Solution 3-1

1)

$$\underset{\substack{\| \\ m_B^2}}{P_B^2} = (P_A - P_C)^2 = m_A^2 + m_C^2 - 2 m_A E_C$$

$$\Rightarrow \boxed{E_C = \frac{m_A^2 + m_C^2 - m_B^2}{2 m_A}}$$

$$\boxed{E_B} = m_A - E_C = \frac{2m_A^2 - (m_A^2 + m_C^2 - m_B^2)}{2 m_A} = \boxed{\frac{m_A^2 + m_B^2 - m_C^2}{2 m_A}}$$

2)

$$|\vec{P}_B| = |\vec{P}_C| = (E_B^2 - m_B^2)^{\frac{1}{2}}$$

$$= \left[\frac{(m_A^2 + m_B^2 - m_C^2)^2 - m_B^2 (2m_A)^2}{(2m_A)^2} \right]^{\frac{1}{2}}$$

$$= \frac{[m_A^4 + m_B^4 + m_C^4 + 2m_A^2 m_B^2 - 2 m_B^2 m_C^2 - 2 m_A^2 m_C^2 - 4 m_A^2 m_B^2]^{\frac{1}{2}}}{2 m_A}$$

$$= \boxed{\frac{(m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2 m_B^2 m_C^2 - 2m_A^2 m_C^2)^{\frac{1}{2}}}{2 m_A}}$$

Solution 3-2

$$\mathcal{H}_{int} = \cancel{\frac{\partial \mathcal{L}_{int}}{\partial(\partial\phi/\partial t)} \frac{\partial \phi}{\partial t}} - \mathcal{L}_{int}$$
$\to 0$

$$= \frac{\lambda}{4!}\phi^4$$

$$\phi(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}})\left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^{+} e^{ip\cdot x}\right) d^3\vec{p}$$

For the process $\phi(q_1) + \phi(q_2) \longrightarrow \phi(q_3) + \phi(q_4)$

$$|i\rangle = C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} a_{\vec{q}_1}^{+} a_{\vec{q}_2}^{+} |0\rangle$$

$$\langle f| = \langle 0| a_{\vec{q}_3} a_{\vec{q}_4} C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

To the leading order,

$$\langle f|iT|i\rangle = \langle f|(-i)\int d^4x :\mathcal{H}_{int}: |i\rangle$$

$$= \langle f|(-i)\int d^4x \frac{\lambda}{4!} : \int_{-\infty}^{+\infty} C(E_{\vec{p}_1})[a_{\vec{p}_1} e^{-ip_1\cdot x} + a_{\vec{p}_1}^{+} e^{ip_1\cdot x}) d^3\vec{p}_1$$
$$\times \int_{-\infty}^{+\infty} C(E_{\vec{p}_2})[a_{\vec{p}_2} e^{-ip_2\cdot x} + a_{\vec{p}_2}^{+} e^{ip_2\cdot x}] d^3\vec{p}_2$$
$$\times \int_{-\infty}^{+\infty} C(E_{\vec{p}_3})[a_{\vec{p}_3} e^{-ip_3\cdot x} + a_{\vec{p}_3}^{+} e^{ip_3\cdot x}] d^3\vec{p}_3$$
$$\times \int_{-\infty}^{+\infty} C(E_{\vec{p}_4})[a_{\vec{p}_4} e^{-ip_4\cdot x} + a_{\vec{p}_4}^{+} e^{ip_4\cdot x}] d^3\vec{p}_4 : |i\rangle$$

Because there're two a^{+} in $|i\rangle$, and two a in $\langle f|$, we need to have, in : :, two a to kill $|i\rangle$ and two a^{+} to kill $\langle f|$.

