Examples of calculations involving scalar particles only

Let's consider the following Lagrangian for two real scalar fields,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}\mu^2\sigma^2 + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2$$
$$- g\phi^4 - g'\sigma^2\phi^2 - g''\sigma^4 - \lambda\sigma\phi^2 - \lambda'\sigma^3$$
$$= \mathcal{L}_0 + \mathcal{L}_{int}$$

where $\mathcal{L}_0$ is the first line, and $\mathcal{L}_{int}$ is the second line.

$\underbrace{g,\ g',\ g''}_{\text{dimensionless real numbers}}$, $\underbrace{\lambda \text{ and } \lambda'}_{\text{mass dimension 1 real numbers}}$ are coupling constants.

Note that there could be some other interaction terms between σ and ϕ, e.g., $g'''\sigma\phi^3$, $g''''\sigma^3\phi$, $\lambda''\phi^3$, $\lambda'''\sigma^2\phi$, where g''' and g'''' are dimensionless, λ'' and λ''' are mass dimension 1.

However, let's <u>assume</u> that $\sigma(x)$ is a scalar field, while $\phi(x)$ is a pseudoscalar field, and that $\mathcal{L}$ preserve P-parity (that is, under a parity transformation, $\sigma(t,\vec{x}) \to \sigma(t,-\vec{x}) = +\sigma(t,\vec{x})$, while $\phi(t,\vec{x}) \to \phi(t,-\vec{x}) = -\phi(t,\vec{x})$), so that there is no interaction terms with odd power of ϕ.

Also, let's assume that we only care about terms with coupling constants with mass dimension zero or one.

(Note that although σ and ϕ both behave as scalar field under proper Lorentz transformation, the parity transformation differentiates σ as scalar field and ϕ as pseudoscalar field).

Anyway, for now let's just take the $\mathcal{L}_{int}$ as the given one, and we start the calculations from it.

First, recall that $\mathcal{H}(\pi_i, \varphi_i) = \sum_i (\pi_i \frac{\partial \varphi_i}{\partial t}) - \mathcal{L}$

where the conjugate momentum for a generic field φ_i is

$$\pi_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial \varphi_i / \partial t)}$$

From $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$,

$$\Rightarrow \mathcal{H} = \sum_i \left(\frac{\partial(\mathcal{L}_0 + \mathcal{L}_{int})}{\partial(\partial\varphi_i/\partial t)} \frac{\partial \varphi_i}{\partial t} \right) - \mathcal{L}_0 - \mathcal{L}_{int}$$

$$= \underline{\sum_i \left(\frac{\partial \mathcal{L}_0}{\partial(\partial\varphi_i/\partial t)} \frac{\partial\varphi_i}{\partial t}\right) - \mathcal{L}_0}$$

$$+ \sum_i \left(\frac{\partial \mathcal{L}_{int}}{\partial(\partial\varphi_i/\partial t)} \frac{\partial\varphi_i}{\partial t}\right) - \mathcal{L}_{int}$$

$$= \mathcal{H}_0 + \mathcal{H}_{int}$$

Since our $\mathcal{L}_{int}$ contains no time derivatives of ϕ and σ, our $\mathcal{H}_{int}$ is just $\mathcal{H}_{int} = -\mathcal{L}_{int} = g\phi^4 + g'\sigma^2\phi^2 + g''\sigma^4 + \lambda\sigma\phi^2 + \lambda'\sigma^3$.

Assume $\mu > 2m$, let's calcute the decay rate for the process $\sigma(k) \longrightarrow \phi(q_1) + \phi(q_2)$, in the rest frame of the decaying particle.

We have already worked out the phase space part, so all we need to know is $|M|^2 \equiv |\langle f|M|i\rangle|^2$,

where $|i\rangle$ is the one particle state with momentum k, i.e.,

$f(\vec{k})\, \alpha^\dagger_{\vec{k}}|0\rangle$, where $f(\vec{k}) = \underbrace{C(E_{\vec{k}})}_{\text{real}}(2\pi)^3 2E_{\vec{k}}$

and $\langle f|$ is $\langle 0|\beta_{\vec{q}_1}\beta_{\vec{q}_2} f(\vec{q}_1) f(\vec{q}_2)$

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$$\sigma(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}}) \left[\alpha_{\vec{p}} e^{-ip\cdot x} + \alpha_{\vec{p}}^{+} e^{ip\cdot x} \right] d^3\vec{p}$$

$$\phi(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}}) \left[\beta_{\vec{p}} e^{-ip\cdot x} + \beta_{\vec{p}}^{+} e^{ip\cdot x} \right] d^3\vec{p}$$

$$[\beta_{\vec{p}}, \beta_{\vec{p}'}^{+}] = [\alpha_{\vec{p}}, \alpha_{\vec{p}'}^{+}] = \frac{1}{(2\pi)^3 2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})} \right)^2 \delta^3(\vec{p} - \vec{p}'),$$

all other commutators are zero. Note that $[\alpha_{\vec{p}}, \beta_{\vec{p}'}^{+}] = 0$, $[\alpha_{\vec{p}}, \beta_{\vec{p}'}] = 0$ etc., since σ and ϕ are independent fields

Recall that $\mathcal{M}$ is just the one in

$$T = (2\pi)^4 \delta^4 (\Sigma p_i^{\mu} - \Sigma p_f^{\mu}) \mathcal{M}$$

and

$$\hat{S} = \mathbb{1} + iT$$

and $\hat{S} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n T(H_{int}(x_1) H_{int}(x_2) \cdots H_{int}(x_n))$

(we have dropped the subscript "I" in $H_I^{int}(x)$)

$\sigma(x)$ and $\phi(x)$ fields should be understood as $\sigma_I(x)$ and $\phi_I(x)$, respectively.

$$\Rightarrow iT = \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n T(H_{int}(x_1) H_{int}(x_2) \cdots H_{int}(x_n))$$

$$\langle f | iT | i \rangle = (2\pi)^4 \delta^4 (\Sigma p_i^{\mu} - \Sigma p_f^{\mu}) i \langle f | \mathcal{M} | i \rangle$$

$$= \underbrace{\langle f | \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n T(H_{int}(x_1) H_{int}(x_2) \cdots H_{int}(x_n)) | i \rangle}$$

