## propagator

The contraction in the Wick's theorem is just the propagator.

① For a real scalar field,

$$\underbrace{\hat{\varphi}(x)\,\hat{\varphi}(y)} \equiv \langle 0|\,T(\hat{\varphi}(x)\,\hat{\varphi}(y))\,|0\rangle$$

recall that the field operators in Wick's theorem are the ones in interaction picture, and the expressions are identical to the free field operators in Heisenberg picture we are familiar with,

$$\hat{\varphi}(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \left( \hat{a}_{\vec{p}}\, e^{-ip\cdot x} + \hat{a}^{+}_{\vec{p}}\, e^{ip\cdot x} \right)$$

where $[\,a_{\vec{p}},\ a^{+}_{\vec{k}}\,] = \frac{1}{(2\pi)^3\, 2E_{\vec{p}}} \cdot \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p}-\vec{k})$

$$\Rightarrow \langle 0|\,T(\hat{\varphi}(x)\,\hat{\varphi}(y))\,|0\rangle$$

$$= \langle 0|\, \theta(x_0 - y_0)\,\hat{\varphi}(x)\,\hat{\varphi}(y) + \theta(y_0 - x_0)\,\hat{\varphi}(y)\,\hat{\varphi}(x)\,|0\rangle$$

Since $\hat{a}|0\rangle = 0$, $\langle 0|\hat{a}^{+} = 0$,

then
$$\langle 0|\,\hat{\varphi}(x)\,\hat{\varphi}(y)\,|0\rangle$$
$$= \langle 0|\,\hat{\varphi}^{(+)}(x)\,\hat{\varphi}^{(-)}(y)\,|0\rangle$$
$$= \langle 0|\int_{-\infty}^{+\infty} d^3\vec{p}\int_{-\infty}^{+\infty} d^3\vec{k}\; C(E_{\vec{p}})\,C(E_{\vec{k}})\; e^{-ip\cdot x}\, e^{ik\cdot y}\; \hat{a}_{\vec{p}}\, a^{+}_{\vec{k}}\,|0\rangle$$

where
$$\langle 0|\,\hat{a}_{\vec{p}}\,\hat{a}^{+}_{\vec{k}}\,|0\rangle$$
$$= \langle 0|\,[\hat{a}_{\vec{p}},\,\hat{a}^{+}_{\vec{k}}]\,|0\rangle$$
$$= \frac{1}{(2\pi)^3\, 2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p}-\vec{k})\; \underbrace{\langle 0|0\rangle}_{=1}$$

$$\Rightarrow \langle 0|\,\hat{\varphi}(x)\,\hat{\varphi}(y)\,|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\; \frac{1}{(2\pi)^3\, 2E_{\vec{p}}}\; e^{-ip\cdot(x-y)}$$

$$\Rightarrow \underbrace{\hat{\varphi}(x)\,\hat{\varphi}(y)} = \theta(x_0 - y_0)\int_{-\infty}^{+\infty} d^3\vec{p}\; \frac{1}{(2\pi)^3\, 2E_{\vec{p}}}\; e^{-ip\cdot(x-y)} + \theta(y_0 - x_0)\int_{-\infty}^{+\infty} d^3\vec{p}\; \frac{1}{(2\pi)^3\, 2E_{\vec{p}}}\; e^{-ip\cdot(y-x)}$$

where $\int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{-ip\cdot(x-y)} = \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{-iE_{\vec{p}}(x_0-y_0)} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$

$$\int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{-ip\cdot(y-x)} = \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{-iE_{\vec{p}}(y_0-x_0)} e^{i\vec{p}\cdot(\vec{y}-\vec{x})}$$

$$\underset{\vec{p}\to-\vec{p}\,'}{\overset{\uparrow}{=}} \int_{+\infty}^{-\infty} d^3(-\vec{p}\,')\, \frac{1}{(2\pi)^3 2E_{\vec{p}\,'}} e^{-iE_{\vec{p}\,'}(y_0-x_0)} e^{i\vec{p}\,'\cdot(\vec{x}-\vec{y})}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,'\, \frac{1}{(2\pi)^3 2E_{\vec{p}\,'}} e^{-iE_{\vec{p}\,'}(y_0-x_0)} e^{i\vec{p}\,'\cdot(\vec{x}-\vec{y})}$$

$$\underset{\vec{p}\,'\to\vec{p}}{\overset{\uparrow}{=}} \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{-iE_{\vec{p}}(y_0-x_0)} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$\Rightarrow \underbrace{\hat{\varphi}(x)\hat{\varphi}(y)} = \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left[\theta(x_0-y_0)e^{-iE_{\vec{p}}(x_0-y_0)} + \theta(y_0-x_0)e^{-iE_{\vec{p}}(y_0-x_0)}\right]$$

Note that $E_{\vec{p}} = (|\vec{p}|^2+m^2)^{\frac{1}{2}}$.

