QED calculations

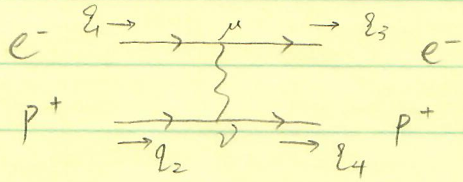
We have studied several QED processes before:
① $e^- p^+ \longrightarrow e^- p^+$, ② $e^+ e^- \longrightarrow \mu^+ \mu^-$, ③ $e^- \gamma \rightarrow e^- \gamma$.

The QED Lagrangian is
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \quad \text{, where}$$
$$D_\mu \psi = \partial_\mu \psi + i|e| q A_\mu \psi \quad \text{, where}$$
$q = -1$ when ψ describes electron-positron field,
$q = +1$ when ---------- proton-antiproton —— ,
$q = +\frac{2}{3}$ ---------- up quark-antiup quark —— , etc...

When there are more than one fermion fields involved, then
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{j=1}^{N} \bar{\psi}_j i \gamma^\mu D_\mu \psi_j - \sum_{j=1}^{N} m_j \bar{\psi}_j \psi_j \quad \text{, where}$$
$$D_\mu \psi_j = \partial_\mu \psi_j + i|e| q_j A_\mu \psi_j$$

page 85 of particle physics I lecture notes

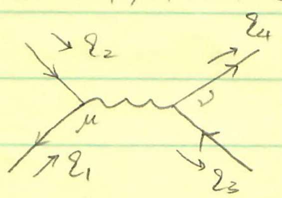
① $e^-(q_1, s_1) + p^+(q_2, s_2) \longrightarrow e^-(q_3, s_3) + p^+(q_4, s_4)$



$$iM_{fi} = \frac{i(-g_{\mu\nu})}{(q_1 - q_3)^2 + i\varepsilon} \left[\bar{u}_e(\vec{q}_3, s_3)(-i|e|(-1))\gamma^\mu u_e(\vec{q}_1, s_1) \right]$$
$$\times \left[\bar{u}_p(\vec{q}_4, s_4)(-i|e|(+1))\gamma^\nu u_p(\vec{q}_2, s_2) \right]$$

page 114

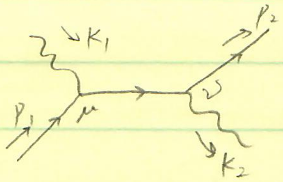
② $e^+(q_1, s_1) + e^-(q_2, s_2) \longrightarrow \mu^+(q_3, s_3) + \mu^-(q_4, s_4)$

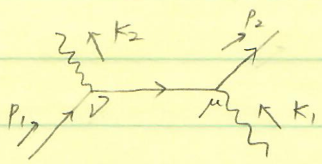


$$iM_{fi} = \left[\bar{v}_e(\vec{q}_1, s_1)(-i|e|(-1))\gamma^\mu u_e(\vec{q}_2, s_2) \right]$$
$$\times \left[\bar{u}_m(\vec{q}_4, s_4)(-i|e|(-1))\gamma^\nu v_m(\vec{q}_3, s_3) \right]$$
$$\times \frac{i(-g_{\mu\nu})}{(q_1 + q_2)^2 + i\varepsilon}$$

page 116

(3) $e^-(p_1, s_1) + \gamma(k_1, r_1) \longrightarrow e^-(p_2, s_2) + \gamma(k_2, r_2)$





$$i\mathcal{M}_{fi} = \bar{u}(\vec{p}_2, s_2)\left(-i|e|(-1)\right)\gamma^\nu \frac{i(\not{p}_1 + \not{k}_1 + m)}{(p_1+k_1)^2 - m^2 + i\varepsilon}$$

$$\cdot\left(-i|e|(-1)\right)\gamma^\mu u(\vec{p}_1, s_1)\ \varepsilon_\mu(\vec{k}_1, r_1)\ \varepsilon_\nu^*(\vec{k}_2, r_2)$$

