Interaction picture.

In Quantum Mechanics, we have learned two equivalent "pictures" or "representations" — the Schrödinger picture and the Heisenberg picture.

"S" picture

"H" picture.

In the Schrödinger picture, the operators are time independent, while the state vectors are time dependent;

In the Heisenberg picture, the operators are time dependent, while the state vectors are time independent.

The reason that we can formulate the same physics in different, but equivalent pictures is due to that we can only observe physical observables, which are expectation values or matrix elements of operators $\hat{O}$ evaluated with state vectors $|\alpha\rangle$, i.e., we can only observe the "sandwichs" like

$$\langle\alpha|\hat{O}|\alpha\rangle \quad \text{and} \quad \langle\beta|\hat{O}|\alpha\rangle,$$

rather than the $\hat{O}$, $|\alpha\rangle$ and $|\beta\rangle$ themselves.

The equivalence of the Schrödinger and Heisenberg pictures means

$$\langle\beta^S|\hat{O}^S|\alpha^S\rangle = \langle\beta^H|\hat{O}^H|\alpha^H\rangle,$$

and we call the special case when $|\beta\rangle = |\alpha\rangle$ the expectation value of the operator $\hat{O}$ with state vector $|\alpha\rangle$, i.e., $\langle\alpha^S|\hat{O}^S|\alpha^S\rangle = \langle\alpha^H|\hat{O}^H|\alpha^H\rangle$.

The Heisenberg and Schrödinger pictures are related by a unitary transformation:

$$\hat{O}^H(t) = e^{i\hat{H}_s(t-t_0)}\,\hat{O}^H(t_0)\,e^{-i\hat{H}_s(t-t_0)}$$

$$|\alpha, t\rangle^H = |\alpha, t_0\rangle^H \equiv |\alpha\rangle^H = e^{i\hat{H}_s(t-t_0)}|\alpha, t\rangle^S$$

where $\hat{O}^H(t_0) \equiv \hat{O}^S$ and $|\alpha\rangle^H = |\alpha, t_0\rangle^S$, that is, the two pictures agree at time $t = t_0$.

Therefore, ${}^S\langle \beta, t|\,\hat{O}^S\,|\alpha, t\rangle^S = {}^H\langle \beta|\,e^{i\hat{H}_s(t-t_0)}\,\hat{O}^H(t_0)\,e^{-i\hat{H}_s(t-t_0)}|\alpha\rangle^H$

$$= {}^H\langle \beta|\,\hat{O}^H(t)\,|\alpha\rangle^H.$$

where we have used the fact the $\hat{H}_s^{+} = H_s$.

Also note that

$$\hat{H}^H(t) = e^{i\hat{H}_s(t-t_0)}\,\hat{H}^H(t_0)\,e^{-i\hat{H}_s(t-t_0)}$$

$$= e^{i\hat{H}_s(t-t_0)}\,\hat{H}_s\,e^{-i\hat{H}_s(t-t_0)}$$

$$= \hat{H}_s$$

That is, the Hamiltonian in both pictures are the same. Note that we have assumed that $\hat{H}_s$ is time independent.

The operators in the Heisenberg picture satisfy the Heisenberg equation:

$$i\frac{\partial}{\partial t}\hat{O}^H(t) = -\hat{H}_s\hat{O}^H(t) + \hat{O}^H(t)\hat{H}_s = [\hat{O}^H(t), \hat{H}_s]$$

The state vectors in the Schrödinger picture satisfy the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\alpha, t\rangle^S = i\frac{\partial}{\partial t}e^{-i\hat{H}_s(t-t_0)}|\alpha\rangle^H = \hat{H}_s|\alpha, t\rangle^S$$

Note that the commutation relations between the two pictures are invariant, that is,

if $[\hat{A}^H(t), \hat{B}^H(t)]_\pm = \hat{C}^H(t)$, where the subscript "$\pm$" means commutation and anti-commutation relations, then

$$[\hat{A}^s, \hat{B}^s]_\pm = \hat{A}^s\hat{B}^s \pm \hat{B}^s\hat{A}^s = e^{-i\hat{H}_s(t-t_0)}\hat{A}^H(t)e^{i\hat{H}_s(t-t_0)}e^{-i\hat{H}_s(t-t_0)}\hat{B}^H(t)e^{i\hat{H}_s(t-t_0)} \pm (B \leftrightarrow A)$$

$$= e^{-i\hat{H}_s(t-t_0)}[\hat{A}^H(t), \hat{B}^H(t)]_\pm e^{i\hat{H}_s(t-t_0)}$$

$$= e^{-i\hat{H}_s(t-t_0)}\hat{C}^H(t)\, e^{i\hat{H}_s(t-t_0)}$$

$$= \hat{C}^s$$

In the free field theories we have studied in the QFT I course, we were actually working in the Heisenberg picture: recall that all the field operators were time dependent, as can be readily seen from their decompositions, for example,

for a real scalar field, $\hat{\phi}(\vec{x},t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})(\hat{a}_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^{+} e^{ip\cdot x})$;

and the state vectors are time independent, for example, the one particle state $|p\rangle$ for a real scalar field, $|p\rangle = C(E_p)(2\pi)^3 2E_p a_{\vec{p}}^{+}|0\rangle$.

Note that the creation & annihilation operators $\hat{a}_{\vec{p}}^{+}$ & $\hat{a}_{\vec{p}}$ are time independent. All the time dependence of the field operator, $\hat{\phi}(\vec{x},t)$, is in the exponential $e^{\pm ip\cdot x} = e^{\pm(iE_p t - i\vec{p}\cdot\vec{x})}$, since they are the solutions of Klein-Gordon equation, $(\Box + m^2)\hat{\phi}(\vec{x},t) = 0$, satisfied by the free scalar field (and all the other free field we have learned in QFT I).

To study the interaction of different fields (or the self-interactions of a field, e.g., ϕ^4 interaction), we first write the Hamiltonian as

$$\hat{H}_S = \hat{H}_S^0 + \hat{H}_S^{int},$$

where the subscript "s" indicates Schrödinger picture as usual, and the superscript "int" means interaction (NOTE: not means the interaction picture which will be introduced below).

In this expression, $\hat{H}_S^0$ is the Hamiltonian of free fields.

Note that still we have $\hat{H}^H(t) = \hat{H}_S$.

Let's now introduce the interaction picture, labeled by "I".
The relation between the "I" and "S" pictures is

$$\hat{O}^I(t) = e^{i\hat{H}_S^0(t-t_0)}\,\hat{O}^S\,e^{-i\hat{H}_S^0(t-t_0)}$$

$$|\alpha, t\rangle^I = e^{i\hat{H}_S^0(t-t_0)}|\alpha, t\rangle^S$$

Therefore, the physical observables, i.e., the sandwiches, remain the same:

$$^I\langle\beta, t|\hat{O}^I(t)|\alpha, t\rangle^I = {}^S\langle\beta, t|\,e^{-i\hat{H}_S^0(t-t_0)}\,\hat{O}^I(t)\,e^{i\hat{H}_S^0(t-t_0)}|\alpha, t\rangle^S$$

$$= {}^S\langle\beta, t|\hat{O}^S|\alpha, t\rangle^S.$$

Using the relation between the "S" and "H" pictures, we get the relation between the "I" and "H" pictures:

$$\hat{O}^I(t) = e^{i\hat{H}_S^0(t-t_0)}\,e^{-i\hat{H}_S(t-t_0)}\,\hat{O}^H(t)\,e^{i\hat{H}_S(t-t_0)}\,e^{-i\hat{H}_S^0(t-t_0)}$$

See page A1–A3.

$$\stackrel{?}{=} e^{-i\hat{H}_S^{int}(t-t_0)}\,\hat{O}^H(t)\,e^{i\hat{H}_S^{int}(t-t_0)}$$

$$|\alpha, t\rangle^I = e^{i\hat{H}_S^0(t-t_0)}\,e^{-i\hat{H}_S(t-t_0)}|\alpha, t\rangle^H$$

$$\stackrel{?}{=} e^{-i\hat{H}_S^{int}(t-t_0)}|\alpha, t\rangle^H$$

Therefore, if $\hat{H}_S^{int} = 0$, then $\hat{H}_S^0 = \hat{H}_S$, $\Rightarrow \hat{O}^I(t) = \hat{O}^H(t)$, $|\alpha, t\rangle^I = |\alpha, t\rangle^H$, that is, the "I" and "H" pictures are the same when $\hat{H}_S^{int} = 0$.

At $t = t_0$, all three pictures agree, that is,

$$\hat{O}^I(t_0) = \hat{O}^H(t_0) = \hat{O}^S,$$

$$|\alpha, t_0\rangle^I = |\alpha, t_0\rangle^H = |\alpha\rangle^H = |\alpha, t_0\rangle^S.$$

The state vector in the "I" picture evolues as

$$i\frac{\partial}{\partial t}|\alpha,t\rangle^I = i\frac{\partial}{\partial t}\left(e^{i\hat{H}_s^o(t-t_0)}|\alpha,t\rangle^s\right)$$

$$=\left(i\frac{\partial}{\partial t}e^{i\hat{H}_s^o(t-t_0)}\right)|\alpha,t\rangle^s + e^{i\hat{H}_s^o(t-t_0)}i\frac{\partial}{\partial t}|\alpha,t\rangle^s$$

$$= -\hat{H}_s^o e^{i\hat{H}_s^o(t-t_0)}|\alpha,t\rangle^s + e^{i\hat{H}_s^o(t-t_0)}\hat{H}_s|\alpha,t\rangle^s$$

$$= -\hat{H}_s^o|\alpha,t\rangle^I + e^{i\hat{H}_s^o(t-t_0)}\hat{H}_s e^{-i\hat{H}_s^o(t-t_0)}e^{i\hat{H}_s^o(t-t_0)}|\alpha,t\rangle^s$$

$$= -\hat{H}_s^o|\alpha,t\rangle^I + \hat{H}_I(t)|\alpha,t\rangle^I$$

Since $\hat{H}_I^o = e^{i\hat{H}_s^o(t-t_0)}\hat{H}_s^o e^{-i\hat{H}_s^o(t-t_0)} = \hat{H}_s^o$

then $i\frac{\partial}{\partial t}|\alpha,t\rangle^I = (\hat{H}_I - \hat{H}_I^o)|\alpha,t\rangle^I$

$$= \hat{H}_I^{int}(t)|\alpha,t\rangle^I$$

where $\hat{H}_I^{int}(t) = \hat{H}_I(t) - \hat{H}_I^o$

That is to say, in the "I" picture, the evolution of the state vector is determined by the interaction Hamiltonian.

The operator in the "I" picture evolues as

$$i\frac{\partial}{\partial t}\hat{O}^I(t) = i\frac{\partial}{\partial t}\left(e^{i\hat{H}_s^o(t-t_0)}\hat{O}^s e^{-i\hat{H}_s^o(t-t_0)}\right)$$

$$=\left(i\frac{\partial}{\partial t}e^{i\hat{H}_s^o(t-t_0)}\right)\hat{O}^s e^{-i\hat{H}_s^o(t-t_0)}$$

$$+ e^{i\hat{H}_s^o(t-t_0)}\hat{O}^s\left(i\frac{\partial}{\partial t}e^{-i\hat{H}_s^o(t-t_0)}\right)$$

$$= -\hat{H}_s^o\hat{O}^I(t) + \hat{O}^I(t)\hat{H}_s^o$$

$$= [\hat{O}^I(t), \hat{H}_s^o].$$

That is to say, in the "I" picture, the evolution of the operator is determined by the free field Hamiltonian $\hat{H}_s^o (= \hat{H}_I^o)$

In particular, the field operator in the "I" picture evolves as

$$i\frac{\partial}{\partial t}\hat{\Phi}^{I}(t) = [\,\hat{\Phi}^{I}(t),\ \hat{H}_s^{0}].$$

On the other hand, the free-field operator $\hat{\Phi}_0^H(t)$, in the "H" picture when $\hat{H}_s^{int} = 0$, satisfies

$$i\frac{\partial}{\partial t}\hat{\Phi}_0^H(t) = [\,\hat{\Phi}_0^H(t),\ \hat{H}_s^0]$$

where $\hat{H}_s^0 = \underset{\sim}{\hat{H}_0^H(t)}$ since $\hat{H}_0^H(t) = e^{i\hat{H}_s^0(t-t_0)}\hat{H}_0^H(t_0)e^{-i\hat{H}_s^0(t-t_0)}$

(note that actually, it is time independent, since it is a conserved Noether charge.) $= e^{i\hat{H}_s^0(t-t_0)}\hat{H}_s^0 e^{-i\hat{H}_s^0(t-t_0)}$

$= \hat{H}_s^0$

The above equation can be checked by writing $\hat{\Phi}_0^H(t)$ and $\hat{H}_0^H(t)$ explicitly:

e.g.