To calculate [ ], we use the integral representation of the step function.

$$\theta(k) = \begin{cases} 1, & \text{if } k>0 \\ 0, & \text{if } k<0 \end{cases}$$

(note that the $k$ & $x$ here have nothing to do with the symbols used above)

$$= \lim_{\varepsilon\to0^+} \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} dx\, \frac{e^{-ikx}}{x+i\varepsilon}$$

$$= \lim_{\varepsilon\to0^+} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{x-i\varepsilon}$$

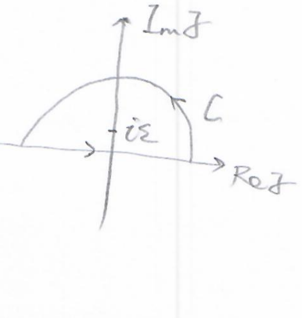
Since the step function is real, its complex conjugate equals itself.

To convince ourselves, let's show that the intergral indeed give the step function. So let's evaluate $I(k,\varepsilon) \equiv \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{x-i\varepsilon}$, where $\varepsilon>0$.

when $k>0$, let's evaluate the contour integral

$$I(k,\varepsilon) = \frac{1}{2\pi i}\int_C dz\, \frac{e^{ikz}}{z-i\varepsilon}$$

where the contour radius is taken to infinity.



For the integral along the semicircular part,

$$I_R = \frac{1}{2\pi i}\int_0^{\pi} \frac{1}{Re^{i\theta}-i\varepsilon}\, e^{ik(R\cos\theta + iR\sin\theta)}\, Re^{i\theta} i\, d\theta$$

$z = Re^{i\theta}$

letting $R$ be sufficiently large, $\left|\frac{1}{Re^{i\theta}-i\varepsilon}\right| < \varsigma$ a small number, which goes to zero when $R\to\infty$

$$|I_R| < \left(\frac{1}{2\pi}\right)\varsigma R\int_0^{\pi} e^{-kR\sin\theta}\, d\theta$$

$$= \left(\frac{1}{2\pi}\right)2\varsigma R\int_0^{\frac{\pi}{2}} e^{-kR\sin\theta}\, d\theta$$

$$\le \left(\frac{1}{2\pi}\right)2\varsigma R\int_0^{\frac{\pi}{2}} e^{-kR\frac{2}{\pi}\theta}\, d\theta = \left(\frac{1}{2\pi}\right)2\varsigma R\,\frac{e^{-kR}-1}{-kR\frac{2}{\pi}} = \left(\frac{1}{2\pi}\right)\frac{\varsigma\pi}{k}(1-e^{-kR})$$

$\frac{2}{\pi}\theta \le \sin\theta$ in the range $[0, \frac{\pi}{2}]$

$\to 0$ as $R\to\infty$

So the integral along the semicircular part does not contribute.

$$\Rightarrow \frac{1}{2\pi i}\int_C dz\, \frac{e^{ikz}}{z-i\varepsilon} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{x-i\varepsilon}$$

Residue theorem $\to$ $\|$

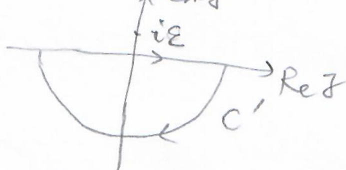
$$\frac{1}{2\pi i}\, e^{ik(i\varepsilon)}\, 2\pi i$$

$\|$

$$e^{-k\varepsilon}$$

when $k<0$, lets close the contour in the lower half plane.

$$I(k,\varepsilon) = \frac{1}{2\pi i}\int_{C'} dz\, \frac{e^{ikz}}{z-i\varepsilon}$$




for the integral along the semicircular part,

again $\left|\frac{1}{Re^{i\theta}-i\varepsilon}\right| < \varsigma$, which goes to zero when $R\to\infty$.

$$I_R = \left(\frac{1}{2\pi i}\right)\int_0^{-\pi} \frac{1}{Re^{i\theta}-i\varepsilon}\, e^{ik(R\cos\theta+iR\sin\theta)}\, Re^{i\theta} i\, d\theta \Rightarrow |I_R| \le \frac{1}{2\pi}\int_{-\pi}^{0} \frac{R}{|Re^{i\theta}-i\varepsilon|}\, e^{-kR\sin\theta}\, d\theta$$

$$|I_R| < \left(\frac{1}{2\pi}\right)\varsigma R\int_{-\pi}^{0} e^{-kR\sin\theta}\, d\theta \underset{\theta\to-\theta}{=} \left(\frac{1}{2\pi}\right)(-\varsigma R)\int_{\pi}^{0} e^{kR\sin\theta}\, d\theta = \left(\frac{1}{2\pi}\right)2\varsigma R\int_0^{\frac{\pi}{2}} e^{kR\sin\theta}\, d\theta$$

$$\le \left(\frac{1}{2\pi}\right)\varsigma R\int_0^{\frac{\pi}{2}} e^{kR\frac{2}{\pi}\theta}\, d\theta = \left(\frac{1}{2\pi}\right)2\varsigma R\,\frac{1}{kR\frac{2}{\pi}}\left(e^{kR}-1\right) = \left(\frac{1}{2\pi}\right)\frac{\varsigma\pi}{k}\left(e^{kR}-1\right) \to 0 \text{ as } R\to\infty$$

so the integral along the semicircular part does not contribute.

$$\Rightarrow \frac{1}{2\pi i}\int_{C'} dz \frac{e^{ikz}}{z-i\varepsilon} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x-i\varepsilon}$$

