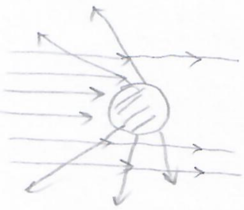
cross sections & decay rates

$$\sigma \equiv \frac{\overbrace{\text{number of particles scattered}}^{N}}{\underbrace{\text{time}}_{T} \times \underbrace{\text{number density in beam} \times \text{velocity of beam}}_{\Phi}} = \frac{1}{T}\,\frac{1}{\Phi}\,N$$

Φ → number of particle go through per unit area per unit time, called flux.

It's also natural to measure the differential cross section $\frac{d\sigma}{d\Omega}$, as this gives the information (classically) about the shape of the object or the form of the potential responsible for the scattering to happen.



While classically a scatter either happen or not, quantum mechanically it has a probability for scattering.

$$d\sigma = \frac{1}{T}\,\frac{1}{\Phi}\,dP$$

where Φ is now normalized as if the beam has just one particle, P is the probability of scattering.

The differential number of scattering events measured in a collider experiment is

$$dN \equiv L \times d\sigma.$$

where L is the luminosity. (unit is area^{-1}).

The experimental data is often presented as $\frac{dN}{dE}$ vs. E, then if we can calculate $\frac{d\sigma}{dE}$ from theory, then multiply by L, we can get the theoretical plot for $\frac{d\sigma}{dE} \times L$ vs. E, to compare with experimental data.

Now let's relate the $d\sigma$ to S-matrix element. $\langle f|S|i\rangle$.

Assume $|i\rangle$ is a two particle state, and $|f\rangle$ is a n particle state, so

$2 \to n$ process

$$p_1 + p_2 \to \{p_j\}, \quad j = 1, \cdots, n.$$

In the rest frame of one of the collidering particle, the flux is

$$\Phi = \frac{|\vec{v}|}{V},$$

where $|\vec{v}|$ is the magnitude of the incoming particle, V is the "big box" in which the whole experiment is taking place.

In a different frame, such as the center-of-mass frame, $|\vec{v}|$ should be changed to $|\vec{v}_1 - \vec{v}_2|$

So we have

$$d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP,$$

where T is the time the experiment lasts. T is large since we assume $|i\rangle$ and $|f\rangle$ are states at $T \to -\infty$ & $T \to +\infty$, respectively

$$dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} d\Pi$$

where $|\langle f|\hat{S}|i\rangle|^2$ is the transition probability density from $|i\rangle$ to $|f\rangle$ due to interaction described by $\hat{S}$ (recall that $\hat{S}$ has interaction Hamiltonian in it).

Note that since $|f\rangle$ is specified by definite momenta of the final state, and considering that momentum is a continuous quantity, we need $d\Pi$ to get from probability density to probability.

Consider a large cubic $V=L^3$, the de Broglie wavelength is

$$\lambda = \frac{h}{P} = \frac{L}{n}, \quad n \text{ is integer}$$

$$\Rightarrow \frac{2\pi\hbar}{p} = \frac{L}{n} \Rightarrow \Delta n \underset{\hbar=1}{\overset{\uparrow}{=}} \frac{\Delta P}{2\pi} L$$

In three dimension and go to $\underbrace{\text{continuous limit}}_{V\to\infty}$, $dn_x\, dn_y\, dn_z = \frac{d^3\vec{P}}{(2\pi)^3} V$

So, the number of states within $d^3\vec{P}$ is $dn_x\, dn_y\, dn_z = \frac{d^3\vec{P}}{(2\pi)^3} V$.

Do the same for every particle in the final state.

$$\Rightarrow d\Pi = \prod_{j=1}^{n} \frac{V}{(2\pi)^3} d^3\vec{P}_j$$

Since for one particle state,

$$\langle p|q\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{P}-\vec{q}) \quad (\text{for scalar field})$$

$$\langle p,r|q,s\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{P}-\vec{q})\, \delta_{rs} \quad (\text{for spinor and vector field})$$

we have $\langle p|p\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(0)$

and $\langle p,r|p,r\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(0)$.

Since $\int_{-\infty}^{+\infty} d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{P})$

then $\int_{-\infty}^{+\infty} d^3\vec{x} = (2\pi)^3 \delta^3(0)$

$$\Rightarrow \delta^3(0) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x} = \lim_{V\to\infty} \frac{V}{(2\pi)^3}$$

Similarly, $\int_{-\infty}^{+\infty} dt\, e^{-iE_{\vec{p}}t} = (2\pi)\,\delta(E_{\vec{p}})$

then $\int_{-\infty}^{+\infty} dt = (2\pi)\,\delta(0)$

$$\Rightarrow \delta(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt = \lim_{T\to\infty} \frac{T}{2\pi}$$

$$\Rightarrow \delta^4(0) = \lim_{V,T\to\infty} \frac{TV}{(2\pi)^4}$$

Keep in mind that at the end we need to go to $V, T \to \infty$,

we can write
$$\langle p | p \rangle = 2E_{\vec{p}} V$$
$$\langle p, r | p, r \rangle = 2E_{\vec{p}} V.$$

So the normalization factor $\langle f|f\rangle$ and $\langle i|i\rangle$ are independent of the types of the fields, we can just use the momentum alone to label the state for the purpose here
$$|i\rangle = |p_1\rangle |p_2\rangle \text{ and } |f\rangle = \prod_j |p_j\rangle$$
$$\Rightarrow \langle i|i\rangle = (2E_1 V)(2E_2 V)$$
$$\langle f|f\rangle = \prod_{j=1}^{n} (2E_j V)$$

For $|\langle f|\hat{S}|i\rangle|^2$, since $\hat{S} = \mathbb{1} + \sum_{l=1}^{\infty} \frac{1}{l!}(-i)^l \int d^4x_1 d^4x_2 \cdots d^4x_l \, T(\hat{H}_I^{int}(x_1) \hat{H}_I^{int}(x_2) \cdots \hat{H}_I^{int}(x_n))$ then we can write it as $\hat{S} = \mathbb{1} + iT$, where $\mathbb{1}$ is the part where there is no interaction, and T is called the transition matrix, then factor out the energy-momentum conservation condition, we can write
$$T \equiv (2\pi)^4 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \mathcal{M}$$

So, we are interested in calculating the interaction related S-matrix element
$$\langle f | \hat{S} - \mathbb{1} | i \rangle = (2\pi)^4 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \underbrace{i}_{\text{imaginary unit}} \langle f | \mathcal{M} | \underbrace{i}_{\text{initial state}} \rangle$$
where $\langle f|\mathcal{M}|i\rangle$ is usually called matrix element.

