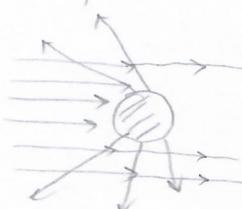


## Cross sections & decay rates

$$\sigma \equiv \frac{\text{number of particles scattered}}{\cancel{\text{time}} \times \cancel{\text{number density in beam}} \times \cancel{\text{velocity of beam}}} = \frac{1}{T} \frac{1}{\Phi} N$$

→ number of particle go through per unit area per unit time, called flux.

It's also natural to measure the differential cross section  $\frac{d\sigma}{d\Omega}$ , as this gives the information (classically) about the shape of the object or the form of the potential responsible for the scattering to happen.



While classically a scatter either happen or not, quantum mechanically it has a probability for scattering.

$$d\sigma = \frac{1}{T} \frac{1}{\Phi} dP$$

where  $\Phi$  is now normalized as if the beam has just one particle,  $P$  is the probability of scattering.

The differential number of scattering events measured in a collider experiment is

$$dN \equiv L \times d\sigma$$

where  $L$  is the luminosity. (unit is  $\text{area}^{-1}$ ).

The experimental data is often presented as  $\frac{dN}{dE}$  vs.  $E$ , then if we can calculate  $\frac{d\sigma}{dE}$  from theory, then multiply by  $L$ , we can get the theoretical plot for  $\frac{d\sigma}{dE} \times L$  vs.  $E$ , to compare with experimental data.

Now let's relate the  $d\sigma$  to S-matrix element.  $\langle f | S | i \rangle$ .

Assume  $|i\rangle$  is a two particle state, and  $|f\rangle$  is a  $n$  particle state, so  $2 \rightarrow n$  process.

$$P_1 + P_2 \rightarrow \{P_j\}, j=1, \dots, n.$$

In the rest frame of one of the colliding particle, the flux is

$$\phi = \frac{|\vec{v}|}{V},$$

where  $|\vec{v}|$  is the magnitude of the incoming particle,  $V$  is the "big box" in which the whole experiment is taking place.

In a different frame, such as the center-of-mass frame,  $|\vec{v}|$  should be changed to  $|\vec{v}_1 - \vec{v}_2|$ .

So we have

$$d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP,$$

where  $T$  is the time the experiment lasts.  $T$  is large since we assume  $|i\rangle$  and  $|f\rangle$  are states at  $T \rightarrow -\infty$  &  $T \rightarrow +\infty$ , respectively.

$$dP = \frac{|\langle f | \hat{S} | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle} d\Omega.$$

where  $|\langle f | \hat{S} | i \rangle|^2$  is the transition probability density from  $|i\rangle$  to  $|f\rangle$  due to interaction described by  $\hat{S}$  (recall that  $\hat{S}$  has interaction Hamiltonian in it).

Note that since  $|f\rangle$  is specified by definite momenta of the final state, and considering that momentum is a continuous quantity, we need  $d\Omega$  to get from probability density to probability.

Consider a large cubic  $V=L^3$ , the de Broglie wavelength is

$$\lambda = \frac{h}{P} = \frac{L}{n}, n \text{ is integer}$$

$$\Rightarrow \frac{2\pi k}{P} = \frac{L}{n} \Rightarrow \Delta n = \frac{\Delta P}{2\pi} L$$

In three dimension and go to continuous limit,  $d\mathbf{n}_x d\mathbf{n}_y d\mathbf{n}_z = \frac{d^3 \vec{P}}{(2\pi)^3} V$

So, the number of states within  $\frac{d^3 \vec{P}}{(2\pi)^3} V$  is  $d\mathbf{n}_x d\mathbf{n}_y d\mathbf{n}_z = \frac{d^3 \vec{P}}{(2\pi)^3} V$ .  
Do the same for every particle in the final state.

$$\Rightarrow d\Pi = \prod_{j=1}^n \frac{V}{(2\pi)^3} d^3 \vec{P}_j$$

Since for one particle state,

$$\langle P | \mathcal{E} \rangle = (2\pi)^3 2E_{\vec{P}} \delta^3(\vec{P} - \vec{\mathcal{E}}) \quad (\text{for scalar field})$$

$$\langle P, \Gamma | \mathcal{E}, S \rangle = (2\pi)^3 2E_{\vec{P}} \delta^3(\vec{P} - \vec{\mathcal{E}}) \delta_{rs} \quad (\text{for spinor and vector field})$$

we have  $\langle P | P \rangle = (2\pi)^3 2E_{\vec{P}} \delta^3(0)$

and  $\langle P, \Gamma | P, \Gamma \rangle = (2\pi)^3 2E_{\vec{P}} \delta^3(0)$ .

Since  $\int_{-\infty}^{+\infty} d^3 \vec{x} e^{-i\vec{P} \cdot \vec{x}} = (2\pi)^3 \delta^3(\vec{P})$

then  $\int_{-\infty}^{+\infty} d^3 \vec{x} = (2\pi)^3 \delta^3(0)$

$$\Rightarrow \delta^3(0) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3 \vec{x} = \lim_{V \rightarrow \infty} \frac{V}{(2\pi)^3}$$

Similarly,  $\int_{-\infty}^{+\infty} dt e^{-iE_{\vec{P}} t} = (2\pi) \delta(E_{\vec{P}})$

then  $\int_{-\infty}^{+\infty} dt = (2\pi) \delta(0)$

$$\Rightarrow \delta(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt = \lim_{T \rightarrow \infty} \frac{T}{2\pi}$$

$$\Rightarrow \delta^4(0) = \lim_{V, T \rightarrow \infty} \frac{TV}{(2\pi)^4}$$

Keep in mind that at the end we need to go to  $V, T \rightarrow \infty$ , we can write

$$\langle p | p \rangle = 2E_p V$$

$$\langle p, r | p, r \rangle = 2E_p V.$$

So the normalization factor  $\langle f | f \rangle$  and  $\langle i | i \rangle$  are independent of the types of the fields, we can just use the momentum alone to label the state for the purpose here

$$|i\rangle = |p_1\rangle |p_2\rangle \text{ and } |f\rangle = \prod_j |p_j\rangle$$

$$\Rightarrow \langle i | i \rangle = (2E_1 V) (2E_2 V)$$

$$\langle f | f \rangle = \prod_{j=1}^n (2E_j V)$$

For  $|k_f | \hat{S} | i \rangle|^2$ , since  $\hat{S} = \mathbb{I} + \sum_{i=1}^{\infty} (-i)^4 \int dx_1 dx_2 \dots dx_n T(H_1^{int}(x_1) H_2^{int}(x_2) \dots H_n^{int}(x_n))$   
 then we can write it as  $\hat{S} = \mathbb{I} + iT$ , where  $\mathbb{I}$  is the part where there is no interaction, and  $T$  is called the transition matrix, then factor out the energy-momentum conservation condition, we can write

$$T = (2\pi)^4 \delta^4(\sum p_i^u - \sum p_f^u) M$$

So, we are interested in calculating the interaction related S-matrix element  $\langle f | \hat{S} - \mathbb{I} | i \rangle = (2\pi)^4 \delta^4(\sum p_i^u - \sum p_f^u) i \langle f | M | i \rangle$

where  $\langle f | M | i \rangle$  is usually called matrix element.

Since we are interested in the transition,  $|f\rangle \neq |i\rangle$ , then  $\langle f | i \rangle = 0$ .

$$\text{So } \langle f | \hat{S} | i \rangle = \langle f | \hat{S} - \mathbb{I} | i \rangle$$

$$\Rightarrow | \langle f | \hat{S} | i \rangle |^2 = (2\pi)^4 \delta^4(\sum p_i^u - \sum p_f^u) \delta^4(0) | \langle f | M | i \rangle |^2$$

$$(\text{since } (\delta^4(\sum p_i^u - \sum p_f^u))^2 = \delta^4(\sum p_i^u - \sum p_f^u) \delta^4(0))$$

$$\text{relative } S^4(0) = \frac{V T}{(2\pi)^4}$$

$$\Rightarrow |f| |\hat{S}| |i\rangle|^2 = (2\pi)^4 S^4(\sum P_i^\mu - \sum P_f^\mu) V T |M|^2,$$

$$\text{where } |M|^2 = |\langle f | M | i \rangle|^2$$

$$\Rightarrow dP = \frac{(2\pi)^4 S^4(\sum P_i^\mu - \sum P_f^\mu) V T |M|^2}{(2E_1 V)(2E_2 V) \prod_{j=1}^n (2E_j V)} \prod_{j=1}^n \frac{V}{(2\pi)^3} d^3 \vec{p}_j$$

$$= \frac{T}{V} \frac{1}{(2E_1)(2E_2)} |M|^2 d\Omega_{LIPS}$$

where  $d\Omega_{LIPS} \equiv \prod_{\text{final state } j} \frac{d^3 \vec{p}_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 S^4(\sum P_i^\mu - \sum P_f^\mu)$

is called the Lorentz-invariant phase space (LIPS)

$$\Rightarrow d\Gamma = \frac{V}{T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} |M|^2 d\Omega_{LIPS}$$

we see that no  $V$  or  $T$  left, so it is trivial to take  $V, T \rightarrow \infty$ .