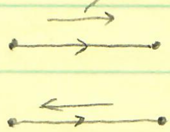
$$+ \bar{u}(\vec{p}_2, s_2)\left(-i|e|(-1)\right)\gamma^\mu \frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1-k_2)^2 - m^2 + i\varepsilon}\left(-i|e|(-1)\right)\gamma^\nu$$

$$\cdot\, u(\vec{p}_1, s_1)\ \varepsilon_\mu(\vec{k}_1, r_1)\ \varepsilon_\nu^*(\vec{k}_2, r_2)$$

$$\equiv i\mathcal{M}_1 + i\mathcal{M}_2$$

For the Feynman rules (not just for QED) about spinors, Nuclear Physics B 387 (1992) 467 – 481, A. Denner et. al., gives a nice method, which can be used for fermion-number-violating interactions as well.

fermion flow. (arbitrarily chosen)



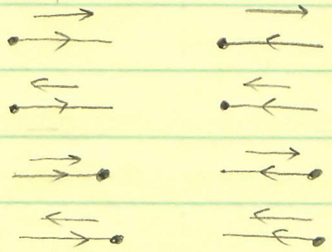
$$iS(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

$$iS(-p) = \frac{i(-\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

(Fig 2. of A. Denner et. al.) Feynman rule for propagator.

The momentum p flows from left to right.

fermion flow (arbitrarily chosen)



$\bar{u}(p, s)$

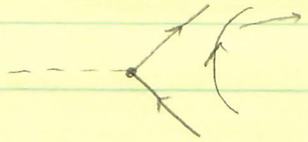
$v(p, s)$

$u(p, s)$

$\bar{v}(p, s)$

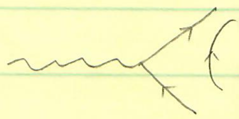
The Feynman rules for external fermion lines. (Fig. 3 of A. Denner et. al.)

The momentum p flows from left to right.



fermion flow (arbitrarily chosen)

$i\Gamma$

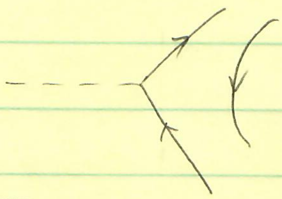


$i\Gamma$

(Fig. 1 of A. Denner et. al.) Feynman rules for fermionic vertex

Method: (i) Draw all possible Feynman diagrams for a given process
(tree-level) (ii) Fix an arbitrary orientation (i.e., fermion flow) for each fermion chain.
(iii) Start at an external leg and write down the appropriat expressions proceeding opposite to the chosen orientation through the chain.
(iv) Multiply by the permutation factor of the external fermion legs with respect to some reference order

Also, in case the following appears, then use

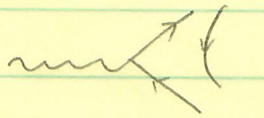


$$i\Gamma' = i C \Gamma^T C^{-1}$$

recall that in the Standard representation, we have used $C = i\gamma^2\gamma^0$

where the charge-conjugation matrix C satisfies $C^\dagger = C^{-1}$, $C^T = -C$, $C\Gamma_i^T C^{-1} = \eta_i \Gamma_i$, where

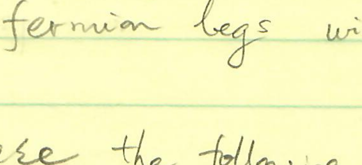
$$\eta_i = \begin{cases} 1, & \text{for } \Gamma_i = 1, i\gamma_5, \gamma_\mu \\ -1, & \text{for } \Gamma_i = \gamma_\mu, \sigma_{\mu\nu} \end{cases}$$



$$i\Gamma' = i C \Gamma^T C^{-1}$$

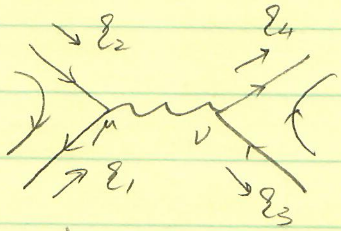
$$u(\vec{p}, s) = C\bar{v}^T(\vec{p}, s), \quad v(\vec{p}, s) = C\bar{u}^T(\vec{p}, s)$$