Residue theorem $\longrightarrow \|$

$0$

$$\Rightarrow I(k,\varepsilon) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dx \frac{e^{ikx}}{x-i\varepsilon} = \begin{cases} e^{-k\varepsilon} & , k>0 \\ 0 & , k<0 \end{cases}$$

$$\Rightarrow \lim_{\varepsilon\to 0^+} I(k,\varepsilon) = \begin{cases} 1 & , k>0 \\ 0 & , k<0 \end{cases}$$

which is $\theta(k)$

$$\text{So, } [\ \ ] = \frac{1}{2\pi i}\left\{\int_{-\infty}^{+\infty} dk_0 \frac{e^{i(x_0-y_0)k_0}}{k_0-i\varepsilon} e^{-iE_{\vec p}(x_0-y_0)} + \int_{-\infty}^{+\infty} dk_0 \frac{e^{i(y_0-x_0)k_0}}{k_0-i\varepsilon} e^{-iE_{\vec p}(y_0-x_0)}\right\}, \text{ where } \varepsilon\to 0^+$$

$$\overset{k_0\to E_{\vec p}+k_0'}{=} \frac{1}{2\pi i}\left\{\int_{-\infty}^{+\infty} dk_0' \frac{e^{i(x_0-y_0)k_0'}}{E_{\vec p}+k_0'-i\varepsilon} + \int_{-\infty}^{+\infty} dk_0' \frac{e^{i(y_0-x_0)k_0'}}{E_{\vec p}+k_0'-i\varepsilon}\right\}$$

where the second integral $\int_{-\infty}^{+\infty} dk_0' \frac{e^{i(y_0-x_0)k_0'}}{E_{\vec p}+k_0'-i\varepsilon}$

$$\overset{k_0'\to -k_0''}{=} \int_{+\infty}^{-\infty} d(-k_0'') \frac{e^{-i(y_0-x_0)k_0''}}{E_{\vec p}-k_0''-i\varepsilon} = \int_{-\infty}^{+\infty} dk_0'' \frac{e^{i(x_0-y_0)k_0''}}{E_{\vec p}-k_0''-i\varepsilon}$$

$$\overset{k_0''\to k_0'}{=} \int_{-\infty}^{+\infty} dk_0' \frac{e^{i(x_0-y_0)k_0'}}{E_{\vec p}-k_0'-i\varepsilon}$$

$$\Rightarrow [\ \ ] = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dk_0' e^{i(x_0-y_0)k_0'}\left(\frac{1}{E_{\vec p}+k_0'-i\varepsilon} + \frac{1}{E_{\vec p}-k_0'-i\varepsilon}\right)$$

use $\lim_{\varepsilon\to 0^+}\left(\frac{1}{E_{\vec p}+k_0'-i\varepsilon} + \frac{1}{E_{\vec p}-k_0'-i\varepsilon}\right) = \lim_{\varepsilon'\to 0^+}\frac{2E_{\vec p}}{E_{\vec p}^2-k_0'^2-i\varepsilon'}$

$$\Rightarrow [\ \ ] = \frac{1}{2\pi i} 2E_{\vec p}\int_{-\infty}^{+\infty} dk_0' e^{i(x_0-y_0)k_0'} \frac{1}{E_{\vec p}^2-k_0'^2-i\varepsilon'}$$

(Note that as long as $(X+Y)>0$, we have $\lim_{\varepsilon\to 0^+}\left(\frac{1}{X-i\varepsilon}+\frac{1}{Y-i\varepsilon}\right) = \lim_{\varepsilon'\to 0^+}\frac{X+Y}{XY-i\varepsilon'}$ )

$$\Rightarrow \quad \underbrace{\hat\varphi(x)\,\hat\varphi(y)} = \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec p}}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\, \frac{E_{\vec p}}{\pi i}\int_{-\infty}^{+\infty} dk_0'\, e^{i(x_0-y_0)k_0'}\, \frac{1}{E_{\vec p}^2-k_0'^2-i\varepsilon'}$$

$$= \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^4}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\, \frac{1}{i}\int_{-\infty}^{+\infty} dk_0'\, \frac{e^{i(x_0-y_0)k_0'}}{E_{\vec p}^2-k_0'^2-i\varepsilon'}$$

use $E_{\vec p}^2 - k_0'^2 = |\vec{p}|^2 + m^2 - k_0'^2$ and define

$$k^\mu \equiv (k_0', \vec{p})$$

$$\Rightarrow \quad k^2 = k_0'^2 - |\vec{p}|^2$$

$$\Rightarrow \quad \underbrace{\hat\varphi(x)\,\hat\varphi(y)} = \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)}\, \frac{i}{k^2-m^2+i\varepsilon'}$$

$$= \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i}{k^2-m^2+i\varepsilon'}$$

Since $\varepsilon'$ is also an infinitesimal positive number, we can write it as $\varepsilon$

$$\boxed{\underbrace{\hat\varphi(x)\,\hat\varphi(y)} = \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i}{k^2-m^2+i\varepsilon}}$$