[1] for a real scalar field,

$$\begin{cases} \hat{\Phi}_0^H(t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\left(a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^{+}e^{ip\cdot x}\right) \\ \hat{H}_0^H(t) = \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right]a_{\vec{p}}^{+}a_{\vec{p}}E_{\vec{p}} + \text{zero point energy.}\end{cases}$$

$$\Rightarrow [\hat{\Phi}_0^H(t),\ \hat{H}_0^H(t)] = \Big[\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})(a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^{+}e^{ip\cdot x}),$$

$$\int_{-\infty}^{+\infty} d^3\vec{k}\left[(C(E_{\vec{k}}))^2(2\pi)^3 2E_{\vec{k}}\right]a_{\vec{k}}^{+}a_{\vec{k}}E_{\vec{k}} + \text{zero point energy}\Big]$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\,(E_{\vec{p}}a_{\vec{p}}e^{-ip\cdot x} - E_{\vec{p}}a_{\vec{p}}^{+}e^{ip\cdot x})$$

use $[a_{\vec{p}}, a_{\vec{k}}^{+}] = \frac{1}{(2\pi)^3 2E_{\vec{p}}}\left[\frac{1}{C(E_{\vec{p}})}\right]^2\delta^3(\vec{p}-\vec{k})$, $[a_{\vec{p}}, a_{\vec{k}}] = [a_{\vec{p}}^{+}, a_{\vec{k}}^{+}] = 0$

and $[\hat{A}, \hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$

while $i\frac{\partial}{\partial t}\hat{\Phi}_0^H(t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})E_{\vec{p}}\,(a_{\vec{p}}e^{-ip\cdot x} - a_{\vec{p}}^{+}e^{ip\cdot x})$

$\Rightarrow\ i\frac{\partial}{\partial t}\hat{\Phi}_0^H(t) = [\hat{\Phi}_0^H(t),\ \hat{H}_0^H(t)]$ ✓

[2] for a complex scalar field,

$$\begin{cases} \hat{\Phi}_0^H(t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\left(a_{\vec{p}}e^{-ip\cdot x} + b_{\vec{p}}^{+}e^{ip\cdot x}\right) \\ \hat{H}_0^H(t) = \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right](a_{\vec{p}}^{+}a_{\vec{p}} + b_{\vec{p}}^{+}b_{\vec{p}})E_{\vec{p}} + \text{zero point energy}\end{cases}$$

$$\Rightarrow [\hat{\Phi}_0^H(t), \hat{H}_0^H(t)] = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(E_{\vec{p}}\, a_{\vec{p}}\, e^{-ip\cdot x} - E_{\vec{p}}\, b_{\vec{p}}^{+}\, e^{ip\cdot x} \right)$$

$$= i\frac{\partial}{\partial t} \hat{\Phi}^H(t) \quad \checkmark$$

[3] for electromagnetic field in Coulomb gauge,

$$\begin{cases} \hat{\Phi}_0^H(t) \equiv \hat{A}_{\mu 0}^H(x) = \int_{-\infty}^{+\infty} d^3\vec{k} \sum_{\lambda=1}^{2} \left(\varepsilon_\mu(\vec{k},\lambda)\, a_{\vec{k},\lambda}\, e^{-ik\cdot x} + \varepsilon_\mu^*(\vec{k},\lambda)\, a_{\vec{k},\lambda}^{+}\, e^{ik\cdot x} \right) C(E_{\vec{k}}) \\ \hat{H}_0^H(t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, \left[(C(E_{\vec{p}}))^2 (2\pi)^3 2E_{\vec{p}} \right] E_{\vec{p}} \sum_{s=1}^{2} \left(a_{\vec{p},s}^{+}\, a_{\vec{p},s} \right) + \text{zero point energy} \end{cases}$$

$$\Rightarrow [\hat{A}_{\mu 0}^H(x), \hat{H}_0^H(t)] = \int_{-\infty}^{+\infty} d^3\vec{k} \sum_{\lambda=1}^{2} \left(\varepsilon_\mu(\vec{k},\lambda)\, E_{\vec{k}}\, a_{\vec{k},\lambda}\, e^{-ik\cdot x} - \varepsilon_\mu^*(\vec{k},\lambda)\, E_{\vec{k}}\, a_{\vec{k},\lambda}^{+}\, e^{ik\cdot x} \right) C(E_{\vec{k}})$$

$$= i\frac{\partial}{\partial t} \hat{A}_{\mu 0}^H(x) \quad \checkmark$$

[4] for Dirac field,

$$\begin{cases} \hat{\Phi}_0^H(t) \equiv \hat{\psi}_0^H(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left(u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, d_{\vec{p},s}^{+}\, e^{ip\cdot x} \right) \\ \hat{H}_0^H(t) = \int_{-\infty}^{+\infty} \left[(C(E_{\vec{p}}))^2 (2\pi)^3 2E_{\vec{p}} \right] E_{\vec{p}} \sum_s \left(b_{\vec{p},s}^{+}\, b_{\vec{p},s} + d_{\vec{p},s}^{+}\, d_{\vec{p},s} \right) + \text{zero point energy} \end{cases}$$

$$\Rightarrow [\hat{\psi}_0^H(x), \hat{H}_0^H(t)] \underset{\substack{\uparrow \\ \text{use } [\hat{A},\hat{B}\hat{C}] = \{\hat{A},\hat{B}\}\hat{C} - \hat{B}\{\hat{A},\hat{C}\}}}{=} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_s \left[E_{\vec{p}}\, u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} - E_{\vec{p}}\, v(\vec{p},s)\, d_{\vec{p},s}^{+}\, e^{ip\cdot x} \right]$$

$$= i\frac{\partial}{\partial t} \hat{\psi}_0^H(x) \quad \checkmark$$

Since $\hat{\Phi}^I(t)$ and $\hat{\Phi}_0^H(t)$ satisfies the same first order differential equation, and we have $\hat{\Phi}^I(t) = \hat{\Phi}_0^H(t)$ when $\hat{H}_s^{int} = 0$, and the evolution of $\hat{\Phi}^I(t)$ is determined by $\hat{H}_s^a$, we conclude that $\hat{\Phi}^I(t)$ is the same as the free field operator in the "H" picture.

It can be also understood like this: before "switch on" the interaction, $\hat{\Phi}^I(t) = \hat{\Phi}_0^H(t)$, then "switch on" the interaction, $\hat{\Phi}^I(t)$ still equals $\hat{\Phi}_0^H(t)$ since the evolution of $\hat{\Phi}^I(t)$ does not depend on $\hat{H}_s^{int}$.

Let's look at the time evolution of the state vector $|\alpha, t\rangle^I$.

Define the time-evolution operator $\hat{U}(t_2, t_1)$, which describes the connection between the state vector at t_2 and t_1,

$$|\alpha, t_2\rangle^I \equiv \hat{U}(t_2, t_1)|\alpha, t_1\rangle^I, \quad \text{so that } \hat{U}(t,t) = 1$$

Since $|\alpha, t_2\rangle^I = e^{i\hat{H}_s^\circ(t_2 - t_0)}|\alpha, t_2\rangle^s$

$|\alpha, t_1\rangle^I = e^{i\hat{H}_s^\circ(t_1 - t_0)}|\alpha, t_1\rangle^s$ (recall that at $t = t_0$, the three pictures agree.)

where $|\alpha, t_2\rangle^s$ and $|\alpha, t_1\rangle^s$ is connected through the Schrödinger equation

$$i\frac{\partial}{\partial t}|\alpha, t\rangle^s = \hat{H}_s|\alpha, t\rangle$$

$\Rightarrow$ formally, $|\alpha, t\rangle^s = e^{-i\hat{H}_s(t - t_1)}|\alpha, t_1\rangle^s$ assume $\hat{H}_s$ is time independent, as before.

so that $|\alpha, t_2\rangle^s = e^{-i\hat{H}_s(t_2 - t_1)}|\alpha, t_1\rangle^s$

$$\Rightarrow |\alpha, t_2\rangle^I = e^{i\hat{H}_s^\circ(t_2 - t_0)}e^{-i\hat{H}_s(t_2 - t_1)}e^{-i\hat{H}_s^\circ(t_1 - t_0)}|\alpha, t_1\rangle^I$$

$$\Rightarrow \hat{U}(t_2, t_1) = e^{i\hat{H}_s^\circ(t_2 - t_0)}e^{-i\hat{H}_s(t_2 - t_1)}e^{-i\hat{H}_s^\circ(t_1 - t_0)}$$

From $i\frac{\partial}{\partial t}|\alpha, t\rangle^I = \hat{H}_I^{int}(t)|\alpha, t\rangle^I$

and $|\alpha, t\rangle^I = \hat{U}(t, t')|\alpha, t'\rangle^I$

$$\Rightarrow i\frac{\partial}{\partial t}\hat{U}(t, t') = \hat{H}_I^{int}(t)\hat{U}(t, t'), \quad \text{since } i\frac{\partial}{\partial t}|\alpha, t'\rangle^I = 0.$$

$\hat{U}(t', t') = 1$ is the boundary condition.

Some properties of the time-evolution operator:

① $\hat{U}(t, t) = 1$ for any t.

② $\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = \hat{U}(t_3, t_1)$.

(check: $\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = e^{i\hat{H}_s^\circ(t_3 - t_0)}e^{-i\hat{H}_s(t_3 - t_2)}e^{-i\hat{H}_s^\circ(t_2 - t_0)} e^{i\hat{H}_s^\circ(t_2 - t_0)}e^{-i\hat{H}_s(t_2 - t_1)}e^{-i\hat{H}_s^\circ(t_1 - t_0)}$

$= e^{i\hat{H}_s^\circ(t_3 - t_0)}e^{-i\hat{H}_s(t_3 - t_1)}e^{-i\hat{H}_s^\circ(t_1 - t_0)} = \hat{U}(t_3, t_1)$)

③ from ① & ② $\Rightarrow$ $\hat{U}(t_1, t_2)\,\hat{U}(t_2, t_1) = \hat{U}(t_1, t_1) = 1$

$$\Rightarrow \boxed{\hat{U}^{-1}(t_2, t_1) = \hat{U}(t_1, t_2)}$$

④ $\hat{U}^{+}(t_2, t_1) \underset{\uparrow}{=} e^{i\hat{H}_s^{0}(t_1-t_0)}\, e^{i\hat{H}_s(t_2-t_1)}\, e^{-i\hat{H}_s^{0}(t_2-t_0)}$

$\hat{H}_s^{0}$ and $\hat{H}_s$ are Hermitian operator.

while $\hat{U}^{-1}(t_2, t_1) = \hat{U}(t_1, t_2) = e^{i\hat{H}_s^{0}(t_1-t_0)}\, e^{-i\hat{H}_s(t_1-t_2)}\, e^{-i\hat{H}_s^{0}(t_2-t_0)}$

$$\Rightarrow \hat{U}^{+}(t_2, t_1) = U^{-1}(t_2, t_1) = \hat{U}(t_1, t_2)$$

$$\Rightarrow \hat{U}^{+}(t_2, t_1)\,\hat{U}(t_2, t_1) = 1, \quad \hat{U}(t_2, t_1)\,\hat{U}^{+}(t_2, t_1) = 1$$

That is, $\boxed{\text{the time-evolution operator is a unitary operator.}}$

Therefore, ${}^{I}\langle \beta, t_2 | \alpha, t_2\rangle^{I} = {}^{I}\langle \beta, t_1 | \hat{U}^{+}(t_2, t_1)\, U(t_2, t_1) | \alpha, t_1\rangle^{I}$

$$= {}^{I}\langle \beta, t_1 | \alpha, t_1\rangle^{I}$$

⑤ For the special case $\hat{U}(t, t_0) = e^{i\hat{H}_s^{0}(t-t_0)}\, e^{-i\hat{H}_s(t-t_0)}\, e^{-i\hat{H}_s^{0}(t_0-t_0)}$

$$= e^{i\hat{H}_s^{0}(t-t_0)}\, e^{-i\hat{H}_s(t-t_0)}$$

it is just the one connect "H" and "I" picture, namely,

$$\hat{O}^{I}(t) = \hat{U}(t, t_0)\,\hat{O}^{H}(t)\,\hat{U}^{-1}(t, t_0)$$

$$|\alpha, t\rangle^{I} = \hat{U}(t, t_0)\,|\alpha, t\rangle^{H}.$$

Again, recall that all three pictures agree at $t = t_0$.