Since we are interested in the transition, $|f\rangle \neq |i\rangle$, then $\langle f|i\rangle = 0$,

So $\langle f|\hat{S}|i\rangle = \langle f|\hat{S} - \mathbb{1}|i\rangle$

$$\Rightarrow |\langle f|\hat{S}|i\rangle|^2 = (2\pi)^8 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \delta^4(0) |\langle f|\mathcal{M}|i\rangle|^2$$
$$\left(\text{Since } \left(\delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)\right)^2 = \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)\delta^4(0)\right)$$

while $\delta^4(0) = \frac{VT}{(2\pi)^4}$

$$\Rightarrow |\langle f|\hat{S}|i\rangle|^2 = (2\pi)^4 \delta^4(\Sigma P_i^\mu - \Sigma P_f^\mu)\, VT\, |\mathcal{M}|^2,$$

where $|\mathcal{M}|^2 \equiv |\langle f|\mathcal{M}|i\rangle|^2$

$$\Rightarrow dP = \frac{(2\pi)^4 \delta^4(\Sigma P_i^\mu - \Sigma P_f^\mu)\, VT\, |\mathcal{M}|^2}{(2E_1 V)(2E_2 V)\prod_{j=1}^{n}(2E_j V)} \prod_{j=1}^{n} \frac{V}{(2\pi)^3} d^3\vec{P}_j$$

$$= \frac{T}{V} \frac{1}{(2E_1)(2E_2)} |\mathcal{M}|^2 d\Pi_{LIPS}$$

where $\boxed{d\Pi_{LIPS} \equiv \prod_{\text{final state } j} \frac{d^3\vec{P}_j}{(2\pi)^3} \frac{1}{2E_{\vec{P}_j}} (2\pi)^4 \delta^4(\Sigma P_i^\mu - \Sigma P_f^\mu)}$

is called the Lorentz-invariant phase space (LIPS)

$$\Rightarrow \boxed{d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - v_2|} dP = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} |\mathcal{M}|^2 d\Pi_{LIPS}}$$

we see that no V or T left, so it is trivial to take $V, T \to \infty$.

A decay is a $1 \to n$ process. the differential decay rate is

$$d\Gamma = \frac{1}{T} dP$$

where $dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi$

$\langle i|i\rangle = 2E_1 V$

$|\langle f|\hat{S}|i\rangle|^2$, $\langle f|f\rangle$ and $d\Pi$ are the same as when deriving $d\sigma$.

$$\Rightarrow d\Gamma = \frac{1}{T} \frac{(2\pi)^4 \delta^4(\Sigma P_i^\mu - \Sigma P_f^\mu)\, VT\, |\mathcal{M}|^2}{(2E_1 V)\prod_{j=1}^{n}(2E_j V)} \prod_{j=1}^{n} \frac{V}{(2\pi)^3} d^3\vec{P}_j$$

$$= \frac{1}{2E_1} |\mathcal{M}|^2 d\Pi_{LIPS}$$

Now we prove that $d\Pi_{LIPS}$ is indeed Lorentz invariant.

$$\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3\vec{p}}{(2\pi)^3}\frac{dp^0}{2p^0}\delta(p^0-E_{\vec{p}})$$, note that it should be understood as an integration for p_0 in this and the following steps.

$$= \frac{d^4p}{(2\pi)^4}\frac{2\pi}{2p^0}[\delta(p^0-E_{\vec{p}})+\delta(p^0+E_{\vec{p}})]\theta(p_0)$$

$$= \frac{d^4p}{(2\pi)^4}2\pi\,\delta(p^{0\,2}-E_{\vec{p}}^2)\,\theta(p^0)$$

$$\delta(x^2-\alpha^2)=\frac{1}{2|\alpha|}(\delta(x+\alpha)+\delta(x-\alpha))$$

$$= \frac{d^4p}{(2\pi)^4}\cdot 2\pi\,\delta(p^2-m^2)\,\theta(p^0)$$

$$E_{\vec{p}}^2=|\vec{p}|^2+m^2$$

$$p^2=p^{0\,2}-|\vec{p}|^2$$

Also, we can write

$$\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3\vec{p}}{(2\pi)^3}\frac{dp^0}{(-2p^0)}\delta(p^0+E_{\vec{p}})$$

$$= \frac{d^4p}{(2\pi)^4}\frac{2\pi}{(-2p^0)}[\delta(p^0+E_{\vec{p}})+\delta(p^0-E_{\vec{p}})]\theta(-p^0)$$

$$= \frac{d^4p}{(2\pi)^4}\cdot 2\pi\,\delta(p^{0\,2}-E_{\vec{p}}^2)\,\theta(-p^0)$$

$$= \frac{d^4p}{(2\pi)^4}\cdot 2\pi\,\delta(p^2-m^2)\,\theta(-p^0)$$

Therefore, it is clear that the role of $\theta(p^0)$ or $\theta(-p^0)$ together with $\delta(p^2-m^2)$ when do the integration is just to select one of the two possible values $p^0=\pm\sqrt{|\vec{p}|^2+m^2}=\pm E_{\vec{p}}$, and the result of the integration doesn't depends on which one to choose.

