<u>bound state</u>

1. relativistic or non-relativistic system.

The analysis of bound state is relatively simple when the constituents travel at speeds substaintially less than c, because then we can use non-relativistic quantum mechanics to do the analysis of the system. That's why the theory of charmonium and bottomonium is well developed. On the other hand, the hadronic states made out of u-, d- and s-quarks are more difficult to handel, because they are intrinsically relativistic, while the techniques of QFT usually assume that the particles are initially free, and become free again after some brief interaction, whereas in bound state the particles interact continuously over an extended period.

The <u>creterion</u> to tell whether a bound state is relativistic or not: if the binding energy is small compared to the rest energies of the constituents, then the system is nonrelativistic. (in general, the total energy of a composite system is the sum of the rest energy of the constituents, the kinetic energy of the constituents and the potential energy of the configuration. The latter two are comparable in size due to virial theorem) For example, the binding energy of hydrogen is 13.6 eV, which is far less than the rest energy of the electron (0.511MeV), so the system is clearly non relativistic. While for a proton or a pion, the quark - (anti) quark binding energies are of the order of hundred MeV, which is comparable to the effective rest masses (i.e., constituent quark masses) of the u and d quarks. On the other hand, the hundreds of MeV binding energies are much smaller than the rest masses in the charmonium and bottomonium system.

2. Hydrogen atom.

(1)

$$V(r) = -\frac{\alpha}{r}, \qquad \left(V(r) = -\frac{1}{r}\frac{e^2}{4\pi\varepsilon_0}, \text{ in SI units} \right)$$

$$E_n = -\alpha^2 m \frac{1}{2}\frac{1}{n^2} = -\frac{13.6\,eV}{n^2}, \quad n = 1, 2, 3, \ldots$$

$a_0 = \frac{1}{m\alpha}$, (Bohr radius: $a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{me^2}$)

$$\left(E_n = -\frac{\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 m}{\hbar^2}\frac{1}{2}\frac{1}{n^2}, \text{ in SI units} \right)$$

For each principle quantum number n, the orbital angular momentum quantum number, l, can take values from 0 to $n-1$, and for each l, the z-component of the orbital angular momentum quantum number, m_l, can take values from $-l$ to $+l$.

So, the total number of distinct states that share the same principle quantum number n is

$$\sum_{l=0}^{n-1}(2l+1) = n^2.$$

This is the degeneracy of the nth energy level.

(2) fine structure of the spectrum

due to ① relatiristic correction. ② spin-orbit coupling

$\left(\vec{\mu}_e = -\frac{|e|}{m}\vec{S}\right.$ in SI unit, where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ ↳ pauli matrices$\left.\right)$

$$\text{Kinetic energy} = E - m = (m^2+|\vec{P}|^2)^{\frac{1}{2}} - m$$
$$\simeq \frac{1}{2}\frac{|\vec{P}|^2}{m} - \frac{1}{8}\frac{|\vec{P}|^4}{m^3}$$

$\hbar \sim P a_0$

where $$\frac{|\vec{P}|^4}{m^3} \sim \left(\frac{\hbar}{a_0}\right)^4 \frac{1}{m^3}$$
$$= \frac{(\alpha m)^4}{m^3} = \alpha^4 m.$$

the dipole moment of the electron $\vec{\mu}_e = -\frac{|e|}{m_e}\vec{S}$.
Then, from the electron's perspective the orbiting proton sets up a magnetic field $\vec{B}$,
$\Rightarrow$ magnetic energy $= -\vec{\mu}_e \cdot \vec{B}$

$\left(\right.$ in SI units,

$\hbar \sim m v a_0$

$$B = \frac{\mu_0 I}{2a_0} = \frac{\mu_0 |e|}{2a_0 \; \frac{2\pi a_0}{v}} \sim \frac{\mu_0 |e|}{(2a_0)^2\pi}\frac{\hbar}{m a_0}$$

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$$\stackrel{\Downarrow}{=} \frac{\frac{1}{\varepsilon_0 c^2}|e|\hbar}{4\pi \; m \; a_0^3}$$

$$\Rightarrow |-\vec{\mu}_e\cdot\vec{B}| = \frac{1}{\varepsilon_0 c^2}\frac{|e|\hbar}{4\pi m a_0^3}\frac{|e|}{m}\frac{\hbar}{2}$$
$$= \frac{e^2\hbar^2}{8\pi\varepsilon_0 c^2 m^2 a_0^3}$$
$$= \frac{\alpha}{2m^2\left(\frac{1}{m\alpha}\right)^3} = \frac{\alpha^4 m}{2}\left.\right)$$

Indeed, these two effects give

$$\Delta E_{fs} = -\alpha^4 m \frac{1}{4n^4}\left(\frac{2n}{j+\frac{1}{2}} - \frac{3}{2}\right), \text{ where } j = l \pm \frac{1}{2}$$

(note that for $l=0$, $j = 0+\frac{1}{2} = \frac{1}{2}$)

Therefore, the nth Bohr level E_n splits into n sublevels.

So, $\frac{\Delta E_{fs}}{E_n} \propto \alpha^2$

(3) Lamb Shift

In fine structure formula, two different values of l share the same energy, e.g. the $2S_{1/2}$ and $2P_{1/2}$ states remain perfectly degenerate

$$\underset{n}{2}\underset{l=0}{S}_{\underset{j=\frac{1}{2}}{1/2}} \qquad \underset{l=1}{P}$$

This degeneracy is lifted by Lamb shift (resulted from loop effect of QED), since ΔE_{lamb} depends on l.

$\Delta E_{lamb} \propto \alpha^5 m$

(4) Hyperfine spitting

The proton also has spin, so it has a magnetic dipole moment

$$\vec{\mu}_p = \gamma_p \frac{|e|}{m_p} \vec{S}_p = \gamma_p \frac{|e|}{m_p} \frac{\hbar}{2} \vec{\sigma} \quad , \text{ in SI units}$$

where $\gamma_p \doteq 2.7928$ due to that proton itself is a composite object.

Then, from the proton's prospective the orbiting electron sets up a magnetic field, so that there is a magnetic energy.

Moreover, there is also the interaction due to proton-electron spin spin coupling.

These two effects together give the hyperfine splitting.

$$\Delta E_{hf} = \left(\frac{m}{m_p}\right) \alpha^4 m \frac{\gamma_p}{2n^3} \frac{\pm 1}{(f+\frac{1}{2})(l+\frac{1}{2})} \quad \text{for } f = j \pm \tfrac{1}{2}$$

where f is the $\underbrace{\text{total angular momentum}}_{\vec{L}+\vec{S}_e+\vec{S}_p}$ quantum number.

Because of the factor $\frac{m}{m_p} \simeq \frac{1}{1836}$, the hyperfine effects are about $O(10^{-3})$ times smaller than fine structure effects.

