Examples of calculations in QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^\mu D_\mu \psi - m \bar{\psi} \psi$$

where $D_\mu \psi = \partial_\mu \psi + i|e| q A_\mu \psi$

where q is the charge of the particle (not antiparticle) described by the field ψ. For example, $q = -1$ when ψ describes the electron-positron field, $q = +1$ when ψ describes the proton-antiproton field.

put $D_\mu \psi$ in $\mathcal{L}$, we get

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi - |e| q \bar{\psi} \gamma^\mu \psi A_\mu$$

If we have more charged spinor fields in the system, we will just write the last three terms for each of them.

Recall that in classical electromagnetism,

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

and from this $\mathcal{L}_{EM}$, we can get the Maxwell equations by writing Euler-Lagrangian equation:

$$\frac{\partial \mathcal{L}_{EM}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}_{EM}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\Rightarrow \quad -j^\nu - \partial_\mu \frac{\partial(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = 0$$

where

$$\frac{\partial(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4} \frac{\partial[(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{4} \times 2 (\partial^\alpha A^\beta - \partial^\beta A^\alpha)(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$$

$$= -\frac{1}{2}[(\partial^\mu A^\nu - \partial^\nu A^\mu) - (\partial^\nu A^\mu - \partial^\mu A^\nu)]$$

$$= -F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = j^\nu \quad \Rightarrow \quad \begin{cases} \vec{\nabla} \cdot \vec{E} = \rho \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \end{cases}$$

, where $j^\mu = (\rho, \vec{j})$,

$E^i = F^{i0}$

$B^i = -\frac{1}{2} \varepsilon^{ijk} F^{jk}$

The other two Maxwell equations come from the identity

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = \partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda)$$
$$= 0$$

$$\Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0 \end{cases}$$

From $\partial_\mu F^{\mu\nu} = j^\nu$

we have $\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=0} = \partial_\nu j^\nu$

For $\mathcal{L}$, the Euler-Lagrangian equation for A_μ gives

$\partial_\mu F^{\mu\nu} = \sum_i |e| \varepsilon_i \bar{\psi}_i \gamma^\nu \psi_i$, where i is for different Dirac fields.

So, we can identify $j^\nu = \sum_i |e| \varepsilon_i \bar{\psi}_i \gamma^\nu \psi_i$

In fact, for Dirac field, the Noether current from internal phase transformation $\psi_i \to \psi_i' = e^{-i\delta \alpha_i}\psi_i$ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}(-i\psi_i) + i\bar{\psi}_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_i)} = \bar{\psi}_i \gamma^\mu \psi_i$$

So the above identification for the conserved current $j^\nu = \sum_i |e| \varepsilon_i \bar{\psi}_i \gamma^\nu \psi_i$ indeed self-consistent (we merely multiply each of the conserved Noether current by a constant $|e|\varepsilon_i$)

In the free field theory of A_μ, i.e., $\mathcal{L}_{\text{freeEM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$,
we can choose the Coulomb gauge $\begin{cases}\vec{\nabla}\cdot\vec{A} = 0 \\ A^0 = 0\end{cases}$ to reduce the number of independent solutions of A_μ from four to two, that is, we can work with the two physical transversal polarizations only.

Since $\Pi^0 = \dfrac{\partial \mathcal{L}_{\text{freeEM}}}{\partial(\partial A_0/\partial t)} = 0$ (since there is no $\partial_0 A^0$ term in $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$)

we only have the commutation relation

$$[A_i(\vec{x},t),\ \dot{A}_j(\vec{y},t)] = i\int_{-\infty}^{+\infty}\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$[A_i(\vec{x},t),\ A_j(\vec{y},t)] = [\dot{A}_i(\vec{x},t),\ \dot{A}_j(\vec{y},t)] = 0$$

and
$$\vec{A}(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}})\sum_{\lambda=1}^{2}\left(\vec{e}(\vec{k},\lambda)\,a_{\vec{k},\lambda}\,e^{-ik\cdot x}+\vec{e}^{\,*}(\vec{k},\lambda)\,a^{+}_{\vec{k},\lambda}\,e^{ik\cdot x}\right)$$

where
$$[a_{\vec{p},r},\ a^{+}_{\vec{k},s}] = \delta_{rs}\,\delta^3(\vec{p}-\vec{k})\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2\frac{1}{(2\pi)^3}.$$

$$[a_{\vec{p},r},\ a_{\vec{k},s}] = [a^{+}_{\vec{p},r},\ a^{+}_{\vec{k},s}] = 0.$$

$$\underbrace{A^i(x)A^j(y)} = \underbrace{A^j(y)A^i(x)} = \int_{-\infty}^{+\infty} d^4k\, e^{-ik\cdot(x-y)}\frac{1}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}\left(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}\right)$$

Also recall that
$$\vec{e}^{\,*}(\vec{k},\lambda)\cdot\vec{e}(\vec{k},\lambda') = \delta_{\lambda\lambda'},\ (\lambda,\lambda'=1,2)$$

$$\vec{k}\cdot\vec{e}(\vec{k},\lambda) = 0$$

$$\sum_{\lambda=1}^{2}(\vec{e}^{\,*}(\vec{k},\lambda))^i(\vec{e}(\vec{k},\lambda))^j = \delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}$$

Now, from $\partial_\mu F^{\mu\nu} = j^\nu$, we get.

$$\partial_\mu F^{\mu i} = \partial_\mu(\partial^\mu A^i-\partial^i A^\mu) = \Box A^i + \partial_i(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\partial_i A^0 = j^i$$

$$\Rightarrow\quad \partial_i\left(\Box A^i + \partial_i(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\partial_i A^0\right) = \partial_i j^i$$

$$\Rightarrow\quad \underbrace{\Box(\vec{\nabla}\cdot\vec{A}) + \partial_i\partial_i(\vec{\nabla}\cdot\vec{A})} + \frac{\partial}{\partial t}\nabla^2 A^0 = \vec{\nabla}\cdot\vec{j}$$

note that
$\nabla^2 = \partial_i\partial_i$
$\Box = \frac{\partial^2}{\partial t^2} - \partial_i\partial_i$

$$\Rightarrow\quad \frac{\partial^2}{\partial t^2}(\vec{\nabla}\cdot\vec{A}) + \frac{\partial}{\partial t}\nabla^2 A^0 = \vec{\nabla}\cdot\vec{j}$$

If we <u>assume</u> that $\nabla^2 A^0 = -j^0$,

then $\frac{\partial}{\partial t}\nabla^2 A^0 = -\frac{\partial}{\partial t}j^0$

due to current conservation from $\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \partial_\mu j^\mu = 0$

$$\Rightarrow\quad \frac{\partial^2}{\partial t^2}(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}j^0 = \partial_\mu j^\mu = 0$$

Therefore, if $\vec{\nabla}\cdot\vec{A}=0$ and $\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=0$ holds at one time, it will hold at all time.

then $\partial_\mu F^{\mu 0} = j^0 \Rightarrow \partial_\mu(\partial^\mu A^0 - \partial^0 A^\mu) = \partial_i(\partial^i A^0 - \partial^0 A^i) = j^0$

$$\Rightarrow \quad -\nabla^2 A^0 - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = j^0$$

$$\Rightarrow \quad \nabla^2 A^0 = -j^0 .$$

So, indeed, our above assumption is consistent.

That is to say, if we can have $\vec{\nabla}\cdot\vec{A}=0$ and $\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=0$ holds at one time, then we can have the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$ held at all time, and we have $\nabla^2 A^0 = -j^0$.

In fact, due to gauge symmetry, we can ensure that $\vec{\nabla}\cdot\vec{A}=0$. Since if $\vec{\nabla}\cdot\vec{A}\neq 0$, we can do a gauge transformation $\vec{A}' = \vec{A} - \vec{\nabla}\Lambda$, such that $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda \underset{\substack{\uparrow \\ \text{require}}}{\equiv} 0$, and we know that this equation can be solved to get Λ.

Therefore, it is always possible to choose the Coulomb gauge, for which we have

$$\begin{cases} \vec{\nabla}\cdot\vec{A} = 0 \\ \nabla^2 A^0 = -j^0 \end{cases}$$

The solution of $\underbrace{\nabla^2 A^0 = -j^0}_{\text{Possion's equation.}}$ is $A^0(\vec{x},t) = \frac{1}{4\pi}\int d^3\vec{x}' \frac{j^0(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$, which is just the Coulomb's law. ($j^0 \equiv \rho$)

Note that when there is no source, i.e., $j^\mu = 0$, we have $A^0(\vec{x},t)=0$, and this is consistent with our result for the free field theory of photon field

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Now let's re-write the original QED Lagrangian in the Coulomb gauge.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu + \underbrace{\text{terms with no } A^\mu}_{\sum_i(\bar{\psi}_i i\gamma^\mu\partial_\mu\psi_i - m_i\bar{\psi}_i\psi_i)}$$

$$= -\frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial^0 A^i - \partial^i A^0) - \frac{1}{4}(\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i)$$
$$- j^0 A^0 + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

$$= -\frac{1}{2}\left[(\partial_0 A_i)\partial^0 A^i - 2(\partial_0 A_i)(\partial^i A^0) + (\partial_i A_0)(\partial^i A^0)\right]$$

$$-\frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) - j^0 A^0 + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

use $|\vec{B}|^2 = \vec{B}\cdot\vec{B} = B^i B^i = (-\frac{1}{2})^2 \varepsilon^{ijk}\varepsilon^{iab} F^{jk}F^{ab}$

$$= \frac{1}{4}(\delta^{ja}\delta^{kb} - \delta^{jb}\delta^{ka})F^{jk}F^{ab}$$

$$= \frac{1}{4}(F^{ab}F^{ab} - F^{ba}F^{ab})$$

$$= \frac{1}{2}F^{ij}F^{ij} = \frac{1}{2}F_{ij}F^{ij}$$

$$= \frac{1}{2}(\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i)$$

$$= \partial_i A_j(\partial^i A^j - \partial^j A^i)$$

note:
$\varepsilon^{123} = \varepsilon_{123} = 1$
$\varepsilon_{ijk}\varepsilon^{imn} = \delta^m_j\delta^n_k - \delta^n_j\delta^m_k$
$\varepsilon_{jmn}\varepsilon^{imn} = 2\delta^i_j$
$\varepsilon_{ijk}\varepsilon^{ijk} = 6$

and Define $\vec{E}_\perp \equiv -\partial_0\vec{A}$, $\Rightarrow |\vec{E}_\perp|^2 = (\partial_0\vec{A})\cdot(\partial^0\vec{A}) = -(\partial_0 A_i)(\partial^0 A^i)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}|\vec{E}_\perp|^2 - \frac{1}{2}|\vec{B}|^2 - (\partial_0 A_i)(\partial_i A^0) + \frac{1}{2}(\partial_i A^0)(\partial_i A^0)$$

$$- j^0 A^0 + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

For the term $\frac{1}{2}(\partial_i A^0)(\partial_i A^0)$, it is

$$\frac{1}{2}(\partial_i A^0)(\partial_i A^0) = \frac{1}{2}\partial_i(A^0\partial_i A^0) - \frac{1}{2}A^0(\partial_i\partial_i A^0)$$

$$= \frac{1}{2}\vec{\nabla}\cdot(A^0\vec{\nabla}A^0) - \frac{1}{2}A^0\nabla^2 A^0$$

where the contribution of the first term to $L = \int d^3\vec{x}\,\mathcal{L}$ vanishes since $\int d^3\vec{x}\,\vec{\nabla}\cdot(A^0\vec{\nabla}A^0) = \iint_S (A^0\vec{\nabla}A^0)\cdot d\vec{S} = 0$ when impose the condition that the field vanishes at infinity, which is true since the source of A^0 (i.e., j^0) distributes in a finite volumn, so that $A^0(t, \vec{x}\to\infty) \to 0$ (i.e., the Coulomb potential goes to zero at infinity).

Therefore, we can drop the $\frac{1}{2}\vec{\nabla}\cdot(A^0\vec{\nabla}A^0)$ term in our $\mathcal{L}$.

Similarly, the term

$$-(\partial_0 A_i)(\partial_i A^0) = -\partial_i(A^0\partial_0 A_i) + A^0\partial_0\partial_i A_i$$
$$= +\vec{\nabla}\cdot(A^0\frac{\partial}{\partial t}\vec{A}) - A^0\frac{\partial}{\partial t}\cancel{(\vec{\nabla}\cdot\vec{A})}$$

and $\int d^3\vec{x}\ \vec{\nabla}\cdot(A^0\frac{\partial}{\partial t}\vec{A}) = \iint_S (A^0\frac{\partial}{\partial t}\vec{A})\cdot d\vec{S} = 0$; 0 by Coulomb gauge.

Therefore,

$$\mathcal{L} = \frac{1}{2}(|\vec{E}_\perp|^2 - |\vec{B}|^2) \underbrace{-\frac{1}{2}A^0\nabla^2 A^0 - j^0A^0}_{\underbrace{+\frac{1}{2}A^0 j^0 - j^0 A^0}_{-\frac{1}{2}j^0A^0}} + \vec{j}\cdot\vec{A} + \text{terms with no } A^\mu$$

$$= \frac{1}{2}(|\vec{E}_\perp|^2 - |\vec{B}|^2) + \sum_l(\bar{\psi}_l i\gamma^\mu\partial_\mu\psi_l - m_l\bar{\psi}_l\psi_l)$$
$$-\frac{1}{2}j^0A^0 + \vec{j}\cdot\vec{A}.$$

where the first line is $\mathcal{L}_0$, and the second line is

$$\mathcal{L}_{int} = -\frac{1}{2}j^0A^0 + \vec{j}\cdot\vec{A}$$
$$= -\frac{1}{2}j^0(\vec{x},t)\frac{1}{4\pi}\int d^3\vec{x}'\frac{j^0(\vec{x}',t)}{|\vec{x}'-\vec{x}|} + \vec{j}\cdot\vec{A}$$
$$= -\frac{1}{8\pi}\int d^3\vec{x}'\frac{j^0(\vec{x}',t)j^0(\vec{x},t)}{|\vec{x}'-\vec{x}|} + \vec{j}\cdot\vec{A}$$

put in $j^0 = \sum_l |e|\varepsilon_l\bar{\psi}_l\gamma^0\psi_l$ and $j^i = \sum_l |e|\varepsilon_l\bar{\psi}_l\gamma^i\psi_l$ in it, we get

$$\mathcal{L}_{int} = -\frac{e^2}{8\pi}\int d^3\vec{x}'\frac{\sum_{lk}\varepsilon_l\varepsilon_k\bar{\psi}_l(\vec{x},t)\gamma^0\psi_l(\vec{x},t)\bar{\psi}_k(\vec{x}',t)\gamma^0\psi_k(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$
$$+|e|\sum_l\varepsilon_l\bar{\psi}_l\gamma^i\psi_l A^i$$

$\Rightarrow \mathcal{H}_{int} = -\mathcal{L}_{int}$, note that there is no field derivative in $\mathcal{L}_{int}$

Example 1

Let's first consider electron-proton scattering

$$e^-(q_1, s_1) + p(q_2, s_2) \rightarrow e^-(q_3, s_3) + p(q_4, s_4)$$

$$|i\rangle = C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1} C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2} b^+_{\vec{q}_1, s_1} B^+_{\vec{q}_2, s_2} |0\rangle$$

$$\langle f| = \langle 0| b_{\vec{q}_3, s_3} B_{\vec{q}_4, s_4} C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3} C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

$$\psi_e(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} (u_e(\vec{p}, s) b_{\vec{p},s} e^{-ip\cdot x} + v_e(\vec{p}, s) d^+_{\vec{p},s} e^{ip\cdot x})$$

$$\psi_p(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_{s=\pm\frac{1}{2}} (u_p(\vec{k}, s) B_{\vec{k},s} e^{-ik\cdot x} + v_p(\vec{k}, s) D^+_{\vec{k},s} e^{ik\cdot x})$$

where we use b & b^+ to denote the annihilation & creation operators for electron, B & B^+ to denote the annihilation & creation operators for proton, d & d^+ to denote the annihilation & creation operators for positron, D & D^+ to denote the annihilation & creation operators for anti-proton.

