Homework final solution 1

1) $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - ig\phi\bar{\psi}\gamma^5\psi$

$$\mathcal{L}_{int} = -ig\phi\bar{\psi}\gamma^5\psi$$

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = ig\phi\bar{\psi}\gamma^5\psi$$

Let's consider the scattering

fermion (P_1, S_1) + scalar (P_2) $\longrightarrow$ fermion (P_3, S_3) + scalar (P_4)

$$\phi(x) = \int d^3\vec{p}\, C(E_{\vec{p}})\left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x}\right)$$

$$\psi(x) = \int d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left[u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, d^\dagger_{\vec{p},s}\, e^{ip\cdot x}\right]$$

$$\bar{\psi}(x) = \int d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left[\bar{u}(\vec{p},s)\, b^\dagger_{\vec{p},s}\, e^{ip\cdot x} + \bar{v}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right]$$

Let's use b & $b^\dagger$ for the annihilation & creation operators of fermion, and d & $d^\dagger$ for the annihilation & creation operators of anti-fermion.

$$|i\rangle = (2\pi)^3 2E_{\vec{p}_1} C(E_{\vec{p}_1})\, (2\pi)^3 2E_{\vec{p}_2} C(E_{\vec{p}_2})\, b^\dagger_{\vec{p}_1, s_1}\, a^\dagger_{\vec{p}_2} |0\rangle$$

$$\langle f| = (2\pi)^3 2E_{\vec{p}_3} C(E_{\vec{p}_3})\, (2\pi)^3 2E_{\vec{p}_4} C(E_{\vec{p}_4})\, \langle 0|\, a_{\vec{p}_4}\, b_{\vec{p}_3, s_3}$$

At tree-level,

$$\langle f| iT |i\rangle = \langle f| \frac{(-i)^2}{2!} \int d^4x\, d^4y\, T(:\mathcal{H}_{int}(x)::\mathcal{H}_{int}(y):) |i\rangle$$

$$= \langle f| \frac{(-i)^2}{2!} \int d^4x\, d^4y\, (ig)^2 \begin{pmatrix} :\phi(x)\bar{\psi}(x)\gamma^5 \underbrace{\psi(x)\phi(y)\bar{\psi}(y)}\gamma^5\psi(y): \\ + :\phi(x)\underbrace{\bar{\psi}(x)\gamma^5\psi(x)\phi(y)\bar{\psi}(y)\gamma^5\psi(y)}: \end{pmatrix} |i\rangle$$

where $\int d^4x\, d^4y : \phi(x)\underbrace{\bar{\psi}(x)\gamma^5\psi(x)\phi(y)\bar{\psi}(y)\gamma^5\psi(y)}:$

$$= \int d^4x\, d^4y : \phi(x)\underbrace{\bar{\psi}_a(x)\gamma^5_{ab}\psi_b(x)\phi(y)\bar{\psi}_c(y)\gamma^5_{cd}\psi_d(y)}:$$

$$\overset{x\leftrightarrow y}{=} \int d^4x\, d^4y : \phi(y)\underbrace{\bar{\psi}_a(y)\gamma^5_{ab}\psi_b(y)\phi(x)\bar{\psi}_c(x)\gamma^5_{cd}\psi_d(x)}:$$

$$\equiv X$$

Use $\underbrace{\psi_a(x)\bar{\psi}_b(y)} = -\underbrace{\bar{\psi}_b(y)\psi_a(x)}$

$$\Rightarrow * = -\int d^4x\, d^4y : \phi(y)\, \underbrace{\psi_d(x)\, \gamma^5_{ab}\, \psi_b(y)\, \phi(x)\, \bar{\psi}_c(x)\, \gamma^5_{cd}\, \bar{\psi}_a(y)} :$$

$$= -\int d^4x\, d^4y\, \underbrace{\psi_d(x)\bar{\psi}_a(y)} : \phi(y)\, \gamma^5_{ab}\psi_b(y)\, \phi(x)\, \bar{\psi}_c(x)\, \gamma^5_{cd} :$$

$$= \int d^4x\, d^4y\, \underbrace{\psi_d(x)\bar{\psi}_a(y)} : \phi(x)\, \bar{\psi}_c(x)\, \gamma^5_{cd}\ \phi(y)\, \gamma^5_{ab}\, \psi_b(y) :$$

