Bilinear combinations of fermion fields under P, C, and T

By imposing that the free fermion Lagrangian is invariant under seperate transformations of P, C, and T, we derive how the bilinear combinations of fermion fields transform under P, C, and T.

1. Parity

$x \to \tilde{x}$: means $t \to t$, $\vec{x} \to -\vec{x}$.

$$\mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x)$$

From $\psi(x) \longrightarrow \eta A \psi(\tilde{x})$, where η is a phase ($|\eta|^2 = 1$), and A is a 4×4 matrix independent of space-time coordinate.

$$\Rightarrow \psi^+(x) \longrightarrow \eta^* \psi^+(\tilde{x}) A^+$$

$$\Rightarrow i\,\eta^*\,\psi^+(\tilde{x}) A^+ \gamma^0 \gamma^\mu \partial_\mu(\eta A \psi(\tilde{x})) - m\,\eta^* \psi^+(\tilde{x}) A^+ \gamma^0 \eta A \psi(\tilde{x})$$

$$\overset{\text{require}}{\equiv} i\bar{\psi}(\tilde{x})\gamma^\mu \tilde{\partial}_\mu \psi(\tilde{x}) - m\bar{\psi}(\tilde{x})\psi(\tilde{x})$$

note that since the action integrates over all $\vec{x}$, we can change $\vec{x} \to -\vec{x}$ and the action will be invariant.

$$\Rightarrow \begin{cases} \gamma^0 A^+ \gamma^0 \gamma^\mu A = \tilde{\gamma}^\mu = \begin{cases} \gamma^0, & \mu = 0 \\ -\gamma^i, & \mu = 1,2,3 \end{cases} & ① \\ \gamma^0 A^+ \gamma^0 A = 1 \end{cases}$$

From ①, $\begin{cases} \gamma^0 A^+ \gamma^0 \gamma^0 A = \gamma^0 & ② \\ \gamma^0 A^+ \gamma^0 \gamma^i A = -\gamma^i & ③ \end{cases}$ $\Rightarrow A^+ A = 1 \overset{\text{right times } A^+}{\Rightarrow} A^+ A A^+ = A^+$

$\Downarrow$ left times $(A^+)^{-1}$

$\underbrace{(A^+)^{-1} A^+}_{=1} A A^+ = (A^+)^{-1} A^+$

$\Downarrow$

$AA^+ = 1$

So, A is a unitary matrix.

From ② $\overset{\text{left times } \gamma^0}{\Rightarrow} A^+ \gamma^0 A = \gamma^0$

From ③ $\Rightarrow \gamma^0 A^+ \gamma^0 \underbrace{A A^+}_{\text{use } A \text{ is unitary.}} \gamma^i A = -\gamma^i \Rightarrow A^+ \gamma^i A = -\gamma^i$

$$\Rightarrow \boxed{A^+ \gamma^\mu A = \tilde{\gamma}^\mu = \gamma_\mu}$$

① (checked)

Therefore, $\bar{\psi}_1(x)\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) \psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\gamma^\mu\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 \underbrace{\ }_{AA^\dagger} \gamma^\mu A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) \gamma^0 \gamma_\mu \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}(\tilde{x}) \gamma_\mu \psi_2(\tilde{x})$$

$\bar{\psi}_1(x) i\gamma_5 \psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 \underbrace{\ }_{AA^\dagger} i\gamma_5 A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) A^\dagger i\gamma_5 A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) A^\dagger i\, i\gamma^0 \underbrace{\ }_{AA^\dagger} \gamma^1 \underbrace{\ }_{AA^\dagger} \gamma^2 \underbrace{\ }_{AA^\dagger} \gamma^3 A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\, i^2 \gamma^0 \gamma_1 \gamma_2 \gamma_3 \psi_2(\tilde{x})$$

$$= -\eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x})\, i\gamma_5 \psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\gamma^\mu\gamma_5\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 \underbrace{\ }_{AA^\dagger} \gamma^\mu \overbrace{\ }^{AA^\dagger} \gamma_5 A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) \gamma_\mu (-\gamma_5) \psi_2(\tilde{x})$$

$$= -\eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) \gamma_\mu \gamma_5 \psi_2(\tilde{x})$$

$\bar{\psi}_1(x)\sigma^{\mu\nu}\psi_2(x)$

$$\to \eta_1^*\eta_2\, \psi_1^\dagger(\tilde{x}) A^\dagger \gamma^0 \overbrace{\ }^{AA^\dagger} \frac{i}{2}(\gamma^\mu \overbrace{\ }^{AA^\dagger} \gamma^\nu - \gamma^\nu \overbrace{\ }^{AA^\dagger} \gamma^\mu) A \psi_2(\tilde{x})$$

$$= \eta_1^*\eta_2\, \bar{\psi}_1(\tilde{x}) \sigma_{\mu\nu} \psi_2(\tilde{x})$$

© (checked)

2. Charge conjugation

since $\mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x)$

$$= \partial_\mu[i\bar{\psi}(x)\gamma^\mu\psi(x)] - i(\partial_\mu\bar{\psi}(x))\gamma^\mu\psi(x) - m\bar{\psi}(x)\psi(x)$$

then it is OK for $\mathcal{L}'(x)$ takes the form $-i(\partial_\mu\bar{\psi}(x))\gamma^\mu\psi(x) - m\bar{\psi}(x)\psi(x)$ to fulfill the invariance of charge conjugation

Also, since charge conjugation change particle to antiparticle, ψ should transform to ψ^*; considering that $\bar{\psi}$ is more frequently used, the transformation rule is

$$\psi(x) \rightarrow \varepsilon B\bar{\psi}^T(x)$$, where ε is a phase, $|\varepsilon|^2 = 1$, B is a 4x4 matrix independent of spacetime

$$\Rightarrow \psi^\dagger(x) \rightarrow \varepsilon^*(B\bar{\psi}^T(x))^\dagger = \varepsilon^*\bar{\psi}^*(x)B^\dagger = \varepsilon^*(\psi^\dagger(x)\gamma^0)^*B^\dagger$$

$$= \varepsilon^*\psi^T(x)\gamma^{0*}B^\dagger$$

$$= \varepsilon^*\psi^T(x)\gamma^{0T}B^\dagger.$$

$$\Rightarrow i\,\varepsilon^*\psi^T(x)\gamma^{0T}B^\dagger\gamma^0\gamma^\mu\partial_\mu(\varepsilon B\bar{\psi}^T(x)) - m\varepsilon^*\psi^T(x)\gamma^{0T}B^\dagger\gamma^0\varepsilon B\bar{\psi}^T(x)$$

$$= i\psi^T(x)\gamma^{0T}B^\dagger\gamma^0\gamma^\mu B\,\partial_\mu\bar{\psi}^T(x) - m\,\psi^T(x)\gamma^{0T}B^\dagger\gamma^0 B\bar{\psi}^T(x)$$

$$= i\left(\psi^T(x)\gamma^{0T}B^\dagger\gamma^0\gamma^\mu B\,\partial_\mu\bar{\psi}^T(x)\right)^T - m\left(\psi^T(x)\gamma^{0T}B^\dagger\gamma^0 B\bar{\psi}^T(x)\right)^T$$

$$= -i(\partial_\mu\bar{\psi}(x))B^T\gamma^{\mu T}\gamma^{0T}B^*\gamma^0\psi(x) + m\bar{\psi}(x)B^T\gamma^{0T}B^*\gamma^0\psi(x)$$

due to exchange the position of two fermion operator

$$\overset{\text{require}}{\equiv} -i(\partial_\mu\bar{\psi}(x))\gamma^\mu\psi(x) - m\bar{\psi}(x)\psi(x)$$

right times γ^0 on both sides

$$\Rightarrow \begin{cases} B^T\gamma^{0T}B^*\gamma^0 = -1 \Rightarrow B^T\gamma^{0T}B^* = -\gamma^0 \Rightarrow B^\dagger\gamma^{0\dagger}B = -\gamma^{0*} & \cdots (1) \\ B^T\gamma^{\mu T}\gamma^{0T}B^*\gamma^0 = \gamma^\mu & \cdots (2) \end{cases}$$

do complex conjugate

use $\begin{cases} \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger} \\ \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \end{cases} \Rightarrow \gamma^0 = \gamma^{0\dagger},\ \gamma^{i\dagger} = -\gamma^i \Rightarrow \gamma^{0*} = \gamma^{0T},\ \gamma^{i*} = -\gamma^{iT}$

from (1) $\Rightarrow B^\dagger\gamma^0 B = -\gamma^{0T}$, from (2) $\Rightarrow \begin{cases} B^T\gamma^{0T}\gamma^{0T}B^*\gamma^0 = \gamma^0 \Rightarrow B^TB^* = 1 \Rightarrow B^\dagger B = 1, \text{ so } B \text{ is a unitary matrix} \\ B^T\gamma^{iT}\gamma^{0T}B^*\gamma^0 = \gamma^i \end{cases}$

(2), (checked)

From $B^T \gamma^{iT} \gamma^{0T} B^* \gamma^0 = \gamma^i$

$$\Rightarrow B^\dagger \gamma^{i\dagger} \gamma^{0\dagger} B \gamma^{0*} = \gamma^{i*} \Rightarrow -B^\dagger \gamma^i \underbrace{\,}_{BB^\dagger} \gamma^0 B \gamma^{0T} = -\gamma^{iT} \Rightarrow B^\dagger \gamma^i B (-\gamma^{0T}) \gamma^{0T} = \gamma^{iT}$$

$\uparrow$ do complex conjugate

$$\Rightarrow B^\dagger \gamma^i B = -\gamma^{iT}$$

$$\Rightarrow \boxed{B^\dagger \gamma^\mu B = -\gamma^{\mu T}}$$

Therefore, $\bar\psi_1(x)\psi_2(x)$

$$\to \varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{0T} B^\dagger \gamma^0 B \bar\psi_2^T(x) = \varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{0T}(-\gamma^{0T}) \bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 \psi_1^T(x) \bar\psi_2^T(x) = -\varepsilon_1^*\varepsilon_2 (\psi_1^T(x)\, \bar\psi_2^T(x))^T$$

$$= \varepsilon_1^*\varepsilon_2 \bar\psi_2(x) \psi_1(x)$$

$\bar\psi_1(x)\gamma^\mu\psi_2(x)$

$$\to \varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{0T} B^\dagger \gamma^0 \underbrace{\,}_{BB^\dagger} \gamma^\mu B \bar\psi_2^T(x) = \varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{0T}(-\gamma^{0T})(-\gamma^{\mu T}) \bar\psi_2^T(x)$$

$$= \varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{\mu T} \bar\psi_2^T(x) = \varepsilon_1^*\varepsilon_2 (\psi_1^T(x) \gamma^{\mu T} \bar\psi_2^T(x))^T$$

$$= -\varepsilon_1^*\varepsilon_2 \bar\psi_2(x) \gamma^\mu \psi_1(x)$$

$\bar\psi_1(x) i\gamma_5 \psi_2(x)$

$$\to \varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{0T} B^\dagger \gamma^0 \underbrace{\,}_{BB^\dagger} i\gamma_5 B \bar\psi_2^T(x) = -\varepsilon_1^*\varepsilon_2 \psi_1^T(x) \gamma^{0T} \gamma^{0T} i B^\dagger \gamma_5 B \bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 \psi_1^T(x)\, i B^\dagger i \gamma^0 \underbrace{\,}_{BB^\dagger} \gamma^1 \underbrace{\,}_{BB^\dagger} \gamma^2 \underbrace{\,}_{BB^\dagger} \gamma^3 B \bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 \psi_1^T(x)\, i^2 \gamma^{0T} \gamma^{1T} \gamma^{2T} \gamma^{3T} \bar\psi_2^T(x)$$