$$\Rightarrow \langle f|iT|i\rangle = \langle 0| a_{\vec{q}_3} a_{\vec{q}_4} C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$
$$\times (-i)\frac{\lambda}{4!}\int d^4x \int_{-\infty}^{+\infty} C(E_{\vec{p}_1}) d^3\vec{p}_1 \int_{-\infty}^{+\infty} C(E_{\vec{p}_2}) d^3\vec{p}_2 \int_{-\infty}^{+\infty} C(E_{\vec{p}_3}) d^3\vec{p}_3 \int_{-\infty}^{+\infty} C(E_{\vec{p}_4}) d^3\vec{p}_4$$
$$\times \left(a_{\vec{p}_1}^{+} e^{ip_1\cdot x} a_{\vec{p}_2}^{+} e^{ip_2\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot x} \right.$$
$$+ a_{\vec{p}_1}^{+} e^{ip_1\cdot x} a_{\vec{p}_3}^{+} e^{ip_3\cdot x} a_{\vec{p}_2} e^{-ip_2\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot x}$$
$$+ a_{\vec{p}_1}^{+} e^{ip_1\cdot x} a_{\vec{p}_4}^{+} e^{ip_4\cdot x} a_{\vec{p}_2} e^{-ip_2\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot x}$$

$$+ a^\dagger_{\vec{p}_2} e^{ip_2\cdot x} a^\dagger_{\vec{p}_3} e^{ip_3\cdot x} a_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot x}$$

$$+ a^\dagger_{\vec{p}_2} e^{ip_2\cdot x} a^\dagger_{\vec{p}_4} e^{ip_4\cdot x} a_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot x}$$

$$+ a^\dagger_{\vec{p}_3} e^{ip_3\cdot x} a^\dagger_{\vec{p}_4} e^{ip_4\cdot x} a_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_2} e^{-ip_2\cdot x} \Big)$$

$$\times \; C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} \, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} \, a^\dagger_{\vec{q}_1} a^\dagger_{\vec{q}_2} |0\rangle$$

Using $\quad [a_{\vec{p}}, a^\dagger_{\vec{k}}] = \frac{1}{(2\pi)^3 2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p}-\vec{k})$

$$\Rightarrow \langle 0| a_{\vec{q}_3} a_{\vec{q}_4} a^\dagger_{\vec{p}_i} a^\dagger_{\vec{p}_j} a_{\vec{p}_r} a_{\vec{p}_s} a^\dagger_{\vec{q}_1} a^\dagger_{\vec{q}_2} |0\rangle$$

$$= \langle 0| a_{\vec{q}_3} \left(a^\dagger_{\vec{p}_i} a_{\vec{q}_4} + \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2 \delta^3(\vec{p}_i - \vec{q}_4) \right) a^\dagger_{\vec{p}_j}$$

$$\times \; a_{\vec{p}_r} \left(a^\dagger_{\vec{q}_1} a_{\vec{p}_s} + \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{p}_s - \vec{q}_1) \right) a^\dagger_{\vec{q}_2} |0\rangle$$

$$= \langle 0| \Big[a_{\vec{q}_3} a^\dagger_{\vec{p}_i} a_{\vec{q}_4} a^\dagger_{\vec{p}_j} + \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2 \delta^3(\vec{p}_i - \vec{q}_4) \, a_{\vec{q}_3} a^\dagger_{\vec{p}_j} \Big]$$

$$\times \Big[a_{\vec{p}_r} a^\dagger_{\vec{q}_1} a_{\vec{p}_s} a^\dagger_{\vec{q}_2} + \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{p}_s - \vec{q}_1) \, a_{\vec{p}_r} a^\dagger_{\vec{q}_2} \Big] |0\rangle$$

$$= \langle 0| a_{\vec{q}_3} a^\dagger_{\vec{p}_i} a_{\vec{q}_4} a^\dagger_{\vec{p}_j} a_{\vec{p}_r} a^\dagger_{\vec{q}_1} a_{\vec{p}_s} a^\dagger_{\vec{q}_2} |0\rangle$$

$$+ \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2 \delta^3(\vec{p}_i - \vec{q}_4) \langle 0| a_{\vec{q}_3} a^\dagger_{\vec{p}_j} a_{\vec{p}_r} a^\dagger_{\vec{q}_1} a_{\vec{p}_s} a^\dagger_{\vec{q}_2} |0\rangle$$

$$+ \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{p}_s - \vec{q}_1) \langle 0| a_{\vec{q}_3} a^\dagger_{\vec{p}_i} a_{\vec{q}_4} a^\dagger_{\vec{p}_j} a_{\vec{p}_r} a^\dagger_{\vec{q}_2} |0\rangle$$