The $\mathcal{H}_{int}$ relavent for this process at the lowest order is

$$\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x) = \frac{e^2}{8\pi}\int d^3\vec{x}' \frac{(-1)(+1)\bar{\psi}_e(\vec{x},t)\gamma^0\psi_e(\vec{x},t)\bar{\psi}_p(\vec{x}',t)\gamma^0\psi_p(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

$$+ \frac{e^2}{8\pi}\int d^3\vec{x}' \frac{(+1)(-1)\bar{\psi}_p(\vec{x},t)\gamma^0\psi_p(\vec{x},t)\bar{\psi}_e(\vec{x}',t)\gamma^0\psi_e(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

$$- |e|(-\bar{\psi}_e(x)\gamma^i\psi_e(x)A^i(x) + \bar{\psi}_p(x)\gamma^i\psi_p(x)A^i(x))$$

(note that the terms $\frac{e^2}{8\pi}\int \frac{(+1)(+1)\bar{\psi}_p(\vec{x},t)\gamma^0\psi_p(\vec{x},t)\bar{\psi}_p(\vec{x}',t)\gamma^0\psi_p(\vec{x}',t)}{|\vec{x}'-\vec{x}|} d^3\vec{x}'$ and

$\frac{e^2}{8\pi}\int \frac{(-1)(-1)\bar{\psi}_e(\vec{x},t)\gamma^0\psi_e(\vec{x},t)\bar{\psi}_e(\vec{x}',t)\gamma^0\psi_e(\vec{x}',t)}{|\vec{x}'-\vec{x}|} d^3\vec{x}'$

do not contribute at the lowest order)

Let's denote the first two terms as $\mathcal{H}_{inst}$, so we have

$$H^{(+)}_{inst} = \int d^3\vec{x}\, \mathcal{H}_{inst}(x) = \frac{e^2}{8\pi}\int d^3\vec{x}\, d^3\vec{x}' \frac{(-1)(+1)\bar{\psi}_e(\vec{x},t)\gamma^0\psi_e(\vec{x},t)\bar{\psi}_p(\vec{x}',t)\gamma^0\psi_p(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

$$+ \frac{e^2}{8\pi}\int d^3\vec{x}\, d^3\vec{x}' \frac{(+1)(-1)\bar{\psi}_p(\vec{x},t)\gamma^0\psi_p(\vec{x},t)\bar{\psi}_e(\vec{x}',t)\gamma^0\psi_e(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

$$= \frac{e^2}{4\pi}\int d^3\vec{x}\,d^3\vec{x}'\ \frac{-\bar{\psi}_e(\vec{x},t)\gamma^0\psi_e(\vec{x},t)\ \bar{\psi}_p(\vec{x}',t)\gamma^0\psi_p(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

The reason to denote it as H_{inst} is because it represents the instantaneous Hamiltonian, meaning that the potential at time t is produced instantaneously by the charge distribution at the same instant.

$$\Rightarrow \langle f|(-i)\int d^4x : \mathcal{H}_{inst}(x) :|i\rangle$$

$$= \langle f|(-i)\int_{-\infty}^{+\infty} dt\ : H_{inst}(x) :|i\rangle \equiv **$$

use

$$\langle 0|b_{\vec{q}_3,s_3}B_{\vec{q}_4,s_4}(-i)\int_{-\infty}^{+\infty} dt : H_{inst}(x) : b^{+}_{\vec{q}_1,s_1}B^{+}_{\vec{q}_2,s_2}|0\rangle$$

$$= \langle 0|b_{\vec{q}_3,s_3}B_{\vec{q}_4,s_4}(-i)\left(-\frac{e^2}{4\pi}\right)\int d^4x\,d^3\vec{x}'\ \frac{1}{|\vec{x}'-\vec{x}|}\times\int_{-\infty}^{+\infty} d^3\vec{k}_1\,C(E_{\vec{k}_1})$$

$$\times\int_{-\infty}^{+\infty} d^3\vec{k}_2\,C(E_{\vec{k}_2})\int_{-\infty}^{+\infty} d^3\vec{k}_3\,C(E_{\vec{k}_3})\int_{-\infty}^{+\infty} d^3\vec{k}_4\,C(E_{\vec{k}_4}) : \sum_{r_1,r_2,r_3,r_4}\Big(\bar{u}_e(\vec{k}_1,r_1)\,b^{+}_{\vec{k}_1,r_1}e^{ik_1\cdot x}$$

$$\cdot\gamma^0 u_e(\vec{k}_2,r_2)\,b_{\vec{k}_2,r_2}e^{-ik_2\cdot x}\ \bar{u}_p(\vec{k}_3,r_3)B^{+}_{\vec{k}_3,r_3}e^{ik_3\cdot x'}\gamma^0 u_p(\vec{k}_4,r_4)B_{\vec{k}_4,r_4}$$

$$\cdot e^{-ik_4\cdot x'}\Big) : b^{+}_{\vec{q}_1,s_1}B^{+}_{\vec{q}_2,s_2}|0\rangle$$

(note that $x'^0 = x^0 = t$)

use

$$\{b_{\vec{k},r},\, b^{+}_{\vec{p},s}\} = \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2\delta_{sr}\,\delta^3(\vec{p}-\vec{k}) = \{B_{\vec{k},r},\, B^{+}_{\vec{p},s}\}$$

$$0 = \{b_{\vec{k},r},\, b_{\vec{p},s}\} = \{b^{+}_{\vec{k},r},\, b^{+}_{\vec{p},s}\} = \{B_{\vec{k},r},\, B_{\vec{p},s}\} = \{B^{+}_{\vec{k},r},\, B^{+}_{\vec{p},s}\}$$

$$0 = \{b_{\vec{k},r},\, B_{\vec{p},s}\} = \{b_{\vec{k},r},\, B^{+}_{\vec{p},s}\} = \{b^{+}_{\vec{k},r},\, B_{\vec{p},s}\} = \{b^{+}_{\vec{k},r},\, B^{+}_{\vec{p},s}\}$$

$$\Rightarrow \langle 0|b_{\vec{q}_3,s_3}B_{\vec{q}_4,s_4} : b^{+}_{\vec{k}_1,r_1}b_{\vec{k}_2,r_2}B^{+}_{\vec{k}_3,r_3}B_{\vec{k}_4,r_4} : b^{+}_{\vec{q}_1,s_1}B^{+}_{\vec{q}_2,s_2}|0\rangle$$

$$= -\langle 0|b_{\vec{q}_3,s_3}B_{\vec{q}_4,s_4}\,b^{+}_{\vec{k}_1,r_1}B^{+}_{\vec{k}_3,r_3}\,b_{\vec{k}_2,r_2}B_{\vec{k}_4,r_4}\,b^{+}_{\vec{q}_1,s_1}B^{+}_{\vec{q}_2,s_2}|0\rangle$$

$$= (-1)(-1)^2\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_3}}\left(\frac{1}{C(E_{\vec{q}_3})}\right)^2\delta_{s_3,r_1}\delta^3(\vec{k}_1-\vec{q}_3)\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_4}}\left(\frac{1}{C(E_{\vec{q}_4})}\right)^2\delta_{s_4,r_3}\delta^3(\vec{k}_3-\vec{q}_4)$$

$$\times\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2\delta_{s_2,r_4}\delta^3(\vec{k}_4-\vec{q}_2)\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_1}}\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2\delta_{s_1,r_2}\delta^3(\vec{k}_2-\vec{q}_1)$$

$$\Rightarrow ** = (-1)(-i)\left(-\frac{e^2}{4\pi}\right)\int d^4x\, d^3\vec{x}'\, \frac{1}{|\vec{x}'-\vec{x}|}$$

$$\times\ \bar{u}_e(\vec{q}_3, s_3)\, e^{iq_3\cdot x}\, \gamma^0\, u_e(\vec{q}_1, s_1)\, e^{-iq_1\cdot x}$$

$$\times\ \bar{u}_p(\vec{q}_4, s_4)\, e^{iq_4\cdot x'}\, \gamma^0\, u_p(\vec{q}_2, s_2)\, e^{-iq_2\cdot x'}$$

$$= (-i)\frac{e^2}{4\pi}\,(2\pi)\,\delta(E_{\vec{q}_3} - E_{\vec{q}_1} + E_{\vec{q}_4} - E_{\vec{q}_2}) \times \left(\bar{u}_e(\vec{q}_3, s_3)\gamma^0 u_e(\vec{q}_1, s_1)\right)$$

$$\times \left(\bar{u}_p(\vec{q}_4, s_4)\, \gamma^0\, u_p(\vec{q}_2, s_2)\right)$$

$$\times \int_{-\infty}^{+\infty} d^3\vec{x} \int_{-\infty}^{+\infty} d^3\vec{x}'\, \frac{1}{|\vec{x}'-\vec{x}|}\, e^{-i\vec{q}_3\cdot\vec{x} + i\vec{q}_1\cdot\vec{x}}\, e^{-i\vec{q}_4\cdot\vec{x}' + i\vec{q}_2\cdot\vec{x}'}$$

where $\displaystyle\int_{-\infty}^{+\infty} d^3\vec{x} \int_{-\infty}^{+\infty} d^3\vec{x}'\, \frac{1}{|\vec{x}'-\vec{x}|}\, e^{-i(\vec{q}_3-\vec{q}_1)\cdot\vec{x}}\, e^{-i(\vec{q}_4-\vec{q}_2)\cdot\vec{x}'}$

$$\overset{\vec{x}' \equiv \vec{r}+\vec{x}}{=} \int_{-\infty}^{+\infty} d^3\vec{x} \int_{-\infty}^{+\infty} d^3\vec{r}\, \frac{1}{|\vec{r}|}\, e^{-i(\vec{q}_3-\vec{q}_1)\cdot\vec{x}}\, e^{-i(\vec{q}_4-\vec{q}_2)\cdot(\vec{r}+\vec{x})}$$

$$= (2\pi)^3\, \delta^3(\vec{q}_3 - \vec{q}_1 + \vec{q}_4 - \vec{q}_2) \int_{-\infty}^{+\infty} d^3\vec{r}\, \frac{1}{|\vec{r}|}\, e^{-i(\vec{q}_4-\vec{q}_2)\cdot\vec{r}}$$

where $\displaystyle\int_{-\infty}^{+\infty} d^3\vec{r}\, \frac{1}{|\vec{r}|}\, e^{-i(\vec{q}_4-\vec{q}_2)\cdot\vec{r}}$

$$= 2\pi \int_0^{+\infty} |\vec{r}|^2\, d|\vec{r}|\, \frac{1}{|\vec{r}|} \int_{-1}^{1} d(\cos\theta)\, e^{-i|\vec{q}_4-\vec{q}_2||\vec{r}|\cos\theta}$$

$$= 2\pi \int_0^{+\infty} |\vec{r}|\, d|\vec{r}|\, \frac{1}{(-i)|\vec{q}_4-\vec{q}_2||\vec{r}|} \left(e^{-i|\vec{q}_4-\vec{q}_2||\vec{r}|} - e^{i|\vec{q}_4-\vec{q}_2||\vec{r}|}\right)$$

$$= 2\pi\, \frac{1}{(-i)|\vec{q}_4-\vec{q}_2|} \underbrace{\int_0^{+\infty} \left(e^{-i|\vec{q}_4-\vec{q}_2||\vec{r}|} - e^{i|\vec{q}_4-\vec{q}_2||\vec{r}|}\right) d|\vec{r}|}_{***}$$

where $\displaystyle *** = \lim_{\mu\to 0^+} \int_0^{+\infty} \left(e^{-i|\vec{q}_4-\vec{q}_2||\vec{r}|} - e^{i|\vec{q}_4-\vec{q}_2||\vec{r}|}\right) e^{-\mu|\vec{r}|}\, d|\vec{r}|$

$$= \lim_{\mu\to 0^+} \left[\frac{1}{-i|\vec{q}_4-\vec{q}_2| - \mu} \left(e^{-i|\vec{q}_4-\vec{q}_2||\vec{r}| - \mu|\vec{r}|} \Big|_0^{+\infty} \right) - \frac{1}{i|\vec{q}_4-\vec{q}_2| - \mu} \left(e^{i|\vec{q}_4-\vec{q}_2||\vec{r}| - \mu|\vec{r}|} \Big|_0^{+\infty} \right) \right]$$

$$= \lim_{\mu \to 0^+} \left[- \frac{1}{-i\,|\vec{q}_4 - \vec{q}_2| - \mu} + \frac{1}{i\,|\vec{q}_4 - \vec{q}_2| - \mu} \right]$$

$$= \frac{2}{i\,|\vec{q}_4 - \vec{q}_2|}$$

$$\Rightarrow \int_{-\infty}^{+\infty} d^3\vec{r}\, \frac{1}{|\vec{r}|}\, e^{-i(\vec{q}_4 - \vec{q}_2)\cdot\vec{r}}$$

$$= 2\pi \frac{1}{(-i)\,|\vec{q}_4 - \vec{q}_2|} \cdot \frac{2}{i\,|\vec{q}_4 - \vec{q}_2|}$$

$$= \frac{4\pi}{|\vec{q}_4 - \vec{q}_2|^2}$$

$$\Rightarrow ** = (-i)\frac{e^2}{4\pi}(2\pi)\,\delta(E_{\vec{q}_3} - E_{\vec{q}_1} + E_{\vec{q}_4} - E_{\vec{q}_2})$$
$$\times \left(\bar{u}_e(\vec{q}_3, s_3)\gamma^0 u_e(\vec{q}_1, s_1)\right)\left(\bar{u}_p(\vec{q}_4, s_4)\gamma^0 u_p(\vec{q}_2, s_2)\right)$$
$$\times (2\pi)^3 \delta^3(\vec{q}_3 - \vec{q}_1 + \vec{q}_4 - \vec{q}_2)\frac{4\pi}{|\vec{q}_4 - \vec{q}_2|^2}$$
$$= (2\pi)^4 \delta^4(q_1 + q_2 - q_3 - q_4)(-i)\, e^2 \frac{1}{|\vec{q}_4 - \vec{q}_2|^2}$$
$$\times \left(\bar{u}_e(\vec{q}_3, s_3)\gamma^0 u_e(\vec{q}_1, s_1)\right)\left(\bar{u}_p(\vec{q}_4, s_4)\gamma^0 u_p(\vec{q}_2, s_2)\right)$$

Since the u and $\bar{u}$ spinors here satisfies Dirac equations, i.e.,

since they are the external particles' spinors.

$$(\not{q}_1 - m_e)\, u_e(\vec{q}_1, s_1) = 0.$$
$$\bar{u}_e(\vec{q}_3, s_3)(\not{q}_3 - m_e) = 0$$
$$(\not{q}_2 - m_p)\, u_p(\vec{q}_2, s_2) = 0$$
$$\bar{u}_p(\vec{q}_4, s_4)(\not{q}_4 - m_p) = 0$$

$$\Rightarrow \bar{u}_e(\vec{q}_3, s_3)(\not{q}_3 - \not{q}_1)\, u_e(\vec{q}_1, s_1)$$
$$= \bar{u}_e(\vec{q}_3, s_3)(m_e - m_e)\, u_e(\vec{q}_1, s_1)$$
$$= 0$$

$$\Rightarrow \bar{u}_e(\vec{q}_3, s_3)(q_3^0 - q_1^0)\gamma^0 u_e(\vec{q}_1, s_1) = \bar{u}_e(\vec{q}_3, s_3)(\vec{q}_3 - \vec{q}_1)\cdot\vec{\gamma}\, u_e(\vec{q}_1, s_1)$$

$$\Rightarrow \bar{u}_e(\vec{q}_3,s_3)(q_2^0-q_4^0)\gamma^0 u_e(\vec{q}_1,s_1)=\bar{u}_e(\vec{q}_3,s_3)(\vec{q}_2-\vec{q}_4)\cdot\vec{\gamma}\, u_e(\vec{q}_1,s_1)$$

Similarly,

$$\bar{u}_p(\vec{q}_4,s_4)(\not{q}_4-\not{q}_2)u_p(\vec{q}_2,s_2)=\bar{u}_p(\vec{q}_4,s_4)(m_p-m_p)u_p(\vec{q}_2,s_2)$$

$$=0$$

$$\Rightarrow \bar{u}_p(\vec{q}_4,s_4)(q_4^0-q_2^0)\gamma^0 u_p(\vec{q}_2,s_2)$$

$$=\bar{u}_p(\vec{q}_4,s_4)(\vec{q}_4-\vec{q}_2)\cdot\vec{\gamma}\, u_p(\vec{q}_2,s_2)$$