So, we just need to evaluate this quantity, factor out $(2\pi)^4 \delta^4 (\Sigma p_i^{\mu} - \Sigma p_f^{\mu})$, and the remaining is just $i\langle f | \mathcal{M} | i \rangle \equiv i\mathcal{M}_{fi}$

Let's evaluate

$$* \equiv \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2} C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$

$$\times \Big\{ (-i)\int d^4x :(g\phi^4(x) + g'\sigma^2(x)\phi^2(x) + g''\sigma^4(x) + \lambda\sigma(x)\phi^2(x) + \lambda'\sigma^3(x)):$$

$$+ \frac{1}{2!}(-i)^2 \int d^4x\, d^4y\, T[:(g\phi^4(x) + \cdots + \lambda'\sigma^3(x))::(g\phi^4(y) + \cdots + \lambda'\sigma^3(y)):]$$

$$+ \cdots \Big\}$$

$$\times \quad C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}} |0\rangle$$

To the lowest possible order, the non-vanishing contribution to $*$ comes from the term $(-i)\int d^4x\, \lambda\sigma(x)\phi^2(x)$, and let's calculate this contribution only.

$$** \equiv \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2} C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} (-i)\int d^4x :\lambda\sigma(x)\phi^2(x):$$
$$\times\ C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}} |0\rangle.$$

$$= \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2} C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} (-i)\lambda \,.$$

$$\times \int d^4x : \int_{-\infty}^{+\infty} C(E_{\vec{p}_1})[\alpha_{\vec{p}_1} e^{-ip_1\cdot x} + \alpha^+_{\vec{p}_1} e^{ip_1\cdot x}]\, d^3\vec{p}_1 \int_{-\infty}^{+\infty} C(E_{\vec{p}_2})[\beta_{\vec{p}_2} e^{-ip_2\cdot x} + \beta^+_{\vec{p}_2} e^{ip_2\cdot x}]\, d^3\vec{p}_2$$

$$\int_{-\infty}^{+\infty} C(E_{\vec{p}_3})[\beta_{\vec{p}_3} e^{-ip_3\cdot x} + \beta^+_{\vec{p}_3} e^{ip_3\cdot x}]\, d^3\vec{p}_3 : C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}} |0\rangle$$

$$= \quad C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}$$

$$\times (-i)\lambda \quad \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2}$$

$$\times \int d^4x \int_{-\infty}^{+\infty} d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3\, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) \left(e^{ip_2\cdot x} e^{ip_3\cdot x} e^{-ip_1\cdot x} \right)$$

$$\times \quad \beta^+_{\vec{p}_2} \beta^+_{\vec{p}_3} \alpha_{\vec{p}_1}$$

$$\times \quad \alpha^+_{\vec{k}} |0\rangle$$

using $\langle 0| \beta_{\vec{q}_1} \beta_{\vec{q}_2} \beta^{+}_{\vec{p}_2} \beta^{+}_{\vec{p}_3} \alpha_{\vec{p}_1} \alpha^{+}_{\vec{k}} |0\rangle$

$$= \langle 0| \beta_{\vec{q}_1} \beta_{\vec{q}_2} \beta^{+}_{\vec{p}_2} \beta^{+}_{\vec{p}_3} |0\rangle \ \langle 0| \alpha_{\vec{p}_1} \alpha^{+}_{\vec{k}} |0\rangle$$

where $\langle 0| \beta_{\vec{q}_1} \beta_{\vec{q}_2} \beta^{+}_{\vec{p}_2} \beta^{+}_{\vec{p}_3} |0\rangle$

$$= \langle 0| \beta_{\vec{q}_1} \beta^{+}_{\vec{p}_3} |0\rangle \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2 - \vec{p}_2)$$

$$+ \langle 0| \beta_{\vec{q}_1} \beta^{+}_{\vec{p}_2} \beta_{\vec{q}_2} \beta^{+}_{\vec{p}_3} |0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{q}_1 - \vec{p}_3) \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2 - \vec{p}_2)$$

$$+ \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{q}_1 - \vec{p}_2) \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2 - \vec{p}_3)$$

and $\langle 0| \alpha_{\vec{p}_1} \alpha^{+}_{\vec{k}} |0\rangle$

$$= \frac{1}{(2\pi)^3 2E_{\vec{k}}} \left(\frac{1}{C(E_{\vec{k}})}\right)^2 \delta^3(\vec{k} - \vec{p}_1)$$

$$\Rightarrow ** \underset{\uparrow}{=} (-i)\lambda \int d^4x \left[e^{iq_2\cdot x} e^{iq_1\cdot x} e^{-ik\cdot x} + e^{iq_1\cdot x} e^{iq_2\cdot x} e^{-ik\cdot x} \right]$$

note that e.g., $\int_{-\infty}^{+\infty} d^3\vec{p}_1 \, e^{-ip_1\cdot x} \delta^3(\vec{k} - \vec{p}_1) = \int_{-\infty}^{+\infty} d^3\vec{p}_1 \, e^{-ip_1^0 x^0 + i\vec{p}_1\cdot\vec{x}} \delta^3(\vec{k} - \vec{p}_1)$

$$\underset{\uparrow}{=} e^{-i(|\vec{k}|^2+\mu^2)^{\frac{1}{2}} x^0} e^{i\vec{k}\cdot\vec{x}} \underset{\uparrow}{=} e^{-ik\cdot x}$$

since $p_1^0 = (|\vec{p}_1|^2 + \mu^2)^{\frac{1}{2}}$ since $k^0 = (|\vec{k}|^2 + \mu^2)^{\frac{1}{2}}$

$$\Rightarrow ** = (-i)\lambda \ (2\pi)^4 (k - q_1 - q_2) \ 2$$

$$\Rightarrow \ i\mathcal{M}_{fi} = 2(-i)\lambda$$

$$\Rightarrow \ |\mathcal{M}_{fi}|^2 = |\langle f|\mathcal{M}|i\rangle|^2 = |i\mathcal{M}_{fi}|^2 = 4\lambda^2$$

$$\Rightarrow \ \left(\frac{d\Gamma}{d\Omega}\right)_{cm} = \frac{\lambda^{\frac{1}{2}}(\mu^2, m^2, m^2)}{64\pi^2\mu^3} 4\lambda^2 \frac{1}{2!} = \frac{(\mu^4 + 2m^4 - 4m^2\mu^2 - 2m^4)^{\frac{1}{2}}}{64\pi^2\mu^3} 4\lambda^2 \frac{1}{2}$$

$$\Rightarrow \quad \Gamma = \frac{1}{64\pi^2 \mu}\left[1-\left(\frac{2m}{\mu}\right)^2\right]^{\frac{1}{2}} 4\lambda^2 \cdot 4\pi \,\frac{1}{2}$$

$$= \frac{\lambda^2}{8\pi\mu}\left[1-\left(\frac{2m}{\mu}\right)^2\right]^{\frac{1}{2}}$$