$$+ \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2 \delta^3(\vec{p}_i - \vec{q}_4) \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{p}_s - \vec{q}_1) \langle 0| a_{\vec{q}_3} a^\dagger_{\vec{p}_j} a_{\vec{p}_r} a^\dagger_{\vec{q}_2} |0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}} \left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big[\delta^3(\vec{p}_r - \vec{q}_1) \delta^3(\vec{p}_s - \vec{q}_2) \delta^3(\vec{p}_i - \vec{q}_3) \delta^3(\vec{p}_j - \vec{q}_4)$$

$$+ \delta^3(\vec{p}_r - \vec{q}_1) \delta^3(\vec{p}_s - \vec{q}_2) \delta^3(\vec{p}_j - \vec{q}_3) \delta^3(\vec{p}_i - \vec{q}_4)$$

$$+\delta^3(\vec{P}_s-\vec{q}_1)\,\delta^3(\vec{P}_r-\vec{q}_2)\,\delta^3(\vec{P}_i-\vec{q}_3)\,\delta^3(\vec{P}_j-\vec{q}_4)$$

$$+\delta^3(\vec{P}_s-\vec{q}_1)\,\delta^3(\vec{P}_r-\vec{q}_2)\,\delta^3(\vec{P}_j-\vec{q}_3)\,\delta^3(\vec{P}_i-\vec{q}_4)\Big]$$

$$\Rightarrow \langle 0|\, a_{\vec{q}_3}\, a_{\vec{q}_4}\, C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

$$\times \int d^4x \int_{-\infty}^{+\infty} C(E_{\vec{P}_i})\,d^3\vec{P}_i \int_{-\infty}^{+\infty} C(E_{\vec{P}_j})\,d^3\vec{P}_j \int_{-\infty}^{+\infty} C(E_{\vec{P}_r})\,d^3\vec{P}_r \int_{-\infty}^{+\infty} C(E_{\vec{P}_s})\,d^3\vec{P}_s$$

$$\times \left(a^{+}_{\vec{P}_i}\, e^{iP_i\cdot x}\, a^{+}_{\vec{P}_j}\, e^{iP_j\cdot x}\, a_{\vec{P}_r}\, e^{-iP_r\cdot x}\, a_{\vec{P}_s}\, e^{-iP_s\cdot x}\right)$$

$$\times\; C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\; C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\; a^{+}_{\vec{q}_1}\, a^{+}_{\vec{q}_2}\, |0\rangle$$

$$= \int d^4x\, 4e^{i(q_3+q_4-q_1-q_2)\cdot x} = 4(2\pi)^4\delta^4(q_3+q_4-q_1-q_2)$$

$$\Rightarrow \langle f|\, iT\,|i\rangle$$

$$= (-i)\frac{\lambda}{4!}\; 6\times 4\,(2\pi)^4\delta^4(q_3+q_4-q_1-q_2)$$

$$= (2\pi)^4\delta^4(q_3+q_4-q_1-q_2)\,(-i)\lambda$$

$$\Rightarrow \boxed{i\mathcal{M}_{fi} = -i\lambda}$$

Solution 3-3.

$$\mathcal{H}_{int} = \cancelto{0}{\frac{\partial \mathcal{L}_{int}}{\partial(\partial\phi/\partial t)}\frac{\partial\phi}{\partial t}} + \cancelto{0}{\frac{\partial \mathcal{L}_{int}}{\partial(\partial\sigma/\partial t)}\frac{\partial\sigma}{\partial t}} - \mathcal{L}_{int}$$

$$= g\phi^4 + g'\sigma^2\phi^2 + g''\sigma^4 + \lambda\sigma\phi^2 + \lambda'\sigma^3$$

$$\phi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\left(a_{\vec{p}}\, e^{-ip\cdot x} + a_{\vec{p}}^{+} e^{ip\cdot x}\right)$$

$$\sigma(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\left(b_{\vec{p}}\, e^{-ip\cdot x} + b_{\vec{p}}^{+} e^{ip\cdot x}\right)$$

$$|i\rangle = C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^{+}_{\vec{q}_1} a^{+}_{\vec{q}_2}|0\rangle$$

$$\langle f| = \langle 0|\, a_{\vec{q}_3} a_{\vec{q}_4}\, C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

Since there are two a in $\langle f|$ and two a^+ in $|i\rangle$, we need four not-contracted-ϕ. Therefore, at the first order in the expansion of iT, we only need the term $g\phi^4$

At second order in the expansion of iT, up to tree-level, we can have $\lambda\,\sigma(x)\phi^2(x)\,\lambda\,\sigma(y)\phi^2(y)$.

