Solution

1. $P = (-1)^{L+1}$ , $C = (-1)^{L+S}$

$S = \frac{1}{2} + \frac{1}{2} = 1$ or $S = \frac{1}{2} - \frac{1}{2} = 0$.

$\Rightarrow J = \begin{cases} L+1 \\ L \\ L-1 \end{cases}$ (for $S=1$) $\quad J = L$ (for $S=0$)

$J \geq 0$ , $L \geq 0$

Therefore, we can list all possible states for $J = 0, 1, 2$ as

| $J$ | $L$ | $S$ | $J^{PC}$ |
|---|---|---|---|
| 0 | 0 | 0 | $0^{-+}$ |
| 0 | 1 | 1 | $0^{++}$ |
| 1 | 1 | 0 | $1^{+-}$ |
| 1 | 0 | 1 | $1^{--}$ |
| 1 | 1 | 1 | $1^{++}$ |
| 1 | 2 | 1 | $1^{--}$ |
| 2 | 2 | 0 | $2^{-+}$ |
| 2 | 1 | 1 | $2^{++}$ |
| 2 | 2 | 1 | $2^{--}$ |
| 2 | 3 | 1 | $2^{++}$ |

So, the states that cannot be realized as a fermion-antifermion system are $\boxed{0^{+-}, 0^{--}, 1^{-+} \text{ and } 2^{+-}}$

## solution

2.1) for spin 1 $\boxed{\hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$,

check: $\hat{S}_z\begin{pmatrix}1\\0\\0\end{pmatrix} = \hbar\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \hat{S}_z\begin{pmatrix}0\\1\\0\end{pmatrix} = 0, \quad \hat{S}_z\begin{pmatrix}0\\0\\1\end{pmatrix} = -\hbar\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\hat{S}_\pm |s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\ |s\ m\pm1\rangle$$

$$\hat{S}_+\begin{pmatrix}1\\0\\0\end{pmatrix} = 0, \quad \hat{S}_+\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \hat{S}_+\begin{pmatrix}0\\0\\1\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\Rightarrow \quad \hat{S}_{+i1} = 0$ for $i = 1,2,3$, $\quad \hat{S}_{+12} = \sqrt{2}\hbar$, $\hat{S}_{+22} = 0$, $\hat{S}_{+32} = 0$, $\quad \hat{S}_{13} = 0$, $\hat{S}_{23} = \sqrt{2}\hbar$, $\hat{S}_{33} = 0$

$$\Rightarrow \quad \hat{S}_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

similarly,

$$\hat{S}_-\begin{pmatrix}1\\0\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \hat{S}_-\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\hat{S}_-\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$$

$\Rightarrow \quad \hat{S}_{-11} = 0$, $\hat{S}_{-21} = \sqrt{2}\hbar$, $\hat{S}_{-31} = 0$, $\quad \hat{S}_{-12} = 0$, $\hat{S}_{-22} = 0$, $\hat{S}_{-32} = \sqrt{2}\hbar$, $\quad \hat{S}_{-i3} = 0$ for $i = 1,2,3$

$$\Rightarrow \quad \hat{S}_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \hat{S}_x &= \frac{\hat{S}_+ + \hat{S}_-}{2} = \frac{\sqrt{2}\hbar}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \hat{S}_y &= \frac{\hat{S}_+ - \hat{S}_-}{2i} = \frac{\sqrt{2}\hbar}{2i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{aligned}}$$

For spin $\frac{3}{2}$, start with the eigenvector of $\hat{S}_z$: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

for $\frac{3}{2}\hbar$, for $\frac{1}{2}\hbar$, for $-\frac{\hbar}{2}$, for $-\frac{3}{2}\hbar$

$$\boxed{\hat{S}_z = \frac{3}{2}\hbar\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{3} & 0 & 0\\ 0 & 0 & -\frac{1}{3} & 0\\ 0 & 0 & 0 & -1\end{pmatrix}}$$

$$\hat{S}_+\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0 \quad\Rightarrow\quad S_{11}=0,\ S_{21}=0,\ S_{31}=0,\ S_{41}=0$$

$$\hat{S}_+\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\frac{3}{2}\cdot\frac{1}{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \sqrt{3}\hbar\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Rightarrow S_{12}=\sqrt{3}\hbar,\ S_{22}=0,\ S_{32}=0,\ S_{42}=0$$

$$\hat{S}_+\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 2\hbar\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow S_{13}=0,\ S_{23}=2\hbar,\ S_{33}=0,\ S_{43}=0$$

$$\hat{S}_+\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \sqrt{3}\hbar\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \Rightarrow S_{14}=0,\ S_{24}=0,\ S_{34}=\sqrt{3}\hbar,\ S_{44}=0$$

$$\Rightarrow \hat{S}_+ = \hbar\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\hat{S}_-\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\frac{3}{2}\cdot\frac{1}{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \sqrt{3}\hbar\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow S_{21}=\sqrt{3}\hbar,\ S_{i1}=0,\ i\neq 2$$

$$\hat{S}_-\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\frac{1}{2}\left(-\frac{1}{2}\right)}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 2\hbar\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \Rightarrow S_{32}=2\hbar,\ S_{i2}=0,\ i\neq 3$$

$$\hat{S}_-\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}-\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \sqrt{3}\hbar\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \Rightarrow S_{43}=\sqrt{3}\hbar,\ S_{i3}=0,\ i\neq 4$$

$$\hat{S}_-\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 0 \Rightarrow S_{i4}=0,\ \text{for all } i$$

$$\Rightarrow S_- = \hbar\begin{pmatrix}0 & 0 & 0 & 0\\ \sqrt{3} & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \hat{S}_x &= \frac{\hat{S}_+ + \hat{S}_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \\ \hat{S}_y &= \frac{\hat{S}_+ - \hat{S}_-}{2i} = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} \end{aligned}}$$

2)

For spin 1.

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1\times(1+1)\hbar^2 I_{3\times3}$$

For spin $\frac{3}{2}$

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{15}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{3}{2}\times\left(\frac{3}{2}+1\right)\hbar^2 I_{4\times4}$$

solution

3

$$\pi^- + P: \left|1\ -1\right\rangle \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$\pi^+ + P: \left|1\ \ 1\right\rangle \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle$$

$$K^0 + \Sigma^0: \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \left|1\ \ 0\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$K^+ + \Sigma^-: \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \left|1\ -1\right\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$K^+ + \Sigma^+ \quad |\tfrac{1}{2}\ \tfrac{1}{2}\rangle |1\ 1\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle$

$$M_a = \frac{\sqrt{2}}{3} M_3 - \frac{\sqrt{2}}{3} M_1$$

$$M_b = \frac{1}{3} M_3 + \frac{2}{3} M_1$$

$$M_c = M_3$$

$$\Rightarrow \quad \sigma_a : \sigma_b : \sigma_c = 2|M_3 - M_1|^2 : |M_3 + 2M_1|^2 : 9|M_3|^2.$$

1) If $I = \frac{3}{2}$ channel dominates, that is, if $|M_3| >> |M_1|$

$$\Rightarrow \quad \sigma_a : \sigma_b : \sigma_c = 2 : 1 : 9$$

2) If $I = \frac{1}{2}$ channel dominates, that is, if $|M_1| >> |M_3|$

$$\Rightarrow \quad \sigma_a : \sigma_b : \sigma_c = 2 : 4 : 0.$$

that is $\sigma_c << \sigma_a, \sigma_b$, and $\sigma_a : \sigma_b = 1 : 2$.