Examples of calculations in QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \gamma^\mu D_\mu \psi - m\bar{\psi}\psi$$

where $D_\mu \psi = \partial_\mu \psi + i|e| q A_\mu \psi$

where q is the charge of the particle (not antiparticle) described by the field ψ. For example, $q=-1$ when ψ describes the electron-positron field, $q=+1$ when ψ describes the proton-antiproton field.

put $D_\mu\psi$ in $\mathcal{L}$, we get

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - |e| q \bar{\psi}\gamma^\mu\psi A_\mu$$

If we have more charged spinor fields in the system, we will just write the last three terms for each of them.

Recall that in classical electromagnetism,

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

and from this $\mathcal{L}_{EM}$, we can get the Maxwell equations by writing Euler-Lagrangian equation:

$$\frac{\partial \mathcal{L}_{EM}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}_{EM}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\Rightarrow \quad -j^\nu - \partial_\mu \frac{\partial(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = 0$$

where

$$\frac{\partial(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\frac{\partial[(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{4}\times 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha)(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)$$

$$= -\frac{1}{2}[(\partial^\mu A^\nu - \partial^\nu A^\mu) - (\partial^\nu A^\mu - \partial^\mu A^\nu)]$$

$$= -F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \end{cases}$$

, where $j^\mu = (\rho, \vec{j})$,

$E^i = F^{i0}$

$B^i = -\frac{1}{2}\varepsilon^{ijk}F^{jk}$

The other two Maxwell equations come from the identity

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = \partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda)$$
$$= 0$$

$$\Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \end{cases}$$

From $\partial_\mu F^{\mu\nu} = j^\nu$

we have $\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=0} = \partial_\nu j^\nu$

For $\mathcal{L}$, the Euler-Lagrangian equation for A_μ gives

$\partial_\mu F^{\mu\nu} = \sum_i |e| q_i \bar{\psi}_i \gamma^\nu \psi_i$, where i is for different Dirac fields.

So, we can identify $j^\nu = \sum_i |e| q_i \bar{\psi}_i \gamma^\nu \psi_i$

In fact, for Dirac field, the Noether current from internal phase transformation $\psi_i \to \psi_i' = e^{-i\delta\alpha_i}\psi_i$ is

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu \psi_i)}(-i\psi_i) + i\bar{\psi}_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu \bar{\psi}_i)} = \bar{\psi}_i \gamma^\mu \psi_i$$

So the above identification for the conserved current $j^\nu = \sum_i |e| q_i \bar{\psi}_i \gamma^\nu \psi_i$ indeed self-consistent (we merely multiply each of the conserved Noether current by a constant $|e| q_i$)

In the free field theory of A_μ, i.e., $\mathcal{L}_{\text{free EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$,

we can choose the Coulomb gauge $\begin{cases} \vec{\nabla}\cdot\vec{A} = 0 \\ A^0 = 0 \end{cases}$ to reduce the number of independent solutions of A_μ from four to two, that is, we can work with the two physical transversal polarizations only.

Since $\Pi^0 = \dfrac{\partial\mathcal{L}_{\text{free EM}}}{\partial(\partial A_0/\partial t)} = 0$ (since there is no $\partial_0 A^0$ term in $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$)

we only have the commutation relation

$$[A_i(\vec{x},t),\ \dot{A}_j(\vec{y},t)] = i\int_{-\infty}^{+\infty}\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_{ij}-\frac{k_i k_j}{|\vec{k}|^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$[A_i(\vec{x},t),A_j(\vec{y},t)] = [\dot{A}_i(\vec{x},t),\dot{A}_j(\vec{y},t)] = 0$$

and $\vec{A}(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}})\sum_{\lambda=1}^{2}\left(\vec{e}(\vec{k},\lambda)\,a_{\vec{k},\lambda}\,e^{-ik\cdot x}+\vec{e}^{*}(\vec{k},\lambda)\,a^{+}_{\vec{k},\lambda}\,e^{ik\cdot x}\right)$

where $[a_{\vec{p},r},a^{+}_{\vec{k},s}] = \delta_{rs}\,\delta^3(\vec{p}-\vec{k})\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2\frac{1}{(2\pi)^3}$.

$$[a_{\vec{p},r},a_{\vec{k},s}] = [a^{+}_{\vec{p},r},a^{+}_{\vec{k},s}] = 0.$$

$$\underbrace{A^i(x)A^j(y)} = \underbrace{A^j(y)A^i(x)} = \int_{-\infty}^{+\infty} d^4k\, e^{-ik\cdot(x-y)}\frac{1}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\left(\delta^{ij}-\frac{k^i k^j}{|\vec{k}|^2}\right)$$

Also recall that $\vec{e}^{*}(\vec{k},\lambda)\cdot\vec{e}(\vec{k},\lambda') = \delta_{\lambda\lambda'},\ (\lambda,\lambda'=1,2)$

$$\vec{k}\cdot\vec{e}(\vec{k},\lambda)=0$$

$$\sum_{\lambda=1}^{2}(\vec{e}^{*}(\vec{k},\lambda))^i(\vec{e}(\vec{k},\lambda))^j = \delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}$$

