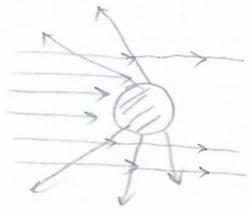
cross sections & decay rates

$$\sigma \equiv \frac{\overbrace{\text{number of particles scattered}}^{N}}{\underbrace{\text{time}}_{T} \times \underbrace{\text{number density in beam} \times \text{velocity of beam}}_{\Phi}} = \frac{1}{T}\frac{1}{\Phi}N$$

$\Phi \to$ number of particle go through per unit area per unit time, called flux.

It's also natural to measure the differential cross section $\frac{d\sigma}{d\Omega}$, as this gives the information (classically) about the shape of the object or the form of the potential responsible for the scattering to happen.



While classically a scatter either happen or not, quantum mechanically it has a probability for scattering.

$$d\sigma = \frac{1}{T}\frac{1}{\Phi}dP$$

where Φ is now normalized as if the beam has just one particle, P is the probability of scattering.

The differential number of scattering events measured in a collider experiment is

$$dN \equiv L \times d\sigma.$$

where L is the luminosity. (unit is area^{-1}).

The experimental data is often presented as $\frac{dN}{dE}$ vs. E, then if we can calculate $\frac{d\sigma}{dE}$ from theory, then multiply by L, we can get the theoretical plot for $\frac{d\sigma}{dE} \times L$ vs. E, to compare with experimental data.

Now let's relate the $d\sigma$ to S-matrix element. $\langle f|S|i\rangle$.

Assume $|i\rangle$ is a two particle state, and $|f\rangle$ is a n particle state, so

$2 \rightarrow n$ process

$$p_1 + p_2 \rightarrow \{p_j\}, \quad j=1, \cdots, n.$$

In the rest frame of one of the collidering particle, the flux is

$$\Phi = \frac{|\vec{v}|}{V},$$

where $|\vec{v}|$ is the magnitude of the incoming particle, V is the "big box" in which the whole experiment is taking place.

In a different frame, such as the center-of-mass frame, $|\vec{v}|$ should be changed to $|\vec{v}_1 - \vec{v}_2|$

So we have

$$d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP,$$

where T is the time the experiment lasts. T is large since we assume $|i\rangle$ and $|f\rangle$ are states at $T \rightarrow -\infty$ & $T \rightarrow +\infty$, respectively

$$dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} d\Pi$$

where $|\langle f|\hat{S}|i\rangle|^2$ is the transition probability density from $|i\rangle$ to $|f\rangle$ due to interaction described by $\hat{S}$ (recall that $\hat{S}$ has interaction Hamiltonian in it).

Note that since $|f\rangle$ is specified by definite momenta of the final state, and considering that momentum is a continuous quantity, we need $d\Pi$ to get from probability density to probability.

Consider a large cubic $V = L^3$, the de Broglie wavelength is

$$\lambda = \frac{h}{P} = \frac{L}{n} \quad , \ n \text{ is integer}$$

$$\Rightarrow \frac{2\pi\hbar}{p} = \frac{L}{n} \Rightarrow \Delta n \underset{\hbar=1}{\overset{\uparrow}{=}} \frac{\Delta P}{2\pi} L$$

In three dimension and go to $\underbrace{\text{continuous limit}}_{V\to\infty}$, $dn_x\,dn_y\,dn_z = \frac{d^3\vec{P}}{(2\pi)^3} V$

So, the number of states within $d^3\vec{P}$ is $dn_x\,dn_y\,dn_z = \frac{d^3\vec{P}}{(2\pi)^3} V$.

Do the same for every particle in the final state.

$$\Rightarrow \quad d\Pi = \prod_{j=1}^{n} \frac{V}{(2\pi)^3} d^3\vec{P}_j$$

Since for one particle state,

$$\langle p | q \rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{p} - \vec{q}) \quad \text{(for scalar field)}$$

$$\langle p, r | q, s \rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{p} - \vec{q})\, \delta_{rs} \quad \text{(for spinor and vector field)}$$

we have $\langle p | p \rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(0)$

and $\langle p, r | p, r \rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(0)$.

Since $\int_{-\infty}^{+\infty} d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{P})$

then $\int_{-\infty}^{+\infty} d^3\vec{x} = (2\pi)^3 \delta^3(0)$

$$\Rightarrow \delta^3(0) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3\vec{x} = \lim_{V\to\infty} \frac{V}{(2\pi)^3}$$

Similarly, $\int_{-\infty}^{+\infty} dt\, e^{-iE_{\vec{p}}t} = (2\pi)\,\delta(E_{\vec{p}})$

then $\int_{-\infty}^{+\infty} dt = (2\pi)\,\delta(0)$

$$\Rightarrow \delta(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt = \lim_{T\to\infty} \frac{T}{2\pi}$$

$$\Rightarrow \delta^4(0) = \lim_{V,T\to\infty} \frac{TV}{(2\pi)^4}$$

Keep in mind that at the end we need to go to $V, T \to \infty$, we can write

$$\langle p | p \rangle = 2E_{\vec{p}} V$$
$$\langle p, r | p, r \rangle = 2E_{\vec{p}} V.$$