$$\overset{d\leftrightarrow b,\, a\leftrightarrow c}{=} \int d^4x\, d^4y\, \underbrace{\psi_b(x)\bar{\psi}_c(y)} : \phi(x)\, \bar{\psi}_a(x)\gamma^5_{ab}\, \phi(y)\, \gamma^5_{cd}\psi_d(y) :$$

while $\int d^4x\, d^4y : \phi(x)\, \bar{\psi}(x)\, \gamma^5 \underbrace{\psi(x)\, \phi(y)\, \bar{\psi}(y)}\, \gamma^5 \psi(y) :$

$$= \int d^4x\, d^4y : \phi(x)\, \bar{\psi}_a(x)\, \gamma^5_{ab}\, \underbrace{\psi_b(x)\, \phi(y)\, \bar{\psi}_c(y)}\, \gamma^5_{cd}\, \psi_d(y) :$$

$$= \int d^4x\, d^4y\, \underbrace{\psi_b(x)\, \bar{\psi}_c(y)} : \phi(x)\, \bar{\psi}_a(x)\, \gamma^5_{ab}\, \phi(y)\, \gamma^5_{cd}\, \psi_d(y) :$$

$\Rightarrow \langle f | iT | i \rangle$

$$= \langle f | (-i)^2 (ig)^2 \int d^4x\, d^4y : \phi(x)\, \bar{\psi}(x)\, \gamma^5\, \underbrace{\psi(x)\, \phi(y)\, \bar{\psi}(y)}\, \gamma^5 \psi(y) : | i \rangle$$

$$= \langle f | (-i)^2 (ig)^2 \int d^4x\, d^4y\, \underbrace{\psi_b(x)\, \bar{\psi}_c(y)} : \phi(x)\, \bar{\psi}_a(x)\gamma^5_{ab}\, \phi(y)\, \gamma^5_{cd}\, \psi_d(y) : | i \rangle$$

where $\psi_b(x)\, \bar{\psi}_c(y) = \int_{-\infty}^{+\infty} d^4k \frac{1}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i(\not{k}+m)_{bc}}{k^2-m^2+i\varepsilon}$

using $\langle 0 | a_{\vec{p}_4} b_{\vec{p}_3, s_3} : \phi(x)\, \bar{\psi}_a(x)\, \phi(y)\, \psi_d(y) : b^{\dagger}_{\vec{p}_1, s_1}\, a^{\dagger}_{\vec{p}_2} | 0 \rangle$

$$= \langle 0 | a_{\vec{p}_4} b_{\vec{p}_3, s_3} \int d^3\vec{q}_1\, C(E_{\vec{q}_1}) \int d^3\vec{q}_2\, C(E_{\vec{q}_2}) \int d^3\vec{q}_3\, C(E_{\vec{q}_3}) \int d^3\vec{q}_4\, C(E_{\vec{q}_4})$$

$$: \Big[(a_{\vec{q}_1} e^{-iq_1\cdot x} + a^{\dagger}_{\vec{q}_1} e^{iq_1\cdot x})(a_{\vec{q}_2} e^{-iq_2\cdot y} + a^{\dagger}_{\vec{q}_2} e^{iq_2\cdot y})$$

$$\sum_{r,\lambda} (\bar{u}_a(\vec{q}_3, r)\, b^{\dagger}_{\vec{q}_3, r}\, e^{iq_3\cdot x})(u_d(\vec{q}_4, \lambda)\, b_{\vec{q}_4,\lambda}\, e^{-iq_4\cdot y}) \Big] :$$

$$b^{\dagger}_{\vec{p}_1, s_1}\, a^{\dagger}_{\vec{p}_2} | 0 \rangle$$

$$= \langle 0| a_{\vec{p}_4} b_{\vec{p}_3, s_3} \left(\prod_{i=1}^{4} \int d^3\vec{q}_i C(E_{\vec{q}_i}) \right) \left(a^\dagger_{\vec{q}_1} e^{iq_1\cdot x} a_{\vec{q}_2} e^{-iq_2\cdot y} + a^\dagger_{\vec{q}_2} e^{iq_2\cdot y} a_{\vec{q}_1} e^{-iq_1\cdot x} \right)$$

$$\sum_{r,\lambda} \left(\bar{u}_a(\vec{q}_3, r) b^\dagger_{\vec{q}_3, r} e^{iq_3\cdot x} \right) \left(u_d(\vec{q}_4, \lambda) b_{\vec{q}_4,\lambda} e^{-iq_4\cdot y} \right) b^\dagger_{\vec{p}_1, s_1} a^\dagger_{\vec{p}_2} |0\rangle$$

