Feynman calculus

1. Decays & scattering

For the three experimental probes of elementary particle interactions — bound states, decay, and scattering — nonrelativistic quantum mechanics is well adapted to handle the first, while QFT is the tool to handel the second & third.

(1) Decays

the item of greatest interest in the lifetime, or more accurately, the mean lifetime, τ.

The decay rate Γ is the probability per unit time that any given particle will disintegrate.

$$dN = -\Gamma N\, dt$$

$$\Rightarrow \quad N(t) = N(t=0)\, e^{-\Gamma t}$$

$$\Rightarrow \quad \tau = \frac{\int_0^\infty N(t)\, dt}{N(t=0)} = \int_0^\infty e^{-\Gamma t}\, dt = -\frac{1}{\Gamma} e^{-\Gamma t}\Big|_0^\infty = \frac{1}{\Gamma}.$$

that is, the mean lifetime is the reciprocal of the decay rate.

For more than one decay channels,

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i \,, \quad \tau = \frac{1}{\Gamma_{tot}}.$$

branching ratio for ith decay channel $= \dfrac{\Gamma_i}{\Gamma_{tot}}$.

(2) Scatterings

$$\sigma_{tot} = \sum_{i=1}^{n} \sigma_i$$

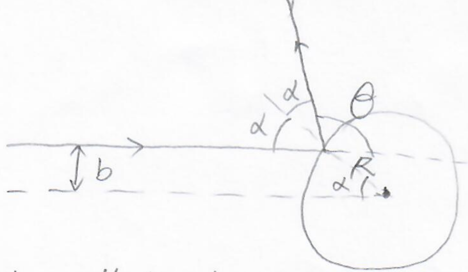
σ_i → the cross section for process i (called "exclusive" cross section)

σ_{tot} → (called "inclusive" cross section)

At the most naive level, we could expect that the cross section to be proportional to the amount of time the incident particle spends in the vicinity of the target, that is, $\sigma \propto \frac{1}{v}$. This behavior is dramatically altered in the neighborhood of a "resonance" — a special energy at which the particles involved "like" to interact by forming a short-lived semibound state before breaking apart. Searching these "bump" in the σ vs. v (or more commonly plotted, σ vs. E) graph is the principle means to discover short-lived particles.

We can use the following two examples to introduce the key concepts in scattering:

Example 1. Hard-sphere Scattering



b is called the impact parameter, θ is the scattering angle.

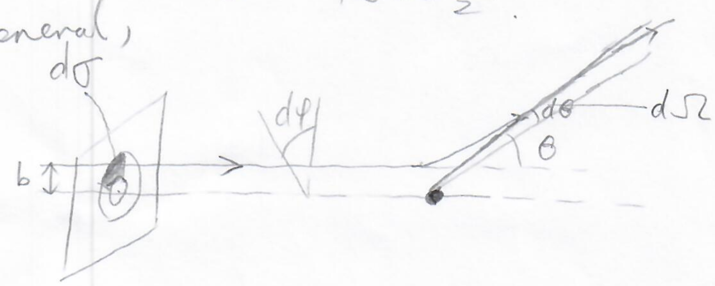
$$b = R \sin\alpha$$

$$2\alpha + \theta = \pi$$

$$\Rightarrow \quad \sin\alpha = \sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2}$$

$$\Rightarrow \quad b = R\cos\frac{\theta}{2}.$$

In general,



If the particle comes in with an impact parameter between b and $b+db$, it will emerge with a scattering angle between θ and $\theta + d\theta$. Therefore If it passes through an infinitesimal area $d\sigma$, it will scatter into a corresponding solid angle $d\Omega$.

$$d\sigma = D(\theta,\varphi)\, d\Omega$$

↳ called the differential cross section.

