Solution 2-1

1) $1 \text{ picobarn} = 10^{-36} \text{ cm}^2$

$1 \text{ year} = 365 \times 24 \times 60 \times 60 \text{ s}$

$\Rightarrow$ the number of Higgs can be produced per year is

$$20 \times 10^{-36} \times 365 \times 24 \times 60 \times 60 \times 10^{33} \simeq \boxed{6.3 \times 10^5}$$

2) The total energy of one beam is

$$2808 \times 1.15 \times 10^{11} \times 7 \text{ TeV}$$

and $1 \text{ TeV} = 10^{12} \text{ eV} = 10^{12} \times 1.6022 \times 10^{-19} \text{ J}$

the latent heat of ice to watter is $333.55 \text{ kJ/kg} = 333.55 \times 10^3 \text{ J/kg}$

$\Rightarrow$ the mass of ice can be melted is

$$\frac{2808 \times 1.15 \times 10^{11} \times 7 \times 10^{12} \times 1.6022 \times 10^{-18}}{333.55 \times 10^3} \simeq \boxed{1.1 \times 10^3 \text{ kg}}$$

3) The proton mass is 938.272 MeV

$E = 7 \text{ TeV}$

$$\gamma = \frac{E}{m} = (1 - v^2)^{-\frac{1}{2}}$$

$$\Rightarrow v = \left[1 - \left(\frac{m}{E}\right)^2\right]^{\frac{1}{2}} = \left[1 - \left(\frac{938.272}{7 \times 10^6}\right)^2\right]^{\frac{1}{2}}$$

$$\simeq \boxed{0.999999991}$$

solution 2-2.

1) $E = m\gamma = m(1-v^2)^{-\frac{1}{2}}$

$\Rightarrow r \equiv \dfrac{m+\frac{1}{2}mv^2}{m(1-v^2)^{-\frac{1}{2}}} = \dfrac{1+\frac{1}{2}v^2}{(1-v^2)^{-\frac{1}{2}}} = (1-v^2)^{\frac{1}{2}}(1+\frac{1}{2}v^2)$

Since $\dfrac{dr}{dv} = \frac{1}{2}(1-v^2)^{-\frac{1}{2}}(-2v)(1+\frac{1}{2}v^2) + (1-v^2)^{\frac{1}{2}}v$

$= \dfrac{-v(1+\frac{1}{2}v^2)+(1-v^2)v}{\sqrt{1-v^2}} = -\dfrac{3v^3}{2\sqrt{1-v^2}}$

then $\dfrac{dr}{dv} < 0$ for $v \in (0,1)$

So, r decreases monotonically as v increases from 0 to 1.

(See the plot from Mathematica)

when $v=0 \Rightarrow r=1$;

when $v \to 1 \Rightarrow r \to 0$

when $r = 1-1\%$. $\Rightarrow$ $\boxed{v \simeq 0.40}$

2) $E = m\gamma$

$|\vec{p}| = mv\gamma$

$\Rightarrow r \equiv \dfrac{|\vec{p}|}{E} = v$

when $r = 1-1\%$. $\Rightarrow$ $\boxed{v = 0.99}$

solution 2-3

1) $$d\Pi_3 = \frac{d(q^2)}{2\pi}\, d\Pi_2(q\,;\,P_1, P_2)\, d\Pi_2(\Sigma P_{\text{initial}}\,;\,q, P_3)$$

let $m_3 = m$, $m_1 = m_2 = 0$

$$\Rightarrow \int d\Pi_2(q; P_1, P_2) = \frac{2|\vec{P}_1^*|}{\sqrt{q^2}\, 8\pi}\int \frac{d(\cos\theta^*)}{2}\,\frac{d\varphi^*}{2\pi}$$

where $|\vec{P}_1^*| = \dfrac{\lambda^{\frac{1}{2}}(q^2, 0, 0)}{2\sqrt{q^2}} = \dfrac{q^2}{2\sqrt{q^2}} = \dfrac{\sqrt{q^2}}{2}$,