Now consider that the $\mathcal{L}$ is enlarged to include complex scalar,

$$\mathcal{L} = \mathcal{L}_{\text{page 72}} + \partial_\mu \varphi^+ \partial^\mu \varphi - m^2 \varphi^+ \varphi$$
$$-4g(\varphi^+\varphi)^2 - 4g\,\varphi^+\varphi\,\phi^2 - 2g'\sigma^2\varphi^+\varphi$$
$$-2\lambda\sigma\,\varphi^+\varphi$$

Note that we can build this Lagrangian by a real triplet ϕ_a, $(a=1,2,3)$, and identify $\underbrace{\varphi \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \Rightarrow \varphi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)}$, $\phi = \phi_3$

$$\Rightarrow \quad \phi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^+)$$
$$\phi_2 = \frac{i}{\sqrt{2}}(\varphi - \varphi^+)$$

$$\Rightarrow \frac{1}{2}\sum_{a=1}^{3}(\partial_\mu \phi_a \partial^\mu \phi_a) = \frac{1}{2}\partial_\mu\phi_3\partial^\mu\phi_3 + \frac{1}{2}\partial_\mu\left[\frac{1}{\sqrt{2}}(\varphi+\varphi^+)\right]\partial^\mu\left[\frac{1}{\sqrt{2}}(\varphi+\varphi^+)\right]$$
$$+\frac{1}{2}\partial_\mu\left[\frac{i}{\sqrt{2}}(\varphi-\varphi^+)\right]\partial^\mu\left[\frac{i}{\sqrt{2}}(\varphi-\varphi^+)\right]$$
$$= \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\varphi^+\partial^\mu\varphi$$

$$-\frac{1}{2}m^2\sum_{a=1}^{3}\phi_a^2 = -\frac{1}{2}m^2\phi_3^2 - \frac{1}{2}m^2(\phi_1^2+\phi_2^2)$$
$$= -\frac{1}{2}m^2\phi^2 - m^2\varphi^+\varphi$$

$$-g\left(\sum_{a=1}^{3}\phi_a^2\right)^2 = -g(\phi_1^2+\phi_2^2+\phi_3^2)^2 = -g(2\varphi^+\varphi+\phi_3^2)^2$$
$$= -g\phi^4 - 4g(\varphi^+\varphi)^2 - 4g(\varphi^+\varphi)\phi^2$$

$$-\lambda\sigma\sum_{a=1}^{3}\phi_a^2 = -\lambda\sigma(\phi_1^2+\phi_2^2+\phi_3^2) = -\lambda\sigma\phi^2 - 2\lambda\sigma\varphi^+\varphi$$
$$-g'\sigma^2\sum_{a=1}^{3}\phi_a^2 = -g'\sigma^2\phi^2 - 2g'\sigma^2\varphi^+\varphi$$

Again, since $\mathcal{L}_{int}$ (the new one include φ & φ^+) contains no time derivatives of ϕ, σ, φ & φ^+, $\mathcal{H}_{int}$ is just $-\mathcal{L}_{int}$.

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = g\phi^4 + g'\sigma^2\phi^2 + g''\sigma^4 + \lambda\sigma\phi^2 + \lambda'\sigma^3 + 4g(\varphi^+\varphi)^2 + 4g\varphi^+\varphi\phi^2$$
$$+2g'\sigma^2\varphi^+\varphi + 2\lambda\sigma\varphi^+\varphi$$

Let's calculate $\sigma(k) \to \pi^+(q_1)\,\pi^-(q_2)$

Again, let's only calculate to the lowest order.

$$|i\rangle = C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}} |0\rangle$$

The complex scalar field φ is

$$\varphi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \left[a_{\vec{p}}\, e^{-ip\cdot x} + b^+_{\vec{p}}\, e^{ip\cdot x} \right]$$

$$\varphi^+(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \left[a^+_{\vec{p}}\, e^{ip\cdot x} + b_{\vec{p}}\, e^{-ip\cdot x} \right]$$

$$\langle f| = \langle 0|\, a_{\vec{q}_1} b_{\vec{q}_2}\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$

Note that we have assigned $C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}}\, a^+_{\vec{p}}|0\rangle$ as the one-particle state for $\pi^+(p)$, i.e., $|\pi^+(p)\rangle$, and $C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}}\, b^+_{\vec{p}}|0\rangle$ as the one-particle state for $\pi^-(p)$, i.e., $|\pi^-(p)\rangle$

$$\langle f|iT|i\rangle = \langle 0|\, a_{\vec{q}_1} b_{\vec{q}_2}\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$
$$\times \left\{ -i \int d^4x :2\lambda\, \sigma(x)\, \varphi^+(x)\varphi(x): \right\}$$
$$\times\; C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}}|0\rangle$$

$$= \langle 0|\, a_{\vec{q}_1} b_{\vec{q}_2}\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$
$$\times (-i)(2\lambda) \int d^4x : \int_{-\infty}^{+\infty} d^3\vec{p}_1\, C(E_{\vec{p}_1})\, \alpha_{\vec{p}_1} e^{-ip_1\cdot x} \int_{-\infty}^{+\infty} d^3\vec{p}_2\, C(E_{\vec{p}_2})\, a^+_{\vec{p}_2} e^{ip_2\cdot x}$$
$$\times \int_{-\infty}^{+\infty} d^3\vec{p}_3\, C(E_{\vec{p}_3})\, b^+_{\vec{p}_3} e^{ip_3\cdot x} :$$
$$\times\; C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}}|0\rangle$$