From $i\frac{\partial}{\partial t}\hat{U}(t, t') = \hat{H}_I^{int}(t)\,\hat{U}(t, t')$

and the boundary condition $\hat{U}(t', t') = 1$

$$\Rightarrow \hat{U}(t, t') = 1 + (-i)\int_{t'}^{t} dt_1\, \hat{H}_I^{int}(t_1)\,\hat{U}(t_1, t')$$

$$\Rightarrow \hat{U}(t,t') = 1 + (-i)\int_{t'}^{t} dt_1\, \hat{H}_I^{int}(t_1)\left(1 + (-i)\int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_2)\hat{U}(t_2,t')\right)$$

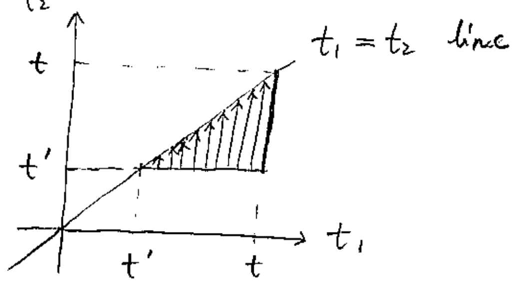
$$= 1 + (-i)\int_{t'}^{t} dt_1\, \hat{H}_I^{int}(t_1)$$
$$+ (-i)^2\int_{t'}^{t} dt_1\int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)\,\hat{U}(t_2,t')$$

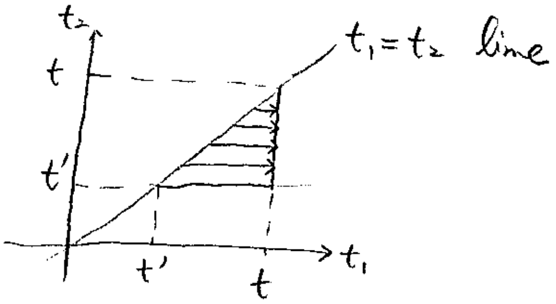
$$= 1 + (-i)\int_{t'}^{t} dt_1\, \hat{H}_I^{int}(t_1)$$
$$+ (-i)^2\int_{t'}^{t} dt_1\int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)$$
$$+ (-i)^3\int_{t'}^{t} dt_1\int_{t'}^{t_1} dt_2\int_{t'}^{t_2} dt_3\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)\hat{H}_I^{int}(t_3)\,\hat{U}(t_3,t')$$

$$= 1 + (-i)\int_{t'}^{t} dt_1\, \hat{H}_I^{int}(t_1)$$
$$+ (-i)^2\int_{t'}^{t} dt_1\int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)$$
$$+ \cdots$$
$$+ (-i)^n\int_{t'}^{t} dt_1\int_{t'}^{t_1} dt_2\cdots\int_{t'}^{t_{n-1}} dt_n\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)\cdots\hat{H}_I^{int}(t_n)$$
$$+ \cdots$$

Let's make the integration limit the same range $\int_{t'}^{t}$ for all integrals.

Take the term $\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\,\hat{H}_I^{int}(t_2)$ as an example.





$$\int_{t'}^{t} dt_1\int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2) = \int_{t'}^{t} dt_2\int_{t_2}^{t} dt_1\, \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)$$
$$\overset{t_1 \leftrightarrow t_2}{=} \int_{t'}^{t} dt_1\int_{t_1}^{t} dt_2\, \hat{H}_I^{int}(t_2)\hat{H}_I^{int}(t_1)$$

$$\Rightarrow \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)$$

$$= \frac{1}{2}\left(\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) + \int_{t'}^{t} dt_1 \int_{t_1}^{t} dt_2\, \hat{H}_I^{int}(t_2)\, \hat{H}_I^{int}(t_1) \right)$$

$$= \frac{1}{2}\left(\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)\, \theta(t_1 - t_2) + \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, \hat{H}_I^{int}(t_2)\, \hat{H}_I^{int}(t_1)\, \theta(t_2 - t_1) \right)$$

where $\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$.

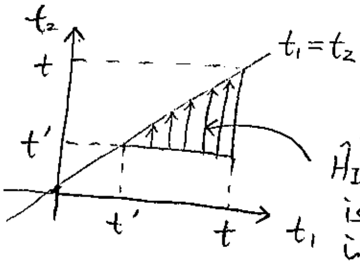
Define time-ordered product as

$$T\left(\hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)\right) \equiv \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)\, \theta(t_1 - t_2) + \hat{H}_I^{int}(t_2)\, \hat{H}_I^{int}(t_1)\, \theta(t_2 - t_1)$$

then $\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)$

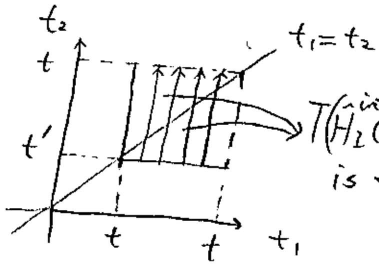
$$= \frac{1}{2} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, T\left(\hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)\right)$$

That is,



$\hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)$ is the integrand.

$= \frac{1}{2}$



$T(\hat{H}_I(t_1), \hat{H}_I^{int}(t_2))$ is the integrand.

Since on the line $t_1 = t_2$, $\hat{H}_I^{int}(t)\hat{H}_I^{int}(t)$ is finite, the result of the two dimensional integration does not change even remove the line from the plane.

Therefore, with the integration, the different definitions of the time-ordered product in the literature, i.e., the definition above vs. the definition

$$T\left(\hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)\right) \equiv \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)\theta(t_1-t_2) + \hat{H}_I^{int}(t_2)\hat{H}_I^{int}(t_1)\theta(t_2-t_1)$$

but with $\theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, or $\theta(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$, all give the same result.

The time-ordering at one time is just defined as $T(\hat{H}_I^{int}(t)) \equiv \hat{H}_I^{int}(t)$

Now it is easy to see that

$$\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n \; \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)$$

$$= \frac{1}{n!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \; T\left(\hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)\right), \quad \text{for } n \geq 2$$

where $T\left(\hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)\right)$

$$= \sum_P \theta(t_{p_1}, t_{p_2}, \cdots, t_{p_n})\, \hat{H}_I^{int}(t_{p_1})\, \hat{H}_I^{int}(t_{p_2}) \cdots \hat{H}_I^{int}(t_{p_n})$$

where the sum runs over all permutations of the set $t_1, t_2, \cdots, t_n$, and the θ enforces the condition $t_{p_1} > t_{p_2} > \cdots > t_{p_n}$ (or in some literature, $t_{p_1} \geq t_{p_2} \geq \cdots \geq t_{p_n}$, and the result is the same with the integration)

Proof: $\frac{1}{n!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \; T\left(\hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)\right)$

$$= \frac{1}{n!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \cdot \sum_P \theta(t_{p_1}, t_{p_2}, \cdots, t_{p_n})\, \hat{H}_I^{int}(t_{p_1})\, \hat{H}_I^{int}(t_{p_2}) \cdots \hat{H}_I^{int}(t_{p_n})$$

$$= \frac{1}{n!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \underbrace{\left\{ \begin{aligned} &\theta(t_1-t_2)\theta(t_2-t_3)\theta(t_3-t_4)\cdots\theta(t_{n-1}-t_n)\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2)\, \hat{H}_I^{int}(t_3) \cdots \hat{H}_I^{int}(t_n) \\ &+\theta(t_1-t_3)\theta(t_3-t_2)\theta(t_2-t_4)\cdots\theta(t_{n-1}-t_n)\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_3)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n) \\ &+\theta(t_1-t_2)\theta(t_2-t_3)\cdots\theta(t_i-t_k)\theta(t_k-t_j)\theta(t_j-t_l)\cdots\theta(t_{n-1}-t_n) \\ &\qquad \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_i)\, \hat{H}_I^{int}(t_k)\, \hat{H}_I^{int}(t_j) \cdots \hat{H}_I^{int}(t_n) \\ &+ \cdots \end{aligned} \right\}}_{n!\ \text{terms}}$$

$n!$ terms, and they are the same since for example the 2nd line goes back to the 1st line by $t_3 \leftrightarrow t_2$

$$= \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \; \theta(t_1-t_2)\theta(t_2-t_3)\theta(t_3-t_4)\cdots\theta(t_{n-1}-t_n)\, \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)$$

$$= \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n \; \hat{H}_I^{int}(t_1)\, \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)$$

Done the proof.

$$\Rightarrow \quad \hat{U}(t,t') = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \, T\left(\hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n)\right)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n \, T\left(\hat{H}_I^{int}(t_1) \cdots \hat{H}_I^{int}(t_n)\right)$$

$$\underset{\uparrow}{\equiv} \; T \exp\left(-i \int_{t'}^{t} dt'' \hat{H}_I^{int}(t'')\right)$$

just a formal definition, should be understood from the above two lines

using $\hat{H}_I^{int}(t) = \int d^3\vec{x} \, \mathcal{H}_I^{int}(t, \vec{x})$

$$\Rightarrow \quad \hat{U}(t,t') = T \exp\left(-i \int_{t'}^{t} dt'' \int d^3\vec{x} \, \mathcal{H}_I^{int}(t'', \vec{x})\right)$$

where $\mathcal{H}_I^{int}(t'', \vec{x})$ is the interaction Hamiltonian density in the interaction picture.

(Note that <u>if</u> $\hat{H}_I^{int}(t)$ at different time commute, then time ordering is not needed, and we have

$$\hat{U}(t,t') = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n \, \hat{H}_I^{int}(t_1) \cdots \hat{H}_I^{int}(t_n)$$

$$= \exp\left(-i \int_{t'}^{t} dt'' \hat{H}_I^{int}(t'')\right)$$

$$= \exp\left(-i \int_{t'}^{t} dt'' \int d^3\vec{x} \, \mathcal{H}_I^{int}(t'', \vec{x})\right)$$
)

Since the time evolution operator $\hat{U}(t_2, t_1)$ describes the connection between the state vector at t_2 and t_1,

$$|\alpha, t_2\rangle^I = \hat{U}(t_2, t_1)|\alpha, t_1\rangle^I.$$

we can use it to find the probability amplitude for the transition from a certain initial state at t_1 to a certain final state at t_2. Let's call the initial state $|i\rangle$ and the final state $|f\rangle$.

Let's imagine that $|i\rangle$ is defined long before the interaction occurs, and can be specified with a definite number of particles which have definite properties (e.g., spin polarizations, momenta) and are far apart from each other so that they do not interact. Then the particles come close together and interaction is switched on so that some interaction happens and then these particles (including newly produced particles due to the interaction) fly apart again — this process is described by $\hat{U}(t_2, t_1)$ (recall that $\hat{H}_I^{int}$ is inside $\hat{U}(t_2, t_1)$).

The final state $|f\rangle$ is defined long after the interaction occurs, and again is specified with a definite number of particles which have definite properties. Since from $|i\rangle$, $\hat{U}(t_2, t_1)|i\rangle$ can lead to many different final state, that is, all possible final states with definite properties are contained in $\hat{U}(t_2, t_1)|i\rangle$, the projection to the specific final state is just $\langle f|\hat{U}(t_2, t_1)|i\rangle$, and this is the probability amplitude for the transition from $|i\rangle$ to $|f\rangle$ due to the interaction described by $\hat{H}_I^{int}$.

Formally, to be sure of the switch-off the interaction, $|i\rangle$ is defined at $t_1 \to -\infty$ and $|f\rangle$ is defined at $t_2 \to +\infty$, and the transition amplitude is

$$S_{fi} \equiv \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \langle f | \hat{U}(t_2, t_1) | i \rangle$$

and we introduce the so called S-matrix,

$$\hat{S} \equiv \hat{U}(+\infty, -\infty)$$

and therefore $S_{fi} = \langle f | \hat{S} | i \rangle$.

$$\hat{S} = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \cdots \int_{-\infty}^{+\infty} dt_n \, T\left(\hat{H}_I^{int}(t_1) \hat{H}_I^{int}(t_2) \cdots \hat{H}_I^{int}(t_n) \right)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_n \, T\left(\hat{H}_I^{int}(t_1) \cdots \hat{H}_I^{int}(t_n) \right)$$

$$\equiv T \exp\left(-i \int_{-\infty}^{+\infty} dt \, \hat{H}_I^{int}(t) \right)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 \, d^4x_2 \cdots d^4x_n \, T\left(\hat{H}_I^{int}(x_1) \hat{H}_I^{int}(x_2) \cdots \hat{H}_I^{int}(x_n) \right)$$

Since we do not know whether $[\hat{H}_S^0, \hat{H}_S] = 0$, the "?" steps in the $\hat{O}^I(t)$ and $|\alpha, t\rangle^I$ expressions are not valid in general.

In fact, we can use the Baker-Campbell-Hausdorff relations to understand the math here.