$$= \prod_{i=1}^{4} \left(\frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \left(\frac{1}{C E_{\vec{p}_i}} \right) \right) \left(e^{ip_4\cdot x} e^{-ip_2\cdot y} + e^{ip_4\cdot y} e^{-ip_2\cdot x} \right)$$

$$\times \left(\bar{u}_a(\vec{p}_3, s_3)\, u_d(\vec{p}_1, s_1)\, e^{ip_3\cdot x} e^{-ip_1\cdot y} \right)$$

$$\Rightarrow \langle f | iT | i \rangle$$

$$= (-i)^2 (ig)^2 \int d^4x \int d^4y \int d^4k \frac{1}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\varepsilon}$$

$$\cdot \left[\bar{u}_a(\vec{p}_3, s_3) \gamma^5_{ab} (\not{k} + m)_{bc} \gamma^5_{cd} u_d(\vec{p}_1, s_1) \right]$$

$$\cdot \left(e^{ip_4\cdot x} e^{-ip_2\cdot y} + e^{ip_4\cdot y} e^{-ip_2\cdot x} \right) e^{ip_3\cdot x} e^{-ip_1\cdot y}$$

$$= (-i)^2 (ig)^2 \int d^4k \frac{1}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

$$\times \left[\bar{u}(\vec{p}_3, s_3) \gamma^5 (\not{k} + m) \gamma^5 u(\vec{p}_1, s_1) \right]$$

$$\times \left[(2\pi)^4 \delta^4(-k + p_4 + p_3) (2\pi)^4 \delta^4(k - p_2 - p_1) + (2\pi)^4 \delta^4(-k - p_2 + p_3) (2\pi)^4 \delta^4(k + p_4 - p_1) \right]$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)\, ig^2\, \bar{u}(\vec{p}_3, s_3) \gamma^5 \left(\frac{\not{p}_1 + \not{p}_2 + m}{(p_1+p_2)^2 - m^2 + i\varepsilon} + \frac{\not{p}_1 - \not{p}_4 + m}{(p_1 - p_4)^2 - m^2 + i\varepsilon} \right) \gamma^5 u(\vec{p}_1, s_1)$$

$$\Rightarrow \boxed{iM_{fi} = ig^2 \bar{u}(\vec{p}_3, s_3) \gamma^5 \left[\frac{\not{p}_1 + \not{p}_2 + m}{(p_1+p_2)^2 - m^2 + i\varepsilon} + \frac{\not{p}_1 - \not{p}_4 + m}{(p_1 - p_4)^2 - m^2 + i\varepsilon} \right] \gamma^5 u(\vec{p}_1, s_1)}$$

Using $(\not{p}_1 - m) u(\vec{p}_1, s_1) = 0$

$$\not{p}_1 \gamma^5 = -\gamma^5 \not{p}_1$$

$$\bar{u}(\vec{p}_3, s_3)(\not{p}_3 - m) = 0$$

$$\not{p}_1 - \not{p}_4 = \not{p}_3 - \not{p}_2$$

$$\not{p}_3 \gamma^5 = -\gamma^5 \not{p}_3$$

$$\Rightarrow p_1^2 = p_3^2 = m^2,\ p_2^2 = p_4^2 = 0,\ (\gamma^5)^2 = 1$$

$$\Rightarrow \boxed{iM_{fi} = ig^2\, \bar{u}(\vec{p}_3, s_3)\gamma^5\left(\frac{\cancel{p}_2}{2p_1\cdot p_2} + \frac{-\cancel{p}_2}{-2p_1\cdot p_4}\right)\gamma^5 u(\vec{p}_1, s_1)}$$

$$\boxed{= \left(-\frac{i}{2}\right)g^2\, \bar{u}(\vec{p}_3, s_3)\cancel{p}_2 u(\vec{p}_1, s_1)\left(\frac{1}{p_1\cdot p_2} + \frac{1}{p_1\cdot p_4}\right)}$$

If we consider antifermion (p_1, s_1) + scalar $(p_2) \to$ antifermion (p_3, s_3) + scalar (p_4), then we just need to change $b^{+}_{\vec{p}_1, s_1}$ in $|i\rangle$ to $d^{+}_{\vec{p}_1, s_1}$,

$b_{\vec{p}_3, s_3}$ in $\langle f|$ to $d_{\vec{p}_3, s_3}$.

