Examples of calculations involving scalar particles only

Let's consider the following Lagrangian for two real scalar fields,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{2}\mu^2\sigma^2 + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$
$$- g\phi^4 - g'\sigma^2\phi^2 - g''\sigma^4 - \lambda\sigma\phi^2 - \lambda'\sigma^3$$
$$= \mathcal{L}_0 + \mathcal{L}_{int}$$

where $\mathcal{L}_0$ is the first line, and $\mathcal{L}_{int}$ is the second line.

$\underbrace{g,\ g',\ g''}_{\text{dimensionless real numbers}},\ \underbrace{\lambda \text{ and } \lambda'}_{\text{mass dimension 1. real numbers}}$ are coupling constants.

Note that there could be some other interaction terms between σ and ϕ, e.g., $g'''\sigma\phi^3$, $g''''\sigma^3\phi$, $\lambda''\phi^3$, $\lambda'''\sigma^2\phi$, where g''' and g'''' are dimensionless, λ'' and λ''' are mass dimension 1.

However, let's <u>assume</u> that $\sigma(x)$ is a scalar field, while $\phi(x)$ is a pseudoscalar field, and that $\mathcal{L}$ preserve P-parity (that is, under a parity transformation, $\sigma(t,\vec{x}) \to \sigma(t,-\vec{x}) = +\sigma(t,\vec{x})$, while $\phi(t,\vec{x}) \to \phi(t,-\vec{x}) = -\phi(t,\vec{x})$), so that there is no interaction terms with odd power of ϕ.

Also, let's assume that we only care about terms with coupling constants with mass dimension zero or one.

(Note that although σ and ϕ both behave as scalar field under proper Lorentz transformation, the parity transformation differentiates σ as scalar field and ϕ as pseudoscalar field).

Anyway, for now let's just take the $\mathcal{L}_{int}$ as the given one, and we start the calculations from it.

First, recall that $\mathcal{H}(\Pi_i, \varphi_i) = \sum_i \left(\Pi_i \frac{\partial \varphi_i}{\partial t}\right) - \mathcal{L}$

where the conjugate momentum for a generic field φ_i is

$$\Pi_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial \varphi_i/\partial t)}$$

From $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$,

$$\Rightarrow \mathcal{H} = \sum_i \left(\frac{\partial(\mathcal{L}_0 + \mathcal{L}_{int})}{\partial(\partial\varphi_i/\partial t)} \frac{\partial \varphi_i}{\partial t}\right) - \mathcal{L}_0 - \mathcal{L}_{int}$$

$$= \underline{\sum_i \left(\frac{\partial \mathcal{L}_0}{\partial(\partial\varphi_i/\partial t)} \frac{\partial \varphi_i}{\partial t}\right) - \mathcal{L}_0}$$

$$+ \underbrace{\sum_i \left(\frac{\partial \mathcal{L}_{int}}{\partial(\partial\varphi_i/\partial t)} \frac{\partial \varphi_i}{\partial t}\right) - \mathcal{L}_{int}}$$

$$= \underline{\mathcal{H}_0} + \underbrace{\mathcal{H}_{int}}$$

Since our $\mathcal{L}_{int}$ contains no time derivatives of ϕ and σ, our $\mathcal{H}_{int}$ is just $\mathcal{H}_{int} = -\mathcal{L}_{int} = g\phi^4 + g'\sigma^2\phi^2 + g''\sigma^4 + \lambda\sigma\phi^2 + \lambda'\sigma^3$.

Assume $\mu > 2m$, let's calcute the decay rate for the process $\sigma(k) \longrightarrow \phi(q_1) + \phi(q_2)$, in the rest frame of the decaying particle.

We have already worked out the phase space part, so all we need to know is $|\mathcal{M}|^2 = |\langle f|\mathcal{M}|i\rangle|^2$,

where $|i\rangle$ is the one particle state with momentum k, i.e.,

$f(\vec{k})\, \alpha^\dagger_{\vec{k}}|0\rangle$, where $f(\vec{k}) = \underbrace{C(E_{\vec{k}})}_{\text{real}}(2\pi)^3 2E_{\vec{k}}$.

and $\langle f|$ is $\langle 0|\beta_{\vec{q}_1}\beta_{\vec{q}_2} f(\vec{q}_1) f(\vec{q}_2)$

$$\sigma(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}}) \left[\alpha_{\vec{p}} e^{-ip\cdot x} + \alpha_{\vec{p}}^{+} e^{ip\cdot x} \right] d^3\vec{p}$$

$$\phi(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}}) \left[\beta_{\vec{p}} e^{-ip\cdot x} + \beta_{\vec{p}}^{+} e^{ip\cdot x} \right] d^3\vec{p}$$

$$[\beta_{\vec{p}}, \beta_{\vec{p}'}^{+}] = [\alpha_{\vec{p}}, \alpha_{\vec{p}'}^{+}] = \frac{1}{(2\pi)^3 2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})} \right)^2 \delta^3(\vec{p} - \vec{p}'),$$

all other commutators are zero. Note that $[\alpha_{\vec{p}}, \beta_{\vec{p}'}^{+}] = 0$, $[\alpha_{\vec{p}}, \beta_{\vec{p}'}] = 0$ etc., since σ and ϕ are independent fields.

Recall that $\mathcal{M}$ is just the one in

$$T = (2\pi)^4 \delta^4(\Sigma p_i^{\mu} - \Sigma p_f^{\mu}) \mathcal{M}$$

and

$$\hat{S} = \mathbb{1} + iT$$

and $\hat{S} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n T(H_{int}(x_1) H_{int}(x_2) \cdots H_{int}(x_n))$

(we have dropped the subscript "I" in $H_I^{int}(x)$)

$\sigma(x)$ and $\phi(x)$ fields should be understood as $\sigma_I(x)$ and $\phi_I(x)$, respectively.

$$\Rightarrow iT = \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n T(H_{int}(x_1) H_{int}(x_2) \cdots H_{int}(x_n))$$

$$\langle f | iT | i \rangle = (2\pi)^4 \delta^4(\Sigma p_i^{\mu} - \Sigma p_f^{\mu}) i \langle f | \mathcal{M} | i \rangle$$

$$= \underbrace{\langle f | \sum_{n=1}^{\infty} \frac{1}{n!} (-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n T(H_{int}(x_1) H_{int}(x_2) \cdots H_{int}(x_n)) | i \rangle}$$

So, we just need to evaluate this quantity, factor out $(2\pi)^4 \delta^4(\Sigma p_i^{\mu} - \Sigma p_f^{\mu})$, and the remaining is just $i\langle f | \mathcal{M} | i \rangle \equiv i\mathcal{M}_{fi}$