$$= -\varepsilon_1^*\varepsilon_2 (\psi_1^T(x)\, i^2 \gamma^{0T} \gamma^{1T} \gamma^{2T} \gamma^{3T} \bar\psi_2^T(x))^T$$

$$= \varepsilon_1^*\varepsilon_2 \bar\psi_2(x)\, i^2 \gamma^3 \gamma^2 \gamma^1 \gamma^0 \psi_1(x)$$

$$= \varepsilon_1^*\varepsilon_2 \bar\psi_2\, i\gamma_5 \psi_1(x)$$

$$\bar\psi_1(x)\gamma^\mu\gamma_5\psi_2(x)$$

$$\begin{aligned}
&\rightarrow \xi_1^*\xi_2\,\psi_1^T(x)\gamma^{0T}B^\dagger\gamma^0\underbrace{}_{BB^\dagger}\gamma^\mu\overbrace{}^{BB^\dagger}\gamma_5 B\bar\psi_2^T(x)\\
&= \xi_1^*\xi_2\,\psi_1^T(x)\underline{\gamma^{0T}\gamma^{0T}}\gamma^{T\mu}B^\dagger\gamma_5 B\bar\psi_2^T(x)\\
&= \xi_1^*\xi_2\,\psi_1^T(x)\gamma^{T\mu}\, i\gamma^{0T}\gamma^{1T}\gamma^{2T}\gamma^{3T}\bar\psi_2^T(x)\\
&= \xi_1^*\xi_2\left(\psi_1^T(x)\gamma^{T\mu}\, i\gamma^{0T}\gamma^{1T}\gamma^{2T}\gamma^{3T}\bar\psi_2^T(x)\right)^T\\
&= -\xi_1^*\xi_2\,\bar\psi_2(x)\, i\gamma^3\gamma^2\gamma^1\gamma^0\gamma^\mu\psi_1(x)\\
&= -\xi_1^*\xi_2\,\bar\psi_2(x)\gamma_5\gamma^\mu\psi_1(x)\\
&= \xi_1^*\xi_2\,\bar\psi_2(x)\gamma^\mu\gamma_5\psi_1(x)
\end{aligned}$$

$$\bar\psi_1(x)\sigma^{\mu\nu}\psi_2(x)$$

$$\begin{aligned}
&\rightarrow \xi_1^*\xi_2\,\psi_1^T(x)\gamma^{0T}B^\dagger\gamma^0\sigma^{\mu\nu}B\bar\psi_2^T(x)\\
&= \xi_1^*\xi_2\,\psi_1^T(x)\gamma^{0T}B^\dagger\gamma^0\underbrace{}_{BB^\dagger}\frac{i}{2}(\gamma^\mu\underbrace{}_{BB^\dagger}\gamma^\nu-\gamma^\nu\underbrace{}_{BB^\dagger}\gamma^\mu)B\bar\psi_2^T(x)\\
&= -\xi_1^*\xi_2\,\psi_1^T(x)\underline{\gamma^{0T}\gamma^{0T}}\frac{i}{2}(\gamma^{T\mu}\gamma^{T\nu}-\gamma^{T\nu}\gamma^{T\mu})\bar\psi_2^T(x)\\
&= -\xi_1^*\xi_2\left(\psi_1^T(x)\frac{i}{2}(\gamma^{T\mu}\gamma^{T\nu}-\gamma^{T\nu}\gamma^{T\mu})\bar\psi_2^T(x)\right)^T\\
&= \xi_1^*\xi_2\,\bar\psi_2(x)\frac{i}{2}(\gamma^\nu\gamma^\mu-\gamma^\mu\gamma^\nu)\psi_1(x)\\
&= -\xi_1^*\xi_2\,\bar\psi_2(x)\sigma^{\mu\nu}\psi_1(x)
\end{aligned}$$

3. time inversion. $\gamma^\mu \to \tilde{\gamma}^\mu$ $\begin{cases} t \to -t \\ \vec{x} \to \vec{x} \end{cases}$

Since time inversion exchange initial and final state, that is, it changes a 'bra' to a 'ket', then it involves complex conjugate

From $\psi(x) \longrightarrow \varsigma D \psi(\tilde{x})$, where ς is a phase, $|\varsigma|^2 = 1$, D is a 4x4 matrix independent of spacetime

$\Rightarrow \psi^+(x) \longrightarrow \varsigma^* \psi^+(\tilde{x}) D^+$

$$\Rightarrow \mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x)$$

$$\longrightarrow (-i)\varsigma^* \psi^+(\tilde{x}) D^+ \gamma^{0*} \gamma^{\mu*} \partial_\mu(\varsigma D \psi(\tilde{x}))$$

$$- m\varsigma^* \psi^+(\tilde{x}) D^+ \gamma^{0*} \varsigma D \psi(\tilde{x})$$

$$= -i\bar{\psi}(\tilde{x})\gamma^0 D^+ \gamma^{0*} \gamma^{\mu*} D \partial_\mu \psi(\tilde{x})$$

$$- m\bar{\psi}(\tilde{x})\gamma^0 D^+ \gamma^{0*} D \psi(\tilde{x})$$

note that since the action integrates over t, we can change $t \to -t$ and $\Rightarrow$ the action will be invariant.

$$\overset{\text{require}}{\equiv} i\bar{\psi}(\tilde{x})\gamma^\mu \tilde{\partial}_\mu \psi(\tilde{x}) - m\bar{\psi}(\tilde{x})\psi(\tilde{x})$$

$$\begin{cases} -\gamma^0 D^+ \gamma^{0*} \gamma^{\mu*} D = \tilde{\gamma}^\mu = \begin{cases} -\gamma^0, & \mu = 0 \quad \cdots ① \\ \gamma^i, & \mu = 1,2,3 \quad \cdots ② \end{cases} \\ \gamma^0 D^+ \gamma^{0*} D = 1 \quad \cdots ③ \end{cases}$$

From ③ $\Rightarrow D^+ \gamma^{0*} D = \gamma^0 \Rightarrow D^+ \gamma^{0T} D = \gamma^0$

From ① $\Rightarrow D^+ \gamma^{0*} \gamma^{0*} D = 1 \Rightarrow D^+ D = 1$, so D is unitary matrix.

From ② $\Rightarrow -\gamma^0 D^+ \underbrace{\gamma^{0*}}_{DD^+} \gamma^{i*} D = \gamma^i \Rightarrow -\gamma^0 \gamma^0 D^+ \gamma^{i*} D = \gamma^i$

$$\Rightarrow D^+ \gamma^{iT} D = \gamma^i$$

$$\Rightarrow \boxed{D^+ \gamma^{T\mu} D = \gamma^\mu}$$

$$\Rightarrow \boxed{D^+ \gamma^{*\mu} D = \gamma_\mu}$$

Therefore,

$$\bar\psi_1(x)\psi_2(x)$$

$$\to \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger \underbrace{\gamma^{0*}}_{\gamma^0} D\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\psi_1^\dagger\cancel{(x)}\gamma^0\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\bar\psi_1(\tilde x)\psi_2(\tilde x)$$

$$\bar\psi_1(x)\gamma^\mu\psi_2(x)$$

$$\to \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger\gamma^{0*}\underset{DD^\dagger}{\wedge}\gamma^{\mu*}D\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\psi_1^\dagger(\tilde x)\gamma^0\gamma_\mu\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\bar\psi_1(\tilde x)\gamma_\mu\psi_2(\tilde x)$$

$$\bar\psi_1(x)i\gamma_5\psi_2(x)$$

$$\to \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger\gamma^{0*}(i\gamma_5)^*D\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger\gamma^{0*}\underset{DD^\dagger}{\wedge}(-i)(-i)\gamma^{0*}\underset{DD^\dagger}{\wedge}\gamma^{1*}\underset{DD^\dagger}{\wedge}\gamma^{2*}\underset{DD^\dagger}{\wedge}\gamma^{3*}D\psi_2(\tilde x)$$

$$= -\eta_1^*\eta_2\psi_1^\dagger(\tilde x)\gamma^0\gamma^0\gamma_1\gamma_2\gamma_3\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\bar\psi_1(\tilde x)\gamma^0\gamma^1\gamma^2\gamma^3\psi_2(\tilde x)$$

$$= -\eta_1^*\eta_2\bar\psi_1(\tilde x)i\gamma_5\psi_2(\tilde x)$$

$$\bar\psi_1(x)\gamma^\mu\gamma_5\psi_2(x)$$

$$\to \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger\gamma^{0*}\underset{DD^\dagger}{\wedge}\gamma^{\mu*}\underset{DD^\dagger}{\wedge}\gamma_5^*D\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\psi_1^\dagger(\tilde x)\gamma^0\gamma_\mu(-i)\gamma^0\gamma_1\gamma_2\gamma_3\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\bar\psi_1(\tilde x)\gamma_\mu\gamma_5\psi_2(\tilde x)$$

$$\bar\psi_1\sigma^{\mu\nu}\psi_2(x)$$

$$\to \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger\gamma^{0*}(\sigma^{\mu\nu})^*D\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\psi_1^\dagger(\tilde x)D^\dagger\gamma^{0*}\underset{DD^\dagger}{\wedge}\left(-\frac{i}{2}\right)(\gamma^{\mu*}\underset{DD^\dagger}{\wedge}\gamma^{\nu*}-\gamma^{\nu*}\underset{DD^\dagger}{\wedge}\gamma^{\mu*})D\psi_2(\tilde x)$$

$$= \eta_1^*\eta_2\psi_1^\dagger(\tilde x)\gamma^0\left(-\frac{i}{2}\right)(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\psi_2(\tilde x)$$

$$= -\eta_1^*\eta_2\bar\psi_1(\tilde x)\sigma_{\mu\nu}\psi_2(\tilde x)$$

(checked)

To summarize, up to the phase factor $\eta_1^* \eta_2$, $\xi_1^* \xi_2$, $\zeta_1^* \zeta_2$, we have the following transformation properties for bilinear combinations

	P	C	T	CP	CPT
$\bar\psi_1 \psi_2$	$\bar\psi_1 \psi_2$	$\bar\psi_2 \psi_1$	$\bar\psi_1 \psi_2$	$\bar\psi_2 \psi_1$	$\bar\psi_2 \psi_1$
$\bar\psi_1 \gamma^\mu \psi_2$	$\bar\psi_1 \gamma_\mu \psi_2$	$-\bar\psi_2 \gamma^\mu \psi_1$	$\bar\psi_1 \gamma_\mu \psi_2$	$-\bar\psi_2 \gamma_\mu \psi_1$	$-\bar\psi_2 \gamma^\mu \psi_1$
$\bar\psi_1 i\gamma_5 \psi_2$	$-\bar\psi_1 i\gamma_5 \psi_2$	$\bar\psi_2 i\gamma_5 \psi_1$	$-\bar\psi_1 i\gamma_5 \psi_2$	$-\bar\psi_2 i\gamma_5 \psi_1$	$\bar\psi_2 i\gamma_5 \psi_1$
$\bar\psi_1 \gamma^\mu \gamma_5 \psi_2$	$-\bar\psi_1 \gamma_\mu \gamma_5 \psi_2$	$\bar\psi_2 \gamma^\mu \gamma_5 \psi_1$	$\bar\psi_1 \gamma_\mu \gamma_5 \psi_2$	$-\bar\psi_2 \gamma_\mu \gamma_5 \psi_1$	$-\bar\psi_2 \gamma^\mu \gamma_5 \psi_1$
$\bar\psi_1 \sigma^{\mu\nu} \psi_2$	$\bar\psi_1 \sigma_{\mu\nu} \psi_2$	$-\bar\psi_2 \sigma^{\mu\nu} \psi_1$	$-\bar\psi_1 \sigma_{\mu\nu} \psi_2$	$-\bar\psi_2 \sigma_{\mu\nu} \psi_1$	$\bar\psi_2 \sigma^{\mu\nu} \psi_1$

Note that the argument also changes for P: $\vec{x} \to -\vec{x}$, and T: $t \to -t$, CP: $\vec{x} \to -\vec{x}$, CPT: $t \to -t$, $\vec{x} \to -\vec{x}$.