A decay is a  $1 \rightarrow n$  process. the differential decay rate is

$$d\Gamma = \frac{1}{T} dP$$

$$\text{where } dP = \frac{|\langle f | \hat{S} | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle} d\Omega$$

$$\langle i | i \rangle = 2E_1 V$$

$|\langle f | \hat{S} | i \rangle|^2$ ,  $\langle f | f \rangle$  and  $d\Omega$  are the same as when deriving  $d\Gamma$ .

$$\Rightarrow d\Gamma = \frac{1}{T} \frac{(2\pi)^4 S^4(\sum P_i^\mu - \sum P_f^\mu) V T |M|^2}{(2E_1 V) \prod_{j=1}^n (2E_j V)} \prod_{j=1}^n \frac{V}{(2\pi)^3} d^3 \vec{p}_j$$

$$= \frac{1}{2E_1} |M|^2 d\Omega_{LIPS}$$

Now we prove that  $d\pi_{Lips}$  is indeed Lorentz invariant.

$$\begin{aligned}
 \frac{d^3 \vec{P}}{(2\pi)^3 2E_{\vec{P}}} &= \frac{d^3 \vec{P}}{(2\pi)^3} \frac{dp^0}{2p^0} \delta(p^0 - E_{\vec{P}}), \text{ note that it should be understood as} \\
 &\quad \text{an integration for } p^0 \text{ in this and} \\
 &= \frac{d^4 P}{(2\pi)^4} \frac{2\pi}{2p^0} [\delta(p^0 - E_{\vec{P}}) + \delta(p^0 + E_{\vec{P}})] \Theta(p^0) \\
 &= \int \frac{d^4 P}{(2\pi)^4} 2\pi \delta(p^{0^2} - E_{\vec{P}}^2) \Theta(p^0) \\
 \delta(x^2 - d^2) &= \frac{1}{2|x|} [\delta(x+d) + \delta(x-d)] \\
 &= \int \frac{d^4 P}{(2\pi)^4} \cdot 2\pi \delta(p^2 - m^2) \Theta(p^0) \\
 E_{\vec{P}}^2 &= |\vec{P}|^2 + m^2 \\
 p^2 &= p^{0^2} - |\vec{P}|^2
 \end{aligned}$$

Also, we can write

$$\begin{aligned}
 \frac{d^3 \vec{P}}{(2\pi)^3 2E_{\vec{P}}} &= \frac{d^3 \vec{P}}{(2\pi)^3} \frac{dp^0}{(-2p^0)} \delta(p^0 + E_{\vec{P}}) \\
 &= \frac{d^4 P}{(2\pi)^4} \frac{2\pi}{(-2p^0)} [\delta(p^0 + E_{\vec{P}}) + \delta(p^0 - E_{\vec{P}})] \Theta(-p^0) \\
 &= \frac{d^4 P}{(2\pi)^4} \cdot 2\pi \delta(p^{0^2} - E_{\vec{P}}^2) \Theta(-p^0) \\
 &= \frac{d^4 P}{(2\pi)^4} \cdot 2\pi \delta(p^2 - m^2) \Theta(-p^0)
 \end{aligned}$$

Therefore, it is clear that the role of  $\Theta(p^0)$  or  $\Theta(-p^0)$  together with  $\delta(p^2 - m^2)$  when do the integration is just to select one of the two possible values  $p^0 = \pm \sqrt{|\vec{P}|^2 + m^2} = \pm E_{\vec{P}}$ , and the result of the integration doesn't depends on which one to choose.

A Lorentz transformation on  $d\pi_{Lips}$  will make

$$\begin{aligned}
 d\pi_{Lips} &= \pi \frac{d^3 \vec{P}}{(2\pi)^3 2E_{\vec{P}}} \frac{1}{(2\pi)^4} \int^4 (\sum p_i^\mu - \sum p_f^\mu) = \pi \underset{\text{find state } j}{\frac{d^4 P}{(2\pi)^4}} 2\pi \delta(p_j^{0^2} - m_j^2) \Theta(p_j^0) (2\pi)^4 \int^4 (\sum p_i^\mu - \sum p_f^\mu) \\
 \rightarrow d\pi'_{Lips} &= \pi \underset{\text{find state } j}{\frac{d^4 P'}{(2\pi)^4}} 2\pi \delta(p_j'^2 - m_j^2) \Theta(p_j'^0) (2\pi)^4 \delta^4(\sum p_i'^\mu - \sum p_f'^\mu)
 \end{aligned}$$

$$\text{Note that } d^4 p \rightarrow d^4 p' = \frac{\downarrow P'^\mu = \Lambda^\mu_\nu P^\nu}{|\det(\Lambda)|} d^4 p = d^4 p.$$

$|\det(\Lambda)| = 1$  is the property  
of Lorentz transformation.

$$P_j'^2 = P_j^2 \Rightarrow \delta(P_j'^2 - m_j^2) = \delta(P_j^2 - m_j^2)$$

$\hookrightarrow$  since  $P^2$  is a Lorentz scalar.

$$\delta^4(\sum P_i'^\mu - \sum P_f'^\mu) = \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^4} d^4 x' e^{-ix' \cdot (\sum P_i' - \sum P_f')}$$

$$\begin{aligned} d^4 x' &= |\det(\Lambda)| d^4 x, x' \cdot (\sum P_i' - \sum P_f') = x \cdot (\sum P_i - \sum P_f) \\ &\stackrel{\downarrow}{=} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^4} d^4 x e^{-ix \cdot (\sum P_i - \sum P_f)} \end{aligned}$$

$$= \delta^4(\sum P_i^\mu - \sum P_f^\mu)$$

Since  $\frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \Theta(p^0) = \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \Theta(p^0)$ ,

it does not matter whether  $\Theta(p_j^0) = \Theta(p_j^0)$  or  $\Theta(p_j^0) = \Theta(-p_j^0)$ .  
That is, for the theta function, we only care about the sign of  $p^0$ ,  
but it does not matter whether the sign of  $p^0$  is the same as  $p^0$  or  $-p^0$ .

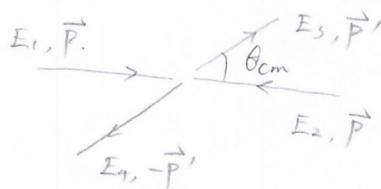
$$\Rightarrow d\Pi'_{LSP} = \prod_{\text{final state}} \frac{d^4 p}{(2\pi)^4} 2\pi \delta(P_j^2 - m_j^2) \Theta(\pm p^0) (2\pi)^4 \delta^4(\sum P_i^\mu - \sum P_f^\mu)$$

$$= d\Pi_{LSP}.$$

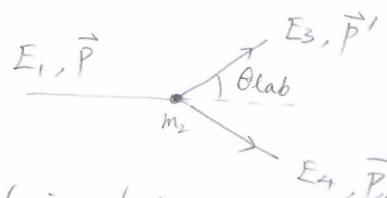
Note that due to the  $\delta(P_j^2 - m_j^2)$ ,  $P_j^0 = \pm \sqrt{P_j^2 + m_j^2} \neq 0$ .  
(if  $m_j = 0$ ,  $|\vec{P}_j|$  cannot be zero, since otherwise  $P_j^0 = 0$ , means that there  
is no  $j$  particle at all).

Some most common cases of scatterings and decays

### ① $2 \rightarrow 2$ scattering



(in center-of-mass frame :  $\vec{P}_1 + \vec{P}_2 = 0 = \vec{P}_3 + \vec{P}_4$ )



(in laboratory frame : particle-2 is at rest)

For these two frames,

note that the four particles has to be in a plane due to conservation of energy-momentum. Therefore,  $d\Omega = 2\pi d(\cos\theta)$ .

The Lorentz invariant phase space is

$$d\Pi_{LIPS} = \frac{d^3\vec{P}_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3\vec{P}_4}{(2\pi)^3} \frac{1}{2E_4} (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4)$$

$$d\sigma = \frac{1}{(2E_1)(2E_2) |\vec{v}_1 - \vec{v}_2|} |M|^2 d\Pi_{LIPS}$$

use  $\delta^3(\vec{P}_1 + \vec{P}_2 - \vec{P}_3 - \vec{P}_4)$  to integrate out  $d^3\vec{P}_4$

and use  $d^3\vec{P}_3 = |\vec{P}_3|^2 d|\vec{P}_3| d\Omega$ , where  $d\Omega = d\varphi d(\cos\theta)$

$\Rightarrow$

$$d\sigma = \frac{1}{(2\pi)^2} \frac{1}{4E_1 E_2} \frac{1}{|\vec{v}_1 - \vec{v}_2|} \frac{1}{4E_3 E_4} |M|^2 |\vec{P}_3|^2 \delta(E_1 + E_2 - E_3 - E_4) d|\vec{P}_3| d\Omega$$

[1] First, let's look at the center-of-mass frame.

$$\vec{P}_1 + \vec{P}_2 = 0 = \vec{P}_3 + \vec{P}_4, \quad \vec{P} = \vec{P}_1, \quad \vec{P}' = \vec{P}_3$$

$$E_3 = (|\vec{P}'|^2 + m_3^2)^{\frac{1}{2}}, \quad E_4 = (|\vec{P}'|^2 + m_4^2)^{\frac{1}{2}}$$

$$(P_1 + P_2)^2 = (E_1 + E_2)^2 = S = (P_3 + P_4)^2 = (E_3 + E_4)^2$$

$$|\vec{v}_1 - \vec{v}_2| = \left| \frac{\vec{P}_1}{E_1} - \frac{\vec{P}_2}{E_2} \right| = |\vec{P}| \left( \frac{1}{E_1} + \frac{1}{E_2} \right) = |\vec{P}| \frac{(E_1 + E_2)}{E_1 E_2}$$

$$d|\vec{P}_3| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_3 + E_4)} d(E_3 + E_4),$$

where  $\frac{d(E_3 + E_4)}{d|\vec{P}'|} = \frac{d((|\vec{P}'|^2 + m_3^2)^{\frac{1}{2}} + (|\vec{P}'|^2 + m_4^2)^{\frac{1}{2}})}{d|\vec{P}'|} = \frac{1}{2} \frac{2|\vec{P}'|}{E_3} + \frac{1}{2} \frac{2|\vec{P}'|}{E_4} = |\vec{P}'| \frac{(E_3 + E_4)}{E_3 E_4}$

$$\Rightarrow \frac{d|\vec{P}'|}{d(E_3 + E_4)} = \frac{1}{|\vec{P}'|} \cdot \frac{E_3 E_4}{(E_3 + E_4)}$$