For two operators $\hat{A}$ and $\hat{B}$, the Baker-Campbell-Hausdorff relations are

(a) $$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots$$

(b) $$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$, provided that $[[\hat{A},\hat{B}],\hat{A}] = [[\hat{A},\hat{B}],\hat{B}] = 0$

proof for (a):

Introduce a continuous auxiliary parameter x, and study $U(x) \equiv e^{x\hat{A}}\hat{B}e^{-x\hat{A}}$

$\Rightarrow$ (drop all the "^" in the following) $\hookrightarrow$ ($\hat{A}$ & $\hat{B}$ are not function of x)

$$\frac{dU(x)}{dx} = Ae^{xA}\hat{B}e^{-xA} + e^{xA}Be^{-xA}(-A) = [A, U(x)]$$

$$\Rightarrow \quad U(x) = B + \int_0^x dy\,[A, U(y)]$$

$$\Rightarrow \quad U(x) = B + \int_0^x dx_1\,[A, U(x_1)]$$

$$= B + \int_0^x dx_1\,[A, B + \int_0^{x_1}[A, U(x_2)]dx_2]$$

$$= B + \int_0^x dx_1[A,B] + \int_0^x dx_1\int_0^{x_1} dx_2\,[A,[A,U(x_2)]]$$

$$= B + \int_0^x dx_1[A,B] + \int_0^x dx_1\int_0^{x_1} dx_2[A,[A, B + \int_0^{x_2} dx_3[A, U(x_3)]]]$$

$$= B + \int_0^x dx_1[A,B] + \int_0^x dx_1\int_0^{x_1} dx_2[A,[A,B]]$$
$$+ \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3\,[A,[A,[A,U(x_3)]]]$$

$$= B + \int_0^x dx_1[A,B] + \int_0^x dx_1\int_0^{x_1} dx_2[A,[A,B]]$$
$$+ \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3\,[A,[A,[A, B + \int_0^{x_3} dx_4[A, U(x_4)]]]]$$

$$= B+\int_0^x dx_1[A,B] + \int_0^x dx_1\int_0^{x_1} dx_2[A,[A,B]]$$
$$+ \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3\,[A,[A,[A,B]]]$$
$$+ \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3\int_0^{x_3} dx_4[A,[A,[A,[A,U(x_4)]]]]$$

$$= B+\int_0^x dx_1[A,B] + \int_0^x dx_1\int_0^{x_1} dx_2[A,[A,B]]$$
$$+ \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3[A,[A,[A,B]]]$$
$$+ \cdots + \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3\cdots\int_0^{x_n} dx_{n+1}\underbrace{[A,[A,[A,[\cdots,[A,}_{\text{there're }(n+1)\text{ "[A"}}\ B\ \underbrace{]]\cdots]}_{\text{there're }(n+1)\text{ "]".}}$$
$$+ \int_0^x dx_1\int_0^{x_1} dx_2\int_0^{x_2} dx_3\cdots\int_0^{x_n} dx_{n+1}\int_0^{x_{n+1}} dx_{n+2}\underbrace{[A,[A,[A,[\cdots,[A,[A,}_{\text{there're }(n+2)\text{ "[A"}}U(x_{n+2})\underbrace{]]\cdots]}_{\text{there're }(n+2)\text{ "]"}}$$

$$= B+[A,B]x+[A,[A,B]]\frac{x^2}{2!}+[A,[A,[A,B]]]\frac{x^3}{3!}$$
$$+\cdots+\underbrace{[A,[A,[A,[\cdots,[A,}_{\text{there're }(n+1)\text{ "[A"}}B\underbrace{]]\cdots]}_{\text{there're }(n+1)\text{ "]"}}\frac{x^{n+1}}{(n+1)!}+\cdots$$

$$\Rightarrow\ U(1)=e^A B e^{-A} = B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]$$
$$+\cdots+\frac{1}{(n+1)!}\underbrace{[A,[A,[A,[\cdots,[A,}_{\text{there're }(n+1)\text{ "[A".}}B\underbrace{]]\cdots]}_{\text{there're }(n+1)\text{ "]"}}+\cdots$$

$\Rightarrow$ (a) is proved.

proof for (b):

Introduce a continuous auxiliary parameter x, and study
$\hookrightarrow$ ($\hat{A}$ & $\hat{B}$ are not function of x)

$$\hat{Q}(x) \equiv e^{+x\hat{A}}e^{-x(\hat{A}+\hat{B})}e^{+x\hat{B}}$$

(drop all the "^" in the following)

From $e^{-x(A+B)} = e^{-xA} Q(x) e^{-xB}$ and do the derivation with respect to x,

$$\Rightarrow -(A+B) e^{-x(A+B)} = -A e^{-x(A+B)} + e^{-xA} \frac{dQ}{dx} e^{-xB} - e^{-xA} Q(x) e^{-xB} B$$

$$\Rightarrow -B e^{-xA} Q e^{-xB} = e^{-xA} \frac{dQ}{dx} e^{-xB} - e^{-xA} Q e^{-xB} B$$

Multiply e^{+xA} from the left and e^{+xB} from the right

$$\Rightarrow -e^{+xA} B e^{-xA} Q = \frac{dQ}{dx} - Q e^{-xB} B e^{+xB}$$

$$\Rightarrow \frac{dQ}{dx} = -e^{+xA} B e^{-xA} Q + Q \underbrace{e^{-xB} B e^{xB}}_{B,\ \text{since } [B,\, e^{xB}] = 0 = [e^{-xB},\, B]}$$

$$= -\left\{ B + [A,B]x + \underbrace{[A,[A,B]]}_{=0} \frac{x^2}{2!} + 0 \right\} Q + QB$$

$$= -[B, Q] - x[A,B]Q$$

Assuming that $[B, Q] = 0$, then

$$\frac{dQ}{dx} = -x[A,B]Q$$

with $Q(0) = e^0 e^{-0} e^0 = 1$.

So the solution is

$$Q(x) = e^{-\frac{1}{2}x^2[A,B]}$$

This solution justifies the assumption $[B, Q] = 0$, since $[[A,B],B] = 0$.

$$\Rightarrow Q(-1) = e^{-A} e^{+(A+B)} e^{-B} = e^{-\frac{1}{2}[A,B]}$$

left times e^A and right times e^B $\Rightarrow$ $e^{A+B} = e^A e^{-\frac{1}{2}[A,B]} e^B \underset{\text{since } [[A,B],B]=0}{=} e^A e^B e^{-\frac{1}{2}[A,B]}$

$\Rightarrow$ (b) is proved.

A3.

Wick's theorem

To calculate S_{fi}, we need to know how to deal with the time-ordered products, $T(\hat{H}_I^{int}(x_1)\hat{H}_I^{int}(x_2)\cdots\hat{H}_I^{int}(x_n))$.

For example, to calculate a two-to-two scattering process,

$$k_1, k_2 \longrightarrow k_3, k_4,$$

we will need to calculate $S_{fi} = \langle f|\hat{S}|i\rangle = \langle k_3, k_4|\hat{S}|k_1, k_2\rangle$

where $|k_1, k_2\rangle \propto a^\dagger_{\vec{k}_1} b^\dagger_{\vec{k}_2}|0\rangle$ and $\langle k_3, k_4| \propto \langle 0| c_{\vec{k}_3} d_{\vec{k}_4}$. Therefore,

in $\hat{S} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n\, T(\hat{H}_I^{int}(x_1)\hat{H}_I^{int}(x_2)\cdots\hat{H}_I^{int}(x_n))$,

terms with the form $f\, c^\dagger d^\dagger a\, b$ will contribute to S_{fi}.

f: a scalar function

a, b: annihilation operator to "kill" $a^\dagger_{\vec{k}_1}$, $b^\dagger_{k_2}$ — through commutator or anticommutator

$c^\dagger$, $d^\dagger$: creation operator to "kill" c_{k_3}, d_{k_4}

Wick theorem is about how to do it and how to get rid of the time-ordering T: by changing it to normal-ordered products and contractions (i.e., propagators)

First, let's recall the definition of normal-ordered product.

Recall that a field operator, $\hat{\Phi}$, can be split into two parts: one part consists of annihilation operator (denote this part as $\hat{\Phi}^{(+)}$) and the other part consists of creation operator (denote this part as $\hat{\Phi}^{(-)}$),

$$\hat{\Phi}(x) = \hat{\Phi}^{(+)}(x) + \hat{\Phi}^{(-)}(x).$$

For two bosonic field operator, the normal product is

$$:\hat{\Phi}(x)\hat{\Phi}(y): = \hat{\Phi}^{(-)}(x)\hat{\Phi}^{(-)}(y) + \hat{\Phi}^{(-)}(x)\hat{\Phi}^{(+)}(y) + \hat{\Phi}^{(+)}(x)\hat{\Phi}^{(+)}(y) + \hat{\Phi}^{(-)}(y)\hat{\Phi}^{(+)}(x)$$

For two fermionic field operators,

$$:\hat{\psi}(x)\hat{\psi}(y): = \hat{\psi}^{(-)}(x)\hat{\psi}^{(-)}(y) + \hat{\psi}^{(-)}(x)\hat{\psi}^{(+)}(y) + \hat{\psi}^{(+)}(x)\hat{\psi}^{(+)}(y) - \hat{\psi}^{(-)}(y)\hat{\psi}^{(+)}(x)$$

Let's in the following use $\hat{\Phi}$ for both bozonic and fermion operators.

$$:\hat{\Phi}_A\hat{\Phi}_B: = :(\underset{A^-}{\hat{\Phi}_A^{(-)}} + \underset{A^+}{\hat{\Phi}_A^{(+)}})(\underset{B^-}{\hat{\Phi}_B^{(-)}} + \underset{B^+}{\hat{\Phi}_B^{(+)}}): = :A^-B^-: + :A^-B^+: + :A^+B^+: + :A^+B^-:$$

Note:

$:A^+B^-: \equiv \varepsilon_{AB}B^-A^+$

$A^+B^- \neq \varepsilon_{AB}B^-A^+$

$A^+B^+ = \varepsilon_{AB}B^+A^+$

$A^-B^- = \varepsilon_{AB}B^-A^-$

$\varepsilon_{AB}\varepsilon_{AB} = 1$

$$= A^-B^- + A^-B^+ + A^+B^+ + \varepsilon_{AB}B^-A^+$$

$$= \varepsilon_{AB}B^-A^- + A^-B^+ + \varepsilon_{AB}B^+A^+ + \varepsilon_{AB}B^-A^+$$

$$= \varepsilon_{AB}(B^-A^- + \varepsilon_{AB}A^-B^+ + B^+A^+ + B^-A^+)$$

$$= \varepsilon_{AB}:\hat{\Phi}_B\hat{\Phi}_A:$$

where $\varepsilon_{AB} = +1$ if $\hat{\Phi}_A$ and $\hat{\Phi}_B$ are bozonic field operators

- - - -1 - - - - - - - - - fermionic - - -

and $\varepsilon_{AB} = +1$ if one is bozonic field operator and the other is fermionic field operator.

In general, let $Q, R, \dots, W$ be operators of the type $\psi^{(+)}, \psi^{(-)}, \phi^{(+)}, \phi^{(-)}, A_\mu^{(+)}, A_\nu^{(-)}$ etc., i.e., each is linear in either creation or annihilation operators (e.g., $\phi^{(+)}(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\; a_{\vec{p}}\, e^{-i p\cdot x}$), then

$$:QR\cdots W: = (-1)^P (Q'R'\cdots W')$$

where $Q', R', \dots, W'$ are the operators $Q, R, \dots, W$ reordered so that all annihilation operators stand to the right of all creation operators. The exponent P is the number of interchanges of neighbouring fermion operators required to change the order $(QR\dots W)$ into $(Q'R'\dots W')$.