Recall that for proper homogeneous Lorentz transformation (which does not include P, T and PT), the transformation properties for bilinear combinations are

$$\bar\psi_1(x)\psi_2(x) \longrightarrow \bar\psi_1(x)\psi_2(x)$$

$$\bar\psi_1(x)\gamma^\mu\psi_2(x) \longrightarrow \Lambda^\mu{}_\nu \bar\psi_1(x)\gamma^\nu\psi_2(x)$$

$$\bar\psi_1(x) i\gamma_5\psi_2(x) \longrightarrow \underbrace{\mathrm{Det}(\Lambda)}_{=1} \bar\psi_1(x) i\gamma_5 \psi_2(x) = \bar\psi_1(x) i\gamma_5\psi_2(x)$$

$$\bar\psi_1(x)\gamma^\mu\gamma_5\psi_2(x) \longrightarrow \Lambda^\mu{}_\nu \underbrace{\mathrm{Det}(\Lambda)}_{=1} \bar\psi_1(x)\gamma^\nu\gamma_5\psi_2(x) = \Lambda^\mu{}_\nu \bar\psi_1(x)\gamma^\nu\gamma_5\psi_2(x)$$

$$\bar\psi_1(x)\sigma^{\mu\nu}\psi_2(x) \longrightarrow \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \bar\psi_1(x)\sigma^{\alpha\beta}\psi_2(x)$$

Since electromagnetic interaction conserve C, P and T, then from the invariance of interaction Lagrangian,

$\mathcal{L}_{int} = -\varepsilon|e|\bar{\psi}\gamma^\mu\psi A_\mu$ (where $|e| = \sqrt{4\pi\alpha}$, and $\varepsilon = -1$ when ψ describes the electron-positron field, $\varepsilon = +1$ when ψ describes the proton-antiproton field, $\varepsilon = +\frac{2}{3}$ when ψ describes the up-quark anti-up-quark field, ect), we derive

$$A^\mu(t,\vec{x}) \xrightarrow{P} A_\mu(t,-\vec{x}), \text{ i.e., } \begin{cases} A^0(t,\vec{x}) \xrightarrow{P} A^0(t,-\vec{x}) \\ \vec{A}(t,\vec{x}) \xrightarrow{P} -\vec{A}(t,-\vec{x}) \end{cases}$$

$$A^\mu(t,\vec{x}) \xrightarrow{C} -A^\mu(t,\vec{x})$$

$$A^\mu(t,\vec{x}) \xrightarrow{T} A_\mu(-t,\vec{x}), \text{ i.e., } \begin{cases} A^0(t,\vec{x}) \xrightarrow{T} A^0(-t,\vec{x}) \\ \vec{A}(t,\vec{x}) \xrightarrow{T} -\vec{A}(-t,\vec{x}) \end{cases}$$

From the Maxwell's equations (which is invarance under C, P and T),

$$\vec{\nabla}\cdot\vec{E} = \rho$$
$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

we got

$$\vec{E}(t,\vec{x}) \xrightarrow{P} -\vec{E}(t,-\vec{x})$$
$$\vec{B}(t,\vec{x}) \xrightarrow{P} \vec{B}(t,-\vec{x})$$
$$\rho(t,\vec{x}) \xrightarrow{P} \rho(t,-\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{P} -\vec{j}(t,-\vec{x})$$
$$\vec{\nabla} \xrightarrow{P} -\vec{\nabla}$$
$$\frac{\partial}{\partial t} \xrightarrow{P} \frac{\partial}{\partial t}$$

$$\rho(t,\vec{x}) \xrightarrow{C} -\rho(t,\vec{x})$$
$$\vec{E}(t,\vec{x}) \xrightarrow{C} -\vec{E}(t,\vec{x})$$
$$\vec{B}(t,\vec{x}) \xrightarrow{C} -\vec{B}(t,\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{C} -\vec{j}(t,\vec{x}) \qquad \vec{\nabla} \xrightarrow{C} \vec{\nabla}$$
$$\frac{\partial}{\partial t} \xrightarrow{C} \frac{\partial}{\partial t},$$

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$$\frac{\partial}{\partial t} \xrightarrow{T} -\frac{\partial}{\partial t}$$

$$\vec{\nabla} \xrightarrow{T} \vec{\nabla}$$

$$\rho(t,\vec{x}) \xrightarrow{T} \rho(-t,\vec{x})$$

$$\vec{E}(t,\vec{x}) \xrightarrow{T} \vec{E}(-t,\vec{x})$$

$$\vec{B}(t,\vec{x}) \xrightarrow{T} -\vec{B}(-t,\vec{x})$$

$$\vec{j}(t,\vec{x}) \xrightarrow{T} -\vec{j}(-t,\vec{x})$$

On the other hand, since $F^{i0} = E^i$, $F^{ij} = -\varepsilon^{ijk}B^k$ (where $\varepsilon^{123} = \varepsilon_{123} = 1$, ε^{ijk} is the Levi-Civita symbol)

and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

then $E^i = \partial^i A^0 - \partial^0 A^i \Rightarrow \vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}$

$$-\varepsilon^{ijk}B^k = F^{ij} \Rightarrow -\varepsilon_{ijl}\varepsilon^{ijk}B^k = \varepsilon_{ijl}F^{ij}$$

$$\Rightarrow -2B^l = \varepsilon_{ijl}F^{ij}$$

$$\Rightarrow \begin{aligned} B^1 &= -\tfrac{1}{2}(F^{23} - F^{32}) = -F^{23} = -(\partial^2 A^3 - \partial^3 A^2) \\ B^2 &= -\tfrac{1}{2}(F^{31} - F^{13}) = -F^{31} = -(\partial^3 A^1 - \partial^1 A^3) \\ B^3 &= -\tfrac{1}{2}(F^{12} - F^{21}) = -F^{12} = -(\partial^1 A^2 - \partial^2 A^1) \end{aligned}$$

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}.$$

Therefore, we can derive the transformation properties of $\vec{E}$ and $\vec{B}$ directly from the ones for A^μ, i.e.,

$$\vec{E}(t,\vec{x}) \xrightarrow{P} -\vec{E}(t,-\vec{x})$$

$$\vec{E}(t,\vec{x}) \xrightarrow{C} -\vec{E}(t,\vec{x})$$

$$\vec{E}(t,\vec{x}) \xrightarrow{T} E(-t,\vec{x})$$

$$\vec{B}(t,\vec{x}) \xrightarrow{P} \vec{B}(t,-\vec{x})$$

$$\vec{B}(t,\vec{x}) \xrightarrow{C} -\vec{B}(t,\vec{x})$$

$$\vec{B}(t,\vec{x}) \xrightarrow{T} -\vec{B}(-t,\vec{x})$$

Consistent with the properties derived from Maxwell's equations.

Also, the electric current for a fermion field is $j^\mu = \frac{q}{|e|}\bar{\psi}\gamma^\mu\psi$.

then from $j^\mu = (\rho, \vec{j})$ and the transformation properties for $\bar{\psi}\gamma^\mu\psi$,

we get
$$\rho(t,\vec{x}) \xrightarrow{P} \rho(t,-\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{P} -\vec{j}(t,-\vec{x})$$
$$\rho(t,\vec{x}) \xrightarrow{C} -\rho(t,\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{C} -\vec{j}(t,\vec{x})$$
(note that $\frac{q}{|e|}$ is a pure number, and it is not change under transformations)
$$\rho(t,\vec{x}) \xrightarrow{T} \rho(-t,\vec{x})$$
$$\vec{j}(t,\vec{x}) \xrightarrow{T} -\vec{j}(-t,\vec{x})$$

Consistent with the properties derived from Maxwell's equations.

We can check that the transformation properties of A^μ makes $\mathcal{L}_{EM} \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ invariant under C, P and T:

$$F^{\mu\nu}(t,\vec{x}) = \partial^\mu A^\nu(t,\vec{x}) - \partial^\nu A^\mu(t,\vec{x})$$

$$F^{\mu\nu}(t,\vec{x}) \xrightarrow{P} \partial_\mu A_\nu(t,-\vec{x}) - \partial_\nu A_\mu(t,-\vec{x}) = F_{\mu\nu}(t,-\vec{x})$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x}) \xrightarrow{P} -\frac{1}{4}F_{\mu\nu}(t,-\vec{x})F^{\mu\nu}(t,-\vec{x})$$

Since the action integrates over all $\vec{x}$, we can change $\vec{x} \to -\vec{x}$ and the action will be invariant.

$$F^{\mu\nu}(t,\vec{x}) \xrightarrow{C} -F^{\mu\nu}(t,\vec{x})$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x}) \xrightarrow{C} -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x})$$

$$F^{\mu\nu}(t,\vec{x}) \xrightarrow{T} -\partial_\mu A_\nu(-t,\vec{x}) + \partial_\nu A_\mu(-t,\vec{x}) = -F_{\mu\nu}(-t,\vec{x})$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}(t,\vec{x})F^{\mu\nu}(t,\vec{x}) \xrightarrow{T} -\frac{1}{4}F_{\mu\nu}(-t,\vec{x})F^{\mu\nu}(-t,\vec{x})$$

Since the action integrates over all t, we can change $t \to -t$ and the action will be invariant.

For an arbitrary current obtained by an internal phase transformation of the fermion field, $\psi \to \psi' = e^{-i\delta\alpha}\psi$, where $\delta\alpha$ is a constant,

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

$$Q = \int d^3\vec{x}\, j^0$$

then from the transformation property of $\bar{\psi}\gamma^\mu\psi$, we have, for charge conjugation,

$$j^\mu_{(t,\vec{x})} \xrightarrow{C} -j^\mu(t,\vec{x})$$

$$Q(t) \xrightarrow{C} -Q(t)$$

Therefore, any charge given by this global phase transformation, e.g., electric charge, baryon number, reverses sign by charge conjugation.

For complex scalar field,

$$\phi(t,\vec{x}) \xrightarrow{P} \eta_B\,\phi(t,-\vec{x}) \quad , \quad |\eta_B|^2 = 1$$

$$\phi^\dagger(t,\vec{x}) \xrightarrow{P} \eta_B^*\,\phi^\dagger(t,-\vec{x}) \quad ,$$

$$\phi(t,\vec{x}) \xrightarrow{C} \varepsilon_B\,\phi^\dagger(t,\vec{x}) \quad , \quad |\varepsilon_B|^2 = 1$$

$$\phi^\dagger(t,\vec{x}) \xrightarrow{C} \varepsilon_B^*\,\phi(t,\vec{x})$$

$$\phi(t,\vec{x}) \xrightarrow{T} \varsigma_B\,\phi(-t,\vec{x}) \quad , \quad |\varsigma_B|^2 = 1$$

$$\phi^\dagger(t,\vec{x}) \xrightarrow{T} \varsigma_B^*\,\phi^\dagger(-t,\vec{x})$$

Apparently, the free field Lagrangian (more accurately, the action)

$$\mathcal{L}(t,\vec{x}) = \partial_\mu\phi(t,\vec{x})\,\partial^\mu\phi^\dagger(t,\vec{x}) - m^2\phi(t,\vec{x})\,\phi^\dagger(t,\vec{x})$$

is unchanged under P, C and T.