Let's evaluate

$$* \equiv \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2} C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$

$$\times \Big\{ (-i) \int d^4x : \big(g\phi^4(x) + g'\sigma^2(x)\phi^2(x) + g''\sigma^4(x) + \lambda\sigma(x)\phi^2(x) + \lambda'\sigma^3(x) \big) :$$

$$+ \frac{1}{2!}(-i)^2 \int d^4x\, d^4y\, T\big[:(g\phi^4(x) + \cdots + \lambda'\sigma^3(x)): :(g\phi^4(y) + \cdots + \lambda'\sigma^3(y)): \big]$$

$$+ \cdots \Big\}$$

$$\times \quad C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}} \alpha^{+}_{\vec{k}} | 0 \rangle$$

To the lowest possible order, the non-vanishing contribution to $*$ comes from the term $(-i)\int d^4x\, \lambda\sigma(x)\phi^2(x)$, and let's calculate this contribution only.

$$** \equiv \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2} C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} (-i) \int d^4x : \lambda\sigma(x)\phi^2(x) :$$

$$\times C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}} \alpha^{+}_{\vec{k}} | 0 \rangle.$$

$$= \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2} C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} (-i)\lambda$$

$$\times \int d^4x : \int_{-\infty}^{+\infty} C(E_{\vec{p}_1}) [\alpha_{\vec{p}_1} e^{-ip_1\cdot x} + \alpha^{+}_{\vec{p}_1} e^{ip_1\cdot x}] d^3\vec{p}_1 \int_{-\infty}^{+\infty} C(E_{\vec{p}_2}) [\beta_{\vec{p}_2} e^{-ip_2\cdot x} + \beta^{+}_{\vec{p}_2} e^{ip_2\cdot x}] d^3\vec{p}_2$$

$$\int_{-\infty}^{+\infty} C(E_{\vec{p}_3}) [\beta_{\vec{p}_3} e^{-ip_3\cdot x} + \beta^{+}_{\vec{p}_3} e^{ip_3\cdot x}] d^3\vec{p}_3 : C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}} \alpha^{+}_{\vec{k}} | 0 \rangle$$

$$= C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}$$

$$\times (-i)\lambda \quad \langle 0 | \beta_{\vec{q}_1} \beta_{\vec{q}_2}$$

$$\times \int d^4x \int_{-\infty}^{+\infty} d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p}_3\, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) \left(e^{ip_2\cdot x} e^{ip_3\cdot x} e^{-ip_1\cdot x} \right)$$

$$\times \quad \beta^{+}_{\vec{p}_2} \beta^{+}_{\vec{p}_3} \alpha_{\vec{p}_1}$$

$$\times \quad \alpha^{+}_{\vec{k}} | 0 \rangle$$

using $\langle 0| \beta_{\vec{q}_1}\beta_{\vec{q}_2}\ \beta^+_{\vec{p}_2}\beta^+_{\vec{p}_3}\alpha_{\vec{p}_1}\alpha^+_{\vec{k}}|0\rangle$

$$= \langle 0| \beta_{\vec{q}_1}\beta_{\vec{q}_2}\beta^+_{\vec{p}_2}\beta^+_{\vec{p}_3}|0\rangle \ \langle 0|\alpha_{\vec{p}_1}\alpha^+_{\vec{k}}|0\rangle$$

where $\langle 0| \beta_{\vec{q}_1}\beta_{\vec{q}_2}\beta^+_{\vec{p}_2}\beta^+_{\vec{p}_3}|0\rangle$

$$= \langle 0|\beta_{\vec{q}_1}\beta^+_{\vec{p}_3}|0\rangle \frac{1}{(2\pi)^3 2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2-\vec{p}_2)$$

$$+ \langle 0|\beta_{\vec{q}_1}\beta^+_{\vec{p}_2}\beta_{\vec{q}_2}\beta^+_{\vec{p}_3}|0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}}\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{q}_1-\vec{p}_3)\frac{1}{(2\pi)^3 2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2-\vec{p}_2)$$

$$+ \frac{1}{(2\pi)^3 2E_{\vec{q}_1}}\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{q}_1-\vec{p}_2)\frac{1}{(2\pi)^3 2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2-\vec{p}_3)$$

and $\langle 0|\alpha_{\vec{p}_1}\alpha^+_{\vec{k}}|0\rangle$

$$= \frac{1}{(2\pi)^3 2E_{\vec{k}}}\left(\frac{1}{C(E_{\vec{k}})}\right)^2 \delta^3(\vec{k}-\vec{p}_1)$$

$$\Rightarrow ** \underset{\uparrow}{=} (-i)\lambda \int d^4x \left[e^{iq_2\cdot x}e^{iq_1\cdot x}e^{-ik\cdot x} + e^{iq_1\cdot x}e^{iq_2\cdot x}e^{-ik\cdot x}\right]$$

note that e.g., $\int_{-\infty}^{+\infty} d^3\vec{p}_1\, e^{-ip_1\cdot x}\,\delta^3(\vec{k}-\vec{p}_1) = \int_{-\infty}^{+\infty} d^3\vec{p}_1\, e^{-ip_1^0\cdot x^0 + i\vec{p}_1\cdot\vec{x}}\,\delta^3(\vec{k}-\vec{p}_1)$

$$\underset{\uparrow}{=} e^{-i(|\vec{k}|^2+\mu^2)^{\frac{1}{2}}\cdot x^0}e^{i\vec{k}\cdot\vec{x}} \underset{\uparrow}{=} e^{-ik\cdot x}$$

since $p_1^0 = (|\vec{p}_1|^2+\mu^2)^{\frac{1}{2}}$ since $k^0 = (|\vec{k}|^2+\mu^2)^{\frac{1}{2}}$

$$\Rightarrow ** = (-i)\lambda\ (2\pi)^4(k-q_1-q_2)\ 2$$

$$\Rightarrow\ i\mathcal{M}_{fi} = 2(-i)\lambda$$

$$\Rightarrow\ |\mathcal{M}_{fi}|^2 = |\langle f|\mathcal{M}|i\rangle|^2 = |i\mathcal{M}_{fi}|^2 = 4\lambda^2$$

$$\Rightarrow\ \left(\frac{d\Gamma}{d\Omega}\right)_{cm} = \frac{\lambda^{\frac{1}{2}}(\mu^2, m^2, m^2)}{64\pi^2\mu^3}4\lambda^2\frac{1}{2!} = \frac{(\mu^4+2m^4-4m^2\mu^2-2m^4)^{\frac{1}{2}}}{64\pi^2\mu^3}4\lambda^2\frac{1}{2}$$

$$\Rightarrow \Gamma = \frac{1}{64\pi^2\mu}\left[1-\left(\frac{2m}{\mu}\right)^2\right]^{\frac{1}{2}} 4\lambda^2 \cdot 4\pi \, \frac{1}{2}$$

$$= \frac{\lambda^2}{8\pi\mu}\left[1-\left(\frac{2m}{\mu}\right)^2\right]^{\frac{1}{2}}$$

Now consider that the $\mathcal{L}$ is enlarged to include complex scalar,

$$\mathcal{L} = \mathcal{L}_{\text{page 72}} + \partial_\mu \varphi^+ \partial^\mu \varphi - m^2 \varphi^+ \varphi$$
$$-4g(\varphi^+\varphi)^2 - 4g\,\varphi^+\varphi\,\phi^2 - 2g'\sigma^2\varphi^+\varphi$$
$$-2\lambda\sigma\,\varphi^+\varphi$$