So the normalization factor $\langle f | f \rangle$ and $\langle i | i \rangle$ are independent of the types of the fields, we can just use the momentum alone to label the state for the purpose here

$$|i\rangle = |p_1\rangle |p_2\rangle \text{ and } |f\rangle = \prod_j |p_j\rangle$$

$$\Rightarrow \langle i | i \rangle = (2E_1 V)(2E_2 V)$$
$$\langle f | f \rangle = \prod_{j=1}^{n} (2E_j V)$$

For $|\langle f | \hat{S} | i \rangle|^2$, since $\hat{S} = \mathbb{1} + \sum_{l=1}^{\infty} \frac{1}{l!} (-i)^l \int d^4x_1 d^4x_2 \cdots d^4x_l \, T(\hat{H}_I^{int}(x_1) \hat{H}_I^{int}(x_2) \cdots \hat{H}_I^{int}(x_n))$

then we can write it as $\hat{S} = \mathbb{1} + iT$, where $\mathbb{1}$ is the part where there is no interaction, and T is called the transition matrix, then factor out the energy-momentum conservation condition, we can write

$$T \equiv (2\pi)^4 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \mathcal{M}$$

So, we are interested in calculating the interaction related S-matrix element

$$\langle f | \hat{S} - \mathbb{1} | i \rangle = (2\pi)^4 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \, \underset{\substack{\downarrow \\ \text{imaginary unit}}}{i} \langle f | \mathcal{M} | \underset{\substack{\downarrow \\ \text{initial state}}}{i} \rangle$$

where $\langle f | \mathcal{M} | i \rangle$ is usually called matrix element.

Since we are interested in the transition, $|f\rangle \neq |i\rangle$, then $\langle f | i \rangle = 0$,

So $\langle f | \hat{S} | i \rangle = \langle f | \hat{S} - \mathbb{1} | i \rangle$

$$\Rightarrow |\langle f | \hat{S} | i \rangle|^2 = (2\pi)^8 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \delta^4(0) |\langle f | \mathcal{M} | i \rangle|^2$$

(since $(\delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu))^2 = \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu) \delta^4(0)$)

while $\delta^4(0) = \frac{VT}{(2\pi)^4}$

$$\Rightarrow |\langle f|\hat{S}|i\rangle|^2 = (2\pi)^4 \delta^4(\sum P_i^\mu - \sum P_f^\mu) VT |\mathcal{M}|^2,$$

where $|\mathcal{M}|^2 \equiv |\langle f|\mathcal{M}|i\rangle|^2$

$$\Rightarrow dP = \frac{(2\pi)^4 \delta^4(\sum P_i^\mu - \sum P_f^\mu) VT |\mathcal{M}|^2}{(2E_1 V)(2E_2 V) \prod_{j=1}^{n} (2E_j V)} \prod_{j=1}^{n} \frac{V}{(2\pi)^3} d^3\vec{P}_j$$

$$= \frac{T}{V} \frac{1}{(2E_1)(2E_2)} |\mathcal{M}|^2 d\Pi_{LIPS}$$

where $\boxed{d\Pi_{LIPS} \equiv \prod_{\text{final state } j} \frac{d^3\vec{P}_j}{(2\pi)^3} \frac{1}{2E_{\vec{P}_j}} (2\pi)^4 \delta^4(\sum P_i^\mu - \sum P_f^\mu)}$

is called the Lorentz-invariant phase space (LIPS)

$$\Rightarrow \boxed{d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_1 - v_2|} dP = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} |\mathcal{M}|^2 d\Pi_{LIPS}}$$

we see that no V or T left, so it is trivial to take $V, T \to \infty$.

A decay is a $1 \to n$ process. the differential decay rate is

$$d\Gamma = \frac{1}{T} dP$$

where $dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi$

$\langle i|i\rangle = 2E_1 V$

$|\langle f|\hat{S}|i\rangle|^2$, $\langle f|f\rangle$ and $d\Pi$ are the same as when deriving $d\sigma$.

$$\Rightarrow d\Gamma = \frac{1}{T} \frac{(2\pi)^4 \delta^4(\sum P_i^\mu - \sum P_f^\mu) VT |\mathcal{M}|^2}{(2E_1 V) \prod_{j=1}^{n} (2E_j V)} \prod_{j=1}^{n} \frac{V}{(2\pi)^3} d^3\vec{P}_j$$

$$= \frac{1}{2E_1} |\mathcal{M}|^2 d\Pi_{LIPS}$$

Now we prove that $d\Pi_{LIPS}$ is indeed Lorentz invariant.

$$\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3\vec{p}}{(2\pi)^3}\frac{dp^0}{2p^0}\delta(p^0-E_{\vec{p}})$$, note that it should be understood as an integration for p_0 in this and the following steps.