The scalar QED interaction Lagrangian

$$\mathcal{L}(t,\vec{x}) = -i\,q|e|\left[\phi^\dagger_{(t,\vec{x})}(\partial^\mu\phi_{(t,\vec{x})}) - (\partial^\mu\phi^\dagger_{(t,\vec{x})})\phi_{(t,\vec{x})}\right]A_{\mu\,(t,\vec{x})} + (q|e|)^2 A_{\mu\,(t,\vec{x})}A^\mu_{(t,\vec{x})}\phi^\dagger_{(t,\vec{x})}\phi_{(t,\vec{x})},$$

transforms as

$$\mathcal{L}(t,\vec{x}) \xrightarrow{P} -i\,q|e|\left[\phi^\dagger(t,-\vec{x})(\partial_\mu\phi(t,-\vec{x})) - (\partial_\mu\phi^\dagger(t,-\vec{x}))\phi(t,-\vec{x})\right]A^\mu(t,-\vec{x})$$
$$+ (q|e|)^2 A^\mu(t,-\vec{x})A_\mu(t,-\vec{x})\,\phi^\dagger(t,-\vec{x})\phi(t,-\vec{x})$$
$$= \mathcal{L}(t,-\vec{x})$$

$$\mathcal{L}(t,\vec{x}) \xrightarrow{C} -i\,q|e|\left[\phi(t,\vec{x})(\partial^\mu\phi^\dagger(t,\vec{x})) - (\partial^\mu\phi(t,\vec{x}))\phi^\dagger(t,\vec{x})\right](-A_\mu(t,\vec{x}))$$
$$+ (q|e|)^2(-A_\mu(t,\vec{x}))(-A^\mu(t,\vec{x}))\,\phi(t,\vec{x})\phi^\dagger(t,\vec{x})$$
$$= \mathcal{L}(t,\vec{x})$$

$$\mathcal{L}(t,\vec{x}) \xrightarrow{T} \underbrace{(-i)^*}_{\text{note this}} q|e|\left[\phi^\dagger(-t,\vec{x})(-\partial_\mu\phi(-t,\vec{x})) - (-\partial_\mu\phi^\dagger(-t,\vec{x}))\phi(-t,\vec{x})\right]$$
$$\cdot A^\mu(-t,\vec{x}) + (q|e|)^2 A^\mu(-t,\vec{x})A_\mu(-t,\vec{x})\,\phi^\dagger(-t,\vec{x})\phi(-t,\vec{x})$$
$$= \mathcal{L}(-t,\vec{x})$$

So the action is unchanged under P, C, and T.

For an arbitrary current obtained by an internal phase transformation of the complex scalar field, $\varphi \to \varphi' = e^{-i\delta\alpha}\varphi$, where $\delta\alpha$ is a constant,

$$j^\mu = i[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi]$$
$$Q = \int d^3\vec{x}\; j^0$$

then

$$j^\mu(t,\vec{x}) \xrightarrow{C} -j^\mu(t,\vec{x})$$
$$Q(t) \xrightarrow{C} -Q(t)$$

as expected, and the same interpretation as the fermion field current and charge

 (checked) under C

For a real scaler field,

$$\phi(t,\vec{x}) \xrightarrow{P} \eta_B \phi(t,-\vec{x}) \quad , \quad \eta_B = \pm 1.$$

$$\phi(t,\vec{x}) \xrightarrow{C} \varepsilon_B \phi(t,\vec{x}) \quad , \quad \varepsilon_B = \pm 1$$

$$\phi(t,\vec{x}) \xrightarrow{T} \varphi_B \phi(-t,\vec{x}) \quad , \quad \varphi_B = \pm 1$$

The free field Lagrangian (more accurately, the action).

$$\mathcal{L}(t,\vec{x}) = \frac{1}{2}\partial_\mu \phi(t,\vec{x})\partial^\mu \phi(t,\vec{x}) - \frac{1}{2}m^2\phi^2(t,\vec{x})$$

is unchanged under P, C and T.

Now let's look at the quantized fields (i.e., the particle creation and annihilation operators) behavior under P, C, and T.

To aid analysis, we choose the standard representation of gamma matrices,

$$\gamma^0 = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and in this representation the solutions of Dirac equation are

$$u(\vec{p},s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} \quad , \quad v(\vec{p},s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s \\ \eta_s \end{pmatrix},$$

where $s = \pm\frac{1}{2}$, χ_s and η_s are 2×1 columns and satisfy $\chi_s^+ \chi_{s'} = \delta_{ss'} = \eta_s^+ \eta_{s'}$ and $\sum_s \chi_s \chi_s^+ = \sum_s \eta_s \eta_s^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[1] then for Parity transformation, $A^+ \gamma^\mu A = \gamma_\mu$, we see that $A = \gamma^0$ satisfies this relation, then

from $\psi(t,\vec{x}) \xrightarrow{P} \eta \gamma^0 \psi(t,-\vec{x})$, and the decomposition

$$\psi(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left(u(\vec{p},s) b_{\vec{p},s} e^{-ip\cdot x} + v(\vec{p},s) d^+_{\vec{p},s} e^{ip\cdot x} \right)$$

we have

$$\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u(\vec{p},s) b_{\vec{p},s} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + v(\vec{p},s) d^+_{\vec{p},s} e^{i(E_{\vec{p}}t-\vec{p}\cdot\vec{x})} \right]$$

$$\xrightarrow{P} \underbrace{\eta \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[\gamma^0 u(\vec{p},s) b_{\vec{p},s} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + \gamma^0 v(\vec{p},s) d^+_{\vec{p},s} e^{i(E_{\vec{p}}t+\vec{p}\cdot\vec{x})} \right]}_{\equiv **}$$

using $\gamma^0 u(\vec{p},s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} = u(-\vec{p},s)$

and $\gamma^0 v(\vec{p},s) = (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s \\ -\eta_s \end{pmatrix} = -v(-\vec{p},s)$

$$\Rightarrow ** = \eta \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u(-\vec{p},s) b_{\vec{p},s} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} - v(-\vec{p},s) d^+_{\vec{p},s} e^{i(E_{\vec{p}}t+\vec{p}\cdot\vec{x})} \right]$$

note that $E_{\vec{p}} = E_{-\vec{p}}$

$$\overset{\downarrow}{=} \eta \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u(\vec{p},s) b_{-\vec{p},s} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} - v(\vec{p},s) d^+_{-\vec{p},s} e^{i(E_{\vec{p}}t-\vec{p}\cdot\vec{x})} \right]$$

$$\Rightarrow \quad b_{\vec{p},s} \xrightarrow{P} \eta b_{-\vec{p},s}, \quad d^+_{\vec{p},s} \xrightarrow{P} \eta(-d^+_{-\vec{p},s})$$

Similarly, from $\psi^+(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s (u^+(\vec{p},s) b^+_{\vec{p},s} e^{ip\cdot x} + v^+(\vec{p},s) d_{\vec{p},s} e^{-ip\cdot x})$

and $\psi^+(t,\vec{x}) \xrightarrow{P} \eta^* \psi^+(t,-\vec{x}) A^+ = \eta^* \psi^+(t,-\vec{x})\gamma^0$

we have

$$\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u^+(\vec{p},s) b^+_{\vec{p},s} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + v^+(\vec{p},s) d_{\vec{p},s} e^{-i(E_{\vec{p}}t-\vec{p}\cdot\vec{x})} \right]$$

$$\xrightarrow{P} \underbrace{\eta^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u^+(\vec{p},s)\gamma^0 b^+_{\vec{p},s} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + v^+(\vec{p},s)\gamma^0 d_{\vec{p},s} e^{-i(E_{\vec{p}}t+\vec{p}\cdot\vec{x})} \right]}_{\equiv ***}$$

using $\gamma^0 u(\vec{p},s) = u(-\vec{p},s) \Rightarrow u^+(\vec{p},s)\gamma^0 = u^+(-\vec{p},s)$

$\gamma^0 v(\vec{p},s) = -v(-\vec{p},s) \Rightarrow v^+(\vec{p},s)\gamma^0 = -v^+(-\vec{p},s)$

$$\Rightarrow *** = \eta^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u^+(-\vec{p},s) b^+_{\vec{p},s} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} - v^+(-\vec{p},s) d_{\vec{p},s} e^{-i(E_{\vec{p}}t+\vec{p}\cdot\vec{x})} \right]$$

$$= \eta^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[u^+(\vec{p},s) b^+_{-\vec{p},s} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} - v^+(\vec{p},s) d_{-\vec{p},s} e^{-i(E_{\vec{p}}t-\vec{p}\cdot\vec{x})} \right]$$

$$\Rightarrow \quad b^+_{\vec{p},s} \xrightarrow{P} \eta^* b^+_{-\vec{p},s}, \quad d_{\vec{p},s} \xrightarrow{P} \eta^*(-d_{-\vec{p},s})$$

B (checked)

Moreover, since

$$\psi(t,\vec{x}) \xrightarrow{P} \eta A \psi(t,-\vec{x}) \xrightarrow{P} \eta^2 A^2 \psi(t,\vec{x}),$$

$$\psi^\dagger(t,\vec{x}) \xrightarrow{P} \eta^* \psi^\dagger(t,-\vec{x}) A^\dagger \xrightarrow{P} \eta^{*2} \psi^\dagger(t,\vec{x}) (A^\dagger)^2,$$

then $\eta^2 A^2 = 1$ is imposed, so that $P^2 = +1$

Then from $A = \gamma^0$, $\Rightarrow A^2 = 1 \Rightarrow \eta^2 = 1 \Rightarrow \eta = \pm 1$

Therefore,

$$b_{\vec{p},s} \xrightarrow{P} \eta b_{-\vec{p},s} \xrightarrow{P} \eta^2 b_{\vec{p},s} = b_{\vec{p},s}$$

$$d^\dagger_{\vec{p},s} \xrightarrow{P} \eta(-d^\dagger_{-\vec{p},s}) \xrightarrow{P} \eta^2 d^\dagger_{\vec{p},s} = d^\dagger_{\vec{p},s}$$

$$b^\dagger_{\vec{p},s} \xrightarrow{P} \eta^* b^\dagger_{-\vec{p},s} = \eta b^\dagger_{-\vec{p},s} \xrightarrow{P} \eta^2 b^\dagger_{\vec{p},s} = b^\dagger_{\vec{p},s}$$

$$d_{\vec{p},s} \xrightarrow{P} \eta^*(-d_{-\vec{p},s}) = \eta(-d_{-\vec{p},s}) = \eta^2 d_{\vec{p},s} = d_{\vec{p},s},$$

as they should be.

In sum, $b_{\vec{p},s} \xrightarrow{P} \eta b_{-\vec{p},s}$, $b^\dagger_{\vec{p},s} \xrightarrow{P} \eta b^\dagger_{-\vec{p},s}$,

$d_{\vec{p},s} \xrightarrow{P} -\eta d_{-\vec{p},s}$, $d^\dagger_{\vec{p},s} \xrightarrow{P} -\eta d^\dagger_{-\vec{p},s}$

where $\eta = \pm 1$.

parity conservation in parity conserving reactions

No matter whether $\eta = +1$ or $\eta = -1$, we have the conclusion that the parity of a Dirac fermion must be opposite to that of the corresponding antiparticle. Also note that parity reverse momentum, without change spin.