then use  $\delta(\sqrt{s} - E_3 - E_4)$  to integrate out  $d|\vec{P}'|$

$$\begin{aligned} \Rightarrow \left( \frac{d\sigma}{d\Omega_{cm}} \right) &= \frac{1}{(2\pi)^2} \frac{1}{4E_1 E_2} \frac{1}{|\vec{P}'| \frac{(E_1 + E_2)}{E_1 E_2}} \frac{1}{4E_3 E_4} \cdot |\vec{P}'|^2 \frac{1}{|\vec{P}'|} \frac{E_3 E_4}{(E_3 + E_4)} |M|^2 \\ &= \frac{1}{(2\pi)^2} \frac{1}{4} \frac{|\vec{P}'|}{|\vec{P}|} \frac{1}{s} \frac{1}{4} |M|^2 \\ &= \frac{|M|^2}{64\pi^2 s} \frac{|\vec{P}'|}{|\vec{P}|} \end{aligned}$$

use  $P_1^2 = [(P_1 + P_2) - P_2]^2$

$$\begin{aligned} \Rightarrow m_1^2 &= s - 2P_2 \cdot (P_1 + P_2) + m_2^2 \\ &= s - 2E_2 \sqrt{s} + m_2^2 \\ \Rightarrow E_2 &= \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \end{aligned}$$

similarly,  $P_2^2 = [(\vec{P}_1 + \vec{P}_2) - \vec{P}_1]^2 = s - 2P_1 \cdot (P_1 + P_2) + P_2^2$

$$\begin{aligned} \Rightarrow m_2^2 &= s - 2E_1 \sqrt{s} + m_1^2 \\ \Rightarrow E_1 &= \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \end{aligned}$$

$$\begin{aligned} \Rightarrow |\vec{P}| = |\vec{P}_1| &= (E_1^2 - m_1^2)^{\frac{1}{2}} = \left[ \left( \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \right)^2 - m_1^2 \right]^{\frac{1}{2}} \\ &= \frac{(s^2 + m_1^4 + m_2^4 + 2m_1^2 s - 2m_2^2 s - 2m_1^2 m_2^2 - 4sm_1^2)^{\frac{1}{2}}}{2\sqrt{s}} \\ &= \frac{(s^2 + m_1^4 + m_2^4 - 2m_1^2 s - 2m_2^2 s - 2m_1^2 m_2^2)^{\frac{1}{2}}}{2\sqrt{s}} \\ &= \frac{\Delta(s, m_1^2, m_2^2)^{\frac{1}{2}}}{2\sqrt{s}} \end{aligned}$$

where  $\Delta(a, b, c) = (a - b - c)^2 - 4bc = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$

$$\text{Similarly, } P_3^2 = [(P_3 + P_4) - P_4]^2 = S - 2P_4 \cdot (P_3 + P_4) + P_4^2 = S - 2E_4\sqrt{S} + m_4^2$$

$$\Rightarrow E_4 = \frac{S + m_4^2 - m_3^2}{2\sqrt{S}}$$

$$E_3 = \sqrt{S} - E_4 = \frac{2S - (S + m_4^2 - m_3^2)}{2\sqrt{S}} = \frac{S + m_3^2 - m_4^2}{2\sqrt{S}}$$

$$|\vec{P}'| = |\vec{P}_3| = (E_3^2 - m_3^2)^{\frac{1}{2}} = \frac{(S^2 + m_3^4 + m_4^4 + 2m_3^2S - 2m_4^2S - 2m_3^2m_4^2 - 4m_3^2S)^{\frac{1}{2}}}{2\sqrt{S}}$$

$$= \frac{\lambda^{\frac{1}{2}}(S, m_3^2, m_4^2)}{2\sqrt{S}}$$

$$\Rightarrow \left[ \frac{d\sigma}{d\Omega} \right]_{cm} = \frac{|M|^2}{64\pi^2 S} \frac{\lambda^{\frac{1}{2}}(S, m_3^2, m_4^2)}{\lambda^{\frac{1}{2}}(S, m_1^2, m_2^2)}$$

use

$$t = (P_1 - P_3)^2 = m_1^2 + m_3^2 - 2P_1 \cdot P_3 = m_1^2 + m_3^2 - 2E_1 E_3 + 2|\vec{P}_1||\vec{P}_3|\cos\theta$$

recall that  
s, t, u are variables.

$$= m_1^2 + m_3^2 - 2E_1 E_3 + 2|\vec{P}_1||\vec{P}'|\cos\theta_{cm}$$

Mandelstam note that  $E_1, E_3, |\vec{P}|, |\vec{P}'|$  only depends on  $S$ , which is fixed.  
and for a given scattering, then

$$dt = 2|\vec{P}_1||\vec{P}'| d(\cos\theta_{cm})$$

$$\Rightarrow \left( \frac{d\sigma}{d\Omega} \right)_{cm} = \left( \frac{d\sigma}{2\pi d(\cos\theta_{cm})} \right)_{cm} = \left( \frac{d\sigma}{2\pi dt} \frac{2|\vec{P}_1||\vec{P}'|}{|M|^2} \right)_{cm}$$

$$\Rightarrow \left[ \frac{d\sigma}{dt} \right]_{cm} = \frac{1}{2|\vec{P}_1||\vec{P}'|} \frac{|M|^2}{64\pi^2 S} \frac{|\vec{P}'|}{|M|^2}$$

$$= \frac{|M|^2}{64\pi S |\vec{P}|^2} = \frac{1}{16\pi \lambda(S, m_1^2, m_2^2)}$$

Since  $0 \leq \cos\theta_{cm} \leq \pi$ , then  $t \in [m_1^2 + m_3^2 - 2E_1 E_3 - 2|\vec{P}_1||\vec{P}'|, m_1^2 + m_3^2 - 2E_1 E_3 + 2|\vec{P}_1||\vec{P}'|]$ , that is  $t \in [(E_1 - E_3)^2 - (|\vec{P}_1| + |\vec{P}'|)^2, (E_1 - E_3)^2 - (|\vec{P}_1| - |\vec{P}'|)^2]$ ,

(check:  $(E_1 - E_3)^2 - (|\vec{P}_1| + |\vec{P}'|)^2 = E_1^2 + E_3^2 - 2E_1 E_3 - |\vec{P}_1|^2 - |\vec{P}'|^2 - 2|\vec{P}_1||\vec{P}'| = m_1^2 + m_3^2 - 2E_1 E_3 - 2|\vec{P}_1||\vec{P}'|$ )

$$(E_1 - E_3)^2 - (|\vec{P}_1| - |\vec{P}'|)^2 = m_1^2 + m_3^2 - 2E_1 E_3 + 2|\vec{P}_1||\vec{P}'|$$

that is,  $\left[ t \in \left[ \frac{m_1^2 + m_3^2 - m_2^2 + m_4^2}{2\sqrt{S}} - (|\vec{P}_1| + |\vec{P}'|)^2, \frac{m_1^2 + m_3^2 - m_2^2 + m_4^2}{2\sqrt{S}} - (|\vec{P}_1| - |\vec{P}'|)^2 \right] \right]$

[2] In the laboratory frame, where particle 2 is at rest,

$$P_1^{\mu} = (E_1, \vec{P}), P_2^{\mu} = (m_2, 0), P_3^{\mu} = (E_3, \vec{P}'), P_4^{\mu} = (E_4, \vec{P}_4)$$

$$E_1 + E_2 = E_3 + E_4 \Rightarrow E_3 + E_4 = (\|\vec{P}\|^2 + m_1^2)^{\frac{1}{2}} + m_2$$

$$\vec{P} = \vec{P}' + \vec{P}_4, E_3 + E_4 = (\|\vec{P}_3\|^2 + m_3^2)^{\frac{1}{2}} + (\|\vec{P}_4\|^2 + m_4^2)^{\frac{1}{2}}$$

$$d|\vec{P}_3| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_3 + E_4)} = \frac{(\|\vec{P}'\|^2 + m_3^2)^{\frac{1}{2}} + (\|\vec{P} - \vec{P}'\|^2 + m_4^2)^{\frac{1}{2}}}{(\|\vec{P}'\|^2 + m_3^2)^{\frac{1}{2}} + (\|\vec{P}'\|^2 + \|\vec{P}'\|^2 - 2|\vec{P}'||\vec{P}| \cos \theta_{lab})^{\frac{1}{2}}} + m_4^2]^{\frac{1}{2}}$$

where

$$\frac{d(E_3 + E_4)}{d|\vec{P}'|} = \frac{1}{2} \frac{2|\vec{P}'|}{E_3} + \frac{1}{2} \frac{2|\vec{P}'| - 2|\vec{P}| \cos \theta_{lab}}{E_4}$$

for given  $|\vec{P}'|$  and  $\theta_{lab}$ .