Note that we don't need to worry about $g'\sigma^2(x)\phi^2(x)\, g'\sigma^2(y)\phi^2(y)$, because we would have to contract $\sigma^2(x)\sigma^2(y)$ and this gives a loop.

Also, we don't need to worry about $g\phi^4(x)\, g\phi^4(y)$, since we would have to contract $\phi^2(x)\phi^2(y)$ and again this gives a loop.

(In fact, for a tree-level process, we have $V = I + 1$, where V is the number of vertices and I is the number of propagators.

Proof: in general, a Feynman diagram satisfies

$$L = I - (V-1).$$

where L is the number of loops.

The reason for this expression is that each vertex gives a $\delta^4(\sum_{i=1}^{n} p_i)$, where n is the number of lines attached to this vertex, and p_i are their momenta, Each propagator gives $\int d^4k$. Out of all the V δ^4 function, $(V-1)$ of them will be integrated out by the $\int d^4k$ from the propagators, and the 1 leftover will be the $\delta^4(\sum P_{\text{initial}} - \sum P_{\text{final}})$ for the process. Therefore, $I-(V-1)$ equals the number of $\int d^4k$ leftover, that is, the number of loops.

So for tree digram, $I-(V-1)=0 \Rightarrow V=I+1$

For the problem at hand, we have vertice having three fields, i.e., $\lambda\sigma\phi^2$ and $\lambda'\sigma^3$, and we have vertice having four fields, i.e., $g\phi^4$, $g'\sigma^2\phi^2$ and $g''\sigma^4$.

So, for a process with 4 external lines, we have

$$4+2I = 3V_3 + 4V_4 = 3V + V_4$$

where I is the number of propagators, V_3 is the number of vertice having three fields, V_4 is the number of vertices having four fields, and $V = V_3 + V_4$. Note that the factor of 2 in front of I means that each propagator connects two vertice.

$\Rightarrow$ For tree-level process,

$$4 + 2(V-1) = 3V + V_4$$

$$\Rightarrow \quad 2 = V + V_4 = V_3 + 2V_4$$

Since $V_3, V_4 = 0, 1, 2, \ldots,$

We can only have $\begin{cases} V_3 = 2 \\ V_4 = 0 \end{cases}$ or $\begin{cases} V_3 = 0 \\ V_4 = 1 \end{cases}$ as the solutions.

So the topology of the diagram can be only

 or 

So for the process $\phi(q_1)+\phi(q_2) \to \phi(q_3)+\phi(q_4)$,

we can only choose $g\phi^4$ for , and $\lambda\sigma\phi^2$ for

$$\Rightarrow \langle f | iT | i \rangle$$

$$= \langle f | (-i) \int d^4x :g\phi^4(x):$$

$$+ \frac{1}{2!}(-i)^2 \int d^4x \int d^4y :\lambda\sigma(x)\phi^2(x): :\lambda\sigma(y)\phi^2(y): | i \rangle$$

$$= \langle 0 | a_{\vec{q}_3} a_{\vec{q}_4} \; C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} \; C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

$$\times \Big\{ (-i) g \int d^4x \int_{-\infty}^{+\infty} d^3\vec{p}_1 \, d^3\vec{p}_2 \, d^3\vec{p}_3 \, d^3\vec{p}_4 \, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) C(E_{\vec{p}_4})$$

$$\times \Big(a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} e^{i(p_1+p_2-p_3-p_4)\cdot x} + a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_2} a_{\vec{p}_4} e^{i(p_1+p_3-p_2-p_4)\cdot x}$$

$$+ a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_4} a_{\vec{p}_2} a_{\vec{p}_3} e^{i(p_1+p_4-p_2-p_3)\cdot x} + a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_1} a_{\vec{p}_4} e^{i(p_2+p_3-p_1-p_4)\cdot x}$$

$$+ a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_3} e^{i(p_2+p_4-p_1-p_3)\cdot x} + a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_2} e^{i(p_3+p_4-p_1-p_2)\cdot x} \Big)$$

$$+ \frac{1}{2!}(-i)^2 \lambda^2 \int d^4x \int d^4y \; \underbrace{\sigma(x)\,\sigma(y)} \; :\phi(x)\phi(x)\phi(y)\phi(y): \Big\}$$

