow \text{the action will be invariant} \quad \leftarrow \text{require.} \quad \equiv i \bar{\psi}(\bar{x}) \gamma^\mu \partial_\mu \psi(x) - m \bar{\psi}(\bar{x}) \psi(x)$$

$$\left\{ \begin{array}{l} -\gamma^0 D^+ \gamma^* \gamma^\mu D = \bar{\gamma}^\mu = \begin{cases} -\gamma^0, \mu=0 \\ \gamma^i, \mu=1,2,3 \end{cases} \\ \gamma^0 D^+ \gamma^* D = 1 \end{array} \right. \quad \begin{array}{l} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array}$$

$$\text{note that } \leftarrow \text{From } \textcircled{2} \Rightarrow D^+ \gamma^* \gamma^0 D = \gamma^0 \Rightarrow D^+ \gamma^0 D = \gamma^0$$

$$\text{From } \textcircled{1} \Rightarrow D^+ \gamma^0 \gamma^* \gamma^0 D = 1 \Rightarrow D^+ D = 1, \text{ so } D \text{ is unitary matrix}$$

$$\text{From } \textcircled{3} \Rightarrow -\gamma^0 D^+ \gamma^0 \gamma^i D = \gamma^i \Rightarrow -\gamma^0 \gamma^0 D^+ \gamma^i D = \gamma^i$$

$$\Rightarrow D^+ \gamma^i D = \gamma^i$$

$$\boxed{\begin{array}{l} D^+ \gamma^0 D = \gamma^0 \\ D^+ \gamma^i D = \gamma^i \end{array}}$$

note that
 T acts on Hilbert space operator only
 (and perform complex conjugate)
 not acts on ∂_μ here

Therefore, $\bar{f}_1(x) f_2(x)$ ← it means $T \bar{f}_1(x) f_2(x) T^\dagger$

$$\rightarrow \xi_1^* \xi_2 f_1^+(x) D^+ \gamma^\mu \gamma^\nu D \bar{f}_2(\bar{x}) = \xi_1^* \xi_2 \bar{f}_1(\bar{x}) f_2(\bar{x})$$

$$= \xi_1^* \xi_2 f_1^+ \gamma^\mu \gamma^\nu f_2(\bar{x})$$

$$= \xi_1^* \xi_2 \bar{f}_1(\bar{x}) f_2(\bar{x})$$

$$\bar{f}_1(x) \gamma^\mu f_2(x)$$

$$\rightarrow \xi_1^* \xi_2 f_1^+(x) D^+ \gamma^\mu \gamma^\nu \underset{DD^+}{\hat{\gamma}} D \bar{f}_2(\bar{x})$$

$$= \xi_1^* \xi_2 f_1^+(\bar{x}) \gamma^\mu \gamma_\mu f_2(\bar{x})$$

$$= \xi_1^* \xi_2 \bar{f}_1(\bar{x}) \gamma_\mu f_2(\bar{x})$$

$$\bar{f}_1(x) i \gamma_5 f_2(x)$$

$$\rightarrow \xi_1^* \xi_2 f_1^+(\bar{x}) D^+ \gamma^\mu (i \gamma_5)^\# D \bar{f}_2(\bar{x})$$

$$= - \xi_1^* \xi_2 f_1^+(\bar{x}) D^+ \gamma^\mu \underset{DD^+}{\hat{\gamma}} \gamma^\nu \underset{DD^+}{\hat{\gamma}} \gamma^\lambda \underset{DD^+}{\hat{\gamma}} \gamma^\rho \underset{DD^+}{\hat{\gamma}} D \bar{f}_2(\bar{x})$$

$$= - \xi_1^* \xi_2 f_1^+(\bar{x}) \gamma^\mu \gamma^\nu \gamma_\lambda \gamma_\rho f_2(\bar{x})$$

$$= - \xi_1^* \xi_2 \bar{f}_1(\bar{x}) i \gamma_5 f_2(\bar{x})$$

$$\bar{f}_1(x) \gamma^\mu \gamma_5 f_2(x)$$

$$\rightarrow \xi_1^* \xi_2 f_1^+(\bar{x}) D^+ \gamma^\mu \underset{DD^+}{\hat{\gamma}} \gamma^\nu \underset{DD^+}{\hat{\gamma}} \gamma_5^\# D \bar{f}_2(\bar{x})$$

$$= \xi_1^* \xi_2 f_1^+(\bar{x}) \gamma^\mu \underset{DD^+}{\hat{\gamma}} (-i) \gamma^\nu \underset{DD^+}{\hat{\gamma}} \gamma_5^\# \gamma_\lambda \gamma_\rho f_2(\bar{x})$$

$$= \xi_1^* \xi_2 \bar{f}_1(\bar{x}) \gamma_\mu \gamma_5 f_2(\bar{x})$$

$$\bar{f}_1 \Gamma^{\mu\nu} f_2(x)$$

$$\rightarrow \xi_1^* \xi_2 f_1^+(\bar{x}) D^+ \gamma^\mu (\Gamma^{\mu\nu})^\# D \bar{f}_2(\bar{x})$$

$$= \xi_1^* \xi_2 f_1^+(\bar{x}) D^+ \gamma^\mu \underset{DD^+}{\hat{\gamma}} (-\frac{i}{2}) (\gamma^\nu \underset{DD^+}{\hat{\gamma}} \gamma^\lambda - \gamma^\lambda \underset{DD^+}{\hat{\gamma}} \gamma^\nu) D \bar{f}_2(\bar{x})$$

$$= \xi_1^* \xi_2 f_1^+(\bar{x}) \gamma^\mu (-\frac{i}{2}) (\gamma_\nu \gamma_\lambda - \gamma_\lambda \gamma_\nu) f_2(\bar{x})$$

$$= - \xi_1^* \xi_2 \bar{f}_1(\bar{x}) \Gamma_{\mu\nu} f_2(\bar{x})$$

⑥ (checked)

To summarize, up to the phase factor $\gamma_1^*\gamma_2$, $\epsilon_1^*\epsilon_2$, $\phi_1^*\phi_2$, we have the following transformation properties for bilinear combinations

P	C	T	CP	CPT
$\bar{\psi}_1 \psi_2$	$\bar{\psi}_1 \psi_2$	$\bar{\psi}_2 \psi_1$	$\bar{\psi}_1 \psi_2$	$\bar{\psi}_2 \psi_1$
$\bar{\psi}_1 \gamma^\mu \psi_2$	$\bar{\psi}_1 \gamma_\mu \psi_2$	$-\bar{\psi}_2 \gamma^\mu \psi_1$	$\bar{\psi}_1 \gamma_\mu \psi_2$	$-\bar{\psi}_2 \gamma_\mu \psi_1$
$\bar{\psi}_1 i\gamma_5 \psi_2$	$-\bar{\psi}_1 i\gamma_5 \psi_2$	$\bar{\psi}_2 i\gamma_5 \psi_1$	$-\bar{\psi}_1 i\gamma_5 \psi_2$	$-\bar{\psi}_2 i\gamma_5 \psi_1$
$\bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2$	$-\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2$	$\bar{\psi}_2 \gamma^\mu \gamma_5 \psi_1$	$\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2$	$-\bar{\psi}_2 \gamma_\mu \gamma_5 \psi_1$
$\bar{\psi}_1 \sigma^{\mu\nu} \psi_2$	$\bar{\psi}_1 \sigma_{\mu\nu} \psi_2$	$-\bar{\psi}_2 \sigma^{\mu\nu} \psi_1$	$-\bar{\psi}_1 \sigma_{\mu\nu} \psi_2$	$-\bar{\psi}_2 \sigma_{\mu\nu} \psi_1$

Note that the argument also changes for P : $\vec{x} \rightarrow -\vec{x}$, and T : $t \rightarrow -t$, CP : $\vec{x} \rightarrow -\vec{x}$, CPT : $t \rightarrow -t$, $\vec{x} \rightarrow -\vec{x}$.

Recall that for proper homogeneous Lorentz transformation (which does not include P, T and PT), the transformation properties for bilinear combinations are

$$\bar{\psi}_1(x) \psi_2(x) \rightarrow \bar{\psi}_1(x) \psi_2(x)$$

$$\bar{\psi}_1(x) \gamma^\mu \psi_2(x) \rightarrow \lambda^\mu{}_\nu \bar{\psi}_1(x) \gamma^\nu \psi_2(x)$$

$$\bar{\psi}_1 i\gamma_5 \psi_2(x) \rightarrow \underbrace{\text{Det}(\lambda)}_{''} \bar{\psi}_1(x) i\gamma_5 \psi_2(x) = \bar{\psi}_1(x) i\gamma_5 \psi_2(x)$$

$$\bar{\psi}_1(x) \gamma^\mu \gamma_5 \psi_2(x) \rightarrow \lambda^\mu{}_\nu \underbrace{\text{Det}(\lambda)}_{''} \bar{\psi}_1(x) \gamma^\nu \gamma_5 \psi_2(x) = \lambda^\mu{}_\nu \bar{\psi}_1(x) \gamma^\nu \gamma_5 \psi_2(x)$$

$$\bar{\psi}_1(x) \sigma^{\mu\nu} \psi_2(x) \rightarrow \lambda^\mu{}_\alpha \lambda^\nu{}_\beta \underbrace{\text{Det}(\lambda)}_{''} \bar{\psi}_1(x) \sigma^{\alpha\beta} \psi_2(x)$$

Since electromagnetic interaction conserve C, P and T, then from the invariance of interaction Lagrangian,

$L_{int} = -\frac{e}{c\hbar} \gamma^\mu A_\mu$ (where $|e| = \sqrt{4\pi\epsilon}$, and $\epsilon = -1$ when γ describes the electron-positron field, $\epsilon = +1$ when γ describes the proton-antiproton field, $\epsilon = +\frac{2}{3}$ when γ describes the up-quark anti-up-quark field, etc.), we derive

it means $P A^\mu(t, \vec{x}) P^{-1} = A_\mu(t, -\vec{x})$

$$A^\mu(t, \vec{x}) \xrightarrow{P} A_\mu(t, -\vec{x}), \text{ i.e., } \begin{cases} A^0(t, \vec{x}) \xrightarrow{P} A^0(t, -\vec{x}) \\ \vec{A}(t, \vec{x}) \xrightarrow{P} -\vec{A}(t, -\vec{x}) \end{cases}$$

$$A^\mu(t, \vec{x}) \xrightarrow{C} -A^\mu(t, \vec{x})$$

$$A^\mu(t, \vec{x}) \xrightarrow{T} A_\mu(-t, \vec{x}), \text{ i.e., } \begin{cases} A^0(t, \vec{x}) \xrightarrow{T} A^0(-t, \vec{x}) \\ \vec{A}(t, \vec{x}) \xrightarrow{T} -\vec{A}(-t, \vec{x}) \end{cases}$$

From the Maxwell's equations (which is invariance under, C, P and T),

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

we get

$$\vec{E}(t, \vec{x}) \xrightarrow{P} -\vec{E}(t, -\vec{x})$$

$$\vec{B}(t, \vec{x}) \xrightarrow{P} \vec{B}(t, -\vec{x})$$

$$\rho(t, \vec{x}) \xrightarrow{P} \rho(t, -\vec{x})$$

$$\vec{j}(t, \vec{x}) \xrightarrow{P} -\vec{j}(t, -\vec{x})$$

$$\vec{\nabla} \xrightarrow{P} -\vec{\nabla}$$

$$\frac{\partial}{\partial t} \xrightarrow{P} \frac{\partial}{\partial t}$$

$$\rho(t, \vec{x}) \xrightarrow{C} -\rho(t, \vec{x})$$

$$\vec{E}(t, \vec{x}) \xrightarrow{C} -\vec{E}(t, \vec{x})$$

$$\vec{B}(t, \vec{x}) \xrightarrow{C} -\vec{B}(t, \vec{x})$$

$$\vec{j}(t, \vec{x}) \xrightarrow{C} -\vec{j}(t, \vec{x})$$

$$\frac{\partial}{\partial t} \xrightarrow{C} \frac{\partial}{\partial t}, \quad \vec{\nabla} \xrightarrow{C} \vec{\nabla}$$