Note: to determine the parity of particles, it is necessary to first impose the parities of a minimum number of reference particles (choose e.g. $\eta_{neutron} = \eta_{proton} = \eta_{electron} = +1$) then parities of other particles are determined from arguments based on

For real scalar, the decomposition is

$$\phi(t,\vec{x}) = \int_{-\infty}^{+\infty} d\vec{p}^3\, C(E_{\vec{p}}) [a_{\vec{p}}\, e^{-i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})} + a^\dagger_{\vec{p}}\, e^{i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})}]$$

then $\phi(t,\vec{x}) \xrightarrow{P} \eta_B \phi(t,-\vec{x})$, $\eta_B = \pm 1$

gives

$$\int_{-\infty}^{+\infty} d\vec{p}^3\, C(E_{\vec{p}}) [a_{\vec{p}}\, e^{-i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})} + a^\dagger_{\vec{p}}\, e^{i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})}]$$

$$\xrightarrow{P} \eta_B \int_{-\infty}^{+\infty} d\vec{p}^3\, C(E_{\vec{p}}) [a_{\vec{p}}\, e^{-i(E_{\vec{p}} t + \vec{p}\cdot\vec{x})} + a^\dagger_{\vec{p}}\, e^{i(E_{\vec{p}} t + \vec{p}\cdot\vec{x})}]$$

$$= \eta_B \int_{-\infty}^{+\infty} d\vec{p}^3\, C(E_{\vec{p}}) [a_{-\vec{p}}\, e^{-i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})} + a^\dagger_{-\vec{p}}\, e^{i(E_{\vec{p}} t - \vec{p}\cdot\vec{x})}]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{P} \eta_B a_{-\vec{p}}, \quad a^+_{\vec{p}} \xrightarrow{P} \eta_B a^+_{-\vec{p}} \quad \left(\begin{array}{l} \text{and} \\ \phi(t,\vec{x}) \xrightarrow{P^2} \phi(t,\vec{x}) \\ a_{\vec{p}} \xrightarrow{P^2} a_{\vec{p}}, \; a^+_{\vec{p}} \xrightarrow{P^2} a^+_{\vec{p}} \\ \Rightarrow P^2 = 1 \end{array} \right)$$

For complex scalar, we have

$$\phi(t,\vec{x}) \xrightarrow{P} \eta_B \phi(t,-\vec{x}) \xrightarrow{P} \eta_B^2 \phi(t,\vec{x})$$

$$\phi^+(t,\vec{x}) \xrightarrow{P} \eta_B^* \phi^+(t,-\vec{x}) \xrightarrow{P} (\eta_B^*)^2 \phi^+(t,\vec{x})$$

impose. $\eta_B^2 = 1 \Rightarrow \eta_B = \pm 1$, so that $P^2 = +1$

then the transformation of the decomposition is

$$\phi(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b^+_{\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\xrightarrow{P} \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{\vec{p}} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + b^+_{\vec{p}} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} \right]$$

$$= \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b^+_{-\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{P} \eta_B a_{-\vec{p}}, \quad b^+_{\vec{p}} \xrightarrow{P} \eta_B b^+_{-\vec{p}} \quad \left(\begin{array}{l} \text{and } a_{\vec{p}} \xrightarrow{P^2} a_{\vec{p}} \\ b^+_{\vec{p}} \xrightarrow{P^2} b^+_{\vec{p}} \end{array} \right)$$

$$\phi^+(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\xrightarrow{P} \eta_B^* \phi^+(t,-\vec{x}) = \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} + b_{\vec{p}} e^{-i(E_{\vec{p}}t + \vec{p}\cdot\vec{x})} \right]$$

$$= \eta_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{-\vec{p}} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} \right]$$

$$\Rightarrow \quad a^+_{\vec{p}} \xrightarrow{P} \eta_B a^+_{-\vec{p}}, \quad b_{\vec{p}} \xrightarrow{P} \eta_B b_{-\vec{p}} \quad \left(\begin{array}{l} \text{and } a^+_{\vec{p}} \xrightarrow{P^2} a^+_{\vec{p}} \\ b_{\vec{p}} \xrightarrow{P^2} b_{\vec{p}} \end{array} \right)$$

Therefore, no matter whether $\eta_B = +1$ or $\eta_B = -1$, we have the conclusion that the parity of a complex scalar particle is the same as its antiparticle.

For the photon field, the decomposition in the Coulomb gauge is

$$A^\mu(t,\vec{x}) = (0, \vec{A}(t,\vec{x}))$$

where $\vec{A}(t,\vec{x}) = \int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda \left[\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^*(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$

To aid the analysis, let's use circular polorization,

$$\vec{e}(\vec{k},-) = \frac{1}{\sqrt{2}}(\hat{g} - i\hat{h}), \quad \vec{e}(\vec{k},+) = -\frac{1}{\sqrt{2}}(\hat{g}+i\hat{h}),$$

where $\hat{g}$, $\hat{h}$, $\hat{k}$ are unit vector perpendicular to each other, and $\hat{g}$, $\hat{h}$ and $\hat{k}$ form a right-handed axes system (like $\hat{x}$, $\hat{y}$ and $\hat{z}$).

Therefore, $\vec{e}^*(\vec{k},-) = \frac{1}{\sqrt{2}}(\hat{g}+i\hat{h}) = -\vec{e}(\vec{k},+)$

(check:

$$\begin{cases} \vec{e}^*(\vec{k},+)\cdot\vec{e}(\vec{k},+) = 1 \\ \vec{e}^*(\vec{k},-)\cdot\vec{e}(\vec{k},-) = 1 \\ \vec{e}^*(\vec{k},+)\cdot\vec{e}(\vec{k},-) = -\vec{e}(\vec{k},-)\cdot\vec{e}(\vec{k},-) = 0 \\ \vec{e}^*(\vec{k},-)\cdot\vec{e}(\vec{k},+) = -\vec{e}(\vec{k},+)\cdot\vec{e}(\vec{k},+) = 0 \end{cases}$$

as required by the orthorgonal and completeness requirement for the basis

$$\vec{e}(\vec{k},+) \underset{\text{cross product}}{\times} \vec{e}(\vec{k},-) = -\frac{1}{2}(\hat{g}+i\hat{h})\times(\hat{g}-i\hat{h}) = +\frac{1}{2}(i\hat{k}+i\hat{k}) = +i\hat{k}.$$)

Using

$$\left.\begin{aligned} \vec{e}(-\vec{k},+) &= \vec{e}(\vec{k},-) \\ \vec{e}(-\vec{k},-) &= \vec{e}(\vec{k},+) \end{aligned}\right\} \Rightarrow \vec{e}(-\vec{k},-\lambda) = \vec{e}(\vec{k},\lambda)$$

$$\Rightarrow \vec{e}(-\vec{k},+)\times\vec{e}(-\vec{k},-) = \vec{e}(\vec{k},-)\times\vec{e}(\vec{k},+) = i(-\vec{k})$$

as they should be.

(that is, if choose $\hat{g}$, $\hat{h}$ and $\hat{k}$ as x, y and z direction, respectively, then for the light propogates in the $+z$ direction, if the electric field $\vec{E}$ is $\vec{E} = (E_0 e^{i(kz-\omega t)}, \underbrace{E_0 e^{i(kz-\omega t+\frac{\pi}{2})}}_{=iE_0 e^{i(kz-\omega t)}}, 0)$, then $Re(\vec{E}) = (E_0\cos(kz-\omega t), -E_0\sin(kz-\omega t), 0)$

it is called right-handed polarized, since at a fixed z, for example $z=0$, we have $Re(\vec{E}) = (E_0\cos\omega t, E_0\sin\omega t, 0)$

$\Rightarrow t=0 \Rightarrow Re(\vec{E}) = \underbrace{(E_0, 0, 0)}_{\text{i.e., } +x \text{ direction}}$, $t=\frac{\pi}{2\omega} \Rightarrow Re(\vec{E}) = \underbrace{(0, +E_0, 0)}_{\text{i.e., } +y \text{ direction}}$

$t=\frac{\pi}{\omega} \Rightarrow Re(\vec{E}) = \underbrace{(-E_0, 0, 0)}_{\text{i.e., } -x \text{ direction}}$, $t=\frac{3\pi}{2\omega} \Rightarrow Re(\vec{E}) = \underbrace{(0, -E_0, 0)}_{\text{i.e., } -y \text{ direction}}$;

On the other hand, $\vec{E} = (E_0 e^{i(kz-\omega t)}, \underbrace{E_0 e^{i(kz-\omega t-\frac{\pi}{2})}}_{-iE_0 e^{i(kz-\omega t)}}, 0)$ is left-handed polarized.

18. (checked)

While for the light propagates in the $-z$ direction, if $\vec{E}=(E_0 e^{i(-kz-\omega t)}, \underbrace{E_0 e^{i(-kz-\omega t+\frac{\pi}{2})}}_{iE_0 e^{i(-kz-\omega t)}}, 0)$, then $Re(\vec{E})=(E_0\cos(-kz-\omega t), -E_0\sin(-kz-\omega t), 0)$,

then for $z=0$, we have $Re(\vec{E})=(E_0\cos(\omega t), E_0\sin(\omega t), 0)$, therefore

$t=0 \Rightarrow Re(\vec{E})=(E_0, 0, 0)$; $t=\frac{\pi}{2\omega} \Rightarrow Re(\vec{E})=(0, E_0, 0)$;

$t=\frac{\pi}{\omega} \Rightarrow Re(\vec{E})=(-E_0, 0, 0)$; $t=\frac{3\pi}{2\omega} \Rightarrow Re(\vec{E})=(0, -E_0, 0)$;

so it is left-handed polarized;

on the other hand, $\vec{E}=(E_0 e^{i(-kz-\omega t)}, \underbrace{E_0 e^{i(-kz-\omega t-\frac{\pi}{2})}}_{-iE_0 e^{i(-kz-\omega t)}}, 0)$, it is right-handed polarized.

So, $$\vec{A}(t,\vec{x}) \xrightarrow{P} -\vec{A}(t,-\vec{x}) = -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda \left[\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-i(E_{\vec{k}}t+\vec{k}\cdot\vec{x})} + \vec{e}^*(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{i(E_{\vec{k}}t+\vec{k}\cdot\vec{x})}\right]$$

$$= -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda \left[\vec{e}(-\vec{k},\lambda) a_{-\vec{k},\lambda} e^{-i(E_{\vec{k}}t-\vec{k}\cdot\vec{x})} + \vec{e}^*(-\vec{k},\lambda) a^+_{-\vec{k},\lambda} e^{i(E_{\vec{k}}t-\vec{k}\cdot\vec{x})}\right]$$

$$= -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda \left[\vec{e}(-\vec{k},-\lambda) a_{\vec{k},-\lambda} e^{-ik\cdot x} + \vec{e}^*(-\vec{k},-\lambda) a^+_{\vec{k},-\lambda} e^{ik\cdot x}\right]$$

$$= -\int d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda \left[\vec{e}(\vec{k},\lambda) a_{\vec{k},-\lambda} e^{-ik\cdot x} + \vec{e}^*(\vec{k},\lambda) a^+_{-\vec{k},\lambda} e^{ik\cdot x}\right]$$

$$\Rightarrow \quad a_{\vec{k},\lambda} \xrightarrow{P} -a_{-\vec{k},-\lambda}, \quad a^+_{\vec{k},\lambda} \xrightarrow{P} -a^+_{-\vec{k},-\lambda}$$

Therefore, the photon has a negative parity; both the momentum and helicity of the photon change signs under the parity operation.

Again, $\vec{A}(t,\vec{x}) \xrightarrow{P} -A(t,-\vec{x}) \xrightarrow{P} A(t,\vec{x})$

$a_{\vec{k},\lambda} \xrightarrow{P^2} a_{\vec{k},\lambda}$, $a^+_{\vec{k},\lambda} \xrightarrow{P^2} a^+_{\vec{k},\lambda}$.