$$= \frac{d^4p}{(2\pi)^4}\frac{2\pi}{2p^0}\left[\delta(p^0-E_{\vec{p}})+\delta(p^0+E_{\vec{p}})\right]\theta(p_0)$$

$$= \frac{d^4p}{(2\pi)^4}2\pi\,\delta(p^{0\,2}-E_{\vec{p}}^2)\,\theta(p^0)$$

$\delta(x^2-\alpha^2)=\frac{1}{2|\alpha|}\left(\delta(x+\alpha)+\delta(x-\alpha)\right]$

$$= \frac{d^4p}{(2\pi)^4}\cdot 2\pi\,\delta(p^2-m^2)\,\theta(p^0)$$

$E_{\vec{p}}^2=|\vec{p}|^2+m^2$

$p^2=p^{0\,2}-|\vec{p}|^2$

Also, we can write

$$\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3\vec{p}}{(2\pi)^3}\frac{dp^0}{(-2p^0)}\delta(p^0+E_{\vec{p}})$$

$$= \frac{d^4p}{(2\pi)^4}\frac{2\pi}{(-2p^0)}\left[\delta(p^0+E_{\vec{p}})+\delta(p^0-E_{\vec{p}})\right]\theta(-p^0)$$

$$= \frac{d^4p}{(2\pi)^4}\cdot 2\pi\,\delta(p^{0\,2}-E_{\vec{p}}^2)\,\theta(-p^0)$$

$$= \frac{d^4p}{(2\pi)^4}\cdot 2\pi\,\delta(p^2-m^2)\,\theta(-p^0)$$

Therefore, it is clear that the role of $\theta(p^0)$ or $\theta(-p^0)$ together with $\delta(p^2-m^2)$ when do the integration is just to select one of the two possible values $p^0=\pm\sqrt{|\vec{p}|^2+m^2}=\pm E_{\vec{p}}$, and the result of the integration doesn't depends on which one to choose.

A Lorentz transformation on $d\Pi_{LIPS}$ will make

$$d\Pi_{LIPS}=\prod_{\text{final state } j}\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}_j}}(2\pi)^4\delta^4\left(\sum p_i^\mu-\sum p_f^\mu\right)=\prod_{\text{final state } j}\frac{d^4p}{(2\pi)^4}2\pi\,\delta(p_j^2-m_j^2)\,\theta(p_j^0)(2\pi)^4\delta^4\left(\sum p_i^\mu-\sum p_f^\mu\right)$$

$$\rightarrow d\Pi'_{LIPS}=\prod_{\text{final state } j}\frac{d^4p'}{(2\pi)^4}2\pi\,\delta(p_j'^2-m_j^2)\,\theta(p_j^{0\prime})(2\pi)^4\delta^4\left(\sum p_i'^\mu-\sum p_f'^\mu\right)$$

Note that $d^4p \to d^4p' \overset{p'^\mu = \Lambda^\mu{}_\nu p^\nu}{=} |\det(\Lambda)| d^4p = d^4p.$

$|\det(\Lambda)| = 1$ is the property of Lorentz transformation.

$p_j'^2 = p_j^2 \Rightarrow \delta(p_j'^2 - m_j^2) = \delta(p_j^2 - m_j^2)$

since p^2 is a Lorentz scalar.

$$\delta^4(\Sigma p_i'^\mu - \Sigma p_f'^\mu) = \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^4} d^4x' e^{-ix'\cdot(\Sigma p_i' - \Sigma p_f')}$$

$$\underset{d^4x' = |\det(\Lambda)| d^4x = d^4x,\ x'\cdot(\Sigma p_i' - \Sigma p_f') = x\cdot(\Sigma p_i - \Sigma p_f)}{=} \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^4} d^4x\, e^{-ix\cdot(\Sigma p_i - \Sigma p_f)}$$

$$= \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)$$

Since $\frac{d^4p'}{(2\pi)^4} 2\pi \delta(p'^2 - m^2)\theta(p'^0) = \frac{d^4p}{(2\pi)^4} 2\pi \delta(p^2 - m^2)\theta(p^0)$,

it does not matter whether $\theta(p_j'^0) = \theta(p_j^0)$ or $\theta(p_j'^0) = \theta(-p_j^0)$.

That is, for the theta function, we only care about the sign of p'^0, but it does not matter whether the sign of p'^0 is the same as p^0 or $-p^0$.

$$\Rightarrow d\Pi'_{LISP} = \prod_{\text{final state } j} \frac{d^4p}{(2\pi)^4} 2\pi \delta(p_j^2 - m_j^2)\theta(\pm p^0)\,(2\pi)^4 \delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)$$

$$= d\Pi_{LISP}.$$

Note that due to the $\delta(p_j^2 - m_j^2)$, $p_j^0 = \pm\sqrt{|\vec{p}_j|^2 + m_j^2} \neq 0$

(if $m_j = 0$, $|\vec{p}_j|$ cannot be zero, since otherwise $p_j^0 = 0$, means that there is no j particle at all)