Note that we can build this Lagrangian by a real triplet ϕ_a, $(a=1,2,3)$, and identify $\underbrace{\varphi \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \Rightarrow \varphi^+}_{} \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\phi = \phi_3$

$$\Rightarrow \phi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^+)$$
$$\phi_2 = \frac{i}{\sqrt{2}}(\varphi - \varphi^+)$$

$$\Rightarrow \frac{1}{2}\sum_{a=1}^{3}(\partial_\mu\phi_a\,\partial^\mu\phi_a) = \frac{1}{2}\partial_\mu\phi_3\,\partial^\mu\phi_3 + \frac{1}{2}\partial_\mu\left[\frac{1}{\sqrt{2}}(\varphi+\varphi^+)\right]\partial^\mu\left[\frac{1}{\sqrt{2}}(\varphi+\varphi^+)\right]$$
$$+\frac{1}{2}\partial_\mu\left[\frac{i}{\sqrt{2}}(\varphi-\varphi^+)\right]\partial^\mu\left(\frac{i}{\sqrt{2}}(\varphi-\varphi^+)\right)$$
$$= \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\varphi^+\,\partial^\mu\varphi$$

$$-\frac{1}{2}m^2\sum_{a=1}^{3}\phi_a^2 = -\frac{1}{2}m^2\phi_3^2 - \frac{1}{2}m^2(\phi_1^2+\phi_2^2)$$
$$= -\frac{1}{2}m^2\phi^2 - m^2\varphi^+\varphi$$

$$-g\left(\sum_{a=1}^{3}\phi_a^2\right)^2 = -g(\phi_1^2+\phi_2^2+\phi_3^2)^2 = -g(2\varphi^+\varphi+\phi_3^2)^2$$
$$= -g\phi^4 - 4g(\varphi^+\varphi)^2 - 4g(\varphi^+\varphi)\phi^2$$

$$-\lambda\sigma\sum_{a=1}^{3}\phi_a^2 = -\lambda\sigma(\phi_1^2+\phi_2^2+\phi_3^2) = -\lambda\sigma\phi^2 - 2\lambda\sigma\varphi^+\varphi$$
$$-g'\sigma^2\sum_{a=1}^{3}\phi_a^2 = -g'\sigma^2\phi^2 - 2g'\sigma^2\varphi^+\varphi$$

Again, since $\mathcal{L}_{int}$ (the new one include φ & φ^+) contains no time derivatives of ϕ, σ, φ & φ^+, $\mathcal{H}_{int}$ is just $-\mathcal{L}_{int}$.

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = g\phi^4 + g'\sigma^2\phi^2 + g''\sigma^4 + \lambda\sigma\phi^2 + \lambda'\sigma^3 + 4g(\varphi^+\varphi)^2 + 4g\,\varphi^+\varphi\,\phi^2$$
$$+2g'\sigma^2\varphi^+\varphi + 2\lambda\sigma\varphi^+\varphi$$

Let's calculate $\sigma(k) \to \pi^+(q_1)\,\pi^-(q_2)$

Again, let's only calculate to the lowest order.

$$|i\rangle = C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}} |0\rangle$$

The complex scalar field φ is

$$\varphi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a_{\vec{p}} e^{-ip\cdot x} + b^+_{\vec{p}} e^{ip\cdot x} \right]$$

$$\varphi^+(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left[a^+_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x} \right]$$

$$\langle f| = \langle 0| a_{\vec{q}_1} b_{\vec{q}_2}\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$

Note that we have assigned $C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}}\, a^+_{\vec{p}}|0\rangle$ as the one-particle state for $\pi^+(p)$, i.e., $|\pi^+(p)\rangle$, and $C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}} b^+_{\vec{p}}|0\rangle$ as the one-particle state for $\pi^-(p)$, i.e., $|\pi^-(p)\rangle$

$$\langle f|iT|i\rangle = \langle 0| a_{\vec{q}_1} b_{\vec{q}_2}\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$
$$\times \left\{ -i \int d^4x :2\lambda\, \sigma(x) \varphi^+(x)\varphi(x): \right\}$$
$$\times\ C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}}|0\rangle$$
$$= \langle 0| a_{\vec{q}_1} b_{\vec{q}_2}\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}$$
$$\times(-i)(2\lambda)\int d^4x :\int_{-\infty}^{+\infty} d^3\vec{P}_1\, C(E_{\vec{P}_1})\, \alpha_{\vec{P}_1} e^{-iP_1\cdot x} \int_{-\infty}^{+\infty} d^3\vec{P}_2\, C(E_{\vec{P}_2})\, a^+_{\vec{P}_2} e^{iP_2\cdot x}$$
$$\times \int_{-\infty}^{+\infty} d^3\vec{P}_3\, C(E_{\vec{P}_3})\, b^+_{\vec{P}_3} e^{iP_3\cdot x} :$$
$$\times\ C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, \alpha^+_{\vec{k}}|0\rangle$$

where

$$\langle 0| a_{\vec{q}_1} b_{\vec{q}_2}\, a^+_{\vec{P}_2} b^+_{\vec{P}_3}\, \alpha_{\vec{P}_1}\, \alpha^+_{\vec{k}} |0\rangle$$
$$= \langle 0| a_{\vec{q}_1} a^+_{\vec{P}_2}|0\rangle \times \langle 0| b_{\vec{q}_2} b^+_{\vec{P}_3}|0\rangle \times \langle 0|\alpha_{\vec{P}_1}\alpha^+_{\vec{k}}|0\rangle$$
$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta^3(\vec{q}_1 - \vec{P}_2)\, \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{q}_2 - \vec{P}_3)\, \frac{1}{(2\pi)^3 2E_{\vec{k}}} \left(\frac{1}{C(E_{\vec{k}})}\right)^2 \delta^3(\vec{P}_1 - \vec{k})$$

$$\Rightarrow \langle f|iT|i\rangle = (-i)(2\lambda)\int d^4x\, e^{-ik\cdot x} e^{iq_1\cdot x} e^{iq_2\cdot x}$$
$$= (-i)(2\lambda)(2\pi)^4 \delta^4(k - q_1 - q_2)$$

$$\Rightarrow iM_{fi} = (-i)2\lambda$$

$$\Rightarrow \Gamma = \frac{\lambda^2}{4\pi\mu}\left[1-\left(\frac{2m}{\mu}\right)^2\right]^{\frac{1}{2}}$$