A Lorentz transformation on $d\Pi_{LIPS}$ will make

$$d\Pi_{LIPS}=\prod_{\text{final state } j}\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}_j}}(2\pi)^4\delta^4(\Sigma P_i^\mu-\Sigma P_f^\mu)=\prod_{\text{final state } j}\frac{d^4p}{(2\pi)^4}2\pi\,\delta(P_j^2-m_j^2)\,\theta(P_j^0)(2\pi)^4\delta^4(\Sigma P_i^\mu-\Sigma P_f^\mu)$$

$$\rightarrow d\Pi'_{LIPS}=\prod_{\text{final state } j}\frac{d^4p'}{(2\pi)^4}2\pi\,\delta(P_j'^2-m_j^2)\,\theta(P_j^{0\prime})(2\pi)^4\delta^4(\Sigma P_i'^\mu-\Sigma P_f'^\mu)$$

Note that $d^4p \to d^4p' \overset{p'^\mu = \Lambda^\mu{}_\nu p^\nu}{=} |\det(\Lambda)|\, d^4p = d^4p.$

$|\det(\Lambda)| = 1$ is the property of Lorentz transformation.

$p_j'^2 = p_j^2 \Rightarrow \delta(p_j'^2 - m_j^2) = \delta(p_j^2 - m_j^2)$

since p^2 is a Lorentz scalar.

$$\delta^4(\sum p_i'^\mu - \sum p_f'^\mu) = \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^4} d^4x' e^{-ix'\cdot(\sum p_i' - \sum p_f')}$$

$$\overset{d^4x' = |\det(\Lambda)|\, d^4x = d^4x,\ x'\cdot(\sum p_i' - \sum p_f') = x\cdot(\sum p_i - \sum p_f)}{=} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^4} d^4x\, e^{-ix\cdot(\sum p_i - \sum p_f)}$$

$$= \delta^4(\sum p_i^\mu - \sum p_f^\mu)$$

Since $\frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(p^0) = \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(p^0)$,

it does not matter whether $\theta(p_j'^0) = \theta(p_j^0)$ or $\theta(p_j'^0) = \theta(-p_j^0)$

That is, for the theta function, we only care about the sign of p'^0, but it does not matter whether the sign of p'^0 is the same as p^0 or $-p^0$.

$$\Rightarrow d\Pi'_{LIPS} = \prod_{\text{final state } j} \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p_j^2 - m_j^2)\, \theta(\pm p^0)\, (2\pi)^4 \delta^4(\sum p_i^\mu - \sum p_f^\mu)$$

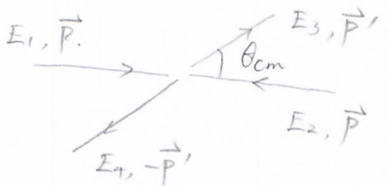
$$= d\Pi_{LIPS}.$$

Note that due to the $\delta(p_j^2 - m_j^2)$, $p_j^0 = \pm\sqrt{|\vec{p}_j|^2 + m_j^2} \neq 0$

(if $m_j = 0$, $|\vec{p}_j|$ cannot be zero, since otherwise $p_j^0 = 0$, means that there is no j particle at all)

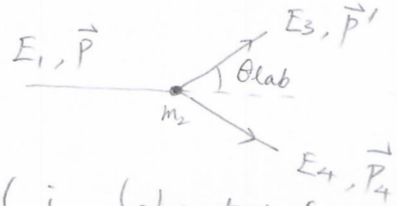
Some most common cases of scatterings and decays

① $2 \to 2$ scattering



For these two frames, note that the four particles has to be in a plane due to conservation of energy-momentum. Therefore, $d\Omega = 2\pi\, d(\cos\theta)$.

(in center-of-mass frame : $\vec{P}_1 + \vec{P}_2 = 0 = \vec{P}_3 + \vec{P}_4$)



(in laboratory frame : particle-2 is at rest).

The Lorentz invariant phase space is

$$d\Pi_{LIPS} = \frac{d^3\vec{P}_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3\vec{P}_4}{(2\pi)^3}\frac{1}{2E_4}(2\pi)^4\delta^4(P_1+P_2-P_3-P_4)$$

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1-\vec{v}_2|}|\mathcal{M}|^2\, d\Pi_{LIPS}$$

use $\delta^3(\vec{P}_1+\vec{P}_2-\vec{P}_3-\vec{P}_4)$ to integrate out $d^3\vec{P}_4$

and use $d^3\vec{P}_3 = |\vec{P}_3|^2 d|\vec{P}_3|\, d\Omega$, where $d\Omega = d\varphi\, d(\cos\theta)$

$\Rightarrow$

$$d\sigma = \frac{1}{(2\pi)^2}\frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1-\vec{v}_2|}\frac{1}{4E_3E_4}|\mathcal{M}|^2|\vec{P}_3|^2\delta(E_1+E_2-E_3-E_4)\, d|\vec{P}_3|\, d\Omega$$

[1] First, let's look at the center-of-mass frame

$$\vec{P}_1+\vec{P}_2 = 0 = \vec{P}_3+\vec{P}_4 \;,\quad \vec{P} = \vec{P}_1 \;,\quad \vec{P}' = \vec{P}_3$$

$$E_3 = (|\vec{P}'|^2 + m_3^2)^{\frac{1}{2}} \;,\quad E_4 = (|\vec{P}'|^2 + m_4^2)^{\frac{1}{2}}$$

$$(P_1+P_2)^2 = (E_1+E_2)^2 = s = (P_3+P_4)^2 = (E_3+E_4)^2$$

$$|\vec{v}_1 - \vec{v}_2| = \left|\frac{\vec{P}_1}{E_1} - \frac{\vec{P}_2}{E_2}\right| = |\vec{P}|\left(\frac{1}{E_1}+\frac{1}{E_2}\right) = |\vec{P}|\frac{(E_1+E_2)}{E_1E_2}$$

$$d|\vec{P}_3| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_3+E_4)}d(E_3+E_4) \;,$$

where $\frac{d(E_3+E_4)}{d|\vec{P}'|} = \frac{d\left((|\vec{P}'|^2+m_3^2)^{\frac{1}{2}} + (|\vec{P}'|^2+m_4^2)^{\frac{1}{2}}\right)}{d|\vec{P}'|} = \frac{1}{2}\frac{2|\vec{P}'|}{E_3} + \frac{1}{2}\frac{2|\vec{P}'|}{E_4} = |\vec{P}'|\frac{(E_3+E_4)}{E_3E_4}$