(checked)

$$\frac{\partial}{\partial t} \xrightarrow{T} -\frac{\partial}{\partial t}$$

$$\vec{\nabla} \xrightarrow{T} \vec{\nabla}$$

$$E(t, \vec{x}) \xrightarrow{T} E(-t, \vec{x})$$

$$\vec{E}(t, \vec{x}) \xrightarrow{T} \vec{E}(-t, \vec{x})$$

$$\vec{B}(t, \vec{x}) \xrightarrow{T} -\vec{B}(-t, \vec{x})$$

$$\vec{j}(t, \vec{x}) \xrightarrow{T} -\vec{j}(-t, \vec{x})$$

On the other hand, since $F^{io} = E^i$, $F^{ij} = -\epsilon^{ijk}B^k$ (where $\epsilon^{123} = \epsilon_{231} = 1$, ϵ^{ijk} is the Levi-Civita symbol) and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\text{then } E^i = \partial^i A^0 - \partial^0 A^i \Rightarrow \vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t}$$

$$-\epsilon^{ijk} B^k = F^{ij} \Rightarrow -\epsilon_{ijl} \epsilon^{ijk} B^k = \epsilon_{ijl} F^{ij}$$

$$\Rightarrow -2B^l = \epsilon_{ijl} F^{ij}$$

$$\Rightarrow B^1 = -\frac{1}{2}(F^{23} - F^{32}) = -F^{23} = -(\partial^2 A^3 - \partial^3 A^2)$$

$$B^2 = -\frac{1}{2}(F^{31} - F^{13}) = -F^{31} = -(\partial^3 A^1 - \partial^1 A^3)$$

$$B^3 = -\frac{1}{2}(F^{12} - F^{21}) = -F^{12} = -(\partial^1 A^2 - \partial^2 A^1)$$

$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

Therefore, we can derive the transformation properties of \vec{E} and \vec{B} directly from the ones for A^μ , i.e.,

$$\vec{E}(t, \vec{x}) \xrightarrow{P} -\vec{E}(t, -\vec{x})$$

$$\vec{E}(t, \vec{x}) \xrightarrow{C} -\vec{E}(t, \vec{x})$$

$$\vec{E}(t, \vec{x}) \xrightarrow{T} E(-t, \vec{x})$$

$$\vec{B}(t, \vec{x}) \xrightarrow{P} \vec{B}(t, -\vec{x})$$

$$\vec{B}(t, \vec{x}) \xrightarrow{C} -\vec{B}(t, \vec{x})$$

$$\vec{B}(t, \vec{x}) \xrightarrow{T} -\vec{B}(-t, \vec{x})$$

② Consistent with the properties derived from Maxwell's equations.

Also, the electric current for a fermion field is $j^\mu = \frac{8ie}{\sqrt{4}\gamma^{\mu} 4}$, then from $j^\mu = (\rho, \vec{j})$ and the transformation properties for $\bar{\gamma}^{\mu} 4$,

we get

$$\begin{aligned} \rho(t, \vec{x}) &\xrightarrow{P} \rho(t, -\vec{x}) \\ \vec{j}(t, \vec{x}) &\xrightarrow{P} -\vec{j}(t, -\vec{x}) \end{aligned}$$

$$\rho(t, \vec{x}) \xrightarrow{C} -\rho(t, \vec{x})$$

$$\vec{j}(t, \vec{x}) \xrightarrow{C} -\vec{j}(t, \vec{x})$$

$$\rho(t, \vec{x}) \xrightarrow{T} \rho(t, \vec{x})$$

$$\vec{j}(t, \vec{x}) \xrightarrow{T} -\vec{j}(-t, \vec{x})$$

(note that $\frac{8ie}{\sqrt{4}}$ is a pure number, and it is not charge under transformations)

Consistent with the properties derived from Maxwell's equations.

We can check that the transformation properties of A^μ makes

$L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ invariant under C, P and T:

note that
P does not
act on ∂^μ here

$$F^{\mu\nu}(t, \vec{x}) = \partial^\mu A^\nu(t, \vec{x}) - \partial^\nu A^\mu(t, \vec{x})$$

$$F^{\mu\nu}(t, \vec{x}) \xrightarrow{P} \underline{\underline{\partial^\mu A_\nu(t, -\vec{x})}} - \underline{\underline{\partial^\nu A_\mu(t, -\vec{x})}} = \tilde{\partial}_\mu A_\nu(t, -\vec{x}) - \tilde{\partial}_\nu A_\mu(t, -\vec{x})$$

$$\Rightarrow -\frac{1}{4} F_{\mu\nu}(t, \vec{x}) F^{\mu\nu}(t, \vec{x}) \xrightarrow{P} -\frac{1}{4} (\tilde{\partial}^\mu A^\nu(t, -\vec{x}) - \tilde{\partial}^\nu A^\mu(t, -\vec{x})) (\tilde{\partial}_\mu A_\nu(t, -\vec{x}) - \tilde{\partial}_\nu A_\mu(t, -\vec{x}))$$

Since the action integrates over all \vec{x} , we can change $\vec{x} \rightarrow -\vec{x}$ and the action will be invariant.

$$F^{\mu\nu}(t, \vec{x}) \xrightarrow{C} -F^{\mu\nu}(t, \vec{x})$$

$$\Rightarrow -\frac{1}{4} F_{\mu\nu}(t, \vec{x}) F^{\mu\nu}(t, \vec{x}) \xrightarrow{C} -\frac{1}{4} F_{\mu\nu}(t, \vec{x}) F^{\mu\nu}(t, \vec{x})$$

$$F^{\mu\nu}(t, \vec{x}) \xrightarrow{T} -\underline{\underline{\partial^\mu A_\nu(t, \vec{x})}} - \underline{\underline{\partial^\nu A_\mu(t, \vec{x})}} = -\tilde{\partial}_\mu A_\nu(t, \vec{x}) + \tilde{\partial}_\nu A_\mu(t, \vec{x})$$

$$\Rightarrow -\frac{1}{4} F_{\mu\nu}(t, \vec{x}) F^{\mu\nu}(t, \vec{x}) \xrightarrow{T} -\frac{1}{4} (\tilde{\partial}^\mu A^\nu(t, \vec{x}) - \tilde{\partial}^\nu A^\mu(t, \vec{x})) (\tilde{\partial}_\mu A_\nu(t, \vec{x}) - \tilde{\partial}_\nu A_\mu(t, \vec{x}))$$

Since the action integrates over all t , we can change $t \rightarrow -t$ and the action will be invariant.

① (checked)

For an arbitrary current obtained by an internal phase transformation of the fermion field, $\psi \rightarrow \psi' = e^{-i\delta} \psi$, where δ is a constant,

$$j^\mu = \bar{\psi}' \gamma^\mu \psi.$$

$$Q = \int d^3x j^0$$

then from the transformation property of $\bar{\psi}' \gamma^\mu \psi$, we have, for charge conjugation,

$$j^\mu_{(t, \vec{x})} \xrightarrow{C} -j^\mu(t, \vec{x})$$

$$Q(t) \xrightarrow{C} -Q(t)$$

Therefore, any charge given by this global phase transformation, e.g., electric charge, baryon number, reverses sign by charge conjugation.

For complex scalar field,

$$\phi(t, \vec{x}) \xrightarrow{P} \gamma_B \phi(t, -\vec{x}), \quad |\gamma_B|^2 = 1$$

$$\phi^+(t, \vec{x}) \xrightarrow{P} \gamma_B^* \phi^+(t, -\vec{x}),$$

$$\phi(t, \vec{x}) \xrightarrow{C} \epsilon_B \phi^+(t, \vec{x}), \quad |\epsilon_B|^2 = 1$$

$$\phi^+(t, \vec{x}) \xrightarrow{C} \epsilon_B^* \phi(t, \vec{x})$$

$$\phi(t, \vec{x}) \xrightarrow{T} \gamma_B \phi(-t, \vec{x}), \quad |\gamma_B|^2 = 1$$

$$\phi^+(t, \vec{x}) \xrightarrow{T} \gamma_B^* \phi^+(-t, \vec{x})$$

Apparently, the free field Lagrangian (more accurately, the action)

$$L(t, \vec{x}) = \partial_\mu \phi(t, \vec{x}) \partial^\mu \phi^+(t, \vec{x}) - m^2 \phi(t, \vec{x}) \phi^+(t, \vec{x})$$

is unchanged under P, C and T.

(2) (checked)

The scalar QED interaction Lagrangian

$$L(t, \vec{x}) = -i\epsilon/e [\phi^+ (\partial^\mu \phi) - (\partial^\mu \phi^+) \phi] A_\mu + (\epsilon/e)^2 A_\mu A^\mu \phi^+ \phi$$

transforms as

$$\begin{aligned} L(t, \vec{x}) &\xrightarrow{P} -i\epsilon/e [\phi^+(t, -\vec{x}) (\tilde{\partial}_\mu \phi(t, -\vec{x})) - (\tilde{\partial}_\mu \phi^+(t, -\vec{x}))] A^\mu(t, -\vec{x}) \\ &\quad + (\epsilon/e)^2 A^\mu(t, -\vec{x}) A_\mu(t, -\vec{x}) \phi^+(t, -\vec{x}) \phi(t, -\vec{x}) \\ &= L(t, -\vec{x}) \end{aligned}$$

$$\begin{aligned} L(t, \vec{x}) &\xrightarrow{C} -i\epsilon/e [\phi(t, \vec{x}) (\partial^\mu \phi^+(t, \vec{x})) - (\partial^\mu \phi(t, \vec{x})) \phi^+(t, \vec{x})] (-A_\mu(t, \vec{x})) \\ &\quad + (\epsilon/e)^2 (-A_\mu(t, \vec{x})) (-A^\mu(t, \vec{x})) \phi(t, \vec{x}) \phi^+(t, \vec{x}) \\ &= L(t, \vec{x}) \end{aligned}$$

$$\begin{aligned} L(t, \vec{x}) &\xrightarrow{T} \underbrace{(-i)^* \epsilon/e [\phi^+(t, \vec{x}) (-\tilde{\partial}_\mu \phi(-t, \vec{x})) - (-\tilde{\partial}_\mu \phi^+(-t, \vec{x})) \phi(-t, \vec{x})]}_{\text{note this}} \\ &\quad A^\mu(-t, \vec{x}) + (\epsilon/e)^2 A^\mu(-t, \vec{x}) A_\mu(-t, \vec{x}) \phi^+(-t, \vec{x}) \\ &\quad \phi(-t, \vec{x}) \end{aligned}$$

it means

$$T L(t, \vec{x}) T^{-1} = L(-t, \vec{x})$$

So the action is unchanged under P, C, and T.