For the ground state ($n=0$, $l=0$), the hyperfine spitting gives rise the famous 21 cm line, due to the two values, $f = \frac{1}{2} + \frac{1}{2} = 1$ (i.e., spin triplet), and $f = \frac{1}{2} - \frac{1}{2} = 0$ (i.e., spin singlet) $\Rightarrow$ $\Delta E \equiv E_{triplet} - E_{singlet} = \left(\frac{m}{m_p}\right) \alpha^4 m \frac{\gamma_p}{2} \frac{1}{(\frac{1}{2})} \left(\frac{1}{\frac{3}{2}} + \frac{1}{\frac{1}{2}}\right)$

$$= \frac{32}{3} \frac{\gamma_p}{m_p} (13.6\,eV)^2$$

$$\Rightarrow \lambda = \frac{2\pi}{\Delta E} = 21 \text{ cm}$$

3. positronium

In particle physics, positronium serves as the model for quarkonium.

Substitute the electron mass to reduced mass of the positronium system, that is, $m \to \frac{m}{2}$:

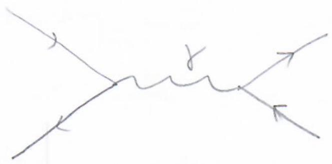
$$\Rightarrow E_n^{\text{positronium}} = -\alpha^2 \frac{m}{2} \frac{1}{2} \frac{1}{n^2}, \quad n=1,2,3,\cdots$$

$$a_{\text{Bohr for positronium}} = \frac{1}{\alpha \frac{m}{2}}.$$

Similar to hydrogen, positronium also has fine structure, Lamb shift etc..

The difference is that now the hyperfine structure contribution is the same as fine structure, since there is no suppress factor $\frac{m_{\text{electron}}}{m_{\text{proton}}}$.

There is a new effect due to virtual photon diagram of the positronium,



this process raises the energy of the spin triplet state by

$$\Delta E_{\text{ann}} = \alpha^4 m \frac{1}{4n^3}, \quad (l=0, \; S=1)$$

the temperarily annihilation of the electron and positron requires that the wave function is non-zero at the origin, i.e, $|\psi(0)|^2 > 0$, so that l has to be 0; the photon's spin is 1, so that the electron and positron has to be in the $S=1$ triplet.

Another difference between positronium and hydrogen is that positronium can annihilation decay (or just call it "decay") to produce two or more real photons.

The ground state ($l=0$, $S=0$) positronium can typically decay into two photons, and its lifetime is

$$\tau = \frac{2}{\alpha^5 m} = 1.2 \times 10^{-10} \text{ sec.}$$

4. quarkonium

First of all, the analogue of positronium applies to heavy-quark mesons, i.e., $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$, not to the light-quark (u, d, s) mesons. Secondly, because the potential energy is such a substantial fraction of the total, we regard the various energy levels as representing different particles. Third, our knowledge of the strong force, which bounds the quarks to form mesons, is not as good as that of the electromagnetic force, which ressponsible for positronium and hydrogen.

Because in QCD, the short-distance behavior is dominated by one-gluon exchange, just as in QED it is dominated by one-photon exchange, we expect a Coulomblike potential; on the other hand, we don't know much about the long-distance potential, apart from that we know it must increase without limit in order to account for quark confinement. Nevertheless, many of the speculative potential can fit the data reasonably well ($\sim r^2$ potential, $\sim \ln(r)$ potential, $\sim r$ potential etc) because they do not differ substantially over the narrow range of the distance for which we have sensitive probes of the bound state system.

For example, $$V(r) = -\underbrace{\frac{4}{3}}_{\text{color factor}} \frac{\alpha_s}{r} + F_0 r$$

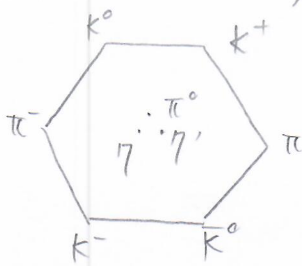
α_s → strong coupling

F_0 → numerical constants to fit the data.

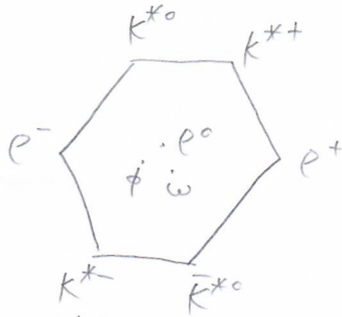
that is, use this potential to solve for the Schrödingen equation and obtains energy levels, and then use measured data to fit to get F_0.

5. Light quark mesons (made out of light quark and antiquark; u, d, s)

For the lowest lying states ($l=0$), we have



pseudoscalar nonet ($S=\frac{1}{2}-\frac{1}{2}=0$)



vector nonet ($S=\frac{1}{2}+\frac{1}{2}=1$)

The nonet is actually an octet and a singlet

SU(3) flavor : $3 \otimes \bar{3} = 8 \oplus 1$

Since the pseudoscalar and vector mesons differ only in the relative orientation of the quark spins, the difference in their masses may be attributed to a spin-spin interaction, and indeed a mass formula

$$M = m_1 + m_2 + A\frac{\vec{S}_1\cdot\vec{S}_2}{m_1 m_2},$$

where A is a constant, gives a good fit to many (not all) of the particle masses in pseudoscalar nonet & vector nonet.

Note that from $\vec{S}=\vec{S}_1+\vec{S}_2 \Rightarrow \vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(S^2-S_1^2-S_2^2)$

then use $S^2 \equiv S(S+1)\hbar^2 = \begin{cases} 1\times2=2\hbar^2, & \text{for vector mesons} \\ 0\times1=0, & \text{for pseudoscalars.} \end{cases}$

take it as an operator

and $S_1^2 = S_2^2 \equiv \frac{1}{2}\times(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$

$$\Rightarrow \quad \vec{S}_1\cdot\vec{S}_2 = \begin{cases} \frac{1}{4}\hbar^2, & \text{for vector mesons} \\ -\frac{3}{4}\hbar^2, & \text{for pseudoscalars} \end{cases}$$

using $A=(2m_u)^2\times159$ MeV, $m_s = 483$ MeV, $m_u = m_d = 308$ MeV for example,

$$\Rightarrow m_\pi = 308\times2 + (2\times308)^2\frac{159}{308^2}\left(-\frac{3}{4}\right) = 139\text{ MeV} \quad (\text{observation: } 138\text{ MeV})$$

$$m_\rho = 308\times2 + (2\times308)^2\frac{159}{308^2}\left(\frac{1}{4}\right) = 775\text{ MeV} \quad (\text{observation: } 776\text{ MeV})$$

$$m_K = 308+483+(2\times308)^2\times159\,\frac{1}{308\times483}\left(-\frac{3}{4}\right) = 487\text{ MeV}$$

(observation: 496 MeV)

6. baryons

6.1 baryon wave functions (made by u, d, s quarks)

First of all, we'll concentrate on ground state, which has no orbital angular momentum.