where

$$\langle 0|\, a_{\vec{q}_1} b_{\vec{q}_2}\, a^+_{\vec{p}_2} b^+_{\vec{p}_3}\, \alpha_{\vec{p}_1} \alpha^+_{\vec{k}} |0\rangle$$
$$= \langle 0| a_{\vec{q}_1} a^+_{\vec{p}_2}|0\rangle \times \langle 0| b_{\vec{q}_2} b^+_{\vec{p}_3}|0\rangle \times \langle 0| \alpha_{\vec{p}_1} \alpha^+_{\vec{k}}|0\rangle$$
$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{q_1})}\right)^2 \delta^3(\vec{q}_1 - \vec{p}_2)\, \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{q_2})}\right)^2 \delta^3(\vec{q}_2 - \vec{p}_3)\, \frac{1}{(2\pi)^3 2E_{\vec{k}}} \left(\frac{1}{C(E_{\vec{k}})}\right)^2 \delta^3(\vec{p}_1 - \vec{k})$$

$$\Rightarrow \langle f|iT|i\rangle = (-i)(2\lambda) \int d^4x\, e^{-ik\cdot x} e^{iq_1\cdot x} e^{iq_2\cdot x}$$
$$= (-i)(2\lambda)(2\pi)^4 \delta^4(k - q_1 - q_2)$$

$\Rightarrow iM_{fi} = (-i)2\lambda$

$$\Rightarrow \Gamma = \frac{\lambda^2}{4\pi\mu}\left[1-\left(\frac{2m}{\mu}\right)^2\right]^{\frac{1}{2}}$$

So $$\frac{\Gamma_{\sigma\to\pi^+\pi^-}}{\Gamma_{\sigma\to\phi\phi}} = 2$$

Now consider scattering process to the second order in g's and λ's

$$\pi^+(q_1) + \pi^-(q_2) \to \pi^+(q_3) + \pi^-(q_4)$$

$$\langle f| = \langle 0|C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}\, a_{\vec{q}_3} b_{\vec{q}_4}$$

$$|i\rangle = C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^\dagger_{\vec{q}_1} b^\dagger_{\vec{q}_2}|0\rangle$$

$$\Rightarrow \langle f|iT|i\rangle = \langle 0| a_{\vec{q}_3} b_{\vec{q}_4}\, C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

$$\times \Big\{(-i)\int d^4x :\big(g\phi^4(x) + g'\sigma^2(x)\phi^2(x) + g''\sigma^4(x) + \lambda\sigma(x)\phi^2(x) \underset{+\lambda\sigma^3(x)}{+} 4g(\varphi^\dagger(x)\varphi(x))^2$$
$$+ 4g\varphi^\dagger(x)\varphi(x)\phi^2(x) + 2g'\sigma^2(x)\varphi^\dagger(x)\varphi(x) + 2\lambda\sigma(x)\varphi^\dagger(x)\varphi(x)\big):$$
$$+ \frac{1}{2!}(-i)^2\int d^4x\, d^4y\, T\Big[:\big(g\phi^4(x) + \cdots + 2\lambda\sigma(x)\varphi^\dagger(x)\varphi(x)\big):$$
$$:\big(g\phi^4(y) + \cdots + 2\lambda\sigma(y)\varphi^\dagger(y)\varphi(y)\big):\Big]\Big\}$$

$$\times\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^\dagger_{\vec{q}_1} b^\dagger_{\vec{q}_2}|0\rangle$$

In the $\{\ \}$, the contribution to the process we are considering are

$$(-i)\int d^4x :4g(\varphi^\dagger(x)\varphi(x))^2:$$

$$+ \frac{1}{2!}(-i)^2\int d^4x\, d^4y\, T\Big[:(4g\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)):(4g\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y)):$$
$$+ :(4g\varphi^\dagger(x)\varphi(x)\phi(x)\phi(x) + 2g'\sigma(x)\sigma(x)\varphi^\dagger(x)\varphi(x)$$
$$+ 2\lambda\sigma(x)\varphi^\dagger(x)\varphi(x)):$$
$$:(4g\varphi^\dagger(y)\varphi(y)\phi(y)\phi(y) + 2g'\sigma(y)\sigma(y)\varphi^\dagger(y)\varphi(y)$$
$$+ 2\lambda\sigma(y)\varphi^\dagger(y)\varphi(y)):\Big]$$

Note that the terms like $T[:g\phi^4(x):\ :4g(\varphi^+(y)\varphi(y))^2:]$ does not contribute because although we need a φ^+ to kill the $a_{\vec{q}_3}$ in $\langle f|$, a φ to kill $b_{\vec{q}_4}$ in $\langle f|$, a φ to kill $a^+_{\vec{q}_1}$ in $|i\rangle$, a φ^+ to kill $b^+_{\vec{q}_2}$ in $|i\rangle$, we do not consider equal time contraction, we cannot get rid of $:g\phi^4(x):$

For the first term,

$$* \equiv \langle f|(-i)\int d^4x :4g\,\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x):|i\rangle$$

$$= (-i)\cdot 4g \langle 0| C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}\, a_{\vec{q}_3} b_{\vec{q}_4}$$

$$\times \int d^4x \int_{-\infty}^{+\infty} d^3\vec{p}_1\, C(E_{\vec{p}_1})\, d^3\vec{p}_2\, C(E_{\vec{p}_2})\, d^3\vec{p}_3\, C(E_{\vec{p}_3})\, d^3\vec{p}_4\, C(E_{\vec{p}_4})$$

$$\times :(a^+_{\vec{p}_1} e^{ip_1\cdot x} + b_{\vec{p}_1} e^{-ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + b^+_{\vec{p}_2} e^{ip_2\cdot x})$$
$$(a^+_{\vec{p}_3} e^{ip_3\cdot x} + b_{\vec{p}_3} e^{-ip_3\cdot x})(a_{\vec{p}_4} e^{-ip_4\cdot x} + b^+_{\vec{p}_4} e^{ip_4\cdot x}):$$

$$\times\ C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

where $\langle 0|a_{\vec{q}_3} b_{\vec{q}_4} :(a^+_{\vec{p}_1} e^{ip_1\cdot x} + b_{\vec{p}_1} e^{-ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + b^+_{\vec{p}_2} e^{ip_2\cdot x})$
$$(a^+_{\vec{p}_3} e^{ip_3\cdot x} + b_{\vec{p}_3} e^{-ip_3\cdot x})(a_{\vec{p}_4} e^{-ip_4\cdot x} + b^+_{\vec{p}_4} e^{ip_4\cdot x}):$$