$P^2=1$

[2] For charge conjugation,

for Dirac fermion field, in the Standard reprentation,

$$B = i\gamma^2\gamma^0 \text{ satisfies.}$$

$$B^\dagger \gamma^\mu B = -\gamma^{\mu T}$$

$$\left(\text{check: } (\gamma^2\gamma^0)^\dagger \gamma^\mu (\gamma^2\gamma^0) = -\gamma^0\gamma^2\gamma^\mu\gamma^2\gamma^0 = \begin{cases} -\gamma^0 = -\gamma^{0T}, & \mu=0 \\ \gamma^1 = \gamma^{1*} = -\gamma^{1T}, & \mu=1 \\ -\gamma^2 = -\gamma^{2T}, & \mu=2 \\ \gamma^3 = \gamma^{3*} = -\gamma^{3T}, & \mu=3 \end{cases} \right)$$

from $\psi(x) \xrightarrow{C} \varepsilon B \bar{\psi}^T_{(x)}$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [u(\vec{p},s) b_{\vec{p},s} e^{-ip\cdot x} + v(\vec{p},s) d^\dagger_{\vec{p},s} e^{ip\cdot x}]$$

$$\xrightarrow{C} \varepsilon \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [i\gamma^2\gamma^0 \bar{u}^T(\vec{p},s) b^\dagger_{\vec{p},s} e^{ip\cdot x} + i\gamma^2\gamma^0 \bar{v}^T(\vec{p},s) d_{\vec{p},s} e^{-ip\cdot x}]$$

where

$$i\gamma^2\gamma^0 \bar{u}^T(\vec{p},s) = i\gamma^2\gamma^0 (u^\dagger\gamma^0)^T = i\gamma^2\gamma^0\gamma^{0T} u^* = i\gamma^2\gamma^0\gamma^0 u^* = i\gamma^2 u^*$$

$$= i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s^* \\ \frac{\vec{\sigma}^*\cdot\vec{p}}{E+m}\chi_s^* \end{pmatrix}$$

$$= (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}(-i\sigma^2)\chi_s^* \\ -i\sigma^2\chi_s^* \end{pmatrix}$$

using $\chi_s^\dagger \chi_{s'} = \delta_{ss'}$, we have $[(-i\sigma^2)\chi_s^*]^\dagger(-i\sigma^2\chi_{s'}^*)$

$$= \chi_s^T (i\sigma^2)(-i\sigma^2)\chi_{s'}^*$$

$$= \chi_s^T \chi_{s'}^* = (\chi_s^\dagger \chi_{s'})^* = \delta_{ss'}$$

also, $i\gamma^2\gamma^0 \bar{v}^T(\vec{p},s) = i\gamma^2 v^* = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}^*}{E+m}\eta_s^* \\ \eta_s^* \end{pmatrix}$

$$= (E+m)^{\frac{1}{2}} \begin{pmatrix} i\sigma^2\eta_s^* \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} i\sigma^2\eta_s^* \end{pmatrix}$$

we can choose $-i\sigma^2\chi_s^* = \eta_s$, then $\chi_s^* = i\sigma^2\eta_s \Rightarrow \chi_s = i\sigma^2\eta_s^*$

20 (checked)

$$\Rightarrow \quad i\gamma^2\gamma^0\bar{u}^T(\vec{p},s) = v(\vec{p},s)$$
$$i\gamma^2\gamma^0\bar{v}^T(\vec{p},s) = u(\vec{p},s)$$

$$\Rightarrow \quad b_{\vec{p},s} \xrightarrow{C} \varepsilon d_{\vec{p},s}$$
$$d^+_{\vec{p},s} \xrightarrow{C} \varepsilon b^+_{\vec{p},s}$$

note:
$$\psi(x) \xrightarrow{C} \varepsilon B\bar{\psi}^T(x) = \varepsilon B(\psi^+(x)\gamma^0)^T$$
$$\xrightarrow{C} \varepsilon B\left(\varepsilon^*\psi^T(x)\gamma^{0T}B^+\gamma^0\right)^T$$
$$= |\varepsilon|^2 B\gamma^{0T}B^*\gamma^0\psi(x) \equiv ****$$
where $B\gamma^{0T}B^*\gamma^0 = i\gamma^2\gamma^0\gamma^{0T}(i\gamma^2\gamma^0)^*\gamma^0$
$$= -\gamma^2\gamma^0\gamma^0\gamma^2\gamma^0\gamma^0$$
$$= 1$$
$$\Rightarrow **** = \psi(x)$$

Also, from $\psi^+(x) \xrightarrow{C} \varepsilon^*\psi^T(x)\gamma^{0T}B^+$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\sum_s \left(u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} + v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)$$

$$\xrightarrow{C} \varepsilon^*\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\sum_s \Big(u^T(\vec{p},s)\gamma^{0T}(-i)(\gamma^2\gamma^0)^+ b_{\vec{p},s}\, e^{-ip\cdot x}$$
$$+ v^T(\vec{p},s)\gamma^{0T}(-i)(\gamma^2\gamma^0)^+ d^+_{\vec{p},s}\, e^{ip\cdot x}\Big)$$

using $u^T(\vec{p},s)\gamma^{0T}(-i)(\gamma^2\gamma^0)^+$
$$= u^T(\vec{p},s)\gamma^{0T}(+i)\gamma^0\gamma^2$$
$$= u^T(\vec{p},s)\, i\gamma^2$$
$$= \left(i\gamma^2 u^*(\vec{p},s)\right)^+ = v^+(\vec{p},s)$$

and $v^T(\vec{p},s)\gamma^{0T}(-i)(\gamma^2\gamma^0)^+$
$$= v^T(\vec{p},s)\gamma^{0T}(+i)\gamma^0\gamma^2$$
$$= v^T(\vec{p},s)\, i\gamma^2$$
$$= \left(i\gamma^2 v^*(\vec{p},s)\right)^+$$
$$= u^+(\vec{p},s)$$

$$\Rightarrow b_{\vec{p},s} \xrightarrow{C^2} b_{\vec{p},s}$$
$$d^+_{\vec{p},s} \xrightarrow{C^2} d^+_{\vec{p},s}$$
$$b^+_{\vec{p},s} \xrightarrow{C^2} b^+_{\vec{p},s}$$
$$d_{\vec{p},s} \xrightarrow{C^2} d_{\vec{p},s}$$

$$\Rightarrow \quad b^+_{\vec{p},s} \xrightarrow{C} \varepsilon^* d^+_{\vec{p},s}, \quad d_{\vec{p},s} \xrightarrow{C} \varepsilon^* b_{\vec{p},s}$$

Therefore, we conclude that charge conjugation change a Dirac fermion to its antifermion, without changing spin and momentum.

note: $\psi^+(x) \xrightarrow{C} \varepsilon^*\psi^T(x)\gamma^{0T}B^+ \xrightarrow{C} \varepsilon^*(\varepsilon B\bar{\psi}^T(x))^T\gamma^{0T}B^+ = |\varepsilon|^2\bar{\psi}(x)B^T\gamma^{0T}B^+ \equiv ***$,
where $B^T\gamma^{0T}B^+ = (i\gamma^2\gamma^0)^T\gamma^{0T}(i\gamma^2\gamma^0)^+ = \gamma^{0T}\gamma^{2T}\gamma^{0T}\gamma^0(-\gamma^2) = -\gamma^0\gamma^2\gamma^0\gamma^0\gamma^2 = \gamma^0$
$\Rightarrow *** = \psi^+(x)$

So, $C^2 = 1$.

21 (checked)

For real scalar field, we can immediately see that (from $\phi(x) \xrightarrow{C} \varepsilon_B \phi(x)$),

$$a_{\vec{p}} \xrightarrow{C} \varepsilon_B a_{\vec{p}}, \quad a^+_{\vec{p}} \xrightarrow{C} \varepsilon_B a^+_{\vec{p}}, \quad \text{where } \varepsilon_B = \pm 1.$$

$$\left(\varepsilon_B,\ \phi(x) \xrightarrow{C^2} \phi(x),\quad a_{\vec{p}} \xrightarrow{C^2} a_{\vec{p}},\quad a^+_{\vec{p}} \xrightarrow{C^2} a^+_{\vec{p}},\quad C^2 = 1 \right)$$

For a complex scalar field, from $\phi(x) \xrightarrow{C} \varepsilon_B \phi^+(x)$ and $\phi^+(x) \longrightarrow \varepsilon_B^* \phi(x)$, we have

$$\phi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(a_{\vec{p}} e^{-ip\cdot x} + b^+_{\vec{p}} e^{ip\cdot x} \right]$$

$$\xrightarrow{C} \varepsilon_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x} \right]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{C} \varepsilon_B b_{\vec{p}}$$

$$b^+_{\vec{p}} \xrightarrow{C} \varepsilon_B a^+_{\vec{p}}.$$

$$\phi^+(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x} \right]$$

$$\xrightarrow{C} \varepsilon_B^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(a_{\vec{p}} e^{-ip\cdot x} + b^+_{\vec{p}} e^{ip\cdot x} \right]$$

$$\Rightarrow \quad a^+_{\vec{p}} \xrightarrow{C} \varepsilon_B^* b^+_{\vec{p}}$$

$$b_{\vec{p}} \xrightarrow{C} \varepsilon_B^* a_{\vec{p}}$$

So, for a complex scalar, charge conjugation transforms to its antiparticle without changing momentum.

$$\left(\phi(x) \xrightarrow{C^2} \phi(x),\ \phi^+(x) \xrightarrow{C^2} \phi^+(x);\quad a_{\vec{p}} \xrightarrow{C^2} a_{\vec{p}},\ a^+_{\vec{p}} \xrightarrow{C^2} a^+_{\vec{p}},\ b_{\vec{p}} \xrightarrow{C^2} b_{\vec{p}},\ b^+_{\vec{p}} \xrightarrow{C^2} b^+_{\vec{p}},\quad C^2 = 1 \right)$$

For the photon field, we can immediately see that (from $A^\mu(x) \xrightarrow{C} -A^\mu(x)$),

$$a_{\vec{k},\lambda} \xrightarrow{C} -a_{\vec{k},\lambda}, \quad a^+_{\vec{k},\lambda} \longrightarrow -a^+_{\vec{k},\lambda}$$

$$\left(A^\mu(x) \xrightarrow{C^2} A^\mu(x),\quad a_{\vec{k},\lambda} \xrightarrow{C^2} a_{\vec{k},\lambda},\quad a^+_{\vec{k},\lambda} \xrightarrow{C^2} a^+_{\vec{k},\lambda},\quad C^2 = 1 \right)$$

So, the photon is odd under charge conjugation (i.e., it has charge conjugation number -1).

For a state of scalar-antiscalar particles with orbital angular momentum l:

$$|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^\dagger_{\vec{p}}\, b^\dagger_{-\vec{p}}\, |0\rangle$$

, where $F_l(\vec{p})$ is an eigenstate of obital angular momentum l.

Apply charge conjugation,

$$C|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, b^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\, |0\rangle$$

$$\overset{\text{exchange } b^\dagger \text{ and } a^\dagger}{=} \int d^3\vec{p}\, F_l(\vec{p})\, a^\dagger_{-\vec{p}}\, b^\dagger_{\vec{p}}\, |0\rangle$$

$$= \int d^3\vec{p}\, F_l(-\vec{p})\, a^\dagger_{\vec{p}}\, b^\dagger_{-\vec{p}}\, |0\rangle$$

$$= \int d^3\vec{p}\, (-1)^l F_l(\vec{p})\, a^\dagger_{\vec{p}}\, b^\dagger_{-\vec{p}}\, |0\rangle$$

$$= (-1)^l |\phi_l(s\bar{s})\rangle .$$

$\boxed{\text{So, this state has charge conjugation number } (-1)^l}$

An example of such system is $\pi^+\pi^-$.