So $$\frac{\Gamma_{\sigma\to\pi^+\pi^-}}{\Gamma_{\sigma\to\phi\phi}} = 2$$

Now consider scattering process to the second order in g's and λ's

$$\pi^+(q_1) + \pi^-(q_2) \longrightarrow \pi^+(q_3) + \pi^-(q_4)$$

$$\langle f| = \langle 0| C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}\, a_{\vec{q}_3} b_{\vec{q}_4}$$

$$|i\rangle = C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^\dagger_{\vec{q}_1} b^\dagger_{\vec{q}_2}|0\rangle$$

$$\Rightarrow \langle f|iT|i\rangle = \langle 0| a_{\vec{q}_3} b_{\vec{q}_4}\, C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

$$\times\Big\{(-i)\int d^4x : (g\phi^4(x) + g'\sigma^2(x)\phi^2(x) + g''\sigma^4(x) + \lambda\sigma(x)\phi^2(x) + 4g(\varphi^\dagger(x)\varphi(x))^2$$
$$+ 4g\varphi^\dagger(x)\varphi(x)\phi^2(x) + 2g'\sigma^2(x)\varphi^\dagger(x)\varphi(x) + 2\lambda\sigma(x)\varphi^\dagger(x)\varphi(x)):$$
$$+\frac{1}{2!}(-i)^2\int d^4x\, d^4y\, T\big[:(g\phi^4(x) + \cdots + 2\lambda\sigma(x)\varphi^\dagger(x)\varphi(x)):$$
$$:(g\phi^4(y) + \cdots + 2\lambda\sigma(y)\varphi^\dagger(y)\varphi(y)):\big]\Big\}$$

$$\times\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^\dagger_{\vec{q}_1} b^\dagger_{\vec{q}_2}|0\rangle$$

In the $\{\ \}$, the contribution to the process we are considering are

$$(-i)\int d^4x : 4g(\varphi^\dagger(x)\varphi(x))^2 :$$

$$+\frac{1}{2!}(-i)^2\int d^4x\, d^4y\, T\big[:(4g\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)):(4g\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y)):$$
$$+ :(4g\varphi^\dagger(x)\varphi(x)\phi(x)\phi(x) + 2g'\sigma(x)\sigma(x)\varphi^\dagger(x)\varphi(x)$$
$$+ 2\lambda\sigma(x)\varphi^\dagger(x)\varphi(x)):$$
$$:(4g\varphi^\dagger(y)\varphi(y)\phi(y)\phi(y) + 2g'\sigma(y)\sigma(y)\varphi^\dagger(y)\varphi(y)$$
$$+ 2\lambda\sigma(y)\varphi^\dagger(y)\varphi(y)):\big]$$

Note that the terms like $T[:g\phi^4(x)::4g(\varphi^+(y)\varphi(y))^2:]$ does not contribute because although we need a φ^+ to kill the $a_{\vec{q}_3}$ in $\langle f|$, a φ to kill $b_{\vec{q}_4}$ in $\langle f|$, a φ to kill $a^+_{\vec{q}_1}$ in $|i\rangle$, a φ^+ to kill $b^+_{\vec{q}_2}$ in $|i\rangle$, we do not consider equal time contraction, we cannot get rid of $:g\phi^4(x):$

For the first term,

$$* \equiv \langle f|(-i)\int d^4x :4g\,\varphi^+(x)\varphi(x)\varphi^+(x)\varphi(x):|i\rangle$$

$$= (-i)\cdot 4g \langle 0| C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}\, a_{\vec{q}_3} b_{\vec{q}_4}$$

$$\times \int d^4x \int_{-\infty}^{+\infty} d^3\vec{P}_1 C(E_{\vec{P}_1}) d^3\vec{P}_2 C(E_{\vec{P}_2}) d^3\vec{P}_3 C(E_{\vec{P}_3}) d^3\vec{P}_4 C(E_{\vec{P}_4})$$

$$\times :(a^+_{\vec{P}_1} e^{iP_1\cdot x} + b_{\vec{P}_1} e^{-iP_1\cdot x})(a_{\vec{P}_2} e^{-iP_2\cdot x} + b^+_{\vec{P}_2} e^{iP_2\cdot x})$$

$$(a^+_{\vec{P}_3} e^{iP_3\cdot x} + b_{\vec{P}_3} e^{-iP_3\cdot x})(a_{\vec{P}_4} e^{-iP_4\cdot x} + b^+_{\vec{P}_4} e^{iP_4\cdot x}):$$

$$\times C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

where

$$\langle 0|a_{\vec{q}_3} b_{\vec{q}_4} :(a^+_{\vec{P}_1} e^{iP_1\cdot x} + b_{\vec{P}_1} e^{-iP_1\cdot x})(a_{\vec{P}_2} e^{-iP_2\cdot x} + b^+_{\vec{P}_2} e^{iP_2\cdot x})$$

$$(a^+_{\vec{P}_3} e^{iP_3\cdot x} + b_{\vec{P}_3} e^{-iP_3\cdot x})(a_{\vec{P}_4} e^{-iP_4\cdot x} + b^+_{\vec{P}_4} e^{iP_4\cdot x}):$$

$$\times a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

$$= \langle 0| a_{\vec{q}_3} b_{\vec{q}_4} \Big(a^+_{\vec{P}_1} e^{iP_1\cdot x} b^+_{\vec{P}_2} e^{iP_2\cdot x} b_{\vec{P}_3} e^{-iP_3\cdot x} a_{\vec{P}_4} e^{-iP_4\cdot x}$$

$$+ a^+_{\vec{P}_1} e^{iP_1\cdot x} b^+_{\vec{P}_4} e^{iP_4\cdot x} a_{\vec{P}_2} e^{-iP_2\cdot x} b_{\vec{P}_3} e^{-iP_3\cdot x}$$

$$+ a^+_{\vec{P}_3} e^{iP_3\cdot x} b^+_{\vec{P}_2} e^{iP_2\cdot x} b_{\vec{P}_1} e^{-iP_1\cdot x} a_{\vec{P}_4} e^{-iP_4\cdot x}$$

$$+ a^+_{\vec{P}_3} e^{iP_3\cdot x} b^+_{\vec{P}_4} e^{iP_4\cdot x} b_{\vec{P}_1} e^{-iP_1\cdot x} a_{\vec{P}_2} e^{-iP_2\cdot x} \Big)\, a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}}\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}}\left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}}\left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big(\delta^3(\vec{P}_1-\vec{q}_3)\delta^3(\vec{P}_2-\vec{q}_4)\delta^3(\vec{P}_3-\vec{q}_2)\delta^3(\vec{P}_4-\vec{q}_1) e^{i(P_1+P_2-P_3-P_4)\cdot x}$$

$$+ (P_2 \leftrightarrow P_4) + (P_1 \leftrightarrow P_3) + (P_1 \leftrightarrow P_3,\ P_2 \leftrightarrow P_4) \Big)$$

$$\Rightarrow \; * = (-i)\, 4g\, 4\, (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)$$

For the second term, we need to first use the Wick's theorem to contract out the fields not needed to kill the initial and final state operators.