Also,

use $$\frac{1}{|\vec{q}_4-\vec{q}_2|^2}=\frac{1}{(q_4-q_2)^2+i\varepsilon}\,\frac{(q_4^0-q_2^0)^2-|\vec{q}_4-\vec{q}_2|^2+i\varepsilon}{|\vec{q}_4-\vec{q}_2|^2}$$

$$\Rightarrow ** = (2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(-i)e^2$$

$$\times\frac{1}{(q_4-q_2)^2+i\varepsilon}\left[(q_4^0-q_2^0)^2-|\vec{q}_4-\vec{q}_2|^2+i\varepsilon\right]\frac{1}{|\vec{q}_4-\vec{q}_2|^2}$$

$$\times\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^0 u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^0 u_p(\vec{q}_2,s_2)\right)$$

$$=(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(-i)e^2\frac{1}{(q_4-q_2)^2+i\varepsilon}$$

$$\times\Big[-\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^0 u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^0 u_p(\vec{q}_2,s_2)\right)$$

$$+\frac{\left(\bar{u}_e(\vec{q}_3,s_3)(q_4^0-q_2^0)\gamma^0 u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^0(q_4^0-q_2^0)u_p(\vec{q}_2,s_2)\right)}{|\vec{q}_4-\vec{q}_2|^2}$$

$$+\frac{i\varepsilon}{|\vec{q}_4-\vec{q}_2|^2}\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^0 u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^0 u_p(\vec{q}_2,s_2)\right)\Big]$$

$$=(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(-i)e^2\frac{1}{(q_4-q_2)^2+i\varepsilon}$$

$$\times\Big[-\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^0 u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^0 u_p(\vec{q}_2,s_2)\right)$$

$$+\left(\bar{u}_e(\vec{q}_3,s_3)\frac{(\vec{q}_4-\vec{q}_2)\cdot\vec{\gamma}}{|\vec{q}_4-\vec{q}_2|}u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\frac{(\vec{q}_4-\vec{q}_2)\cdot\vec{\gamma}}{|\vec{q}_4-\vec{q}_2|}u_p(\vec{q}_2,s_2)\right)\Big]$$

note that we have dropped the $i\varepsilon$ term in the numerator, anticipating that in the final result we will make $\varepsilon\to 0^+$, and the $i\varepsilon$ term in the numerator is not singular (i.e., it goes to zero)

Now let's consider the contribution from the other term in $H_{int}(x)$, that is $H_{rad}^{(x)} \equiv -|e|\left(-\bar\psi_e(x)\gamma^i\psi_e(x)A^i(x) + \bar\psi_p(x)\gamma^i\psi_p(x)A^i(x)\right)$

The lowest order contribution to our process is

$$* \equiv \langle f|\frac{(-i)^2}{2!}\int d^4x\int d^4y\, T(:H_{rad}(x)::H_{rad}(y):)|i\rangle$$

$$= \langle f|\frac{(-i)^2}{2!}|e|^2\int d^4x\int d^4y\, T\Big[:(-\bar\psi_e(x)\gamma^i\psi_e(x)A^i(x)+\bar\psi_p(x)\gamma^i\psi_p(x)A^i(x)):$$
$$:(-\bar\psi_e(y)\gamma^j\psi_e(y)A^j(y)+\bar\psi_p(y)\gamma^j\psi_p(y)A^j(y)):\Big]$$
$$\cdot|i\rangle$$

$$= \langle f|\frac{(-i)^2}{2!}e^2\int d^4x\int d^4y\, T\Big[:(-\bar\psi_e(x)\gamma^i\psi_e(x)A^i(x))::(\bar\psi_p(y)\gamma^j\psi_p(y)A^j(y)):$$
$$+:(\bar\psi_p(x)\gamma^i\psi_p(x)A^i(x))::(-\bar\psi_e(y)\gamma^j\psi_e(y)A^j(y)):\Big]$$
$$\cdot i\rangle$$

$$= \langle f|\frac{(-i)^2}{2!}e^2(-1)\int d^4x\int d^4y\, \underbrace{A^i(x)A^j(y)}\Big[:\bar\psi_e(x)\gamma^i\psi_e(x)\bar\psi_p(y)\gamma^j\psi_p(y):$$
$$+:\bar\psi_p(x)\gamma^i\psi_p(x)\bar\psi_e(y)\gamma^j\psi_e(y):\Big]|i\rangle$$

Since $\underbrace{A^i(x)A^j(y)} = \underbrace{A^j(y)A^i(x)} = \int_{-\infty}^{+\infty} d^4k\, e^{-ik\cdot(x-y)}\frac{1}{(2\pi)^4}$
$$\times\frac{i}{k^2+i\varepsilon}\left(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}\right)$$

$$\Rightarrow * = \langle f|(-i)^2e^2(-1)\int d^4x\int d^4y\,\underbrace{A^i(x)A^j(y)}:\bar\psi_e(x)\gamma^i\psi_e(x)\bar\psi_p(y)\gamma^j\psi_p(y):$$
$$|i\rangle$$

where $\langle f|:\bar\psi_e(x)\gamma^i\psi_e(x)\bar\psi_p(y)\gamma^j\psi_p(y):|i\rangle$

$$= \langle f|\int d^3\vec{k}_1 C(E_{\vec{k}_1})\, d^3\vec{k}_2 C(E_{\vec{k}_2})\, d^3\vec{k}_3 C(E_{\vec{k}_3})\, d^4\vec{k}_4 C(E_{\vec{k}_4}):\Big(\sum_{r_1,r_2,r_3,r_4}\bar u_e(\vec{k}_1,r_1)b^+_{\vec{k}_1,r_1}$$
$$\cdot e^{ik_1\cdot x}\gamma^i u_e(\vec{k}_2,r_2)b_{\vec{k}_2,r_2}\, e^{-ik_2\cdot x}\,\bar u_p(\vec{k}_3,r_3)B^+_{\vec{k}_3,r_3}\, e^{ik_3\cdot y}\gamma^j u_p(\vec{k}_4,r_4)$$
$$\cdot B_{\vec{k}_4,r_4}\, e^{-ik_4\cdot y}|i\rangle$$

using $\langle 0| b_{\vec{q}_3,s_3} B_{\vec{q}_4,s_4} : b^\dagger_{\vec{k}_1,r_1} b_{\vec{k}_2,r_2} B^\dagger_{\vec{k}_3,r_3} B_{\vec{k}_4,r_4} : b^\dagger_{\vec{q}_1,s_1} B^\dagger_{\vec{q}_2,s_2} |0\rangle$

$$= (-1)\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_3}}\left(\frac{1}{C(E_{\vec{q}_3})}\right)^2 \delta_{s_3,r_1}\delta^3(\vec{k}_1-\vec{q}_3)$$
$$\times\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_4}}\left(\frac{1}{C(E_{\vec{q}_4})}\right)^2 \delta_{s_4,r_3}\delta^3(\vec{k}_3-\vec{q}_4)$$
$$\times\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_2}}\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta_{s_2,r_4}\delta^3(\vec{k}_4-\vec{q}_2)$$
$$\times\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_1}}\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \delta_{s_1,r_2}\delta^3(\vec{k}_2-\vec{q}_1)$$

$$\Rightarrow \langle f| : \bar{\psi}_e(x)\gamma^i\psi_e(x)\,\bar{\psi}_p(y)\gamma^j\psi_p(y)) : |i\rangle$$
$$= (-1)\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^i u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^j u_p(\vec{q}_2,s_2)\right)$$
$$\times\; e^{iq_3\cdot x}e^{-iq_1\cdot x}e^{iq_4\cdot y}e^{-iq_2\cdot y}$$

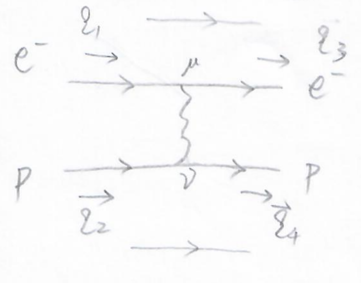
$$\Rightarrow * = (-i)^2 e^2(-1)(-1)\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^i u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^j u_p(\vec{q}_2,s_2)\right)$$
$$\int d^4x\int d^4y\int_{-\infty}^{+\infty} d^4k\, e^{-ik\cdot(x-y)}\frac{1}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}\left(\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}\right)e^{-i(q_1-q_3)\cdot x}e^{-i(q_2-q_4)\cdot y}$$
$$=(-i)^2e^2\int d^4k\Big[\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^i u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^i u_p(\vec{q}_2,s_2)\right)$$
$$-\frac{1}{|\vec{k}|^2}\left(\bar{u}_e(\vec{q}_3,s_3)\vec{\gamma}\cdot\vec{k}\, u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\vec{\gamma}\cdot\vec{k}\, u_p(\vec{q}_2,s_2)\right)\Big]$$
$$\times\frac{i}{k^2+i\varepsilon}\times\frac{1}{(2\pi)^4}(2\pi)^4\delta^4(q_1-q_3+k)(2\pi)^4\delta^4(q_2-q_4-k)$$
$$=(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(-i)^2e^2\frac{i}{(q_4-q_2)^2+i\varepsilon}$$
$$\times\Big[-\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^i u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma_i u_p(\vec{q}_2,s_2)\right)$$
$$-\frac{1}{|\vec{q}_4-\vec{q}_2|^2}\left(\bar{u}_e(\vec{q}_3,s_3)\vec{\gamma}\cdot(\vec{q}_4-\vec{q}_2)u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\vec{\gamma}\cdot(\vec{q}_4-\vec{q}_2)\,u_p(\vec{q}_2,s_2)\right)\Big]$$

$$\Rightarrow \langle f|iT|i\rangle = * + ** = (2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(-i)^2e^2\frac{i}{(q_4-q_2)^2+i\varepsilon}$$
$$\times(-1)\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^\mu u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma_\mu u_p(\vec{q}_2,s_2)\right)$$

$$\Rightarrow i\mathcal{M}_{fi} = \frac{i(-g_{\mu\nu})}{(q_1-q_3)^2+i\varepsilon}\left(\bar{u}_e(\vec{q}_3,s_3)(-i|e|(-1))\gamma^\mu u_e(\vec{q}_1,s_1)\right)$$

$$\times\left(\bar{u}_p(\vec{q}_4,s_4)(-i|e|(+1))\gamma^\nu u_p(\vec{q}_2,s_2)\right)$$

$$\times(-1)$$



Note that the (-1) in the third line is not important, since it is not there if we were to switch the $b^\dagger_{\vec{q}_1,s_1}$ and $B^\dagger_{\vec{q}_2,s_2}$ when we define the initial state, or if we were to switch the $b_{\vec{q}_3,s_3}$ and $B_{\vec{q}_4,s_4}$ when we define the final state.

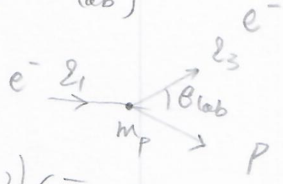
Now let's calculate the unpolarized cross section, that is, for unpolarized initial state particles and for undetected spins of final state particles, we sum over all final spin states and average over all initial spin states. In other words, we now calculate the total cross section to all possible final spin states for an averaged initial spin states, i.e.,

$$\underbrace{\left[\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\sum_{s_1,s_2}\right]}_{\substack{\text{average over}\\ \text{initial states}}}\underbrace{\sum_{s_3,s_4}}_{\substack{\text{sum over}\\ \text{final states.}}}(i\mathcal{M}_{fi})(i\mathcal{M}_{fi})^* \equiv |\overline{\mathcal{M}}|^2$$

and
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\overline{\mathcal{M}}|^2}{64\pi^2 s}\,\frac{\lambda^{\frac{1}{2}}(s,m_e^2,m_p^2)}{\lambda^{\frac{1}{2}}(s,m_e^2,m_p^2)} = \frac{|\overline{\mathcal{M}}|^2}{64\pi^2 s}$$

$$\left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab} = \frac{\overline{|M|^2}}{64\pi^2 m_p}\,\frac{|\vec{q}_3|}{|\vec{q}_1|}\,\frac{1}{\left[(E_{q_1}+m_p)-\frac{|\vec{q}_1|}{|\vec{q}_3|}E_{q_3}\cos\theta_{lab}\right]}$$

in the rest frame of the initial proton



where $$\overline{|M|^2} = \frac{1}{4}\frac{e^4}{(q_1-q_3)^4}\sum_{s_1,s_2,s_3,s_4}\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^\mu u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_p(\vec{q}_4,s_4)\gamma^\nu u_p(\vec{q}_2,s_2)\right)$$
$$\times\left(\bar{u}_e(\vec{q}_1,s_1)\gamma^\alpha u_e(\vec{q}_3,s_3)\right)\left(\bar{u}_p(\vec{q}_2,s_2)\gamma^\beta u_p(\vec{q}_4,s_4)\right)$$
$$\times\; g_{\mu\nu}\,g_{\alpha\beta}$$

(note : $$\big(\underbrace{\bar{u}_1\gamma^\mu u_2}_{\text{a 1x1 number, so transpose does not do anything}}\big)^* = (\bar{u}_1\gamma^\mu u_2)^\dagger = (u_1^\dagger\gamma^0\gamma^\mu u_2)^\dagger = u_2^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u_1$$

$$\underset{\text{use } \gamma^{0\,2}=1 \text{ and } \gamma^{0\dagger}=\gamma^0}{=} u_2^\dagger\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 u_1 \underset{\text{use } \gamma^0\gamma^{\mu\dagger}\gamma^0=\gamma^\mu}{=} \bar{u}_2\gamma^\mu u_1$$)

use
$$\sum_{s_1,s_3}\left(\bar{u}_e(\vec{q}_3,s_3)\gamma^\mu u_e(\vec{q}_1,s_1)\right)\left(\bar{u}_e(\vec{q}_1,s_1)\gamma^\alpha u_e(\vec{q}_3,s_3)\right)$$
$$= \sum_{s_1,s_3}\left(\bar{u}_{e_A}(\vec{q}_3,s_3)\gamma^\mu_{AB}u_{e_B}(\vec{q}_1,s_1)\right)\left(\bar{u}_{e_C}(\vec{q}_1,s_1)\gamma^\alpha_{CD}u_{e_D}(\vec{q}_3,s_3)\right)$$
$$= \sum_{s_1 s_3}\left(u_{e_B}(\vec{q}_1,s_1)\bar{u}_{e_C}(\vec{q}_1,s_1)\right)\gamma^\alpha_{CD}\left(u_{e_D}(\vec{q}_3,s_3)\bar{u}_{e_A}(\vec{q}_3,s_3)\right)\gamma^\mu_{AB}$$
$$= (\not{q}_1+m_e)_{BC}\,\gamma^\alpha_{CD}\,(\not{q}_3+m_e)_{DA}\,\gamma^\mu_{AB}$$

use $\sum_{s_1} u_e(\vec{q}_1,s_1)\bar{u}_e(\vec{q}_1,s_1) = \not{q}_1+m_e$
and $\sum_{s_3} u_e(\vec{q}_3,s_3)\bar{u}_e(\vec{q}_3,s_3) = \not{q}_3+m_e$

$$= \mathrm{Tr}\left[(\not{q}_1+m_e)\gamma^\alpha(\not{q}_3+m_e)\gamma^\mu\right]$$

use $\mathrm{Tr}[AB] = \sum_{ij}A_{ij}B_{ji}$

$$= \mathrm{Tr}(\not{q}_1\gamma^\alpha\not{q}_3\gamma^\mu) + m_e^2\,\mathrm{Tr}(\gamma^\alpha\gamma^\mu)$$
$$= 4(q_1^\alpha q_3^\mu + q_1^\mu q_3^\alpha - g^{\alpha\mu}q_1\cdot q_3) + m_e^2\,4g^{\alpha\mu}$$

and $\sum_{s_2,s_4} \left(\bar{u}_p(\vec{q}_4,s_4)\gamma^\nu u_p(\vec{q}_2,s_2)\right)\left(\bar{u}_p(\vec{q}_2,s_2)\gamma^\beta u_p(\vec{q}_4,s_4)\right)$

$$= \mathrm{Tr}\left[(\not{q}_2 + m_p)\gamma^\beta(\not{q}_4 + m_p)\gamma^\nu\right]$$

$$= \mathrm{Tr}(\not{q}_2\gamma^\beta\not{q}_4\gamma^\nu) + m_p^2\,\mathrm{Tr}(\gamma^\beta\gamma^\nu)$$

$$= 4(q_2^\beta q_4^\nu + q_2^\nu q_4^\beta - g^{\beta\nu} q_2\cdot q_4) + m_p^2\, 4 g^{\beta\nu}$$

$$\Rightarrow |\bar{M}|^2 = \frac{1}{4}\frac{e^4}{(q_1-q_3)^4}\, 4(q_1^\alpha q_3^\mu + q_1^\mu q_3^\alpha - g^{\alpha\mu} q_1\cdot q_3 + m_e^2 g^{\alpha\mu})$$
$$\times 4(q_2^\beta q_4^\nu + q_2^\nu q_4^\beta - g^{\beta\nu} q_2\cdot q_4 + m_p^2 g^{\beta\nu})$$
$$\times\ g_{\mu\nu}\, g_{\alpha\beta}$$

$$= \frac{4e^4}{(q_1-q_3)^4}(q_1^\alpha q_3^\mu + q_1^\mu q_3^\alpha - g^{\alpha\mu} q_1\cdot q_3 + m_e^2 g^{\alpha\mu})$$
$$\times(q_{2_\alpha} q_{4_\mu} + q_{2\mu} q_{4\alpha} - g_{\alpha\mu} q_2\cdot q_4 + m_p^2 g_{\alpha\mu})$$

$$= \frac{4e^4}{(q_1-q_3)^4}\Big[(q_1\cdot q_2)(q_3\cdot q_4) + (q_1\cdot q_4)(q_3\cdot q_2) - (q_1\cdot q_3)(q_2\cdot q_4)$$
$$+ m_p^2(q_1\cdot q_3) + (q_1\cdot q_4)(q_3\cdot q_2) + (q_1\cdot q_2)(q_3\cdot q_4)$$
$$- (q_1\cdot q_3)(q_2\cdot q_4) + m_p^2(q_1\cdot q_3) - (q_2\cdot q_4)(q_1\cdot q_3)$$
$$- (q_2\cdot q_4)(q_1\cdot q_3) + 4(q_1\cdot q_3)(q_2\cdot q_4) - 4m_p^2(q_1\cdot q_3)$$
$$+ m_e^2(q_2\cdot q_4) + m_e^2(q_2\cdot q_4) - 4m_e^2(q_2\cdot q_4)$$
$$+ 4m_e^2 m_p^2\Big]$$

$$= \frac{4e^4}{(q_1-q_3)^4}\Big[2(q_1\cdot q_2)(q_3\cdot q_4) + 2(q_1\cdot q_4)(q_2\cdot q_3) - 2m_p^2(q_1\cdot q_3)$$
$$- 2m_e^2(q_2\cdot q_4) + 4m_e^2 m_p^2\Big]$$

using $(q_1\cdot q_2) = [(q_1+q_2)^2 - q_1^2 - q_2^2]/2 = \dfrac{s - m_e^2 - m_p^2}{2}$