$\downarrow$  will be integrated

a fixed value for a given scattering

$$\Rightarrow d|\vec{P}_3| = \frac{1}{\left( \frac{|\vec{P}'|}{E_3} + \frac{|\vec{P}'| - |\vec{P}| \cos \theta_{lab}}{E_4} \right)} d(E_3 + E_4)$$

$$= \frac{E_3 E_4}{|\vec{P}'| E_4 + |\vec{P}'| E_3 - |\vec{P}| E_3 \cos \theta_{lab}} d(E_3 + E_4)$$

integrate out  $d|\vec{P}_3|$  using  $\delta(E_1 + E_2 - E_3 - E_4)$

$$\begin{aligned} \boxed{\frac{d\sigma}{d\Omega_{lab}}} &= \frac{1}{(2\pi)^3} \frac{1}{4E_1 E_2} \frac{1}{|\vec{D}_1|} \frac{1}{4E_3 E_4} |M|^2 |\vec{P}'|^2 \frac{E_3 E_4}{|\vec{P}'|(E_3 + E_4) - |\vec{P}| E_3 \cos \theta_{lab}} \\ |\vec{D}_1| &= \frac{|\vec{P}|}{E_1}, E_3 + E_4 = E_1 + m_2, E_2 = m_2 \\ &\stackrel{\downarrow}{=} \frac{1}{(2\pi)^3} \frac{1}{4E_1 m_2} \frac{1}{|\vec{P}|} \frac{1}{4} |M|^2 |\vec{P}'|^2 \frac{|\vec{P}'|^2}{|\vec{P}'|(E_1 + m_2) - |\vec{P}| E_3 \cos \theta_{lab}} \\ &= \frac{|M|^2}{64\pi^2 m_2} \frac{|\vec{P}'|}{|\vec{P}|} \frac{1}{[(E_1 + m_2) - \frac{|\vec{P}|}{|\vec{P}'|} E_3 \cos \theta_{lab}]} \end{aligned}$$

Note that  $|\vec{P}'|$  and  $E_3$  can be expressed through  $|\vec{P}|$  and  $\theta_{lab}$ .

$$\text{Since } E_3 + E_4 = (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} + m_2$$

$$(\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}} + [\lvert \vec{P} \rvert^2 + \lvert \vec{P}' \rvert^2 - 2\lvert \vec{P} \rvert \lvert \vec{P}' \rvert \cos\theta_{\text{lab}} + m_4^2]^{\frac{1}{2}}$$

$\Rightarrow \lvert \vec{P}' \rvert$  can be expressed as a function of  $\lvert \vec{P} \rvert$  and  $\theta_{\text{lab}}$ ,

and  $E_3 = (\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}}$  can be also expressed as a function of  $\lvert \vec{P} \rvert$  and  $\theta_{\text{lab}}$ .

$$\text{from } \left( \frac{d\sigma}{ds} \right)_{\text{lab}} = \frac{d\sigma}{2\pi d(\cos\theta_{\text{lab}})}$$

$$\Rightarrow \left( \frac{d\sigma}{d \cos\theta} \right)_{\text{lab}} = \frac{1}{32\pi m_2} \frac{\lvert M \rvert^2}{\lvert \vec{P} \rvert} \frac{1}{[(E_1 + m_2) - \frac{\lvert \vec{P} \rvert}{\lvert \vec{P}' \rvert} E_3 \cos\theta_{\text{lab}}]}$$

$$\text{use } E_3 + E_4 = (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} + m_2$$

$$\Rightarrow \underbrace{\lvert \vec{P} \rvert^2}_{m_1^2} + \underbrace{\lvert \vec{P}' \rvert^2}_{m_3^2} - 2 \lvert \vec{P} \rvert \lvert \vec{P}' \rvert \cos\theta_{\text{lab}} + m_4^2 = \left[ (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} + m_2 - (\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}} \right]^2$$

$$= \underbrace{\lvert \vec{P} \rvert^2}_{m_1^2} + \underbrace{m_2^2}_{m_2^2} + \underbrace{\lvert \vec{P}' \rvert^2}_{m_3^2} + \underbrace{m_3^2}_{m_3^2} + 2m_2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} - 2m_2 (\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}} \\ - 2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} (\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}}$$

$$\Rightarrow \underbrace{2m_2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}}}_{\equiv a} + \underbrace{m_1^2 + m_2^2 + m_3^2 - m_4^2}_{m_1^2 + m_2^2 + m_3^2 - m_4^2} + 2 \lvert \vec{P} \rvert \lvert \vec{P}' \rvert \cos\theta_{\text{lab}} \\ = 2m_2 (\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}} + 2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} (\lvert \vec{P}' \rvert^2 + m_3^2)^{\frac{1}{2}}$$

$$\Rightarrow a^2 + 4a \lvert \vec{P} \rvert \lvert \vec{P}' \rvert \cos\theta_{\text{lab}} + 4 \lvert \vec{P} \rvert^2 \lvert \vec{P}' \rvert^2 \cos^2\theta_{\text{lab}}$$

$$= 4m_2^2 (\lvert \vec{P}' \rvert^2 + m_3^2) + 4(\lvert \vec{P} \rvert^2 + m_1^2) (\lvert \vec{P}' \rvert^2 + m_3^2) + 8m_2 (\lvert \vec{P} \rvert^2 + m_3^2) (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}}$$

$$= 4 \lvert \vec{P}' \rvert^2 [m_2^2 + \lvert \vec{P} \rvert^2 + m_1^2 + 2m_2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}}]$$

$$+ 4m_2^2 m_3^2 + 4 \lvert \vec{P} \rvert^2 m_3^2 + 4m_1^2 m_3^2 + 8m_2 m_3^2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}}$$

$$\Rightarrow \lvert \vec{P}' \rvert^2 A + B \lvert \vec{P}' \rvert + C = 0$$

$$\text{where } A = 4(m_2^2 + \lvert \vec{P} \rvert^2 + m_1^2 + 2m_2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} - \lvert \vec{P} \rvert^2 \cos^2\theta_{\text{lab}})$$

$$B = -4 \lvert \vec{P} \rvert \cos\theta_{\text{lab}} [2m_2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} + m_1^2 + m_2^2 + m_3^2 - m_4^2]$$

$$C = 4m_2^2 m_3^2 + 4 \lvert \vec{P} \rvert^2 m_3^2 + 4m_1^2 m_3^2 + 8m_2 m_3^2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} \\ - [2m_2 (\lvert \vec{P} \rvert^2 + m_1^2)^{\frac{1}{2}} + m_1^2 + m_2^2 + m_3^2 - m_4^2]^2$$

②  $1 \rightarrow 2$  decay. ( $m_1 > (m_2 + m_3)$ )



Similarly to the  $2 \rightarrow 2$  scattering, the phase space is

$$d\Gamma_{LIPS} = \frac{1}{(2\pi)^2} \frac{1}{4E_2 E_3} |\vec{P}_2|^2 S(E_1 - E_2 - E_3) d|\vec{P}_2| d\Omega_2, \text{ where } d\Omega_2 \text{ is the solid angle of particle 2, } d\Omega_2 = d\theta d\phi d\varphi.$$

[1] In the rest frame of particle 1, i.e., the center-of-mass frame

$$\vec{P}_1 = 0, \vec{P}' = \vec{P}_2 = -\vec{P}_3$$

$$E_1 = m_1$$

$$d|\vec{P}_2| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_2 + E_3)} d(E_2 + E_3)$$

$$\text{where } \frac{d(E_2 + E_3)}{d|\vec{P}'|} = \frac{d[(|\vec{P}'|^2 + m_2^2)^{\frac{1}{2}} + (|\vec{P}'|^2 + m_3^2)^{\frac{1}{2}}]}{d|\vec{P}'|} = \frac{1}{2} \frac{2|\vec{P}'|}{E_2} + \frac{1}{2} \frac{2|\vec{P}'|}{E_3} = |\vec{P}'| \frac{(E_2 + E_3)}{E_2 E_3}$$

$$\Rightarrow \frac{d|\vec{P}'|}{d(E_2 + E_3)} = \frac{1}{|\vec{P}'|} \frac{E_2 E_3}{(E_2 + E_3)}$$

$$\Rightarrow \frac{d\Gamma_{LIPS}}{d\Omega_2} = \frac{1}{(2\pi)^2} \frac{1}{4E_2 E_3} |\vec{P}'|^2 \frac{1}{|\vec{P}'|} \frac{E_2 E_3}{m_1} = \frac{|\vec{P}'|}{16\pi^2} \frac{1}{m_1}$$

use  $S(m_1, -E_2 - E_3)$  to integrate out  $d|\vec{P}'|$

$$\text{Since } \vec{P}_3^2 = (\vec{P}_1 - \vec{P}_2)^2 = m_1^2 + m_2^2 - 2m_1 E_2$$

$$\begin{aligned} \Rightarrow E_2 &= \frac{m_1^2 + m_2^2 - m_3^2}{2m_1} \\ \Rightarrow |\vec{P}'| &= (E_2 - m_2^2)^{\frac{1}{2}} = \frac{[(m_1^2 + m_2^2 - m_3^2)^2 - 4m_1^2 m_2^2]^{\frac{1}{2}}}{2m_1} \\ &= \frac{\lambda(m_1^2, m_2^2, m_3^2)}{2m_1} \end{aligned}$$

$$\begin{aligned} \Rightarrow \left[ \frac{d\Gamma}{d\Omega_2} \right]_{cm} &= \frac{1}{2m_1} |M|^2 \frac{\lambda^{\frac{1}{2}}(m_1^2, m_2^2, m_3^2)}{2m_1^2 \times 16\pi^2} \\ &= \frac{\lambda^{\frac{1}{2}}(m_1^2, m_2^2, m_3^2)}{64\pi^2 m_1^3} |M|^2 \end{aligned}$$

[2] In the lab frame where  $P_1^{\mu} = (E_1, \vec{P})$ ,  $P_2^{\mu} = (E_2, \vec{P}')$ ,  $P_3^{\mu} = (E_3, \vec{P}_3)$ ,

$$E_1 = E_2 + E_3,$$

$$\vec{P} = \vec{P}' + \vec{P}_3$$

$$\Rightarrow d|\vec{P}_2| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_2 + E_3)} d(E_2 + E_3)$$

$$\text{where } \frac{d(E_2 + E_3)}{d|\vec{P}'|} = \frac{d[(|\vec{P}'|^2 + m_2^2)^{\frac{1}{2}} + (|\vec{P}|^2 + |\vec{P}'|^2 - 2|\vec{P}||\vec{P}'|\cos\theta_{lab} + m_3^2)^{\frac{1}{2}}]}{d|\vec{P}'|}$$