Go back to the hard-sphere scattering example,

$$d\sigma = |b\, db\, d\varphi| ,$$

$$d\Omega = |\sin\theta\, d\theta\, d\varphi|$$

positive definite, so take absolute value.

$$\Rightarrow D(\theta,\varphi) \underset{\substack{\uparrow \\ \text{since no } \varphi \text{ dependence}}}{=} D(\theta) = \left| \frac{b\, db}{\sin\theta\, d\theta} \right|$$

using $b = R\cos\frac{\theta}{2}$

$$\Rightarrow db = -R\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\, d\theta$$

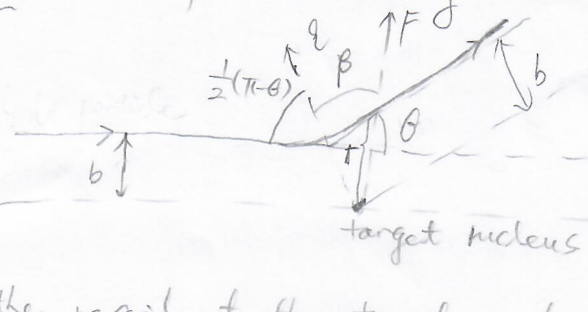
$$\Rightarrow D(\theta,\varphi) = \left| \frac{R\cos\frac{\theta}{2}}{\sin\theta}\left(-\frac{1}{2}R\sin\frac{\theta}{2}\right) \right|$$

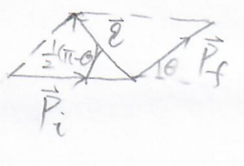
$$= \frac{R^2}{4}$$

$$\Rightarrow \sigma = \int d\sigma = \int D(\theta,\varphi)\, d\Omega = 4\pi\frac{R^2}{4} = \pi R^2 .$$

The result is as expected: any particle within this πR^2 area will scatter, and any outside will pass by unaffected.

Example 2. Rutherford Scattering





Neglect the recoil of the target nucleus and assume elastic scattering, then $|\vec{P}_i| = |\vec{P}_f|$, $\vec{P}_f = \vec{P}_i + \vec{q}$.

$$\frac{|\vec{q}|}{\sin\theta} = \frac{|\vec{P}_i|}{\sin\left(\frac{\pi-\theta}{2}\right)}$$

where the direction of the vector $\vec{q}$ is along the line joining the nucleus to the point of closest approach of the incoming particle, and $\beta = 0$ chosen to be the point of closest approach.

By angular momentum conservation,

$$m b v = m r^2 \frac{d\beta}{dt}$$, where m is mass of the incoming particle, v is its incoming velocity.

and $\vec{F}_{\vec{q}} = \frac{d\vec{q}}{dt}$

$$\Rightarrow \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\cos\beta = \frac{d|\vec{q}|}{dt} = \frac{d|\vec{q}|}{d\beta}\frac{d\beta}{dt} = \frac{d|\vec{q}|}{d\beta}\frac{bv}{r^2}$$

$$\Rightarrow |\vec{q}| = \int_{-\frac{\pi-\theta}{2}}^{\frac{\pi-\theta}{2}} d\beta \, \frac{q_1 q_2}{4\pi\epsilon_0 \, b v}\cos\beta$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{bv} 2\sin\left(\frac{\pi-\theta}{2}\right)$$

using $$\frac{|\vec{q}|}{\sin\theta} = \frac{|\vec{P}_i|}{\sin\left(\frac{\pi-\theta}{2}\right)} = \frac{\sqrt{2mT}}{\sin\left(\frac{\pi-\theta}{2}\right)}$$, where T is the kinetic energy of the incoming particle.

and $\frac{1}{2}mv^2 = T \Rightarrow v = \sqrt{\frac{2T}{m}}$

$$\Rightarrow \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{b\sqrt{\frac{2T}{m}}} 2\sin\left(\frac{\pi-\theta}{2}\right) = \frac{\sin\theta\cdot\sqrt{2mT}}{\sin\left(\frac{\pi-\theta}{2}\right)}$$

$$\Rightarrow b = \frac{q_1 q_2}{4\pi\epsilon_0 \, 2T}\frac{2\sin^2\left(\frac{\pi-\theta}{2}\right)}{\sin\theta}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0 \, 2T}\cot\left(\frac{\theta}{2}\right)$$ (in SI unit)