$\Rightarrow \frac{d|\vec{P}'|}{d(E_3+E_4)} = \frac{1}{|\vec{P}'|}\cdot\frac{E_3E_4}{(E_3+E_4)}$

then use $\delta(\sqrt{s}-E_3-E_4)$ to integrate out $d|\vec{P}'|$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{(2\pi)^2}\frac{1}{4E_1E_2}\frac{1}{|\vec{P}|\frac{(E_1+E_2)}{E_1E_2}}\frac{1}{4E_3E_4}\cdot|\vec{P}'|^2\frac{1}{|\vec{P}'|}\frac{E_3E_4}{(E_3+E_4)}|\mathcal{M}|^2$$

$$= \frac{1}{(2\pi)^2}\frac{1}{4}\frac{|\vec{P}'|}{|\vec{P}|}\frac{1}{s}\frac{1}{4}|\mathcal{M}|^2$$

$$= \frac{|\mathcal{M}|^2}{64\pi^2 s}\cdot\frac{|\vec{P}'|}{|\vec{P}|}$$

use $P_1^2 = [(P_1+P_2)-P_2]^2$

$$\Rightarrow m_1^2 = s - 2P_2\cdot(P_1+P_2) + m_2^2$$

$$= s - 2E_2\sqrt{s} + m_2^2$$

$$\Rightarrow E_2 = \frac{s+m_2^2-m_1^2}{2\sqrt{s}}$$

similarly, $P_2^2 = [(P_1+P_2)-P_1]^2 = s - 2P_1\cdot(P_1+P_2) + P_1^2$

$$\Rightarrow m_2^2 = s - 2E_1\sqrt{s} + m_1^2$$

$$\Rightarrow E_1 = \frac{s+m_1^2-m_2^2}{2\sqrt{s}}$$

$$\Rightarrow |\vec{P}| = |\vec{P}_1| = (E_1^2-m_1^2)^{\frac{1}{2}} = \left[\left(\frac{s+m_1^2-m_2^2}{2\sqrt{s}}\right)^2 - m_1^2\right]^{\frac{1}{2}}$$

$$= \frac{(s^2+m_1^4+m_2^4+2m_1^2s-2m_2^2s-2m_1^2m_2^2-4sm_1^2)^{\frac{1}{2}}}{2\sqrt{s}}$$

$$= \frac{(s^2+m_1^4+m_2^4-2m_1^2s-2m_2^2s-2m_1^2m_2^2)^{\frac{1}{2}}}{2\sqrt{s}}$$

$$= \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}$$

where $\lambda(a,b,c) = (a-b-c)^2 - 4bc = a^2+b^2+c^2-2ab-2ac-2bc$

Similarly, $p_3^2 = [(p_3+p_4) - p_4]^2 = s - 2p_4\cdot(p_3+p_4) + p_4^2 = s - 2E_4\sqrt{s} + m_4^2$

$$\Rightarrow \quad E_4 = \frac{s+m_4^2-m_3^2}{2\sqrt{s}}$$

$$E_3 = \sqrt{s} - E_4 = \frac{2s-(s+m_4^2-m_3^2)}{2\sqrt{s}} = \frac{s+m_3^2-m_4^2}{2\sqrt{s}}$$

$$|\vec{p}'| = |\vec{p}_3| = (E_3^2 - m_3^2)^{\frac{1}{2}} = \frac{(s^2+m_3^4+m_4^4+2m_3^2 s-2m_4^2 s-2m_3^2m_4^2-4m_3^2 s)^{\frac{1}{2}}}{2\sqrt{s}}$$

$$= \frac{\lambda^{\frac{1}{2}}(s, m_3^2, m_4^2)}{2\sqrt{s}}$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 s}\,\frac{\lambda^{\frac{1}{2}}(s, m_3^2, m_4^2)}{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}}$$

use

$$t = (p_1-p_3)^2 = m_1^2+m_3^2-2p_1\cdot p_3 = m_1^2+m_3^2-2E_1E_3+2|\vec{p}_1||\vec{p}_3|\cos\theta$$

$$= m_1^2+m_3^2-2E_1E_3+2|\vec{p}||\vec{p}'|\cos\theta_{cm}$$

recall that s, t, u are Mandelstam variables, and $s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$ where $u=(p_1-p_4)^2$

note that $E_1, E_3, |\vec{p}|, |\vec{p}'|$ only depends on s, which is fixed for a given scattering, then

$$dt = 2|\vec{p}||\vec{p}'|\,d(\cos\theta_{cm})$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left(\frac{d\sigma}{2\pi\, d(\cos\theta_{cm})}\right)_{cm} = \left(\frac{d\sigma}{2\pi\, dt}\,2|\vec{p}|\,|\vec{p}'|\right)_{cm}$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{dt}\right)_{cm}} = 2\pi\frac{1}{2|\vec{p}||\vec{p}'|}\,\frac{|\mathcal{M}|^2}{64\pi^2 s}\,\frac{|\vec{p}'|}{|\vec{p}|}$$

$$= \boxed{\frac{|\mathcal{M}|^2}{64\pi s|\vec{p}|^2} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_1^2, m_2^2)}}$$

Since $0 \le \cos\theta_{cm} \le \pi$, then $t \in [m_1^2+m_2^2-2E_1E_3-2|\vec{p}||\vec{p}'|,\ m_1^2+m_2^2-2E_1E_3+2|\vec{p}||\vec{p}'|]$

that is $t \in [(E_1-E_3)^2-(|\vec{p}|+|\vec{p}'|)^2,\ (E_1-E_3)^2-(|\vec{p}|-|\vec{p}'|)^2]$,

(check: $(E_1-E_3)^2-(|\vec{p}|+|\vec{p}'|)^2 = E_1^2+E_3^2-2E_1E_3-|\vec{p}|^2-|\vec{p}'|^2-2|\vec{p}||\vec{p}'| = m_1^2+m_3^2-2E_1E_3-2|\vec{p}||\vec{p}'|$