$$** \equiv \langle f | \frac{1}{2!}(-i)^2 \int d^4x\, d^4y\; T[:\,:\,:\,:] | i \rangle$$

$$
\begin{aligned}
&= \frac{1}{2!}(-i)^2 \langle f | (4g)(4g) \Big(:\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y): \\
&\qquad + :\varphi^\dagger(x)\varphi(x)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\varphi^\dagger(y)\varphi(y):
\end{aligned}
$$

$$+ :\varphi^{+}(x)\varphi(x)\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y)\varphi^{+}(y)\varphi(y):$$

$$+ :\varphi^{+}(x)\varphi(x)\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y)\varphi^{+}(y)\varphi(y):$$

$$+ :\varphi^{+}(x)\varphi(x)\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y)\varphi^{+}(y)\varphi(y):$$

$$+ :\varphi^{+}(x)\varphi(x)\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y)\varphi^{+}(y)\varphi(y):\Big)$$

$$+ (4g)(4g)\Big(:\varphi^{+}(x)\varphi(x)\phi(x)\phi(x)\varphi^{+}(y)\varphi(y)\phi(y)\phi(y):$$

$$+ :\varphi^{+}(x)\varphi(x)\phi(x)\phi(x)\varphi^{+}(y)\varphi(y)\phi(y)\phi(y):\Big)$$

$$+ (2g')(2g')\Big(:\sigma(x)\sigma(x)\varphi^{+}(x)\varphi(x)\sigma(y)\sigma(y)\varphi^{+}(y)\varphi(y):$$

$$+ :\sigma(x)\sigma(x)\varphi^{+}(x)\varphi(x)\sigma(y)\sigma(y)\varphi^{+}(y)\varphi(y):\Big)$$

$$+ (2\lambda)(2\lambda)\Big(:\sigma(x)\varphi^{+}(x)\varphi(x)\sigma(y)\varphi^{+}(y)\varphi(y):\Big)$$

$$|i\rangle$$

using
$$\underbracket{\varphi(x)\varphi^{+}(y)} = \underbracket{\varphi^{+}(x)\varphi(y)} = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-m^2+i\varepsilon}$$

$$\underbracket{\phi(x)\phi(y)} = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-m_3^2+i\varepsilon}$$, note that $m_3=m$.

$$\underbracket{\sigma(x)\sigma(y)} = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-\mu^2+i\varepsilon}$$

$$\Rightarrow ** = \frac{1}{2!}(-i)^2\langle f|\int d^4x\,d^4y\Big\{(4g)^2\Big[16\,\underbracket{\varphi^{+}(x)\varphi(y)}\,\underbracket{\varphi(x)\varphi^{+}(y)}\,:\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):$$

$$+2\,\underbracket{\varphi(x)\varphi^{+}(y)}\,\underbracket{\varphi(x)\varphi^{+}(y)}\,:\varphi^{+}(x)\varphi^{+}(x)\varphi(y)\varphi(y):$$

$$+2\,\underbracket{\varphi^{+}(x)\varphi(y)}\,\underbracket{\varphi^{+}(x)\varphi(y)}\,:\varphi(x)\varphi(x)\varphi^{+}(y)\varphi^{+}(y):\Big]$$

$$+(4g)^2\,2\,\underbracket{\phi(x)\phi(y)}\,\underbracket{\phi(x)\phi(y)}\,:\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):$$

$$+(2g')^2\,2\,\underbracket{\sigma(x)\sigma(y)}\,\underbracket{\sigma(x)\sigma(y)}\,:\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):$$

$$+(2\lambda)^2\,\underbracket{\sigma(x)\sigma(y)}\,:\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):\Big\}$$

$$|i\rangle$$

use $\langle f | :\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):|i\rangle$

$$= \langle 0 | C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}\, a_{\vec{q}_3}\, b_{\vec{q}_4}$$

$$\times \int_{-\infty}^{+\infty} d^3\vec{P}_1\, C(E_{\vec{P}_1})\, d^3\vec{P}_2\, C(E_{\vec{P}_2})\, d^3\vec{P}_3\, C(E_{\vec{P}_3})\, d^4\vec{P}_3\, C(E_{\vec{P}_4})$$

$$\times :(a^{+}_{\vec{P}_1} e^{iP_1\cdot x} + b_{\vec{P}_1} e^{-iP_1\cdot x})(a_{\vec{P}_2} e^{-iP_2\cdot x} + b^{+}_{\vec{P}_2} e^{iP_2\cdot x})$$

$$(a^{+}_{\vec{P}_3} e^{iP_3\cdot y} + b_{\vec{P}_3} e^{-iP_3\cdot y})(a_{\vec{P}_4} e^{-iP_4\cdot y} + b^{+}_{\vec{P}_4} e^{iP_4\cdot y}):$$

$$\times\, C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, a^{+}_{\vec{q}_1} b^{+}_{\vec{q}_2} |0\rangle$$

where $\langle 0| a_{\vec{q}_3} b_{\vec{q}_4} :(a^{+}_{\vec{P}_1} e^{iP_1\cdot x} + b_{\vec{P}_1} e^{-iP_1\cdot x})(a_{\vec{P}_2} e^{-iP_2\cdot x} + b^{+}_{\vec{P}_2} e^{iP_2\cdot x})$

$$(a^{+}_{\vec{P}_3} e^{iP_3\cdot y} + b_{\vec{P}_3} e^{-iP_3\cdot y})(a_{\vec{P}_4} e^{-iP_4\cdot y} + b^{+}_{\vec{P}_4} e^{iP_4\cdot y}):\, a^{+}_{\vec{q}_1} b^{+}_{\vec{q}_2} |0\rangle$$

$$= \langle 0 | a_{\vec{q}_3} b_{\vec{q}_4} \big(a^{+}_{\vec{P}_1} e^{iP_1\cdot x} b^{+}_{\vec{P}_2} e^{iP_2\cdot x}\, b_{\vec{P}_3} e^{-iP_3\cdot y} a_{\vec{P}_4} e^{-iP_4\cdot y}$$

$$+ a^{+}_{\vec{P}_1} e^{iP_1\cdot x} b^{+}_{\vec{P}_4} e^{iP_4\cdot y}\, a_{\vec{P}_2} e^{-iP_2\cdot x} b_{\vec{P}_3} e^{-iP_3\cdot y}$$

$$+ a^{+}_{\vec{P}_3} e^{iP_3\cdot y} b^{+}_{\vec{P}_2} e^{iP_2\cdot x}\, b_{\vec{P}_1} e^{-iP_1\cdot x} a_{\vec{P}_4} e^{-iP_4\cdot y}$$

$$+ a^{+}_{\vec{P}_3} e^{iP_3\cdot y} b^{+}_{\vec{P}_4} e^{iP_4\cdot y}\, b_{\vec{P}_1} e^{-iP_1\cdot x} a_{\vec{P}_2} e^{-iP_2\cdot x} \big)\, a^{+}_{\vec{q}_1} b^{+}_{\vec{q}_2} |0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}} \left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big(\delta^3(\vec{P}_1-\vec{q}_3)\,\delta^3(\vec{P}_2-\vec{q}_4)\,\delta^3(\vec{P}_3-\vec{q}_2)\,\delta^3(\vec{P}_4-\vec{q}_1)\, e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y}$$

$$+ \delta^3(\vec{P}_1-\vec{q}_3)\,\delta^3(\vec{P}_4-\vec{q}_4)\,\delta^3(\vec{P}_3-\vec{q}_2)\,\delta^3(\vec{P}_2-\vec{q}_1)\, e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ \delta^3(\vec{P}_3-\vec{q}_3)\,\delta^3(\vec{P}_2-\vec{q}_4)\,\delta^3(\vec{P}_1-\vec{q}_2)\,\delta^3(\vec{P}_4-\vec{q}_1)\, e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y}$$

$$+ \delta^3(\vec{P}_3-\vec{q}_3)\,\delta^3(\vec{P}_4-\vec{q}_4)\,\delta^3(\vec{P}_1-\vec{q}_2)\,\delta^3(\vec{P}_2-\vec{q}_1)\, e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y} \Big)$$