(2) For a complex scalar field

$$\hat\varphi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p})\left(\hat a_{\vec p}\, e^{-ip\cdot x} + \hat b_{\vec p}^{+}\, e^{ip\cdot x}\right), \quad \hat\varphi^{+}(y) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p})\left(\hat a_{\vec p}^{+} e^{ip\cdot y} + b_{\vec p}\, e^{-ip\cdot y}\right)$$

since $\langle 0|\hat\varphi(x)\,\hat\varphi(y)|0\rangle = 0$ (note that $\langle 0|a_{\vec p}\, b_{\vec k}^{+}|0\rangle = \langle 0|b_{\vec k}^{+} a_{\vec p}|0\rangle = 0$

and $\langle 0|\hat\varphi^{+}(x)\,\hat\varphi^{+}(y)|0\rangle = 0$ (note that $\langle 0|b_{\vec p}\, a_{\vec k}^{+}|0\rangle = \langle 0|a_{\vec k}^{+} b_{\vec p}|0\rangle = 0$ )

we only need to worry $\underbrace{\hat\varphi(x)\,\hat\varphi^{+}(y)}$ and $\underbrace{\hat\varphi^{+}(x)\,\hat\varphi(y)}$

Since $\underbrace{\hat\varphi^{+}(x)\,\hat\varphi(y)} = \langle 0|T(\hat\varphi^{+}(x)\varphi(y))|0\rangle = \langle 0|\theta(x_0-y_0)\,\hat\varphi^{+}(x)\,\hat\varphi(y) + \theta(y_0-x_0)\,\hat\varphi(y)\,\hat\varphi^{+}(x)|0\rangle$

and $\underbrace{\hat\varphi(y)\,\hat\varphi^{+}(x)} = \langle 0|T(\hat\varphi(y)\,\hat\varphi^{+}(x))|0\rangle = \langle 0|\theta(y_0-x_0)\,\hat\varphi(y)\,\hat\varphi^{+}(x) + \theta(x_0-y_0)\,\hat\varphi^{+}(x)\varphi(y)|0\rangle$

$$= \underbrace{\hat\varphi^{+}(x)\,\varphi(y)}$$

we just need to worry about $\underbrace{\varphi(x)\,\hat\varphi^{+}(y)}$.

In fact, in general, $\underbrace{\hat A(x)\,\hat B(y)} = \langle 0|\theta(x_0-y_0)\hat A(x)\hat B(y) + \varepsilon_{AB}\,\theta(y_0-x_0)\,\hat B(y)\hat A(x)|0\rangle = \varepsilon_{AB}\underbrace{\hat B(y)\,\hat A(x)}$

$$\underbrace{\hat{\varphi}(x)\,\hat{\varphi}^+(y)} = \langle 0|T(\hat{\varphi}(x)\,\hat{\varphi}^+(y))|0\rangle = \langle 0|\theta(x_0-y_0)\,\hat{\varphi}(x)\,\hat{\varphi}^+(y) + \theta(y_0-x_0)\,\hat{\varphi}^+(y)\,\hat{\varphi}(x)|0\rangle$$

where $\langle 0|\hat{\varphi}(x)\,\hat{\varphi}^+(y)|0\rangle$

$$= \langle 0|\hat{\varphi}^{(+)}(x)\,\hat{\varphi}^{+(-)}(y)|0\rangle = \langle 0|\int_{-\infty}^{+\infty} d^3\vec{p}\int_{-\infty}^{+\infty} d^3\vec{k}\; C(E_{\vec{p}})\,C(E_{\vec{k}})\,e^{-ip\cdot x}\,e^{ik\cdot y}\,\hat{a}_{\vec{p}}\,\hat{a}^+_{\vec{k}}|0\rangle$$

where $\langle 0|\hat{a}_{\vec{p}}\,\hat{a}^+_{\vec{k}}|0\rangle = \frac{1}{(2\pi)^3 2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p}-\vec{k})$

$$\Rightarrow \langle 0|\hat{\varphi}(x)\,\hat{\varphi}^+(y)|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\,\frac{1}{(2\pi)^3 2E_{\vec{p}}}\,e^{-ip\cdot(x-y)}$$

$$\langle 0|\hat{\varphi}^+(y)\,\hat{\varphi}(x)|0\rangle = \langle 0|\hat{\varphi}^{+(+)}(y)\,\hat{\varphi}^{(-)}(x)|0\rangle = \langle 0|\int_{-\infty}^{+\infty} d^3\vec{p}\int_{-\infty}^{+\infty} d^3\vec{k}\; C(E_{\vec{p}})\,C(E_{\vec{k}})\,e^{-ip\cdot y}\,e^{ik\cdot x}\,\hat{b}_{\vec{p}}\,\hat{b}^+_{\vec{k}}|0\rangle$$

where $\langle 0|\hat{b}_{\vec{p}}\,\hat{b}^+_{\vec{k}}|0\rangle = \langle 0|[\hat{b}_{\vec{p}},\hat{b}^+_{\vec{k}}]|0\rangle$

$$= \frac{1}{(2\pi)^3 2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p}-\vec{k})$$

$$\Rightarrow \langle 0|\hat{\varphi}^+(y)\,\hat{\varphi}(x)|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\,\frac{1}{(2\pi)^3 2E_{\vec{p}}}\,e^{-ip\cdot(y-x)}$$

$$\Rightarrow \underbrace{\hat{\varphi}(x)\,\hat{\varphi}^+(y)} = \int_{-\infty}^{+\infty} d^3\vec{p}\,\frac{1}{(2\pi)^3 2E_{\vec{p}}}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left[\theta(x_0-y_0)\,e^{-iE_{\vec{p}}(x_0-y_0)} + \theta(y_0-x_0)\,e^{-iE_{\vec{p}}(y_0-x_0)}\right]$$

exactly the same as the real scalar case

$$\Rightarrow \boxed{\underbrace{\hat{\varphi}(x)\,\hat{\varphi}^+(y)} = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2-m^2+i\varepsilon} = \underbrace{\hat{\varphi}^+(y)\,\hat{\varphi}(x)}}$$