$(E_1-E_3)^2-(|\vec{p}|-|\vec{p}'|)^2 = m_1^2+m_3^2-2E_1E_3+2|\vec{p}||\vec{p}'|$)

that is, $\boxed{t \in \left[\left(\frac{m_1^2-m_2^2-m_3^2+m_4^2}{2\sqrt{s}}\right)^2-(|\vec{p}|+|\vec{p}'|)^2,\ \left(\frac{m_1^2-m_2^2-m_3^2+m_4^2}{2\sqrt{s}}\right)^2-(|\vec{p}|-|\vec{p}'|)^2\right]}$

[2] In the laboratory frame, where particle 2 is at rest,

$$p_1^\mu = (E_1, \vec{p}), \quad p_2^\mu = (m_2, 0), \quad p_3^\mu = (E_3, \vec{p}\,'), \quad p_4^\mu = (E_4, \vec{p}_4)$$

$$E_1 + E_2 = E_3 + E_4 \Rightarrow E_3 + E_4 = (|\vec{p}|^2 + m_1^2)^{\frac{1}{2}} + m_2$$

$$\vec{p} = \vec{p}\,' + \vec{p}_4, \qquad E_3 + E_4 = (|\vec{p}_3|^2 + m_3^2)^{\frac{1}{2}} + (|\vec{p}_4|^2 + m_4^2)^{\frac{1}{2}}$$

$$= (|\vec{p}\,'|^2 + m_3^2)^{\frac{1}{2}} + (|\vec{p} - \vec{p}\,'|^2 + m_4^2)^{\frac{1}{2}}$$

$$= (|\vec{p}\,'|^2 + m_3^2)^{\frac{1}{2}} + [|\vec{p}|^2 + |\vec{p}\,'|^2 - 2|\vec{p}||\vec{p}\,'|\cos\theta_{lab} + m_4^2]^{\frac{1}{2}}$$

$$d|\vec{p}_3| = d|\vec{p}\,'| = \frac{d|\vec{p}\,'|}{d(E_3 + E_4)} \, d(E_3 + E_4)$$

where

$$\frac{d(E_3 + E_4)}{d|\vec{p}\,'|} = \frac{1}{2}\frac{2|\vec{p}\,'|}{E_3} + \frac{1}{2}\frac{2|\vec{p}\,'| - 2|\vec{p}|\cos\theta_{lab}}{E_4}$$

for given $|\vec{p}|$ and θ_{lab}

$\downarrow$ a fixed value for a given scattering ($|\vec{p}|$)

$\llcorner$ will be integrated (θ_{lab})

$$\Rightarrow d|\vec{p}_3| = \frac{1}{\left(\frac{|\vec{p}\,'|}{E_3} + \frac{|\vec{p}\,'| - |\vec{p}|\cos\theta_{lab}}{E_4}\right)} d(E_3 + E_4)$$

$$= \frac{E_3 E_4}{|\vec{p}\,'|E_4 + |\vec{p}\,'|E_3 - |\vec{p}|E_3\cos\theta_{lab}} d(E_3 + E_4)$$

integrate out $d|\vec{p}_3|$ using $\delta(E_1 + E_2 - E_3 - E_4)$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{lab}} = \frac{1}{(2\pi)^2}\frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1|}\frac{1}{4E_3E_4}|\mathcal{M}|^2|\vec{p}\,'|^2\frac{E_3E_4}{|\vec{p}\,'|(E_3+E_4) - |\vec{p}|E_3\cos\theta_{lab}}$$

$|\vec{v}_1| = \frac{|\vec{p}|}{E_1}$, $E_3 + E_4 = E_1 + m_2$, $E_2 = m_2$

$$\Downarrow \quad \frac{\frac{1}{(2\pi)^2}\frac{1}{4E_1m_2}\frac{1}{\frac{|\vec{p}|}{E_1}}\frac{1}{4}|\mathcal{M}|^2|\vec{p}\,'|^2}{|\vec{p}\,'|(E_1 + m_2) - |\vec{p}|E_3\cos\theta_{lab}}$$

$$= \boxed{\frac{|\mathcal{M}|^2}{64\pi^2 m_2}\frac{|\vec{p}\,'|}{|\vec{p}|}\frac{1}{\left[(E_1 + m_2) - \frac{|\vec{p}|}{|\vec{p}\,'|}E_3\cos\theta_{lab}\right]}}$$

Note that $|\vec{p}\,'|$ and E_3 can be expressed through $|\vec{p}|$ and θ_{lab}.

Since $E_3+E_4 = (|\vec{P}|^2+m_1^2)^{\frac{1}{2}} + m_2$

$$\| $$

$$(|\vec{P}'|^2+m_3^2)^{\frac{1}{2}} + [\,|\vec{P}|^2+|\vec{P}'|^2 - 2|\vec{P}||\vec{P}'|\cos\theta_{lab} + m_4^2]^{\frac{1}{2}}$$

$\Rightarrow$ $|\vec{P}'|$ can be expressed as a function of $|\vec{P}|$ and θ_{lab},

and $E_3 = (|\vec{P}'|^2+m_3^2)^{\frac{1}{2}}$ can be also expressed as a function of $|\vec{P}|$ and θ_{lab}

from $\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{d\sigma}{2\pi\, d(\cos\theta_{lab})}$

$$\Rightarrow \left(\frac{d\sigma}{d\cos\theta}\right)_{lab} = \frac{|\mathcal{M}|^2}{32\pi m_2}\frac{|\vec{P}'|}{|\vec{P}|}\frac{1}{[(E_1+m_2)-\frac{|\vec{P}|}{|\vec{P}'|}E_3\cos\theta_{lab}]}$$

use $E_3+E_4 = (|\vec{P}|^2+m_1^2)^{\frac{1}{2}}+m_2$

$$\Rightarrow |\vec{P}|^2 + |\vec{P}'|^2 - 2|\vec{P}||\vec{P}'|\cos\theta_{lab} + m_4^2 = \left[(|\vec{P}|^2+m_1^2)^{\frac{1}{2}} + m_2 - (|\vec{P}'|^2+m_3^2)^{\frac{1}{2}}\right]^2$$

$$= |\vec{P}|^2 + m_1^2 + m_2^2 + |\vec{P}'|^2 + m_3^2 + 2m_2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}} - 2m_2(|\vec{P}'|^2+m_3^2)^{\frac{1}{2}}$$
$$-2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}}(|\vec{P}'|^2+m_3^2)^{\frac{1}{2}}$$