The quark spins can combine to give a total of $\frac{3}{2}$ or $\frac{1}{2}$. Since each quark can have spin up, $\uparrow$, and spin down, $\downarrow$, there're eight possible states. Let's group them as

$$\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = (\uparrow\uparrow\uparrow)$$
$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = (\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)/\sqrt{3}$$
$$\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = (\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)/\sqrt{3}$$
$$\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = (\downarrow\downarrow\downarrow)$$

spin $\frac{3}{2}$ (ψ_s)
completely symmetric

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle_{12} = (\uparrow\downarrow - \downarrow\uparrow)\uparrow/\sqrt{2}$$
$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_{12} = (\uparrow\downarrow - \downarrow\uparrow)\downarrow/\sqrt{2}$$

spin $\frac{1}{2}$ (ψ_{12})
antisymmetric in quarks 1 and 2

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle_{23} = \uparrow(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$$
$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_{23} = \downarrow(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$$

spin $\frac{1}{2}$ (ψ_{23})
antisymmetric in quarks 2 and 3

we can also have

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle_{13} = (\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)/\sqrt{2}$$
$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle_{13} = (\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow)/\sqrt{2}$$

spin $\frac{1}{2}$ (ψ_{13})
antisymmetric in quarks 1 and 3

but $|\ \rangle_{13}$, $|\ \rangle_{12}$ and $|\ \rangle_{23}$ are not independent, since $|\ \rangle_{13} = |\ \rangle_{12} + |\ \rangle_{23}$

The above spin states can be directly constructed by first combine particles 1 and 2 as $|1\,1\rangle$, $|1\,0\rangle$, $|1\,-1\rangle$ and $|0\,0\rangle$ by using the $\frac{1}{2}\times\frac{1}{2}$ C-G coefficients, and then combine these combinations with particle 3 by using the $1\times\frac{1}{2}$ and $0\times\frac{1}{2}$ C-G coefficients, that is,

first $|1\,1\rangle_{12} = \uparrow\uparrow$, $|1\,0\rangle_{12} = \frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow)$, $|1\,-1\rangle_{12} = \downarrow\downarrow$

$|0\,0\rangle_{12} = \frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)$

then using $1\times\frac{1}{2}$: $|\frac{3}{2}\,\frac{3}{2}\rangle = |1\,1\rangle_{12}|\frac{1}{2}\,\frac{1}{2}\rangle_3 = (\uparrow\uparrow\uparrow)$ ✓

$$|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|1\,1\rangle_{12}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_3 + \sqrt{\tfrac{2}{3}}|1\,0\rangle_{12}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_3$$
$$= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow) + \sqrt{\tfrac{2}{3}}\tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)$$
$$= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow+\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)$$ ✓

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|1\,1\rangle_{12}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_3 - \sqrt{\tfrac{1}{3}}|1\,0\rangle_{12}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_3$$
$$= \sqrt{\tfrac{2}{3}}\uparrow\uparrow\downarrow - \sqrt{\tfrac{1}{6}}(\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)$$

$$|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|1\,0\rangle_{12}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_3 + \sqrt{\tfrac{1}{3}}|1\,-1\rangle_{12}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_3$$
$$= \sqrt{\tfrac{1}{3}}(\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow) + \sqrt{\tfrac{1}{3}}(\downarrow\downarrow\uparrow)$$
$$= (\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow+\downarrow\downarrow\uparrow)/\sqrt{3}.$$ ✓

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|1\,0\rangle_{12}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_3 - \sqrt{\tfrac{2}{3}}|1\,-1\rangle_{12}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_3$$
$$= \sqrt{\tfrac{1}{6}}(\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow) - \sqrt{\tfrac{2}{3}}(\downarrow\downarrow\uparrow)$$

$$|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = |1\,-1\rangle_{12}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_3$$
$$= (\downarrow\downarrow\downarrow)$$ ✓

using $0\times\frac{1}{2}$:

$$\left.\begin{aligned} |\tfrac{1}{2}\,\tfrac{1}{2}\rangle &= |0\,0\rangle_{12}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_3 = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\uparrow \\ |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle &= |0\,0\rangle_{12}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_3 = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\downarrow \end{aligned}\right\}$$ these are just ψ_{12}

The $|\frac{1}{2}\ \frac{1}{2}\rangle$ from $1 \times \frac{1}{2}$ is actually the normalized

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle_{23} + |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_{13}$$

The $|\frac{1}{2}\ -\frac{1}{2}\rangle$ from $1 \times \frac{1}{2}$ is actually the normalized

$$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_{23} + |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_{13}.$$

They are in fact symmetric in quarks 1 and 2.

In the Language of group theory, the direct product of three fundamental (two dimensional) representations of SU(2) decomposes into the direct sum of four-dimensional representation and two two-dimensional representations.

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2.$$

If we treat all quarks, regardless of color and flavor, as different states of a single particle, then the wave function of a baryon

$$\psi = \psi(\text{space})\,\psi(\text{spin})\,\psi(\text{flavor})\,\psi(\text{color})$$

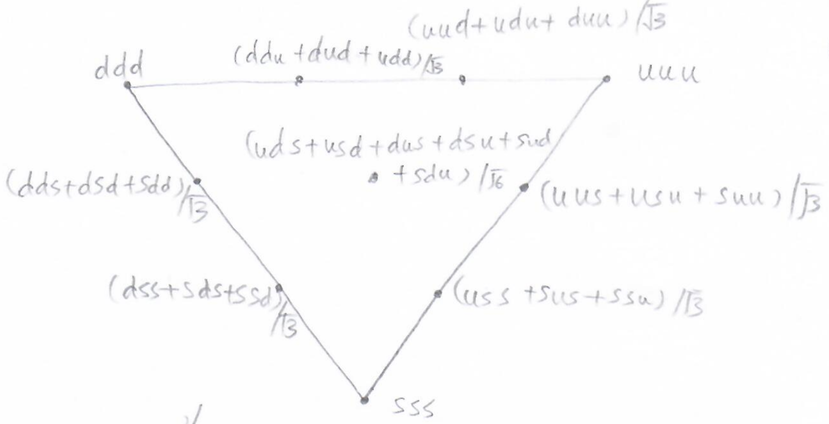
as a whole must be antisymmetric under the exchange of any two quarks.

Since we are concentrating the zero orbital angular momentum case, $\psi(\text{space})$ is symmetric.

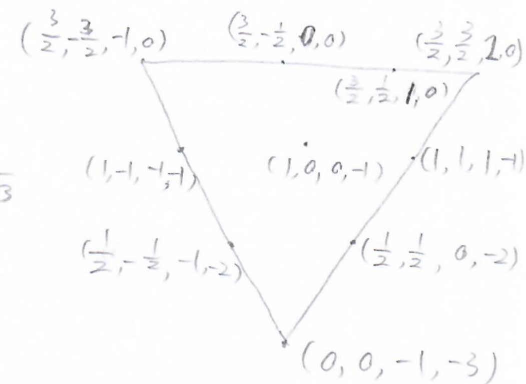
Now let's look at the flavor part, $\psi(\text{flavor})$.

The $3 \times 3 \times 3 = 27$ possibilities: uuu, uud, udu, ..., sss, are reshuffle into symmetric, antisymmetric, and mixed combinations. These are displayed in Eightfold-Way patterns: a decuplet, a singlet and two octets.