Since $\underbrace{\hat{\varphi}^+(x)\,\varphi(y)} = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(y-x)}\,\frac{i}{k^2-m^2+i\varepsilon}$

$$\overset{k\to -k}{=} \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2-m^2+i\varepsilon}$$

$$= \underbrace{\hat{\varphi}(x)\,\hat{\varphi}^+(y)}$$

we have $\boxed{\underbrace{\hat{\varphi}(x)\,\hat{\varphi}^+(y)} = \underbrace{\hat{\varphi}^+(x)\,\varphi(y)}}$

③ For a Dirac fermion field.

$$\hat{\psi}(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left( u(\vec{p},s)\, \hat{b}_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, \hat{d}^{+}_{\vec{p},s}\, e^{ip\cdot x} \right)$$

$$\hat{\bar{\psi}}(y) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left( \bar{u}(\vec{p},s)\, \hat{b}^{+}_{\vec{p},s}\, e^{ip\cdot x} + \bar{v}(\vec{p},s)\, \hat{d}_{\vec{p},s}\, e^{-ip\cdot x} \right)$$

$$\{ b_{\vec{p},s}, b^{+}_{\vec{k},r} \} = \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{rs}\, \delta^3(\vec{p}-\vec{k}) = \{ d_{\vec{p},s}, d^{+}_{\vec{k},r} \}$$

other $\{\quad\} = 0$

$$\Rightarrow \quad \langle 0 | \psi_a(x)\, \psi_b(y) | 0 \rangle = 0, \quad \langle 0 | \bar{\psi}_a(x)\, \bar{\psi}_b(y) | 0 \rangle = 0.$$

$$\underbrace{\psi_a(x)\, \bar{\psi}_b(y)} = \langle 0 | T(\psi_a(x)\, \bar{\psi}_b(y)) | 0 \rangle$$

$$= \langle 0 | \theta(x_0 - y_0)\, \psi_a(x)\bar{\psi}_b(y) - \theta(y_0 - x_0)\, \bar{\psi}_b(y)\, \psi_a(x) | 0 \rangle$$

$$= - \underbrace{\bar{\psi}_b(y)\, \psi_a(x)}$$

where $\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$

$$= \langle 0 | \psi_a^{(+)}(x)\, \bar{\psi}_b^{(-)}(y) | 0 \rangle = \int_{-\infty}^{+\infty} d^3\vec{p} \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}})\, e^{-ip\cdot x}\, e^{ik\cdot y}$$

$$\times \langle 0 | \sum_{s,r} \left( u_a(\vec{p},s)\, b_{\vec{p},s}\, \bar{u}_b(\vec{k},r)\, b^{+}_{\vec{k},r} \right) | 0 \rangle$$

where $\langle 0 | b_{\vec{p},s}\, b^{+}_{\vec{k},r} | 0 \rangle = \langle 0 | \{ b_{\vec{p},s}, b^{+}_{\vec{k},r} \} | 0 \rangle = \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{rs}\, \delta^3(\vec{p}-\vec{k})$

$$\Rightarrow \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3\, 2E_{\vec{p}}}\, e^{-ip\cdot(x-y)} \sum_s \left( u_a(\vec{p},s)\, \bar{u}_b(\vec{p},s) \right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3\, 2E_{\vec{p}}}\, e^{-ip\cdot(x-y)}\, (\not{p} + m)_{ab}$$

(recall that $\sum_s u(\vec{p},s)\bar{u}(\vec{p},s) = \not{p} + m$, $\sum_s v(\vec{p},s)\bar{v}(\vec{p},s) = \not{p} - m$)

and. $\langle 0 | \bar{\psi}_b(y)\, \psi_a(x) | 0 \rangle = \int_{-\infty}^{+\infty} d^3\vec{p} \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}})\, e^{-ik\cdot y}\, e^{ip\cdot x}$

$$\times \langle 0 | \sum_{s,r} \left( \bar{v}_b(\vec{k},r)\, v_a(\vec{p},s)\, d_{\vec{k},r}\, d^{+}_{\vec{p},s} \right) | 0 \rangle$$

where $\langle 0 | d_{\vec{k},r}\, d^{+}_{\vec{p},s} | 0 \rangle = \langle 0 | \{ d_{\vec{k},r}, d^{+}_{\vec{p},s} \} | 0 \rangle = \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{rs}\, \delta^3(\vec{p}-\vec{k})$

$$\Rightarrow \langle 0 | \bar{\psi}_b(y)\, \psi_a(x) | 0 \rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3\, 2E_{\vec{p}}}\, e^{ip\cdot(x-y)} \sum_s \underbrace{\left( \bar{v}_b(\vec{p},s)\, v_a(\vec{p},s) \right)}_{= v_a(\vec{p},s)\, \bar{v}_b(\vec{p},s)}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{ip\cdot(x-y)} (\not{p}-m)_{ab} = \int_{-\infty}^{+\infty} d^3\vec{p} \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{iE_{\vec{p}}(x_0-y_0) - i\vec{p}\cdot(\vec{x}-\vec{y})} \underbrace{(\not{p}-m)_{ab}}_{(E_{\vec{p}}\gamma^0 - \vec{p}\cdot\vec{\gamma} - m)_{ab}}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{iE_{\vec{p}}(x_0-y_0) + i\vec{p}\cdot(\vec{x}-\vec{y})} (E_{\vec{p}}\gamma^0 + \vec{p}\cdot\vec{\gamma} - m)_{ab}$$