Another example is $\pi^0\pi^0$, where $b^\dagger_{\vec{p}}$ in the above should be substitute by $a^\dagger_{\vec{p}}$, then $|\phi_l(\pi^0\pi^0)\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\, |0\rangle \Rightarrow C|\phi_l(\pi^0\pi^0)\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\, |0\rangle = |\phi_l(\pi^0$

$$= \int d^3\vec{p}\, F_l(\vec{p})\, a^\dagger_{-\vec{p}}\, a^\dagger_{\vec{p}}\, |0\rangle$$

$$= \int d^3\vec{p}\, F_l(-\vec{p})\, a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\, |0\rangle$$

$$= (-1)^l \int d^3\vec{p}\, F_l(\vec{p})\, a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\, |0\rangle$$

$$= (-1)^l \phi(\pi^0\pi^0).$$

$\Rightarrow$ l is even for $\pi^0\pi^0$ system (consistent with the requirement of wavefunction when exchange two identical bosons)

For a state of fermion-antifermion particles, and neglect the effects of spin-orbit coupling, we have the state being

$$|\psi_{l,s}(f\bar{f})\rangle = \int d^3\vec{p} \sum_{s_1,s_2} F^{s_1 s_2}_{ls}(\vec{p})\, b^\dagger_{\vec{p},s_1}\, d^\dagger_{-\vec{p},s_2}\, |0\rangle ,$$

then

$$C|\psi_{l,s}(f\bar{f})\rangle = \int d^3\vec{p} \sum_{s_1,s_2} F^{s_1 s_2}_{ls}(\vec{p})\, d^\dagger_{\vec{p},s_1}\, b^\dagger_{-\vec{p},s_2}\, |0\rangle$$

$$\overset{\text{interchange } b^\dagger \text{ and } d^\dagger}{=} -\int d^3\vec{p} \sum_{s_1,s_2} F^{s_1 s_2}_{ls}(\vec{p})\, b^\dagger_{-\vec{p},s_2}\, d^\dagger_{\vec{p},s_1}\, |0\rangle$$

$$\overset{\text{interchange } \vec{p} \leftrightarrow -\vec{p},\ s_1 \leftrightarrow s_2}{=} -\int d^3\vec{p} \sum_{s_1,s_2} F^{s_2 s_1}_{ls}(-\vec{p})\, b^\dagger_{\vec{p},s_1}\, d^\dagger_{-\vec{p},s_2}\, |0\rangle$$

using $F_{ls}^{S_2 S_1}(-\vec{p}) = (-1)^{l+s+1} F_{ls}^{S_1 S_2}(\vec{p})$

(note: S=1 is spin triplet, which is symmetric under the exchange of the two spins; S=0 is spin singlet, which is antisymmetric under the exchange of the two spins.

$F_{ls}^{S_1 S_2}(\vec{p})$ is the product of orbital wave function and total spin wave function).

$$\Rightarrow C|\psi_{l,s}(f\bar{f})\rangle = (-1)^{l+s} |\psi_{l,s}(f\bar{f})\rangle$$

Therefore, the charge conjugation number for fermion-antifermion system is $(-1)^{l+s}$.

So, for a pseudoscalar meson with $l=S=0$, its charge conjugation number is $(-1)^{0+0} = +1$, π^0 is an example; for a vector meson with $l=0$ and $S=1$, its charge conjugation number is $(-1)^{0+1} = -1$, ρ^0 is an example.

[3] For time inversion:

for Dirac fermion field, in the standard representation,

$$D = \gamma^1\gamma^3 \text{ satisfies}$$

$$D^+ \gamma^{*\mu} D = \gamma_\mu .$$

(check: $(\gamma^1\gamma^3)^+ \gamma^{*0} (\gamma^1\gamma^3) = \gamma^{3+}\gamma^{1+}\gamma^0\gamma^1\gamma^3 = \gamma^3\gamma^1\gamma^0\gamma^1\gamma^3 = \gamma_0$ ✓

$(\gamma^1\gamma^3)^+ \gamma^{*1} (\gamma^1\gamma^3) = \gamma^3\gamma^1\gamma^1\gamma^1\gamma^3 = -\gamma^1 = \gamma_1$ ✓

$(\gamma^1\gamma^3)^+ \gamma^{*2} (\gamma^1\gamma^3) = -\gamma^3\gamma^1\gamma^2\gamma^1\gamma^3 = -\gamma^2 = \gamma_2$ ✓

$(\gamma^1\gamma^3)^+ \gamma^{*3} (\gamma^1\gamma^3) = \gamma^3\gamma^1\gamma^3\gamma^1\gamma^3 = -\gamma^3 = \gamma_3$ ✓).

from $\psi(x) \xrightarrow{T} \varsigma D \psi(\tilde{x})$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [u(\vec{p},s) b_{\vec{p},s} e^{-ip\cdot x} + v(\vec{p},s) d^+_{\vec{p},s} e^{ip\cdot x}]$$

$$\xrightarrow{T} \varsigma \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [\gamma^1\gamma^3 u^*(\vec{p},s) b_{\vec{p},s} e^{+i(-E_p t - \vec{p}\cdot\vec{x})} + \gamma^1\gamma^3 v^*(\vec{p},s) d^+_{\vec{p},s} e^{-i(-E_p t - \vec{p}\cdot\vec{x})}]$$

$$= \varsigma \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s [\gamma^1\gamma^3 u^*(-\vec{p},s) b_{-\vec{p},s} e^{-ip\cdot x} + \gamma^1\gamma^3 v^*(-\vec{p},s) d^+_{-\vec{p},s} e^{ip\cdot x}]$$

$= **$

note that complex conjugate is applied on u, v and exponential in ψ.

using

$$\gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

$$\Rightarrow \gamma^1\gamma^3 u^*(\vec{p},s) = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s^* \\ \frac{\vec{\sigma}^*\cdot\vec{p}}{E+m}\chi_s^* \end{pmatrix}$$

$$= (E+m)^{\frac{1}{2}} \cdot \begin{pmatrix} i\sigma^2\chi_s^* \\ -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} i\sigma^2\chi_s^* \end{pmatrix}$$

 (checked)

we can choose $i\sigma^2 \chi_s^* = -(-1)^{\frac{1}{2}-s} \chi_{-s}$

$$\Rightarrow \quad \gamma^1 \gamma^3 u^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} \, u(-\vec{p}, -s)$$

Also,

$$\gamma^1 \gamma^3 v^*(\vec{p}, s) = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}^*}{E+m} \eta_s^* \\ \eta_s^* \end{pmatrix}$$

$$= (E+m)^{\frac{1}{2}} \begin{pmatrix} -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} i\sigma^2 \eta_s^* \\ i\sigma^2 \eta_s^* \end{pmatrix}$$

we can choose $i\sigma^2 \eta_s^* = -(-1)^{\frac{1}{2}-s} \eta_{-s}$

$$\Rightarrow \quad \gamma^1 \gamma^3 v^*(\vec{p}, s) = -(-1)^{\frac{1}{2}-s} \, v(-\vec{p}, -s)$$

$$\Rightarrow ** = \xi \int_{-\infty}^{+\infty} d^3\vec{p} \, C(E_{\vec{p}}) \sum_s \Big[\overbrace{u(\vec{p}, -s)}^{-(-1)^{\frac{1}{2}-s}} b_{-\vec{p}, s} \, e^{-ip\cdot x} + \overbrace{v(\vec{p}, -s)}^{-(-1)^{\frac{1}{2}-s}} d^\dagger_{-\vec{p}, s} \, e^{ip\cdot x} \Big]$$

$$= \xi \int_{-\infty}^{+\infty} d^3\vec{p} \, C(E_{\vec{p}}) \sum_s \Big[\overbrace{u(\vec{p}, s)}^{-(-1)^{\frac{1}{2}+s}} b_{-\vec{p}, -s} \, e^{-ip\cdot x} + \overbrace{v(\vec{p}, s)}^{-(-1)^{\frac{1}{2}+s}} d^\dagger_{-\vec{p}, -s} \, e^{ip\cdot x} \Big]$$

 (checked)

$\Rightarrow$ using $-(-1)^{\frac{1}{2}+s} = (-1)^{\frac{1}{2}-s}$

$$\Rightarrow \quad b_{\vec{p},s} \xrightarrow{T} \varsigma(-1)^{\frac{1}{2}-s} b_{-\vec{p},-s} \Rightarrow b_{\vec{p},s} \xrightarrow{T^2} (-1)^{\frac{1}{2}-s}(-1)^{\frac{1}{2}+s} b_{\vec{p},s} = -b_{\vec{p},s}$$

$$d^\dagger_{\vec{p},s} \xrightarrow{T} \varsigma(-1)^{\frac{1}{2}-s} d^\dagger_{-\vec{p},-s} \Rightarrow d^\dagger_{\vec{p},s} \xrightarrow{T^2} -d^\dagger_{\vec{p},s}$$

from $\psi^\dagger(x) \xrightarrow{T} \varsigma^*\psi^\dagger(\tilde{x})D^\dagger$

$$\Rightarrow \psi^\dagger(t,\vec{x}) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left(u^\dagger(\vec{p},s)\, b^\dagger_{\vec{p},s}\, e^{ip\cdot x} + v^\dagger(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)$$

$$\xrightarrow{T} \varsigma^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \Big(u^T(\vec{p},s) \underbrace{b^\dagger_{\vec{p},s}}_{(\gamma^1\gamma^3)^\dagger} e^{-i(-E_{\vec{p}}t-\vec{p}\cdot\vec{x})} + v^T(\vec{p},s) \underbrace{d_{\vec{p},s}}_{(\gamma^1\gamma^3)^\dagger} e^{i(-E_{\vec{p}}t-\vec{p}\cdot\vec{x})}\Big)$$

$\equiv ***$

using $\gamma^1\gamma^3 u^*(\vec{p},s) = -(-1)^{\frac{1}{2}-s} u(-\vec{p},-s)$

$$\Rightarrow [\gamma^1\gamma^3 u^*(\vec{p},s)]^\dagger = u^T(\vec{p},s)(\gamma^1\gamma^3)^\dagger = -(-1)^{\frac{1}{2}-s} u^\dagger(-\vec{p},-s)$$

using $\gamma^1\gamma^3 v^*(\vec{p},s) = -(-1)^{\frac{1}{2}-s} v(-\vec{p},-s)$

$$\Rightarrow [\gamma^1\gamma^3 v^*(\vec{p},s)]^\dagger = v^T(\vec{p},s)(\gamma^1\gamma^3)^\dagger = -(-1)^{\frac{1}{2}-s} v^\dagger(-\vec{p},-s)$$

$$\Rightarrow *** = \varsigma^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s (-1)^{\frac{1}{2}+s}\Big[u^\dagger(-\vec{p},-s)\, b^\dagger_{\vec{p},s}\, e^{-i(-E_{\vec{p}}t-\vec{p}\cdot\vec{x})} + v^\dagger(-\vec{p},-s)\, d_{\vec{p},s}\, e^{i(-E_{\vec{p}}t-\vec{p}\cdot\vec{x})}\Big]$$

$$= \varsigma^* \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s (-1)^{\frac{1}{2}-s}\Big[u^\dagger(\vec{p},s)\, b^\dagger_{-\vec{p},-s}\, e^{ip\cdot x} + v^\dagger(\vec{p},s)\, d_{-\vec{p},-s}\, e^{-ip\cdot x}\Big]$$

$$\Rightarrow \quad b^\dagger_{\vec{p},s} \xrightarrow{T} \varsigma^*(-1)^{\frac{1}{2}-s} b^\dagger_{-\vec{p},-s} \Rightarrow b^\dagger_{\vec{p},s} \xrightarrow{T^2} -b^\dagger_{\vec{p},s}$$

$$d_{\vec{p},s} \xrightarrow{T} \varsigma^*(-1)^{\frac{1}{2}-s} d_{-\vec{p},-s} \Rightarrow d_{\vec{p},s} \xrightarrow{T^2} -d_{\vec{p},s}$$

In sum. $\psi(t,\vec{x}) \xrightarrow{T} \varsigma D\psi(-t,\vec{x}) \xrightarrow{T} \varsigma^* D^* \varsigma D\psi(t,\vec{x}) = D^*D\psi(t,\vec{x})$

from $D = \gamma^1\gamma^3 \Rightarrow D^* = \gamma^1\gamma^3 \Rightarrow D^*D = \gamma^1\gamma^3\gamma^1\gamma^3 = -1$

$\Rightarrow \psi(t,\vec{x}) \xrightarrow{T^2} -\psi(t,\vec{x})$

also, $\psi^\dagger(t,\vec{x}) \xrightarrow{T} \varsigma^*\psi^\dagger(-t,\vec{x})D^\dagger \xrightarrow{T} \varsigma\varsigma^*\psi^\dagger(t,\vec{x})D^\dagger D^T$, then from $(D^\dagger D^T)^\dagger = D^*D = -1$

$\Rightarrow \psi^\dagger(t,\vec{x}) \xrightarrow{T^2} -\psi^\dagger(t,\vec{x})$.

that is, $T^2 = -1$ for Dirac fermion field.