$$\times a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

$$= \langle 0| a_{\vec{q}_3} b_{\vec{q}_4} \Big(a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_2} e^{ip_2\cdot x} b_{\vec{p}_3} e^{-ip_3\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot x}$$
$$+ a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot x} a_{\vec{p}_2} e^{-ip_2\cdot x} b_{\vec{p}_3} e^{-ip_3\cdot x}$$
$$+ a^+_{\vec{p}_3} e^{ip_3\cdot x} b^+_{\vec{p}_2} e^{ip_2\cdot x} b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot x}$$
$$+ a^+_{\vec{p}_3} e^{ip_3\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot x} b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_2} e^{-ip_2\cdot x} \Big) a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}}\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}}\left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}}\left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big(\delta^3(\vec{p}_1 - \vec{q}_3)\,\delta^3(\vec{p}_2 - \vec{q}_4)\,\delta^3(\vec{p}_3 - \vec{q}_2)\,\delta^3(\vec{p}_4 - \vec{q}_1)\, e^{i(p_1+p_2-p_3-p_4)\cdot x}$$
$$+ (p_2 \leftrightarrow p_4) + (p_1 \leftrightarrow p_3) + (p_1 \leftrightarrow p_3,\ p_2 \leftrightarrow p_4) \Big)$$

So

$$\Rightarrow \; * = (-i)\, 4g\, 4\, (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)$$

For the second term, we need to first use the Wick's theorem to contract out the fields not needed to kill the initial and final state operators.

$$** \equiv \langle f | \frac{1}{2!}(-i)^2 \int d^4x\, d^4y\; T[:\,:\,:\,:]\, | i \rangle$$

$$= \frac{1}{2!}(-i)^2 \langle f | \int d^4x\, d^4y\, (4g)(4g) \Big(:\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y): + :\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y)\varphi^+(y)\varphi(y):$$

$$+ :\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(y)\,\varphi(y)\,\varphi^{+}(y)\,\varphi(y):$$

$$+ :\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(y)\,\varphi(y)\,\varphi^{+}(y)\,\varphi(y):$$

$$+ :\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(y)\,\varphi(y)\,\varphi^{+}(y)\,\varphi(y):$$

$$+ :\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(x)\,\varphi(x)\,\varphi^{+}(y)\,\varphi(y)\,\varphi^{+}(y)\,\varphi(y):\Big)$$

$$+ (4g)(4g)\Big(:\varphi^{+}(x)\,\varphi(x)\,\phi(x)\,\phi(x)\,\varphi^{+}(y)\,\varphi(y)\,\phi(y)\,\phi(y):$$

$$+ :\varphi^{+}(x)\,\varphi(x)\,\phi(x)\,\phi(x)\,\varphi^{+}(y)\,\varphi(y)\,\phi(y)\,\phi(y):\Big)$$

$$+ (2g')(2g')\Big(:\sigma(x)\,\sigma(x)\,\varphi^{+}(x)\,\varphi(x)\,\sigma(y)\,\sigma(y)\,\varphi^{+}(y)\,\varphi(y):$$

$$+ :\sigma(x)\,\sigma(x)\,\varphi^{+}(x)\,\varphi(x)\,\sigma(y)\,\sigma(y)\,\varphi^{+}(y)\,\varphi(y):\Big)$$

$$+ (2\lambda)(2\lambda)\Big(:\sigma(x)\,\varphi^{+}(x)\,\varphi(x)\,\sigma(y)\,\varphi^{+}(y)\,\varphi(y):\Big)\Big\}$$

$$|i\rangle$$

using $$\varphi(x)\varphi^{+}(y) = \varphi^{+}(x)\varphi(y) = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2-m^2+i\varepsilon}$$

$$\phi(x)\phi(y) = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2-m_3^2+i\varepsilon}$$

, note that $m_3=m$. but let's temperarily use m_3 to distinguish the terms coming from different fields

$$\sigma(x)\sigma(y) = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2-\mu^2+i\varepsilon}$$

$$\Rightarrow ** = \frac{1}{2!}(-i)^2\langle f|\int d^4x\,d^4y\,\Big\{(4g)^2\Big[16\,\varphi^{+}(x)\varphi(y)\,\varphi(x)\varphi^{+}(y) :\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):$$

$$+ 2\,\varphi(x)\varphi^{+}(y)\,\varphi(x)\varphi^{+}(y) :\varphi^{+}(x)\varphi^{+}(x)\varphi(y)\varphi(y):$$

$$+ 2\,\varphi^{+}(x)\varphi(y)\,\varphi^{+}(x)\varphi(y) :\varphi(x)\varphi(x)\varphi^{+}(y)\varphi^{+}(y):\Big]$$

$$+ (4g)^2\,2\,\phi(x)\phi(y)\,\phi(x)\phi(y) :\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):$$

$$+ (2g')^2\,2\,\sigma(x)\sigma(y)\,\sigma(x)\sigma(y) :\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):$$

$$+ (2\lambda)^2\,\sigma(x)\sigma(y) :\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):\Big\}$$

$$|i\rangle$$

use $\langle f| : \varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y) : |i\rangle$

$$= \langle 0| C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}\, a_{\vec{q}_3} b_{\vec{q}_4}$$

$$\times \int_{-\infty}^{+\infty} d^3\vec{p}_1 C(E_{\vec{p}_1})\, d^3\vec{p}_2 C(E_{\vec{p}_2})\, d^3\vec{p}_3 C(E_{\vec{p}_3})\, d^4\vec{p}_3 C(E_{\vec{p}_4})$$

$$\times :(a^+_{\vec{p}_1} e^{ip_1\cdot x} + b_{\vec{p}_1} e^{-ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + b^+_{\vec{p}_2} e^{ip_2\cdot x})$$

$$(a^+_{\vec{p}_3} e^{ip_3\cdot y} + b_{\vec{p}_3} e^{-ip_3\cdot y})(a_{\vec{p}_4} e^{-ip_4\cdot y} + b^+_{\vec{p}_4} e^{ip_4\cdot y}):$$

$$\times\ C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^+_{\vec{q}_1} b^+_{\vec{q}_2} |0\rangle$$