$$\times C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} \; C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} \; a^\dagger_{\vec{q}_1} a^\dagger_{\vec{q}_2} | 0 \rangle$$

where $$\underbrace{\sigma(x)\,\sigma(y)} = \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m_1^2 + i\varepsilon}$$

$$\Rightarrow \langle f | \frac{1}{2!}(-i)^2 \lambda^2 \int d^4x \, d^4y \; \underbrace{\sigma(x)\,\sigma(y)} \; :\phi(x)\phi(x)\phi(y)\phi(y): | i \rangle$$

$$= \frac{1}{2!}(-i)^2 \lambda^2 \int d^4x \int d^4y \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m_1^2 + i\varepsilon} \langle f | :\phi(x)\phi(x)\phi(y)\phi(y): | i \rangle$$

where $\langle f | :\phi(x)\phi(x)\phi(y)\phi(y): | i \rangle$

$$= \langle 0 | a_{\vec{q}_3} a_{\vec{q}_4} C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4} \int_{-\infty}^{+\infty} d^3\vec{p}_1 \, d^3\vec{p}_2 \, d^3\vec{p}_3 \, d^3\vec{p}_4 \, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) C(E_{\vec{p}_4})$$

$$\times :\left(a_{\vec{p}_1} e^{-ip_1\cdot x} + a^\dagger_{\vec{p}_1} e^{ip_1\cdot x}\right)\left(a_{\vec{p}_2} e^{-ip_2\cdot x} + a^\dagger_{\vec{p}_2} e^{ip_2\cdot x}\right)\left(a_{\vec{p}_3} e^{-ip_3\cdot y} + a^\dagger_{\vec{p}_3} e^{ip_3\cdot y}\right)$$

$$\left(a_{\vec{p}_4} e^{-ip_4\cdot y} + a^\dagger_{\vec{p}_4} e^{ip_4\cdot y}\right): C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} \; a^\dagger_{\vec{q}_1} a^\dagger_{\vec{q}_2} | 0 \rangle$$

$$= \prod_{i=1}^{4} \left[C(E_{\vec{q}_i})(2\pi)^3 2E_{\vec{q}_i} \right] \times \langle 0 | a_{\vec{q}_3} a_{\vec{q}_4} \int_{-\infty}^{+\infty} d^3\vec{p}_1 \, d^3\vec{p}_2 \, d^3\vec{p}_3 \, d^3\vec{p}_4 \, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) C(E_{\vec{p}_4})$$

$$\times \Big(a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} e^{i(p_1+p_2)\cdot x} e^{i(-p_3-p_4)\cdot y} + a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_2} a_{\vec{p}_4} e^{i(p_1-p_2)\cdot x} e^{i(p_3-p_4)\cdot y}$$

$$+ a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_4} a_{\vec{p}_2} a_{\vec{p}_3} e^{i(p_1-p_2)\cdot x} e^{i(-p_3+p_4)\cdot y} + a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_1} a_{\vec{p}_4} e^{i(-p_1+p_2)\cdot x} e^{i(p_3-p_4)\cdot y}$$

$$+ a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_3} e^{i(-p_1+p_2)\cdot x} e^{i(-p_3+p_4)\cdot y} + a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_2} e^{i(-p_1-p_2)\cdot x} e^{i(p_3+p_4)\cdot y}$$

$$\times\ a^\dagger_{\vec q_1} a^\dagger_{\vec q_2} |0\rangle$$

using $\langle 0| a_{\vec q_3} a_{\vec q_4}\ a^\dagger_{\vec p_i} a^\dagger_{\vec p_j} a_{\vec p_k} a_{\vec p_r}\ a^\dagger_{\vec q_1} a^\dagger_{\vec q_2} |0\rangle$

$$= \langle 0| a_{\vec q_3} \left(\frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_4}} \left(\frac{1}{C(E_{\vec q_4})}\right)^2 \delta^3(\vec p_i - \vec q_4) + a^\dagger_{\vec p_i} a_{\vec q_4} \right) a^\dagger_{\vec p_j}$$
$$a_{\vec p_k} \left(\frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_1}} \left(\frac{1}{C(E_{\vec q_1})}\right)^2 \delta^3(\vec p_r - \vec q_1) + a^\dagger_{\vec q_1} a_{\vec p_r} \right) a^\dagger_{\vec q_2} |0\rangle$$