↑  
for given  $|\vec{P}|$  and  $\theta_{lab}$

$$= \frac{|\vec{P}'|}{E_2} + \frac{|\vec{P}'| - |\vec{P}|\cos\theta_{lab}}{E_3}$$

$$\Rightarrow d|\vec{P}'| = \frac{1}{\frac{|\vec{P}'|}{E_2} + \frac{|\vec{P}'| - |\vec{P}|\cos\theta_{lab}}{E_3}} d(E_2 + E_3)$$

$$= \frac{E_2 E_3}{|\vec{P}'| E_3 + E_2 (|\vec{P}'| - |\vec{P}|\cos\theta_{lab})} d(E_2 + E_3)$$

Integrate out  $d|\vec{P}_2|$  using  $f(E_1, -E_2, -E_3)$

$$\Rightarrow \left( \frac{dT}{d\Omega} \right)_{lab} = \frac{1}{2E_1} |M|^2 \frac{1}{(2\pi)^2} \frac{1}{4E_2 E_3} |\vec{P}'|^2 \frac{E_2 E_3}{|\vec{P}'|(E_2 + E_3) - E_2 |\vec{P}|\cos\theta_{lab}}$$

$$= \frac{|M|^2}{32\pi^2 E_1} \frac{|\vec{P}'|}{\left( E_1 - E_2 \frac{|\vec{P}|}{|\vec{P}'|} \cos\theta_{lab} \right)}$$

$$\text{Again, from } \begin{cases} E_1 = E_2 + E_3 \\ \vec{P} = \vec{P}' + \vec{P}_3 \end{cases} \Rightarrow (|\vec{P}|^2 + m_1^2)^{\frac{1}{2}} = (|\vec{P}'|^2 + m_2^2)^{\frac{1}{2}} + (|\vec{P}|^2 + |\vec{P}'|^2 - 2|\vec{P}||\vec{P}'|\cos\theta_{lab} + m_3^2)^{\frac{1}{2}}$$

$$\Rightarrow |\vec{P}'| \text{ and } E_2, \text{ given } |\vec{P}| \text{ and } \theta_{lab}.$$

Since the two-body phase space is the basis of computing higher body phase space, let's look at it a bit more.

Let's compute it [in the rest frame of the two-body system,

$$K^\mu = (p_1 + p_2)^\mu \equiv (\sqrt{s}, 0, 0, 0), \quad p^\mu \equiv p_1^\mu$$

$$\begin{aligned} \int d\Omega_{LSP} &= \int \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3 \vec{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(K^\mu - p_1^\mu - p_2^\mu) \\ &= \int \frac{d^3 \vec{P}}{(2\pi)^3 2E_1 2E_2} (2\pi) \cdot \delta(\sqrt{s} - E_1 - E_2) \\ &= \int \frac{|\vec{P}|^2 d|\vec{P}| d(\cos\theta) d\varphi}{(2\pi)^2 2(|\vec{P}|^2 + m_1^2)^{\frac{1}{2}} 2(|\vec{P}|^2 + m_2^2)^{\frac{1}{2}}} \delta(\sqrt{s} - |\vec{P}|^2 + m_1^2)^{\frac{1}{2}} - (|\vec{P}|^2 + m_2^2)^{\frac{1}{2}} \end{aligned}$$

Since  $\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$

then  $\int d|\vec{P}| \delta(\sqrt{s} - (|\vec{P}|^2 + m_1^2)^{\frac{1}{2}} - (|\vec{P}|^2 + m_2^2)^{\frac{1}{2}}) = \left( \frac{|\vec{P}|}{(|\vec{P}|^2 + m_1^2)^{\frac{1}{2}}} + \frac{|\vec{P}|}{(|\vec{P}|^2 + m_2^2)^{\frac{1}{2}}} \right)^{-1}$

$$\Rightarrow \boxed{\int d\Omega_{LSP} = \int \frac{|\vec{P}|^2 d(\cos\theta) d\varphi}{(2\pi)^2 2E_1 2E_2} \frac{1}{|\vec{P}|} \frac{E_1 E_2}{\sqrt{s}} = \int \frac{|\vec{P}| d(\cos\theta) d\varphi}{16\pi^2 \sqrt{s}}}$$

$$= \frac{\bar{\beta}}{8\pi} \int \frac{d(\cos\theta)}{2} \frac{d\varphi}{2\pi}$$

where  $\boxed{\bar{\beta} = \frac{2|\vec{P}|}{\sqrt{s}} = \frac{2\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{\sqrt{s} 2\sqrt{s}} = \frac{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{s}}$

It is consistent with the  $2 \rightarrow 2$  scattering in the center-of-mass frame, and  $1 \rightarrow 2$  decay in the rest frame of the decaying particle.

It's worthwhile looking at three special cases.

a) When  $m_1 = m_2 = m$ , so that  $E_1 = E_2 = \frac{\sqrt{s}}{2}$ ,  $v_1 = v_2 = \frac{|\vec{p}|}{(\frac{\sqrt{s}}{2})}$

$$\bar{\beta} = \frac{(s^2 + 2m^4 - 2sm^2 - 2sm^2 - 2m^4)^{\frac{1}{2}}}{s} = \frac{(s^2 - 4sm^2)^{\frac{1}{2}}}{s} = \left(1 - \frac{4m^2}{s}\right)^{\frac{1}{2}}$$

$$= \left(1 - \frac{4m^2}{4E^2}\right)^{\frac{1}{2}} = v_1 \quad (\text{hence the notation } \bar{\beta})$$

$$= v_2$$

b<sub>1</sub>) when  $m_2 = 0$ ,

$$\bar{\beta} = \frac{(s^2 + m_1^4 - 2m_1^2 s)^{\frac{1}{2}}}{s} = \frac{(s - m_1^2)}{s} = 1 - \frac{m_1^2}{s}$$

b<sub>2</sub>) when  $m_1 = 0$ ,

$$\bar{\beta} = \frac{(s^2 + m_2^4 - 2m_2^2 s)^{\frac{1}{2}}}{s} = 1 - \frac{m_2^2}{s}$$

c) when  $m_1 = m_2 = 0$

$$\bar{\beta} = 1$$

For  $n$ -body phase space  $d\Omega_n = \prod_{i=1}^n \frac{d^3 \vec{p}_i}{(2\pi)^3 2E_{p_i}} (2\pi)^4 \int^4 (\sum P_i^{\mu} - \sum_{i=1}^n p_i^{\mu})$ ,

let's derive a recursion relation to decompose it.

For  $1 \leq m < n$ ,

using

$$I = \int \frac{d^4 \vec{e}}{(2\pi)^4} (2\pi)^4 \int^4 (\vec{e}^{\mu} - \sum_{j=1}^m p_j^{\mu}) \theta(\vec{e}^{\circ})$$

(since  $\sum_{j=1}^m E_{p_j} > 0$ , the  $\theta(\vec{e}^{\circ})$  does not do anything).

and

$$I = \int \frac{dS_m}{2\pi} 2\pi \delta(S_m - \vec{e}^2)$$

$$\Rightarrow I = \int \frac{d^4 \vec{e}}{(2\pi)^4} \frac{dS_m}{2\pi} (2\pi)^4 \int^4 (\vec{e}^{\mu} - \sum_{j=1}^m p_j^{\mu}) \theta(\vec{e}^{\circ}) 2\pi \delta(S_m - \vec{e}^2)$$

$$= \int \frac{d^3 \vec{e}}{(2\pi)^3} \underbrace{\frac{d\vec{e}^{\circ}}{2\pi}}_{\frac{dS_m}{2\pi}} \frac{dS_m}{2\pi} (2\pi)^4 \int^4 (\vec{e}^{\mu} - \sum_{j=1}^m p_j^{\mu}) \theta(\vec{e}^{\circ}) 2\pi \delta(S_m - \vec{e}^2 + |\vec{e}|^2)$$

$$\delta(x^2 - d^2) = \frac{1}{2id} [\delta(x+d) + \delta(x-d)]$$

$$= \int \frac{d^3 \vec{e}}{(2\pi)^3} \underbrace{\frac{d\vec{e}^{\circ}}{2\pi}}_{\frac{dS_m}{2\pi}} \frac{dS_m}{2\pi} (2\pi)^4 \int^4 (\vec{e}^{\mu} - \sum_{j=1}^m p_j^{\mu}) \theta(\vec{e}^{\circ}) \underbrace{2\pi}_{\frac{1}{2|S_m + |\vec{e}|^2|^{\frac{1}{2}}}} [\delta(\vec{e}^{\circ} + (S_m + |\vec{e}|^2)^{\frac{1}{2}}) + \delta(\vec{e}^{\circ} - (S_m + |\vec{e}|^2)^{\frac{1}{2}})]$$

$$= \int \frac{d^3 \vec{e}}{(2\pi)^3} \frac{d(\vec{e}^2)}{2\pi} (2\pi)^4 \int^4 (\vec{e}^{\mu} - \sum_{j=1}^m p_j^{\mu}) \frac{1}{2|\vec{e}|}$$