$(q_3\cdot q_4) = [(q_3+q_4)^2 - q_3^2 - q_4^2]/2 = \dfrac{s - m_e^2 - m_p^2}{2}$

$(q_1\cdot q_4) = -\dfrac{(q_1-q_4)^2 - q_1^2 - q_4^2}{} = -\dfrac{u - m_e^2 - m_p^2}{2}$

$(q_2\cdot q_3) = -\dfrac{(q_2-q_3)^2 - q_2^2 - q_3^2}{2} = -\dfrac{u - m_e^2 - m_p^2}{2}$

$$(q_1 \cdot q_3) = -\frac{(q_1-q_3)^2 - q_1^2 - q_3^2}{2} = -\frac{t-2m_e^2}{2}$$

$$(q_2 \cdot q_4) = -\frac{(q_2-q_4)^2 - q_2^2 - q_4^2}{2} = -\frac{t-2m_p^2}{2}$$

$$\Rightarrow \overline{|M|}^2 = \frac{8e^4}{t^2}\left[\left(\frac{s-m_e^2-m_p^2}{2}\right)^2 + \left(\frac{u-m_e^2-m_p^2}{2}\right)^2 + \frac{m_p^2(t-2m_e^2)}{2} + \frac{m_e^2(t-2m_p^2)}{2} + 2m_e^2 m_p^2\right]$$

$$\underset{\uparrow}{=} \frac{8e^4}{t^2}\left\{\left[\frac{s-(m_e^2+m_p^2)}{2}\right]^2 + \left[\frac{(m_e^2+m_p^2)-t-s}{2}\right]^2 + \frac{t}{2}(m_e^2+m_p^2)\right\}$$

use $u = 2m_e^2 + 2m_p^2 - t - s$

Let's consider $\left(\frac{d\sigma}{d\Omega}\right)_{lab}$.

In the lab frame where the initial proton is at rest, we have

$$\vec{q}_3 + \vec{q}_4 = \vec{q}_1, \quad E_{\vec{q}_3} + E_{\vec{q}_4} = E_{\vec{q}_1} + m_p$$

$$\Rightarrow (|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}} + \left[|\vec{q}_1|^2 + |\vec{q}_3|^2 - 2|\vec{q}_1||\vec{q}_3|\cos\theta_{lab} + m_p^2\right]^{\frac{1}{2}}$$
$$= (|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} + m_p$$

$$\Rightarrow |\vec{q}_1|^2 + |\vec{q}_3|^2 - 2|\vec{q}_1||\vec{q}_3|\cos\theta_{lab} + m_p^2$$
$$= m_p^2 + |\vec{q}_1|^2 + m_e^2 + |\vec{q}_3|^2 + m_e^2 + 2m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}$$
$$- 2m_p(|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}} - 2(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}(|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}}$$

$$\Rightarrow -(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}(|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}} + m_p\left[(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} - (|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}}\right]$$
$$= -|\vec{q}_1||\vec{q}_3|\cos\theta_{lab} - m_e^2$$

$$\Rightarrow m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} + m_e^2 + |\vec{q}_1||\vec{q}_3|\cos\theta_{lab}$$
$$= m_p(|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}} + (|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}(|\vec{q}_3|^2 + m_e^2)^{\frac{1}{2}}$$

$$\Rightarrow \left[m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} + m_e^2\right]^2 + 2\left[m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} + m_e^2\right]|\vec{q}_1||\vec{q}_3|\cos\theta_{lab}$$
$$+ |\vec{q}_1|^2|\vec{q}_3|^2\cos^2\theta_{lab} = \left[m_p + (|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}\right]^2(|\vec{q}_3|^2 + m_e^2)$$

$$\Rightarrow \quad |\vec{q}_3|^2\left[\left(m_p + (|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}\right)^2 - |\vec{q}_1|^2\cos^2\theta_{lab}\right]$$

$$+\ |\vec{q}_3|\left[-2\left(m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}+m_e^2\right)|\vec{q}_1|\cos\theta_{lab}\right]$$

$$+\ \underbrace{\left\{\left[m_p+(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}\right]^2 m_e^2-\left[m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}+m_e^2\right]^2\right\}}_{\underbrace{[m_p^2+|\vec{q}_1|^2+m_e^2+2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}]m_e^2-(m_p^2(|\vec{q}_1|^2+m_e^2)+m_e^4+2m_e^2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}})}_{|\vec{q}_1|^2(m_e^2-m_p^2)}}$$

$$=0$$

while
$$A\equiv\left[m_p+(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}\right]^2-|\vec{q}_1|^2\cos^2\theta_{lab}$$
$$= m_e^2+m_p^2+(1-\cos^2\theta_{lab})|\vec{q}_1|^2+2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}$$
$$>0$$

$$C\equiv|\vec{q}_1|^2(m_e^2-m_p^2)\leq 0$$

$$\Rightarrow |\vec{q}_3| = \frac{-B+(B^2-4AC)^{\frac{1}{2}}}{2A}$$

where $B=-2\left[m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}+m_e^2\right]|\vec{q}_1|\cos\theta_{lab}$.

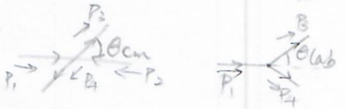
and $E_{\vec{q}_3}=(|\vec{q}_3|^2+m_e^2)^{\frac{1}{2}}$

$$s=(q_1+q_2)^2=q_1^2+q_2^2+2q_1\cdot q_2$$
$$= m_e^2+m_p^2+2m_pE_{\vec{q}_1}$$
$$= m_e^2+m_p^2+2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}$$

$$t=(q_1-q_3)^2=q_1^2+q_3^2-2q_1\cdot q_3$$
$$=2m_e^2-2(E_{\vec{q}_1}E_{\vec{q}_3}-\vec{q}_1\cdot\vec{q}_3)$$
$$=2m_e^2-2\left[(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}(|\vec{q}_3|^2+m_e^2)^{\frac{1}{2}}-|\vec{q}_1||\vec{q}_3|\cos\theta_{lab}\right]$$

So, we have written all quantities $\underbrace{}_{\text{in } \left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab}}$ in terms of $\cos\theta_{lab}$, $|\vec{q}_1|$, m_e & m_p

This page gives the relation between the θ_{lab} and θ_{cm}. The P_1 and P_2 here should be understood as P_3 and P_4 respectively, in the picture



Therefore, the result should be understood as

$$\tan\theta_{lab} = \frac{|\vec{P_3}|_{cm} \sin\theta_{cm}}{\gamma(vE_{3_{cm}} + |\vec{P_3}|_{cm}\cos\theta_{cm})}$$

where
$$\gamma = \frac{(E_3+E_4)_{lab}}{(E_3+E_4)_{cm}} = \frac{(E_1+E_2)_{lab}}{(E_1+E_2)_{cm}} = \frac{m_2 + E_{1_{lab}}}{\sqrt{s}}$$

$v = (1-\gamma^{-2})^{\frac{1}{2}}$

In the center-of-mass frame

$$s = (P_1+P_2)^2$$

Since in the lab frame $(P_1+P_2)^2 = P^2 = E^2 - |\vec{P}|^2$

$$\Rightarrow \quad s = E^2 - |\vec{P}|^2$$

The Lorentz boost along z-axis is

$$\begin{pmatrix} E \\ 0 \\ 0 \\ |\vec{P}| \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad E = \gamma\sqrt{s}$$
$$|\vec{P}| = v\gamma\sqrt{s} = vE$$

Therefore, $P^{\mu}_{1_{lab}} = (E_1, |\vec{P_1}|\sin\theta_{lab}\cos\varphi_{lab}, |\vec{P_1}|\sin\theta_{lab}\sin\varphi_{lab}, |\vec{P_1}|\cos\theta_{lab})$

relates $P^{\mu}_{1_{cm}} = \left(\frac{s+m_1^2-m_2^2}{2\sqrt{s}}, \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\sin\theta_{cm}\cos\varphi_{cm}, \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\sin\theta_{cm}\sin\varphi_{cm}, \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\cos\theta_{cm}\right)$

as

$$E_1 = \gamma\left(\frac{s+m_1^2-m_2^2}{2\sqrt{s}} + v\frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\cos\theta_{cm}\right)$$

$$|\vec{P_1}|\cos\theta_{lab} = \gamma\left(v\frac{s+m_1^2-m_2^2}{2\sqrt{s}} + \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\cos\theta_{cm}\right)$$

note that $\frac{s+m_1^2-m_2^2}{2\sqrt{s}}$ is the energy of particle-1 in the center-of-mass frame, and $\frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}$ is the magnitude of the three-momentum of particle-1 in the center-of-mass frame.

$$\tan\theta_{lab} = \frac{\left[(|\vec{P_1}|\sin\theta_{lab}\sin\varphi_{lab})^2 + (|\vec{P_1}|\sin\theta_{lab}\cos\varphi_{lab})^2\right]^{\frac{1}{2}}}{|\vec{P_1}|\cos\theta_{lab}}$$

$$= \frac{\frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\sin\theta_{cm}}{\gamma\left(v\frac{s+m_1^2-m_2^2}{2\sqrt{s}} + \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}\cos\theta_{cm}\right)}$$

(note that $P^x_{1_{lab}} = P^x_{1_{cm}}$, $P^y_{1_{lab}} = P^y_{1_{cm}}$)

(checked)

The range of θ_{lab} can be determined from

$$\tan\theta_{lab} = \frac{\frac{\lambda^{\frac{1}{2}}(s, m_e^2, m_p^2)}{2\sqrt{s}} \sin\theta_{cm}}{\gamma\left(v \frac{s + m_e^2 - m_p^2}{2\sqrt{s}} + \frac{\lambda^{\frac{1}{2}}(s, m_e^2, m_p^2)}{2\sqrt{s}} \cos\theta_{cm}\right)}$$

where $\theta_{cm} \in [0, \pi]$

$$\gamma = \frac{(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} + m_p}{\sqrt{s}}$$

$$v = (1 - \gamma^{-2})^{\frac{1}{2}}$$

$$= \left[1 - \frac{s}{|\vec{q}_1|^2 + m_e^2 + m_p^2 + 2m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{|\vec{q}_1|^2 + m_e^2 + m_p^2 + 2m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} - [m_e^2 + m_p^2 + 2m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}]}{|\vec{q}_1|^2 + m_e^2 + m_p^2 + 2m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

$$= \frac{|\vec{q}_1|}{m_p + (m_e^2 + |\vec{q}_1|^2)^{\frac{1}{2}}}$$

using

$$\begin{aligned}
\lambda(s, m_e^2, m_p^2) &= s^2 + m_e^4 + m_p^4 - 2m_e^2 s - 2m_p^2 s - 2m_e^2 m_p^2 \\
&= m_e^4 + m_p^4 + 4m_p^2(|\vec{q}_1|^2 + m_e^2) + 2m_e^2 m_p^2 \\
&\quad + 4m_e^2 m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} + 4m_p^2 m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} \\
&\quad + m_e^4 + m_p^4 - 2(m_e^2 + m_p^2)(m_e^2 + m_p^2 + 2m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}) \\
&\quad - 2m_e^2 m_p^2 \\
&= 2m_e^4 + 2m_p^4 + 4m_p^2(|\vec{q}_1|^2 + m_e^2) + 4m_e^2 m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} \\
&\quad + 4m_p^2 m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} \\
&\quad - 2(m_e^4 + m_p^4 + 2m_e^2 m_p^2 + 2m_e^2 m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}} \\
&\quad\quad + 2m_p^2 m_p(|\vec{q}_1|^2 + m_e^2)^{\frac{1}{2}}) \\
&= 4m_p^2[|\vec{q}_1|^2 + m_e^2 - m_e^2] \\
&= 4m_p^2|\vec{q}_1|^2
\end{aligned}$$

$$\Rightarrow \tan\theta_{lab} = \frac{2m_p|\vec{q}_1|\sin\theta_{cm}}{\frac{(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}+m_p}{\sqrt{m_e^2+m_p^2+2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}}}\left(\frac{|\vec{q}_1|(2m_e^2+2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}})}{m_p+(m_e^2+|\vec{q}_1|^2)^{\frac{1}{2}}}+2m_p|\vec{q}_1|\cos\theta_{cm}\right)}$$

$$= \frac{m_p\sqrt{m_e^2+m_p^2+2m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}}\sin\theta_{cm}}{m_e^2+m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}+(m_p^2+m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}})\cos\theta_{cm}}$$

From the denomenator, since $m_e < m_p$, then

when $\cos\theta_{cm} = -\frac{m_e^2+m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}}{m_p^2+m_p(|\vec{q}_1|^2+m_e^2)^{\frac{1}{2}}}$

the denomenator becomes zero, and this happens when θ_{cm} is somewhere between $\frac{\pi}{2}$ to π.

From the numerator, when $\theta_{cm}=0$ or $\theta_{cm}=\pi$, the numerator becomes zero.

$\Rightarrow$ $\tan\theta_{lab}$ can take all values between $-\infty$ to $+\infty$.

$\Rightarrow$ $\theta_{lab} \in [0, \pi]$

Note that ① if $m_e > m_p$, then the denomenator is positive definite, so that $0 \le \theta_{lab} < \frac{\pi}{2}$.

② If $m_e = m_p > 0$, then $\tan\theta_{lab} = \frac{m_p\sqrt{m_p^2+m_p^2+2m_p(|\vec{q}_1|^2+m_p^2)^{\frac{1}{2}}}\sin\theta_{cm}}{m_p^2+m_p(|\vec{q}_1|^2+m_p^2)^{\frac{1}{2}}+(m_p^2+m_p(|\vec{q}_1|^2+m_p^2)^{\frac{1}{2}})\cos\theta_{cm}}$

$$= \frac{m_p\sqrt{2}\sin\theta_{cm}}{[m_p^2+m_p(|\vec{q}_1|^2+m_p^2)^{\frac{1}{2}}]^{\frac{1}{2}}(1+\cos\theta_{cm})}$$

$\Rightarrow$ $\theta_{lab} \in [0, \frac{\pi}{2}]$

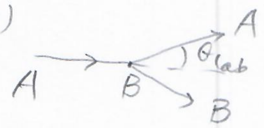
③ if $m_e = 0$, $m_p > 0$, then $\tan\theta_{lab} = \frac{m_p\sqrt{m_p^2+2m_p|\vec{q}_1|}\sin\theta_{cm}}{m_p|\vec{q}_1|+(m_p^2+m_p|\vec{q}_1|)\cos\theta_{cm}}$

$$= \frac{\sqrt{m_p}\sqrt{m_p+2|\vec{q}_1|}\sin\theta_{cm}}{|\vec{q}_1|+(m_p+|\vec{q}_1|)\cos\theta_{cm}}$$

So still $\theta_{lab} \in [0, \pi]$

Of course, the above analysis for the range of θ_{lab} is valid for all $2\to 2$ "elastic" scattering (i.e., $A+B \to A+B$).