where $\langle 0| a_{\vec{q}_3} b_{\vec{q}_4} : (a^+_{\vec{p}_1} e^{ip_1\cdot x} + b_{\vec{p}_1} e^{-ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + b^+_{\vec{p}_2} e^{ip_2\cdot x})$

$$(a^+_{\vec{p}_3} e^{ip_3\cdot y} + b_{\vec{p}_3} e^{-ip_3\cdot y})(a_{\vec{p}_4} e^{-ip_4\cdot y} + b^+_{\vec{p}_4} e^{ip_4\cdot y}) : a^+_{\vec{q}_1} b^+_{\vec{q}_2} |0\rangle$$

$$= \langle 0| a_{\vec{q}_3} b_{\vec{q}_4} \big(a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_2} e^{ip_2\cdot x}\, b_{\vec{p}_3} e^{-ip_3\cdot y} a_{\vec{p}_4} e^{-ip_4\cdot y}$$

$$+ a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot y}\, a_{\vec{p}_2} e^{-ip_2\cdot x} b_{\vec{p}_3} e^{-ip_3\cdot y}$$

$$+ a^+_{\vec{p}_3} e^{ip_3\cdot y} b^+_{\vec{p}_2} e^{ip_2\cdot x}\, b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot y}$$

$$+ a^+_{\vec{p}_3} e^{ip_3\cdot y} b^+_{\vec{p}_4} e^{ip_4\cdot y}\, b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_2} e^{-ip_2\cdot x} \big) a^+_{\vec{q}_1} b^+_{\vec{q}_2} |0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}} \left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big(\delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_4)\delta^3(\vec{p}_3-\vec{q}_2)\delta^3(\vec{p}_4-\vec{q}_1) e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_4-\vec{q}_4)\delta^3(\vec{p}_3-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_1) e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ \delta^3(\vec{p}_3-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_4)\delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_4-\vec{q}_1) e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_3-\vec{q}_3)\delta^3(\vec{p}_4-\vec{q}_4)\delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_1) e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y} \Big)$$

$\Rightarrow \langle f| : \varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y) : |i\rangle$

$$= e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y} + e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} + e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y}$$

and for $\langle f| :\varphi^+(x)\varphi^+(x)\varphi(y)\varphi(y): |i\rangle$

in which $\langle 0| a_{\vec{q}_3} b_{\vec{q}_4} :(a^+_{\vec{p}_1} e^{ip_1\cdot x} + b_{\vec{p}_1} e^{-ip_1\cdot x})(a^+_{\vec{p}_2} e^{ip_2\cdot x} + b_{\vec{p}_2} e^{-ip_2\cdot x})$

$(a_{\vec{p}_3} e^{-ip_3\cdot y} + b^+_{\vec{p}_3} e^{ip_3\cdot y})(a_{\vec{p}_4} e^{-ip_4\cdot y} + b^+_{\vec{p}_4} e^{ip_4\cdot y}): a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$

$$= \langle 0| a_{\vec{q}_3} b_{\vec{q}_4} \Big(a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_3} e^{ip_3\cdot y} b_{\vec{p}_2} e^{-ip_2\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot y}$$

$$+ a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot y} b_{\vec{p}_2} e^{-ip_2\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot y}$$

$$+ a^+_{\vec{p}_2} e^{ip_2\cdot x} b^+_{\vec{p}_3} e^{ip_3\cdot y} b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot y}$$

$$+ a^+_{\vec{p}_2} e^{ip_2\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot y} b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot y}\Big) a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}} \left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big(\delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_1)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_4)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_1)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_4)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}\Big)$$

$$\Rightarrow \langle f| :\varphi^+(x)\varphi^+(x)\varphi(y)\varphi(y): |i\rangle$$

$$= 4 e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

so, $\langle f| :\varphi(x)\varphi(x)\varphi^+(y)\varphi^+(y): |i\rangle$

$$= \langle f| :\varphi^+(y)\varphi^+(y)\varphi(x)\varphi(x): |i\rangle$$

$$= 4 e^{i(q_3-q_2)\cdot y} e^{i(q_4-q_1)\cdot x}$$

$$\Rightarrow ** = \frac{1}{2!}(-i)^2 \Bigg\{ (4g)^2 \int d^4x\, d^4y \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon} \int_{-\infty}^{+\infty} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\varepsilon}$$

$$\times \Big[16 \big(e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y} + e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} + e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y} \big)$$

$$+ 2\times 4\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ 2\times 4\, e^{i(q_3-q_2)\cdot y} e^{i(q_4-q_1)\cdot x} \Big]$$

$$+ (4g)^2 \times 2 \int d^4x\, d^4y \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m_3^2+i\varepsilon} \int_{-\infty}^{+\infty} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m_3^2+i\varepsilon}$$

$$\times \big(e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y} + e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} + e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y} \big)$$

$$+ (2g')^2 \times 2 \int d^4x\, d^4y \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-\mu^2+i\varepsilon} \int_{-\infty}^{+\infty} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-\mu^2+i\varepsilon}$$

$$\times \big(e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y} + e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} + e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y} \big)$$

$$+ (2\lambda)^2 \int d^4x\, d^4y \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-\mu^2+i\varepsilon}$$

$$\times \big(e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y} + e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} + e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y} \big) \Bigg\}$$

using $$\int d^4x\, d^4y \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} e^{-ik\cdot(x-y)} e^{-ip\cdot(x-y)} \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{p^2-m_2^2+i\varepsilon} e^{ir\cdot x} e^{is\cdot y}$$

$$= \int d^4x\, d^4y \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} e^{-i(k+p-r)\cdot x} e^{-i(-k-p-s)\cdot y} \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{p^2-m_2^2+i\varepsilon}$$

$$= \int d^4k\, d^4p\, \delta^4(k+p-r)\, \delta^4(k+p+s) \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{p^2-m_2^2+i\varepsilon}$$

$$= \int d^4k\, \delta^4(k+r-k+s) \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{(r-k)^2-m_2^2+i\varepsilon}$$

$$= (2\pi)^4 \delta^4(r+s) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{(r-k)^2-m_2^2+i\varepsilon} = (2\pi)^4 \delta^4(r+s) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{(r+k)^2-m_2^2+i\varepsilon}$$