using

$$\langle 0|a_{\vec{p}_4} d_{\vec{p}_3, s_3} : \phi(x)\bar{\psi}_a(x)\phi(y)\psi_d(y): d^{+}_{\vec{p}_1, s_1} a^{+}_{\vec{p}_2}|0\rangle$$

$$= \langle 0|a_{\vec{p}_4} d_{\vec{p}_3, s_3} \prod_{i=1}^{4}\int d^3q_i\, C(E_{\vec{q}_i})$$

$$:[(a_{\vec{q}_1} e^{-iq_1\cdot x} + a^{+}_{\vec{q}_1} e^{iq_1\cdot x})(a_{\vec{q}_2} e^{-iq_2\cdot y} + a^{+}_{\vec{q}_2} e^{iq_2\cdot y})$$

$$\sum_{r,\lambda}(\bar{v}_a(\vec{q}_3, r) d_{\vec{q}_3, r} e^{-iq_3\cdot x})(v_d(\vec{q}_4, \lambda) d^{+}_{\vec{q}_4, \lambda} e^{iq_4\cdot y})]:$$

$$d^{+}_{\vec{p}_1, s_1} a^{+}_{\vec{p}_2}|0\rangle$$

$$= -\langle 0|a_{\vec{p}_4} d_{\vec{p}_3, s_3} \prod_{i=1}^{4}\int d^3q_i\, C(E_{\vec{q}_i})(a^{+}_{\vec{q}_1} e^{iq_1\cdot x} a_{\vec{q}_2} e^{-iq_2\cdot y} + a^{+}_{\vec{q}_2} e^{iq_2\cdot y} a_{\vec{q}_1} e^{-iq_1\cdot x})$$

$$\sum_{r,\lambda}(\bar{v}_a(\vec{q}_3, r) d^{+}_{\vec{q}_4, \lambda} d_{\vec{q}_3, r} e^{-iq_3\cdot x} v_d(\vec{q}_4, \lambda) e^{iq_4\cdot y}) d^{+}_{\vec{p}_1, s_1} a^{+}_{\vec{p}_2}|0\rangle$$

$$= -\prod_{i=1}^{4}\left(\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}_i}}\left(\frac{1}{C(E_{\vec{p}_i})}\right)\right)(e^{ip_4\cdot x}e^{-ip_2\cdot y} + e^{ip_4\cdot y}e^{-ip_2\cdot x})$$

$$\times\ (\bar{v}_a(\vec{p}_1, s_1) v_d(\vec{p}_3, s_3) e^{-ip_1\cdot x} e^{ip_3\cdot y})$$

$$\Rightarrow \langle f|iT|i\rangle$$

$$= -(-i)^2(ig)^2\int d^4x\int d^4y\int d^4k\, \frac{1}{(2\pi)^4} e^{-ik\cdot(x-y)}\frac{i}{k^2-m^2+i\varepsilon}$$

$$\cdot[\bar{v}_a(\vec{p}_1, s_1)\gamma^5_{ab}(\cancel{k}+m)_{bc}\gamma^5_{cd} v_d(\vec{p}_3, s_3)]$$

$$\cdot(e^{ip_4\cdot x}e^{-ip_2\cdot y} + e^{ip_4\cdot y}e^{-ip_2\cdot x}) e^{-ip_1\cdot x} e^{ip_3\cdot y}$$

$$= -(-i)^2 (ig)^2 \int d^4k \frac{1}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon}$$

$$\times \left[\bar{v}(\vec{p}_1, s_1) \gamma^5 (\not{k} + m) \gamma^5 v(\vec{p}_3, s_3) \right]$$

$$\times \Big[(2\pi)^4 \delta^4(-k + p_4 - p_1) (2\pi)^4 \delta^4(k - p_2 + p_3)$$

$$+ (2\pi)^4 \delta^4(-k - p_2 - p_1) (2\pi)^4 \delta^4(k + p_4 + p_3) \Big]$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) (-i) g^2 \cdot \bar{v}(\vec{p}_1, s_1) \gamma^5 \left(\frac{\not{p}_2 - \not{p}_3 + m}{(p_2-p_3)^2 - m^2 + i\varepsilon} + \frac{-\not{p}_1 - \not{p}_2 + m}{(p_1+p_2)^2 - m^2 + i\varepsilon} \right)$$

$$\cdot \gamma^5 v(\vec{p}_3, s_3)$$

$$\Rightarrow \boxed{ iM_{fi} = (-i) g^2 \bar{v}(\vec{p}_1, s_1) \gamma^5 \left[\frac{\not{p}_2 - \not{p}_3 + m}{(p_2-p_3)^2 - m^2 + i\varepsilon} + \frac{-\not{p}_1 - \not{p}_2 + m}{(p_1+p_2)^2 - m^2 + i\varepsilon} \right] \gamma^5 v(\vec{p}_3, s_3) }$$