$$= \langle 0| a_{\vec q_3} a^\dagger_{\vec p_i} a_{\vec q_4} a^\dagger_{\vec p_j} a_{\vec p_k} a^\dagger_{\vec q_1} a_{\vec p_r} a^\dagger_{\vec q_2} |0\rangle$$
$$+ \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_4}} \left(\frac{1}{C(E_{\vec q_4})}\right)^2 \langle 0| a_{\vec q_3} a^\dagger_{\vec p_j} a_{\vec p_k} a^\dagger_{\vec q_1} a_{\vec p_r} a^\dagger_{\vec q_2} |0\rangle \delta^3(\vec p_i - \vec q_4)$$
$$+ \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_1}} \left(\frac{1}{C(E_{\vec q_1})}\right)^2 \delta^3(\vec p_r - \vec q_1) \langle 0| a_{\vec q_3} a^\dagger_{\vec p_i} a_{\vec q_4} a^\dagger_{\vec p_j} a_{\vec p_k} a^\dagger_{\vec q_2} |0\rangle$$
$$+ \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_4}} \left(\frac{1}{C(E_{\vec q_4})}\right)^2 \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_1}} \left(\frac{1}{C(E_{\vec q_1})}\right)^2 \delta^3(\vec p_i - \vec q_4) \delta^3(\vec p_r - \vec q_1)$$
$$\langle 0| a_{\vec q_3} a^\dagger_{\vec p_j} a_{\vec p_k} a^\dagger_{\vec q_2} |0\rangle$$

$$= \prod_{s=1}^{4} \left[\frac{1}{(2\pi)^3} \frac{1}{2E_{\vec q_s}} \left(\frac{1}{C(E_{\vec q_s})}\right)^2 \right] \Big[\delta^3(\vec p_i - \vec q_3) \delta^3(\vec p_j - \vec q_4) \delta^3(\vec p_k - \vec q_1) \delta^3(\vec p_r - \vec q_2)$$
$$+ \delta^3(\vec p_i - \vec q_4) \delta^3(\vec p_j - \vec q_3) \delta^3(\vec p_k - \vec q_1) \delta^3(\vec p_r - \vec q_2)$$
$$+ \delta^3(\vec p_i - \vec q_3) \delta^3(\vec p_j - \vec q_4) \delta^3(\vec p_k - \vec q_2) \delta^3(\vec p_r - \vec q_1)$$
$$+ \delta^3(\vec p_i - \vec q_4) \delta^3(\vec p_j - \vec q_3) \delta^3(\vec p_k - \vec q_2) \delta^3(\vec p_r - \vec q_1) \Big]$$

$$\Rightarrow \langle f| : \phi(x) \phi(x) \phi(y) \phi(y) : |i\rangle$$
$$= \left(4 e^{i(q_3+q_4)\cdot x} e^{i(-q_1-q_2)\cdot y}\right) + \Big(e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y} + e^{i(q_4-q_1)\cdot x} e^{i(q_3-q_2)\cdot y}$$
$$+ e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y} + e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} \Big)$$
$$+ \Big(e^{i(q_3-q_1)\cdot x} e^{i(-q_2+q_4)\cdot y} + e^{i(q_4-q_1)\cdot x} e^{i(-q_2+q_3)\cdot y} + e^{i(q_3-q_2)\cdot x} e^{i(-q_1+q_4)\cdot y}$$
$$+ e^{i(q_4-q_2)\cdot x} e^{i(-q_1+q_3)\cdot y} \Big)$$
$$+ \Big(e^{i(-q_1+q_3)\cdot x} e^{i(q_4-q_2)\cdot y} + e^{i(-q_1+q_4)\cdot x} e^{i(q_3-q_2)\cdot y}$$
$$+ e^{i(-q_2+q_3)\cdot x} e^{i(q_4-q_1)\cdot y} + e^{i(-q_2+q_4)\cdot x} e^{i(q_3-q_1)\cdot y} \Big)$$
$$+ \Big(e^{i(-q_1+q_3)\cdot x} e^{i(-q_2+q_4)\cdot y} + e^{i(-q_1+q_4)\cdot x} e^{i(-q_2+q_3)\cdot y}$$
$$+ e^{i(-q_2+q_3)\cdot x} e^{i(-q_1+q_4)\cdot y} + e^{i(-q_2+q_4)\cdot x} e^{i(-q_1+q_3)\cdot y} \Big)$$
$$+ \left(4 e^{i(-q_1-q_2)\cdot x} e^{i(q_3+q_4)\cdot y}\right)$$