For an arbitrary current obtained by an internal phase transformation of the complex scalar field, $\varphi \rightarrow \varphi' = e^{-i\delta_2} \varphi$, where δ_2 is a constant,

$$j^\mu = i [\phi^+ \partial^\mu \phi - (\partial^\mu \phi^+) \phi]$$

$$\begin{aligned} j^\mu(t, \vec{x}) &\xrightarrow{T} -i [\phi^+(-t, \vec{x}) \partial^\mu \phi(-t, \vec{x}) \\ &\quad - (\partial^\mu \phi^+(-t, \vec{x})) \phi(-t, \vec{x})] \\ \Rightarrow j^\mu(t, \vec{x}) &= i [\phi^+(t, \vec{x}) - \delta_2 \phi(t, \vec{x})] \end{aligned}$$

$$\begin{aligned} j^\mu(t, \vec{x}) &\xrightarrow{C} -j^\mu(t, \vec{x}) \\ &= -i [\phi^+(-t, \vec{x}) \partial^\mu \phi(-t, \vec{x}) \\ &\quad - (\partial^\mu \phi^+(-t, \vec{x})) \phi(-t, \vec{x})] \end{aligned}$$

$$Q = \int d^3 \vec{x} j^0$$

$$\begin{aligned} j^\mu(t, \vec{x}) &\xrightarrow{P} i [\phi^+(t, -\vec{x}) \partial^\mu \phi(t, -\vec{x}) \\ &\quad - (\partial^\mu \phi^+(t, -\vec{x})) \phi(t, -\vec{x})] \\ \Rightarrow j^0(t, \vec{x}) &\xrightarrow{P} j^0(t, -\vec{x}) \\ j(t, \vec{x}) &\xrightarrow{P} -j(t, -\vec{x}) \end{aligned}$$

$$\text{where } -j^0(t, \vec{x}) = -i [\phi^+(t, -\vec{x}) - \partial^\mu \phi^+(t, -\vec{x})] \phi(t, -\vec{x})$$

as expected, and the same interpretation as the fermion field current and charge

③ (checked) under C.

For a real scalar field,

$$\phi(t, \vec{x}) \xrightarrow{P} \gamma_B \phi(t, -\vec{x}), \quad \gamma_B = \pm 1.$$

$$\phi(t, \vec{x}) \xrightarrow{C} \epsilon_B \phi(t, \vec{x}), \quad \epsilon_B = \pm 1$$

$$\phi(t, \vec{x}) \xrightarrow{T} \gamma_B \phi(-t, \vec{x}), \quad \gamma_B = \pm 1$$

The free field Lagrangian (more accurately, the action).

$$L(t, \vec{x}) = \frac{1}{2} \partial_\mu \phi(t, \vec{x}) \partial^\mu \phi(t, \vec{x}) - \frac{1}{2} m^2 \phi^2(t, \vec{x})$$

is unchanged under P , C and T .

Now let's look at the quantized fields (i.e., the particle creation and annihilation operators) behavior under P , C , and T .

To aid analysis, we choose the standard representation of gamma matrices,

$$\gamma^0 = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and in this representation the solutions of Dirac equation are

$$u(\vec{p}, s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi_s \end{pmatrix}, \quad v(\vec{p}, s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \vec{p} \cdot \vec{\sigma} \\ E+m \end{pmatrix} \gamma_s,$$

where $s = \pm \frac{1}{2}$, χ_s and γ_s are 2×1 columns and satisfy $\chi_s^\dagger \chi_{s'} = \delta_{ss'} = \gamma_s^\dagger \gamma_{s'}$. and $\sum_s \chi_s \chi_s^\dagger = \sum_s \gamma_s \gamma_s^\dagger = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$

[1] then for parity transformation, $A^\dagger \gamma^\mu A = \gamma_\mu$, we see that $A = \gamma^0$ satisfies this relation, then

from $\psi(t, \vec{x}) \xrightarrow{P} \gamma \gamma^0 \psi(t, -\vec{x})$, and the decomposition

$$\psi(t, \vec{x}) = \int_{-\infty}^{+\infty} d\vec{p} C(E_p) \sum_s (u(\vec{p}, s) b_{\vec{p}, s} e^{-ip \cdot x} + v(\vec{p}, s) d_{\vec{p}, s}^\dagger e^{ip \cdot x})$$

(# checked)

we have

$$\int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u(\vec{P}, s) b_{\vec{P}, s} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + v(\vec{P}, s) d_{\vec{P}, s}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{\vec{P}} 7 \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [\gamma^0 u(\vec{P}, s) b_{\vec{P}, s} e^{-i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})} + \gamma^0 v(\vec{P}, s) d_{\vec{P}, s}^+ e^{i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})}]$$

$$\text{using } \gamma^0 u(\vec{P}, s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ -\frac{\vec{P} \cdot \vec{P}}{E+m} \chi_s \end{pmatrix} = u(-\vec{P}, s)$$

$$\text{and } \gamma^0 v(\vec{P}, s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{P} \cdot \vec{P}}{E+m} \eta_s \\ -\eta_s \end{pmatrix} = -v(-\vec{P}, s)$$

$$\Rightarrow * = 7 \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u(-\vec{P}, s) b_{\vec{P}, s} e^{-i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})} - v(-\vec{P}, s) d_{\vec{P}, s}^+ e^{i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})}]$$

note that

$$E_{\vec{P}} = E_{-\vec{P}} \xrightarrow{\vec{P}} 7 \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u(\vec{P}, s) b_{-\vec{P}, s} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} - v(\vec{P}, s) d_{-\vec{P}, s}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow b_{\vec{P}, s} \xrightarrow{\vec{P}} \gamma b_{-\vec{P}, s}, \quad d_{\vec{P}, s}^+ \xrightarrow{\vec{P}} \gamma (-d_{-\vec{P}, s}^+)$$

$$\text{Similarly, from } 4^+(t, \vec{x}) = \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u^+(\vec{P}, s) b_{\vec{P}, s}^+ e^{i\vec{P} \cdot \vec{x}} + v^+(\vec{P}, s) d_{\vec{P}, s}^+ e^{-i\vec{P} \cdot \vec{x}}]$$

$$\text{and } 4^+(t, \vec{x}) \xrightarrow{\vec{P}} \gamma^* 4^+(t, -\vec{x}) A^+ = \gamma^* 4^+(t, -\vec{x}) \gamma^*$$

we have

$$\int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u^+(\vec{P}, s) b_{\vec{P}, s}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + v^+(\vec{P}, s) d_{\vec{P}, s}^+ e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{\vec{P}} \gamma^* \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u^+(\vec{P}, s) \gamma^0 b_{\vec{P}, s}^+ e^{i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})} + v^+(\vec{P}, s) \gamma^0 d_{\vec{P}, s}^+ e^{-i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})}]$$

$$\text{using } \gamma^0 u(\vec{P}, s) = u(-\vec{P}, s) \Rightarrow u^+(\vec{P}, s) \gamma^0 = u^+(-\vec{P}, s)$$

$$\gamma^0 v(\vec{P}, s) = -v(-\vec{P}, s) \Rightarrow v^+(\vec{P}, s) \gamma^0 = -v^+(-\vec{P}, s)$$

$$\Rightarrow ** = \gamma^* \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u^+(-\vec{P}, s) b_{\vec{P}, s}^+ e^{i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})} - v^+(-\vec{P}, s) d_{\vec{P}, s}^+ e^{-i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})}]$$

$$= \gamma^* \int_{-\infty}^{+\infty} d\vec{P} C(E_{\vec{P}}) \sum_s [u^+(\vec{P}, s) b_{-\vec{P}, s}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} - v^+(\vec{P}, s) d_{-\vec{P}, s}^+ e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow b_{\vec{P}, s}^+ \xrightarrow{\vec{P}} \gamma^* b_{-\vec{P}, s}^+, \quad d_{\vec{P}, s}^+ \xrightarrow{\vec{P}} \gamma^* (-d_{-\vec{P}, s}^+)$$

Moreover, since

$$\begin{aligned} \psi(t, \vec{x}) &\xrightarrow{P} \eta A \psi(t, -\vec{x}) \xrightarrow{P} \eta^2 A^2 \psi(t, \vec{x}), \\ \psi^+(t, \vec{x}) &\xrightarrow{P} \eta^* \psi^+(t, -\vec{x}) A^+ \xrightarrow{P} \eta^{*2} \psi^+(t, \vec{x}) (A^+)^2, \end{aligned}$$

then $\eta^2 A^2 = 1$ is imposed, so that $P^2 = +1$

Then from $A = 8^\circ$, $\Rightarrow A^2 = 1 \Rightarrow \eta^2 = 1 \Rightarrow \eta = \pm 1$

Therefore,

$$\begin{aligned} b_{\vec{p},s} &\xrightarrow{P} \eta b_{-\vec{p},s} \xrightarrow{P} \eta^2 b_{\vec{p},s} = b_{\vec{p},s} \\ d_{\vec{p},s}^+ &\xrightarrow{P} \eta(-d_{-\vec{p},s}^+) \xrightarrow{P} \eta^2 d_{\vec{p},s}^+ = d_{\vec{p},s}^+ \\ b_{\vec{p},s}^+ &\xrightarrow{P} \eta^* b_{-\vec{p},s}^+ = \eta b_{-\vec{p},s}^+ \xrightarrow{P} \eta^2 b_{\vec{p},s}^+ = b_{\vec{p},s}^+ \\ d_{\vec{p},s}^- &\xrightarrow{P} \eta^*(-d_{-\vec{p},s}^-) = \eta(-d_{-\vec{p},s}^-) = \eta^2 d_{\vec{p},s}^- = d_{\vec{p},s}^- \end{aligned}$$

as they should be ✓ it means $P b_{\vec{p},s} P^{-1} = \eta b_{-\vec{p},s}$

In sum, $b_{\vec{p},s} \xrightarrow{P} \eta b_{-\vec{p},s}$, $b_{\vec{p},s}^+ \xrightarrow{P} \eta b_{-\vec{p},s}^+$
 $d_{\vec{p},s} \xrightarrow{P} -\eta d_{-\vec{p},s}$, $d_{\vec{p},s}^+ \xrightarrow{P} -\eta d_{-\vec{p},s}^+$

where $\eta = \pm 1$.

therefore, $P(d_{\vec{p},s}^+ | 0\rangle) = P d_{\vec{p},s}^+ P^{-1} | 0\rangle = -\eta d_{-\vec{p},s}^+ | 0\rangle$
parity conservation
in parity conserving
reactions

No matter whether $\eta = +1$ or $\eta = -1$, we have the conclusion that the parity of a Dirac fermion must be opposite to that of the corresponding antiparticle. Also note that parity reverse momentum, without change spin.

Note: to determine the parity of particles, it is necessary to first impose the parities of a minimum number of reference particles (choose e.g., $\text{Proton} = \eta_{\text{Proton}} = \eta_{\text{Electron}} = +1$)

For real scalar, the decomposition is

$$\phi(t, \vec{x}) = \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + a_{\vec{p}}^+ e^{i(E_{\vec{p}} t - \vec{p} \cdot \vec{x})}]$$

then $\phi(t, \vec{x}) \xrightarrow{P} \eta_B \phi(t, -\vec{x})$, $\eta_B = \pm 1$.

gives $\int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + a_{\vec{p}}^+ e^{i(E_{\vec{p}} t - \vec{p} \cdot \vec{x})}]$

$$\xrightarrow{P} \eta_B \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}} t + \vec{p} \cdot \vec{x})} + a_{\vec{p}}^+ e^{i(E_{\vec{p}} t + \vec{p} \cdot \vec{x})}]$$

$$= \eta_B \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + a_{\vec{p}}^+ e^{i(E_{\vec{p}} t - \vec{p} \cdot \vec{x})}]$$

(16) (checked)

$$\Rightarrow a_{\vec{P}} \xrightarrow{P} \gamma_B a_{-\vec{P}}, \quad a_{\vec{P}}^+ \xrightarrow{P} \gamma_B a_{-\vec{P}}^+ \quad \left(\begin{array}{l} \text{and} \\ \phi(t, \vec{x}) \xrightarrow{P^2} \phi(t, \vec{x}) \\ a_{\vec{P}} \xrightarrow{P^2} a_{\vec{P}}, \quad a_{\vec{P}}^+ \xrightarrow{P^2} a_{\vec{P}}^+ \\ \Rightarrow P^2 = 1 \end{array} \right)$$