That is, in the initial B rest frame ($m_B > 0$)



① if $m_A > m_B$, then $0 \le \theta_{lab} < \frac{\pi}{2}$

② if $m_A = m_B$, then $0 \le \theta_{lab} \le \frac{\pi}{2}$

③ if $0 \le m_A < m_B$, then $0 \le \theta_{lab} \le \pi$.

Now let's go back to our $e^- p \to e^- p$ scattering.

First of all, note that the range of t is

$$t \in \left[-\frac{\lambda(s, m_e^2, m_p^2)}{s}, 0 \right]$$

so $\overline{|M|^2}$ diverges when $t \to 0$, this corresponds to when $\theta_{cm} = 0$, that is, when $\theta_{lab} = 0$. This is called infrared divergence, which we don't discuss here.

[1] In the limit that $m_p \gg |\vec{q}_1|, m_e$, we can treat the proton mass as effectively infinite. In the lab frame, to the leading order,

$$q_1^\mu = (E_{q_1}, \vec{q}_1), \quad q_2^\mu = (m_p, 0), \quad q_3^\mu = (E_{q_3}, \vec{q}_3), \quad q_4^\mu \doteq (m_p, 0)$$

where $E_{q_3} \doteq E_{q_1} \equiv E$, $|\vec{q}_3| \doteq |\vec{q}_1| \equiv |\vec{p}|$, and $\vec{q}_1 \cdot \vec{q}_3 \doteq |\vec{p}|^2 \cos\theta_{lab}$

$$\Rightarrow \quad t = (q_1 - q_3)^2 = (E-E)^2 - (\vec{q}_1 - \vec{q}_3)^2 = -(|\vec{q}_1|^2 + |\vec{q}_3|^2 - 2|\vec{q}_1||\vec{q}_3|\cos\theta_{lab})$$

$$= -(2|\vec{p}|^2 - 2|\vec{p}|^2\cos\theta_{lab}) = -2|\vec{p}|^2(1-\cos\theta_{lab})$$

Also, in this limit, from the formula for $\tan\theta_{lab}$, we can see that

$$\lim_{m_p \gg m_e, |\vec{q}_1|} \tan\theta_{lab} = \tan\theta_{cm} \quad \Rightarrow \quad \theta_{lab} = \theta_{cm} \equiv \theta$$

$$s = (\mathcal{E}_1 + \mathcal{E}_2)^2 = (E + m_p)^2 - |\vec{P}|^2 = m_p^2 + 2m_p E + m_e^2$$

$$\Rightarrow \overline{|\mathcal{M}|^2} = \frac{8e^4}{4|\vec{P}|^4(1-\cos\theta)^2}\left\{\left[\frac{m_p^2+2m_pE+m_e^2-(m_e^2+m_p^2)}{2}\right]^2 + \left[\frac{(m_e^2+m_p^2)+2|\vec{P}|^2(1-\cos\theta)-m_p^2-2m_pE-m_e^2}{2}\right]^2 - \frac{2|\vec{P}|^2(1-\cos\theta)}{2}(m_e^2+m_p^2)\right\}$$

$$= \frac{2e^4}{|\vec{P}|^4(1-\cos\theta)^2}\left\{m_p^2E^2 + \left(|\vec{P}|^2(1-\cos\theta) - m_pE\right)^2 - \frac{2|\vec{P}|^2(1-\cos\theta)}{2}(m_e^2+m_p^2)\right\}$$

Keep leading order of m_p in $\{\ \}$

$$\simeq \frac{2e^4}{|\vec{P}|^4(1-\cos\theta)^2}\left\{2m_p^2E^2 - m_p^2|\vec{P}|^2(1-\cos\theta)\right\}$$

use $\vec{v}_1 = \frac{\vec{\mathcal{E}}_1}{E_{\mathcal{E}_1}} \Rightarrow v_e \equiv |\vec{v}_1| = \frac{|\vec{\mathcal{E}}_1|}{E_{\mathcal{E}_1}} = \frac{|\vec{P}|}{E}$

$$\Rightarrow \overline{|\mathcal{M}|^2} = \frac{2e^4 m_p^2 E^2}{|\vec{P}|^4(1-\cos\theta)^2}\left(2 - v_e^2(1-\cos\theta)\right)$$

$$= \frac{e^4 m_p^2}{v_e^4 E^2 \sin^4\frac{\theta}{2}}\left(1 - v_e^2 \sin^2\frac{\theta}{2}\right)$$

$$\Rightarrow \left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 m_p}\underbrace{\frac{|\vec{\mathcal{E}}_3|}{|\vec{\mathcal{E}}_1|}}_{1}\underbrace{\frac{1}{\left(E_{\mathcal{E}_1} + m_p - \frac{|\vec{\mathcal{E}}_1|}{|\vec{\mathcal{E}}_3|}E_{\mathcal{E}_3}\cos\theta_{lab}\right)}}_{\simeq \frac{1}{m_p}}$$

$$\simeq \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 m_p^2}$$

while $\left(\frac{d\bar{\sigma}}{d\Omega}\right)_{cm} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2(m_p^2+2m_pE+m_e^2)} \simeq \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 m_p^2}$

So in this limit, $\left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab} = \left(\frac{d\bar{\sigma}}{d\Omega}\right)_{cm}$

This makes sense since in this limit $\theta_{lab} = \theta_{cm}$

That is, the center of mass frame is the same as the lab frame in this limit

$$\Rightarrow \left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab} = \frac{e^4}{64\pi^2}\frac{1}{V_e^4 E^2 \sin^4\frac{\theta}{2}}\left(1 - V_e^2 \sin^2\frac{\theta}{2}\right)$$

$$\overset{\frac{e^2}{4\pi}=\alpha}{=} \frac{\alpha^2}{4 V_e^2 |\vec{p}|^2 \sin^4\frac{\theta}{2}}\left(1 - V_e^2 \sin^2\frac{\theta}{2}\right),$$

This is the $\boxed{\text{Mott formula}}$

Note that the result does not depend on the mass of m_p.

In the limit $V_e \ll 1$, we have $|\vec{p}| \ll E \simeq m_e$, and $|\vec{p}| \simeq m_e V_e$

$$\Rightarrow \left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab} = \frac{e^4}{64\pi^2}\frac{E^2}{V_e^4 E^4 \sin^4\frac{\theta}{2}}\left(1 - V_e^2 \sin^2\frac{\theta}{2}\right)$$

$$\simeq \frac{\alpha^2 m_e^2}{4|\vec{p}|^4 \sin^4\frac{\theta}{2}}$$

$$\simeq \frac{\alpha^2}{4 m_e^2 V_e^4 \sin^4\frac{\theta}{2}}$$

$$= \left(\frac{\alpha}{4E_K \sin^2\frac{\theta}{2}}\right)^2$$

where $E_K = \frac{1}{2} m_e V_e^2$ is the kinetic energy of the incoming electron.

This is the $\boxed{\text{Rutherford formula.}}$

[2] In the limit $|\vec{q}_1| \gg m_e$, and we neglect the mass of electron, we get $q_1^\mu = (E_{\vec{q}_1}, \vec{q}_1)$, $q_2^\mu = (m_p, 0)$, $q_3^\mu = (E_{\vec{q}_3}, \vec{q}_3)$,

$$\Rightarrow \quad |\vec{q}_1| = E_{\vec{q}_1}, \quad |\vec{q}_3| = E_{\vec{q}_3}$$

$$\Rightarrow \quad t = (q_1 - q_3)^2 = (E_{\vec{q}_1} - E_{\vec{q}_3})^2 - (\vec{q}_1 - \vec{q}_3)^2 = (E_{\vec{q}_1} - E_{\vec{q}_3})^2 - (E_{\vec{q}_1}^2 + E_{\vec{q}_3}^2 - 2E_{\vec{q}_1}E_{\vec{q}_3}\cdot\cos\theta_{lab})$$

$$= -2E_{\vec{q}_1}E_{\vec{q}_3}(1 - \cos\theta_{lab})$$

$$s = (q_1 + q_2)^2 = m_p^2 + \cancel{m_e^2}^{\to 0} + 2m_p E_{\vec{q}_1}$$

$$\Rightarrow \overline{|\mathcal{M}|^2} = \frac{8e^4}{4E_{\vec{q}_1}^2 E_{\vec{q}_3}^2 (1-\cos\theta_{lab})^2} \Bigg\{ \left(\frac{m_p^2 + 2m_p E_{\vec{q}_1} - (\cancelto{0}{m_e^2} + m_p^2)}{2} \right)^2 + \left(\frac{(\cancelto{0}{m_e^2} + m_p^2) + 2E_{\vec{q}_1} E_{\vec{q}_3}(1-\cos\theta_{lab}) - m_p^2 - 2m_p E_{\vec{q}_1}}{2} \right)^2$$

$$+ \frac{(-2E_{\vec{q}_1} E_{\vec{q}_3})}{2} (1-\cos\theta_{lab})(\cancelto{0}{m_e^2} + m_p^2) \Bigg\}$$

$$= \frac{2e^4}{E_{\vec{q}_1}^2 E_{\vec{q}_3}^2 (1-\cos\theta_{lab})^2} \Big\{ m_p^2 E_{\vec{q}_1}^2 + \left[E_{\vec{q}_1} E_{\vec{q}_3}(1-\cos\theta_{lab}) - m_p E_{\vec{q}_1} \right]^2 - m_p^2 E_{\vec{q}_1} E_{\vec{q}_3}(1-\cos\theta_{lab}) \Big\}$$

In the lab frame (no limit has taken yet)

$$E_{\vec{q}_1} + m_p = E_{\vec{q}_3} + (|\vec{q}_4|^2 + m_p^2)^{\frac{1}{2}}$$

where $|\vec{q}_4|^2 = |\vec{q}_1 + \cancelto{0}{\vec{q}_2} - \vec{q}_3|^2 = |\vec{q}_1|^2 + |\vec{q}_3|^2 - 2|\vec{q}_1||\vec{q}_3|\cos\theta_{lab}$

$$= E_{\vec{q}_1}^2 + E_{\vec{q}_3}^2 - 2m_e^2 - 2|\vec{q}_1||\vec{q}_3|\cos\theta_{lab}$$

$$\Rightarrow E_{\vec{q}_1}^2 + E_{\vec{q}_3}^2 - 2m_e^2 - 2|\vec{q}_1||\vec{q}_3|\cos\theta_{lab} + m_p^2 = (E_{\vec{q}_1} + m_p - E_{\vec{q}_3})^2$$

$$= E_{\vec{q}_1}^2 + E_{\vec{q}_3}^2 + m_p^2 + 2E_{\vec{q}_1} m_p \underbrace{- 2E_{\vec{q}_3} m_p - 2E_{\vec{q}_1} E_{\vec{q}_3}}$$

$$\Rightarrow E_{\vec{q}_3}(E_{\vec{q}_1} + m_p) - |\vec{q}_1||\vec{q}_3|\cos\theta_{lab} = E_{\vec{q}_1} m_p + m_e^2$$

Then in the limit $m_e \to 0$, we have

$$E_{\vec{q}_3}(E_{\vec{q}_1} + m_p) - E_{\vec{q}_1} E_{\vec{q}_3}\cos\theta_{lab} = E_{\vec{q}_1} m_p \Rightarrow t = -2m_p(E_{\vec{q}_1} - E_{\vec{q}_3})$$

$$\Rightarrow m_p(E_{\vec{q}_1} - E_{\vec{q}_3}) = E_{\vec{q}_1} E_{\vec{q}_3}(1-\cos\theta_{lab}) = 2E_{\vec{q}_1} E_{\vec{q}_3}\sin^2\frac{\theta_{lab}}{2}$$

$$\Rightarrow \overline{|\mathcal{M}|^2} = \frac{2e^4}{4E_{\vec{q}_1}^2 E_{\vec{q}_3}^2 \sin^4\frac{\theta_{lab}}{2}} \Big\{ \underbrace{m_p^2 E_{\vec{q}_1}^2 + m_p^2 E_{\vec{q}_3}^2 - 2m_p^2 E_{\vec{q}_1} E_{\vec{q}_3}\sin^2\frac{\theta_{lab}}{2}}_{m_p^2 \left[\underbrace{E_{\vec{q}_1}^2 + E_{\vec{q}_3}^2 - 2E_{\vec{q}_1}E_{\vec{q}_3}}_{(E_{\vec{q}_1} - E_{\vec{q}_3})^2} \underbrace{+ 2E_{\vec{q}_1}E_{\vec{q}_3} - 2E_{\vec{q}_1}E_{\vec{q}_3}\sin^2\frac{\theta_{lab}}{2}}_{= (E_{\vec{q}_1} - E_{\vec{q}_3}) \frac{2E_{\vec{q}_1}E_{\vec{q}_3}\sin^2\frac{\theta_{lab}}{2}}{m_p}} \right]} \Big\}$$

$$= \frac{2e^4 m_p^2 \, 2E_{\vec{q}_1}E_{\vec{q}_3}}{4E_{\vec{q}_1}^2 E_{\vec{q}_3}^2 \sin^4\frac{\theta_{lab}}{2}} \left(\frac{E_{\vec{q}_1}-E_{\vec{q}_3}}{m_p}\sin^2\frac{\theta_{lab}}{2} + \cos^2\frac{\theta_{lab}}{2}\right)$$

$$\Rightarrow \left(\frac{d\bar{\sigma}}{d\Omega}\right)_{lab} = \frac{\overline{|M|}^2}{64\pi^2 m_p}\frac{|\vec{q}_3|}{|\vec{q}_1|}\frac{1}{\left[(E_{\vec{q}_1}+m_p) - \frac{|\vec{q}_1|}{|\vec{q}_3|}E_{\vec{q}_3}\cos\theta_{lab}\right]}$$

$$= \frac{\overline{|M|}^2}{64\pi^2 m_p}\frac{E_{\vec{q}_3}}{E_{\vec{q}_1}}\frac{1}{\left[\underbrace{E_{\vec{q}_1}(1-\cos\theta_{lab})+m_p}_{}\right]}$$

$$= \frac{e^4 m_p^2 E_{\vec{q}_1}E_{\vec{q}_3}}{E_{\vec{q}_1}^2 E_{\vec{q}_3}^2 \sin^4\frac{\theta_{lab}}{2}}\left(\frac{E_{\vec{q}_1}-E_{\vec{q}_3}}{m_p}\sin^2\frac{\theta_{lab}}{2} + \cos^2\frac{\theta_{lab}}{2}\right)\frac{1}{64\pi^2 m_p}\left(\frac{E_{\vec{q}_3}}{E_{\vec{q}_1}}\right)$$

$$\times \frac{1}{m_p} \times \frac{1}{\left(1+\frac{2E_{\vec{q}_1}}{m_p}\sin^2\frac{\theta_{lab}}{2}\right)}$$

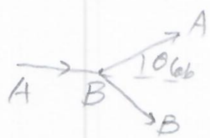
$$= \frac{\alpha^2}{4E_{\vec{q}_1}^2}\left(\frac{\cos^2\frac{\theta_{lab}}{2}}{\sin^4\frac{\theta_{lab}}{2}}\right)\left(1+\frac{E_{\vec{q}_1}-E_{\vec{q}_3}}{m_p}\tan^2\theta_{lab}\right)\frac{1}{\left(1+\frac{2E_{\vec{q}_1}}{m_p}\sin^2\frac{\theta_{lab}}{2}\right)}$$

$$= \frac{\alpha^2}{4E_{\vec{q}_1}^2}\frac{\cos^2\frac{\theta_{lab}}{2}}{\sin^4\frac{\theta_{lab}}{2}}\left(\frac{1-\frac{t}{2m_p^2}\tan^2\theta_{lab}}{1+\frac{2E_{\vec{q}_1}}{m_p}\sin^2\frac{\theta_{lab}}{2}}\right)$$

when $m_p \to +\infty$, the above expression becomes $\frac{\alpha^2}{4E_{\vec{q}_1}^2}\frac{\cos^2\frac{\theta_{lab}}{2}}{\sin^4\frac{\theta_{lab}}{2}}$, Consictent with the Mott formula when $v_e \to 1$.