Note also that $:RS\dots + VW\dots: = :RS\dots: + :VW\dots:$

For the time-ordered product, $T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2))$, of two field operators

① if $t_1 > t_2$, then

$$T(\underbrace{\hat{\Phi}_A(x_1)}_{A_{x_1}}\underbrace{\hat{\Phi}_B(x_2)}_{B_{x_2}}) = A_{x_1}B_{x_2} = (A^+_{x_1} + A^-_{x_1})(B^+_{x_2} + B^-_{x_2})$$

$$= A^+_{x_1}B^+_{x_2} + A^-_{x_1}B^-_{x_2} + A^-_{x_1}B^+_{x_2} + \underbrace{A^+_{x_1}B^-_{x_2}}_{\varepsilon_{AB}B^-_{x_2}A^+_{x_1} + [A^+_{x_1}, B^-_{x_2}]_{\mp}}$$

where $\varepsilon_{AB} = -1$ when both $\hat{\Phi}_A(x_1)$ and $\hat{\phi}_B(x_2)$ are fermionic field operator, and for this case the anti-commutator $[\quad]_+$ is taken; otherwise $\varepsilon_{AB} = +1$ and commutator $[\quad]_-$ is taken.

$$\Rightarrow T(\hat{\Phi}_A(x_1)\hat{\phi}_B(x_2)) = :\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2): + [\hat{\Phi}^{(+)}_A(x_1), \hat{\Phi}^{(-)}_B(x_2)]_{\mp}$$

Since the commutator or anticommutator of a annihilation operator and a creation operator is a "c number", i.e., $[a_{\vec{k}}, b^+_{\vec{p}}]_{\mp} \propto \delta^3(\vec{k}-\vec{p})$ if $b = a$, and otherwise $[a_{\vec{k}}, b^+_{\vec{p}}]_{\mp} = 0$, then

$$[\hat{\Phi}^{(+)}_A(x_1), \hat{\Phi}^{(-)}_B(x_2)]_{\mp} = \langle 0|[\hat{\Phi}^{(+)}_A(x_1), \hat{\Phi}^{(-)}_B(x_2)]_{\mp}|0\rangle \underset{\uparrow}{=} \langle 0|\hat{\Phi}^{(+)}_A(x_1)\hat{\Phi}^{-}_B(x_2)|0\rangle$$

use $\langle 0|\hat{\Phi}^{(-)}_B(x_2) = 0$ and $\hat{\Phi}^{(+)}_A(x_1)|0\rangle = 0$.

$$\underset{\uparrow}{=} \langle 0|\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)|0\rangle = \langle 0|T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2))|0\rangle$$

use $\langle 0|\hat{\Phi}^{(+)}_A(x_1)\hat{\Phi}^{(+)}_B(x_2)|0\rangle = 0$

$\langle 0|\hat{\Phi}^{(-)}_A(x_1)\hat{\Phi}^{(+)}_B(x_2)|0\rangle = 0$

$\langle 0|\hat{\Phi}^{(-)}_A(x_1)\hat{\Phi}^{(-)}_B(x_2)|0\rangle = 0$

$$\Rightarrow T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)) = :\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2): + \langle 0|T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2))|0\rangle$$

② if $t_1 < t_2$, then

$$T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)) \equiv \varepsilon_{AB}\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1)$$

Note that we didn't introduce the ε_{AB} yet when we talk about $T(\hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)) \equiv \hat{H}_I^{int}(t_1)\hat{H}_I^{int}(t_2)\theta(t_1-t_2) + \hat{H}_I^{int}(t_2)\hat{H}_I^{int}(t_1)\theta(t_2-t_1)$,

Since $\hat{H}_I^{int}$ must have even number of fermionic field operator in it to make it a scalar type rather than a column or a row.

$$= \varepsilon_{AB}(B^+_{x_2} + B^-_{x_2})(A^+_{x_1} + A^-_{x_1}) = \varepsilon_{AB}(B^+_{x_2}A^+_{x_1} + B^-_{x_2}A^-_{x_1} + B^-_{x_2}A^+_{x_1} + \underbrace{B^+_{x_2}A^-_{x_1}}_{=\ \varepsilon_{AB}A^-_{x_1}B^+_{x_2} + [B^+_{x_2}, A^-_{x_1}]_{\mp}})$$

$$= \varepsilon_{AB} : \hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1) : + \varepsilon_{AB}[\hat{\Phi}_B^{(+)}(x_2), \hat{\Phi}_A^{(-)}(x_1)]_{\mp}$$

using $[\hat{\Phi}_B^{(+)}(x_2), \hat{\Phi}_A^{(-)}(x_1)]_{\mp} = \langle 0|[\hat{\Phi}_B^{(+)}(x_2), \hat{\Phi}_A^{(-)}(x_1)]_{\mp}|0\rangle$

$$= \langle 0|\hat{\Phi}_B^{(+)}(x_2)\hat{\Phi}_A^{(-)}(x_1)|0\rangle = \langle 0|\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1)|0\rangle$$

$$= \langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1))|0\rangle$$

$$\Rightarrow T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)) = \varepsilon_{AB} : \hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1) : + \varepsilon_{AB}\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1))|0\rangle$$

While for any relation between t_1 and t_2 with $t_1 \neq t_2$

$$T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)) \equiv \hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\theta(t_1-t_2) + \varepsilon_{AB}\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1)\theta(t_2-t_1)$$

$$= \varepsilon_{AB}(\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1)\theta(t_2-t_1) + \varepsilon_{AB}\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\theta(t_1-t_2))$$

$$= \varepsilon_{AB}T(\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1))$$

and we have shown $\varepsilon_{AB} : \hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1) : = : \hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2) :$ regardless of t_1 & t_2

$$\Rightarrow T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)) = : \hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2) : + \langle 0|T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2))|0\rangle$$

Therefore, for any relation between t_1 and t_2 with $t_1 \neq t_2$, we have

$$T(\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)) = :\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2): + \langle 0 | T(\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)) | 0 \rangle$$

We denote contraction of $\hat{\Phi}_A(x_1)$ and $\hat{\Phi}_B(x_2)$ as

$$\underbrace{\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)}_{} \equiv \langle 0 | T(\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)) | 0 \rangle .$$

then we have

$$T(\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)) = :\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2): + \underbrace{\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)}_{}$$

$$= :\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2): + :\underbrace{\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)}_{}:$$

$\uparrow$ since $\underbrace{\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)}_{}$ is a c number, it does not change after putting it between $:\ :$

Now let's consider three field operators, $T(\hat{\Phi}_A(x_1)\,\hat{\Phi}_B(x_2)\,\hat{\Phi}_C(x_3))$.

① if $t_1, t_2 > t_3$, then

$$T(\underbrace{\hat{\Phi}_A(x_1)}_{\downarrow\, A_{x_1}}\,\underbrace{\hat{\Phi}_B(x_2)}_{\downarrow\, B_{x_2}}\,\underbrace{\hat{\Phi}_C(x_3)}_{\downarrow\, C_{x_3}}) = T(A_{x_1} B_{x_2})\, C_{x_3} = \left(:A_{x_1} B_{x_2}: + :\underbrace{A_{x_1} B_{x_2}}_{}:\right) C_{x_3}$$

where $:A_{x_1} B_{x_2}:\, C_{x_3} = (A^-_{x_1} B^-_{x_2} + A^+_{x_1} B^+_{x_2} + A^-_{x_1} B^+_{x_2} + \varepsilon_{AB} B^-_{x_2} A^+_{x_1})(C^+_{x_3} + C^-_{x_3})$

in which

$$A^-_{x_1} B^+_{x_2} C^-_{x_3} = \varepsilon_{BC} A^-_{x_1} C^-_{x_3} B^+_{x_2} + A^-_{x_1} [B^+_{x_2}, C^-_{x_3}]_\mp$$

$$\varepsilon_{AB} B^-_{x_2} A^+_{x_1} C^-_{x_3} = \varepsilon_{AB}\varepsilon_{AC} B^-_{x_2} C^-_{x_3} A^+_{x_1} + \varepsilon_{AB} B^-_{x_2} [A^+_{x_1}, C^-_{x_3}]_\mp$$

$$A^+_{x_1} B^+_{x_2} C^-_{x_3} = A^+_{x_1} (\varepsilon_{BC} C^-_{x_3} B^+_{x_2} + [B^+_{x_2}, C^-_{x_3}]_\mp)$$

$$= \varepsilon_{BC}\varepsilon_{AC} C^-_{x_3} A^+_{x_1} B^+_{x_2} + \varepsilon_{BC} [A^+_{x_1}, C^-_{x_3}]_\mp B^+_{x_2} + A^+_{x_1} [B^+_{x_2}, C^-_{x_3}]$$

note that $\varepsilon_{BC} [A^+_{x_1}, C^-_{x_3}]_\mp \neq 0$ if $[A^+_{x_1}, C^-_{x_3}]_\mp \neq 0$, that is, if A and C are of the same type of field operators.

then $\varepsilon_{BC} [A^+_{x_1}, C^-_{x_3}]_\mp = \varepsilon_{AB} [A^+_{x_1}, C^-_{x_3}] \Longrightarrow \varepsilon_{AB} B^-_{x_2} [A^+_{x_1}, C^-_{x_3}]_\mp + \varepsilon_{BC} [A^+_{x_1}, C^-_{x_3}]_\mp B^+_{x_2}$

$$= \varepsilon_{AB} B_{x_2} [A^+_{x_1}, C^-_{x_3}]_\mp$$

$$\Rightarrow \; :A_{x_1} B_{x_2}: C_{x_3} = \; :A_{x_1} B_{x_2} C_{x_3}: + A_{x_1}[B^+_{x_2}, C^-_{x_3}]_{\mp} + \varepsilon_{AB} B_{x_2}[A^+_{x_1}, C^-_{x_3}]_{\mp}$$

$$= \; :A_{x_1} B_{x_2} C_{x_3}: + A_{x_1} \langle 0| T(B_{x_2} C_{x_3})|0\rangle + \varepsilon_{AB} B_{x_2} \langle 0| T(A_{x_1} C_{x_3})|0\rangle$$

$$\Rightarrow \; T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)) = \; :\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3): + :\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)}\hat{\Phi}_C(x_3): + :\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}: + \varepsilon_{AB} :\hat{\Phi}_B(x_2)\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}:$$

② if $t_1, t_3 > t_2$, then

$$T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)) = \varepsilon_{BC} T(A_{x_1} C_{x_3}) B_{x_2}$$

$$= \varepsilon_{BC} (:A_{x_1} C_{x_3}: + :\underbrace{A_{x_1} C_{x_3}}:) B_{x_2}$$

where

$$:A_{x_1} C_{x_3}: B_{x_2} = \; :A_{x_1} C_{x_3} B_{x_2}: + A_{x_1}[C^+_{x_3}, B^-_{x_2}]_{\mp} + \varepsilon_{AC} C_{x_3}[A^+_{x_1}, B^-_{x_2}]_{\mp}$$

$$= \; :A_{x_1} C_{x_3} B_{x_2}: + A_{x_1} \langle 0| T(C_{x_3} B_{x_2})|0\rangle + \varepsilon_{AC} C_{x_3} \langle 0| T(A_{x_1} B_{x_2})|0\rangle$$

$$\Rightarrow \; T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)) = \varepsilon_{BC} :\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)\hat{\Phi}_B(x_2):$$

$$+ \varepsilon_{BC} :\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_C(x_3)\hat{\Phi}_B(x_2)}: + \varepsilon_{BC} :\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}\hat{\Phi}_B(x_2):$$

$$+ \varepsilon_{BC}\varepsilon_{AC} :\hat{\Phi}_C(x_3)\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)}:$$

③ if $t_2, t_3 > t_1$, then

$$T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)) = \varepsilon_{AB}\varepsilon_{AC} T(B_{x_2} C_{x_3}) A_{x_1}$$

$$= \varepsilon_{AB}\varepsilon_{AC} (:B_{x_2} C_{x_3}: + :\underbrace{B_{x_2} C_{x_3}}:) A_{x_1}$$

where

$$:B_{x_2} C_{x_3}: A_{x_1} = \; :B_{x_2} C_{x_3} A_{x_1}: + :B_{x_2} \underbrace{C_{x_2} A_{x_1}}: + \varepsilon_{BC} :C_{x_3} \underbrace{B_{x_2} A_{x_1}}:$$

$$\Rightarrow \; T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)) = \varepsilon_{AB}\varepsilon_{AC} :\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)\hat{\Phi}_A(x_1): + \varepsilon_{AB}\varepsilon_{AC} :\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}\hat{\Phi}_A(x_1):$$

$$+ \varepsilon_{AB}\varepsilon_{AC} :\hat{\Phi}_B(x_2)\underbrace{\hat{\Phi}_C(x_3)\hat{\Phi}_A(x_1)}:$$

$$+ \varepsilon_{AB}\varepsilon_{AC}\varepsilon_{BC} :\hat{\Phi}_C(x_3)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1)}:$$

Since for any relation between t_1 and t_2 with $t_1 \neq t_2$, we have shown

$$T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)) = \varepsilon_{AB}\, T(\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1))$$

$$\Rightarrow \langle 0|T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2))|0\rangle = \varepsilon_{AB}\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1))|0\rangle$$

$$\Rightarrow \hat{\Phi}_A(x_1)\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3))|0\rangle = \varepsilon_{BC}\hat{\Phi}_A(x_1)\langle 0|T(\hat{\Phi}_C(x_3)\hat{\Phi}_B(x_2))|0\rangle$$

$$\Rightarrow :\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}: = \varepsilon_{BC}:\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_C(x_3)\hat{\Phi}_B(x_2)}:,$$

for any t_1, t_2, t_3 relations

Also, $\hat{\Phi}_A(x_1)\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3))|0\rangle = \langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3))|0\rangle\hat{\Phi}_A(x_1)$