$$\Rightarrow \underbrace{\psi_a(x)\bar{\psi}_b(y)} = \int_{-\infty}^{+\infty} d^3\vec{p} \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \Big[\theta(x_0-y_0) e^{-iE_{\vec{p}}(x_0-y_0)} (E_{\vec{p}}\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)_{ab}$$
$$- \theta(y_0-x_0) e^{iE_{\vec{p}}(x_0-y_0)} (E_{\vec{p}}\gamma^0 + \vec{p}\cdot\vec{\gamma} - m)_{ab} \Big]$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (-\vec{p}\cdot\vec{\gamma} + m)_{ab} \left[\theta(x_0-y_0) e^{-iE_{\vec{p}}(x_0-y_0)} + \theta(y_0-x_0) e^{iE_{\vec{p}}(x_0-y_0)}\right]$$
$$+ \int_{-\infty}^{+\infty} d^3\vec{p} \frac{1}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (E_{\vec{p}}\gamma^0)_{ab} \left[\theta(x_0-y_0) e^{-iE_{\vec{p}}(x_0-y_0)} - \theta(y_0-x_0) e^{iE_{\vec{p}}(x_0-y_0)}\right]$$

where the first [ ] is the same as the scalar case,

the second [ ] $= \theta(x_0-y_0) e^{-iE_{\vec{p}}(x_0-y_0)} - \theta(y_0-x_0) e^{iE_{\vec{p}}(x_0-y_0)}$

$$= \frac{1}{2\pi i} \left\{ \int_{-\infty}^{+\infty} dk_0 \frac{e^{i(x_0-y_0)k_0}}{k_0 - i\varepsilon} e^{-iE_{\vec{p}}(x_0-y_0)} - \int_{-\infty}^{+\infty} dk_0 \frac{e^{i(y_0-x_0)k_0}}{k_0 - i\varepsilon} e^{iE_{\vec{p}}(x_0-y_0)} \right\}$$

where $\varepsilon \to 0^+$

$$\overset{k_0 \to E_{\vec{p}} + k_0'}{=} \frac{1}{2\pi i} \left\{ \int_{-\infty}^{+\infty} dk_0' \frac{e^{i(x_0-y_0)k_0'}}{E_{\vec{p}} + k_0' - i\varepsilon} - \int_{-\infty}^{+\infty} dk_0' \frac{e^{i(y_0-x_0)k_0'}}{E_{\vec{p}} + k_0' - i\varepsilon} \right\}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dk_0' \, e^{i(x_0-y_0)k_0'} \underbrace{\left( \frac{1}{E_{\vec{p}} + k_0' - i\varepsilon} - \frac{1}{E_{\vec{p}} - k_0' - i\varepsilon} \right)}$$

$$\| \quad \frac{(E_{\vec{p}} - k_0' - i\varepsilon) - (E_{\vec{p}} + k_0' - i\varepsilon)}{(E_{\vec{p}} - i\varepsilon)^2 - k_0'^2} = \frac{-2k_0'}{E_{\vec{p}}^2 - k_0'^2 - \varepsilon^2 - 2i\varepsilon E_{\vec{p}}}$$

$$= \frac{-2k_0'}{E_{\vec{p}}^2 - k_0'^2 - i\varepsilon}$$

$\uparrow$ $2\varepsilon E_{\vec{p}} \to \varepsilon$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dk_0' \, e^{i(x_0-y_0)k_0'} \frac{(-2k_0')}{E_{\vec{p}}^2 - k_0'^2 - i\varepsilon}$$

$$\Rightarrow \underbrace{\psi_a(x)\bar{\psi}_b(y)} = \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (-\vec{p}\cdot\vec{\gamma}+m)_{ab}\, \frac{2E_{\vec{p}}}{2\pi i} \int_{-\infty}^{+\infty} dk_0'\, e^{i(x_0-y_0)k_0'}\, \frac{1}{E_{\vec{p}}^2-k_0'^2-i\varepsilon}$$

$$+ \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^3 2E_{\vec{p}}}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (E_{\vec{p}}\gamma^0)_{ab}\, \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dk_0'\, e^{i(x_0-y_0)k_0'}\, \frac{-2k_0'}{E_{\vec{p}}^2-k_0'^2-i\varepsilon}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^4}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (-\vec{p}\cdot\vec{\gamma}+m)_{ab}\, \frac{1}{i} \int_{-\infty}^{+\infty} dk_0'\, e^{i(x_0-y_0)k_0'}\, \frac{1}{E_{\vec{p}}^2-k_0'^2-i\varepsilon}$$

$$+ \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^4}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\, \frac{1}{i} \int_{-\infty}^{+\infty} dk_0'\, (-k_0'\gamma^0)_{ab}\, e^{i(x_0-y_0)k_0'}\, \frac{1}{E_{\vec{p}}^2-k_0'^2-i\varepsilon}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^4}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} (\vec{p}\cdot\vec{\gamma}+m)_{ab}\, \frac{1}{i} \int_{-\infty}^{+\infty} dk_0'\, e^{i(x_0-y_0)k_0'}\, \frac{1}{E_{\vec{p}}^2-k_0'^2-i\varepsilon}$$

$$+ \int_{-\infty}^{+\infty} d^3\vec{p}\, \frac{1}{(2\pi)^4}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\, \frac{1}{i} \int_{-\infty}^{+\infty} dk_0'\, (-k_0'\gamma^0)_{ab}\, e^{i(x_0-y_0)k_0'}\, \frac{1}{E_{\vec{p}}^2-k_0'^2-i\varepsilon}$$