For example, $e^+(q_1, s_1) + e^-(q_2, s_2) \longrightarrow \mu^+(q_3, s_3) + \mu^-(q_4, s_4)$



$$iM_{fi} = \left[\bar{v}_e(\vec{q}_1, s_1)(-i|e|(-1))\gamma^\mu u(\vec{q}_2, s_2)\right] \cdot \left[\bar{u}_m(\vec{q}_3, s_3)(-i|e|(-1))(-\gamma^\nu) v_m(\vec{q}_4, s_4)\right] \frac{i(-g_{\mu\nu})}{(q_1+q_2)^2 + i\varepsilon}$$

$\equiv$ **

While the expression for  was written as

$$iM_{fi} = \left[\bar{v}_e(\vec{q}_1, s_1)\,(-i|e|(-1))\,\gamma^\mu\, u_e(\vec{q}_2, s_2)\right]$$

$$\cdot \left[\bar{u}_m(\vec{q}_4, s_4)\,(-i|e|(-1))\,\gamma^\nu\, v_m(\vec{q}_3, s_3)\right] \cdot \frac{i(-g_{\mu\nu})}{(q_1+q_2)^2+i\epsilon}$$

$$\equiv ***$$

using $\bar{u}_m(\vec{q}_3, s_3)\,\gamma^\nu\, v_m(\vec{q}_4, s_4)$

$$= \left[\bar{u}_m(q_3, s_3)\,\gamma^\nu\, v_m(\vec{q}_4, s_4)\right]^T$$

$$= v_m^T(\vec{q}_4, s_4)\,\gamma^{\nu T}\,\bar{u}_m^T(q_3, s_3)$$

$$= \bar{u}_m(\vec{q}_4, s_4)\,C^T\,\gamma^{\nu T}\,C^{-1}\,v_m(\vec{q}_3, s_3)$$

use $C\gamma^{\nu T}C^{-1} = -\gamma^\nu$

$$= \bar{u}_m(\vec{q}_4, s_4)\,C^T(-C^{-1}\gamma^\nu)\,v_m(\vec{q}_3, s_3)$$

$$= \bar{u}_m(\vec{q}_4, s_4)\,C C^{-1}\,\gamma^\nu\,v_m(\vec{q}_3, s_3)$$

$$= \bar{u}_m(\vec{q}_4, s_4)\,\gamma^\nu\,v_m(\vec{q}_3, s_3)$$

$$\Rightarrow ** = \left[\bar{v}_e(\vec{q}_1, s_1)\,(-i|e|(-1))\,\gamma^\mu\, u(\vec{q}_2, s_2)\right]$$

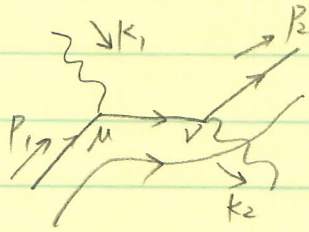
$$\cdot \left[\bar{u}_m(\vec{q}_4, s_4)\,(-i|e|(-1))\,(-\gamma^\nu)\, v_m(\vec{q}_3, s_3)\right] \frac{i(-g_{\mu\nu})}{(q_1+q_2)^2+i\epsilon}$$

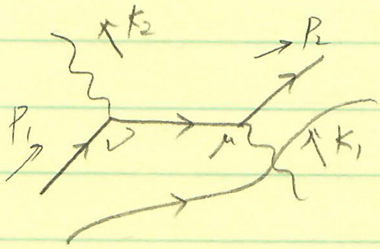
$$= -(***)$$

The "$-$" sign will be compensated by the rule (iv), i.e., for **, it is 1 2 3 4, while for *** it is 1 2 4 3.

Another example,

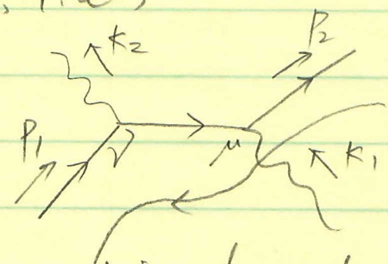
$$e^-(p_1, s_1) + \gamma(k_1, r_1) \longrightarrow e^-(p_2, s_2) + \gamma(k_2, r_2)$$





For these fermion flow orientation, the expression is given before. In particular, since for both diagrams the order of the external fermion legs is 21, the relative sign betw these two diagrams is positive.