Since $\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3))|0\rangle$ vanishes unless B and C are of the same type of field operators, then

$$\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3))|0\rangle\hat{\Phi}_A(x_1) = \varepsilon_{AB}\varepsilon_{AC}\langle 0|T(\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3))|0\rangle\hat{\Phi}_A(x_1)$$

$$\Rightarrow :\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}: = \varepsilon_{AB}\varepsilon_{AC}:\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}\hat{\Phi}_A(x_1):$$

$$\Rightarrow \varepsilon_{AB}\varepsilon_{AC}:\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}: = :\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}\hat{\Phi}_A(x_1):$$

$$\Rightarrow \varepsilon_{CA}\varepsilon_{CB}:\hat{\Phi}_C(x_3)\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)}: = :\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)}\hat{\Phi}_C(x_3):$$

$$\| $$

$$\varepsilon_{CA}\varepsilon_{CB}\varepsilon_{AB}:\hat{\Phi}_C(x_3)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_A(x_1)}:$$

and

$$\varepsilon_{AB}:\hat{\Phi}_B(x_2)\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}: = \varepsilon_{AB}\varepsilon_{BA}\varepsilon_{BC}:\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}\hat{\Phi}_B(x_2):$$

$$\|$$

$$\varepsilon_{AB}\varepsilon_{AC}:\hat{\Phi}_B(x_2)\underbrace{\hat{\Phi}_C(x_3)\hat{\Phi}_A(x_1)}: \qquad = \varepsilon_{BC}:\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}\hat{\Phi}_B(x_2):$$

Also,

$$:\underbrace{\hat{\Phi}_A(x_1)}_{A}\underbrace{\hat{\Phi}_B(x_2)}_{B}\underbrace{\hat{\Phi}_C(x_3)}_{C}: = :(A^+ + A^-)(B^+ + B^-)(C^+ + C^-):$$

$$= :A^+B^+C^+: + :A^+B^+C^-: + :A^+B^-C^+: + :A^+B^-C^-: + :A^-B^+C^+: + :A^-B^+C^-: + :A^-B^-C^+: + :A^-B^-C^-:$$

$$= A^+B^+C^+ + \varepsilon_{BC}\varepsilon_{AC}C^-A^+B^+ + \varepsilon_{AB}B^-A^+C^+ + \varepsilon_{AB}\varepsilon_{AC}B^-C^-A^+ + A^-B^+C^+ + \varepsilon_{BC}A^-C^-B^+ + A^-B^-C^+ + A^-B^-C^-$$

$$= \varepsilon_{BC}\Big(A^+C^+B^+ + \varepsilon_{AC}C^-A^+B^+ + \varepsilon_{AB}\varepsilon_{BC}B^-A^+C^+ + \varepsilon_{AB}\varepsilon_{AC}C^-B^-A^+$$
$$+ A^-C^+B^+ + A^-C^-B^+ + \varepsilon_{BC}A^-B^-C^+ + A^-C^-B^-\Big)$$
$$= \varepsilon_{BC} : \hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)\hat{\Phi}_B(x_2) :$$

from the second "="

$$\overset{\downarrow}{=} \varepsilon_{AB}\varepsilon_{AC}\Big(B^+C^+A^+ + \varepsilon_{BC}C^-B^+A^+ + B^-C^+A^+ + B^-C^-A^+$$
$$+ \varepsilon_{AB}\varepsilon_{AC}A^-B^+C^+ + \varepsilon_{AB}\varepsilon_{AC}\varepsilon_{BC}A^-C^-B^+ + \varepsilon_{AB}\varepsilon_{AC}A^-B^-C^+$$
$$+ B^-C^-A^-\Big)$$
$$= \varepsilon_{AB}\varepsilon_{AC} : \hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)\hat{\Phi}_A(x_1) :$$

So, for any relations of t_1, t_2 and t_3, with $t_1 \neq t_2 \neq t_3$, we have

$$T\big(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)\big) = :\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3):$$
$$+ :\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)}\hat{\Phi}_C(x_3):$$
$$+ :\hat{\Phi}_A(x_1)\underbrace{\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)}:$$
$$+ \varepsilon_{AB} :\hat{\Phi}_B(x_2)\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}:$$

we can write the last term as $:\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C}(x_3):$

$$\equiv \varepsilon_{AB} : \hat{\Phi}_B(x_2)\underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C(x_3)}:$$
$$= \varepsilon_{BC} : \underbrace{\hat{\Phi}_A(x_1)\hat{\Phi}_C}(x_3)\hat{\Phi}_B(x_2):$$

For four field operators, $T(\hat{\Phi}_A(x_1)\hat{\Phi}_B(x_2)\hat{\Phi}_C(x_3)\hat{\Phi}_D(x_4))$,

① if $t_1, t_2, t_3 > t_4$, then

$$T(A_{x_1}B_{x_2}C_{x_3}D_{x_4}) = T(A_{x_1}B_{x_2}C_{x_3})D_{x_4} = \Big(:A_{x_1}B_{x_2}C_{x_3}: + :\underbrace{A_{x_1}B_{x_2}}C_{x_3}: + :A_{x_1}\underbrace{B_{x_2}C_{x_3}}: + :\underbrace{A_{x_1}B_{x_2}C_{x_3}}:\Big)D_{x_4}$$

where $:\underbrace{A_{x_1}}_{A}\underbrace{B_{x_2}}_{B}\underbrace{C_{x_3}}_{C}:\underbrace{D_{x_4}}_{D}$

$$= (A^-B^-C^- + A^-B^-C^+ + \varepsilon_{BC}A^-C^-B^+ + A^-B^+C^+ + \varepsilon_{AB}B^-A^+C^+ + \varepsilon_{AB}\varepsilon_{AC}B^-C^-A^+ + \varepsilon_{AB}\varepsilon_{AC}\varepsilon_{BC}C^-B^+A^+ + A^+B^+C^+)(D^- + D^+)$$

$$= :ABCD: + A^-B^-[C^+,D^-]_\mp + \varepsilon_{BC}A^-C^-[B^+,D^-]_\mp$$
$$+ \varepsilon_{CD}A^-[B^+,D^-]_\mp C^+ + A^-B^+[C^+,D^-]_\mp + \varepsilon_{AB}\varepsilon_{CD}B^-[A^+,D^-]_\mp C^+ + \varepsilon_{AB}B^-A^+[C^+,D^-]_\mp$$
$$+ \varepsilon_{AB}\varepsilon_{AC}B^-C^-[A^+,D^-]_\mp + \varepsilon_{AB}\varepsilon_{AC}\varepsilon_{BC}\varepsilon_{AD}C^-[B^+,D^-]_\mp A^+ + \varepsilon_{AB}\varepsilon_{AC}\varepsilon_{BC}C^-B^+[A^+,D^-]_\mp$$
$$+ \varepsilon_{CD}\varepsilon_{BD}[A^+,D^-]_\mp B^+C^+ + \varepsilon_{CD}A^+[B^+,D^-]_\mp C^+ + A^+B^+[C^+,D^-]_\mp$$

where we used $A^-B^+C^+D^- = A^-B^+D^-C^+\varepsilon_{CD} + A^-B^+[C^+,D^-]_\mp$

$$= \varepsilon_{CD}\varepsilon_{BD}A^-D^-B^+C^+ + \varepsilon_{CD}A^-[B^+,D^-]_\mp C^+ + A^-B^+[C^+,D^-]_\mp$$

$$A^+B^+C^+D^- = \varepsilon_{CD}A^+B^+D^-C^+ + A^+B^+[C^+,D^-]_\mp$$
$$= \varepsilon_{CD}\varepsilon_{BD}A^+D^-B^+C^+ + \varepsilon_{CD}A^+[B^+,D^-]_\mp C^+ + A^+B^+[C^+,D^-]_\mp$$
$$= \varepsilon_{CD}\varepsilon_{BD}\varepsilon_{AD}D^-A^+B^+C^+ + \varepsilon_{CD}\varepsilon_{BD}[A^+,D^-]_\mp B^+C^+ + \varepsilon_{CD}A^+[B^+,D^-]_\mp C^+ + A^+B^+[C^+,D^-]_\mp$$

$$\Rightarrow :ABC:D = :ABCD: + :AB\underbrace{CD}: + \varepsilon_{BC}:AC\underbrace{BD}: + \varepsilon_{AB}\varepsilon_{AC}:BC\underbrace{AD}:$$

and
$$:\underbrace{AB}C:D = \underbrace{AB}(C^- + C^+)(D^- + D^+) = \underbrace{AB}(C^-D^- + C^+D^+ + C^-D^+ + \varepsilon_{CD}D^-C^+ + [C^+,D^-]_\mp)$$
$$= \underbrace{AB}:CD: + \underbrace{AB}\,\underbrace{CD} = :\underbrace{AB}CD: + :\underbrace{AB}\,\underbrace{CD}:$$

$$:A\underbrace{BC}:D = \underbrace{BC}:AD: + \underbrace{BC}\,\underbrace{AD} = :\underbrace{BC}AD: + :\underbrace{BC}\,\underbrace{AD}:$$

$$:\underbrace{ABC}:D = \varepsilon_{AB}(\underbrace{AC}:BD: + \underbrace{AC}\,\underbrace{BD}) = \varepsilon_{AB}:\underbrace{AC}BD: + \varepsilon_{AB}:\underbrace{AC}\,\underbrace{BD}:$$
$$= \varepsilon_{BC}:ACBD: + \varepsilon_{BC}:ACBD:$$

28.

$$\Rightarrow T(A_{x_1}B_{x_2}C_{x_3}D_{x_4}) = :ABCD:$$
$$+ :\wick{\c A \c B CD}: + :\wick{A \c B \c C D}: + :\wick{AB \c C \c D}: + :\wick{A \c B C \c D}:$$
$$+ :\wick{\c A B \c C D}: + :\wick{\c A BC \c D}: + :\wick{\c1 A \c1 B \c2 C \c2 D}: + :\wick{\c1 A \c2 B \c2 C \c1 D}: + :\wick{\c1 A \c2 B \c1 C \c2 D}:$$

where
$$:\wick{A \c B C \c D}: \equiv \varepsilon_{BC} :\wick{AC \c B \c D}: = \varepsilon_{CD} :\wick{A \c B \c D C}:$$
$$:\wick{\c A B \c C D}: \equiv \varepsilon_{BC} :\wick{\c A \c C BD}: = \varepsilon_{AB} :\wick{B \c A \c C D}:$$
$$:\wick{\c A BC \c D}: \equiv \varepsilon_{AB}\varepsilon_{AC} :\wick{BC \c A \c D}:$$
$$:\wick{\c1 A \c2 B \c2 C \c1 D}: \equiv :\wick{\c1 B \c1 C \c2 A \c2 D}: = \varepsilon_{AB}\varepsilon_{AC} :\wick{\c1 B \c1 C \c2 A \c2 D}:$$
$$:\wick{\c1 A \c2 B \c1 C \c2 D}: \equiv \varepsilon_{BC} :\wick{\c1 A \c1 C \c2 B \c2 D}:$$

② if $t_1, t_3, t_4 > t_2$, then

$$T(A_{x_1}B_{x_2}C_{x_3}D_{x_4}) = \varepsilon_{BC}\varepsilon_{BD}\, T(A_{x_1}C_{x_3}D_{x_4})B_{x_2}$$
$$= \varepsilon_{BC}\varepsilon_{BD}\big(:A_{x_1}C_{x_3}D_{x_4}: + :\wick{\c A_{x_1} \c C_{x_3}} D_{x_4}: + :A_{x_1}\wick{\c C_{x_3} \c D_{x_4}}: + :\wick{\c A_{x_1} C_{x_3} \c D_{x_4}}: \big) B_{x_2}$$
$$= \varepsilon_{BC}\varepsilon_{BD}\Big(:ACDB: + :\wick{\c A \c C DB}: + :\wick{AC \c D \c B}: + :\wick{A \c C \c D B}: + :\wick{\c A C \c D B}:$$
$$+ :\wick{A \c C D \c B}: + :\wick{\c A CD \c B}:$$
$$+ :\wick{\c1 A \c1 C \c2 D \c2 B}: + :\wick{\c1 A \c2 C \c2 D \c1 B}: + :\wick{\c1 A \c2 C \c1 D \c2 B}: \Big)$$

where
$$\varepsilon_{BC}\varepsilon_{BD} :\wick{\c A \c C DB}: = \varepsilon_{BC} :\wick{\c A \c C BD}: = :\wick{\c A B \c C D}:$$
$$\varepsilon_{BC}\varepsilon_{BD} :\wick{AC \c D \c B}: = \varepsilon_{BC} :\wick{AC \c B \c D}: = :\wick{A \c B C \c D}:$$
$$\varepsilon_{BC}\varepsilon_{BD} :\wick{A \c C \c D B}: = :\wick{AB \c C \c D}:$$
$$\varepsilon_{BC}\varepsilon_{BD} :\wick{\c A C \c D B}: = \varepsilon_{BC}\varepsilon_{BD}\varepsilon_{CD} :\wick{\c A \c D CB}: = \varepsilon_{AB}\varepsilon_{AC} :\wick{BC \c A \c D}: = :\wick{\c A BC \c D}:$$
$$\varepsilon_{BC}\varepsilon_{BD} :\wick{A \c C D \c B}: = \varepsilon_{BC}\varepsilon_{BD}\varepsilon_{CD} :\wick{AD \c C \c B}: = :\wick{A \c B \c C D}:$$
$$\varepsilon_{BC}\varepsilon_{BD} :\wick{\c A CD \c B}: = \varepsilon_{BC}\varepsilon_{BD}\varepsilon_{AC}\varepsilon_{AD} :\wick{CD \c A \c B}: = :\wick{\c A \c B CD}:$$