and $\displaystyle\int \frac{d(\cos\theta^*)}{2}\,\frac{d\varphi^*}{2\pi} = 1$, where θ^* and φ^* are the polar and azimuthal angle of particle-1 in the center-of-mass frame of particle-1 and -2,

$$\Rightarrow \int d\Pi_2(q; P_1, P_2) = \frac{2\,\frac{\sqrt{q^2}}{2}}{\sqrt{q^2}\, 8\pi} = \frac{1}{8\pi}$$

$$\int d\Pi_2(\Sigma P_{\text{initial}}; q, P_3) = \frac{2|\vec{P}_3|}{\sqrt{s}\, 8\pi}\int \frac{d\cos\theta}{2}\,\frac{d\varphi}{2\pi}$$

where $|\vec{P}_3| = \dfrac{\lambda^{\frac{1}{2}}(s, m^2, q^2)}{2\sqrt{s}} = \dfrac{(s^2+m^4+q^4-2q^2s-2q^2m^2-2m^2s)^{\frac{1}{2}}}{2\sqrt{s}}$

and $\displaystyle\int \frac{d\cos\theta}{2}\,\frac{d\varphi}{2\pi} = 1$, where θ and φ are the polar and azimuthal angle of particle-3 in the center-of-mass frame of the whole three-body system (or say, the center-of-mass frame of the initial particles)

$$\Rightarrow \int d\Pi_2(\Sigma P_{\text{initial}}; q, P_3) = \frac{1}{8\pi}\,\frac{(s^2+m^4+q^4-2q^2s-2q^2m^2-2m^2s)^{\frac{1}{2}}}{s}$$

$$\Rightarrow d\Pi_3 = \frac{d(q^2)}{2\pi s}\left(\frac{1}{8\pi}\right)^2 (s^2+m^4+q^4-2q^2s-2q^2m^2-2m^2s)^{\frac{1}{2}}$$

Since $q^2 = (\Sigma P_{\text{initial}} - P_3)^2 = (P_1+P_2)^2$

$\Rightarrow$ in the center-of-mass frame of particle-1 and -2,

$q^2 = (E_1^* + E_2^*)^2 \Rightarrow (q^2)_{min} = (m_1+m_2)^2 = 0$

while in the center-of-mass frame of the initial particles,

$$q^2 = s + m_3^2 - 2\sqrt{s}\,E_3$$

$$\Rightarrow \quad (q^2)_{max} = s + m_3^2 - 2\sqrt{s}\, m_3 = (\sqrt{s} - m_3)^2 = (\sqrt{s} - m)^2$$

$$\Rightarrow \int d\Pi_3 = \int_0^{(\sqrt{s}-m)^2} \frac{d(q^2)}{2\pi s} \left(\frac{1}{8\pi}\right)^2 \left(s^2 + m^4 + (q^2)^2 - 2q^2 s - 2q^2 m^2 - 2m^2 s\right)^{\frac{1}{2}}$$

Define $\quad q^2 \equiv (\sqrt{s} - m)^2 x$

$$\Rightarrow \quad dq^2 = (\sqrt{s} - m^2)\, dx$$

and the range of x is $[0, 1]$

$$s^2 + m^4 + (q^2)^2 - 2q^2 s - 2q^2 m^2 - 2m^2 s$$

$$= s^2 + m^4 + (\sqrt{s} - m)^4 x^2 - 2(s + m^2)(\sqrt{s} - m)^2 x - 2m^2 s$$

$$= (s - m^2)^2 + (\sqrt{s} - m)^4 x^2 - 2(s + m^2)(\sqrt{s} - m)^2 x$$

$$= (\sqrt{s} - m)^2 \left[(\sqrt{s} + m)^2 + (\sqrt{s} - m)^2 x^2 - 2(s + m^2) x \right]$$