If use the opposite orientation for the second diagram, i.e.,



then its matrix element expression is

$$iM_2' = \bar{v}(\vec{p}_1, s_1)(-i|e|(-1))(\underset{\triangle}{-}\gamma^\nu)\frac{i[\underset{\triangle}{-}(\not{p}_1 - \not{k}_2)+m]}{(p_1-k_2)^2 - m^2 + i\epsilon}$$

$$\cdot(-i|e|(-1))(\underset{\triangle}{-}\gamma^\mu)\, v(\vec{p}_2, s_2)$$

$$\cdot\ \epsilon_\mu(\vec{k}_1, r_1)\,\epsilon^*_\nu(\vec{k}_2, r_2)$$

Since $\bar{v}(\vec{p}_1, s_1)(-\gamma^\nu)[-(\not{p}_1 - \not{k}_2)+m](-\gamma^\mu)\, v(\vec{p}_2, s_2)$

$$= [\bar{v}(\vec{p}_1, s_1)\gamma^\nu[-(\not{p}_1 - \not{k}_2)+m]\gamma^\mu v(\vec{p}_2, s_2)]^T$$

$$= v^T(\vec{p}_2, s_2)\gamma^{\mu T}[-(\not{p}_1 - \not{k}_2)+m]^T\gamma^{\nu T}\bar{v}^T(\vec{p}_1, s_1))$$

$$= \bar{u}(\vec{p}_2, s_2)C^T\gamma^{\mu T}\underbrace{\,}_{C^{-1}C}[-(\not{p}_1 - \not{k}_2)+m]^T\underbrace{\,}_{C^{-1}C}\gamma^{\nu T}C^{-1}u(\vec{p}_1, s_1)$$

$$= \bar{u}(\vec{p}_2, s_2)(-1)C\gamma^{\mu T}C^{-1}C[-(\not{p}_1 - \not{k}_2)+m]^T C^{-1}C\gamma^{\nu T}C^{-1}u(\vec{p}_1, s_1)$$

$$= \bar{u}(\vec{p}_2, s_2)(-1)(-\gamma^\mu)[(\not{p}_1 - \not{k}_2) + m](-\gamma^\nu)\, u(\vec{p}_1, s_1)$$

$$= -\bar{u}(\vec{p}_2, s_2)\gamma^\mu[(\not{p}_1 - \not{k}_2) + m]\gamma^\nu u(\vec{p}_1, s_1)$$

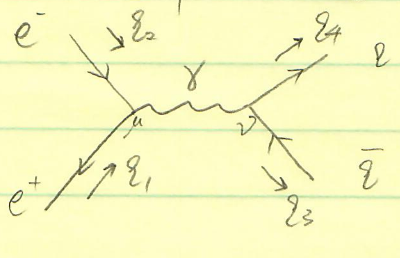
This minus sign is compensated by the order of the external legs relative for the one of $i\mathcal{M}_1$, that is,

$$i\mathcal{M}_{fi} = i\mathcal{M}_1 - i\mathcal{M}_2'$$

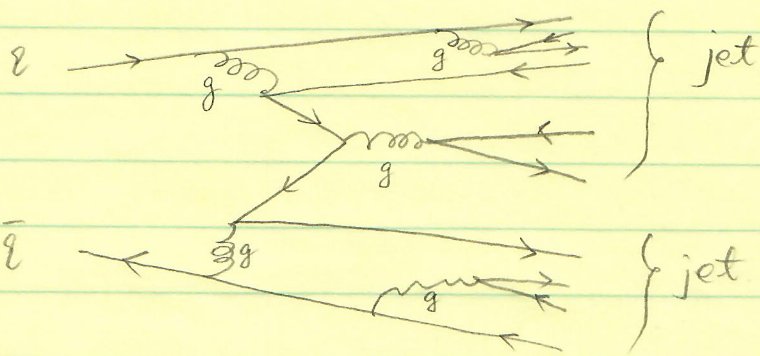
$$= i\mathcal{M}_1 + i\mathcal{M}_2$$

Another motivation to consider color

Consider the process $e^+ + e^- \xrightarrow{\gamma^*} q + \bar{q}$



In fact, what we observe in the laboratory are hadrons rather than free quark & antiquark. This is achieved by a process called hadronization.