In the language of group theory, $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$



the (I, I_3, Q, S) numbers are

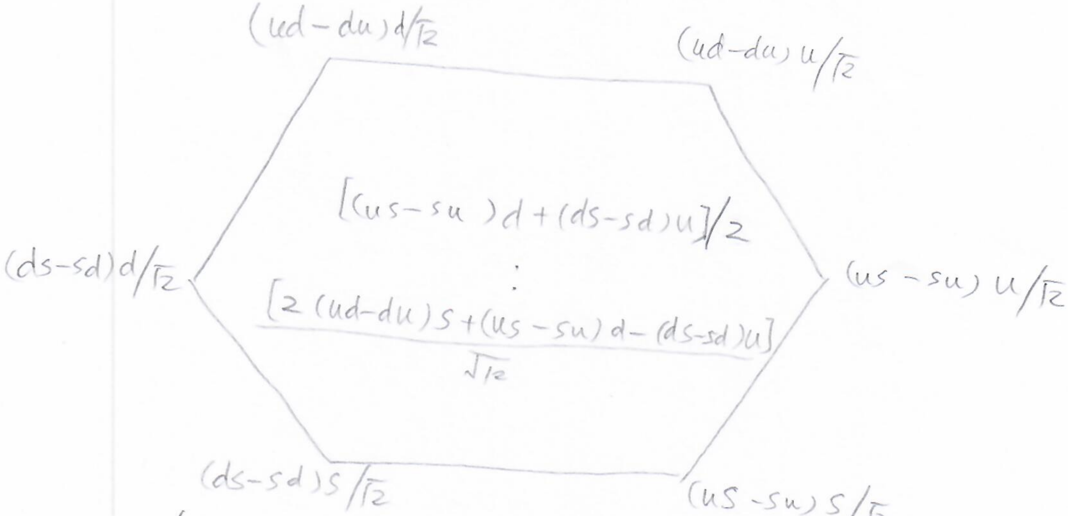


ψ_S : completely symmetric states

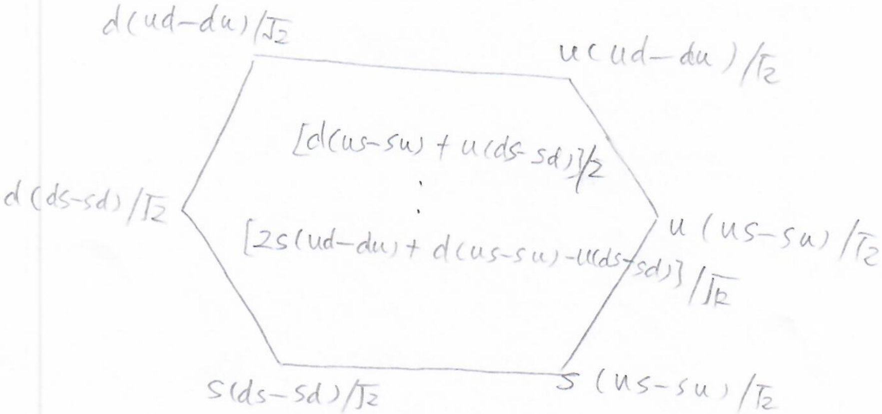
$(uds - usd + dsu - dus + sud - sdu)/\sqrt{6}$

ψ_A : completely antisymmetric state.

$(I, I_3, Q, S) = (0, 0, 0, -1)$

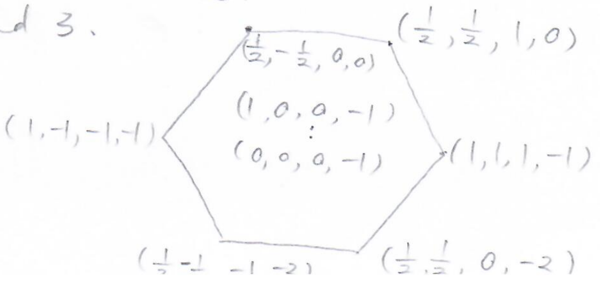


ψ_{12} : antisymmetric in 1 and 2.

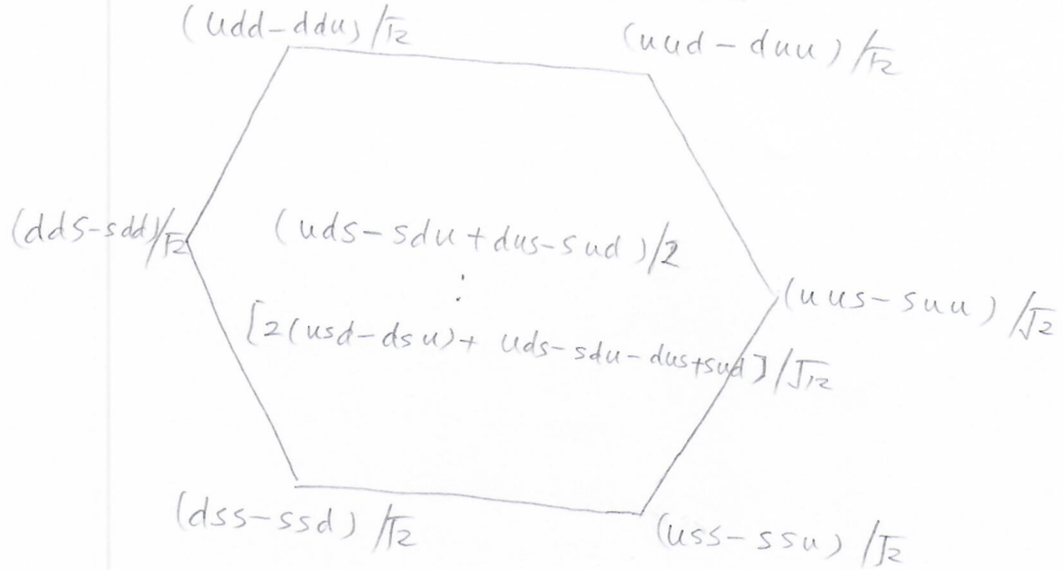


ψ_{23} : antisymmetric in 2 and 3.

the (I, I_3, Q, S) for octets are



We can also construct ψ_{13}: antisymmetric in 1 and 3, but it can be formed by $\psi_{13} = \psi_{12} + \psi_{23}$, so not independent