use

$$(\not{p}_3 + m) v(\vec{p}_3, s_3) = 0$$

$$\bar{v}(\vec{p}_1, s_1)(\not{p}_1 + m) = 0$$

$$(p_2 - p_3)^2 - m^2 = (p_1 - p_4)^2 - m^2 = -2 p_1 \cdot p_4$$

$$(p_1 + p_2)^2 - m^2 = 2 p_1 \cdot p_2$$

$$\Rightarrow \boxed{iM_{fi}} = (-i) g^2 \bar{v}(\vec{p}_1, s_1) \gamma^5 \left[\frac{\not{p}_2}{-2p_1\cdot p_4} + \frac{-\not{p}_2}{2 p_1 \cdot p_2} \right] \gamma^5 v(\vec{p}_3, s_3)$$

$$= \boxed{ \left(-\frac{i}{2}\right) g^2 \bar{v}(\vec{p}_1, s_1) \not{p}_2 v(\vec{p}_3, s_3) \left(\frac{1}{p_1\cdot p_2} + \frac{1}{p_1 \cdot p_4} \right) }$$

2) $\overline{|M|^2} = \frac{1}{2} \sum_{s_1, s_3} |M|^2$

where we have $\sum_{s_1,s_3} \bar{u}(\vec{p}_3, s_3) \not{p}_2 u(\vec{p}_1, s_1) \bar{u}(\vec{p}_1, s_1) \not{p}_2 u(\vec{p}_3, s_3)$

$$= \mathrm{Tr}\left[(\not{p}_3 + m) \not{p}_2 (\not{p}_1 + m) \not{p}_2 \right]$$

$$= \mathrm{Tr}(\not{p}_3 \not{p}_2 \not{p}_1 \not{p}_2) + m^2 \,\mathrm{Tr}(\not{p}_2 \not{p}_2)$$

or we have $\sum_{s_1,s_3} \bar{v}(\vec{p}_1, s_1) \not{p}_2 v(\vec{p}_3, s_3) \bar{v}(\vec{p}_3, s_3) \not{p}_2 v(\vec{p}_1, s_1)$

$$= \mathrm{Tr}\left[(\not{p}_1 - m) \not{p}_2 (\not{p}_3 - m) \not{p}_2 \right]$$

$$= \mathrm{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_2) + m^2 \,\mathrm{Tr}(\not{p}_2 \not{p}_2)$$

$$= \mathrm{Tr}(\not{p}_3 \not{p}_2 \not{p}_1 \not{p}_2) + m^2 \,\mathrm{Tr}(\not{p}_2 \not{p}_2)$$

So, it is clear that no matter we consider fermion or antifermion scattering process, the $\overline{|M|}^2$ is the same, which is

$$\overline{|M|}^2 = \frac{1}{2}\left[T_r(\not{P_3}\not{P_2}\not{P_1}\not{P_2}) + m^2 T_r(\not{P_2}\not{P_2})\right] \frac{1}{4} g^4 \left(\frac{1}{P_1\cdot P_2} + \frac{1}{P_1\cdot P_4}\right)^2$$

$$= \frac{1}{2}\left[8(P_3\cdot P_2)(P_2\cdot P_1) - 4(P_3\cdot P_1)\underset{0}{\cancel{P_2^2}} + 4m^2\underset{0}{\cancel{P_2^2}}\right]\frac{1}{4} g^4\left(\frac{1}{P_1\cdot P_2} + \frac{1}{P_1\cdot P_4}\right)^2$$

$$= g^4 (P_3\cdot P_2)(P_2\cdot P_1)\left(\frac{1}{P_1\cdot P_2} + \frac{1}{P_1\cdot P_4}\right)^2$$

use $\quad P_3\cdot P_2 = -\frac{(P_3-P_2)^2 - m^2}{2} = -\frac{(P_1-P_4)^2-m^2}{2} = P_1\cdot P_4$

$$\Rightarrow \overline{|M|}^2 = g^4 (P_1\cdot P_4)(P_1\cdot P_2)\left[\left(\frac{1}{P_1\cdot P_2}\right)^2 + \left(\frac{1}{P_1\cdot P_4}\right)^2 + \frac{2}{(P_1\cdot P_2)(P_1\cdot P_4)}\right]$$