$$\varepsilon_{BC}\varepsilon_{BD} :ACDB: \; = \; \varepsilon_{BC} :ACBD: \; = \; :ABCD:$$

$$\varepsilon_{BC}\varepsilon_{BD} :ACDB: \; = \; \varepsilon_{BC}\varepsilon_{BD}\varepsilon_{CD} :ADCB: \; = \; \varepsilon_{BD}\varepsilon_{CD} :BCAD: \; = \; :BCAD: \; = \; :ABCD:$$

$$\varepsilon_{BC}\varepsilon_{BD} :ACDB: \; = \; \varepsilon_{BC}\varepsilon_{BD} :CDAB: \; = \; :ABCD:$$

$$\begin{aligned}
\varepsilon_{BC}\varepsilon_{BD} :ACDB: = \varepsilon_{BC}\varepsilon_{BD} \big(& A^-C^-D^-B^- + A^-C^-D^-B^+ + A^-C^-D^+B^+ + \varepsilon_{BD} A^-C^-B^-D \\
& + \varepsilon_{CD}\varepsilon_{CB} A^-D^-B^-C^+ + \varepsilon_{CD} A^-D^-C^+B^+ + A^-C^+D^+B^+ \\
& + \varepsilon_{BC}\varepsilon_{BD} A^-B^-C^+D^+ \\
& + \varepsilon_{AC}\varepsilon_{AD}\varepsilon_{AB} C^-D^-B^-A^+ + \varepsilon_{AC}\varepsilon_{AD} C^-D^-A^+B^+ + \varepsilon_{AC} C^-A^+D^+B^+ \\
& + \varepsilon_{BD}\varepsilon_{AC}\varepsilon_{AB} C^-B^-A^+D^+ \\
& + \varepsilon_{CD}\varepsilon_{CB}\varepsilon_{AD}\varepsilon_{AB} D^-B^-A^+C^+ + \varepsilon_{CD}\varepsilon_{AD} D^-A^+C^+B^+ \\
& + A^+C^+D^+B^+ + \varepsilon_{BC}\varepsilon_{BD}\varepsilon_{AB} B^-A^+C^+D^+ \big)
\end{aligned}$$

$$= \; :ABCD:$$

So also get the same expression as in ①.

Now we show that in general,

$$\boxed{:A \cdots i\, j \cdots B: \; = \; \varepsilon_{ij} :A \cdots j\, i \cdots B:}$$

proof: $\text{LHS} = \; :A \cdots (i^+ + i^-)(j^+ + j^-) \cdots B:$

$= \; :A \cdots i^+ j^+ \cdots B: + :A \cdots i^- j^- \cdots B: + :A \cdots i^- j^+ \cdots B:$

$\quad + :A \cdots i^+ j^- \cdots B:$

$= \varepsilon_{ij} \big(:A \cdots j^+ i^+ \cdots B: + :A \cdots j^- i^- \cdots B: + :A \cdots j^+ i^- \cdots B:$

$\quad + A \cdots j^- i^+ \cdots B: \big)$

$\uparrow$ definition of normal ordering

$= \varepsilon_{ij} :A \cdots j\, i \cdots B:$

$\boxed{\text{Also, all possible contractions (no matter how many) of } :A \cdots i\, j \cdots B: = \text{the corresponding ones in } \varepsilon_{ij} :A \cdots j\, i \cdots B:}$

proof: ① if the contractions are not include i and j, then it is true by noticing that the uncontracted operators satisfy $:\cdots ij \cdots: = \varepsilon_{ij} :\cdots ji \cdots:$,

② if the contractions include ij, then using $ij = \varepsilon_{ij} ji$, then it goes back to reasoning of ①

③ if the contractions include i and j seperately (i.e., $:\cdots k \cdots i\, j \cdots l \cdots:$, $:\cdots k \cdots i\, j \cdots l \cdots:$, $:\cdots k \cdots l \cdots i\, j \cdots:$, $:\cdots k \cdots l \cdots i\, j \cdots:$, $:\cdots i\, j \cdots k \cdots l \cdots:$, $:\cdots i\, j \cdots k \cdots l \cdots:$) or only i is contracted (i.e., $:\cdots k \cdots i\, j \cdots:$, $:\cdots i\, j \cdots k \cdots:$) or only j is contracted (i.e., $:\cdots k \cdots i\, j \cdots:$, $:\cdots i\, j \cdots k \cdots:$), then we can move k and l next to $i\,j$ (i.e., $:\cdots k\, i\, j\, l \cdots:$, $:\cdots k\, i\, j\, l \cdots:$, $:\cdots k\, l\, i\, j \cdots:$, $:\cdots k\, l\, i\, j \cdots:$, $:\cdots i\, j\, k\, l \cdots:$, $:\cdots i\, j\, k\, l \cdots:$, $:\cdots k\, i\, j \cdots$, $:\cdots i\, j\, k \cdots:$, $:\cdots k\, i\, j \cdots:$, $:\cdots i\, j\, k \cdots:$), by noticing that (shown before)

$$k\,i\,j\,l = \varepsilon_{ij}\, k\,j\,i\,l,$$

$$k\,i\,j\,l = \varepsilon_{ij}\, k\,j\,i\,l$$

$$k\,l\,i\,j = k\,j\,l\,i = \varepsilon_{jl}\, k\,l\,j\,i = \varepsilon_{ij}\, k\,l\,j\,i$$

$$k\,l\,i\,j = \varepsilon_{il}\, k\,i\;l\,j = \varepsilon_{ij}\, k\,l\,j\,i$$

$$i\,j\,k\,l = \varepsilon_{jk}\, i\,k\,j\,l = \varepsilon_{ij}\, j\,i\,k\,l$$

$$i\,j\,k\,l = j\,k\;i\,l = \varepsilon_{ik}\, j\,i\,k\,l = \varepsilon_{ij}\, j\,i\,k\,l$$

$$k\,i\,j = \varepsilon_{ij}\, k\,i\,j\,, \quad i\,j\,k = \varepsilon_{ij}\, j\,i\,k\,, \quad k\,i\,j = \varepsilon_{ij}\, k\,j\,i,$$

$$i\,j\,k = \varepsilon_{ij}\, j\,i\,k$$

in this way, i and j switched position and we get the factor ε_{ij}.
done the proof.

We now show in general

$$\boxed{T(A_{x_1}\cdots P_{x_i}Q_{x_j}\cdots B_{x_n}) = \varepsilon_{PQ}T(A_{x_1}\cdots Q_{x_j}P_{x_i}\cdots B_{x_n})}$$

proof: For any $\cdots > t_i > \cdots > t_j > \cdots$,

$$T(A_{x_1}\cdots P_{x_i}Q_{x_j}\cdots B_{x_n}) = \cdots \; P_{x_i}\cdots P_{x_j}\cdots$$

and $T(A_{x_1}\cdots Q_{x_j}P_{x_i}\cdots B_{x_n}) = \varepsilon_{PQ}(\cdots \; P_{x_i}\cdots P_{x_j}\cdots)$

Similarly, for any $\cdots > t_j > \cdots > t_i > \cdots$,

$$T(A_{x_1}\cdots P_{x_i}Q_{x_j}\cdots B_{x_n}) = \cdots \; P_{x_j}\cdots P_{x_i}\cdots$$

and $T(A_{x_1}\cdots Q_{x_j}P_{x_i}\cdots B_{x_n}) = \varepsilon_{PQ}(\cdots P_{x_j}\cdots P_{x_i}\cdots)$

So done the proof.

Therefore, for case ③ if $t_1, t_2, t_4 > t_3$ and case ④ if $t_2, t_3, t_4 > t_1$, we still have the same result as case ①.

By noticing that any permutation can be constructed as interchanges of neighbouring operators, we have

$$:A_1 A_2 \cdots A_n: = \underbrace{\varepsilon_{ij}\varepsilon_{li}\varepsilon_{mn}\cdots}_{\text{the interchanges to achieve the permutation}} :\underbrace{\qquad\qquad}_{\text{a permutation of } A_1 A_2 \cdots A_n}:$$

$$= (-1)^P :\qquad\qquad:$$

where P is the number of interchanges of neighbouring fermion field operators.

If there are same contractions, the same relation holds, i.e.,

$$:A_1 A_2 \cdots A_n: \;=\; \varepsilon_{ij}\,\varepsilon_{li}\,\varepsilon_{mn}\cdots :\overline{\qquad\qquad\qquad}:$$

a permutation of $A_1 A_2 \cdots A_n$ with contractions of the same operators as the left-hand-side

the interchanges to achieve the permutation

$$= (-1)^P :\underline{\qquad\qquad}:$$

Also, $$T(A^1_{x_1} A^2_{x_2} \cdots A^n_{x_n}) = \varepsilon_{ij}\,\varepsilon_{li}\,\varepsilon_{mn}\cdots T(\overline{\qquad\qquad\qquad})$$

a permutation of $A^1_{x_1} A^2_{x_2} \cdots A^n_{x_n}$

the interchanges to achieve the permutation

$$= (-1)^P\, T(\underline{\qquad\qquad})$$

The results of two, three and four field operators can be generalized to arbitrary number of field operators, and this is the <u>Wick's theorem</u>:

The time-ordered product of a set of field operators can be decomposed into the sum of the corresponding contracted normal products. All contractions of pair of operators that possibly can arise enter this sum.

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proof of Wick's theorem (I follow "Field Quantization" by Greiner & Reinhardt) pp 231–233

First, let's prove the following lemma:

For field operators $\underbrace{A, B, \cdots, Z}_{\text{not means only 26 operators, can be arbitrary number of operators.}}$ with $t_Z < t_A, \cdots, t_Y$

$$:AB\cdots XY:Z = :AB\cdots XYZ: + :\underbrace{AB\cdots XYZ}: + :A\underbrace{B\cdots XYZ}: + \cdots + :AB\cdots X\underbrace{YZ}: \quad \cdots (1)$$

proof: first, let's show that we can assume ① Z only contains creation operator ② A, ⋯, Y only contain annihilation operators, without loss of the generality of the proof.

For ①, for the possible annihilation operator part of Z, call it $Z^{(+)}$, LHS of ① $= :AB\cdots XYZ^{(+)}:$, and all the contraction with $Z^{(+)}$, e.g., $\underbrace{AZ^{(+)}} = \langle 0|T(AZ^{(+)})|0\rangle = \langle 0|AZ^{(+)}|0\rangle = 0$, (since $\hat{a}|0\rangle = 0$),

so that RHS of ① $= :AB\cdots XYZ^{(+)}:$

For ②, if (1) is valid under the assumption of ①&②, then left times the creation-operator-contained A, i.e., $A^{(-)}$ to $:B^{(+)}C^{(+)}\cdots Y^{(+)}:Z^{(-)}$ on the LHS, and also on the RHS, then we have

$$\begin{aligned} A^{(-)}:B^{(+)}C^{(+)}\cdots Y^{(+)}:Z^{(-)} = {} & A^{(-)}:B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}: \\ & + A^{(-)}:\underbrace{B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}}: \\ & + \cdots \\ & + A^{(-)}:B^{(+)}C^{(+)}\cdots \underbrace{Y^{(+)}Z^{(-)}}: \end{aligned}$$

$$\begin{aligned} \Rightarrow :A^{(-)}B^{(+)}C^{(+)}\cdots Y^{(+)}:Z^{(-)} = {} & :A^{(-)}B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}: \\ & + :A^{(-)}\underbrace{B^{+}C^{(+)}\cdots Y^{(+)}Z^{(-)}}: \\ & + \cdots \\ & + :A^{(-)}B^{(+)}C^{(+)}\cdots \underbrace{Y^{(+)}Z^{(-)}}: \end{aligned}$$

together with

$$\begin{aligned} :A^{(+)}B^{(+)}C^{(+)}Y^{(+)}:Z^{(-)} = {} & :A^{(+)}B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}: \\ & + :\underbrace{A^{(+)}B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}}: \\ & + :A^{(+)}\underbrace{B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}}: \\ & + \cdots + :A^{(+)}B^{(+)}C^{(+)}\cdots Y^{(+)}Z^{(-)}: \end{aligned}$$

Also notice that $\underbrace{A^{(-)} Z^{(-)}} = \langle 0 | T(A^{(-)} Z^{(-)}) | 0 \rangle = 0$. (Since $\langle 0 | \hat{a}^{+} = 0$)

add together and use $A = A^{(+)} + A^{(-)}$

$$\Rightarrow \; :A B^{(+)} C^{(+)} \cdots Y^{(+)}: Z^{(-)} = \; :A B^{(+)} C^{(+)} \cdots Y^{(+)} Z^{(-)}:$$
$$+ :\underbrace{A B^{(+)} C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ :A \underbrace{B^{(+)} C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ \cdots$$
$$+ :A B^{(+)} C^{(+)} \cdots \underbrace{Y^{(+)} Z^{(-)}}:$$