For complex scalar field,

from $\phi(t, \vec{x}) \xrightarrow{T} \varsigma_B \phi(-t, \vec{x})$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{\vec{p}}^{+} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\xrightarrow{T} \varsigma_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{i(-E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{\vec{p}}^{+} e^{-i(-E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$= \varsigma_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{-\vec{p}}^{+} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{T} \varsigma_B a_{-\vec{p}}, \quad b_{\vec{p}}^{+} \xrightarrow{T} \varsigma_B b_{-\vec{p}}^{+}$$

from $\phi^{+}(t, \vec{x}) \xrightarrow{T} \varsigma_B^{*} \phi^{+}(-t, \vec{x})$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}}^{+} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\xrightarrow{T} \varsigma_B^{*} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}}^{+} e^{-i(-E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{\vec{p}} e^{i(-E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$= \varsigma_B^{*} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{-\vec{p}}^{+} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + b_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\Rightarrow \quad a_{\vec{p}}^{+} \xrightarrow{T} \varsigma_B^{*} a_{-\vec{p}}^{+}, \quad b_{\vec{p}} \xrightarrow{T} \varsigma_B^{*} b_{-\vec{p}}.$$

where $|\varsigma_B|^2 = 1$.

For real scalar field,

from $\phi(t, \vec{x}) \xrightarrow{T} \varsigma_B \phi(-t, \vec{x})$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + a_{\vec{p}}^{+} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\xrightarrow{T} \varsigma_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{\vec{p}} e^{i(-E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + a_{\vec{p}}^{+} e^{-i(-E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$= \varsigma_B \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) [a_{-\vec{p}} e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} + a_{-\vec{p}}^{+} e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})}]$$

$$\Rightarrow \quad a_{\vec{p}} \xrightarrow{T} \varsigma_B a_{-\vec{p}}, \quad a_{\vec{p}}^{+} \xrightarrow{T} \varsigma_B a_{-\vec{p}}^{+}$$

28 (checked) where $\varsigma_B = \pm 1$

For photon field,

from $\vec{A}(t,\vec{x}) \xrightarrow{T} -\vec{A}(-t,\vec{x})$,

we have, in the Coulomb gauge,

$$\int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\, \vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^*(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{ik\cdot x}]$$

$$\xrightarrow{T} -\int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\vec{e}^*(\vec{k},\lambda) a_{\vec{k},\lambda} e^{i(-E_{\vec{k}}t-\vec{k}\cdot\vec{x})} + \vec{e}(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{-i(-E_{\vec{k}}t-\vec{k}\cdot\vec{x})}]$$

$$= -\int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\, \vec{e}^*(-\vec{k},\lambda) a_{-\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}(-\vec{k},\lambda) a^+_{-\vec{k},\lambda} e^{ik\cdot x}]$$

$\equiv **$

using $\vec{e}(-\vec{k},-\lambda) = \vec{e}(\vec{k},\lambda) = -\vec{e}^*(\vec{k},-\lambda)$

$$\Rightarrow ** = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_\lambda [\, \vec{e}(\vec{k},\lambda) a_{-\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^*(\vec{k},\lambda) a^+_{-\vec{k},\lambda} e^{ik\cdot x}]$$

$$\boxed{\Rightarrow a_{\vec{k},\lambda} \xrightarrow{T} a_{-\vec{k},\lambda}, \quad a^+_{\vec{k},\lambda} \xrightarrow{T} a^+_{-\vec{k},\lambda}}$$

So, the momentum reverse, but helicity does not change.

In sum, for real scalar field,

$$\phi(t,\vec{x}) \xrightarrow{T} \varsigma_B \phi(-t,\vec{x}) \xrightarrow{T} \varsigma_B \varsigma_B^* \phi(t,\vec{x}) = \varsigma_B^2 \phi(t,\vec{x}) = \phi(t,\vec{x})$$

$$a_{\vec{p}} \xrightarrow{T} \varsigma_B a_{-\vec{p}} \xrightarrow{T} a_{\vec{p}}, \quad a^+_{\vec{p}} \xrightarrow{T} \varsigma_B a^+_{-\vec{p}} \xrightarrow{T} a^+_{\vec{p}}$$

for complex scalar field,

$$\phi(t,\vec{x}) \xrightarrow{T} \varsigma_B \phi(-t,\vec{x}) \xrightarrow{T} \phi(t,\vec{x}).$$

$$a_{\vec{p}} \xrightarrow{T} \varsigma_B a_{-\vec{p}} \xrightarrow{T} a_{\vec{p}}, \quad b^+_{\vec{p}} \xrightarrow{T} \varsigma_B b^+_{-\vec{p}} \xrightarrow{T} b^+_{\vec{p}}$$

$$\phi^+(t,\vec{x}) \xrightarrow{T} \varsigma_B^* \phi^+(-t,\vec{x}) \xrightarrow{T} \phi^+(t,\vec{x})$$

$$a^+_{\vec{p}} \xrightarrow{T} \varsigma_B^* a^+_{-\vec{p}} \xrightarrow{T} a^+_{\vec{p}}, \quad b_{\vec{p}} \xrightarrow{T} \varsigma_B^* b_{-\vec{p}} \xrightarrow{T} b_{\vec{p}}$$

for photon field, $\vec{A}(t,\vec{x}) \xrightarrow{T} -\vec{A}(-t,\vec{x}) \xrightarrow{T} A(t,\vec{x})$

$$a_{\vec{k},\lambda} \xrightarrow{T} a_{-\vec{k},\lambda} \xrightarrow{T} a_{\vec{k},\lambda}, \quad a^+_{\vec{k},\lambda} \xrightarrow{T} a^+_{-\vec{k},\lambda} \xrightarrow{T} a_{\vec{k},\lambda}$$

that is, $T^2 = +1$ for $\underbrace{\text{scalar field}}_{\text{both real and complex}}$ and photon field.

(29. (checked))

Some note for C, P and T:

1. We have assumed that the vaccum state $|0\rangle$ is invariant under seperate transformations of C, P and T.

2. For the photon field, we came to the conclusion that the photon has parity -1 , charge conjugation number -1; parity transformation reverse the photon's momentum and helicity, time inverse transformation reserve its momentum but not change its helicity.

3. For the complex scalar field and Dirac field, the parity of the particles described by the fields are determined by parity conservation requirement of the parity conservation reactions, after the parities of a minimum number of particles (usually taken $\eta_{proton} = \eta_{neutron} = \eta_{electron} = +1$) are assigned. Their parities are either $+1$ or -1.
   The parities of particle and antiparticle are the same for a complex scalar field; while they are opposite for a Dirac field.

4. For the real scalar field, the parity number, charge conjugation number and the phase of T transformation, are $+1$ or -1, i.e., the phases for P, C and T transformations are all real.
   For photon field, such phases are determined by the requirement of the invariance of $\bar{\psi}\gamma^\mu\psi A_\mu$ under P, C and T, that is, the photon's parity and charge conjugation number are both -1.

5. For T and C transformations, the phases for complex scalar and Dirac fields are only required to be $|\varepsilon|^2 = 1$ and $|\varsigma|^2 = 1$.

(checked)

6. For a state of scalar – antiscalar particles with orbital angular momentum l:

$$|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^+_{\vec{p}}\, b^+_{-\vec{p}}\, |0\rangle$$, where $F_l(\vec{p})$ is an eigenstate of orbital angular momentum l.

Apply parity transformation,

$$P|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^+_{-\vec{p}}\, b^+_{\vec{p}}\, |0\rangle$$

$$= \int d^3\vec{p}\, F_l(-\vec{p})\, a^+_{\vec{p}}\, b^+_{-\vec{p}}\, |0\rangle$$

$$= (-1)^l \int_{-\infty}^{+\infty} F_l(\vec{p})\, a^+_{\vec{p}}\, b^+_{-\vec{p}}\, |0\rangle$$

$$= (-1)^l |\phi_l(s\bar{s})\rangle$$

(for real scalar, $|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^+_{\vec{p}}\, a^+_{-\vec{p}}\, |0\rangle \Rightarrow P|\phi_l(s\bar{s})\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, a^+_{-\vec{p}}\, a^+_{\vec{p}}\, |0\rangle$

$= \int d^3\vec{p}\, F_l(\vec{p})\, a^+_{\vec{p}}\, a^+_{-\vec{p}}\, |0\rangle$
$= |\phi_l(s\bar{s})\rangle$
$= \int d^3\vec{p}\, F_l(-\vec{p})\, a^+_{\vec{p}}\, a^+_{-\vec{p}}\, |0\rangle$
$= (-1)^l |\phi_l(s\bar{s})\rangle$
$\Rightarrow l$ is even

For a state of fermion - antifermion particles, and neglect the effects of spin - orbit coupling, we have the state being

$$|\psi^{s_1 s_2}_{l,s}(f\bar{f})\rangle = \int d^3\vec{p} \sum_{s_1 s_2} F^{s_1 s_2}_{ls}(\vec{p})\, b^+_{\vec{p},s_1}\, d^+_{-\vec{p},s_2}\, |0\rangle$$

then $$P|\psi^{s_1 s_2}_{l,s}(f\bar{f})\rangle = -\int d^3\vec{p} \sum_{s_1 s_2} F^{s_1 s_2}_{ls}(\vec{p})\, b^+_{-\vec{p},s_1}\, d^+_{\vec{p},s_2}\, |0\rangle$$

$$= -\int d^3\vec{p} \sum_{s_1 s_2} F^{s_1 s_2}_{ls}(-\vec{p})\, b^+_{\vec{p},s_1}\, d^+_{-\vec{p},s_2}\, |0\rangle$$

$$= (-1)^{l+1} |\psi^{s_1 s_2}_{l,s}(f\bar{f})\rangle$$

Therefore, the scalar-antiscalar system has parity number $(-1)^l$, and the fermion-antifermion system has parity number $(-1)^{l+1}$.

In fact, there is no need to require particle - antiparticle system in the above analysis — for state with two particles (and neglect the effects of spin-orbit coupling), we always have the parity of the system $\eta_A \eta_B (-1)^l$, where η_A and η_B are the intrinsic parity of the two particles.
($|\phi_l(AB)\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, A^+_{\vec{p}}\, B^+_{-\vec{p}}\, |0\rangle$, where $A^+_{\vec{p}}$ and $B^+_{\vec{p}}$ are two arbitrary creation operators, with spin label suppressed (since not relevent), then $P|\phi_l(AB)\rangle = \int d^3\vec{p}\, F_l(\vec{p})\, A^+_{-\vec{p}}\, B^+_{\vec{p}}\, |0\rangle\, \eta_A \eta_B = (-1)^l \eta_A \eta_B |\phi_l(AB)\rangle$

31 (checked)