Note: the ψ_S states can be constructed easily by arranging the 10 possible completely antisymmetric states according to (I, I_3, Q, S); the ψ_A state can be constructed easily, since that's the only possible completely antisymmetric state. For the ψ_{12} states, we can start from the upper-right corner $|\frac{1}{2}\frac{1}{2}\rangle = (ud-du)u/\sqrt{2}$, and then use the ladder operator in isospin space to get the other states, that is, $I_- = I_{1-} + I_{2-} + I_{3-}$, where 1, 2, 3 indicate the three quarks, then $I_- \frac{1}{\sqrt{2}}(ud-du)u = (I_{1-}+I_{2-}+I_{3-})\frac{1}{\sqrt{2}}(ud-du)u$

using $I_- u = I_-|\frac{1}{2}\ \frac{1}{2}\rangle = \sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\,|\frac{1}{2}\ -\frac{1}{2}\rangle = |\frac{1}{2}\ -\frac{1}{2}\rangle = d$

and $I_- d = 0 \Rightarrow$

$$I_- \frac{1}{\sqrt{2}}(ud-du)u = \frac{1}{\sqrt{2}}(ddu + udd - dud - ddu)$$

$$= \frac{1}{\sqrt{2}}(ud-du)d,$$

also note that $I_-|\frac{1}{2}\ \frac{1}{2}\rangle = |\frac{1}{2}\ -\frac{1}{2}\rangle$, so we get the upper-left corner; $(|\frac{1}{2}\ -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(ud-du)d.$ For the middle line, the right can be obtained by changing d to s from the upper-right, then again use I_- to get the center triplet state (note that $I_- s = 0$ since s is an isospin singlet), that is $I_- \ (us-su)u/\sqrt{2} = \frac{1}{\sqrt{2}}(dsu + usd - sdu - sud)$ and $I_-|1\ 1\rangle = \sqrt{1\times(1+1) - 1\times(1-1)}\,|1 0\rangle = \sqrt{2}|1 0\rangle$

$\Rightarrow |1\ 0\rangle = \frac{1}{2}[(ds-sd)u + (us-su)d)]$

Note: $I_-|I\ I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}\,|I\ I_3-1\rangle$

Then from $I_-|1\,0\rangle = \sqrt{1\times(1+1) - 0\,(0-1)}\,|1\,-1\rangle = \sqrt{2}\,|1\,-1\rangle$

$= (I_{1-} + I_{2-} + I_{3-})\,\frac{1}{2}\,[\,(ds-sd)u + (us-su)d\,]$

$= \frac{1}{2}\,[\,dsd - sdd + dsd - sdd\,]$

$\Rightarrow$

$|1\,-1\rangle = \frac{1}{\sqrt{2}}\,(ds-sd)d$

Then the lower left corner can be obtained from the middle far east corner by changing a d to s, and the lower right corner can be obtained using $I_+|I\;I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}\;|I\;\;I_3+1\rangle$, so that $I_+u = 0$, $I_+d = \sqrt{\frac{1}{2}(\frac{1}{2}+1) + \frac{1}{2}(-\frac{1}{2}+1)}\;u = u$ and $I_+|\frac{1}{2}\,-\frac{1}{2}\rangle = |\frac{1}{2}\,\frac{1}{2}\rangle$.

also, $I_+s = 0$ (since s is an isospin singlet)

$\Rightarrow$ $I_+(ds-sd)s/\sqrt{2} = (us-su)s/\sqrt{2}$ $\Rightarrow$ the lower-right corner is $\frac{1}{\sqrt{2}}(us-su)s$.

For the isosinglet in the center of the octet ψ_{12}, it has the same (I, I_3, Q, S) number as the ψ_A. It should be orthogonal to ψ_A and the triplet state $|1\,0\rangle = [(us-su)d + (ds-sd)u]/2$. Then we can get the coefficients α, β, γ in $\alpha(ud-du)s + \beta(us-su)d + \gamma(ds-sd)u$ from these two orthogonal relations and the normalization condition, that is

$0 = \langle 0\,0|1\,0\rangle = \langle[\,\alpha(ud-du)s + \beta(us-su)d + \gamma(ds-sd)u\,]\,|\,\frac{1}{2}\,((ds-sd)u + (us-su)d]\rangle$

$= \frac{1}{2}\,[\alpha\times 0 + \beta(1+1) + \gamma(1+1)] \Rightarrow \beta + \gamma = 0$ ①

and $0 = \langle 0\,0|\psi_A\rangle = \langle[\,\alpha(ud-du)s + \beta(us-su)d + \gamma(ds-sd)u\,]\,|\,\frac{1}{\sqrt{6}}\,(uds - usd$
$+ dsu - dus$
$+ sud - sdu)\rangle$

$= \frac{1}{\sqrt{6}}\,(\alpha(1+1) + \beta(-1-1) + \gamma(1+1)]$

$\Rightarrow \alpha - \beta + \gamma = 0$ ②

① and ②
$\Rightarrow$ $\alpha = 2\beta = -2\gamma$

So, $|0\,0\rangle = [\,2(ud-du)s + (us-su)d - (ds-sd)u\,]/\sqrt{12}$

The ψ_{23} and ψ_{13} can be obtained in the same way.

Griffith problem

5.14 extend to ψ_{23} and ψ_{13}.

For ψ_{23}, start from the upper right corner,

$$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = u(ud-du)/\sqrt{2}$$

then $I_- |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}[d(ud-du) + u(dd-dd)] = \frac{1}{\sqrt{2}} d(ud-du)$

$$\Rightarrow |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}} d(ud-du) \quad \checkmark$$

changing d to s from upper-right corner to get the far right middle line. $|1\ 1\rangle = u(us-su)/\sqrt{2}$ ✓

using $I_- |1\ 1\rangle = \sqrt{2}\,|1\ 0\rangle$

$$= \tfrac{1}{\sqrt{2}}[d(us-su) + u(ds-sd)]$$

$$\Rightarrow |1\ 0\rangle = \tfrac{1}{2}[d(us-su) + u(ds-sd)] \quad \checkmark$$

using $I_- |1\ 0\rangle = \sqrt{2}\,|1\ -1\rangle$

$$= \tfrac{1}{2}[d(ds-sd) + d(ds-sd)] = dds - dsd.$$

$$\Rightarrow |1\ -1\rangle = \tfrac{1}{\sqrt{2}} d(ds-sd)$$

changing d to s from the far left middle line to get the lower left corner, $|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = s(ds-sd)/\sqrt{2} + d(ss-ss)/\sqrt{2} = \frac{1}{\sqrt{2}} s(ds-sd)$

then using $I_+ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}} s(us-su)$ to get the lower right corner ✓

For the isosinglet in the middle, let it be

$$|0\ 0\rangle = \alpha u(ds-sd) + \beta d(us-su) + \gamma s(ud-du)$$

then $0 = \langle 0\ 0 | 1\ 0\rangle = \frac{1}{2}[\alpha(1+1) + \beta(1+1)] \Rightarrow \alpha + \beta = 0.$

$$0 = \langle 0\ 0 | \psi_A\rangle = \tfrac{1}{\sqrt{6}}[\alpha(1+1) + \beta(-1-1) + \gamma(1+1)] \Rightarrow \alpha + \gamma = \beta$$

$$\Rightarrow \begin{cases} \gamma = \beta - \alpha = -2\alpha \\ \beta = -\alpha \end{cases}$$

$$\Rightarrow |0\ 0\rangle = \frac{1}{\sqrt{12}}[-u(ds-sd) + d(us-su) + 2s(ud-du)] \quad \checkmark$$

(note that the normalization factor can have a phase, and we choose $\alpha = -\frac{1}{\sqrt{12}}$ here)

13 (checked)

For ψ_{13}, start from the upper right corner,

$|\frac{1}{2}\ \frac{1}{2}\rangle = (uud - duu)/\sqrt{2}$,

then $I_-|\frac{1}{2}\ \frac{1}{2}\rangle = |\frac{1}{2}\ -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(dud + udd - dud - ddu) = \frac{1}{\sqrt{2}}(udd - ddu)$ ✓