$$= (2\pi)^4 \delta^4(r+s) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_1^2+i\varepsilon} \frac{i}{(s+k)^2 \ldots} = (2\pi)^4 \delta^4(r+s) \int \frac{d^4k}{\ldots} \frac{i}{\ldots} \frac{i}{\ldots}$$

and $$\int d^4x\, d^4y \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon} e^{ir\cdot x} e^{is\cdot y}$$

$$= \int d^4x\, d^4y \frac{d^4k}{(2\pi)^4} e^{-i(k-r)\cdot x} e^{-i(-k-s)\cdot y} \frac{i}{k^2-m^2+i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} (2\pi)^4 (2\pi)^4 \cdot \delta^4(k-r)\, \delta^4(k+s) \frac{i}{k^2-m^2+i\varepsilon}$$

$$= (2\pi)^4 \delta^4(r+s) \frac{i}{r^2-m^2+i\varepsilon}$$

$$= (2\pi)^4 \delta^4(r+s) \frac{i}{s^2-m^2+i\varepsilon}$$

$$\Rightarrow ** = \frac{1}{2!}(-i)^2 (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)$$

$$\times \Big\{ (4g)^2 \Big[16 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \Big(\frac{i}{(q_3+q_4-k)^2-m^2+i\varepsilon} + \frac{i}{(q_3-q_1-k)^2-m^2+i\varepsilon}$$
$$+ \frac{i}{(q_4-q_2-k)^2-m^2+i\varepsilon} + \frac{i}{(-q_1-q_2-k)^2-m^2+i\varepsilon} \Big)$$
$$+ 2\times 4 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(q_3-q_2-k)^2-m^2+i\varepsilon}$$
$$+ 2\times 4 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(q_4-q_1-k)^2-m^2+i\varepsilon} \Big]$$
$$+ (4g)^2 \times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_3^2+i\varepsilon} \Big(\frac{i}{(q_3+q_4-k)^2-m_3^2+i\varepsilon} + \frac{i}{(q_3-q_1-k)^2-m_3^2+i\varepsilon}$$
$$+ \frac{i}{(q_4-q_2-k)^2-m_3^2+i\varepsilon} + \frac{i}{(-q_1-q_2-k)^2-m_3^2+i\varepsilon} \Big)$$
$$+ (2g')^2 \times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \Big(\frac{i}{(q_3+q_4-k)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1-k)^2-\mu^2+i\varepsilon}$$
$$+ \frac{i}{(q_4-q_2-k)^2-\mu^2+i\varepsilon} + \frac{i}{(-q_1-q_2-k)^2-\mu^2+i\varepsilon} \Big)$$
$$+ (2\lambda)^2 \Big(\frac{i}{(q_3+q_4)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon} + \frac{i}{(q_4-q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(-q_1-q_2)^2-\mu^2+i\varepsilon} \Big) \Big\}$$

$$= \frac{1}{2!}(-i)^2 (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)$$

$$\times \Bigg\{ (4g)^2 \Bigg[16\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \left(\frac{i}{(q_1+q_2+k)^2-m^2+i\varepsilon} + \frac{i}{(q_3-q_1+k)^2-m^2+i\varepsilon} \right)$$

$$+ 2\times 4\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(q_3-q_2+k)^2-m^2+i\varepsilon} \Bigg]$$

$$+ (4g)^2 \times 2\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_3^2+i\varepsilon} \left(\frac{i}{(q_1+q_2+k)^2-m_3^2+i\varepsilon} + \frac{i}{(q_3-q_1+k)^2-m_3^2+i\varepsilon} \right)$$

$$+ (2g')^2 \times 2\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \left(\frac{i}{(q_1+q_2+k)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1+k)^2-\mu^2+i\varepsilon} \right)$$

$$+ (2\lambda)^2 \times 2 \left(\frac{i}{(q_1+q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon} \right) \Bigg\}$$

Now, let's only calculate the cross section up to the lowest order of g & λ, that is, we only keep the terms of $*$ and the last line in $**$.

So, $i\mathcal{M}_{fi} \equiv i\langle f|\mathcal{M}|i\rangle$

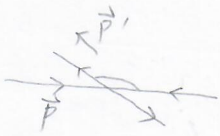
$$= (-i)\,4g\times 4 + \frac{1}{2!}(-i)^2 (2\lambda)^2 \times 2 \left(\frac{i}{(q_1+q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon} \right)$$

$$= 4\times(-i4g) + (-i2\lambda)^2 \left(\frac{i}{(q_1+q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon} \right)$$

Using $s \equiv (q_1+q_2)^2$, $t = (q_1-q_3)^2$

$$\Rightarrow |\mathcal{M}_{fi}|^2 = (i\mathcal{M}_{fi})(i\mathcal{M}_{fi})^* \underset{\varepsilon\to 0}{=} \left[16g + (2\lambda)^2 \left(\frac{1}{s-\mu^2} + \frac{1}{t-\mu^2} \right) \right]^2$$

using $\left(\frac{d\sigma}{dt}\right)_{cm} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s,m^2,m^2)}$



and $t \in \left[-(|\vec{p}|+|\vec{p}\,'|)^2,\ -(|\vec{p}|-|\vec{p}\,'|)^2 \right]$, where $|\vec{p}| = \dfrac{\lambda^{\frac{1}{2}}(s, m^2, m^2)}{2\sqrt{s}} = |\vec{p}\,'|$

$$\Rightarrow \quad t \in [-4|\vec{p}|^2, 0]$$

$$\Rightarrow \quad t \in \left[-4\frac{s^2+2m^4-2m^4-4sm^2}{4s}, 0\right]$$

$$\Rightarrow \quad t \in [-(s-4m^2), 0]$$

$$\Rightarrow \sigma = \int_{-(s-4m^2)}^{0} dt \; \frac{\left[16g + (2\lambda)^2\left(\frac{1}{s-\mu^2} + \frac{1}{t-\mu^2}\right)\right]^2}{16\pi(s^2+2m^4-2m^4-4sm^2)}$$