$\Rightarrow \langle f | :\varphi^{+}(x)\varphi(x)\varphi^{+}(y)\varphi(y):|i\rangle$

$$= e^{i(q_3+q_4)\cdot x} e^{-i(q_1+q_2)\cdot y} + e^{i(q_3-q_1)\cdot x} e^{i(q_4-q_2)\cdot y}$$

$$+ e^{i(q_4-q_2)\cdot x} e^{i(q_3-q_1)\cdot y} + e^{-i(q_1+q_2)\cdot x} e^{i(q_3+q_4)\cdot y}$$

and for $\langle f| : \varphi^+(x)\,\varphi^+(x)\,\varphi(y)\,\varphi(y) : |i\rangle$

in which $\langle 0| a_{\vec{q}_3} b_{\vec{q}_4} : (a^+_{\vec{p}_1} e^{ip_1\cdot x} + b_{\vec{p}_1} e^{-ip_1\cdot x})(a^+_{\vec{p}_2} e^{ip_2\cdot x} + b_{\vec{p}_2} e^{-ip_2\cdot x})$

$(a_{\vec{p}_3} e^{-ip_3\cdot y} + b^+_{\vec{p}_3} e^{ip_3\cdot y})(a_{\vec{p}_4} e^{-ip_4\cdot y} + b^+_{\vec{p}_4} e^{ip_4\cdot y}) : a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$

$$= \langle 0| a_{\vec{q}_3} b_{\vec{q}_4} \Big(a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_3} e^{ip_3\cdot y} b_{\vec{p}_2} e^{-ip_2\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot y}$$

$$+ a^+_{\vec{p}_1} e^{ip_1\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot y} b_{\vec{p}_2} e^{-ip_2\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot y}$$

$$+ a^+_{\vec{p}_2} e^{ip_2\cdot x} b^+_{\vec{p}_3} e^{ip_3\cdot y} b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_4} e^{-ip_4\cdot y}$$

$$+ a^+_{\vec{p}_2} e^{ip_2\cdot x} b^+_{\vec{p}_4} e^{ip_4\cdot y} b_{\vec{p}_1} e^{-ip_1\cdot x} a_{\vec{p}_3} e^{-ip_3\cdot y} \Big) a^+_{\vec{q}_1} b^+_{\vec{q}_2}|0\rangle$$

$$= \frac{1}{(2\pi)^3 2E_{\vec{q}_1}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_3}} \left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \frac{1}{(2\pi)^3 2E_{\vec{q}_4}} \left(\frac{1}{C(E_{\vec{q}_4})}\right)^2$$

$$\times \Big(\delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_1)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_3)\delta^3(\vec{p}_2-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_4)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_3-\vec{q}_4)\delta^3(\vec{p}_4-\vec{q}_1)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

$$+ \delta^3(\vec{p}_1-\vec{q}_2)\delta^3(\vec{p}_2-\vec{q}_3)\delta^3(\vec{p}_3-\vec{q}_1)\delta^3(\vec{p}_4-\vec{q}_4)\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y} \Big)$$

$\Longrightarrow \langle f| : \varphi^+(x)\varphi^+(x)\varphi(y)\varphi(y) : |i\rangle$

$$= 4\, e^{i(q_3-q_2)\cdot x} e^{i(q_4-q_1)\cdot y}$$

so, $\langle f| : \varphi(x)\varphi(x)\varphi^+(y)\varphi^+(y) : |i\rangle$

$$= \langle f| : \varphi^+(y)\varphi^+(y)\varphi(x)\varphi(x) : |i\rangle$$

$$= 4\, e^{i(q_3-q_2)\cdot y} e^{i(q_4-q_1)\cdot x}$$

$$\Rightarrow ** = \frac{1}{2!}(-i)^2\Bigg\{(4g)^2\int d^4x\, d^4y\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-m^2+i\varepsilon}\int_{-\infty}^{+\infty}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-m^2+i\varepsilon}$$

$$\times\Big[16\big(e^{i(q_3+q_4)\cdot x}e^{-i(q_1+q_2)\cdot y}+e^{i(q_3-q_1)\cdot x}e^{i(q_4-q_2)\cdot y}$$

$$+e^{i(q_4-q_2)\cdot x}e^{i(q_3-q_1)\cdot y}+e^{-i(q_1+q_2)\cdot x}e^{i(q_3+q_4)\cdot y}\big)$$

$$+2\times 4\, e^{i(q_3-q_2)\cdot x}e^{i(q_4-q_1)\cdot y}$$

$$+2\times 4\, e^{i(q_3-q_2)\cdot y}e^{i(q_4-q_1)\cdot x}\Big]$$

$$+(4g)^2\times 2\int d^4x\, d^4y\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-m_3^2+i\varepsilon}\int_{-\infty}^{+\infty}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-m_3^2+i\varepsilon}$$

$$\times\big(e^{i(q_3+q_4)\cdot x}e^{-i(q_1+q_2)\cdot y}+e^{i(q_3-q_1)\cdot x}e^{i(q_4-q_2)\cdot y}$$

$$+e^{i(q_4-q_2)\cdot x}e^{i(q_3-q_1)\cdot y}+e^{-i(q_1+q_2)\cdot x}e^{i(q_3+q_4)\cdot y}\big)$$

$$+(2g')^2\times 2\int d^4x\, d^4y\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-\mu^2+i\varepsilon}\int_{-\infty}^{+\infty}\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-\mu^2+i\varepsilon}$$

$$\times\big(e^{i(q_3+q_4)\cdot x}e^{-i(q_1+q_2)\cdot y}+e^{i(q_3-q_1)\cdot x}e^{i(q_4-q_2)\cdot y}$$

$$+e^{i(q_4-q_2)\cdot x}e^{i(q_3-q_1)\cdot y}+e^{-i(q_1+q_2)\cdot x}e^{i(q_3+q_4)\cdot y}\big)$$

$$+(2\lambda)^2\int d^4x\, d^4y\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i}{k^2-\mu^2+i\varepsilon}$$

$$\times\big(e^{i(q_3+q_4)\cdot x}e^{-i(q_1+q_2)\cdot y}+e^{i(q_3-q_1)\cdot x}e^{i(q_4-q_2)\cdot y}$$