$\Rightarrow |\frac{1}{2}\ -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(udd - ddu)$ ✓

Changing d to s from upper right corner to get the far right middle line $|1\ 1\rangle = (uus - suu)/\sqrt{2}$ ✓

using $I_-|1\ 1\rangle = \sqrt{2}|1\ 0\rangle = \frac{1}{\sqrt{2}}(dus + uds - sdu - sud)$

$\Rightarrow |1\ 0\rangle = \frac{1}{2}(uds - sdu + dus - sud)$

using $I_-|1\ 0\rangle = \sqrt{2}|1\ -1\rangle = \frac{1}{2}(dds - sdd + dds - sdd) = dds - sdd \Rightarrow |1\ -1\rangle = \frac{1}{\sqrt{2}}(dds - sdd)$ ✓

Changing d to s from the far left middle line to get the lower left corner, $|\frac{1}{2}\ -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(sds + dss - ssd - sds) = \frac{1}{\sqrt{2}}(dss - ssd)$

then using $I_+|\frac{1}{2}\ -\frac{1}{2}\rangle = |\frac{1}{2}\ \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(uss - ssu)$ ✓ to get the lower right corner

For the isospin singlet in the middle, let it be

$|0\,0\rangle = \alpha(usd - dsu) + \beta(sud - dus) + \gamma(uds - sdu)$

then $0 = \langle 0\ 0|1\ 0\rangle = \frac{1}{2}[\beta(-1-1) + \gamma(1+1)] \Rightarrow \beta = \gamma$

$0 = \langle 0\ 0|\psi_A\rangle = \frac{1}{\sqrt{6}}[\alpha(-1-1) + \beta(1+1) + \gamma(1+1)] \Rightarrow \alpha = \beta + \gamma$

$\Rightarrow |0\ 0\rangle = \frac{1}{\sqrt{12}}[2(usd - dsu) + (sud - dus) + (uds - sdu)]$ ✓

(again, we choose a normalization phase such that $\alpha = \frac{2}{\sqrt{12}}$)

For ψ(color), it can also makes $\overset{\text{color triplet}}{\overset{\downarrow}{3}} \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$.

However, nature only chooses the color singlet. Therefore,

$$\psi(\text{color}) = (rgb - rbg + gbr - grb + brg - bgr)/\sqrt{6}.$$

This color wave function is the same for ALL baryons, and it is antisymmetric, so the ψ(space) ψ(spin) ψ(flavor) has to be symmetric. Then for the ground state, which has ψ(space) symmetric, we need ψ(spin) ψ(flavor) be symmetric.

In particular, the spin $\frac{3}{2}$ baryon decouplet wave function is

$$\psi(\text{baryon decouplet}) = \underbrace{\psi_s(\text{spin})}_{\text{totally symmetric}} \underbrace{\psi_s(\text{flavor})}_{\text{totally symmetric}}$$

(we have dropped the ψ(space) and ψ(color))

For example, the wave function for Δ^+, in the spin state $m_j = -\frac{1}{2}$ is

$$|\Delta^+ : \tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \{(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)/\sqrt{3}\}\{(uud + udu + duu)/\sqrt{3}\}$$

$$= [u\downarrow u\downarrow d\uparrow + u\downarrow u\uparrow d\downarrow + u\uparrow u\downarrow d\downarrow$$
$$+ u\downarrow d\downarrow u\uparrow + u\downarrow d\uparrow u\downarrow + u\uparrow d\downarrow u\downarrow$$
$$+ d\downarrow u\downarrow u\uparrow + d\downarrow u\uparrow u\downarrow + d\uparrow u\downarrow u\downarrow]/3$$

For the baryon octet wave function, first of all we notice that ψ_{ij}(spin) ψ_{ij}(flavor) is symmetric in i & j (since ψ_{ij}(spin) and ψ_{ij}(flavor) are antisymmetric in i & j, respectively), where ij are 12, 23 and 13; then we can check that for both the spin and flavor wave functions, $\psi_{23} \leftrightarrow \psi_{13}$ when do $1 \leftrightarrow 2$, $\psi_{12} \leftrightarrow \psi_{13}$ when do $2 \leftrightarrow 3$, $\psi_{23} \leftrightarrow -\psi_{12}$ when do $1 \leftrightarrow 3$. Therefore, we can make completely symmetric wave functions from ψ_{ij}(spin) and ψ_{ij}(flavor),

$$\Rightarrow \psi(\text{baryon octet}) = \frac{\sqrt{2}}{3}\left[\psi_{12}(\text{spin})\,\psi_{12}(\text{flavor}) + \psi_{23}(\text{spin})\,\psi_{23}(\text{flavor}) + \psi_{13}(\text{spin})\,\psi_{13}(\text{flavor})\right]$$

for the normalization factor, see Griffiths problem 5.16.

So, for example, for a proton with spin up, the spin flavor wave function is

$$|p: \tfrac{1}{2}\ \tfrac{1}{2}\rangle = \frac{\sqrt{2}}{3}\Big\{\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\uparrow\frac{1}{\sqrt{2}}(ud-du)u + \frac{1}{\sqrt{2}}\uparrow(\uparrow\downarrow-\downarrow\uparrow)\frac{1}{\sqrt{2}}u(ud-du) + \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow)\frac{1}{\sqrt{2}}(uud-duu)\Big\}$$

$$= \frac{1}{3\sqrt{2}}\Big\{[2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\downarrow] + [2u\uparrow d\downarrow u\uparrow - u\downarrow d\uparrow u\uparrow - u\uparrow d\uparrow u\downarrow] + [2u\uparrow u\uparrow d\downarrow - u\downarrow u\uparrow d\uparrow - u\uparrow u\downarrow d\uparrow]\Big\}$$

If we don't introduce color wave function, then $\psi(\text{spin})\psi(\text{flavor})$ has to be antisymmetric, then we can make only one spin $\frac{3}{2}$ baryon (instead of ten) from $\psi_S(\text{spin})\,\psi_A(\text{flavor})$. This is a motivation to introduce color.

(Even without introducing color, its still possible to get eight spin $\frac{1}{2}$ baryons by making $\psi(\text{spin})\psi(\text{flavor})$ antisymmetric, and the wave function can be constructed as

$$\psi = \frac{\sqrt{2}}{3\sqrt{3}}\left[\psi_{12}(\text{spin})(\psi_{31}(\text{flavor}) + \psi_{32}(\text{flavor})) + \psi_{23}(s)(\psi_{12}(f)+\psi_{13}(f)) + \psi_{31}(s)(\psi_{23}(f)+\psi_{21}(f))\right],$$

normalization factor

where $\psi_{ij} \equiv -\psi_{ji}$), see Griffiths problem 5.18 and 5.20.

6.2 magnetic moments

In SI units, the magnetic dipole moment of a spin $\frac{1}{2}$ point particle of charge q and mass m is

$$\vec{\mu} = \frac{q}{m}\vec{S} = \frac{q}{m}\frac{\hbar}{2}\vec{\sigma}.$$

It's magnitude is $\mu = \frac{q\hbar}{2m}$, i.e., the value of μ_z in the spin-up state.