For complex scalar, we have

$$\phi(t, \vec{x}) \xrightarrow{P} \gamma_B \phi(t, -\vec{x}) \xrightarrow{P} \gamma_B^2 \phi(t, \vec{x})$$

$$\phi^+(t, \vec{x}) \xrightarrow{P} \gamma_B^* \phi^+(t, -\vec{x}) \xrightarrow{P} (\gamma_B^*)^2 \phi^+(t, \vec{x})$$

impose. $\gamma_B^2 = 1 \Rightarrow \gamma_B = \pm 1$, so that $P^2 = +1$

then the transformation of the decomposition is

$$\phi(t, \vec{x}) = \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) [a_{\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{P} \gamma_B \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) [a_{\vec{P}} e^{-i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})} + b_{\vec{P}}^+ e^{i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})}]$$

$$= \gamma_B \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) [a_{-\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{-\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow a_{\vec{P}} \xrightarrow{P} \gamma_B a_{-\vec{P}}, \quad b_{\vec{P}}^+ \xrightarrow{P} \gamma_B b_{-\vec{P}}^+ \quad \left(\begin{array}{l} \text{and} \\ a_{\vec{P}} \xrightarrow{P^2} a_{\vec{P}} \\ b_{\vec{P}}^+ \xrightarrow{P^2} b_{\vec{P}}^+ \end{array} \right)$$

$$\phi^+(t, \vec{x}) = \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) [a_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{\vec{P}}^- e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{P} \gamma_B^* \phi^+(t, -\vec{x}) = \gamma_B \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) [a_{\vec{P}}^+ e^{i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})} + b_{\vec{P}}^- e^{-i(E_{\vec{P}} t + \vec{P} \cdot \vec{x})}]$$

$$= \gamma_B \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) [a_{-\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{-\vec{P}}^- e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow a_{\vec{P}}^+ \xrightarrow{P} \gamma_B a_{-\vec{P}}^+, \quad b_{\vec{P}}^- \xrightarrow{P} \gamma_B b_{-\vec{P}}^- \quad \left(\begin{array}{l} \text{and} \\ a_{\vec{P}}^+ \xrightarrow{P^2} a_{\vec{P}}^+ \\ b_{\vec{P}}^- \xrightarrow{P^2} b_{\vec{P}}^- \end{array} \right)$$

Therefore, no matter whether $\gamma_B = +1$ or $\gamma_B = -1$, we have the conclusion that the parity of a complex scalar particle is the same as its antiparticle.

For the photon field, the decomposition in the Coulomb gauge is

$$A^\mu(t, \vec{x}) = (0, \vec{A}(t, \vec{x}))$$

where $\vec{A}(t, \vec{x}) = \int d^3 \vec{K} C(E_{\vec{K}}) \sum_{\lambda} [\vec{e}_{(\vec{K}, \lambda)} a_{\vec{K}, \lambda} e^{-ik \cdot x} + \vec{e}_{*(\vec{K}, \lambda)} a_{\vec{K}, \lambda}^+ e^{ik \cdot x}]$

(checked)

To aid the analysis, let's use circular polarization,

$$\vec{e}(\vec{R}, -) = \frac{1}{\sqrt{2}} (\hat{\vec{g}} - i \hat{\vec{h}}), \quad \vec{e}(\vec{R}, +) = -\frac{1}{\sqrt{2}} (\hat{\vec{g}} + i \hat{\vec{h}}),$$

where \hat{g} , \hat{h} , \hat{k} are unit vector perpendicular to each other, and \hat{g} , \hat{h} and \hat{k} form a right-handed axes system (like \hat{x} , \hat{y} and \hat{z}),

$$\text{Therefore, } \vec{e}^*(\vec{r}, -) = \frac{1}{\sqrt{2}} (\hat{\vec{g}} + i \hat{\vec{h}}) = - \vec{e}(\vec{r}, +)$$

check:

$$\left\{ \begin{array}{l} \vec{e}^*(\vec{k},+) \cdot \vec{e}(\vec{k},+) = 1 \\ \vec{e}^*(\vec{k},-) \cdot \vec{e}(\vec{k},-) = 1 \\ \vec{e}^*(\vec{k},+) \cdot \vec{e}(\vec{k},-) = -\vec{e}(\vec{k},-) \cdot \vec{e}(\vec{k},-) = 0 \\ \vec{e}^*(\vec{k},-) \cdot \vec{e}(\vec{k},+) = -\vec{e}(\vec{k},+) \cdot \vec{e}(\vec{k},+) = 0 \end{array} \right.$$

as required by the orthogonality and completeness requirement for the basis

$$\vec{e}(\vec{k}, +) \underset{\text{cross product}}{\underset{\times}{\underset{\downarrow}{\times}}} \vec{e}(\vec{k}, -) = -\frac{1}{2} (\hat{\vec{g}} + i\hat{\vec{h}}) \times (\hat{\vec{g}} - i\hat{\vec{h}}) \\ = +\frac{1}{2} (i\hat{\vec{K}} + i\hat{\vec{K}}) = +i\hat{\vec{K}}.$$

$$\text{Using } \vec{e}(-\vec{k}, +) = \vec{e}(\vec{k}, -) \quad \vec{e}(-\vec{k}, -) = \vec{e}(\vec{k}, +) \quad \left\{ \Rightarrow \vec{e}(-\vec{k}, -\lambda) = \vec{e}(\vec{k}, \lambda) \right.$$

$$\Rightarrow \vec{e}(-\vec{k},+) \times \vec{e}(-\vec{k},-) = \vec{e}(\vec{k},-) \times \vec{e}(\vec{k},+) = i(-\vec{k})$$

as they should be.

(that is, if choose \hat{g} , \hat{h} and \hat{k} as x, y and z direction, respectively, then for the light propagates in the +z direction, if the electric field \vec{E} is $\vec{E} = (E_0 e^{i(kz - \omega t)}, E_0 e^{i(kz - \omega t + \frac{\pi}{2})}, 0)$, then $\text{Re}(\vec{E}) = (E_0 \cos(kz - \omega t), -E_0 \sin(kz - \omega t), 0)$)

if it is called right-handed polarized, since at a fixed θ , for example $\theta=0$, we have
 $\text{Re}(\vec{E}) = (E_0 \cos \omega t, E_0 \sin \omega t, 0)$

$$\Rightarrow t=0 \Rightarrow \text{Re}(\vec{E}) = (\underbrace{E_0, 0, 0}_{\text{i.e., } Fx \text{ direction}}), \quad t = \frac{\pi}{2\omega} \Rightarrow \text{Re}(\vec{E}) = (\underbrace{0, +E_0, 0}_{\text{i.e., } Fy \text{ direction}})$$

On the other hand, $\vec{E} = (E_0 e^{i(\omega t - \frac{\pi}{2})}, -iE_0 e^{i(\omega t - \frac{\pi}{2})}, 0)$ is left-handed polarized.

(18) (checked)

While for the light propagates in the $-z$ direction, if
 $\vec{E} = (E_0 e^{i(-kz-wt)}, \underbrace{E_0 e^{i(-kz-wt+\frac{\pi}{2})}}_{i E_0 e^{i(-kz-wt)}}, 0)$, then $\text{Re}(\vec{E}) = (E_0 \cos(-kz-wt), -E_0 \sin(-kz-wt), 0)$,

then for $z=0$, we have $\text{Re}(\vec{E}) = (E_0 \cos(\omega t), -E_0 \sin(\omega t), 0)$, therefore

$$t=0 \Rightarrow \text{Re}(\vec{E}) = (E_0, 0, 0); t = -\frac{\pi}{2\omega} \Rightarrow \text{Re}(\vec{E}) = (0, -E_0, 0);$$

$$t = \frac{\pi}{\omega} \Rightarrow \text{Re}(\vec{E}) = (-E_0, 0, 0); t = \frac{3\pi}{2\omega} \Rightarrow \text{Re}(\vec{E}) = (0, E_0, 0);$$

so it is left-handed polarized;

on the other hand, $\vec{E} = (E_0 e^{i(-kz-wt)}, \underbrace{E_0 e^{i(-kz-wt-\frac{\pi}{2})}}_{-i E_0 e^{i(-kz-wt)}}, 0)$, it is right-handed polarized.

$$\begin{aligned} \text{So, } \vec{A}(t, \vec{x}) &\xrightarrow{P} -\vec{A}(t, -\vec{x}) = - \int d^3\vec{k} C(E_k) \sum_{\lambda} [\vec{e}(\vec{k}, \lambda) a_{\vec{k}, \lambda} e^{-i(E_k t + \vec{k} \cdot \vec{x})} \\ &\quad + \vec{e}^*(\vec{k}, \lambda) a_{\vec{k}, \lambda}^+ e^{i(E_k t + \vec{k} \cdot \vec{x})}] \\ &= - \int d^3\vec{k} C(E_k) \sum_{\lambda} [\vec{e}(-\vec{k}, \lambda) a_{-\vec{k}, \lambda} e^{-i(E_k t - \vec{k} \cdot \vec{x})} \\ &\quad + \vec{e}^*(-\vec{k}, \lambda) a_{-\vec{k}, \lambda}^+ e^{i(E_k t - \vec{k} \cdot \vec{x})}] \\ &= - \int d^3\vec{k} C(E_k) \sum_{\lambda} [\vec{e}(-\vec{k}, -\lambda) a_{\vec{k}, -\lambda} e^{-i k \cdot x} + \vec{e}^*(-\vec{k}, -\lambda) a_{\vec{k}, -\lambda}^+ e^{i k \cdot x}] \\ &\Rightarrow - \int d^3\vec{k} C(E_k) \sum_{\lambda} [\vec{e}(\vec{k}, \lambda) a_{\vec{k}, \lambda}^+ e^{-i k \cdot x} + \vec{e}^*(\vec{k}, \lambda) a_{\vec{k}, \lambda}^+ e^{i k \cdot x}] \\ \Rightarrow a_{\vec{k}, \lambda} &\xrightarrow{P} -a_{-\vec{k}, -\lambda}, \quad a_{\vec{k}, \lambda}^+ \xrightarrow{P} -a_{-\vec{k}, -\lambda}^+ \end{aligned}$$

Therefore, the photon has a negative parity; both the momentum and helicity of the photon change signs under the parity operation.

$$\begin{aligned} \text{Again, } \vec{A}(t, \vec{x}) &\xrightarrow{P} -\vec{A}(t, -\vec{x}) \xrightarrow{P} \vec{A}(t, \vec{x}) \\ a_{\vec{k}, \lambda} &\xrightarrow{P^2} a_{\vec{k}, \lambda}, \quad a_{\vec{k}, \lambda}^+ \xrightarrow{P^2} a_{\vec{k}, \lambda}^+. \\ P^2 &= 1 \end{aligned}$$

[2] For charge conjugation,

for Dirac fermion field, in the standard representation,
 $B = i\gamma^2\gamma^0$ satisfies.

$$B^\dagger \gamma^\mu B = -\gamma^\mu$$

$$\left(\text{check: } (\gamma^2\gamma^0)^\dagger \gamma^\mu (\gamma^2\gamma^0) = -\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 = \begin{cases} -\gamma^0 = -\gamma^0, & \mu=0 \\ \gamma^1 = \gamma_1^* = -\gamma_1^*, & \mu=1 \\ -\gamma^2 = -\gamma_2^*, & \mu=2 \\ \gamma^3 = \gamma_3^* = -\gamma_3^*, & \mu=3 \end{cases} \right)$$

from $\mathcal{H}(x) \xrightarrow{C} \epsilon B \bar{\psi}_{(x)}^T$ ← note that this formula means that