$$= g^4\left[\frac{P_1\cdot P_4}{P_1\cdot P_2} + \frac{P_1\cdot P_2}{P_1\cdot P_4} + 2\right]$$

use

$$P_1\cdot P_2 = \frac{(P_1+P_2)^2 - m^2}{2} = \frac{s-m^2}{2}$$

$$P_1\cdot P_4 = -\frac{(P_1-P_4)^2-m^2}{2} = -\frac{u-m^2}{2} = -\frac{2m^2-t-s-m^2}{2}$$

$$= \frac{s+t-m^2}{2}$$

$$\Rightarrow \overline{|M|}^2 = g^4\left(2 + \frac{s+t-m^2}{s-m^2} + \frac{s-m^2}{s+t-m^2}\right)$$

$$= g^4\left(4 + \frac{t}{s-m^2} - \frac{t}{s+t-m^2}\right)$$

$$= 4g^4\left[1 + \left(\frac{1}{4}\right)\frac{t(s+t-m^2-s+m^2)}{(s-m^2)(s+t-m^2)}\right]$$

$$= 4g^4\left[1 + \frac{t^2}{4(s-m^2)(s+t-m^2)}\right]$$

$$\left(\frac{d\sigma}{dt}\right)_{cm} = \frac{\overline{|M|}^2}{16\pi\lambda(s,m^2,0)}, \quad \text{where } \lambda(s,m^2,0) = s^2+m^4-2sm^2 = (s-m^2)^2$$

where $t \in \left[-(|\vec{P}_1|_{cm} + |\vec{P}_3|_{cm})^2, \ -(|\vec{P}_1|_{cm} - |\vec{P}_3|_{cm})^2\right]$

where $|\vec{P}_1|_{cm} = \frac{\lambda^{\frac{1}{2}}(s, m^2, 0)}{2\sqrt{s}} = \frac{(s-m^2)}{2\sqrt{s}}$

$$|\vec{P}_3|_{cm} = \frac{\lambda^{\frac{1}{2}}(s, m^2, 0)}{2\sqrt{s}} = \frac{(s-m^2)}{2\sqrt{s}}$$

$$\Rightarrow \quad t \in \left[-\frac{(s-m^2)^2}{s} \, , \, 0 \right]$$

$$\Rightarrow \boxed{\sigma_{cm}} = \int_{-\frac{(s-m^2)^2}{s}}^{0} \frac{1}{16\pi} \frac{1}{(s-m^2)^2} 4g^4 \left[1 + \frac{t^2}{4(s-m^2)(s+t-m^2)} \right] dt$$

$$= \int_{-\frac{(s-m^2)^2}{s}}^{0} \frac{1}{16\pi} \frac{1}{(s-m^2)^2} 4g^4 \left[1 + \frac{1}{4}\left(\frac{t}{s-m^2} - \underbrace{\underbrace{\frac{t}{s+t-m^2}}_{-\left(\frac{s+t-m^2}{s+t-m^2} - \frac{s-m^2}{s+t-m^2}\right)}}_{-\left(1 - \frac{s-m^2}{s+t-m^2}\right)} \right) \right] dt$$

$$= \int_{-\frac{(s-m^2)^2}{s}}^{0} \frac{4g^4}{16\pi} \frac{1}{(s-m^2)^2} \left[1 + \left(\frac{1}{4}\right) \frac{t}{s-m^2} - \frac{1}{4}\left(1 - \frac{s-m^2}{s+t-m^2}\right) \right] dt$$

$$= \frac{g^4}{4\pi} \frac{1}{(s-m^2)^2} \left[\frac{3}{4} \frac{(s-m^2)^2}{s} + \frac{1}{4} \frac{1}{s-m^2} \frac{1}{2}\left(0 - \frac{(s-m^2)^4}{s^2}\right) + \frac{1}{4}(s-m^2) \ln\left(\frac{s-m^2}{s-m^2 - \frac{(s-m^2)^2}{s}} \right) \right]$$

$$= \frac{g^4}{4\pi} \frac{1}{(s-m^2)} \left[\underbrace{\underbrace{\frac{3}{4} \frac{s-m^2}{s} - \frac{1}{8} \frac{(s-m^2)^2}{s^2}}_{\frac{3}{4}\left(1 - \frac{m^2}{s}\right) - \frac{1}{8}\left(1 - \frac{2m^2}{s} + \frac{m^4}{s^2}\right)}}_{\frac{5}{8} - \frac{1}{2}\frac{m^2}{s} - \frac{1}{8}\frac{m^4}{s^2}} + \frac{1}{4} \ln\left(\frac{1}{1 - \frac{s-m^2}{s}} \right) \right]$$