Now, from $\partial_\mu F^{\mu\nu} = j^\nu$, we get.

$$\partial_\mu F^{\mu i} = \partial_\mu(\partial^\mu A^i - \partial^i A^\mu) = \Box A^i + \partial_i(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\partial_i A^0 = j^i$$

$$\Rightarrow\quad \partial_i\left(\Box A^i + \partial_i(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\partial_i A^0\right) = \partial_i j^i$$

$$\Rightarrow\quad \underbrace{\Box(\vec{\nabla}\cdot\vec{A}) + \partial_i\partial_i(\vec{\nabla}\cdot\vec{A})} + \frac{\partial}{\partial t}\nabla^2 A^0 = \vec{\nabla}\cdot\vec{j}$$

note that $\nabla^2 = \partial_i\partial_i$, $\Box = \frac{\partial^2}{\partial t^2} - \partial_i\partial_i$

$$\Rightarrow\quad \frac{\partial^2}{\partial t^2}(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\nabla^2 A^0 = \vec{\nabla}\cdot\vec{j}$$

If we <u>assume</u> that $\nabla^2 A^0 = -j^0$,

then $\frac{\partial}{\partial t}\nabla^2 A^0 = -\frac{\partial}{\partial t}j^0$

due to current conservation from $\partial_\mu F^{\mu\nu}=j^\nu \Rightarrow \partial_\mu j^\mu = 0$

$$\Rightarrow\quad \frac{\partial^2}{\partial t^2}(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}j^0 = \partial_\mu j^\mu = 0$$

Therefore, if $\vec{\nabla}\cdot\vec{A}=0$ and $\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=0$ holds at one time, it will hold at all time.

then $\partial_\mu F^{\mu 0} = j^0 \Rightarrow \partial_\mu(\partial^\mu A^0 - \partial^0 A^\mu) = \partial_i(\partial^i A^0 - \partial^0 A^i) = j^0$

$\Rightarrow \quad -\nabla^2 A^0 - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = j^0$

$\Rightarrow \quad \nabla^2 A^0 = -j^0 .$

So, indeed, our above assumption is consistent.

That is to say, if we can have $\vec{\nabla}\cdot\vec{A}=0$ and $\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=0$ holds at one time, then we can have the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$ held at all time, and we have $\quad \nabla^2 A^0 = -j^0 .$

In fact, due to gauge symmetry, we can ensure that $\vec{\nabla}\cdot\vec{A}=0$, Since if $\vec{\nabla}\cdot\vec{A}\neq 0$, we can do a gauge transformation $\quad \vec{A}' = \vec{A} - \vec{\nabla}\Lambda$, such that $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda \underset{\substack{\uparrow \\ \text{require}}}{\equiv} 0$, and we know that this equation can be solved to get Λ.

Therefore, it is always possible to choose the Coulomb gauge, for which we have

$$\begin{cases} \vec{\nabla}\cdot\vec{A} = 0 \\ \nabla^2 A^0 = -j^0 \end{cases}$$

The solution of $\underbrace{\nabla^2 A^0 = -j^0}_{\text{possion's equation.}}$ is $\quad A^0(\vec{x},t) = \frac{1}{4\pi}\int d^3\vec{x}' \frac{j^0(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$, which is just the Coulomb's law. ($j^0 \equiv \rho$)

Note that when there is no source, i.e., $j^\mu = 0$, we have $A^0(\vec{x},t)=0$, and this is consistent with our result for the free field theory of photon field

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Now let's re-write the original QED Lagrangian in the Coulomb gauge.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu + \underbrace{\text{terms with no } A^\mu}_{\sum_i(\bar{\psi}_i i\gamma^\mu\partial_\mu\psi_i - m_i\bar{\psi}_i\psi_i)}$$

$$= -\frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial^0 A^i - \partial^i A^0) - \frac{1}{4}(\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i)$$
$$- j^0 A^0 + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

$$= -\frac{1}{2}\left[(\partial_0 A_i)\partial^0 A^i - 2(\partial_0 A_i)(\partial^i A^0) + (\partial_i A_0)(\partial^i A^0)\right]$$

$$-\frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) - j^0 A^0 + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

use $|\vec{B}|^2 = \vec{B}\cdot\vec{B} = B^i B^i = (-\frac{1}{2})^2 \varepsilon^{ijk}\varepsilon^{iab} F^{jk}F^{ab}$

$$= \frac{1}{4}(\delta^{ja}\delta^{kb} - \delta^{jb}\delta^{ka})F^{jk}F^{ab}$$

$$= \frac{1}{4}(F^{ab}F^{ab} - F^{ba}F^{ab})$$

$$= \frac{1}{2}F^{ij}F^{ij} = \frac{1}{2}F_{ij}F^{ij}$$

$$= \frac{1}{2}(\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i)$$

$$= \partial_i A_j(\partial^i A^j - \partial^j A^i)$$

note :
$\varepsilon^{123} = \varepsilon_{123} = 1$
$\varepsilon_{ijk}\varepsilon^{imn} = \delta^m_j\delta^n_k - \delta^n_j\delta^m_k$
$\varepsilon_{jmn}\varepsilon^{imn} = 2\delta^i_j$
$\varepsilon_{ijk}\varepsilon^{ijk} = 6$