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{P} C(E_{\vec{P}}) \bar{s} [u(\vec{P}, s) b_{\vec{P}, s} e^{-ip \cdot x} + v(\vec{P}, s) d_{\vec{P}, s}^+ e^{ip \cdot x}] \quad \text{act on } b \text{ and } d^+$$

$$\xrightarrow{C} \epsilon \int_{-\infty}^{+\infty} d^3\vec{P} C(E_{\vec{P}}) \bar{s} [i\gamma^2\gamma^0 \bar{u}(\vec{P}, s) b_{\vec{P}, s}^+ e^{ip \cdot x} + i\gamma^2\gamma^0 \bar{v}(\vec{P}, s) d_{\vec{P}, s} e^{-ip \cdot x}]$$

where

$$\begin{aligned} i\gamma^2\gamma^0 \bar{u}^T(\vec{P}, s) &= i\gamma^2\gamma^0 (u^+ \gamma^0)^T = i\gamma^2\gamma^0 \gamma^0 u^* = i\gamma^2\gamma^0 u^* = i\gamma^2 u^* \\ &= i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} (E+m)^{\frac{1}{2}} \left(\frac{\chi_s^*}{\frac{\vec{P} \cdot \vec{P}}{E+m} \chi_s^*} \chi_s^* \right) \\ &= -(E+m)^{\frac{1}{2}} \left(\frac{\frac{\vec{P} \cdot \vec{P}}{E+m} (-i\sigma^2) \chi_s^*}{-i\sigma^2 \chi_s^*} \right) \end{aligned}$$

$$\text{using } \chi_s^+ \chi_{s'}^- = \delta_{ss'}, \text{ we have } [(-i\sigma^2) \chi_s^*]^+ (-i\sigma^2 \chi_{s'}^*)$$

$$= \chi_s^T (i\sigma^2) (-i\sigma^2) \chi_{s'}^*$$

$$\begin{aligned} \text{also, } i\gamma^2\gamma^0 \bar{v}^T(\vec{P}, s) &= i\gamma^2 v^* = i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} (E+m)^{\frac{1}{2}} \left(\frac{\eta_s^*}{\frac{\vec{P} \cdot \vec{P}}{E+m} \eta_s^*} \eta_s^* \right) \\ &= (E+m)^{\frac{1}{2}} \left(\frac{i\sigma^2 \eta_s^*}{\frac{\vec{P} \cdot \vec{P}}{E+m} i\sigma^2 \eta_s^*} \right) \end{aligned}$$

$$\text{we can choose } -i\sigma^2 \chi_s^* = \eta_s, \text{ then } \chi_s^* = i\sigma^2 \eta_s \Rightarrow \chi_s = i\sigma^2 \eta_s^*$$

(checked)

$$\Rightarrow i\gamma^2 \gamma^0 \bar{U}^T(\vec{p}, s) = V(\vec{p}, s)$$

$$i\gamma^2 \gamma^0 \bar{V}^T(\vec{p}, s) = U(\vec{p}, s)$$

$$\Rightarrow b_{\vec{p}, s} \xrightarrow{C} \sum d_{\vec{p}, s}$$

$$d_{\vec{p}, s}^+ \xrightarrow{C} \sum b_{\vec{p}, s}^+$$

Also, from $\gamma^+(x) \xrightarrow{C} \sum^* \gamma^+(x) \gamma^0 B^+$

$$\Rightarrow \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s (U^+(\vec{p}, s) b_{\vec{p}, s} e^{ip \cdot x} + V^+(\vec{p}, s) d_{\vec{p}, s} e^{-ip \cdot x})$$

$$\xrightarrow{C} \sum^* \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s (U^+(\vec{p}, s) \gamma^0 (-i)(\gamma^2 \gamma^0)^+ b_{\vec{p}, s} e^{-ip \cdot x} + V^+(\vec{p}, s) \gamma^0 (-i)(\gamma^2 \gamma^0)^+ d_{\vec{p}, s} e^{ip \cdot x})$$

using $U^+(\vec{p}, s) \gamma^0 (-i)(\gamma^2 \gamma^0)^+$

$$= U^+(\vec{p}, s) \gamma^0 (+i) \gamma^0 \gamma^2$$

$$= U^+(\vec{p}, s) i \gamma^2$$

$$= (i \gamma^2 U^*(\vec{p}, s))^+ = V^+(\vec{p}, s)$$

and $V^+(\vec{p}, s) \gamma^0 (-i)(\gamma^2 \gamma^0)^+$

$$= V^+(\vec{p}, s) \gamma^0 (+i) \gamma^0 \gamma^2$$

$$= V^+(\vec{p}, s) i \gamma^2$$

$$= (i \gamma^2 V^*(\vec{p}, s))^+$$

$$= U^+(\vec{p}, s)$$

$$\Rightarrow b_{\vec{p}, s} \xrightarrow{C^2} b_{\vec{p}, s}$$

$$d_{\vec{p}, s}^+ \xrightarrow{C^2} d_{\vec{p}, s}^+$$

$$b_{\vec{p}, s}^+ \xrightarrow{C^2} b_{\vec{p}, s}^+$$

$$d_{\vec{p}, s} \xrightarrow{C^2} d_{\vec{p}, s}^+$$

$$\Rightarrow b_{\vec{p}, s}^+ \xrightarrow{C} \sum^* d_{\vec{p}, s}^+, \quad d_{\vec{p}, s} \xrightarrow{C} \sum^* b_{\vec{p}, s}$$

this means $C(b_{\vec{p}, s}^+ | 0\rangle) = C b_{\vec{p}, s}^+ C^{-1} C | 0\rangle = \sum^* d_{\vec{p}, s}^+ | 0\rangle$, (note that $C| 0\rangle = | 0\rangle$)

Therefore, we conclude that charge conjugation change a Dirac fermion to its antifermion, without changing spin and momentum.

note: $\gamma^+(x) \xrightarrow{C} \sum^* \gamma^+(x) \gamma^0 B^+ \xrightarrow{C} \sum^* (\epsilon B \bar{\gamma}^T(x))^+ \gamma^0 B^+ = |\epsilon|^2 \bar{\gamma}(x) B^+ \gamma^0 B^+ \equiv ***$,

where $B^+ \gamma^0 B^+ = (i \gamma^2 \gamma^0)^+ \gamma^0 (i \gamma^2 \gamma^0)^+ = \gamma^0 \gamma^2 \gamma^0 \gamma^0 (-\gamma^2) = -\gamma^0 \gamma^2 \gamma^0 \gamma^2 = \gamma^0$

$\Rightarrow *** = \gamma^+(x)$

(checked), $C^2 = 1$.

For real scalar field, we can immediately see that (from $\phi(x) \xrightarrow{C} \epsilon_B \phi(x)$),
 $a_{\vec{p}} \xrightarrow{C} \epsilon_B a_{\vec{p}}$, $a_{\vec{p}}^+ \xrightarrow{C} \epsilon_B a_{\vec{p}}^+$, where $\epsilon_B = \pm 1$.

For a complex scalar field, from $\phi(x) \xrightarrow{C} \epsilon_B \phi^+(x)$ and
 $\phi^+(x) \rightarrow \epsilon_B^* \phi(x)$, we have

$$\phi(x) = \int_{-\infty}^{+\infty} d^3\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-ip \cdot x} + b_{\vec{p}}^+ e^{ip \cdot x}]$$

$$\xrightarrow{C} \epsilon_B \int_{-\infty}^{+\infty} d^3\vec{p} C(E_{\vec{p}}) [a_{\vec{p}}^+ e^{ip \cdot x} + b_{\vec{p}} e^{-ip \cdot x}]$$

$$\Rightarrow a_{\vec{p}} \xrightarrow{C} \epsilon_B b_{\vec{p}} \\ b_{\vec{p}}^+ \xrightarrow{C} \epsilon_B a_{\vec{p}}^+$$

$$\phi^+(x) = \int_{-\infty}^{+\infty} d^3\vec{p} C(E_{\vec{p}}) [a_{\vec{p}}^+ e^{ip \cdot x} + b_{\vec{p}} e^{-ip \cdot x}]$$

$$\xrightarrow{C} \epsilon_B^* \int_{-\infty}^{+\infty} d^3\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-ip \cdot x} + b_{\vec{p}}^+ e^{ip \cdot x}]$$

$$\Rightarrow a_{\vec{p}}^+ \xrightarrow{C} \epsilon_B^* b_{\vec{p}}^+ \\ b_{\vec{p}} \xrightarrow{C} \epsilon_B a_{\vec{p}}$$

So, for a complex scalar, charge conjugation transforms to its antiparticle, without changing momentum.

$$\begin{aligned} \phi(x) &\xrightarrow{C} \phi(x), \quad \phi^+(x) \xrightarrow{C^2} \phi^+(x) \\ a_{\vec{p}} &\xrightarrow{C^2} a_{\vec{p}}, \quad a_{\vec{p}}^+ \xrightarrow{C^2} a_{\vec{p}}^+, \quad b_{\vec{p}} \xrightarrow{C^2} b_{\vec{p}}, \quad b_{\vec{p}}^+ \xrightarrow{C^2} b_{\vec{p}}^+, \quad C^2 = 1 \end{aligned}$$

For the photon field, we can immediately see that (from $A''(x) \xrightarrow{C} -A''(x)$),

$$a_{k,\lambda} \xrightarrow{C} -a_{k,\lambda}, \quad a_{k,\lambda}^+ \xrightarrow{C} -a_{k,\lambda}^+$$

$$\begin{aligned} A''(x) &\xrightarrow{C^2} A''(x) \\ a_{k,\lambda} &\xrightarrow{C^2} a_{k,\lambda} \\ a_{k,\lambda}^+ &\xrightarrow{C^2} a_{k,\lambda}^+, \quad C^2 = 1 \end{aligned}$$

So, the photon is odd under charge conjugation (i.e., it has
charge conjugation number -1).

For a state of scalar-antiscalar particles with orbital angular momentum ℓ :

$|\phi_c(s\bar{s})\rangle = \int d^3\vec{P} F_c(\vec{P}) a_{\vec{P}}^+ b_{-\vec{P}}^+ |0\rangle$, where $F_c(\vec{P})$ is an eigenstate of orbital angular momentum ℓ .
Apply charge conjugation,

$$\begin{aligned} C|\phi_c(s\bar{s})\rangle &= \underbrace{\int d^3\vec{P} F_c(\vec{P}) C a_{\vec{P}}^+ C^\dagger C b_{-\vec{P}}^+ C^\dagger C |0\rangle}_{\text{exchange } b^+ \text{ and } a^+} \\ &\stackrel{?}{=} \int d^3\vec{P} F_c(\vec{P}) a_{\vec{P}}^+ b_{-\vec{P}}^+ |0\rangle \\ &= \int d^3\vec{P} F_c(-\vec{P}) a_{\vec{P}}^+ b_{-\vec{P}}^+ |0\rangle \\ &= (-1)^\ell |\phi_c(s\bar{s})\rangle \end{aligned}$$

So, this state has charge conjugation number $(-1)^\ell$.

An example of such system is $\pi^+\pi^-$.

Another example is $\pi^0\pi^0$, where $b_{\vec{P}}^+$ in the above should be substitute by $a_{\vec{P}}^+$, then $|\phi_c(\pi^0\pi^0)\rangle = \int d^3\vec{P} F_c(\vec{P}) a_{\vec{P}}^+ a_{-\vec{P}}^+ |0\rangle \Rightarrow C|\phi_c(\pi^0\pi^0)\rangle = \int d^3\vec{P} F_c(\vec{P}) a_{\vec{P}}^+ a_{-\vec{P}}^+ |0\rangle = |\phi_c(\pi^0\pi^0)\rangle$

$$\begin{aligned} \text{But } C a_{\vec{P}}^+ a_{-\vec{P}}^+ |0\rangle &= \int d^3\vec{P} F_c(-\vec{P}) a_{\vec{P}}^+ a_{-\vec{P}}^+ |0\rangle \\ &= \int d^3\vec{P} F_c(-\vec{P}) a_{\vec{P}}^+ a_{-\vec{P}}^+ |0\rangle \\ &= (-1)^\ell \int d^3\vec{P} F_c(\vec{P}) a_{\vec{P}}^+ a_{-\vec{P}}^+ |0\rangle \\ &= (-1)^\ell |\phi_c(\pi^0\pi^0)\rangle \end{aligned}$$

$\Rightarrow \ell$ is even for $\pi^0\pi^0$ system (consistent with the requirement of wavefunction when

For a state of fermion-antifermion particles, and neglect the effects of spin-orbit coupling, we have the state being

$$|\psi_{l,s}(f\bar{f})\rangle = \int d^3\vec{P} \sum_{S_1 S_2}^{S_1 S_2} F_{l,s}^{S_1 S_2}(\vec{P}) b_{\vec{P},s_1}^+ d_{-\vec{P},s_2}^+ |0\rangle, \quad \text{exchange two identical bosons}$$

$$\text{then } C|\psi_{l,s}(f\bar{f})\rangle = \int d^3\vec{P} \sum_{S_1 S_2}^{S_1 S_2} F_{l,s}^{S_1 S_2}(\vec{P}) d_{\vec{P},s_1}^+ b_{-\vec{P},s_2}^+ |0\rangle$$

$$\begin{aligned} &\stackrel{?}{=} - \int d^3\vec{P} \sum_{S_1 S_2}^{S_1 S_2} F_{l,s}^{S_1 S_2}(\vec{P}) b_{-\vec{P},s_2}^+ d_{\vec{P},s_1}^+ |0\rangle \\ &\stackrel{?}{=} - \int d^3\vec{P} \sum_{S_1 S_2}^{S_2 S_1} F_{l,s}^{S_2 S_1}(-\vec{P}) b_{\vec{P},s_1}^+ d_{-\vec{P},s_2}^+ |0\rangle \end{aligned}$$

using

$$F_{l,s}^{S_2 S_1}(-\vec{p}) = (-1)^{l+s+1} F_{l,s}^{S_1 S_2}(\vec{p})$$

(note: $S=1$ is spin triplet, which is symmetric under the exchange of the two spins; $S=0$ is spin singlet, which is antisymmetric under the exchange of the two spins).