$$= \frac{g^4}{32\pi(s-m^2)} \left[5 - 4\frac{m^2}{s} - \frac{m^4}{s^2} + 2\ln\left(\frac{s}{m^2}\right) \right]$$

$$= \frac{g^4}{32\pi s} \frac{1}{\left(1 - \frac{m^2}{s}\right)} \left[\left(1 - \frac{m^2}{s}\right)\left(5 + \frac{m^2}{s}\right) + 2\ln\left(\frac{s}{m^2}\right) \right] = \boxed{\frac{g^4}{32\pi s}\left[5 + \frac{m^2}{s} - \frac{2}{\left(1 - \frac{m^2}{s}\right)} \ln\left(\frac{m^2}{s}\right) \right]}$$

Refine $\mu = \frac{m^2}{s}$

$$\Rightarrow \boxed{\sigma_{cm} = \frac{g^4}{32\pi s}\left(5 + \mu - \frac{2}{1-\mu} \ln \mu \right)}$$

(3)

In the limit $s \gg m^2$, that is $\mu \to 0$.

$$\boxed{\lim_{s \gg m^2} \sigma_{cm}} = \frac{g^4}{32\pi s} 2 \ln \frac{1}{\mu}$$

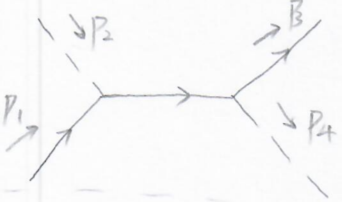
$$= \boxed{\frac{g^4}{16\pi s} \ln \frac{s}{m^2}}$$

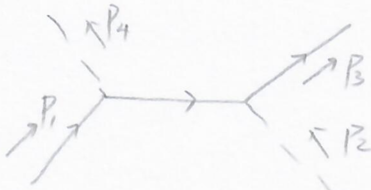
In the limit $s \to m^2$, that is $\mu \to 1$

$$\boxed{\lim_{s \to m^2} \sigma_{cm}} = \frac{g^4}{32\pi m^2}(5+1+2)$$

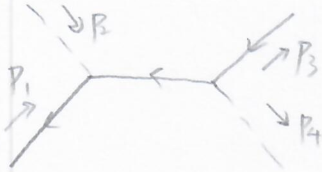
$$= \boxed{\frac{g^4}{4\pi m^2}}$$

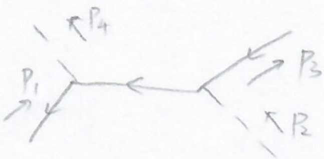
Feynman diagrams are





for fermion + scalar
→ fermion + scalar





for antifermion + scalar
→ antifermion + scalar

Homework final solution 2

Scalar $(P_1) \longrightarrow$ fermion (P_2, S_2) + antifermion (P_3, S_3)

$$i\mathcal{M}_{fi} = \bar{u}(P_2, S_2)\, i(-ig)\gamma^5 v(P_3, S_3)$$

$$= g\,\bar{u}(P_2, S_2)\gamma^5 v(P_3, S_3)$$

let's check this to be sure.

$$|i\rangle = (2\pi)^3 2E_{\vec{P}_1} C(E_{\vec{P}_1})\, a^+_{\vec{P}_1}|0\rangle$$

$$\langle f| = (2\pi)^3 2E_{\vec{P}_2} C(E_{\vec{P}_2})(2\pi)^3 2E_{\vec{P}_3} C(E_{\vec{P}_3})\langle 0|\, b_{\vec{P}_2, S_2} d_{\vec{P}_3, S_3}$$

$$\langle f|iT|i\rangle = \langle f|(-i)\int d^4x :\mathcal{H}_{int}(x):|i\rangle$$

$$= \langle f|(-i)\, ig\int d^4x :\phi(x)\bar{\psi}(x)\gamma^5\psi(x):|i\rangle$$

$$= \prod_{i=1}^{3}\left[(2\pi)^3 2E_{\vec{P}_i} C(E_{\vec{P}_i})\right](-i)(ig)\langle 0|\, b_{\vec{P}_2, S_2} d_{\vec{P}_3, S_3}\int d^4x \int d^3\vec{k}_1 C(E_{\vec{k}_1})\int d^3\vec{k}_2 C(E_{\vec{k}_2})$$