Define $\gamma^2 \equiv \dfrac{s}{4m^2}$, $\quad \omega = \dfrac{\frac{\mu^2}{m^2}}{4\frac{s}{4m^2}} - 1 = \dfrac{\mu^2}{s} - 1$

$$\Rightarrow \sigma = \frac{1}{128\pi m^6 \omega^2 \gamma^6}$$

$$\times \left\{ 512 g^2 m^4 \omega^2 \gamma^4 - 64 g \lambda^2 m^2 \omega \gamma^2 + \lambda^4 \frac{2[-1+2\gamma^2+2\omega^2\gamma^2+\omega(-1+3\gamma^2)]}{(1+\omega)[-1+(2+\omega)\gamma^2]} \right.$$

$$\left. - \frac{4\lambda^2\omega\gamma^2}{\gamma^2-1}(16 g m^2 \omega \gamma^2 - \lambda^2) \ln\left(\frac{[-1+(2+\omega)\gamma^2]}{(1+\omega)\gamma^2}\right) \right\}$$

The reason that we need to consider the proper number of creation & annihilation operators in calculating $\langle f|iT|i\rangle$ is the following:

Since the interchange of creation or annihilation operators among different fields at most introduce a sign (when the interchanged ones are both fermionic operators), and iT is normal ordered (or can be made normal ordered by Wick's theorem), we just need to consider

$$\underbrace{a\cdots a}_{m}\underbrace{a^{+}\cdots a^{+}}_{n}|0\rangle$$

Let's move the right most a to the far right:

$$\underbrace{a\cdots a}_{m}\underbrace{a^{+}\cdots a^{+}}_{n}|0\rangle = \underbrace{a\cdots a}_{m-1}\underbrace{a^{+}\cdots a^{+}}_{n-1}|0\rangle \times \delta \text{ function}$$
$$\pm \underbrace{a\cdots a}_{m-1} a^{+} a \underbrace{a^{+}\cdots a^{+}}_{n-1}|0\rangle$$

where the "$-$" sign is for fermionic operator.

Then

$$\underbrace{a\cdots a}_{m-1} a^{+} a \underbrace{a^{+}\cdots a^{+}}_{n-1}|0\rangle = \underbrace{a\cdots a}_{m-1} \overbrace{a^{+}\underbrace{a^{+}\cdots a^{+}}_{n-2}}^{n-1}|0\rangle \times \delta \text{ function}$$
$$\pm \underbrace{a\cdots a}_{m-1} a^{+} a^{+} a \underbrace{a^{+}\cdots a^{+}}_{n-2}|0\rangle$$

Then

$$\underbrace{a\cdots a}_{m-1} a^{+} a^{+} a \underbrace{a^{+}\cdots a^{+}}_{n-2}|0\rangle = \underbrace{a\cdots a}_{m-1} \overbrace{a^{+} a^{+}\underbrace{a^{+}\cdots a^{+}}_{n-3}}^{n-1}|0\rangle \times \delta \text{ function}$$
$$\pm \underbrace{a\cdots a}_{m-1} a^{+} a^{+} a^{+} a \underbrace{a^{+}\cdots a^{+}}_{n-3}|0\rangle$$

keep doing this, and finally

$$\underbrace{a\cdots a}_{m-1}\underbrace{a^{+}a^{+}\cdots a^{+}}_{n-1} a a^{+}|0\rangle = \underbrace{a\cdots a}_{m-1}\underbrace{a^{+}a^{+}\cdots a^{+}}_{n-1}|0\rangle \times \delta \text{ function}$$

So

$$\underbrace{a\cdots a}_{m}\underbrace{a^{+}\cdots a^{+}}_{n}|0\rangle = \sum\Big[\underbrace{a\cdots a}_{m-1}\underbrace{a^{+}\cdots a^{+}}_{n-1}|0\rangle \times \delta \text{ function}\Big]$$

keep doing this, we will get

$$\underbrace{a\cdots a}_{m}\underbrace{a^{+}\cdots a^{+}}_{n}|0\rangle = \begin{cases} \sum\Big[\underbrace{a^{+}\cdots a^{+}}_{n-m}|0\rangle \times \delta \text{ functions}\Big], & n \geq m \\ 0, & n < m \end{cases}$$

Do conjugation

$$\Rightarrow \langle 0|\underbrace{a\cdots a}_{r}\underbrace{a^{+}\cdots a^{+}}_{s} = \begin{cases} \sum\Big[\langle 0|\underbrace{a\cdots a}_{r-s} \times \delta \text{ functions}\Big], & r \geq s \\ 0, & r < s \end{cases}$$

$$\text{So, for } \overbrace{\langle 0| \underbrace{a\cdots a}_{r}}^{\langle f|} \; \overbrace{\underbrace{a^{+}\cdots a^{+}}_{s} \, \underbrace{a\cdots a}_{m}}^{iT} \; \overbrace{\underbrace{a^{+}\cdots a^{+}}_{n} |0\rangle}^{|i\rangle}$$

$$= \begin{cases} \sum [\langle 0| \underbrace{a\cdots a}_{r-s} \; \underbrace{a^{+}\cdots a^{+}}_{n-m} |0\rangle \times \delta \text{ functions}] , & n \geq m \;\&\; r \geq s \\ 0 , & \text{otherwise} \end{cases}$$

$$\text{while } \underbrace{a\cdots a}_{r-s} \; \underbrace{a^{+}\cdots a^{+}}_{n-m} |0\rangle = \begin{cases} \sum [\underbrace{a^{+}\cdots a^{+}}_{(n-m)-(r-s)} |0\rangle \times \delta \text{ functions}] , & (n-m) \geq (r-s) \\ 0 , & (n-m) < (r-s) \end{cases}$$

$$\text{and } \langle 0| \underbrace{a\cdots a}_{r-s} \; \underbrace{a^{+}\cdots a^{+}}_{n-m} | = \begin{cases} \sum [\langle 0| \underbrace{a\cdots a}_{(r-s)-(n-m)} | \times \delta \text{ functions}] , & (r-s) \geq (n-m) \\ 0 , & (r-s) < (n-m) \end{cases}$$

So, we could get non-zero value for $\langle f | iT | i \rangle$ if

$$\begin{cases} n-m \geq 0 \\ r-s \geq 0 \\ (n-m) \geq (r-s) \\ (r-s) \geq (n-m) \end{cases} \Rightarrow (n-m) = (r-s) \geq 0$$

However, if $(n-m) = (r-s) > 0$, we will have $(n-m)$ particles in the initial state directly become the $(r-s) = (n-m)$ particles in the final state. We are not interested in this situation, since we only care about fully connected diagrams, that is, all external legs connected to each other.

So, we only care about the case when $n=m$ & $r=s$