Note that it is customary to refer to μ, rather than $\vec{\mu}$, as "the magnetic moment" of the particle.

$$\mu_u = \frac{2}{3}\frac{|e|\hbar}{2m_u}, \quad \mu_d = -\frac{1}{3}\frac{|e|\hbar}{2m_d}, \quad \mu_s = -\frac{1}{3}\frac{|e|\hbar}{2m_s}$$

The magnetic moment of a baryon, B, is defined as

$$\mu_B \equiv \langle B\uparrow|(\mu_1 + \mu_2 + \mu_3)|B\uparrow\rangle$$

For example, the magnetic moment of the proton is

$$\mu_p = \langle P: \tfrac{1}{2}\,\tfrac{1}{2}|(\mu_1+\mu_2+\mu_3)|P: \tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$= \left(\frac{1}{3\sqrt{2}}\right)^2 \{ 4(2\mu_u - \mu_d) + \mu_d + \mu_d + 4(2\mu_u - \mu_d) + \mu_d + \mu_d + 4(2\mu_u - \mu_d) + \mu_d + \mu_d \}$$

$$= \frac{1}{18} \times 3 \times (8\mu_u - 2\mu_d)$$

$$= \frac{1}{3}(4\mu_u - \mu_d)$$

using $m_u = m_d = 336\,\mathrm{MeV}$, $m_p = 938.27\,\mathrm{MeV}$

$$\Rightarrow \mu_u = \frac{2}{3}\frac{|e|\hbar}{2m_p}\frac{m_p}{m_u} = \frac{2}{3}\frac{m_p}{m_u}\mu_N, \quad \text{where } \mu_N \equiv \frac{|e|\hbar}{2m_p}$$

$$= 1.862\,\mu_N$$

$$\mu_d = -\frac{1}{3}\frac{|e|\hbar}{2m_d} = -\frac{1}{3}\frac{|e|\hbar}{2m_p}\frac{m_p}{m_d} = -0.931\,\mu_N$$

$\Rightarrow \mu_p \simeq 2.79\,\mu_N$, which is in very good agreement with the experimental value $2.793\,\mu_N$

Also, it can be calculated that $\frac{\mu_n}{\mu_p} = -\frac{2}{3}$, which is again consistent with the experimental result, (-0.685).

On the other hand, if there is no color wave-function, i.e., use

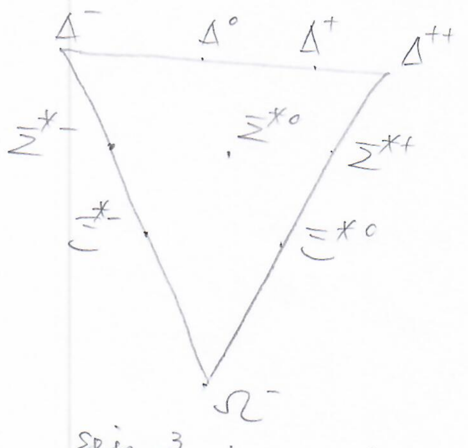
$$\psi = \frac{\sqrt{2}}{3\sqrt{3}}\Big[\psi_{12}(\text{spin})\big(\psi_{31}(\text{flavor}) + \psi_{32}(\text{flavor})\big) + \psi_{23}(s)\big(\psi_{12}(f) + \psi_{13}(f)\big) + \psi_{31}(s)\big(\psi_{23}(f) + \psi_{21}(f)\big)\Big],$$

then $\mu_p = \mu_d < 0$, not consistent with experiment (should be > 0)

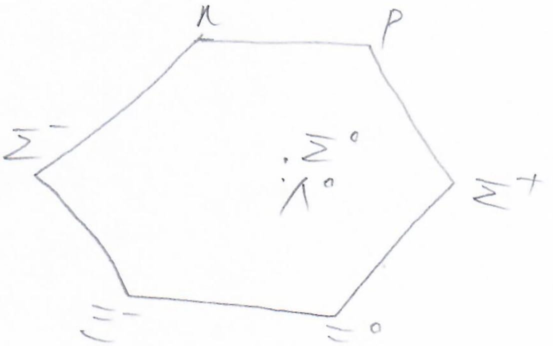
and $\mu_n = \mu_u > 0$, - - - - - - - - - - (- - - < 0)

and $\frac{\mu_n}{\mu_p} = \frac{\mu_u}{\mu_d} = -2$, - - - - - - - - - (should be -0.685)

This is another motivation to introduce color.



spin $\frac{3}{2}$ baryon decuplet



spin $\frac{1}{2}$ baryon octet

With the color wavefunction, from the baryon octet and decuplet wave function, we can calculate the magnetic moments as the following. (Note that for the decuplet, take the spin $|\frac{3}{2}\ \frac{3}{2}\rangle$ state)

Using $m_u = m_d = 336$ MeV, $m_s = 538$ MeV, $m_p = 938.27$ MeV

$\Rightarrow \mu_u = 1.862\mu_N$, $\mu_d = -0.931\mu_N$, $\mu_s = -0.581\mu_N$

Baryon	magnetic moment	prediction	experiment
p	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	2.79	2.793
n	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	−1.86	−1.913
Λ	μ_s	−0.58	−0.613
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.68	2.458
Σ^0	$\frac{2}{3}\mu_u + \frac{2}{3}\mu_d - \frac{1}{3}\mu_s$	0.81	—
Σ^-	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	−1.05	−1.160
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	−1.40	−1.250
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	−0.46	−0.651
Δ^{++}	$3\mu_u$	5.59	3.7~7.5
Δ^+	$2\mu_u + \mu_d$	2.79	—
Δ^0	$\mu_u + 2\mu_d$	0	—
Δ^-	$3\mu_d$	−2.79	—
Σ^{*+}	$2\mu_u + \mu_s$	3.14	—
Σ^{*0}	$\mu_u + \mu_d + \mu_s$	0.35	—
Σ^{*-}	$2\mu_d + \mu_s$	−2.44	—
Ξ^{*0}	$\mu_u + 2\mu_s$	0.70	—
Ξ^{*-}	$\mu_d + 2\mu_s$	−2.09	—
Ω^-	$3\mu_s$	−1.74	−2.2~−1.6

So, reasonably good.

6.3

baryon octet and decuplet masses

The constituent quark model gives the following mass formula, considering that the different spin configuration is responsible to the mass difference of e.g. Nucleon and Δ's :

$$M(\text{baryon}) = m_1 + m_2 + m_3 + A'\left[\frac{\vec{S}_1\cdot\vec{S}_2}{m_1 m_2} + \frac{\vec{S}_1\cdot\vec{S}_3}{m_1 m_3} + \frac{\vec{S}_2\cdot\vec{S}_3}{m_2 m_3}\right]$$

where A' is the fitting parameter to be adjusted to obtain optimal fit to the data; m_1, m_2 and m_3 are constituent quark masses, and we take $m_u = m_d$.