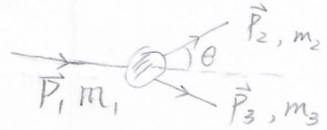
$$\Rightarrow \underbrace{2m_2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}} + m_1^2+m_2^2+m_3^2-m_4^2}_{\equiv a} + 2|\vec{P}||\vec{P}'|\cos\theta_{lab}$$
$$= 2m_2(|\vec{P}'|^2+m_3^2)^{\frac{1}{2}} + 2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}}(|\vec{P}'|^2+m_3^2)^{\frac{1}{2}}$$

$$\Rightarrow a^2 + 4a|\vec{P}||\vec{P}'|\cos\theta_{lab} + 4|\vec{P}|^2|\vec{P}'|^2\cos^2\theta_{lab}$$
$$= 4m_2^2(|\vec{P}'|^2+m_3^2) + 4(|\vec{P}|^2+m_1^2)(|\vec{P}'|^2+m_3^2) + 8m_2(|\vec{P}'|^2+m_3^2)(|\vec{P}|^2+m_1^2)^{\frac{1}{2}}$$
$$= 4|\vec{P}'|^2\left[m_2^2 + |\vec{P}|^2 + m_1^2 + 2m_2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}}\right]$$
$$+ 4m_2^2m_3^2 + 4|\vec{P}|^2m_3^2 + 4m_1^2m_3^2 + 8m_2m_3^2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}}$$

$$\Rightarrow |\vec{P}'|^2A + B|\vec{P}'| + C = 0$$

where
$$A = 4\left(m_2^2 + |\vec{P}|^2 + m_1^2 + 2m_2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}} - |\vec{P}|^2\cos^2\theta_{lab}\right)$$
$$B = -4|\vec{P}|\cos\theta_{lab}\left[2m_2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}} + m_1^2+m_2^2+m_3^2-m_4^2\right]$$
$$C = 4m_2^2m_3^2 + 4|\vec{P}|^2m_3^2 + 4m_1^2m_3^2 + 8m_2m_3^2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}}$$
$$-\left[2m_2(|\vec{P}|^2+m_1^2)^{\frac{1}{2}} + m_1^2+m_2^2+m_3^2-m_4^2\right]^2$$

② $1 \to 2$ decay. $(m_1 > (m_2 + m_3))$



Similarly to the $2\to 2$ scattering, the phase space is

$$d\Pi_{LIPS} = \frac{1}{(2\pi)^2}\frac{1}{4E_2E_3}|\vec{P}_2|^2 \delta(E_1 - E_2 - E_3)\, d|\vec{P}_2|\, d\Omega,$$ where $d\Omega$ is the solid angle of particle 2, $d\Omega = d(\cos\theta)\, d\varphi$.

[1] In the rest frame of particle 1, i.e., the center-of-mass frame

$\vec{P}_1 = 0$, $\vec{P}' \equiv \vec{P}_2 = -\vec{P}_3$

$E_1 = m_1$

$$d|\vec{P}_2| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_2+E_3)}\, d(E_2+E_3)$$

where $$\frac{d(E_2+E_3)}{d|\vec{P}'|} = \frac{d[(|\vec{P}'|^2 + m_2^2)^{\frac{1}{2}} + (|\vec{P}'|^2 + m_3^2)^{\frac{1}{2}}]}{d|\vec{P}'|} = \frac{1}{2}\frac{2|\vec{P}'|}{E_2} + \frac{1}{2}\frac{2|\vec{P}'|}{E_3} = |\vec{P}'|\frac{(E_2+E_3)}{E_2E_3}$$

$$\Rightarrow \frac{d|\vec{P}'|}{d(E_2+E_3)} = \frac{1}{|\vec{P}'|}\frac{E_2E_3}{(E_2+E_3)}$$

$$\Rightarrow \frac{d\Pi_{LIPS}}{d\Omega} = \frac{1}{(2\pi)^2}\frac{1}{4E_2E_3}|\vec{P}'|^2\frac{1}{|\vec{P}'|}\frac{E_2E_3}{m_1} = \frac{|\vec{P}'|}{16\pi^2}\frac{1}{m_1}$$

use $\delta(m_1 - E_2 - E_3)$ to integrate out $d|\vec{P}'|$

Since $$\underset{m_3^2}{\overset{\|}{P_3^2}} = (P_1 - P_2)^2 = m_1^2 + m_2^2 - 2m_1E_2$$

$$\Rightarrow E_2 = \frac{m_1^2 + m_2^2 - m_3^2}{2m_1}$$

$$\Rightarrow |\vec{P}'| = (E_2^2 - m_2^2)^{\frac{1}{2}} = \frac{[(m_1^2 + m_2^2 - m_3^2)^2 - 4m_1^2m_2^2]^{\frac{1}{2}}}{2m_1}$$

$$= \frac{\lambda^{\frac{1}{2}}(m_1^2, m_2^2, m_3^2)}{2m_1}$$

$$\Rightarrow \boxed{\left(\frac{d\Gamma}{d\Omega}\right)_{cm}} = \frac{1}{2m_1}|M|^2\frac{\lambda^{\frac{1}{2}}(m_1^2, m_2^2, m_3^2)}{2m_1^2 \times 16\pi^2}$$

$$= \boxed{\frac{\lambda^{\frac{1}{2}}(m_1^2, m_2^2, m_3^2)}{64\pi^2 m_1^3}|M|^2}$$