$$= 4e^{i(q_3+q_4)\cdot x}e^{i(-q_1-q_2)\cdot y} + 4e^{i(-q_1-q_2)\cdot x}e^{i(q_3+q_4)\cdot y}$$
$$+4e^{i(q_3-q_1)\cdot x}e^{i(q_4-q_2)\cdot y} + 4e^{i(q_4-q_1)\cdot x}e^{i(q_3-q_2)\cdot y}$$
$$+4e^{i(q_3-q_2)\cdot x}e^{i(q_4-q_1)\cdot y} + 4e^{i(q_4-q_2)\cdot x}e^{i(q_3-q_1)\cdot y}$$

and

$$\langle f|:\phi^4(x):|i\rangle$$
$$= 4e^{i(q_3+q_4-q_1-q_2)\cdot x} + 4e^{i(q_3+q_4-q_1-q_2)\cdot x}$$
$$+4e^{i(q_3+q_4-q_1-q_2)\cdot x} + 4e^{i(q_3+q_4-q_1-q_2)\cdot x}$$
$$+4e^{i(q_3+q_4-q_1-q_2)\cdot x} + 4e^{i(q_3+q_4-q_1-q_2)\cdot x}$$
$$= 4\times 6\, e^{i(q_3+q_4-q_1-q_2)\cdot x}$$

$$\Rightarrow \langle f|iT|i\rangle$$
$$= (-i)g\int_{-\infty}^{+\infty} d^4x\,(4\times 6)\,e^{i(q_3+q_4-q_1-q_2)\cdot x}$$
$$+\frac{1}{2!}(-i)^2\lambda^2\int_{-\infty}^{+\infty}d^4x\int_{-\infty}^{+\infty}d^4y\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}e^{ik\cdot y}\frac{i}{k^2-m_1^2+i\varepsilon}\times 4$$
$$\times\Big(e^{i(q_3+q_4)\cdot x}e^{i(-q_1-q_2)\cdot y} + e^{i(-q_1-q_2)\cdot x}e^{i(q_3+q_4)\cdot y}$$
$$+e^{i(q_3-q_1)\cdot x}e^{i(q_4-q_2)\cdot y} + e^{i(q_4-q_1)\cdot x}e^{i(q_3-q_2)\cdot y}$$
$$+e^{i(q_3-q_2)\cdot x}e^{i(q_4-q_1)\cdot y} + e^{i(q_4-q_2)\cdot x}e^{i(q_3-q_1)\cdot y}\Big)$$

$$= (-i)g\times 24\times(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)$$
$$+\frac{1}{2!}(-i)^2\lambda^2\times 4\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\varepsilon}(2\pi)^4(2\pi)^4$$
$$\times\Big[\delta^4(-k+q_3+q_4)\delta^4(k-q_1-q_2) + \delta^4(-k-q_1-q_2)\delta^4(k+q_3+q_4)$$
$$+\delta^4(-k+q_3-q_1)\delta^4(k+q_4-q_2) + \delta^4(-k+q_4-q_1)\delta^4(k+q_3-q_2)$$
$$+\delta^4(-k+q_3-q_2)\delta^4(k+q_4-q_1) + \delta^4(-k+q_4-q_2)\delta^4(k+q_3-q_1)\Big]$$
$$= (-i)g\cdot 24\times(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)$$
$$+(-i)^2\lambda^2\times 2\times(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)\times 2\Big[\frac{i}{(q_1+q_2)^2-m_1^2+i\varepsilon}+\frac{i}{(q_1-q_3)^2-m_1^2+i\varepsilon}+\frac{i}{(q_1-q_4)^2-m_1^2+i\varepsilon}\Big]$$
$$\Rightarrow \boxed{iM_{fi} = (-ig)\times 24 + (i\lambda)^2\times 4\Big(\frac{i}{s-m_1^2+i\varepsilon}+\frac{i}{t-m_1^2+i\varepsilon}+\frac{i}{u-m_1^2+i\varepsilon}\Big)}$$