$$\int d^3\vec{k}_3 C(E_{\vec{k}_3}) : a_{\vec{k}_1} e^{-ik_1\cdot x}\sum_{r,\lambda}\Big[\left(\bar{u}(\vec{k}_2, r)\, b^+_{\vec{k}_2, r}\, e^{ik_2\cdot x}\right)\gamma^5$$

$$\left(v(\vec{k}_3, \lambda)\, d^+_{\vec{k}_3, \lambda}\, e^{ik_3\cdot x}\right)\Big] : a^+_{\vec{P}_1}|0\rangle$$

$$= -(-i)(ig)\int d^4x\ \bar{u}(\vec{P}_2, S_2)\gamma^5 v(\vec{P}_3, S_3)\, e^{-iP_1\cdot x} e^{iP_2\cdot x} e^{iP_3\cdot x}$$

due to exchange of $b_{\vec{P}_2, S_2}$ and $d_{\vec{P}_3, S_3}$

$$= (2\pi)^4\delta^4(P_1 - P_2 - P_3)\ g\ \bar{u}(\vec{P}_2, S_2)\gamma^5 v(\vec{P}_3, S_3)\,(-1)$$

$$\Rightarrow\ i\mathcal{M}_{fi} = g\,\bar{u}(\vec{P}_2, S_2)\gamma^5 v(\vec{P}_3, S_3)\ (-1)$$

The (-1) at the end is due to our definition of $\langle f|$. If we define $\langle f| = (2\pi)^3 2E_{\vec{P}_2} C(E_{\vec{P}_2})(2\pi)^3 2E_{\vec{P}_3} C(E_{\vec{P}_3})\langle 0|\, d_{\vec{P}_3, S_3} b_{\vec{P}_2, S_2}$, then we do not have the (-1) in $i\mathcal{M}_{fi}$. Anyway it is not important since we will calculate $|\mathcal{M}|^2$.

To calculate the total decay rate, we need to calculate

$$\sum_{S_2,S_3} |M|^2 = \sum_{S_2,S_3} (iM_{fi})(iM_{fi})^*$$

$$= \sum_{S_2,S_3} g^2 \, \bar{u}(\vec{P}_2,S_2)\gamma^5 v(\vec{P}_3,S_3)\,\bar{v}(\vec{P}_3,S_3)(-\gamma^5)u(\vec{P}_2,S_2)$$

where $\left(\bar{u}(\vec{P}_2,S_2)\gamma^5 v(\vec{P}_3,S_3)\right)^*$

$$= \left(\bar{u}(\vec{P}_2,S_2)\gamma^5 v(\vec{P}_3,S_3)\right)^\dagger$$

$$= v^\dagger(\vec{P}_3,S_3)\gamma^{5\dagger}\gamma^{0\dagger}u(\vec{P}_2,S_2)$$

$$= \bar{v}(\vec{P}_3,S_3)\gamma^0\gamma^5\gamma^0 u(\vec{P}_2,S_2)$$

$$= -\bar{v}(\vec{P}_3,S_3)\gamma^5 u(\vec{P}_2,S_2)$$

is used.

$$\Rightarrow \sum_{S_1,S_2} |M|^2 = g^2 \, T_r\left[(\not{P}_3 - m)(-\gamma^5)(\not{P}_2 + m)\gamma^5\right]$$

$$= g^2 \cdot T_r\left[(\not{P}_3 + m)(\not{P}_2 + m)\right]$$

$$= g^2(4m^2 + 4P_2\cdot P_3)$$

$$= 4g^2\left[m^2 + \frac{(P_2+P_3)^2 - P_2^2 - P_3^2}{2}\right]$$

$$= 4g^2\left[m^2 + \frac{M^2 - 2m^2}{2}\right]$$

$$= 2g^2M^2$$

$$\Rightarrow \left(\frac{d\Gamma}{d\Omega}\right)_{cm} = \frac{\lambda^{\frac{1}{2}}(M^2, m^2, m^2)}{64\pi^2M^3}\, 2g^2M^2$$

$$= \frac{(M^4 + 2m^4 - 2m^4 - 4m^2M^2)^{\frac{1}{2}}}{32\pi^2 M}\, g^2$$

$$= \frac{g^2}{32\pi^2}(M^2 - 4m^2)^{\frac{1}{2}}$$

$$\Rightarrow \boxed{\Gamma_{cm} = \frac{g^2 M}{8\pi}\left(1 - \frac{4m^2}{M^2}\right)^{\frac{1}{2}}}$$