[2] In the lab frame where $P_1^\mu = (E_1, \vec{P})$, $P_2^\mu = (E_2, \vec{P}')$, $P_3^\mu = (E_3, \vec{P}_3)$,

$$E_1 = E_2 + E_3,$$
$$\vec{P} = \vec{P}' + \vec{P}_3$$

$$\Rightarrow \quad d|\vec{P}_2| = d|\vec{P}'| = \frac{d|\vec{P}'|}{d(E_2+E_3)} d(E_2+E_3)$$

where
$$\frac{d(E_2+E_3)}{d|\vec{P}'|} \underset{\uparrow}{=} \frac{d\left[(|\vec{P}'|^2 + m_2^2)^{\frac{1}{2}} + (|\vec{P}|^2 + |\vec{P}'|^2 - 2|\vec{P}||\vec{P}'|\cos\theta_{lab} + m_3^2)^{\frac{1}{2}}\right]}{d|\vec{P}'|}$$

for given $|\vec{P}|$ and θ_{lab}

$$= \frac{|\vec{P}'|}{E_2} + \frac{|\vec{P}'| - |\vec{P}|\cos\theta_{lab}}{E_3}$$

$$\Rightarrow \quad d|\vec{P}'| = \frac{1}{\frac{|\vec{P}'|}{E_2} + \frac{|\vec{P}'| - |\vec{P}|\cos\theta_{lab}}{E_3}} d(E_2+E_3)$$

$$= \frac{E_2E_3}{|\vec{P}'|E_3 + E_2(|\vec{P}'| - |\vec{P}|\cos\theta_{lab})} d(E_2+E_3)$$

integrate out $d|\vec{P}_2|$ using $\delta(E_1 - E_2 - E_3)$

$$\Rightarrow \quad \left(\frac{d\Gamma}{d\Omega}\right)_{lab} = \frac{1}{2E_1}|\mathcal{M}|^2 \frac{1}{(2\pi)^2}\frac{1}{4E_2E_3}|\vec{P}'|^2 \frac{E_2E_3}{|\vec{P}'|(E_2+E_3) - E_2|\vec{P}|\cos\theta_{lab}}$$

$$= \frac{|\mathcal{M}|^2}{32\pi^2 E_1} \frac{|\vec{P}'|}{\left(E_1 - E_2\frac{|\vec{P}|}{|\vec{P}'|}\cos\theta_{lab}\right)}$$

Again, from $\begin{cases} E_1 = E_2 + E_3 \\ \vec{P} = \vec{P}' + \vec{P}_3 \end{cases} \Rightarrow (|\vec{P}|^2 + m_1^2)^{\frac{1}{2}} = (|\vec{P}'|^2 + m_2^2)^{\frac{1}{2}} + (|\vec{P}|^2 + |\vec{P}'|^2 - 2|\vec{P}||\vec{P}'|\cos\theta_{lab} + m_3^2)^{\frac{1}{2}}$

$\Rightarrow$ $|\vec{P}'|$ and E_2, given $|\vec{P}|$ and θ_{lab}.

Since the two-body phase space is the basis of computing higher body phase space, let's look at it a bit more.

Let's compute it $\boxed{\text{in the rest frame of the two-body system.}}$

$$K^\mu = (P_1 + P_2)^\mu \equiv (\sqrt{s}, 0, 0, 0), \quad p^\mu \equiv P_1^\mu$$

$$\int d\Pi_{LISP} = \int \frac{d^3\vec{P}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{P}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(K^\mu - P_1^\mu - P_2^\mu)$$

$$= \int \frac{d^3\vec{P}}{(2\pi)^3\, 2E_1\, 2E_2} (2\pi) \cdot \delta(\sqrt{s} - E_1 - E_2)$$

$$= \int \frac{|\vec{P}|^2 d|\vec{P}|\, d(\cos\theta)\, d\varphi}{(2\pi)^2\, 2(|\vec{P}|^2 + m_1^2)^{\frac{1}{2}}\, 2(|\vec{P}|^2 + m_2^2)^{\frac{1}{2}}} \delta(\sqrt{s} - (|\vec{P}|^2 + m_1^2)^{\frac{1}{2}} - (|\vec{P}|^2 + m_2^2)^{\frac{1}{2}})$$

Since $\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$

then $\int d|\vec{P}|\, \delta(\sqrt{s} - (|\vec{P}|^2 + m_1^2)^{\frac{1}{2}} - (|\vec{P}|^2 + m_2^2)^{\frac{1}{2}}) = \left(\frac{|\vec{P}|}{(|\vec{P}|^2 + m_1^2)^{\frac{1}{2}}} + \frac{|\vec{P}|}{(|\vec{P}|^2 + m_2^2)^{\frac{1}{2}}}\right)^{-1}$

$$= |\vec{P}|^{-1}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)^{-1} = \frac{1}{|\vec{P}|} \frac{E_1 E_2}{\sqrt{s}}$$

$$\Rightarrow \boxed{\int d\Pi_{LISP}} = \int \frac{|\vec{P}|^2 d(\cos\theta)\, d\varphi}{(2\pi)^2\, 2E_1\, 2E_2} \frac{1}{|\vec{P}|} \frac{E_1 E_2}{\sqrt{s}} = \int \frac{|\vec{P}|\, d(\cos\theta)\, d\varphi}{16\pi^2 \sqrt{s}}$$

$$= \boxed{\frac{\bar{\beta}}{8\pi} \int \frac{d(\cos\theta)}{2} \frac{d\varphi}{2\pi}}$$

where $\boxed{\bar{\beta} = \frac{2|\vec{P}|}{\sqrt{s}} = \frac{2\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{\sqrt{s}\; 2\sqrt{s}} = \frac{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{s}}$

It is consistent with the $2 \to 2$ scattering in the center-of-mass frame, and $1 \to 2$ decay in the rest frame of the decaying particle.