$$\Rightarrow \Pi_3 = \int_0^1 \frac{dx}{2\pi s} (\sqrt{s} - m)^2 \left(\frac{1}{8\pi}\right)^2 (\sqrt{s} - m) \left[(\sqrt{s} + m)^2 + (\sqrt{s} - m)^2 x^2 - 2(s + m^2) x \right]^{\frac{1}{2}}$$

$$= \frac{(\sqrt{s} - m)^3}{2\pi s} \left(\frac{1}{8\pi}\right)^2 \frac{s^2 - m^4 + 2m^2 s \ln\left(\frac{m^2}{s}\right)}{2(\sqrt{s} - m)^3}$$

$$= \frac{1}{256\pi^3 s} \left[s^2 - m^4 + 2m^2 s \ln\left(\frac{m^2}{s}\right) \right]$$

$$\boxed{= \frac{m^2}{128\pi^3} \left[\frac{s}{2m^2} - \frac{m^2}{2s} + \ln\left(\frac{m^2}{s}\right) \right]}$$

2) put in $s = (106\,\text{MeV})^2$, $\quad m = 0.511\,\text{MeV}$

we get $\Pi_3 \simeq \boxed{1.4\,\text{MeV}^2 = 1.4 \times 10^{-6}\,\text{GeV}^2}$

Solution 2-4

1) Let $m=0$, follow the derivation in 3-1)

$$\Pi_3 = \int d\Pi_3 = \int_0^s \frac{d(q^2)}{2\pi s}\left(\frac{1}{8\pi}\right)^2 (s-q^2)$$

$$\overset{q^2 = sx}{=} \int_0^1 \frac{s\,dx}{2\pi s}\left(\frac{1}{8\pi}\right)^2 s(1-x)$$

$$= \frac{1}{2\pi}\left(\frac{1}{8\pi}\right)^2 s \int_0^1 dx\,(1-x)$$

$$= \frac{1}{2\pi}\left(\frac{1}{8\pi}\right)^2 s\left(1-\frac{1}{2}\right)$$

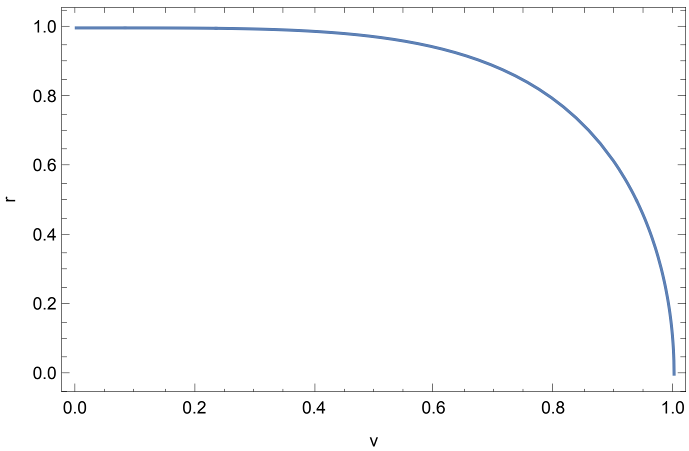
$$\boxed{= \frac{s}{256\pi^3}}$$

2) put in $s=(106\,\mathrm{MeV})^2$

$$\Rightarrow \quad \Pi_3 \simeq \boxed{1.4\ \mathrm{MeV}^2 = 1.4\times10^{-6}\ \mathrm{GeV}^2}$$

In[∘]:=
```
Plot[(1 - v^2)^(1/2) * (1 + 1/2 * v^2), {v, 0, 1},
 Frame → True, PlotRange → All, FrameLabel → {"v", "r"}]
```

Out[∘]=



In[∘]:=
```
NSolve[(1 - v^2)^(1/2) * (1 + 1/2 * v^2) == 0.99, v, Reals]
```

Out[∘]=
```
{{v → -0.398428}, {v → 0.398428}}
```