In calculating $e^+ e^- \to$ hadrons, for not too large $\sqrt{s}$ ($\sqrt{s} << m_Z$), the cross section is given by $e^+ + e^- \xrightarrow{\gamma^*} q + \bar{q}$. The result is the same as $e^+ e^- \xrightarrow{\gamma^*} \mu^+ \mu^-$ apart from that we substitute the charge of muon (i.e., -1) to the charge of quark (i.e., $\frac{2}{3}$ for up-type and $(-\frac{1}{3})$ for down-type).

page 115-6 $$\bar{\sigma}_{cm} = \frac{4\pi\alpha^2}{3s} Q^2 \sqrt{\frac{1-\frac{4M^2}{s}}{1-\frac{4m^2}{s}}} \left[1 + \frac{2(m^2+M^2)}{s} + \frac{4m^2M^2}{s^2} \right]$$

spin averaged cross section in the center of momentum frame

where Q is the quark charge, M is the quark mass, and m is the electron mass.

$$\left(= \left(1 + \frac{1}{2}\frac{m^2}{\left(\frac{s}{4}\right)}\right)\left(1 + \frac{1}{2}\frac{M^2}{\left(\frac{s}{4}\right)}\right) \right)$$

For $s >> m^2$ & M^2,

$$\Rightarrow \bar{\sigma}_{cm} \simeq \frac{4\pi\alpha^2}{3s} Q^2$$

Let's examine the ratio of the rate of hadron production to that for muon pairs:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

In the pre-mentioned limit ($s >> m^2, M^2$),

$R(s) \simeq \sum_i Q_i^2$, where i for each type of possible quark-antiquark pair above the production threshould.

Note that in fact we just need $\sqrt{\frac{s}{4}}$ be modestly larger than M to make the approximation $R(s) \simeq \sum Q_i^2$ a good one, because from the exact expression, we have

$$R = \sum_i \left(1 - \frac{M_i^2}{(\frac{s}{4})}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2}\frac{M_i^2}{(\frac{s}{4})}\right) Q_i^2 \Big/ \left[\left(1 - \frac{m_\mu^2}{\frac{s}{4}}\right)^{\frac{1}{2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{(\frac{s}{4})}\right)\right]$$

$$= \sum_i Q_i^2 \left[1 - \frac{3}{8}\frac{M_i^4 - m_\mu^4}{(\frac{s}{4})^2} + O\left(\left(\frac{m_\mu^2}{(\frac{s}{4})}\right)^3, \left(\frac{M_i^2}{(\frac{s}{4})}\right)^3\right)\right]$$

so the error compared to $\sum_i Q_i^2$ is of order $\left(\frac{M}{\sqrt{\frac{s}{4}}}\right)^4$.

At low energy where only u, d and s quark contribute, we expect.

$$R \simeq \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{2}{3}$$

Between the charm quark threshould and the bottom quark threshould (i.e, $1.4\,\text{GeV} \simeq m_c < \sqrt{\frac{s}{4}} < m_b \simeq 4.2\,\text{GeV}$), we expect

$$R \simeq \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{10}{9}$$

Above the bottom quark threshould and below m_Z (so that far below top quark threshould), we expect

$$R \simeq \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{11}{9}.$$

We expect that apart from the vicinity of resonace (i.e., those meson bound states such as $\phi = s\bar{s}$, $\psi = c\bar{c}$, $\Upsilon = b\bar{b}$), and substract the contribution of the hadrons from the produced tau-antitau which decay into hadrons, the $R = \frac{2}{3}, \frac{10}{9}, \frac{11}{9}$ obtained above should agree with experimental data. However,

it is <u>not</u> unless we multiple R by a factor of 3.
This 3, counts the number of colors.
This is a compelling experimental evidence for the color hypothesis.