$$+e^{i(q_4-q_2)\cdot x}e^{i(q_3-q_1)\cdot y}+e^{-i(q_1+q_2)\cdot x}e^{i(q_3+q_4)\cdot y}\big)\Bigg\}$$

using $\displaystyle\int d^4x\, d^4y\,\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{-ik\cdot(x-y)}e^{-ip\cdot(x-y)}\frac{i}{k^2-m_1^2+i\varepsilon}\,\frac{i}{p^2-m_2^2+i\varepsilon}\,e^{ir\cdot x}e^{is\cdot y}$

$$=\int d^4x\, d^4y\,\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{-i(k+p-r)\cdot x}e^{-i(-k-p-s)\cdot y}\frac{i}{k^2-m_1^2+i\varepsilon}\,\frac{i}{p^2-m_2^2+i\varepsilon}$$

$$=\int d^4k\, d^4p\,\delta^4(k+p-r)\,\delta^4(k+p+s)\frac{i}{k^2-m_1^2+i\varepsilon}\,\frac{i}{p^2-m_2^2+i\varepsilon}$$

$$=\int d^4k\,\delta^4(k+r-k+s)\frac{i}{k^2-m_1^2+i\varepsilon}\,\frac{i}{(r-k)^2-m_2^2+i\varepsilon}$$

$$=(2\pi)^4\delta^4(r+s)\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\varepsilon}\,\frac{i}{(r-k)^2-m_2^2+i\varepsilon}=(2\pi)^4\delta^4(r+s)\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_1^2+i\varepsilon}\,\frac{i}{(r+k)^2-m_2^2+i\varepsilon}$$

$$=(2\pi)^4\delta^4(r+s)\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}\ \cdots$$

and $\int d^4x\, d^4y \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon} e^{ir\cdot x} e^{is\cdot y}$

$$= \int d^4x\, d^4y \frac{d^4k}{(2\pi)^4} e^{-i(k-r)\cdot x} e^{-i(-k-s)\cdot y} \frac{i}{k^2-m^2+i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} (2\pi)^4 (2\pi)^4 \cdot \delta^4(k-r)\, \delta^4(k+s) \frac{i}{k^2-m^2+i\varepsilon}$$

$$= (2\pi)^4 \delta^4(r+s) \frac{i}{r^2-m^2+i\varepsilon}$$

$$= (2\pi)^4 \delta^4(r+s) \frac{i}{s^2-m^2+i\varepsilon}$$

$$\Rightarrow ** = \frac{1}{2!}(-i)^2 (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)$$

$$\times \Big\{ (4g)^2 \Big[16 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \Big(\frac{i}{(q_3+q_4-k)^2-m^2+i\varepsilon} + \frac{i}{(q_3-q_1-k)^2-m^2+i\varepsilon}$$

$$+ \frac{i}{(q_4-q_2-k)^2-m^2+i\varepsilon} + \frac{i}{(-q_1-q_2-k)^2-m^2+i\varepsilon} \Big)$$

$$+ 2\times 4 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(q_3-q_2-k)^2-m^2+i\varepsilon}$$

$$+ 2\times 4 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(q_4-q_1-k)^2-m^2+i\varepsilon} \Big]$$

$$+ (4g)^2 \times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_3^2+i\varepsilon} \Big(\frac{i}{(q_3+q_4-k)^2-m_3^2+i\varepsilon} + \frac{i}{(q_3-q_1-k)^2-m_3^2+i\varepsilon}$$

$$+ \frac{i}{(q_4-q_2-k)^2-m_3^2+i\varepsilon} + \frac{i}{(-q_1-q_2-k)^2-m_3^2+i\varepsilon} \Big)$$

$$+ (2g')^2 \times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \Big(\frac{i}{(q_3+q_4-k)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1-k)^2-\mu^2+i\varepsilon}$$

$$+ \frac{i}{(q_4-q_2-k)^2-\mu^2+i\varepsilon} + \frac{i}{(-q_1-q_2-k)^2-\mu^2+i\varepsilon} \Big)$$

$$+ (2\lambda)^2 \Big(\frac{i}{(q_3+q_4)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon} + \frac{i}{(q_4-q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(-q_1-q_2)^2-\mu^2+i\varepsilon} \Big) \Big\}$$

$$= \frac{1}{2!}(-i)^2 (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)$$

$$\times \Big\{ (4g)^2 \Big[16\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon}\left(\frac{i}{(q_1+q_2+k)^2-m^2+i\varepsilon} + \frac{i}{(q_3-q_1+k)^2-m^2+i\varepsilon}\right)$$

$$+2\times 4\times 2 \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(q_3-q_2+k)^2-m^2+i\varepsilon}\Big]$$

$$+(4g)^2\times 2\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_3^2+i\varepsilon}\left(\frac{i}{(q_1+q_2+k)^2-m_3^2+i\varepsilon} + \frac{i}{(q_3-q_1+k)^2-m_3^2+i\varepsilon}\right)$$

$$+(2g')^2\times 2\times 2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon}\left(\frac{i}{(q_1+q_2+k)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1+k)^2-\mu^2+i\varepsilon}\right)$$

$$+(2\lambda)^2\times 2\left(\frac{i}{(q_1+q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon}\right)\Big\}$$

Now, let's only calculate the cross section up to the lowest order of g & λ, that is, we only keep the terms of $*$ and the last line in $**$.

So, $iM_{fi} \equiv i\langle f|M|i\rangle$

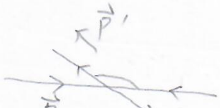
$$= (-i)\,4g\times 4 + \frac{1}{2!}(-i)^2(2\lambda)^2\times 2\left(\frac{i}{(q_1+q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon}\right)$$

$$= 4\times(-i4g) + (-i2\lambda)^2\left(\frac{i}{(q_1+q_2)^2-\mu^2+i\varepsilon} + \frac{i}{(q_3-q_1)^2-\mu^2+i\varepsilon}\right)$$

Using $s \equiv (q_1+q_2)^2$, $t = (q_1-q_3)^2$

$$\Rightarrow |M_{fi}|^2 = (iM_{fi})(iM_{fi})^* \underset{\varepsilon\to 0}{=} \left[16g + (2\lambda)^2\left(\frac{1}{s-\mu^2}+\frac{1}{t-\mu^2}\right)\right]^2$$

using $\left(\frac{d\sigma}{dt}\right)_{cm} = \frac{|M|^2}{16\pi\lambda(s, m^2, m^2)}$



and $t \in \left[-(|\vec{p}|+|\vec{p}\,'|)^2,\ -(|\vec{p}|-|\vec{p}\,'|)^2\right]$, where $|\vec{p}| = \dfrac{\lambda^{\frac{1}{2}}(s, m^2, m^2)}{2\sqrt{s}} = |\vec{p}\,'|$