Continue with the case ① in page 109, i.e., $m_A > m_B > 0$

$$\tan\theta_{lab} = \frac{m_B\sqrt{m_A^2 + m_B^2 + 2m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}}\ \sin\theta_{cm}}{m_A^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}} + \left(m_B^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}\right)\cos\theta_{cm}}$$



let's do the derivative for $Y = \dfrac{C\sin\theta_{cm}}{D + E\cos\theta_{cm}}$.

where $Y \equiv \tan\theta_{lab}$ is a function of θ_{cm}.

$$C = m_B\sqrt{m_A^2 + m_B^2 + 2m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}} > 0$$

$$D \equiv m_A^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}} > 0$$

$$E = m_B^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}} > 0, \quad D > E.$$

$$\Rightarrow \frac{dY}{d\theta_{cm}} = \frac{C\cos\theta_{cm}}{D + E\cos\theta_{cm}} + \frac{C\sin\theta_{cm}\,E\sin\theta_{cm}}{(D + E\cos\theta_{cm})^2}$$

let $\dfrac{dY}{d\theta_{cm}} \equiv 0 \Rightarrow (D + E\cos\theta_{cm})\,C\cos\theta_{cm} + CE\sin^2\theta_{cm} = 0$

$$\Rightarrow CD\cos\theta_{cm} + CE = 0$$

$$\Rightarrow \cos\theta_{cm} = -\frac{E}{D}$$

while

$$\frac{d^2Y}{d\theta_{cm}^2} = \frac{d\left[\dfrac{CD\cos\theta_{cm} + CE}{(D + E\cos\theta_{cm})^2}\right]}{d\theta_{cm}} = \frac{-CD\sin\theta_{cm}}{(D + E\cos\theta_{cm})^2} + \frac{(CD\cos\theta_{cm} + CE)\times 2E\sin\theta_{cm}}{(D + E\cos\theta_{cm})^3}$$

$$= \frac{-CD\sin\theta_{cm}(D + E\cos\theta_{cm}) + 2E\sin\theta_{cm}\,C(E + D\cos\theta_{cm})}{(D + E\cos\theta_{cm})^3}$$

$$= \frac{CDE\sin\theta_{cm}\cos\theta_{cm} - CD^2\sin\theta_{cm} + 2CE^2\sin\theta_{cm}}{(D + E\cos\theta_{cm})^3}$$

$$\Rightarrow \left.\frac{d^2Y}{d\theta_{cm}^2}\right|_{\cos\theta_{cm} = -\frac{E}{D}} = \left.\frac{CE^2\sin\theta_{cm} - CD^2\sin\theta_{cm}}{\left(D - \frac{E^2}{D}\right)^3}\right|_{\cos\theta_{cm} = -\frac{E}{D}} = -\left.\frac{D^3C\sin\theta_{cm}}{(D^2 - E^2)^2}\right|_{\cos\theta_{cm} = -\frac{E}{D}} < 0$$

Therefore, the maximum θ_{lab} is achieved when

$$\cos\theta_{cm} = -\frac{E}{D} = -\frac{m_B^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}}{m_A^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}} = -\frac{m_B^2 + m_B E_{1lab}}{m_A^2 + m_B E_{1lab}}$$

and at the same time,

$$\sin\theta_{cm} = (1 - \cos^2\theta_{cm})^{\frac{1}{2}}$$

$$= \frac{(D^2 - E^2)^{\frac{1}{2}}}{D} = \frac{\left[m_A^4 + m_B^2(|\vec{q}_1|^2 + m_A^2) + 2m_A^2 m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}} - \begin{pmatrix} m_B^4 + m_B^2(|\vec{q}_1|^2 + m_A^2) \\ + 2m_B^3(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}} \end{pmatrix}\right]^{\frac{1}{2}}}{m_A^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}}$$

$$= \frac{\left[m_A^4 - m_B^4 + 2m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}(m_A^2 - m_B^2)\right]^{\frac{1}{2}}}{m_A^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}}$$

$$= \frac{(m_A^2 - m_B^2)^{\frac{1}{2}}\left(m_A^2 + m_B^2 + 2m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}}{m_A^2 + m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}}$$

$$= \frac{(m_A^2 - m_B^2)^{\frac{1}{2}} C}{D\, m_B}$$

$\Rightarrow$ the largest $\tan\theta_{lab}$ is

$$\boxed{(\tan\theta_{lab})_{max}} = \frac{C^2 (m_A^2 - m_B^2)^{\frac{1}{2}}}{D\, m_B\left(D - \frac{E^2}{D}\right)}$$

$$= \frac{C^2}{D^2 - E^2}\left(\frac{m_A^2}{m_B^2} - 1\right)^{\frac{1}{2}}$$

$$= \frac{m_B^2\left(m_A^2 + m_B^2 + 2m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}\right)}{(m_A^2 - m_B^2)\left(m_A^2 + m_B^2 + 2m_B(|\vec{q}_1|^2 + m_A^2)^{\frac{1}{2}}\right)}\left(\frac{m_A^2}{m_B^2} - 1\right)^{\frac{1}{2}}$$

$$= \frac{m_B}{(m_A^2 - m_B^2)^{\frac{1}{2}}} = \boxed{\frac{1}{\sqrt{\left(\frac{m_A}{m_B}\right)^2 - 1}}} \Rightarrow \boxed{(\theta_{lab})_{max} = \arcsin\left(\frac{m_B}{m_A}\right)}$$

This result also covers the case for $m_A = m_B > 0$, which gives $(\theta_{lab})_{max} = \frac{\pi}{2}$.
So, the maximum θ_{lab} only depends on the mass ratio, not on $|\vec{q}_1|$.

Example 2

Electron - positron annihilation.

Consider $e^+(q_1, s_1) + e^-(q_2, s_2) \to \mu^+(q_3, s_3) + \mu^-(q_4, s_4)$

$$|i\rangle = C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, d^+_{\vec{q}_1, s_1} b^+_{\vec{q}_2, s_2} |0\rangle$$

$$\langle f| = \langle 0| D_{\vec{q}_3, s_3} B_{\vec{q}_4, s_4} C(E_{\vec{q}_3})(2\pi)^3 2E_{\vec{q}_3}\, C(E_{\vec{q}_4})(2\pi)^3 2E_{\vec{q}_4}$$

$$\psi_e(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left(u_e(\vec{p}, s) b_{\vec{p},s} e^{-ip\cdot x} + v_e(\vec{p}, s) d^+_{\vec{p},s} e^{ip\cdot x} \right)$$

$$\psi_m(x) = \int_{-\infty}^{+\infty} d^3k\, C(E_{\vec{k}}) \sum_{s=\pm\frac{1}{2}} \left(u_m(\vec{k}, s) B_{\vec{k},s} e^{-ik\cdot x} + v_m(\vec{k}, s) D^+_{\vec{k},s} e^{ik\cdot x} \right)$$

where we use b & b^+ for electron, d & d^+ for positron, B & B^+ for muon, D & D^+ for anti-muon.

The $\mathcal{H}_{int}$ relevent for this process at the lowest order is

$$\mathcal{H}_{int}(x) = \frac{e^2}{8\pi} \int d^3\vec{x}' \frac{(-1)(-1)\, \bar{\psi}_e(\vec{x}, t) \gamma^0 \psi_e(\vec{x}, t) \bar{\psi}_m(\vec{x}', t) \gamma^0 \psi_m(\vec{x}', t)}{|\vec{x}' - \vec{x}|}$$

$$+ \frac{e^2}{8\pi} \int d^3\vec{x}' \frac{(-1)(-1)\, \bar{\psi}_m(\vec{x}, t) \gamma^0 \psi_m(\vec{x}, t) \bar{\psi}_e(\vec{x}', t) \gamma^0 \psi_e(\vec{x}', t)}{|\vec{x}' - \vec{x}|}$$

$$- |e| (-1) \bar{\psi}_e(x) \gamma^i \psi_e(x) A^i(x) - |e| (-1) \bar{\psi}_m(x) \gamma^i \psi_m(x) A^i(x)$$

Again, the first two terms can be combined when we introduce

$$H_{inst} \equiv \int d^3\vec{x}\, \mathcal{H}_{inst}(x) = \frac{e^2}{4\pi} \int d^3\vec{x}' d^3\vec{x} \frac{\bar{\psi}_e(\vec{x}, t) \gamma^0 \psi_e(\vec{x}, t) \bar{\psi}_m(\vec{x}', t) \gamma^0 \psi_m(\vec{x}', t)}{|\vec{x}' - \vec{x}|}$$

$$\Rightarrow \langle f| (-i) \int d^4x : \mathcal{H}_{inst}(x) : |i\rangle$$

$$= \langle f| (-i) \int_{-\infty}^{+\infty} dt : H_{inst}(x) : |i\rangle \equiv **$$

use $\langle 0| D_{\vec{q}_3, s_3} B_{\vec{q}_4, s_4} (-i) \int_{-\infty}^{+\infty} dt : H_{inst}(x) : d^+_{\vec{q}_1, s_1} b^+_{\vec{q}_2, s_2} |0\rangle$

$$= \langle 0| D_{\vec{q}_3, s_3} B_{\vec{q}_4, s_4} (-i) \frac{e^2}{4\pi} \int d^4x\, d^3\vec{x}' \frac{1}{|\vec{x}' - \vec{x}|} \int d^3\vec{p}_1\, C(E_{\vec{p}_1}) \int d^3\vec{p}_2\, C(E_{\vec{p}_2}) \int d^3\vec{p}_3\, C(E_{\vec{p}_3})$$

$$\times \int d^3\vec{p}_4\, C(E_{\vec{p}_4}) \sum_{r_1, r_2, r_3, r_4} : [\bar{v}_e(\vec{p}_1, r_1) d_{\vec{p}_1, r_1} e^{ip_1\cdot x} \gamma^0 u_e(\vec{p}_2, r_2) b_{\vec{p}_2, r_2} e^{-ip_2\cdot x}$$

$$\cdot \bar{u}_m(\vec{p}_3, r_3) B^+_{\vec{p}_3, r_3} e^{ip_3\cdot x'} \gamma^0 v_m(\vec{p}_4, r_4) D^+_{\vec{p}_4, r_4} e^{ip_4\cdot x'}] : d^+_{\vec{q}_1, s_1} b^+_{\vec{q}_2, s_2} |0\rangle$$

Since $\langle 0| D_{\vec{q}_3,s_3} B_{\vec{q}_4,s_4} : d_{\vec{p}_1,r_1} b_{\vec{p}_2,r_2} B^{+}_{\vec{p}_3,r_3} D^{+}_{\vec{p}_4,r_4} : d^{+}_{\vec{q}_1,s_1} b^{+}_{\vec{q}_2,s_2} |0\rangle$

$$= -\prod_{i=1}^{4}\left[\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{q}_i}}\left(\frac{1}{C(E_{\vec{q}_i})}\right)^2\right]\delta_{s_1r_1}\delta^3(\vec{q}_1-\vec{p}_1)\,\delta_{s_2r_2}\delta^3(\vec{q}_2-\vec{p}_2)$$
$$\times\,\delta_{s_3r_4}\delta^3(\vec{q}_3-\vec{p}_4)\,\delta_{s_4r_3}\delta^3(\vec{q}_4-\vec{p}_3)$$

$$\Rightarrow\ ** = -(-i)\frac{e^2}{4\pi}\int d^4x\, d^3\vec{x}'\frac{1}{|\vec{x}'-\vec{x}|}\left(\overline{V}_e(\vec{q}_1,s_1)\gamma^0 U_e(\vec{q}_2,s_2)\right)e^{-i(q_1+q_2)\cdot x}$$
$$\times\left(\overline{U}_m(\vec{q}_4,s_4)\gamma^0 V_m(\vec{q}_3,s_3)\right)e^{i(q_3+q_4)\cdot x'}$$

where $\int d^4x\, d^3\vec{x}'\frac{1}{|\vec{x}'-\vec{x}|}e^{-i(q_1+q_2)\cdot x}e^{i(q_3+q_4)\cdot x'}$

note that $x^0 = x'^0$

$$\overset{\downarrow}{=}\ 2\pi\,\delta(E_{\vec{q}_1}+E_{\vec{q}_2}-E_{\vec{q}_3}-E_{\vec{q}_4})\int_{-\infty}^{+\infty}d^3\vec{x}\int_{-\infty}^{+\infty}d^3\vec{x}'\frac{1}{|\vec{x}'-\vec{x}|}e^{i(\vec{q}_1+\vec{q}_2)\cdot\vec{x}}e^{-i(\vec{q}_3+\vec{q}_4)\cdot\vec{x}'}$$

$$\overset{\vec{x}'\equiv\vec{r}+\vec{x}}{=\!=}\ (2\pi)\,\delta(E_{\vec{q}_1}+E_{\vec{q}_2}-E_{\vec{q}_3}-E_{\vec{q}_4})\int_{-\infty}^{+\infty}d^3\vec{x}\int_{-\infty}^{+\infty}d^3\vec{r}\,\frac{1}{|\vec{r}|}e^{i(\vec{q}_1+\vec{q}_2-\vec{q}_3-\vec{q}_4)\cdot\vec{x}}\times e^{-i(\vec{q}_3+\vec{q}_4)\cdot\vec{r}}$$

$$= (2\pi)^4\delta^4(q_1+q_2-q_3-q_4)\underbrace{\int_{-\infty}^{+\infty}d^3\vec{r}\,\frac{1}{|\vec{r}|}e^{-i(\vec{q}_3+\vec{q}_4)\cdot\vec{r}}}_{\frac{4\pi}{|\vec{q}_3+\vec{q}_4|^2}}$$

$$\Rightarrow\ ** = (2\pi)^4\delta^4(q_1+q_2-q_3-q_4)\,(-i)\,e^2\frac{1}{|\vec{q}_3+\vec{q}_4|^2}$$
$$\times\left(\overline{V}_e(\vec{q}_1,s_1)\gamma^0 U_e(\vec{q}_2,s_2)\right)\left(\overline{U}_m(\vec{q}_4,s_4)\gamma^0 V_m(\vec{q}_3,s_3)\right)$$
$$\times\ (-1)$$

use $\frac{1}{|\vec{q}_3+\vec{q}_4|^2} = \frac{1}{(q_3+q_4)^2+i\varepsilon} \frac{(q_3^0+q_4^0)^2-|\vec{q}_3+\vec{q}_4|^2+i\varepsilon}{|\vec{q}_3+\vec{q}_4|^2}$

and use $(\not{q}_2 - m_e)\, u_e(\vec{q}_2, s_2) = 0$

$\bar{v}_e(\vec{q}_1, s_1)(\not{q}_1 + m_e) = 0$

$\Rightarrow \bar{v}_e(\vec{q}_1, s_1)(\not{q}_1 + \not{q}_2)\, u_e(\vec{q}_2, s_2) = 0$

$\Rightarrow \bar{v}_e(\vec{q}_1, s_1)(q_1^0+q_2^0)\gamma^0 u_e(\vec{q}_2, s_2) = \bar{v}_e(\vec{q}_1, s_1)(\vec{q}_1+\vec{q}_2)\cdot\vec{\gamma}\, u_e(\vec{q}_2, s_2)$

Similarly

$\bar{u}_m(\vec{q}_4, s_4)(\not{q}_4 + \not{q}_3)\, v_m(\vec{q}_3, s_3) = 0$

$\Rightarrow \bar{u}_m(\vec{q}_4, s_4)(q_3^0+q_4^0)\gamma^0 v_m(\vec{q}_3, s_3) = \bar{u}_m(\vec{q}_4, s_4)(\vec{q}_3+\vec{q}_4)\cdot\vec{\gamma}\, v_m(\vec{q}_3, s_3)$

$$\Rightarrow ** = (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)(-i)e^2 \frac{1}{(q_1+q_2)^2+i\varepsilon}$$
$$\times \Big[-\left(\bar{v}_e(\vec{q}_1, s_1)\gamma^0 u_e(\vec{q}_2, s_2)\right)\left(\bar{u}_m(\vec{q}_4, s_4)\gamma^0 v_m(\vec{q}_3, s_3)\right)$$
$$+ \left(\bar{v}_e(\vec{q}_1, s_1)(\vec{q}_1+\vec{q}_2)\cdot\vec{\gamma}\, u_e(\vec{q}_2, s_2)\right)\left(\bar{u}_m(\vec{q}_4, s_4)(\vec{q}_1+\vec{q}_2)\cdot\vec{\gamma}\, v_m(\vec{q}_3, s_3)\right)\Big]$$
$$\times(-1)$$