We can do the similar thing for $B^{(-)}$,

$$\varepsilon_{AB} \; B^{(-)} :A C^{(+)} \cdots Y^{(+)}: Z^{(-)} = \varepsilon_{AB} \big(B^{(-)} :A C^{(+)} \cdots Y^{(+)} Z^{(-)}:$$
$$+ B^{(-)} :\underbrace{A C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ B^{(-)} :A \underbrace{C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ \cdots$$
$$+ B^{(-)} :A C^{(+)} \cdots \underbrace{Y^{(+)} Z^{(-)}}: \big)$$

$$\Rightarrow \; :A B^{(-)} C^{(+)} \cdots Y^{(+)}: Z^{(-)} = \; :A B^{(-)} C^{(+)} \cdots Y^{(+)} Z^{(-)}:$$
$$+ :\underbrace{A B^{(-)} C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ :A B^{(-)} \underbrace{C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ \cdots$$
$$+ :A B^{(-)} C^{(+)} \cdots \underbrace{Y^{(+)} Z^{(-)}}:$$

again, $\underbrace{B^{(-)} Z^{(-)}} = 0$

$$\Rightarrow \; :A B C^{(+)} \cdots Y^{(+)}: Z^{(-)} = \; :A B C^{(+)} \cdots Y^{(+)} Z^{(-)}:$$
$$+ :\underbrace{A B C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ :A \underbrace{B C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ :A B \underbrace{C^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ \cdots$$
$$+ :A B C^{(+)} \cdots \underbrace{Y^{(+)} Z^{(-)}}:$$

we can then do the similar thing for $C^{(-)}$,

$$\varepsilon_{AC} \varepsilon_{BC} \; C^{(-)} :A B D^{(+)} \cdots Y^{(+)}: Z^{(-)} = \varepsilon_{AC} \varepsilon_{BC} \, C^{(-)} \big(:A B D^{(+)} \cdots Y^{(+)} Z^{(-)}:$$
$$+ :\underbrace{A B D^{(+)} \cdots Y^{(+)} Z^{(-)}}:$$
$$+ \cdots$$
$$+ :A B D^{(+)} \cdots \underbrace{Y^{(+)} Z^{(-)}}: \big)$$

$$\Rightarrow\ :ABC^{(-)}D^{(+)}\cdots Y^{(+)}:Z^{(-)} = :ABC^{(-)}D^{(+)}\cdots Y^{(+)}Z^{(-)}:$$
$$+ :\underbrace{ABC^{(-)}D^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ :A\underbrace{BC^{(-)}D^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ :ABC^{(-)}\underbrace{D^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ \cdots$$
$$+ :ABC^{(-)}D^{(+)}\cdots \underbrace{Y^{(+)}Z^{(-)}}:$$

and use $\underbrace{C^{(-)}Z^{(-)}} = 0$

$$\Rightarrow\ :ABCD^{(+)}\cdots Y^{(+)}:Z^{(-)} = :ABCD^{(+)}\cdots Y^{(+)}Z^{(-)}:$$
$$+ :\underbrace{ABCD^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ :A\underbrace{BCD^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ :AB\underbrace{CD^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ :ABC\underbrace{D^{(+)}\cdots Y^{(+)}Z^{(-)}}:$$
$$+ \cdots$$
$$+ :ABCD^{(+)}\cdots \underbrace{Y^{(+)}Z^{(-)}}:$$

Continue in the same way, we get

$$:AB\cdots Y:Z^{(-)} = :AB\cdots YZ^{(-)}: + :\underbrace{AB\cdots YZ^{(-)}}: + \cdots + :AB\cdots \underbrace{YZ^{(-)}}:$$

Therefore, we have shown that without loss of generality, under the assumption of ① & ②, we can prove (1).

Now let's prove (1) under the assumption of ① & ②, by using Mathematical induction.

For two field operator case, $:Y:Z = YZ = T(YZ) = :YZ: + \underbrace{YZ}$

since $t_Z < t_Y$ — Shown explicitly before in page 21.

Now assume that (1) is valid for n field operator

$$:BC\cdots Y:Z = :BC\cdots YZ: + :\underbrace{BC\cdots YZ}: + \cdots + :BC\cdots\underbrace{YZ}:$$

then times from the left an additional annihilation-operator-contained-only operator A, with $t_A > t_Z$, we get.

$$A:BC\cdots Y:Z = A:BC\cdots YZ: + :A\underbrace{BC\cdots YZ}: + \cdots + :ABC\cdots \underbrace{YZ}:$$

Note that $B, C\cdots Y$ contain only annihilation operator, and Z contains only creation operator, so we can move A inside $:\;:$ once Z is contracted away,

where $A:BC\cdots YZ: = (-1)^P AZBC\cdots Y$,

(where P counts the number of neighbouring interchanges involving two fermionic operators when Z is moved cross $BC\cdots Y$, (note that if Z is a bosonic operator, then $(-1)^P = +1$))

$$\overset{\text{since } t_A > t_Z}{=} (-1)^P\, T(AZ)\, BC\cdots Y$$

$$= (-1)^P (:AZ: + \underbrace{AZ})\, BC\cdots Y$$

$$= (-1)^P :AZ:BC\cdots Y + (-1)^P \underbrace{AZ} :BC\cdots Y:$$

$$= (-1)^P \varepsilon_{AZ}\, ZA\, BC\cdots Y + :A\underbrace{BC\cdots YZ}:$$

$$= (-1)^P \varepsilon_{AZ} :ZABC\cdots Y: + :A\underbrace{BC\cdots YZ}:$$

$$= :ABC\cdots YZ: + :\underbrace{ABC\cdots YZ}:$$

Since $LHS = A:BC\cdots Y:Z = :ABC\cdots Y:Z$,

$\Rightarrow$ we have
$$:ABC\cdots Y:Z = :ABC\cdots YZ: + :\underbrace{ABC\cdots YZ}:$$
$$+ :A\underbrace{BC\cdots YZ}: + \cdots + :ABC\cdots \underbrace{YZ}:$$

So done the proof of the lemma.

Now we use this lemma to prove Wick's theorem.

First, we notice that the lemma can be immediately generalized to the case where one or more operators in the normal product $:AB\cdots XY:$ are contracted, since the contracted operators can be pulled out of the normal products on both side of (1), and the remaining is still in the form of (1).

Then still use Mathematical induction to do the proof.

For two operator case, we have shown explicitly that

$$T(AB) = :AB: + :\underbrace{AB}:$$

Assume that Wick's theorem is valid for n operators, i.e.,

$$T(AB\cdots XY) = :AB\cdots XY: + :\underbrace{AB}\cdots XY: + \cdots$$
$$+ :\underbrace{AB}\,\underbrace{CD}\cdots XY: + :\underbrace{A\underbrace{BC}D\cdots XY}: + \cdots$$
$$+ \cdots$$

and multiply this expression from the right by Z, with $t_Z < t_A, \dots, t_Y$, then

$$LHS = T(AB\cdots XY)Z \underset{\uparrow}{=} T(AB\cdots XYZ)$$

since $t_Z < t_A, \dots, t_Z$

and the

$$RHS = :AB\cdots XY:Z + :\underbrace{AB}\cdots XY:Z + \cdots$$
$$+ :\underbrace{AB}\,\underbrace{CD}\cdots XY:Z + :\underbrace{A\underbrace{BC}D\cdots XY}:Z$$
$$+ \cdots$$

Then use the lemma for the RHS, we notice that it exactly give $:AB\cdots XYZ:$ and all possible contractions involving $AB\cdots XY$ and Z.

So we have shown that the Wick's theorem is valid for $(n+1)$ operator.

Finally, we note that the condition in the lemma, $t_Z < t_A, \dots, t_Y$, does not harm the generality of the Wick's theorem. Because for any $T(AB\cdots ij XYZ)$, we can always put the earliest time operator at the far right, say, $t_i < t_A, \dots, t_Z$, so $T(AB\cdots ij\cdots XYZ) = (-1)^P T(AB\cdots j\cdots XYZi)$, where P counts the number of fermionic operators involved for i to go across $j\cdots XYZ$ (if i is a bosonic operator, then $(-1)^P = +1$), then we can use the above result to write

$$\begin{aligned}
T(AB\cdots ij\cdots XYZ) &= (-1)^P\, T(AB\cdots j\cdots XYZi) \\
&= (-1)^P \left\{ :AB\cdots j\cdots XYZi: + \text{all possible contractions in } :AB\cdots j\cdots XYZi: \right\} \\
&= :AB\cdots ij\cdots XYZ: + \text{all possible contractions in } :AB\cdots ij\cdots XYZ:
\end{aligned}$$

So the Wick's theorem is proved.

However, for the time-ordered product we are interected in,

$$T\left(:\hat{H}_I^{int}(x_1):\ :\hat{H}_I^{int}(x_2):\cdots:\hat{H}_I^{int}(x_n):\right)$$

the $\hat{H}_I^{int}$ typicelly contains several field operators, e.g. $\hat{H}_I^{int}(x)=\lambda\hat{\phi}_I(x)\hat{\bar{\psi}}_I(x)\hat{\psi}(x)$

then we need to deal with equal time field operators.

Notice that the $\hat{H}_I^{int}(x)$ are normal-ordered one (to remove vaccum energy from the beginning), i.e., $\hat{H}_I^{int}(x)=:A(x)B(x)\cdots Z(x):$, where $A, B, \cdots Z$ are field operators, we can avoid contractions for equal time field operators by adding a small $\varepsilon>0$ to t for all the creation-operator-contained part and substract $\varepsilon>0$ from t for all annihilation-operator-contained part.

Then

$$:\hat{H}_I^{int}(x):=:\hat{H}_I^{int}(t,\vec{x}):=:(A^{(-)}(t+\varepsilon,\vec{x})+A^{(+)}(t-\varepsilon,\vec{x}))$$
$$(B^{(-)}(t+\varepsilon,\vec{x})+B^{(+)}(t-\varepsilon,\vec{x}))\cdots(Z^{(-)}(t+\varepsilon,\vec{x})+Z^{(+)}(t-\varepsilon,\vec{x})):$$

$$=A^{(-)}B^{(-)}\cdots Z^{(-)}+A^{(-)}B^{(+)}\cdots Z^{(+)}+\varepsilon_{AB}B^{(-)}A^{(+)}\cdots Z^{(+)}+A^{(+)}B^{(+)}\cdots Z^{(+)}$$
$$+\varepsilon_{AB}\varepsilon_{AC}B^{(-)}C^{(-)}A^{(+)}\cdots Z^{(+)}+\cdots$$

we notice that all terms are already time-ordered, since the creation-operator-contained field operators which have a $(t+\varepsilon)$ time argument are already in front of the annihilation-operator-contained field operators which have a $(t-\varepsilon)$ time argument

So, when putting these terms in $T(\hat{H}_I^{int}(x_1)\hat{H}_I^{int}(x_2)\cdots\hat{H}_I^{int}(x_n))$, and use Wick's theorem, all the contractions vanishes: $\underbrace{A^{(-)}B^{(-)}}\propto\langle 0|a^\dagger b^\dagger|0\rangle=0$

$\underbrace{A^{(+)}B^{(+)}}\propto\langle 0|ab|0\rangle=0$, $\underbrace{B^{(-)}A^{(+)}}\propto\langle 0|b^\dagger a|0\rangle=0$ (no $\langle 0|ab^\dagger|0\rangle$ term,

since

$$\underbrace{B^{(-)}A^{(+)}}=\langle 0|T(B^{(-)}A^{(+)})|0\rangle=\theta((t+\varepsilon)-(t-\varepsilon))\langle 0|B^{(-)}A^{(+)}|0\rangle$$
$$+\theta((t-\varepsilon)-(t+\varepsilon))\langle 0|A^{(+)}B^{(-)}|0\rangle=\theta(2\varepsilon)\langle 0|B^{(-)}A^{(+)}|0\rangle+\theta(-2\varepsilon)\langle 0|A^{(+)}B^{(-)}|0\rangle$$
$$=\langle 0|B^{(-)}A^{(+)}|0\rangle$$

So, after finishing the contractions for field operators among different $:\hat{H}_I^{int}(x_i):$ and $:\hat{H}_I^{int}(x_j):$, we do the limit $\varepsilon \to 0^+$, we get the desired result.

$$T\left(:\hat{H}_I^{int}(x_1)::\hat{H}_I^{int}(x_2):\cdots:\hat{H}_I^{int}(x_n):\right) = T\left(\hat{H}_I^{int}(x_1)\,\hat{H}_I^{int}(x_2)\cdots\hat{H}_I^{int}(x_n)\right)_{\text{no e.t.c}}$$

where "no e.t.c" means "no equal-time contractions".