and Define $\vec{E}_\perp \equiv -\partial_0\vec{A}$, $\Rightarrow |\vec{E}_\perp|^2 = (\partial_0\vec{A})\cdot(\partial^0\vec{A}) = -(\partial_0 A_i)(\partial^0 A^i)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}|\vec{E}_\perp|^2 - \frac{1}{2}|\vec{B}|^2 - (\partial_0 A_i)(\partial_i A^0) + \frac{1}{2}(\partial_i A^0)(\partial_i A^0)$$

$$- j^0 A^0 + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

For the term $\frac{1}{2}(\partial_i A^0)(\partial_i A^0)$, it is

$$\frac{1}{2}(\partial_i A^0)(\partial_i A^0) = \frac{1}{2}\partial_i(A^0\partial_i A^0) - \frac{1}{2}A^0(\partial_i\partial_i A^0)$$

$$= \frac{1}{2}\vec{\nabla}\cdot(A^0\vec{\nabla}A^0) - \frac{1}{2}A^0\nabla^2 A^0$$

where the contribution of the first term to $L = \int d^3\vec{x}\,\mathcal{L}$ vanishes since $\int d^3\vec{x}\ \vec{\nabla}\cdot(A^0\vec{\nabla}A^0) = \iint_S (A^0\vec{\nabla}A^0)\,dS = 0$ when impose the condition that the field vanishes at infinity, which is true since the source of A^0 (i.e., j^0) distributes in a finite volumn, so that $A^0(t, \vec{x}\to\infty) \to 0$ (i.e., the Coulomb potential goes to zero at infinity).

Therefore, we can drop the $\frac{1}{2}\vec{\nabla}\cdot(A^0\vec{\nabla}A^0)$ term in our $\mathcal{L}$.

Similarly, the term

$$-(\partial_0 A_i)(\partial_i A^0) = -\partial_i (A^0 \partial_0 A_i) + A^0 \partial_0 \partial_i A_i$$

$$= -\vec{\nabla}\cdot(A^0 \frac{\partial}{\partial t}\vec{A}) + A^0 \frac{\partial}{\partial t}\cancel{(\vec{\nabla}\cdot\vec{A})}$$

and $\int d^3\vec{x}\ \vec{\nabla}\cdot(A^0 \frac{\partial}{\partial t}\vec{A}) = \iint_S (A^0 \frac{\partial}{\partial t}\vec{A})\, dS = 0$ $\quad 0$ by Coulomb gauge.

Therefore,

$$\mathcal{L} = \frac{1}{2}(|\vec{E}_\perp|^2 - |\vec{B}|^2) \underbrace{\underbrace{- \frac{1}{2} A^0 \nabla^2 A^0 - j^0 A^0}_{+\frac{1}{2}A^0 j^0 - j^0 A^0}}_{-\frac{1}{2} j^0 A^0} + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

$$= \frac{1}{2}(|\vec{E}_\perp|^2 - |\vec{B}|^2) + \sum_l (\bar{\psi}_l i\gamma^\mu \partial_\mu \psi_l - m_l \bar{\psi}_l \psi_l)$$

$$- \frac{1}{2} j^0 A^0 + \vec{j}\cdot\vec{A}.$$

where the first line is $\mathcal{L}_0$, and the second line is

$$\mathcal{L}_{int} = -\frac{1}{2} j^0 A^0 + \vec{j}\cdot\vec{A}$$

$$= -\frac{1}{2} j^0(\vec{x},t) \frac{1}{4\pi} \int d^3\vec{x}' \frac{j^0(\vec{x}',t)}{|\vec{x}'-\vec{x}|} + \vec{j}\cdot\vec{A}$$

$$= -\frac{1}{8\pi} \int d^3\vec{x}' \frac{j^0(\vec{x}',t)\, j^0(\vec{x},t)}{|\vec{x}'-\vec{x}|} + \vec{j}\cdot\vec{A}$$

put in $j^0 = \sum_l |e| \varepsilon_l \bar{\psi}_l \gamma^0 \psi_l$ and $j^i = \sum_l |e| \varepsilon_l \bar{\psi}_l \gamma^i \psi_l$ in it, we get

$$\mathcal{L}_{int} = -\frac{e^2}{8\pi} \int d^3\vec{x}' \frac{\sum_{lk} \varepsilon_l \varepsilon_k\ \bar{\psi}_l(\vec{x},t)\gamma^0\psi_l(\vec{x},t)\ \bar{\psi}_k(\vec{x}',t)\gamma^0\psi_k(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

$$+ |e| \sum_l \varepsilon_l \bar{\psi}_l \gamma^i \psi_l A^i$$

$\Rightarrow \mathcal{H}_{int} = -\mathcal{L}_{int}$, note that there is no field derivative in $\mathcal{L}_{int}$