Notice that by  $\delta(S_m - \vec{e}^2)$

$$\Rightarrow (S_m + |\vec{e}|^2)^{\frac{1}{2}} = (\vec{e}^2 + |\vec{e}|^2)^{\frac{1}{2}} = (\vec{e}^2)^{\frac{1}{2}} = |\vec{e}|$$

$$\Rightarrow d\Omega_n = \int \prod_{i=1}^n \frac{d^3 \vec{p}_i}{(2\pi)^3 2E_{p_i}} (2\pi)^4 \int^4 (\sum P_i^{\mu} - \sum_{i=1}^n p_i^{\mu}) \underbrace{\frac{d^3 \vec{e}}{(2\pi)^3} \frac{d(\vec{e}^2)}{2\pi} (2\pi)^4 \int^4 (\vec{e}^{\mu} - \sum_{j=1}^m p_j^{\mu})}_{\sum P_i^{\mu} - \sum_{j=1}^m p_j^{\mu} - \sum_{i=m+1}^n p_i^{\mu}} \frac{1}{2|\vec{e}|}$$

$$= \int \prod_{i=m+1}^n \frac{d^3 \vec{p}_i}{(2\pi)^3 2E_{p_i}} \prod_{j=1}^m \frac{d^3 \vec{p}_j}{(2\pi)^3 2E_{p_j}} \underbrace{\frac{(2\pi)^4 \int^4 (\sum P_i^{\mu} - \vec{e}^{\mu} - \sum_{i=m+1}^n p_i^{\mu})}{d(\vec{e}^2)}}_{\times \frac{d(\vec{e}^2)}{2\pi}} \frac{d^3 \vec{e}}{(2\pi)^3 2|\vec{e}|}$$

$$= \int \frac{d(\vec{e}^2)}{2\pi} d\Omega_m d\Omega_{n-m+1}$$

That is,

$$d\Pi_n(\Sigma P_{\text{initial}}; p_1, \dots, p_n) = \frac{d(\mathcal{E}^2)}{2\pi} d\Pi_m(\mathcal{E}; p_1, \dots, p_m) d\Pi_{n-m+1}(\Sigma P_{\text{initial}}; \mathcal{E}, p_{m+1}, p_{m+2}, \dots, p_n)$$

notation:  $d\Pi_n(\boxed{\quad}; \boxed{\quad})$   
                ↑                      ↑  
                n-body              initial momenta  
                                    final momenta

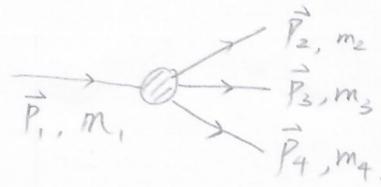
In particular, we can decompose any  $n$ -body phase space into a 2-body phase space and a  $(n-1)$ -body phase space.

$$d\Pi_n(\Sigma P_{\text{initial}}; p_1, \dots, p_n) = \frac{d(\mathcal{E}^2)}{2\pi} d\Pi_{n-1}(\mathcal{E}; p_1, \dots, p_{n-1}) d\Pi_2(\Sigma P_{\text{initial}}; \mathcal{E}, p_n)$$

and we can keep doing so to decompose the  $(n-1)$ -body phase space into a 2-body phase space and a  $(n-2)$ -body phase space, ..., and eventually, we can decompose a  $n$ -body phase space into  $(n-1)$  number of 2-body phase space.

③

$1 \rightarrow 3$  decay  $(m_1 > (m_2 + m_3 + m_4))$



Let's use the recursion relation to decompose the 3-body phase space,

$$d\Pi_3(P_1; P_2, P_3, P_4) = \frac{d(\mathcal{E}^2)}{2\pi} d\Pi_2(\mathcal{E}; P_2, P_3) d\Pi_2(P_1; \mathcal{E}, P_4)$$

$$\text{where } \mathcal{E}^2 = (P_2 + P_3)^2 = m_{23}^2$$

Let's work [in the rest frame of the decaying particle], and use the 2-body phase space result,

$$\text{then } d\Pi_2(P_1; \mathcal{E}, P_4) = \frac{2|\vec{P}_4|}{m_1} \frac{1}{8\pi} \frac{d\Omega_4}{4\pi}$$

where  $|\vec{P}_4|$  &  $\Omega_4$  are the momentum and solid angle of particle-4 in the rest frame of the decaying particle,

$$\text{From } m_{23}^2 = (P_1 - P_4)^2 = m_1^2 + m_4^2 - 2m_1 E_4$$

$$\Rightarrow E_4 = \frac{m_1^2 + m_4^2 - m_{23}^2}{2m_1}$$

$$\Rightarrow |\vec{P}_4| = (E_4^2 - m_4^2)^{\frac{1}{2}} = \frac{(m_1^4 + m_4^4 + m_{23}^4 + 2m_1^2 m_4^2 - 2m_1^2 m_{23}^2 - 2m_4^2 m_{23}^2 - 4m_1^2 m_4^2)^{\frac{1}{2}}}{2m_1}$$

$$= \frac{\chi^{\frac{1}{2}}(m_1^2, m_{23}^2, m_4^2)}{2m_1}$$

$$d\Pi_2(\mathcal{E}; P_2, P_3) = \frac{2|\vec{P}_2^*|}{m_{23}} \frac{1}{8\pi} \frac{d\Omega_2^*}{4\pi}$$

where  $|\vec{P}_2^*|$  &  $\Omega_2^*$  are the momentum and solid angle of particle-2 in the center-of-mass frame of particle-2 and 3.

$$\text{From } P_3^2 = [(P_2 + P_3) - P_2]^2 = m_{23}^2 + m_2^2 - 2m_{23} E_2^*$$

$$\Rightarrow E_2^* = \frac{m_{23}^2 + m_2^2 - m_3^2}{2m_{23}}$$

$$\Rightarrow |\vec{P}_2^*| = (\vec{E}_2^* - m_2^2)^{\frac{1}{2}} = \frac{(m_{23}^4 + m_2^4 + m_3^4 + 2m_{23}^2 m_2^2 - 2m_{23}^2 m_3^2 - 2m_2^2 m_3^2 - 4m_{23}^2 m_2^2)^{\frac{1}{2}}}{2m_{23}}$$

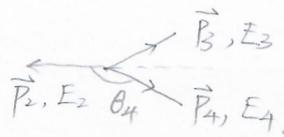
$$= \frac{\chi^{\frac{1}{2}}(m_{23}^2, m_2^2, m_3^2)}{2m_{23}}$$

$$\Rightarrow d\Gamma = \frac{1}{2m_1} |M|^2 \frac{d(m_{23}^2)}{2\pi} \frac{2|\vec{P}_2^*|}{m_{23}} \frac{1}{8\pi} \frac{d\mathcal{L}_2^*}{4\pi} \frac{2|\vec{P}_4|}{m_1} \frac{1}{8\pi} \frac{d\mathcal{L}_4}{4\pi}$$

$$= \frac{1}{(2\pi)^5} \frac{1}{16m_1^2} |M|^2 |\vec{P}_2^*| |\vec{P}_4| dm_{23} d\mathcal{L}_2^* d\mathcal{L}_4$$

Alternatively, we can directly use the 3-body phase space without decomposing into 2-body phase space.

Still [in the rest frame of the decaying particle]



$$d\Pi_3 = \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3 2E_4} (2\pi)^4 S^4 (P_1 - P_2 - P_3 - P_4)$$

$$= \frac{d^3\vec{p}_2}{(2\pi)^5 8E_2 E_3 E_4} d(\cos\theta_2) d\phi_2 \frac{d^3\vec{p}_4}{(2\pi)^5 8E_2 E_3 E_4} d(\cos\theta_4) d\phi_4 S(m_1 - E_2 - E_3 - E_4)$$

$$= \frac{|\vec{P}_2|^2 d|\vec{P}_2| d(\cos\theta_2) d\phi_2 |\vec{P}_4|^2 d|\vec{P}_4| d(\cos\theta_4) d\phi_4}{(2\pi)^5 8E_2 E_3 E_4} S(m_1 - E_2 - E_3 - E_4)$$

$$\vec{P}_2^2 + m_2^2 = E_2^2 \Rightarrow |\vec{P}_2| d|\vec{P}_2| = E_2 dE_2, |\vec{P}_4|^2 + m_4^2 = E_4^2 \Rightarrow |\vec{P}_4| d|\vec{P}_4| = E_4 dE_4$$

$$\underline{\underline{=}} \frac{|\vec{P}_2| E_2 dE_2 d(\cos\theta_2) d\phi_2 |\vec{P}_4| E_4 dE_4 d(\cos\theta_4) d\phi_4}{(2\pi)^5 8E_2 E_3 E_4} S(m_1 - E_2 - E_3 - E_4)$$

using  $\vec{P}_1 = 0$

$$E_3 = (|\vec{P}_3|^2 + m_3^2)^{\frac{1}{2}} = (|\vec{P}_1 - \vec{P}_2 - \vec{P}_4|^2 + m_3^2)^{\frac{1}{2}}$$

$$= (|\vec{P}_2 + \vec{P}_4|^2 + m_3^2)^{\frac{1}{2}}$$

$$= (|\vec{P}_2|^2 + |\vec{P}_4|^2 + 2|\vec{P}_2||\vec{P}_4|\cos\theta_4 + m_3^2)^{\frac{1}{2}}$$

$$= [(E_2^2 - m_2^2) + (E_4^2 - m_4^2) + 2(E_2^2 - m_2^2)^{\frac{1}{2}}(E_4^2 - m_4^2)^{\frac{1}{2}}\cos\theta_4 + m_3^2]^{\frac{1}{2}}$$

$$\Rightarrow \int d(\cos\theta_4) S(m_1 - E_2 - E_3 - E_4) = \int d(\cos\theta_4) S(m_1 - E_2 - E_4 - (E_2^2 - m_2^2) + (E_4^2 - m_4^2) + 2(E_2^2 - m_2^2)^{\frac{1}{2}} \times (E_4^2 - m_4^2)^{\frac{1}{2}} \cos\theta_4 + m_3^2)^{\frac{1}{2}}$$

$$\underline{\underline{=}} \left( \frac{1}{2} \frac{2|\vec{P}_2||\vec{P}_4|}{E_3} \right)^{-1} = \frac{E_3}{|\vec{P}_2||\vec{P}_4|}$$

use  $S(g(x)) = \sum_i \frac{S(x-x_i)}{|g'(x_i)|}$

$$\Rightarrow d\Gamma_3 = \frac{|\vec{P}_2| E_2 dE_2 d(\cos\theta_2) d\varphi_2 |\vec{P}_4| E_4 dE_4 d\varphi_4}{(2\pi)^5 8E_2 E_3 E_4} \frac{E_3}{|\vec{P}_2| |\vec{P}_4|}$$

$$= \frac{dE_2 dE_4 d(\cos\theta_2) d\varphi_2 d\varphi_4}{(2\pi)^5 8}$$

$$\Rightarrow [d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16m_1} |M|^2 dE_2 dE_4 d(\cos\theta_2) d\varphi_2 d\varphi_4]$$

where  $(\theta_2, \varphi_2)$  are the orientation of particle-2 in the rest frame of the decaying particle,

and  $\varphi_4$  is the azimuthal angle of particle-4 around particle-2,  
and  $\theta_4$  is the polar angle of particle-4 relative to particle-2.