$\Rightarrow \quad t \in [-4|\vec{p}|^2, 0]$

$\Rightarrow \quad t \in [-4 \dfrac{s^2+2m^4-2m^4-4sm^2}{4s}, 0]$

$\Rightarrow \quad t \in [-(s-4m^2), 0]$

$$\Rightarrow \sigma = \int_{-(s-4m^2)}^{0} dt \; \frac{[16g + (2\lambda)^2 (\frac{1}{s-\mu^2} + \frac{1}{t-\mu^2})]^2}{16\pi(s^2+2m^4-2m^4-4sm^2)}$$

Define $\gamma \equiv \dfrac{s}{4m^2}$, $\quad \omega = \dfrac{\frac{\mu^2}{m^2}}{4\frac{s}{4m^2}} - 1 = \dfrac{\mu^2}{s} - 1$

$$\Rightarrow \sigma = \frac{1}{128\pi m^6 \omega^2 \gamma^3}$$

$$\times \Big\{ 512 g^2 m^4 \omega^2 \gamma^2 - 64 g \lambda^2 m^2 \omega \gamma + \lambda^4 \frac{2[-1+2\gamma+2\omega^2\gamma+\omega(-1+3\gamma)]}{(1+\omega)[-1+(2+\omega)\gamma]}$$

$$- \frac{4\lambda^2\omega\gamma}{\gamma-1} (16 g m^2 \omega \gamma - \lambda^2) \ln\left(\frac{[-1+(2+\omega)\gamma]}{(1+\omega)\gamma}\right) \Big\}$$

The reason that we need to consider the proper number of creation & annihilation operators in calculating $\langle f | iT | i \rangle$ is the following:

Since the interchange of creation or annihilation operators among different fields at most introduce a sign (when the interchanged ones are both fermionic operators), and iT is normal ordered (or can be made normal ordered by Wick's theorem), we just need to consider

$$\underbrace{a \cdots a}_{m} \underbrace{a^{+} \cdots a^{+}}_{n} |0\rangle$$

Let's move the right most a to the far right:

$$\underbrace{a \cdots a}_{m} \underbrace{a^{+} \cdots a^{+}}_{n} |0\rangle = \underbrace{a \cdots a}_{m-1} \underbrace{a^{+} \cdots a^{+}}_{n-1} |0\rangle \times \delta \text{ function}$$
$$\pm \underbrace{a \cdots a}_{m-1} a^{+} a \underbrace{a^{+} \cdots a^{+}}_{n-1} |0\rangle$$

where the "$-$" sign is for fermionic operator.

Then

$$\underbrace{a \cdots a}_{m-1} a^{+} a \underbrace{a^{+} \cdots a^{+}}_{n-1} |0\rangle = \underbrace{a \cdots a}_{m-1} \overbrace{a^{+} \underbrace{a^{+} \cdots a^{+}}_{n-2}}^{n-1} |0\rangle \times \delta \text{ function}$$
$$\pm \underbrace{a \cdots a}_{m-1} a^{+} a^{+} a \underbrace{a \cdots a^{+}}_{n-2} |0\rangle$$

Then

$$\underbrace{a \cdots a}_{m-1} a^{+} a^{+} a \underbrace{a^{+} \cdots a^{+}}_{n-2} |0\rangle = \underbrace{a \cdots a}_{m-1} \overbrace{a^{+} a^{+} \underbrace{a^{+} \cdots a^{+}}_{n-3}}^{n-1} |0\rangle \times \delta \text{ function}$$
$$\pm \underbrace{a \cdots a}_{m-1} a^{+} a^{+} a^{+} a \underbrace{a^{+} \cdots a^{+}}_{n-3} |0\rangle$$

keep doing this, and finally

$$\underbrace{a \cdots a}_{m-1} \underbrace{a^{+} a^{+} \cdots a^{+}}_{n-1} a\, a^{+} |0\rangle = \underbrace{a \cdots a}_{m-1} \underbrace{a^{+} a^{+} \cdots a^{+}}_{n-1} |0\rangle \times \delta \text{ function}$$

So

$$\underbrace{a \cdots a}_{m} \underbrace{a^{+} \cdots a^{+}}_{n} |0\rangle = \sum \Big[\underbrace{a \cdots a}_{m-1} \underbrace{a^{+} \cdots a^{+}}_{n-1} |0\rangle \times \delta \text{ function} \Big]$$

Keep doing this, we will get

$$\underbrace{a \cdots a}_{m} \underbrace{a^{+} \cdots a^{+}}_{n} |0\rangle = \begin{cases} \sum \Big[\underbrace{a^{+} \cdots a^{+}}_{n-m} |0\rangle \times \delta \text{ functions} \Big], & n \geq m \\ 0, & n < m \end{cases}$$

Do conjugation

$$\Rightarrow \langle 0 | \underbrace{a \cdots a}_{r} \underbrace{a^{+} \cdots a^{+}}_{s} = \begin{cases} \sum \Big[\langle 0 | \underbrace{a \cdots a}_{r-s} \times \delta \text{ functions} \Big], & r \geq s \\ 0, & r < s \end{cases}$$

So, for $\overbrace{\langle 0| \underbrace{a\cdots a}_{r}}^{\langle f|}\ \overbrace{\underbrace{a^+\cdots a^+}_{s}\ \underbrace{a\cdots a}_{m}}^{iT}\ \overbrace{\underbrace{a^+\cdots a^+}_{n}|0\rangle}^{|i\rangle}$

$$= \begin{cases} \sum[\langle 0| \underbrace{a\cdots a}_{r-s}\ \underbrace{a^+\cdots a^+}_{n-m}|0\rangle \times \delta \text{ functions}], & n\geq m \,\&\, r\geq s \\ 0, & \text{otherwise} \end{cases}$$

while $\underbrace{a\cdots a}_{r-s}\ \underbrace{a^+\cdots a^+}_{n-m}|0\rangle = \begin{cases} \sum[\underbrace{a^+\cdots a^+}_{(n-m)-(r-s)}|0\rangle \times \delta \text{ functions}], & (n-m)\geq(r-s) \\ 0, & (n-m)<(r-s) \end{cases}$

and $\langle 0| \underbrace{a\cdots a}_{r-s}\ \underbrace{a^+\cdots a^+}_{n-m}| = \begin{cases} \sum[\langle 0| \underbrace{a\cdots a}_{(r-s)-(n-m)}| \times \delta \text{ functions}], & (r-s)\geq(n-m) \\ 0, & (r-s)<(n-m) \end{cases}$

So, we could get non-zero value for $\langle f| iT |i\rangle$ if

$$\begin{cases} n-m\geq 0 \\ r-s\geq 0 \\ (n-m)\geq(r-s) \\ (r-s)\geq(n-m) \end{cases} \Rightarrow (n-m)=(r-s)\geq 0$$

However, if $(n-m)=(r-s)>0$, we will have $(n-m)$ particles in the initial state directly become the $(r-s)=(n-m)$ particles in the final state. We are not interested in this situation, since we only care about fully connected diagrams, that is, all external legs connected to each other.

So, we only care about the case when $n=m$ & $r=s$