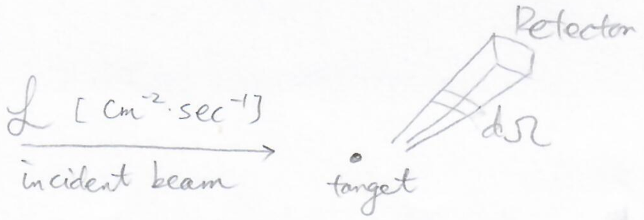
$d\sigma = |b\,db\,d\phi|$, $d\Omega = |\sin\theta\,d\theta\,d\phi|$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left|\frac{b\,db}{\sin\theta\,d\theta}\right| = \frac{b}{\sin\theta}\frac{q_1 q_2}{4\pi\epsilon_0 \, 2T}\frac{1}{2}\csc^2\frac{\theta}{2}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0 \, 2T}\frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}}\frac{1}{\sin\theta}\frac{q_1 q_2}{4\pi\epsilon_0 \, 4T}$$

$$= \frac{(q_1 q_2)^2}{(4\pi\epsilon_0 \, 4T)^2}\frac{1}{\sin^4\frac{\theta}{2}}$$

$$\Rightarrow \sigma = 2\pi\left(\frac{q_1 q_2}{4\pi\epsilon_0 \, 4T}\right)^2\int_0^\pi \frac{1}{\sin^4\frac{\theta}{2}}\sin\theta\,d\theta = \infty$$ ← due to that Coulomb potential has infinite range



The number of particles per unit time passing through area $d\sigma$, and hence also the number of particles per unit time scattered into solid angle $d\Omega$ is

$$dN = \mathcal{L}\, d\sigma = \mathcal{L}\, D(\theta,\varphi)\, d\Omega.$$

$d\Omega$ → determined by detector

dN: experimentalist measure

$\mathcal{L}$ → determined by accelerator

that is, the differential cross section is

$$D(\theta,\varphi) = \frac{d\sigma}{d\Omega} = \frac{dN}{\mathcal{L}\, d\Omega}$$

If the dector completely surrounds the target, then $N = \sigma \mathcal{L}$ (the event rate is the cross section times the luminosity).

2. The Golden Rule.

Fermi's Golden Rule says that a transition rate is given by the product of the phase space and the absolute square of the amplitude.

For the formulas of decay rate and cross section, see my particle physics lecture notes.

For the factor in the $2 \to n$ scattering cross section $E_1 E_2 |\vec{v}_1 - \vec{v}_2|$, it can be written in general as

$$E_1 E_2 |\vec{v}_1 - \vec{v}_2| = E_1 E_2 \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right| = |\vec{p}_1 E_2 - \vec{p}_2 E_1|$$

Since $(p_1 \cdot p_2)^2 - (m_1 m_2)^2 - (\vec{p}_1 \cdot \vec{p}_2)^2 + |\vec{p}_1|^2 |\vec{p}_2|^2$

$$= (E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2)^2 - m_1^2 m_2^2 - (\vec{p}_1 \cdot \vec{p}_2)^2 + |\vec{p}_1|^2 |\vec{p}_2|^2$$

$$= E_1^2 E_2^2 - 2(\vec{p}_1 \cdot \vec{p}_2) E_1 E_2 - m_1^2 m_2^2 + |\vec{p}_1|^2 |\vec{p}_2|^2$$

$$= (m_1^2+|\vec{P}_1|^2)(m_2^2+|\vec{P}_2|^2) - 2(\vec{P}_1\cdot\vec{P}_2)E_1E_2 - m_1^2m_2^2 + |\vec{P}_1|^2|\vec{P}_2|^2$$

$$= m_1^2|\vec{P}_2|^2 + m_2^2|\vec{P}_1|^2 + 2|\vec{P}_1|^2|\vec{P}_2|^2 - 2(\vec{P}_1\cdot\vec{P}_2)E_1E_2$$

while $|\vec{P}_1E_2 - \vec{P}_2E_1|^2 = -2(\vec{P}_1\cdot\vec{P}_2)E_1E_2 + |\vec{P}_1|^2E_2^2 + |\vec{P}_2|^2E_1^2$

$$= -2(\vec{P}_1\cdot\vec{P}_2)E_1E_2 + 2|\vec{P}_1|^2|\vec{P}_2|^2 + m_2^2|\vec{P}_1|^2 + m_1^2|\vec{P}_2|^2$$

$$\Rightarrow \quad E_1E_2|\vec{v}_1-\vec{v}_2| = \left[(P_1\cdot P_2)^2 - (m_1\cdot m_2)^2 - (\vec{P}_1\cdot\vec{P}_2)^2 + |\vec{P}_1|^2|\vec{P}_2|^2\right]^{\frac{1}{2}}$$

The last two terms vanish in the rest frame where $\vec{P}_1$ and $\vec{P}_2$ are collinear or when one of them is zero.