If  $|M|^2$  does not depend on  $\theta_2, \varphi_2$  and  $\varphi_4$ , we can integrate them out,

$$[d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16m_1} |M|^2 dE_2 dE_4 (4\pi)(2\pi)$$

$$= \frac{1}{(2\pi)^3} \frac{1}{8m_1} |M|^2 dE_2 dE_4]$$

Since  $m_{23}^2 \equiv (P_2 + P_3)^2 = (P_1 - P_4)^2 = m_1^2 + m_4^2 - 2m_1 E_4$

then  $d(m_{23}^2) = -2m_1 dE_4$

Since  $m_{34}^2 \equiv (P_3 + P_4)^2 = (P_1 - P_2)^2 = m_1^2 + m_2^2 - 2m_1 E_2$

then  $d(m_{34}^2) = -2m_1 dE_2$

$$\Rightarrow [d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{32m_1^3} |M|^2 d(m_{23}^2) d(m_{34}^2)]$$

For a given value of  $m_{23}^2$ , the range of  $m_{34}^2$  is determined:

using  $P_2^2 = [(P_2 + P_3) - P_3]^2 = m_{23}^2 + m_3^2 - 2E_3^* m_{23}$

where  $E_3^*$  is the energy of particle-3 in the center-of-mass frame of particle 2 and 3.

$$\Rightarrow \boxed{E_3^* = \frac{m_{23}^2 + m_3^2 - m_2^2}{2m_{23}}}$$

Using  $P_1^2 = [(P_2 + P_3) + P_4]^2 = m_{23}^2 + m_4^2 + 2E_4^* m_{23}$

$\parallel$   
 $m_1^2$

where  $E_4^*$  is the energy of particle-4 in the center-of-mass frame of particle 2 and 3.

$$\Rightarrow \boxed{E_4^* = \frac{m_1^2 - m_{23}^2 - m_3^2}{2m_{23}}}$$

Using  $m_{34}^2 = (P_3 + P_4)^2 = (E_3^* + E_4^*)^2 - |\vec{P}_3^* + \vec{P}_4^*|^2$

$$= (E_3^* + E_4^*)^2 - (|\vec{P}_3^*|^2 + |\vec{P}_4^*|^2 + 2|\vec{P}_3^*||\vec{P}_4^*| \cos\theta_{34}^*)$$

$$\Rightarrow \boxed{(m_{34}^2)_{\max} = (E_3^* + E_4^*)^2 - (|\vec{P}_3^*| - |\vec{P}_4^*|)^2}$$

$$= \frac{(E_3^* + E_4^*)^2 - (\sqrt{E_3^{*2} - m_3^2} - \sqrt{E_4^{*2} - m_4^2})^2}{(m_{34}^2)_{\min} = (E_3^* + E_4^*)^2 - (|\vec{P}_3^*| + |\vec{P}_4^*|)^2}$$

$$= \frac{(E_3^* + E_4^*)^2 - (\sqrt{E_3^{*2} - m_3^2} + \sqrt{E_4^{*2} - m_4^2})^2}{}$$

For the range of  $m_{23}^2$ ,

Since  $m_{23}^2 = (P_2 + P_3)^2 = (P_1 - P_4)^2 = P_1^2 + P_4^2 - 2P_1 \cdot P_4 = m_1^2 + m_4^2 - 2m_1 m_4$

in the rest frame of  
the decaying particle

$\parallel \leftarrow$  in the center-of-mass frame of particle 2 and 3.

$$\Rightarrow \boxed{(m_{23}^2)_{\min} = (E_2^* + E_3^*)^2 = (m_2 + m_3)^2}$$

$$(m_{23}^2)_{\max} = m_1^2 + m_4^2 - 2m_1 m_4 = (m_1 - m_4)^2$$

Therefore,

$$\Gamma = \frac{1}{(2\pi)^3} \frac{1}{32m_1^3} \int_{(m_{23}^2)_{\min}}^{(m_{23}^2)_{\max}} d(m_{23}^2) \int_{(m_{34}^2)_{\min}}^{(m_{34}^2)_{\max}} d(m_{34}^2) |M|^2$$

Since  $(P_2 + P_3)^2 = m_{23}^2 = (P_1 - P_4)^2 \stackrel{!}{=} m_1^2 + m_4^2 - 2m_1 E_4$   
 in the decaying particle rest frame

$$\Rightarrow E_4 = \frac{m_1^2 + m_4^2 - m_{23}^2}{2m_1}$$

$$\Rightarrow |\vec{P}_4|_{\max} = [(E_4)_{\max}^2 - m_4^2]^{1/2} = \left[ \left( \frac{m_1^2 + m_4^2 - (m_{23})_{\min}^2}{2m_1} \right)^2 - m_4^2 \right]^{1/2}$$

$$= \left[ \left( \frac{m_1^2 + m_4^2 - (m_2 + m_3)^2}{2m_1} \right)^2 - m_4^2 \right]^{1/2}$$

Since the minimum of  $m_{23}^2$  is  $(m_2 + m_3)^2$ , it means that

$|\vec{P}_4|$  achieve its maximum value when particle 2 and 3 have the same three-velocity (i.e.,  $\vec{v}_2 = \vec{v}_3$ ) in the rest frame of the decaying particle, that is, the configuration is

so that in the center of mass frame of particle 2 and 3, their momenta are zero, so that  $m_{23}^2 = (P_2 + P_3)^2 = (m_2 + m_3)^2$



Similarly, from  $(P_4 + P_3)^2 = m_{34}^2 = (P_1 - P_2)^2 \stackrel{!}{=} m_1^2 + m_2^2 - 2m_1 E_2$  in the decaying particle rest frame

$$\Rightarrow E_2 = \frac{m_1^2 + m_2^2 - m_{34}^2}{2m_1}$$

$$\Rightarrow |\vec{P}_2|_{\max} = [(E_2)_{\max}^2 - m_2^2]^{1/2} = \left[ \left( \frac{m_1^2 + m_2^2 - (m_{34})_{\min}^2}{2m_1} \right)^2 - m_2^2 \right]^{1/2}$$

Since  $(m_{34})_{\min}^2 = (m_3 + m_4)^2$ ,

$$|\vec{P}_2|_{\max} = \left[ \left( \frac{m_1^2 + m_2^2 - (m_3 + m_4)^2}{2m_1} \right)^2 - m_2^2 \right]^{1/2}$$

and it is achieved when particle 3 and 4 have the same three-velocity (i.e.,  $\vec{v}_3 = \vec{v}_4$ ) in the rest frame of the decaying particle, and the configuration is



Similarly, from  $(P_2 + P_4)^2 = m_{24}^2 = (P_1 - P_3)^2 \stackrel{!}{=} m_1^2 + m_3^2 - 2m_1 E_3$  in the decaying particle rest frame

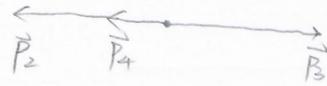
$$\Rightarrow E_3 = \frac{m_1^2 + m_3^2 - m_{24}^2}{2m_1}$$

$$\Rightarrow |\vec{P}_3|_{\max} = [(E_3)_{\max}^2 - m_3^2]^{1/2} = \left[ \left( \frac{m_1^2 + m_3^2 - (m_{24})_{\min}^2}{2m_1} \right)^2 - m_3^2 \right]^{1/2}$$

Since  $(m_{24})_{\min}^2 = (m_2 + m_4)^2$ ,

$$|\vec{P}_3|_{\max} = \left[ \frac{\left( m_1^2 + m_3^2 - (m_2 + m_4)^2 \right)^2}{2m_1} - m_3^2 \right]^{\frac{1}{2}},$$

and it is achieved when particle 2 and 4 have the same three-velocity (i.e.,  $\vec{v}_2 = \vec{v}_4$ ) in the rest frame of the decaying particle, and the configuration is



In general, for a  $n$ -body decay ( $n \geq 3$ ), i.e.,  $1 \rightarrow n$  decay, the maximum three-momentum of the  $i$ -th particle,  $|\vec{P}_i|_{\max}$ , in the rest frame of the decaying particle can be obtained as ( $2 \leq i \leq n+1$ )

$$(P_2 + \dots + P_{i-1} + P_{i+1} + \dots + P_n)^2 = (P_i - P_i)^2 = m_i^2 + m_i^2 - 2m_i E_i$$

$$\Rightarrow E_i = \frac{m_i^2 + m_i^2 - (P_2 + \dots + P_{i-1} + P_{i+1} + \dots + P_n)^2}{2m_i}$$

$$\Rightarrow |\vec{P}_i|_{\max} = [(E_i)_{\max}^2 - m_i^2]^{\frac{1}{2}} = \left[ \frac{\left( m_i^2 + m_i^2 - \left( \sum_{j=2, j \neq i}^{n+1} m_j \right)^2 \right)^2}{2m_i} - m_i^2 \right]^{\frac{1}{2}}$$

and it is achieved when all the decay products,  $2, \dots, i-1, i+1, \dots, n$ , have the same three-velocity (i.e.,  $\vec{v}_2 = \dots = \vec{v}_{i-1} = \vec{v}_{i+1} = \dots = \vec{v}_n$ ) in the rest frame of the decaying particle, and the configuration is