Note that we have dropped the $i\varepsilon$ term in the numerator, anticipating that in the final result we will take $\varepsilon \to 0^+$, and the $i\varepsilon$ term in the numerator is not singular (i.e., it goes to zero)

For the contribution from H_{rad}, we have, to the lowest order,

$$* \equiv \langle f | \frac{(-i)^2}{2!} \int d^4x\, d^4y\, e^2\, T\big(:(\bar{\psi}_e(x)\gamma^i\psi_e(x)A^i(x) + \bar{\psi}_m(x)\gamma^i\psi_m(x)A^i(x)):$$
$$:(\bar{\psi}_e(y)\gamma^j\psi_e(y)A^j(y) + \bar{\psi}_m(y)\gamma^j\psi_m(y)A^j(y)):\big)|i\rangle$$

$$= \langle f | \frac{(-i)^2}{2!} \int d^4x\, d^4y\, e^2\, T\big[:(\bar{\psi}_e(x)\gamma^i\psi_e(x)A^i(x))::(\bar{\psi}_m(y)\gamma^j\psi_m(y)A^j(y)):$$
$$+:(\bar{\psi}_m(x)\gamma^i\psi_m(x)A^i(x)):(\bar{\psi}_e(y)\gamma^j\psi_e(y)A^j(y)):\big]|i\rangle$$

$$= \langle f | \frac{(-i)^2}{2!} \int d^4x\, d^4y\, e^2 \Big[:\bar{\psi}_e(x) \gamma^i \psi_e(x) \bar{\psi}_m(y) \gamma^j \psi_m(y): \underbrace{A^i(x) A^j(y)}$$

$$+ :\bar{\psi}_m(y) \gamma^j \psi_m(y) \bar{\psi}_e(x) \gamma^i \psi_e(x): \underbrace{A^j(y) A^i(x)} \Big] | i \rangle$$

since $\underbrace{A^i(x) A^j(y)} = \underbrace{A^j(y) A^i(x)}$

$$\overset{\downarrow}{=} \langle f | (-i)^2 e^2 \int d^4x \int d^4y \int d^4k\, e^{-ik\cdot(x-y)} \frac{1}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right)$$

$$\times :\bar{\psi}_e(x) \gamma^i \psi_e(x) \bar{\psi}_m(y) \gamma^j \psi_m(y): | i \rangle$$

where $\langle f | :\bar{\psi}_e(x) \gamma^i \psi_e(x) \bar{\psi}_m(y) \gamma^j \psi_m(y): | i \rangle$

$$= \langle 0 | D_{\vec{q}_3, s_3} B_{\vec{q}_4, s_4} \left(\prod_{a=1}^{4} \int d^3k_a\, (E_{\vec{k}_a}) \right) \left(\prod_{b=1}^{4} (2\pi)^3\, 2E_{\vec{q}_b}\, (E_{\vec{q}_b}) \right) \sum_{r_1, r_2, r_3, r_4}$$

$$: \Big[\bar{V}_e(\vec{k}_1, r_1) d_{\vec{k}_1, r_1} e^{-ik_1\cdot x} \gamma^i u_e(\vec{k}_2, r_2) b_{\vec{k}_2, r_2} e^{-ik_2 \cdot x}$$

$$\bar{u}_m(\vec{k}_3, r_3) B^{\dagger}_{\vec{k}_3, r_3} e^{ik_3\cdot y} \gamma^j V_m(\vec{k}_4, r_4) D^{\dagger}_{\vec{k}_4, r_4} e^{ik_4 \cdot y} \Big] :$$

$$d^{\dagger}_{\vec{q}_1, s_1} b^{\dagger}_{\vec{q}_2, s_2} | 0 \rangle$$

$$= (-1)\ \bar{V}_e(\vec{q}_1, s_1) \gamma^i u_e(\vec{q}_2, s_2) e^{-iq_1\cdot x} e^{-iq_2 \cdot x}$$

$$\cdot \bar{u}_m(\vec{q}_4, s_4) \gamma^j V_m(\vec{q}_3, s_3) e^{iq_4 \cdot y} e^{iq_3 \cdot y}$$

$$\Rightarrow * = (-1)(-i)^2 e^2 \int d^4x \int d^4y \int d^4k\, e^{-ik\cdot(x-y)} \frac{1}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right)$$

$$\cdot \left(\bar{V}_e(\vec{q}_1, s_1) \gamma^i u_e(\vec{q}_2, s_2) \right) \left(\bar{u}_m(\vec{q}_4, s_4) \gamma^j V_m(\vec{q}_3, s_3) \right)$$

$$\cdot\, e^{-i(q_1+q_2)\cdot x}\, e^{i(q_4+q_3)\cdot y}$$

$$= (-1)(-i)^2 e^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right) (2\pi)^4 \delta^4(q_1+q_2+k)$$

$$\cdot (2\pi)^4 \delta^4(q_3+q_4+k) \left(\bar{V}_e(\vec{q}_1, s_1) \gamma^i u_e(\vec{q}_2, s_2) \right) \left(\bar{u}_m(\vec{q}_4, s_4) \gamma^j V_m(\vec{q}_3, s_3) \right)$$

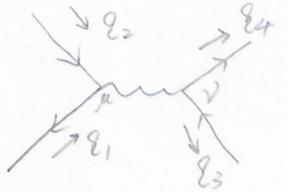
$$= (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)\, (-i) e^2 \frac{1}{(q_1+q_2)^2+i\varepsilon}$$

$$\times \Big[-\left(\bar{V}_e(\vec{q}_1, s_1) \gamma^i u_e(\vec{q}_2, s_2) \right) \left(\bar{u}_m(\vec{q}_4, s_4) \gamma_i V_m(\vec{q}_3, s_3) \right) - \left(\bar{V}_e(\vec{q}_1, s_1) (\vec{q}_1+\vec{q}_2)\cdot\vec{\gamma}\, u_e(\vec{q}_2, s_2) \right)$$

$$\cdot \left(\bar{u}_m(\vec{q}_4, s_4) (\vec{q}_1+\vec{q}_2)\cdot\vec{\gamma}\, V_m(\vec{q}_3, s_3) \right) \Big] (-1)$$

$$\Rightarrow \; * + ** = (2\pi)^4 \delta^4(q_1+q_2-q_3-q_4)\,(-i)\,e^2 \frac{1}{(q_1+q_2)^2+i\varepsilon}$$
$$\cdot \left(\bar{v}_e(\vec{q}_1,s_1)\gamma^\mu u_e(\vec{q}_2,s_2)\right)\left(\bar{u}_m(\vec{q}_4,s_4)\gamma_\mu v_m(\vec{q}_3,s_3)\right)$$

$$\Rightarrow \; iM_{fi} = \bar{v}_e(\vec{q}_1,s_1)\left(-i|e|(-1)\gamma^\mu\right) u_e(\vec{q}_2,s_2)$$
$$\cdot \bar{u}_m(\vec{q}_4,s_4)\left(-i|e|(-1)\gamma^\nu\right) v_m(\vec{q}_3,s_3)\,\frac{i(-g_{\mu\nu})}{(q_1+q_2)^2+i\varepsilon} \times (-1)$$



Note that the (-1) at the end is not important, since we can remove it if we were to switch $d^\dagger_{\vec{q}_1,s_1}$ and $b^\dagger_{\vec{q}_2,s_2}$ in our definition of $|i\rangle$, or to switch $D_{\vec{q}_3,s_3}$ and $B_{\vec{q}_4,s_4}$ in our definition of $\langle f|$.

Now let's calculate the unpolarized $\overline{|M|^2} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sum_{\substack{s_1,s_2,\\ s_3,s_4}} |M|^2$.

$$\overline{|M|^2} = \frac{e^4}{4}\frac{1}{(q_1+q_2)^4}\sum_{\substack{s_1,s_2,\\ s_3,s_4}}\left(\bar{v}_e(\vec{q}_1,s_1)\gamma^\mu u_e(\vec{q}_2,s_2)\right)\left(\bar{u}_e(\vec{q}_2,s_2)\gamma^\alpha v_e(\vec{q}_1,s_1)\right)$$
$$\cdot \left(\bar{u}_m(\vec{q}_4,s_4)\gamma^\nu v_m(\vec{q}_3,s_3)\right)\left(\bar{v}_m(\vec{q}_3,s_3)\gamma^\beta u_m(\vec{q}_4,s_4)\right)$$
$$\cdot\, g_{\mu\nu}\, g_{\alpha\beta}$$

$$= \frac{e^4}{4}\frac{1}{s^2}\, \mathrm{Tr}\left[(\not{q}_2+m)\gamma^\alpha(\not{q}_1-m)\gamma^\mu\right]$$
$$\cdot\, \mathrm{Tr}\left[(\not{q}_3-M)\gamma^\beta(\not{q}_4+M)\gamma^\nu\right]$$
$$\cdot\, g_{\mu\nu}\, g_{\alpha\beta}$$

where the m & M are the masses of electron and muon, respectively.

$$\Rightarrow |\overline{\mathcal{M}}|^2 = \frac{e^4}{4}\frac{1}{s^2}\Big[\mathrm{Tr}(\not{p}_2\gamma^\alpha\not{p}_1\gamma^\mu) - m^2\,\mathrm{Tr}(\gamma^\alpha\gamma^\mu)\Big]$$
$$\Big[\mathrm{Tr}(\not{p}_3\gamma_\alpha\not{p}_4\gamma_\mu) - M^2\,\mathrm{Tr}(\gamma_\alpha\gamma_\mu)\Big]$$

$$= \frac{e^4}{4s^2}\times 4\Big[p_2^\alpha p_1^\mu + p_2^\mu p_1^\alpha - g^{\alpha\mu}\,p_1\cdot p_2 - m^2 g^{\alpha\mu}\Big]$$
$$\times 4\Big[p_{3\alpha}p_{4\mu} + p_{3\mu}p_{4\alpha} - g_{\alpha\mu}\,p_3\cdot p_4 - M^2 g_{\alpha\mu}\Big]$$

$$= \frac{4e^4}{s^2}\Big[(p_2\cdot p_3)(p_1\cdot p_4) + (p_2\cdot p_4)(p_1\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4) - M^2(p_1\cdot p_2)$$
$$+ (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_3)(p_1\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4) - M^2(p_1\cdot p_2)$$
$$- (p_3\cdot p_4)(p_1\cdot p_2) - (p_3\cdot p_4)(p_1\cdot p_2) + 4(p_1\cdot p_2)(p_3\cdot p_4) + 4M^2(p_1\cdot p_2)$$
$$- m^2(p_3\cdot p_4) - m^2(p_3\cdot p_4) + 4m^2(p_3\cdot p_4) + 4m^2M^2\Big]$$

$$= \frac{4e^4}{s^2}\Big[2(p_2\cdot p_3)(p_1\cdot p_4) + 2(p_2\cdot p_4)(p_1\cdot p_3) + 2M^2(p_1\cdot p_2) + 2m^2(p_3\cdot p_4)$$
$$+ 4m^2M^2\Big]$$

using

$$p_1\cdot p_2 = \frac{(p_1+p_2)^2 - p_1^2 - p_2^2}{2} = \frac{s}{2} - m^2$$
$$p_3\cdot p_4 = \frac{(p_3+p_4)^2 - p_3^2 - p_4^2}{2} = \frac{s}{2} - M^2$$
$$p_1\cdot p_3 = \frac{p_1^2 + p_3^2 - (p_1-p_3)^2}{2} = \frac{m^2+M^2-t}{2}$$
$$p_1\cdot p_4 = \frac{p_1^2 + p_4^2 - (p_1-p_4)^2}{2} = \frac{m^2+M^2-u}{2}$$
$$p_2\cdot p_3 = \frac{p_2^2 + p_3^2 - (p_2-p_3)^2}{2} = \frac{m^2+M^2-u}{2}$$
$$p_2\cdot p_4 = \frac{p_2^2 + p_4^2 - (p_2-p_4)^2}{2} = \frac{m^2+M^2-t}{2}$$

$$\Rightarrow |\overline{\mathcal{M}}|^2 = \frac{8e^4}{s^2}\Big[\Big(\frac{m^2+M^2-u}{2}\Big)^2 + \Big(\frac{m^2+M^2-t}{2}\Big)^2 + M^2\Big(\frac{s}{2}-m^2\Big) + m^2\Big(\frac{s}{2}-M^2\Big)$$
$$+ 2m^2M^2\Big]$$

$u = 2m^2+2M^2-t-s$

$$\stackrel{\rightarrow}{=} \frac{8e^4}{s^2}\Big[\Big(\frac{t+s-m^2-M^2}{2}\Big)^2 + \Big(\frac{m^2+M^2-t}{2}\Big)^2 + (M^2+m^2)\frac{s}{2}\Big]$$

$$= \frac{2e^4}{s^2}\Big[t^2+s^2+(m^2+M^2)^2+2ts-2t(m^2+M^2)-2s(m^2+M^2)$$
$$+t^2+(m^2+M^2)^2-2t(m^2+M^2)+2s(M^2+m^2)\Big]$$

$e^2=4\pi\alpha$

$$\stackrel{?}{=} \frac{32\pi^2\alpha^2}{s^2}\left[2t^2+s^2+2ts-4t(m^2+M^2)+2(m^2+M^2)^2\right]$$

use $t=(P_1-P_3)^2 = m^2+M^2-2P_1\cdot P_3 = m^2+M^2-2E_1E_3+2|\vec{P}_1||\vec{P}_3|\cos\theta_{cm}$

$$= (m^2+M^2)-2\frac{s^2}{(2\sqrt{s})(2\sqrt{s})}+\frac{2\lambda^{\frac12}(s,m^2,m^2)\,\lambda^{\frac12}(s,M^2,M^2)}{(2\sqrt{s})(2\sqrt{s})}\cos\theta_{cm}$$

$$= m^2+M^2-\frac{s}{2}+\frac{1}{2s}(s^2+2m^4-4m^2s-2m^4)^{\frac12}(s^2+2M^4-4M^2s-2M^4)^{\frac12}\cos\theta_{cm}$$

$$= m^2+M^2-\frac{s}{2}+\frac12(s-4m^2)^{\frac12}(s-4M^2)^{\frac12}\cos\theta_{cm}$$

$$\Rightarrow u = 2m^2+2M^2-s-t$$
$$= 2m^2+2M^2-s-\left(m^2+M^2-\frac{s}{2}+\frac12(s-4m^2)^{\frac12}(s-4M^2)^{\frac12}\cos\theta_{cm}\right)$$
$$= m^2+M^2-\frac{s}{2}-\frac12(s-4m^2)^{\frac12}(s-4M^2)^{\frac12}\cos\theta_{cm}$$

while $2t^2+2ts-4t(m^2+M^2)$

$$= 2t(t+s-2m^2-2M^2)$$
$$= -2tu$$
$$= -2\left[\left(m^2+M^2-\frac{s}{2}\right)^2-\frac14(s-4m^2)(s-4M^2)\cos^2\theta_{cm}\right]$$

$$\Rightarrow |\overline{M}|^2 = \frac{32\pi^2\alpha^2}{s^2}\left[\frac12(s-4m^2)(s-4M^2)\cos^2\theta_{cm}-2(m^2+M^2)^2-\frac{s^2}{2}\right.$$
$$\left.+2s(m^2+M^2)+s^2+2(m^2+M^2)^2\right]$$

$$= 16\pi^2\alpha^2\left[1+\frac{4}{s}(m^2+M^2)+\left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right)\cos^2\theta_{cm}\right]$$

$$\Rightarrow \left(\frac{d\bar\sigma}{d\Omega}\right)_{cm} = \frac{|\overline{M}|^2}{64\pi^2 s}\,\frac{\lambda^{\frac12}(s,M^2,M^2)}{\lambda^{\frac12}(s,m^2,m^2)}$$
$$= \frac{|\overline{M}|^2}{64\pi^2 s}\,\frac{(s^2-4M^2s)^{\frac12}}{(s^2-4m^2s)^{\frac12}}$$

$$\Rightarrow \overline{\sigma}_{cm} = \frac{2\pi}{64\pi^2 s}\left(\frac{s-4M^2}{s-4m^2}\right)^{\frac{1}{2}} 16\pi^2\alpha^2 \left\{ 2 + \frac{8}{s}(m^2+M^2) + \frac{2}{3}\left(1-\frac{4m^2}{s}\right)\left(1-\frac{4M^2}{s}\right) \right\}$$

$$= \frac{2\times\pi\alpha^2}{3\times 2\, s}\sqrt{\frac{1-\frac{4M^2}{s}}{1-\frac{4m^2}{s}}}\left[3 + \frac{12}{s}(m^2+M^2) + 1 - \frac{4(m^2+M^2)}{s} + \frac{16m^2M^2}{s^2}\right]$$

$$= \frac{4\pi\alpha^2}{3s}\sqrt{\frac{1-\frac{4M^2}{s}}{1-\frac{4m^2}{s}}}\left[1 + \frac{2(m^2+M^2)}{s} + \frac{4m^2M^2}{s^2}\right]$$