It's worthwhile looking at three special cases.

a) When $m_1 = m_2 = m$, so that $E_1 = E_2 = \frac{\sqrt{s}}{2}$, $v_1 = v_2 = \frac{|\vec{p}|}{\left(\frac{\sqrt{s}}{2}\right)}$

$$\bar{\beta} = \frac{(s^2 + 2m^4 + 2sm^2 - 2sm^2 - 2m^4)^{\frac{1}{2}}}{s} = \frac{(s^2 - 4sm^2)^{\frac{1}{2}}}{s} = \left(1 - \frac{4m^2}{s}\right)^{\frac{1}{2}}$$

$$= \left(1 - \frac{4m^2}{4E_1^2}\right)^{\frac{1}{2}} = v_1 \quad \text{(hence the notation } \bar{\beta} \text{)}$$

$$= v_2$$

b_1) when $m_2 = 0$,

$$\bar{\beta} = \frac{(s^2 + m_1^4 - 2m_1^2 s)^{\frac{1}{2}}}{s} = \frac{(s - m_1^2)}{s} = 1 - \frac{m_1^2}{s}$$

b_2) when $m_1 = 0$,

$$\bar{\beta} = \frac{(s^2 + m_2^4 - 2m_2^2 s)^{\frac{1}{2}}}{s} = 1 - \frac{m_2^2}{s}$$

c) when $m_1 = m_2 = 0$

$$\bar{\beta} = 1$$

For n-body phase space $d\Pi_n = \prod_{i=1}^{n} \frac{d^3\vec{P}_i}{(2\pi)^3 2E_{\vec{P}_i}} (2\pi)^4 \delta^4 (\sum P^\mu_{initial} - \sum_{i=1}^{n} P_i^\mu)$,

let's derive a recursion relation to decompose it.

For $1 \le m < n$,

using

$$1 = \int \frac{d^4 q}{(2\pi)^4} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu) \theta(q^0)$$

(since $\sum_{j=1}^{m} E_{\vec{P}_j} > 0$, the $\theta(q^0)$ does not do anything).

and

$$1 = \int \frac{dS_m}{2\pi} 2\pi \delta(S_m - q^2)$$

$$\Rightarrow \quad 1 = \int \frac{d^4 q}{(2\pi)^4} \frac{dS_m}{2\pi} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu) \theta(q^0)\, 2\pi \delta(S_m - q^2)$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{dq^0}{2\pi} \frac{dS_m}{2\pi} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu) \theta(q^0)\, 2\pi \delta(S_m - q^{0\,2} + |\vec{q}|^2)$$

$\delta(x^2 - \alpha^2) = \frac{1}{2|\alpha|}[\delta(x+\alpha) + \delta(x-\alpha)]$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{dq^0}{2\pi} \frac{dS_m}{2\pi} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu) \theta(q^0)\, 2\pi$$
$$\times \frac{1}{2|(S_m + |\vec{q}|^2)^{\frac{1}{2}}|} [\delta(q^0 + (S_m + |\vec{q}|^2)^{\frac{1}{2}}) + \delta(q^0 - (S_m + |\vec{q}|^2)^{\frac{1}{2}})]$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{d(q^2)}{2\pi} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu) \frac{1}{2|q^0|}$$

notice that by $\delta(S_m - q^2)$

$\Rightarrow (S_m + |\vec{q}|^2)^{\frac{1}{2}} = (q^2 + |\vec{q}|^2)^{\frac{1}{2}} = (q^{0\,2})^{\frac{1}{2}} = |q^0|$

$$\Rightarrow \int d\Pi_n = \int \prod_{i=1}^{n} \frac{d^3\vec{P}_i}{(2\pi)^3 2E_{\vec{P}_i}} (2\pi)^4 \delta^4 \underbrace{(\sum P^\mu_{initial} - \sum_{i=1}^{n} P_i^\mu)}_{\sum P^\mu_{initial} - \underbrace{\sum_{j=1}^{m} P_j^\mu}_{q^\mu} - \sum_{i=m+1}^{n} P_i^\mu} \frac{d^3\vec{q}}{(2\pi)^3} \frac{d(q^2)}{2\pi} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu) \frac{1}{2|q^0|}$$

$$= \int \prod_{i=m+1}^{n} \frac{d^3\vec{P}_i}{(2\pi)^3 2E_{\vec{P}_i}} \prod_{j=1}^{m} \frac{d^3\vec{P}_j}{(2\pi)^3 2E_{\vec{P}_j}} (2\pi)^4 \delta^4 (\sum P^\mu_{initial} - q^\mu - \sum_{i=m+1}^{n} P_i^\mu) \frac{d^3\vec{q}}{(2\pi)^3 2|q^0|}$$
$$\times \frac{d(q^2)}{2\pi} (2\pi)^4 \delta^4 (q^\mu - \sum_{j=1}^{m} P_j^\mu)$$

$$= \int \frac{d(q^2)}{2\pi} d\Pi_m \, d\Pi_{n-m+1}$$

That is,

$$d\Pi_n(\Sigma P_{\text{initial}}\,;\,P_1,\cdots,P_n)=\frac{d(q^2)}{2\pi}\,d\Pi_m(q\,;\underbrace{P_1,\cdots,P_m}_{m})\,d\Pi_{n-m+1}(\Sigma P_{\text{initial}}\,;\underbrace{q,P_{m+1},P_{m+2},\cdots,P_n}_{n-m+1})$$

notation : $d\Pi_n(\boxed{}\,;\,\boxed{})$

final momenta

initial momenta

n-body

In particular, we can decompose any n-body phase space into a 2-body phase space and a $(n-1)$-body phase space :

$$d\Pi_n(\Sigma P_{\text{initial}}\,;\,P_1,\cdots,P_n)=\frac{d(q^2)}{2\pi}\,d\Pi_{n-1}(q\,;\,P_1,\cdots,P_{n-1})\,d\Pi_2(\Sigma P_{\text{initial}}\,;\,q,P_n)$$

and we can keep doing so to decompose the $(n-1)$-body phase space into a 2-body phase space and a $(n-2)$-body phase space, $\cdots$, and eventually, we can decompose a n-body phase space into $(n-1)$ number of 2-body phase space.