$k^\mu \equiv (k_0', \vec{p})$

$k^2 = k_0'^2 - |\vec{p}|^2 = k_0'^2 - E_{\vec{p}}^2 + m^2$

$$= \int_{-\infty}^{+\infty} d^4k\, \frac{1}{(2\pi)^4}\, e^{i(x-y)\cdot k}\, \frac{i}{k^2-m^2+i\varepsilon} (-k^0\gamma^0 + k^i\gamma^i + m)_{ab}$$

$$= \int_{-\infty}^{+\infty} d^4k\, \frac{1}{(2\pi)^4}\, e^{i(x-y)\cdot k}\, \frac{i}{k^2-m^2+i\varepsilon} (-\not{k} + m)_{ab}$$

$$= \int_{-\infty}^{+\infty} d^4k\, \frac{1}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(\not{k}+m)_{ab}}{k^2-m^2+i\varepsilon}$$

Often, $\frac{i(\not{k}+m)}{k^2-m^2-i\varepsilon}$ is written as $\frac{i}{\not{k}-m-i\varepsilon}$. However, it is just a short hand notation, since otherwise put the matrix in denomenator is nonsensical.

So,

$$\boxed{\begin{aligned} \underbrace{\psi_a(x)\bar{\psi}_b(y)} &= \int_{-\infty}^{+\infty} d^4k\, \frac{1}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(\not{k}+m)_{ab}}{k^2-m^2+i\varepsilon} \\ \underbrace{\bar{\psi}_b(x)\psi_a(y)} = -\underbrace{\psi_a(y)\bar{\psi}_b(x)} &= -\int_{-\infty}^{+\infty} d^4k\, \frac{1}{(2\pi)^4}\, e^{-ik\cdot(y-x)}\, \frac{i(\not{k}+m)_{ab}}{k^2-m^2+i\varepsilon} \\ &= -\int_{-\infty}^{+\infty} d^4k\, \frac{1}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(-\not{k}+m)_{ab}}{k^2-m^2+i\varepsilon} \end{aligned}}$$

④ For photon field, in the Coulomb gauge we have studied. ($A^0(x)=0$, $\vec{\nabla}\cdot\vec{A}(x)=0$)

$$A^\mu(x)=\int_{-\infty}^{+\infty}d^3\vec{k}\,C(E_{\vec{k}})\sum_{\lambda=1}^{2}\left(\varepsilon^\mu(\vec{k},\lambda)\,a_{\vec{k},\lambda}e^{-ik\cdot x}+\varepsilon^{*\mu}(\vec{k},\lambda)\,a^+_{\vec{k},\lambda}e^{ik\cdot x}\right)$$

$$=(0,\vec{A}(x))$$

$$=\left(0,\int_{-\infty}^{+\infty}d^3\vec{k}\,C(E_{\vec{k}})\sum_{\lambda=1}^{2}\left(\vec{e}(\vec{k},\lambda)a_{\vec{k},\lambda}e^{-ik\cdot x}+\vec{e}^{\,*}(\vec{k},\lambda)\,a^+_{\vec{k},\lambda}e^{ik\cdot x}\right)\right)$$

where $\sum_{\lambda=1}^{2}\left(\vec{e}^{\,*}(\vec{k},\lambda)\right)^i\left(\vec{e}(\vec{k},\lambda)\right)^j=\delta^{ij}-\frac{(\vec{k})^i(\vec{k})^j}{|\vec{k}|^2}$

$$[a_{\vec{k},s},a^+_{\vec{p},r}]=\delta_{sr}\,\delta^3(\vec{p}-\vec{k})\frac{1}{2E_{\vec{p}}}\frac{1}{(2\pi)^3}\left(\frac{1}{C(E_{\vec{p}})}\right)^2,$$

$$[a_{\vec{k},s},a_{\vec{p},r}]=[a^+_{\vec{k},s},a^+_{\vec{p},r}]=0$$

$$\underbrace{A^\mu(x)A^\nu(y)}=\langle 0|T(A^\mu(x)A^\nu(y))|0\rangle=\theta(x_0-y_0)\langle 0|A^\mu(x)A^\nu(y)|0\rangle+\theta(y_0-x_0)\langle 0|A^\nu(y)A^\mu(x)|0\rangle$$

$$=\theta(x_0-y_0)\langle 0|\overset{(+)}{A^\mu}(x)\overset{(-)}{A^\nu}(y)|0\rangle+\theta(y_0-x_0)\langle 0|\overset{(+)}{A^\nu}(y)\overset{(-)}{A^\mu}(x)|0\rangle$$