Let's use recursion relation to decompose  $n$ -body phase space to 2-body phase space product :  
 $(n \geq 3)$

$$\begin{aligned}
 d\Pi_n(\sum P_{\text{initial}}; p_1, \dots, p_n) &= \frac{d(q_1^2)}{2\pi} d\Pi_{n-1}(q_1; p_1, \dots, p_{n-1}) d\Pi_2(\sum P_{\text{initial}}; q_1, p_n) \\
 &= \frac{d(q_1^2)}{2\pi} \frac{d(q_2^2)}{2\pi} d\Pi_{n-2}(q_2; p_1, \dots, p_{n-2}) d\Pi_2(q_1; q_2, p_{n-1}) d\Pi_2(\sum P_{\text{initial}}; q_1, p_n) \\
 &= \frac{d(q_1^2)}{2\pi} \frac{d(q_2^2)}{2\pi} \frac{d(q_3^2)}{2\pi} d\Pi_{n-3}(q_3; p_1, \dots, p_{n-3}) d\Pi_2(q_2; q_3, p_{n-2}) d\Pi_2(q_1; q_2, p_{n-1}) \\
 &= \dots \\
 &= \frac{d(q_1^2)}{2\pi} \frac{d(q_2^2)}{2\pi} \frac{d(q_3^2)}{2\pi} \dots \frac{d(q_{n-2}^2)}{2\pi} d\Pi_2(q_{n-2}; p_1, p_2) d\Pi_2(q_{n-3}; q_{n-2}, p_3) d\Pi_2(q_{n-4}; q_{n-3}, p_4) \\
 &\quad \dots d\Pi_2(q_1; q_2, p_{n-1}) d\Pi_2(\sum P_{\text{initial}}; q_1, p_n)
 \end{aligned}$$

Since

$$q_{n-2}^2 = (p_1 + p_2)^2 = (q_{n-3} - p_3)^2, \text{ where } q_0 = \sum P_{\text{initial}}$$

$$q_{n-3}^2 = (p_1 + q_{n-2})^2 = (p_1 + p_2 + p_3)^2 = (q_{n-4} - p_4)^2$$

$$q_{n-4}^2 = (p_4 + q_{n-3})^2 = (p_1 + p_2 + p_3 + p_4)^2 = (q_{n-5} - p_5)^2$$

$$q_j^2 = (p_{n-j} + q_{j+1})^2 = (p_1 + p_2 + \dots + p_{n-j})^2 = (q_{j-1} - p_{n-j+1})^2$$

$$q_2^2 = (p_{n-2} + q_3)^2 = (p_1 + p_2 + \dots + p_{n-3} + p_{n-2})^2 = (q_1 - p_{n-1})^2$$

$$q_1^2 = (p_{n-1} + q_2)^2 = (p_1 + p_2 + \dots + p_{n-2} + p_{n-1})^2 = (q_0 - p_n)^2$$

$\Rightarrow$

$$(q_1^2)_{\min} = \left( \sum_{i=1}^{n-1} m_i \right)^2, \quad (q_2^2)_{\min} = \left( \sum_{i=1}^{n-2} m_i \right)^2, \quad = (\sum P_{\text{initial}} - p_n)^2$$

$$\dots, (q_{n-2}^2)_{\min} = \left( \sum_{i=1}^{n-n-2} m_i \right)^2 = \left( \sum_{i=1}^2 m_i \right)^2$$

$$\text{that is, } (q_j^2)_{\min} = \left( \sum_{i=1}^{n-j} m_i \right)^2, \quad j = 1, 2, \dots, n-2$$

$$(q_1^2)_{\max} = S + m_n^2 - 2\sqrt{S} m_n = (S - m_n)^2, \text{ where } (\sum P_{\text{initial}})^2 = S$$

For a given  $q_1^2$ , the maximum of  $q_2^2$  is

$$(q_2^2)_{\max} = q_1^2 + m_{n-1}^2 - 2\sqrt{q_1^2} m_{n-1} = (\sqrt{q_1^2} - m_{n-1})^2$$

For a given  $\epsilon_2^2$ , the maximum of  $\epsilon_3^2$  is

$$(\epsilon_3^2)_{\max} = \epsilon_2^2 + m_{n-2}^2 - 2\sqrt{\epsilon_2^2} m_{n-2} = (\sqrt{\epsilon_2^2} - m_{n-2})^2$$

...  
For a given  $\epsilon_{j-1}^2$ , the maximum of  $\epsilon_j^2$  is

$$(\epsilon_j^2)_{\max} = \epsilon_{j-1}^2 + m_{n-j+1}^2 - 2\sqrt{\epsilon_{j-1}^2} m_{n-j+1} = (\sqrt{\epsilon_{j-1}^2} - m_{n-j+1})^2$$

...  
For a given  $\epsilon_{n-3}^2$ , the maximum of  $\epsilon_{n-2}^2$  is

$$(\epsilon_{n-2}^2)_{\max} = \epsilon_{n-3}^2 + m_3^2 - 2\sqrt{\epsilon_{n-3}^2} m_3 = (\sqrt{\epsilon_{n-3}^2} - m_3)^2$$

Since

$$d\pi_2(\sum P_{\text{initial}}; \epsilon_1, p_n) = \frac{\lambda^{\frac{1}{2}}(S, \epsilon_1^2, m_n^2)}{S} \frac{1}{8\pi} \int \frac{d\cos\theta_n d\phi_n}{2} \frac{dp_n}{2\pi}$$

$$d\pi_2(\epsilon_1; \epsilon_2, p_{n-1}) = \frac{\lambda^{\frac{1}{2}}(\epsilon_1^2, \epsilon_2^2, m_{n-1}^2)}{\epsilon_1^2} \frac{1}{8\pi} \int \frac{d\cos\theta_{n-1}^* d\phi_{n-1}^*}{2} \frac{dp_{n-1}^*}{2\pi}$$

where  $\theta_{n-1}^*$  and  $\phi_{n-1}^*$  are the polar and azimuthal angle of particle (n-1) in the center-of-mass frame of particle 1 to (n-1).

$$d\pi_2(\epsilon_j; \epsilon_{j+1}, p_{n-j}) = \frac{\lambda^{\frac{1}{2}}(\epsilon_j^2, \epsilon_{j+1}^2, m_{n-j}^2)}{\epsilon_j^2} \frac{1}{8\pi} \int \frac{d\cos\theta_{n-j}^* d\phi_{n-j}^*}{2} \frac{dp_{n-j}^*}{2\pi}$$

where  $\theta_{n-j}^*$  and  $\phi_{n-j}^*$  are the polar and azimuthal angle of particle (n-j) in the center-of-mass frame of particle 1 to (n-j).

Now, if all the masses,  $m_1, m_2, \dots, m_n$ , are zero, and if  $|\mathcal{M}|^2$  does not depend on the angles,

we can do the integrations and get

$$\int d\pi_2(\sum P_{\text{initial}}; \epsilon_1, p_n) = (1 - \frac{\epsilon_1^2}{S}) \frac{1}{8\pi}$$

$$\int d\pi_2(\epsilon_1; \epsilon_2, p_{n-1}) = (1 - \frac{\epsilon_2^2}{\epsilon_1^2}) \frac{1}{8\pi}$$

$$\int d\pi_2(\epsilon_j; \epsilon_{j+1}, p_{n-j}) = (1 - \frac{\epsilon_{j+1}^2}{\epsilon_j^2}) \frac{1}{8\pi}$$

$$\Rightarrow \Pi_n = \left(\frac{1}{2\pi}\right)^{n-2} \left(\frac{1}{8\pi}\right)^{n-1} \int_0^S d\zeta_1^2 \int_0^{\zeta_1^2} d\zeta_2^2 \int_0^{\zeta_2^2} d\zeta_3^2 \dots \int_0^{\zeta_{j-1}^2} d\zeta_j^2 \dots \int_0^{\zeta_{n-3}^2} d\zeta_{n-2}^2$$

$$\times \left(1 - \frac{\zeta_{n-2}^2}{\zeta_{n-3}^2}\right) \left(1 - \frac{\zeta_{n-3}^2}{\zeta_{n-4}^2}\right) \dots \left(1 - \frac{\zeta_{j+1}^2}{\zeta_j^2}\right) \dots \left(1 - \frac{\zeta_2^2}{\zeta_1^2}\right)$$

Let  $\zeta_j^2 \equiv S x_j \Rightarrow d\zeta_j^2 = S dx_j$

The range of  $x_j$  is  $[0, x_{j-1}]$ , where  $x_0 = 1$

$$\Rightarrow \Pi_n = \left(\frac{1}{2\pi}\right)^{n-2} \left(\frac{1}{8\pi}\right)^{n-1} S^{n-2} \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \dots \int_0^{x_{j-1}} dx_j$$

$$\times \dots \int_0^{x_{n-4}} dx_{n-3} \int_0^{x_{n-3}} dx_{n-2}$$

$$\times \left(1 - \frac{x_{n-2}}{x_{n-3}}\right) \left(1 - \frac{x_{n-3}}{x_{n-4}}\right) \dots \left(1 - \frac{x_{j+1}}{x_j}\right) \dots \left(1 - \frac{x_2}{x_1}\right)$$

$$= \left(\frac{1}{2\pi}\right)^{n-2} \left(\frac{1}{8\pi}\right)^{n-1} \frac{(n-1)}{[(n-1)!]^2} S^{n-2}$$

This expression is also valid for  $n=2$ .

$$\text{So } \Pi_2 = \frac{1}{8\pi}$$

$$\Pi_3 = \frac{1}{2\pi} \left(\frac{1}{8\pi}\right)^2 \frac{2S}{(2!)^2} = \frac{S}{256\pi^3}$$

$$\Pi_4 = \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{8\pi}\right)^3 \frac{3S^2}{(3!)^2} = \frac{S^2}{24576\pi^5}$$

For  $n \geq 3$ ,

$$\frac{\Pi_n}{\Pi_{n-1}} = \left(\frac{1}{2\pi}\right) \left(\frac{1}{8\pi}\right) \frac{S}{(n-2)(n-1)} = \frac{1}{(4\pi)^2} \frac{S}{(n-2)(n-1)}$$