From $\vec{J}^2 = (\vec{S}_1+\vec{S}_2+\vec{S}_3)^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_1\cdot\vec{S}_3)$

$$\Rightarrow \quad \vec{S}_1\cdot\vec{S}_2 + \vec{S}_1\cdot\vec{S}_3 + \vec{S}_2\cdot\vec{S}_3 = \frac{\hbar^2}{2}\left[j(j+1) - \frac{9}{4}\right]$$

$$= \begin{cases} \frac{3}{4}\hbar^2, & \text{for } j=\frac{3}{2} \text{ (decuplet)} \\ -\frac{3}{4}\hbar^2, & \text{for } j=\frac{1}{2} \text{ (octet)} \end{cases}$$

$$\Rightarrow M_N = 3m_u + A'\left[\frac{1}{m_u m_u}\underbrace{(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_1\cdot\vec{S}_3 + \vec{S}_2\cdot\vec{S}_3)}_{\text{use } j=\frac{1}{2}}\right]$$

$$= 3m_u + A'\left(-\frac{3}{4}\hbar^2\right)\frac{1}{m_u^2}$$

$$m_\Delta = 3m_u + A'\left[\frac{1}{m_u m_u}\underbrace{(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_1\cdot\vec{S}_3 + \vec{S}_2\cdot\vec{S}_3)}_{\text{use } j=\frac{3}{2}}\right]$$

$$= 3m_u + A'\left(\frac{3}{4}\hbar^2\right)\frac{1}{m_u^2}$$

$$m_{\Omega^-} = 3m_s + \frac{3}{4}\hbar^2\frac{1}{m_s^2}A'$$

For baryons contain u (or d) and s quarks, we cannot pull out all the denomenator $m_1 m_2$, $m_2 m_3$ and $m_1 m_3$ out of the [ ] since they are different, but we can use the fact that for the decuplet the spins are all parallel, that is, every pair combines to make spin 1, so

$$(\vec{S}_1+\vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2 = 1\times(1+1)\hbar^2 = 2\hbar^2 \Rightarrow \vec{S}_1\cdot\vec{S}_2 = \frac{\hbar^2}{2}\left(2 - 2\times\frac{1}{2}\times\frac{3}{2}\right) = \frac{\hbar^2}{4}$$

Similarly, $\vec{S}_1\cdot\vec{S}_3 = \vec{S}_2\cdot\vec{S}_3 = \frac{\hbar^2}{4}$

Therefore, $M_{\Sigma^*} = 2m_u + m_s + \frac{\hbar^2}{4} A' \left(\frac{1}{m_u^2} + \frac{2}{m_u m_s} \right)$

$$M_{\Xi^*} = m_u + 2m_s + \frac{\hbar^2}{4} A' \left(\frac{2}{m_u m_s} + \frac{1}{m_s^2} \right)$$

While for the rest of octet particles, we can use the following facts: Σ's are isospin triplet, so under the exchange of u and d in the flavor part of the wave function for Σ^0 (while for Σ^+ it has two u, so u switch with the other u is of course do nothing, that is, the flavor wave function is automatically symmetric with $u \leftrightarrow u$; and for Σ^-, $d \leftrightarrow d$ is automatically symmetric for the flavor wave function) should be symmetric, and it is (checked to be so); on the other hand, Λ is isospin singlet, and under the exchange of u and d the flavor part of wave function is antisymmetric (checked to be so). Since the flavor part wave function times the spin part of the wave function should be symmetric under the exchange of any two particles, then we conclude that for the Σ's, the $\underbrace{\vec{S}_u \cdot \vec{S}_u}_{\text{for } \Sigma^+}$, $\underbrace{\vec{S}_d \cdot \vec{S}_d}_{\text{for } \Sigma^-}$ and $\underbrace{\vec{S}_u \cdot \vec{S}_d}_{\text{for } \Sigma^0}$ all equals $\frac{\hbar^2}{2}\left[1 \times (1+1) - 2 \times \frac{1}{2} \times (\frac{1}{2}+1)\right] = \frac{\hbar^2}{4}$

while $\underbrace{\vec{S}_u \cdot \vec{S}_d}_{\text{for } \Lambda}$ equals $\frac{\hbar^2}{2}\left[0 \times (0+1) - 2 \times \frac{1}{2} \times (\frac{1}{2}+1)\right] = -\frac{3}{4}\hbar^2$

$$\Rightarrow m_\Sigma = 2m_u + m_s + A'\left[\frac{\hbar^2}{4}\frac{1}{m_u^2} + \frac{\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3 - \frac{1}{4}\hbar^2}{m_u m_s}\right]$$

$$= 2m_u + m_s + A'\left[\frac{\hbar^2}{4}\frac{1}{m_u^2} + \frac{-\frac{3}{4}\hbar^2 - \frac{1}{4}\hbar^2}{m_u m_s}\right]$$

$$= 2m_u + m_s + \frac{\hbar^2}{4} A' \left(\frac{1}{m_u^2} - \frac{4}{m_u m_s} \right)$$

$$m_\Lambda = 2m_u + m_s + A'\left[-\frac{3\hbar^2}{4}\frac{1}{m_u^2} + \frac{\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_1 \cdot \vec{S}_3 - (-\frac{3}{4}\hbar^2)}{m_u m_s}\right]$$

$$= 2m_u + m_s + A'\left(-\frac{3\hbar^2}{4}\frac{1}{m_u^2} + \frac{-\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2}{m_u m_s}\right]$$

$$= 2m_u + m_s - \frac{3\hbar^2}{4 m_u^2} A'$$

For Ξ^0 and Ξ^-, their flavor part of the wave functions are symmetric under the exchange of the two s's (of course, since $s \leftrightarrow s$ changes nothing), and therefore the spin part of the wave function should be symmetric under the exchange of the spins of the two s's, then

$$\vec{S}_s \cdot \vec{S}_s = \frac{\hbar^2}{2}\left[1\times(1+1) - 2\times\frac{1}{2}\times\left(\frac{1}{2}+1\right)\right] = \frac{\hbar^2}{4}$$

$$\Rightarrow M_\Xi = 2m_s + m_u + A'\left[\frac{\frac{\hbar^2}{4}}{m_s^2} + \frac{\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_3\cdot\vec{S}_1 - \vec{S}_s\cdot\vec{S}_s}{m_u m_s}\right]$$

$$= 2m_s + m_u + A'\left[\frac{\frac{\hbar^2}{4}}{m_s^2} + \frac{-\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4}}{m_u m_s}\right]$$

$$= 2m_s + m_u + \frac{\hbar^2}{4}A'\left[\frac{1}{m_s^2} - \frac{4}{m_u m_s}\right]$$

Using $m_u = m_d = 363$ MeV, $m_s = 538$ MeV, $A' = (2m_u/\hbar)^2 \times 50$ MeV, we get

	Calculated (MeV)	observed (MeV)
N	939	939
Λ	1114	1116
Σ	1179	1193
Ξ	1327	1318
Δ	1239	1232
Σ^*	1381	1385
Ξ^*	1529	1533
Ω	1682	1672

This is another achievement of the constituent quark model!

However, note that the input quark masses are different from the ones for calculating the magnetic moments.

This is a drawback of the constituent quark model.