$F_{l,s}^{S_1 S_2}(\vec{p})$ is the product of orbital wave function and total spin wave function).

$$\Rightarrow C | \psi_{l,s}(f\bar{f}) \rangle = (-1)^{l+s} |\psi_{l,s}(f\bar{f})\rangle$$

Therefore, the charge conjugation number for fermion-antifermion system is $(-1)^{l+s}$.

So, for a pseudoscalar meson with $l=s=0$, its charge conjugation number is $(-1)^{0+0} = +1$; π^0 is an example; for a vector meson with $l=0$ and $s=1$, its charge conjugation number is $(-1)^{0+1} = -1$, ρ^0 is an example.

[3] For time inversion:

for Dirac fermion field, in the standard representation,

$$D = \gamma^1 \gamma^3 \cdot \text{satisfies}$$

$$D^\dagger \gamma^\mu D = \gamma_\mu.$$

(check: $(\gamma^1 \gamma^3)^+ \gamma_0^* (\gamma^1 \gamma^3) = \gamma_3^+ \gamma^1 \gamma^0 \gamma^1 \gamma^3 = \gamma^3 \gamma^1 \gamma^0 \gamma^1 \gamma^3 = \gamma_0$ ✓
 $(\gamma^1 \gamma^3)^+ \gamma_1^* (\gamma^1 \gamma^3) = \gamma_3^+ \gamma^1 \gamma^1 \gamma^1 \gamma^3 = -\gamma^1 = \gamma_1$ ✓
 $(\gamma^1 \gamma^3)^+ \gamma_2^* (\gamma^1 \gamma^3) = -\gamma_3^+ \gamma^1 \gamma^2 \gamma^1 \gamma^3 = -\gamma^2 = \gamma_2$ ✓
 $(\gamma^1 \gamma^3)^+ \gamma_3^* (\gamma^1 \gamma^3) = \gamma_3^+ \gamma^1 \gamma^3 \gamma^1 \gamma^3 = -\gamma^3 = \gamma_3$ ✓).

See modify
pages.

from $\psi(x) \xrightarrow{T} \oint D \psi(\vec{x})$ ← Note that this formula means that $T \psi(x) T^{-1} = \oint D \psi(\vec{x})$, where T act on the Hilbert space operator b and d^+ , and change all the C-number to complex conjugate

$$\Rightarrow \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) \sum_s [u(\vec{P}, s) b_{\vec{P}, s} e^{-ip \cdot x} + v(\vec{P}, s) d_{\vec{P}, s}^+ e^{ip \cdot x}]$$

$$\xrightarrow{\frac{T}{\not{P}}} \oint \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) \sum_s [\gamma^1 \gamma^3 u^*(\vec{P}, s) b_{\vec{P}, s} e^{+i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + \gamma^1 \gamma^3 v^*(\vec{P}, s) d_{\vec{P}, s}^+ e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$= \oint \int_{-\infty}^{+\infty} d^3 \vec{P} C(E_{\vec{P}}) \sum_s [\gamma^1 \gamma^3 u^*(\vec{P}, s) b_{\vec{P}, s} e^{-ip \cdot x} + \gamma^1 \gamma^3 v^*(\vec{P}, s) d_{\vec{P}, s}^+ e^{ip \cdot x}].$$

Note that complex conjugate is applied on u, v and exponential in ψ .

Using

$$\gamma^1 \gamma^3 = \begin{pmatrix} 0 & \Gamma' \\ -\Gamma' & 0 \end{pmatrix} \begin{pmatrix} 0 & \Gamma^3 \\ -\Gamma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1 \sigma^3 & 0 \\ 0 & -\sigma^1 \sigma^3 \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

$$\Rightarrow \gamma^1 \gamma^3 u^*(\vec{P}, s) = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s^* \\ \frac{\vec{\sigma}^* \cdot \vec{P}}{E+m} \chi_s^* \end{pmatrix}$$

$$= (E+m)^{\frac{1}{2}} \cdot \begin{pmatrix} i\sigma^2 \chi_s^* \\ -\frac{\vec{\sigma} \cdot \vec{P}}{E+m} i\sigma^2 \chi_s^* \end{pmatrix}$$

② (checked)

We can take $\chi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\text{Then } i\Gamma^2 \chi_0^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \chi_1.$$

$$i\Gamma^2 \chi_1^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_0.$$

$$\text{Apparently, } \chi_0 \chi_1^* = 1, \quad \chi_1 \chi_0^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

we can choose $i\Gamma^2 \chi_s^* = -(-1)^{\frac{1}{2}-s} \chi_{-s}$

$$\Rightarrow \gamma^1 \gamma^3 u(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} u(-\vec{p}, -s)$$

$$\text{Also, } \gamma^1 \gamma^3 v(\vec{p}, s) = \begin{pmatrix} i\Gamma^2 & 0 \\ 0 & i\Gamma^2 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \vec{p} \cdot \vec{\gamma} \eta_s^* \\ \eta_s^* \end{pmatrix}$$

$$= (E+m)^{\frac{1}{2}} \begin{pmatrix} -\frac{\vec{p} \cdot \vec{\gamma}}{E+m} i\Gamma^2 \eta_s^* \\ i\Gamma^2 \eta_s^* \end{pmatrix}$$

we can take $\eta_0^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta_1^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\text{we can choose } i\Gamma^2 \eta_s^* = -(-1)^{\frac{1}{2}-s} \eta_{-s}$$

$$\Rightarrow \gamma^1 \gamma^3 v^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} v(-\vec{p}, -s)$$

$$\Rightarrow ** = \oint \int_{-\infty}^{+\infty} d^3 \vec{p} (E_{\vec{p}}) \sum_s \left[\overline{u}(\vec{p}, -s) b_{-\vec{p}, s} e^{-i\vec{p} \cdot \vec{x}} + \overline{v(\vec{p}, -s)} d_{-\vec{p}, s}^+ e^{i\vec{p} \cdot \vec{x}} \right]$$

$$= \oint \int_{-\infty}^{+\infty} d^3 \vec{p} (E_{\vec{p}}) \sum_s \left[\overline{u(\vec{p}, s)} b_{-\vec{p}, -s} e^{-i\vec{p} \cdot \vec{x}} + \overline{v(\vec{p}, s)} d_{-\vec{p}, -s}^+ e^{i\vec{p} \cdot \vec{x}} \right]$$

(checked)

$$\Rightarrow \text{using } -(-1)^{\frac{1}{2}+s} = (-1)^{\frac{1}{2}-s} \quad \swarrow \text{ same as modify pages}$$

$$\Rightarrow b_{\vec{p},s} \xrightarrow{T} \mathcal{S}(-1)^{\frac{1}{2}-s} b_{-\vec{p},-s} \Rightarrow b_{\vec{p},s} \xrightarrow{T^2} (-1)^{\frac{1}{2}-s} (-1)^{\frac{1}{2}+s} b_{\vec{p},s} = -b_{\vec{p},s}$$

$$d_{\vec{p},s}^+ \xrightarrow{T} \mathcal{S}(-1)^{\frac{1}{2}-s} d_{-\vec{p},-s}^+ \Rightarrow d_{\vec{p},s}^+ \xrightarrow{T^2} -d_{\vec{p},s}^+$$

from $\psi^+(x) \xrightarrow{T} \mathcal{S}^* \psi^+(\tilde{x}) D^+ \quad \leftarrow \text{see modify pages}$

$$\Rightarrow \psi^+(t, \vec{x}) = \int_{-\infty}^{+\infty} d^3 \vec{p} C(E_{\vec{p}}) \sum_s (u^+(\vec{p}, s) b_{\vec{p},s}^+ e^{ip \cdot x} + v^+(\vec{p}, s) d_{\vec{p},s}^+ e^{-ip \cdot x})$$

$$\xrightarrow{T} \mathcal{S}^* \int_{-\infty}^{+\infty} d^3 \vec{p} C(E_{\vec{p}}) \sum_s (u^T(\vec{p}, s) b_{\vec{p},s}^+ e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + v^T(\vec{p}, s) d_{\vec{p},s}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})})$$

using $\gamma' \gamma^3 u^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} u(-\vec{p}, -s)$

$$\Rightarrow [\gamma' \gamma^3 u^*(\vec{p}, s)]^+ = u^T(\vec{p}, s) (\gamma' \gamma^3)^+ = -(-1)^{\frac{1}{2}-s} u^+(-\vec{p}, -s)$$

using $\gamma' \gamma^3 v^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} v(-\vec{p}, -s)$

$$\Rightarrow [\gamma' \gamma^3 v^*(\vec{p}, s)]^+ = v^T(\vec{p}, s) (\gamma' \gamma^3)^+ = -(-1)^{\frac{1}{2}-s} v^+(-\vec{p}, -s)$$

$$\Rightarrow *** = \mathcal{S}^* \int_{-\infty}^{+\infty} d^3 \vec{p} C(E_{\vec{p}}) \sum_s (-1)^{\frac{1}{2}+s} [u^+(-\vec{p}, -s) b_{-\vec{p},-s}^+ e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + v^+(-\vec{p}, -s) d_{-\vec{p},-s}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})}]$$

$$= \mathcal{S}^* \int_{-\infty}^{+\infty} d^3 \vec{p} C(E_{\vec{p}}) \sum_s (-1)^{\frac{1}{2}-s} [u^+(\vec{p}, s) b_{\vec{p},s}^+ e^{ip \cdot x} + v^+(\vec{p}, s) d_{\vec{p},s}^+ e^{-ip \cdot x}]$$

\Rightarrow $b_{\vec{p},s}^+ \xrightarrow{T} \mathcal{S}^* (-1)^{\frac{1}{2}-s} b_{-\vec{p},-s}^+ \Rightarrow b_{\vec{p},s}^+ \xrightarrow{T^2} -b_{\vec{p},s}^+$
 $d_{\vec{p},s}^+ \xrightarrow{T} \mathcal{S}^* (-1)^{\frac{1}{2}-s} d_{-\vec{p},-s}^+ \Rightarrow d_{\vec{p},s}^+ \xrightarrow{T^2} -d_{\vec{p},s}^+$

In sum: $\psi(t, \vec{x}) \xrightarrow{T} \mathcal{S} D \psi(-t, \vec{x}) \xrightarrow{T} \mathcal{S}^* D^* \mathcal{S} D \psi(t, \vec{x}) = D^* D \psi(t, \vec{x})$

from $D = \gamma' \gamma^3 \Rightarrow D^* = \gamma' \gamma^3 \Rightarrow D^* D = \gamma' \gamma^3 \gamma' \gamma^3 = -1$

$$\Rightarrow \psi(t, \vec{x}) \xrightarrow{T^2} -\psi(t, \vec{x})$$

also, $\psi^+(t, \vec{x}) \xrightarrow{T} \mathcal{S}^* \psi^+(t, \vec{x}) D^+ \xrightarrow{T} \mathcal{S}^* \mathcal{S}^* \psi^+(t, \vec{x}) D^+ D^T$, then from $(D^+ D^T)^+ = D^* D = -1$
 $\Rightarrow \psi^+(t, \vec{x}) \xrightarrow{T^2} -\psi^+(t, \vec{x})$.