where $\langle 0|\overset{(+)}{A^\mu}(x)\overset{(-)}{A^\nu}(y)|0\rangle$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,d^3\vec{k}\,C(E_{\vec{p}})C(E_{\vec{k}})\,e^{-ip\cdot x+ik\cdot y}\sum_{s,r=1}^{2}\varepsilon^\mu(\vec{p},s)\,\varepsilon^{*}_{\nu}(\vec{k},r)\langle 0|a_{\vec{p},s}a^+_{\vec{k},r}|0\rangle$$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,e^{-ip\cdot(x-y)}\sum_{s=1}^{2}\varepsilon^\mu(\vec{p},s)\,\varepsilon^{*\nu}(\vec{p},s)\frac{1}{(2\pi)^3 2E_{\vec{p}}}$$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,e^{-ip\cdot(x-y)}\frac{1}{(2\pi)^3 2E_{\vec{p}}}\times\begin{pmatrix}0&0&0&0\\0&&&\\0&&\delta^{ij}-\frac{p^ip^j}{|\vec{p}|^2}&\\0&&&\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&&&\\0&&\langle 0|\overset{(+)}{A^i}(x)\overset{(-)}{A^j}(y)|0\rangle&\\0&&&\end{pmatrix}$$

and $\langle 0|\overset{(+)}{A^\nu}(y)\overset{(-)}{A^\mu}(x)|0\rangle$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,d^3\vec{k}\,C(E_{\vec{p}})C(E_{\vec{k}})\,e^{-ik\cdot y+ip\cdot x}\sum_{s,r=1}^{2}\varepsilon^\nu(\vec{k},r)\,\varepsilon^{*\mu}(\vec{p},s)\langle 0|a_{\vec{k},r}a^+_{\vec{p},s}|0\rangle$$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,e^{ip\cdot(x-y)}\frac{1}{(2\pi)^3 2E_{\vec{p}}}\sum_{s=1}^{2}\varepsilon^\nu(\vec{p},s)\,\varepsilon^{*\mu}(\vec{p},s)$$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,e^{ip\cdot(x-y)}\frac{1}{(2\pi)^3 2E_{\vec{p}}}\times\begin{pmatrix}0&0&0&0\\0&&&\\0&&\delta^{ji}-\frac{p^jp^i}{|\vec{p}|^2}&\\0&&&\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&&&\\0&&\langle 0|\overset{(+)}{A^j}(y)\overset{(-)}{A^i}(x)|0\rangle&\\0&&&\end{pmatrix}$$

$$=\int_{-\infty}^{+\infty}d^3\vec{p}\,e^{iE_{\vec{p}}(x_0-y_0)}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\frac{1}{(2\pi)^3 2E_{\vec{p}}}\begin{pmatrix}0&0&0&0\\0&&&\\0&&\delta^{ij}-\frac{p^ip^j}{|\vec{p}|^2}&\\0&&&\end{pmatrix}$$

$$\Rightarrow \underbrace{A^\mu(x)\, A^\nu(y)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underbrace{A^i(x)\, A^j(y)} & \\ 0 & & & \end{pmatrix}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{(2\pi)^3 2E_{\vec{p}}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} & \\ 0 & & & \end{pmatrix} \left[\theta(x_0-y_0)\, e^{-iE_{\vec{p}}(x_0-y_0)} + \theta(y_0-x_0)\, e^{iE_{\vec{p}}(x_0-y_0)}\right]$$

where $[\quad] \underset{\uparrow}{=} \frac{1}{2\pi i} 2E_{\vec{p}} \int_{-\infty}^{+\infty} dk_0'\, e^{i(x_0-y_0)k_0'} \frac{1}{E_{\vec{p}}^2 - k_0'^2 - i\varepsilon} = \frac{E_{\vec{p}}}{\pi i} \int_{-\infty}^{+\infty} dk_0'\, e^{-i(x_0-y_0)k_0'} \frac{1}{E_{\vec{p}}^2 - k_0'^2 - i\varepsilon}$

the same as the scalar case , note that $E_{\vec{p}} > 0$

$$\Rightarrow \underbrace{A^\mu(x)\, A^\nu(y)} = \int_{-\infty}^{+\infty} d^3\vec{p}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{(2\pi)^3 2E_{\vec{p}}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2} & \\ 0 & & & \end{pmatrix} \frac{E_{\vec{p}}}{\pi i} \int_{-\infty}^{+\infty} dk_0'\, e^{-i(x_0-y_0)k_0'} \frac{1}{E_{\vec{p}}^2 - k_0'^2 - i\varepsilon}$$

Define $k^\mu \equiv (k_0', \vec{p})$

and use $k^2 = k_0'^2 - |\vec{p}|^2 = k_0'^2 - E_{\vec{p}}^2$

$$\Rightarrow \boxed{\begin{aligned} \underbrace{A^\mu(x) A^\nu(y)} &= \int_{-\infty}^{+\infty} d^4k\, e^{-ik\cdot(x-y)} \frac{1}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} & \\ 0 & & & \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underbrace{A^i(x) A^j(y)} & \\ 0 & & & \end{pmatrix} \\ &= \underbrace{A^\nu(y)\, A^\mu(x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underbrace{A^j(y)\, A^i(x)} & \\ 0 & & & \end{pmatrix} \end{aligned}}$$