② that is, $T^2 = -1$ for Dirac fermion field.

For complex scalar field,

$$\text{from } \phi(t, \vec{x}) \xrightarrow{T} \oint_B \phi(-t, \vec{x})$$

↙ See modify pages

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{T} \oint_B \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}} e^{i(-E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{\vec{P}}^+ e^{-i(-E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$= \oint_B \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{-\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow a_{\vec{P}} \xrightarrow{T} \oint_B a_{-\vec{P}}, \quad b_{\vec{P}}^+ \xrightarrow{T} \oint_B b_{-\vec{P}}^+ \quad \text{↙ Same as modify pages.}$$

$$\text{from } \phi^+(t, \vec{x}) \xrightarrow{T} \oint_B^* \phi^+(-t, \vec{x}) \quad \text{↙ See modify pages}$$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{T} \oint_B^* \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}}^+ e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{\vec{P}} e^{i(-E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$= \oint_B^* \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + b_{-\vec{P}}^+ e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow a_{\vec{P}}^+ \xrightarrow{T} \oint_B^* a_{\vec{P}}^+, \quad b_{\vec{P}} \xrightarrow{T} \oint_B^* b_{-\vec{P}}^+$$

$$\text{where } |\oint_B| = 1.$$

For real scalar field,

↙ See modify pages.

$$\text{from } \phi(t, \vec{x}) \xrightarrow{T} \oint_B \phi(-t, \vec{x})$$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + a_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\xrightarrow{T} \oint_B \int_{-\infty}^{+\infty} d^3\vec{P}(E_{\vec{P}}) [a_{\vec{P}} e^{i(-E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + a_{\vec{P}}^+ e^{-i(-E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$= \oint_B \int_{-\infty}^{+\infty} d^2\vec{P}(E_{\vec{P}}) [a_{\vec{P}} e^{-i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})} + a_{\vec{P}}^+ e^{i(E_{\vec{P}} t - \vec{P} \cdot \vec{x})}]$$

$$\Rightarrow a_{\vec{P}} \xrightarrow{T} \oint_B a_{-\vec{P}}, \quad a_{\vec{P}}^+ \xrightarrow{T} \oint_B a_{-\vec{P}}^+ \quad \text{↙ Same as modify pages.}$$

(checked)

where $\oint_B = \pm 1$

For photon field, \leftarrow see modify pages.

$$\text{from } \vec{A}(t, \vec{x}) \xrightarrow{T} -\vec{A}(-t, \vec{x}),$$

we have, in the Coulomb gauge,

$$\int_{-\infty}^{+\infty} d^3\vec{R} C(E_{\vec{R}}) \sum_{\lambda} [\vec{e}(\vec{R}, \lambda) a_{\vec{R}, \lambda} e^{-ik \cdot x} + \vec{e}^*(\vec{R}, \lambda) a_{\vec{R}, \lambda}^+ e^{ik \cdot x}]$$

$$\xrightarrow{T} - \int_{-\infty}^{+\infty} d^3\vec{R} C(E_{\vec{R}}) \sum_{\lambda} [\vec{e}^*(\vec{R}, \lambda) a_{\vec{R}, \lambda} e^{i(-E_{\vec{R}} t - \vec{R} \cdot \vec{x})} + \vec{e}(\vec{R}, \lambda) a_{\vec{R}, \lambda}^+ e^{-i(-E_{\vec{R}} t - \vec{R} \cdot \vec{x})}]$$

$$= - \int_{-\infty}^{+\infty} d^3\vec{R} C(E_{\vec{R}}) \sum_{\lambda} [\vec{e}^*(-\vec{R}, \lambda) a_{-\vec{R}, \lambda} e^{-ik \cdot x} + \vec{e}(-\vec{R}, \lambda) a_{-\vec{R}, \lambda}^+ e^{ik \cdot x}]$$

$$\text{Using } \vec{e}(-\vec{R}, -\lambda) = \vec{e}(\vec{R}, \lambda) = -\vec{e}^*(\vec{R}, -\lambda)$$

$$\Rightarrow ** = - \int_{-\infty}^{+\infty} d^3\vec{R} C(E_{\vec{R}}) \sum_{\lambda} [\vec{e}(\vec{R}, \lambda) a_{\vec{R}, \lambda} e^{-ik \cdot x} + \vec{e}^*(\vec{R}, \lambda) a_{\vec{R}, \lambda}^+ e^{ik \cdot x}]$$

$$\Rightarrow a_{\vec{R}, \lambda} \xrightarrow{T} a_{-\vec{R}, \lambda}, \quad a_{\vec{R}, \lambda}^+ \xrightarrow{T} a_{-\vec{R}, \lambda}^+ \quad \leftarrow \text{Same as modify pages}$$

So, the momentum reverse, but helicity does not change.

In sum, for real scalar field,

$$\phi(t, \vec{x}) \xrightarrow{T} S_B \phi(-t, \vec{x}) \xrightarrow{T} S_B S_B^* \phi(t, \vec{x}) = S_B^2 \phi(t, \vec{x}) = \phi(t, \vec{x})$$

for complex scalar field, $a_{\vec{P}} \xrightarrow{T} S_B a_{-\vec{P}} \xrightarrow{T} a_{\vec{P}}, \quad a_{\vec{P}}^+ \xrightarrow{T} S_B a_{-\vec{P}}^+ \xrightarrow{T} a_{\vec{P}}^+$

$$\phi(t, \vec{x}) \xrightarrow{T} S_B \phi(-t, \vec{x}) \xrightarrow{T} \phi(t, \vec{x}).$$

$$a_{\vec{P}} \xrightarrow{T} S_B a_{-\vec{P}} \xrightarrow{T} a_{\vec{P}}, \quad b_{\vec{P}}^+ \xrightarrow{T} S_B b_{-\vec{P}}^+ \xrightarrow{T} b_{\vec{P}}^+$$

$$\phi^+(t, \vec{x}) \xrightarrow{T} S_B^* \phi^+(-t, \vec{x}) \xrightarrow{T} \phi^+(t, \vec{x})$$

$$a_{\vec{P}}^+ \xrightarrow{T} S_B^* a_{-\vec{P}}^+ \xrightarrow{T} a_{\vec{P}}^+, \quad b_{\vec{P}} \xrightarrow{T} S_B^* b_{-\vec{P}} \xrightarrow{T} b_{\vec{P}}$$

for photon field, $\vec{A}(t, \vec{x}) \xrightarrow{T} -\vec{A}(-t, \vec{x}) \xrightarrow{T} A(t, \vec{x})$

$$a_{\vec{R}, \lambda} \xrightarrow{T} a_{-\vec{R}, \lambda} \xrightarrow{T} a_{\vec{R}, \lambda}, \quad a_{\vec{R}, \lambda}^+ \xrightarrow{T} a_{-\vec{R}, \lambda}^+ \xrightarrow{T} a_{\vec{R}, \lambda}$$

that is, $T^2 = +1$ for scalar field and photon field:
both real and complex

Q. (checked)

Some note for C, P and T:

1. We have assumed that the vacuum state $|0\rangle$ is invariant under separate transformations of C, P and T.
2. For the photon field, we came to the conclusion that the photon has parity -1 , charge conjugation number -1 ; parity transformation reverse the photon's momentum and helicity, time inverse transformation reverse its momentum but not change its helicity.
3. For the complex scalar field and Dirac field, the parity of the particles described by the fields are determined by parity conservation requirement of the parity conservation reactions, after the parities of a minimum number of particles (usually taken $\gamma_{\text{proton}} = \gamma_{\text{neutron}} = \gamma_{\text{electron}} = +1$) are assigned. Their parities are either $+1$ or -1 .
The parities of particle and antiparticle are the same for a complex scalar field; while they are opposite for a Dirac field.
4. For the real scalar field, the parity number, charge conjugation number and the phase of T transformation, are $+1$ or -1 , i.e., the phases for P, C and T transformations are all real.
For photon field, such phases are determined by the requirement of the invariance of $\bar{\psi} \gamma^\mu \psi A_\mu$ under P, C and T, that is, the photon's parity and charge conjugation number are both -1 .
5. For T and C transformations, the phases for complex scalar and Dirac fields are only required to be $|\mathcal{E}|^2 = 1$ and $|\mathcal{S}|^2 = 1$.

6. For a state of scalar - antiscalar particles with orbital angular momentum ℓ :

$$|\phi_c(s\bar{s})\rangle = \int d^3\vec{p} F_c(\vec{p}) a_{\vec{p}}^+ b_{-\vec{p}}^+ |0\rangle, \text{ where } F_c(\vec{p}) \text{ is an eigenstate of orbital angular momentum } \ell.$$

Apply parity transformation.,

$$\begin{aligned} P|\phi_c(s\bar{s})\rangle &= \int d^3\vec{p} F_c(\vec{p}) a_{-\vec{p}}^+ b_{\vec{p}}^+ |0\rangle \\ &= \int d^3\vec{p} F_c(-\vec{p}) a_{\vec{p}}^+ b_{-\vec{p}}^+ |0\rangle \\ &= (-1)^\ell \int_{-\infty}^{+\infty} F_c(\vec{p}) a_{\vec{p}}^+ b_{-\vec{p}}^+ |0\rangle \\ &= (-1)^\ell |\phi_c(s\bar{s})\rangle \end{aligned}$$

(for real scalar, $|T\phi_c(s\bar{s})\rangle = \int d^3\vec{p} F_c(\vec{p}) a_{\vec{p}}^+ a_{-\vec{p}}^+ |0\rangle \Rightarrow P|\phi_c(s\bar{s})\rangle = \int d^3\vec{p} F_c(\vec{p}) a_{-\vec{p}}^+ a_{\vec{p}}^+ |0\rangle$)

For a state of fermion - antifermion particles, and neglect the effects of spin - orbit coupling, we have the state being

$$|\psi_{c,s}(f\bar{f})\rangle = \int d^3\vec{p} \sum_{s_1 s_2} F_{cs}^{s_1 s_2}(\vec{p}) b_{\vec{p}, s_1}^+ d_{-\vec{p}, s_2}^+ |0\rangle$$

$$\begin{aligned} \text{then } P|\psi_{c,s}^{s_1 s_2}(f\bar{f})\rangle &= - \int d^3\vec{p} \sum_{s_1 s_2} F_{cs}^{s_1 s_2}(\vec{p}) b_{-\vec{p}, s_1}^+ d_{\vec{p}, s_2}^+ |0\rangle \\ &= - \int d^3\vec{p} \sum_{s_1 s_2} F_{cs}^{s_1 s_2}(-\vec{p}) b_{\vec{p}, s_1}^+ d_{-\vec{p}, s_2}^+ |0\rangle \\ &= (-1)^{\ell+1} |\psi_{c,s}^{s_1 s_2}(f\bar{f})\rangle \end{aligned}$$

Therefore, the scalar-antiscalar system has parity number $(-1)^\ell$, and the fermion-antifermion system has parity number $(-1)^{\ell+1}$.

In fact, there is no need to require particle-antiparticle system in the above analysis — for state with two particles (and neglect the effects of spin-orbit coupling) we always have the parity of the system $\eta_A \eta_B (-1)^\ell$, where η_A and η_B are the intrinsic parity of the two particles.

$$(|\phi_c(A\bar{B})\rangle = \int d^3\vec{p} F_c(\vec{p}) A_{\vec{p}}^+ B_{-\vec{p}}^+ |0\rangle, \text{ where } A_{\vec{p}}^+ \text{ and } B_{\vec{p}}^+ \text{ are two arbitrary creation operators with spin label})$$

(3) expressed (since not relevant), then $P|\phi_c(A\bar{B})\rangle = \int d^3\vec{p} F_c(-\vec{p}) A_{\vec{p}}^+ B_{\vec{p}}^+ |0\rangle \eta_A \eta_B = (-1)^\ell \eta_A \eta_B |\phi_c(AB)\rangle$

Modify ④ - ⑧

From $T \mathcal{F}(x) T^{-1} = \mathcal{S} D \mathcal{F}(x)$.

$$\begin{aligned} & \Rightarrow \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [u^*(\vec{p}, s) b_{\vec{p}, s}^T e^{i\vec{p} \cdot \vec{x}} + v^*(\vec{p}, s) d_{\vec{p}, s}^+ e^{-i\vec{p} \cdot \vec{x}}] \\ &= \mathcal{S} \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [\gamma' \gamma^3 u(\vec{p}, s) b_{\vec{p}, s} e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} \\ & \quad + \gamma' \gamma^3 v(\vec{p}, s) d_{\vec{p}, s}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})}] \equiv ** \end{aligned}$$

using $\gamma' \gamma^3 u(\vec{p}, s) \stackrel{\dagger}{=} (\gamma' \gamma^3 u^*(\vec{p}, s))^* = -(-1)^{\frac{1}{2}-s} u^*(-\vec{p}, -s)$

since γ' , γ^3 are real

$$\gamma' \gamma^3 v(\vec{p}, s) \stackrel{\dagger}{=} (\gamma' \gamma^3 v^*(\vec{p}, s))^* = -(-1)^{\frac{1}{2}-s} v^*(-\vec{p}, -s)$$

$$\begin{aligned} & \Rightarrow ** = \mathcal{S} \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [(-(-1))^{\frac{1}{2}-s} [u^*(-\vec{p}, -s) b_{\vec{p}, s} e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} \\ & \quad + v^*(-\vec{p}, -s) d_{\vec{p}, s}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})}] \end{aligned}$$

$$\begin{aligned} &= \mathcal{S} \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [[-(-1)^{\frac{1}{2}+s} [u^*(\vec{p}, s) b_{-\vec{p}, -s}^T e^{i\vec{p} \cdot \vec{x}} \\ & \quad + v^*(\vec{p}, s) d_{-\vec{p}, -s}^+ e^{-i\vec{p} \cdot \vec{x}}]] \end{aligned}$$

$$\begin{aligned} & \Rightarrow T b_{\vec{p}, s}^T T^{-1} = -(-1)^{\frac{1}{2}+s} b_{-\vec{p}, -s}^T \quad \mathcal{S} = \mathcal{S} (-1)^{\frac{1}{2}-s} b_{-\vec{p}, -s}^T \\ & T d_{\vec{p}, s}^+ T^{-1} = -(-1)^{\frac{1}{2}+s} d_{-\vec{p}, -s}^+ \quad \mathcal{S} = \mathcal{S} (-1)^{\frac{1}{2}-s} d_{-\vec{p}, -s}^+ \end{aligned}$$

From $T \mathcal{F}^+(x) T^{-1} = \mathcal{S}^* \mathcal{F}^+(x) D^+$

$$\begin{aligned} & \Rightarrow \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [\overbrace{T u^+(\vec{p}, s)}^{\overbrace{T T^{-1}}^T} \overbrace{b_{\vec{p}, s}^+}^{\overbrace{T^{-1} T}^T} e^{i\vec{p} \cdot \vec{x}} T^{-1} + T v^+(\vec{p}, s) \overbrace{d_{\vec{p}, s}^+}^{\overbrace{T^{-1} T}^T} e^{-i\vec{p} \cdot \vec{x}} T^{-1}] \\ &= \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [u^+(\vec{p}, s) T b_{\vec{p}, s}^+ T^{-1} e^{-i\vec{p} \cdot \vec{x}} + v^+(\vec{p}, s) T d_{\vec{p}, s}^+ T^{-1} e^{i\vec{p} \cdot \vec{x}}] \\ &= \mathcal{S}^* \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [\underbrace{u^+(\vec{p}, s)}_{(\gamma' \gamma^3)^+} b_{\vec{p}, s}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + v^+(\vec{p}, s) d_{\vec{p}, s}^+ e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})}] \\ & \equiv *** \end{aligned}$$

$$\text{using } U^+(\vec{p}, s) (\gamma^1 \gamma^3)^+ = (\gamma^1 \gamma^3 U(\vec{p}, s))^+ = -(-1)^{\frac{1}{2}+s} U^+(\vec{p}, -s)$$

$$V^+(\vec{p}, s) (\gamma^1 \gamma^3)^+ = -(-1)^{\frac{1}{2}-s} V^+(\vec{p}, -s)$$

$$\Rightarrow \text{****} = \oint * \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) \sum_s [-(-1)^{\frac{1}{2}+s} U^+(\vec{p}, s) b_{\vec{p}, -s}^+ e^{-ip \cdot x} \\ - (-1)^{\frac{1}{2}+s} V^+(\vec{p}, s) d_{\vec{p}, -s} e^{ip \cdot x}]$$

$$\Rightarrow T b_{\vec{p}, s}^+ T^{-1} = -(-1)^{\frac{1}{2}+s} b_{\vec{p}, -s}^+ \quad \oint * = \oint (-1)^{\frac{1}{2}-s} b_{\vec{p}, -s}^+$$

$$T d_{\vec{p}, s} T^{-1} = -(-1)^{\frac{1}{2}+s} d_{\vec{p}, -s} \quad \oint * = \oint (-1)^{\frac{1}{2}-s} d_{\vec{p}, -s}$$

For complex scalar field.

$$\text{From } T \phi(t, \vec{x}) T^{-1} = \oint_B \phi(-t, \vec{x})$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [T a_{\vec{p}} e^{-ip \cdot x} T^{-1} + T b_{\vec{p}}^+ e^{ip \cdot x} T^{-1}] \\ &= \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [T a_{\vec{p}} T^{-1} e^{ip \cdot x} + T b_{\vec{p}}^+ T^{-1} e^{-ip \cdot x}] \\ &= \oint_B \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + b_{\vec{p}}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})}] \\ &= \oint_B \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}} e^{ip \cdot x} + b_{\vec{p}}^+ e^{-ip \cdot x}] \end{aligned}$$

$$\Rightarrow T a_{\vec{p}} T^{-1} = \oint_B a_{\vec{p}}, \quad T b_{\vec{p}}^+ T^{-1} = \oint_B b_{\vec{p}}^+$$

$$\text{From } T \phi^+(t, \vec{x}) T^{-1} = \oint_B^* \phi^+(-t, \vec{x})$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [T a_{\vec{p}}^+ e^{ip \cdot x} T^{-1} + T b_{\vec{p}}^- e^{-ip \cdot x} T^{-1}] \\ &= \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [T a_{\vec{p}}^+ T^{-1} e^{-ip \cdot x} + T b_{\vec{p}}^- T^{-1} e^{ip \cdot x}] \\ &= \oint_B^* \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}}^+ e^{i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})} + b_{\vec{p}}^- e^{-i(-E_{\vec{p}} t - \vec{p} \cdot \vec{x})}] \\ &= \oint_B^* \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [a_{\vec{p}}^+ e^{-ip \cdot x} + b_{\vec{p}}^- e^{ip \cdot x}] \\ &\Rightarrow T a_{\vec{p}}^+ T^{-1} = \oint_B^* a_{\vec{p}}^+, \quad T b_{\vec{p}}^- T^{-1} = \oint_B^* b_{\vec{p}}^- \end{aligned}$$

For real scalar field

$$\text{From } T\phi(t, \vec{x}) T^{-1} = S_B \phi(-t, \vec{x})$$

$$\begin{aligned} \Rightarrow & \int_{-\infty}^{+\infty} d\vec{p} C(E_{\vec{p}}) [T a_{\vec{p}} e^{-ip \cdot x} T^{-1} + T a_{\vec{p}}^+ e^{ip \cdot x} T^{-1}] \\ &= \int_{-\infty}^{+\infty} C(E_{\vec{p}}) [T a_{\vec{p}} T^{-1} e^{ip \cdot x} + T a_{\vec{p}}^+ T^{-1} e^{-ip \cdot x}] \\ &= S_B \int_{-\infty}^{+\infty} C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(-E_p t - \vec{p} \cdot \vec{x})} + a_{\vec{p}}^+ e^{i(-E_p t - \vec{p} \cdot \vec{x})}] \\ &= S_B \int_{-\infty}^{+\infty} C(E_{\vec{p}}) [a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^+ e^{-ip \cdot x}] \\ \Rightarrow & T a_{\vec{p}} T^{-1} = S_B a_{\vec{p}} \\ T a_{\vec{p}}^+ T^{-1} &= S_B a_{\vec{p}}^+ \end{aligned}$$

$$\text{From } T \vec{A}(t, \vec{x}) T^{-1} = -\vec{A}(-t, \vec{x})$$

$$\begin{aligned} \Rightarrow & \int_{-\infty}^{+\infty} d\vec{k} C(E_{\vec{k}}) \sum [T \vec{e}(\vec{E}, \lambda) a_{\vec{k}, \lambda} e^{-ik \cdot x} T^{-1} \\ & \quad + T \vec{e}^*(\vec{E}, \lambda) a_{\vec{k}, \lambda}^+ e^{ik \cdot x} T^{-1}] \\ &= \int_{-\infty}^{+\infty} d\vec{k} C(E_{\vec{k}}) \sum [\vec{e}^*(\vec{E}, \lambda) T a_{\vec{k}, \lambda} T^{-1} e^{ik \cdot x} \\ & \quad + \vec{e}(\vec{E}, \lambda) T a_{\vec{k}, \lambda}^+ T^{-1} e^{-ik \cdot x}] \\ &= - \int_{-\infty}^{+\infty} d\vec{k} C(E_{\vec{k}}) \sum [\vec{e}(\vec{E}, \lambda) a_{\vec{k}, \lambda} e^{-i(-E_k t - \vec{k} \cdot \vec{x})} \\ & \quad + \vec{e}^*(\vec{E}, \lambda) a_{\vec{k}, \lambda}^+ e^{i(-E_k t - \vec{k} \cdot \vec{x})}] = ** \end{aligned}$$

$$\text{Using } \vec{e}(-\vec{k}, -\lambda) = \vec{e}(\vec{k}, \lambda) = -\vec{e}^*(\vec{k}, -\lambda)$$

$$\begin{aligned} \Rightarrow ** &= - \int_{-\infty}^{+\infty} d\vec{k} C(E_{\vec{k}}) \sum [\vec{e}(-\vec{k}, -\lambda) a_{\vec{k}, \lambda} e^{-ik \cdot x} + \vec{e}^*(-\vec{k}, \lambda) a_{\vec{k}, \lambda}^+ e^{ik \cdot x}] \\ &= - \int_{-\infty}^{+\infty} d\vec{k} C(E_{\vec{k}}) \sum [\vec{e}(\vec{k}, \lambda) a_{\vec{k}, \lambda} e^{ik \cdot x} + \vec{e}^*(\vec{k}, -\lambda) a_{\vec{k}, \lambda}^+ e^{-ik \cdot x}] \\ \Rightarrow & T a_{\vec{k}, \lambda} T^{-1} = a_{\vec{k}, \lambda}, \quad T a_{\vec{k}, \lambda}^+ T^{-1} = a_{\vec{k}, \lambda}^+ \end{aligned}$$

modify pages ③ (checked)