[1] In the limit $s \gg m^2$, $s \gg M^2$,

$$\overline{\sigma}_{cm} \simeq \frac{4\pi\alpha^2}{3s}$$

[2] In the limit $M \gg m$ (so that $s \gg m^2$),

$$\overline{\sigma}_{cm} \simeq \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4M^2}{s}}\left[1+\frac{2M^2}{s}\right]$$

using $s = (2E_3)^2$, $\beta_f \equiv \frac{|\vec{p}_3|}{E_3} = \frac{(E_3^2-M^2)^{\frac{1}{2}}}{E_3} = \left(1-\frac{M^2}{\frac{s}{4}}\right)^{\frac{1}{2}}$

$$= \left(1-\frac{4M^2}{s}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \beta_f^2 = 1-\frac{4M^2}{s} \Rightarrow 1+\frac{2M^2}{s} = 1+\frac{1}{2}-\frac{1}{2}\beta_f^2$$

$$= \frac{3}{2}-\frac{1}{2}\beta_f^2$$

$$\Rightarrow \overline{\sigma}_{cm} \simeq \frac{4\pi\alpha^2}{3s}\beta_f\left(\frac{3}{2}-\frac{1}{2}\beta_f^2\right)$$

$$= \frac{2\pi\alpha^2}{s}\beta_f\left(1-\frac{\beta_f^2}{3}\right)$$

[3] In the limit $m \gg M$ (so that $s \gg M^2$),

$$\overline{\sigma}_{cm} \doteq \frac{4\pi\alpha^2}{3s}\sqrt{\frac{1}{1-\frac{4m^2}{s}}}\left[1+\frac{2m^2}{s}\right]$$

using $s = (2E_1)^2$, $\beta_i \equiv \frac{|\vec{p}_1|}{E_1} = \frac{(E_1^2-m^2)^{\frac{1}{2}}}{E_1} = \left(1-\frac{m^2}{E_1^2}\right)^{\frac{1}{2}} = \left(1-\frac{4m^2}{s}\right)^{\frac{1}{2}}$,

$$\Rightarrow \beta_i^2 = 1-\frac{4m^2}{s} \Rightarrow 1+\frac{2m^2}{s} = 1+\frac{1}{2}-\frac{1}{2}\beta_i^2 = \frac{3}{2}-\frac{1}{2}\beta_i^2$$

$$\Rightarrow \overline{\sigma}_{cm} \simeq \frac{4\pi\alpha^2}{3s}\beta_i^{-1}\left(\frac{3}{2}-\frac{1}{2}\beta_i^2\right) = \frac{2\pi\alpha^2}{s\beta_i}\left(1-\frac{\beta_i^2}{3}\right)$$

Example 3.

Compton Scattering.

$$e^-(p_1, s_1) + \gamma(k_1, r_1) \longrightarrow e^-(p_2, s_2) + \gamma(k_2, r_2)$$

$$|i\rangle = (2\pi)^3\, 2E_{\vec{p}_1} C(E_{\vec{p}_1}) (2\pi)^3\, 2E_{\vec{k}_1} C(E_{\vec{k}_1})\, b^\dagger_{\vec{p}_1, s_1} a^\dagger_{\vec{k}_1, r_1} |0\rangle$$

$$\langle f| = (2\pi)^3\, 2E_{\vec{p}_2} C(E_{\vec{p}_2}) (2\pi)^3\, 2E_{\vec{k}_2} C(E_{\vec{k}_2}) \langle 0|\, b_{\vec{p}_2, s_2} a_{\vec{k}_2, r_2}$$

$$\psi = \int d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left(u(\vec{p}, s)\, b_{\vec{p}, s}\, e^{-ip\cdot x} + v(\vec{p}, s)\, d^\dagger_{\vec{p}, s}\, e^{ip\cdot x} \right)$$

$$\vec{A}(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k}, \lambda)\, a_{\vec{k}, \lambda}\, e^{-ik\cdot x} + \vec{e}^{\,*}(\vec{k}, \lambda)\, a^\dagger_{\vec{k}, \lambda}\, e^{ik\cdot x} \right)$$

The lowest order result comes from the second order of

$$\mathcal{H}_{int} = -|e|(-1)\, \bar{\psi} \gamma^i \psi A^i$$

$$\langle f| i\mathcal{T} |i\rangle = \langle f| \frac{(-i)^2}{2!} \int d^4x\, d^4y\, e^2\, T\left(:\bar{\psi}(x)\gamma^i\psi(x)A^i(x): :\bar{\psi}(y)\gamma^j\psi(y)A^j(y): \right) |i\rangle$$

$$= \langle f| \frac{(-i)^2}{2!} e^2 \int d^4x\, d^4y \begin{pmatrix} :\bar{\psi}_a(x)\, \gamma^i_{ab}\, \underbrace{\psi_b(x)\, A^i(x)\, \bar{\psi}_c(y)}\, \gamma^j_{cd}\, \psi_d(y)\, A^j(y): \\ + :\underbrace{\bar{\psi}_a(x)\, \gamma^i_{ab}\psi_b(x)\, A^i(x)\, \bar{\psi}_c(y)\, \gamma^j_{cd}\psi_d(y)}\, A^j(y): \end{pmatrix} |i\rangle$$

use

$$\underbrace{\psi_a(x)\bar{\psi}_b}(y) = \int_{-\infty}^{+\infty} d^4q\, \frac{1}{(2\pi)^4}\, e^{-iq\cdot(x-y)}\, \frac{i(\not{q} + m)_{ab}}{q^2 - m^2 + i\varepsilon}$$

$$\underbrace{\bar{\psi}_b(x)\psi_a}(y) = -\int_{-\infty}^{+\infty} d^4q\, \frac{1}{(2\pi)^4}\, e^{-iq\cdot(x-y)}\, \frac{i(-\not{q} + m)_{ab}}{q^2 - m^2 + i\varepsilon}$$

$$\Rightarrow \langle f| i\mathcal{T} |i\rangle = \langle f| \frac{(-i)^2}{2!} e^2 \int d^4x \int d^4y \int d^4q\, \frac{1}{(2\pi)^4}\, e^{-iq\cdot(x-y)}$$

$$\times \Big[:\bar{\psi}_a(x)\, \gamma^i_{ab} A^i(x)\, \gamma^j_{cd}\, \psi_d(y)\, A^j(y): \frac{i(\not{q} + m)_{bc}}{q^2 - m^2 + i\varepsilon}$$

$$+ :\gamma^i_{ab}\psi_b(x)\, A^i(x)\, \bar{\psi}_c(y)\, \gamma^j_{cd} A^j(y): \frac{i(\not{q} - m)_{da}}{q^2 - m^2 + i\varepsilon} \Big] |i\rangle$$

For the second term,

Since $\int d^4x\, d^4y\, d^4q \frac{1}{(2\pi)^4} e^{-iq\cdot(x-y)} : \gamma^i_{ab}\psi_b(x)A^i(x)\bar{\psi}_c(y)\gamma^j_{cd}A^j(y): \cdot \frac{i(\not{q}-m)_{da}}{q^2-m^2+i\epsilon}$

$$\overset{x\leftrightarrow y,\ b\leftrightarrow d,\ a\leftrightarrow c,\ i\leftrightarrow j}{=} \int d^4x\, d^4y\, d^4q \frac{1}{(2\pi)^4} e^{iq\cdot(x-y)} : \gamma^j_{cd}\psi_d(y)A^j(y)\bar{\psi}_a(x)\gamma^i_{ab}A^i(x): \cdot \frac{i(\not{q}-m)_{bc}}{q^2-m^2+i\epsilon}$$

$$\overset{q\to -q}{=} \int d^4x\, d^4y\, d^4q \frac{1}{(2\pi)^4} e^{-iq\cdot(x-y)} : \bar{\psi}_a(x)\gamma^i_{ab}A^i(x)\gamma^j_{cd}\psi_d(y)A^j(y): \times \underbrace{(-1)}_{\substack{\text{due to}\\ \text{switch } \bar{\psi}_a(x) \text{ and } \psi_d(y)}} \times \frac{i(-\not{q}-m)_{bc}}{q^2-m^2+i\epsilon}$$

$$= \int d^4x\, d^4y\, d^4q \frac{1}{(2\pi)^4} e^{-iq\cdot(x-y)} : \bar{\psi}_a(x)\gamma^i_{ab}A^i(x)\gamma^j_{cd}\psi_d(y)A^j(y): \frac{i(\not{q}+m)_{bc}}{q^2-m^2+i\epsilon}$$

$\Rightarrow$ It's the same as the first term.

Therefore,

$$\langle f|iT|i\rangle = \langle f|(-i)^2e^2\int d^4x\, d^4y\, d^4q \frac{1}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{i(\not{q}+m)_{bc}}{q^2-m^2+i\epsilon}$$
$$\times :\bar{\psi}_a(x)\psi_d(y)A^i(x)A^j(y): \gamma^i_{ab}\gamma^j_{cd}|i\rangle$$

use $\langle f| :\bar{\psi}_a(x)\psi_d(y)A^i(x)A^j(y): |i\rangle$

$$= (2\pi)^3 2E_{\vec{p}_1}C(E_{\vec{p}_1})(2\pi)^3 2E_{\vec{p}_2}C(E_{\vec{p}_2})(2\pi)^3 2E_{\vec{k}_1}C(E_{\vec{k}_1})(2\pi)^3 2E_{\vec{k}_2}C(E_{\vec{k}_2})\int d^3q_1\int d^3q_2\int d^3q_3$$
$$\int d^3q_4\, C(E_{\vec{q}_1})C(E_{\vec{q}_2})C(E_{\vec{q}_3})C(E_{\vec{q}_4})\langle 0|b_{\vec{p}_2,s_2}a_{\vec{k}_2,r_2} : \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}\Big(\bar{u}_a(\vec{q}_1,\lambda_1)b^\dagger_{\vec{q}_1,\lambda_1}e^{iq_1\cdot x}$$
$$\cdot\, u_d(\vec{q}_2,\lambda_2)b_{\vec{q}_2,\lambda_2}e^{-iq_2\cdot y}\left(\epsilon^i(\vec{q}_3,\lambda_3)a_{\vec{q}_3,\lambda_3}e^{-iq_3\cdot x}+\epsilon^{*i}(\vec{q}_3,\lambda_3)a^\dagger_{\vec{q}_3,\lambda_3}e^{iq_3\cdot x}\right)$$
$$\cdot\left(\epsilon^j(\vec{q}_4,\lambda_4)a_{\vec{q}_4,\lambda_4}e^{-iq_4\cdot y}+\epsilon^{*j}(\vec{q}_4,\lambda_4)a^\dagger_{\vec{q}_4,\lambda_4}e^{iq_4\cdot y}\right)\Big) : b^\dagger_{\vec{p}_1,s_1}a^\dagger_{\vec{k}_1,r_1}|0\rangle$$

where $\langle\ \ \rangle = \sum_{\substack{\lambda_1,\lambda_2,\\ \lambda_3,\lambda_4}} \Big\{ \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}_2}}\left(\frac{1}{C(E_{\vec{p}_2})}\right)^2 \delta_{\lambda_1 s_2}\delta^3(\vec{q}_1-\vec{p}_2)\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}_1}}\left(\frac{1}{C(E_{\vec{p}_1})}\right)^2$

$\cdot\delta_{\lambda_2 s_1}\delta^3(\vec{q}_2-\vec{p}_1)\ \ \bar{u}_a(\vec{p}_2,s_2)\,u_d(\vec{p}_1,s_1)\,e^{ip_2\cdot x}e^{-ip_1\cdot y}$

$\times\Big[\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{k}_1}}\left(\frac{1}{C(E_{\vec{k}_1})}\right)^2\delta_{\lambda_3 r_1}\delta^3(\vec{q}_3-\vec{k}_1)\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{k}_2}}\left(\frac{1}{C(E_{\vec{k}_2})}\right)^2\delta_{\lambda_4 r_2}\delta^3(\vec{q}_4-\vec{k}_2)$

$\times e^{-ik_1\cdot x}e^{ik_2\cdot y}\ e^{i}(\vec{k}_1,r_1)\,e^{*j}(\vec{k}_2,r_2)$

$+\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{k}_2}}\left(\frac{1}{C(E_{\vec{k}_2})}\right)^2\delta_{\lambda_3 r_2}\delta^3(\vec{q}_3-\vec{k}_2)\frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{k}_1}}\left(\frac{1}{C(E_{\vec{k}_1})}\right)^2\delta_{\lambda_4 r_1}\delta^3(\vec{q}_4-\vec{k}_1)$

$\times e^{ik_2\cdot x}e^{-ik_1\cdot y}\ e^{*i}(\vec{k}_2,r_2)\,e^{j}(\vec{k}_1,r_1)\Big]\Big\}$

$\Rightarrow \langle f|:\bar{\psi}_a(x)\psi_d(y)A^i(x)A^j(y):|i\rangle$

$= \bar{u}_a(\vec{p}_2,s_2)\,u_d(\vec{p}_1,s_1)\,e^{i}(\vec{k}_1,r_1)\,e^{*j}(\vec{k}_2,r_2)\,e^{i(p_2-k_1)\cdot x}e^{i(k_2-p_1)\cdot y}$

$+\bar{u}_a(\vec{p}_2,s_2)\,u_d(\vec{p}_1,s_1)\,e^{j}(\vec{k}_1,r_1)\,e^{*i}(\vec{k}_2,r_2)\,e^{i(p_2+k_2)\cdot x}e^{i(-k_1-p_1)\cdot y}$

$\Rightarrow \langle f|iT|i\rangle$

$= (-i)^2e^2\int d^4x\,d^4y\,d^4q\,\frac{1}{(2\pi)^4}e^{-iq\cdot(x-y)}\frac{i}{q^2-m^2+i\varepsilon}$

$\times\left[\ \bar{u}_a(\vec{p}_2,s_2)\,\gamma^i_{ab}\,(\not{q}+m)_{bc}\,\gamma^j_{cd}\,u_d(\vec{p}_1,s_1)\ \right]$

$\times\Big(e^{i}(\vec{k}_1,r_1)\,e^{*j}(\vec{k}_2,r_2)\,e^{i(p_2-k_1)\cdot x}e^{i(k_2-p_1)\cdot y}$

$+\ e^{j}(\vec{k}_1,r_1)\,e^{*i}(\vec{k}_2,r_2)\,e^{i(p_2+k_2)\cdot x}e^{i(-k_1-p_1)\cdot y}\Big)$

$= (-i)^2e^2\int d^4q\,\frac{1}{(2\pi)^4}\frac{i}{q^2-m^2+i\varepsilon}\left[\bar{u}(\vec{p}_2,s_2)\gamma^i(\not{q}+m)\gamma^j u(\vec{p}_1,s_1)\right]$

$\times\Big(e^{i}(\vec{k}_1,r_1)\,e^{*j}(\vec{k}_2,r_2)\,(2\pi)^4\delta^4(p_2-k_1-q)\,(2\pi)^4\delta^4(k_2-p_1+q)$

$+\ e^{j}(\vec{k}_1,r_1)\,e^{*i}(\vec{k}_2,r_2)\,(2\pi)^4\delta^4(p_2+k_2-q)\,(2\pi)^4\delta^4(-k_1-p_1+q)\Big)$

$= (2\pi)^4\delta^4(p_1+k_1-p_2-k_2)\ (-i)^2e^2$

$\times\Big[\ \bar{u}(\vec{p}_2,s_2)\gamma^i(\not{p}_2-\not{k}_1+m)\gamma^j u(\vec{p}_1,s_1)\,e^{i}(\vec{k}_1,r_1)\,e^{*j}(\vec{k}_2,r_2)\frac{i}{(p_2-k_1)^2-m^2+i\varepsilon}$

$+\ \bar{u}(\vec{p}_2,s_2)\gamma^i(\not{p}_2+\not{k}_2+m)\gamma^j u(\vec{p}_1,s_1)\,e^{j}(\vec{k}_1,r_1)\,e^{*i}(\vec{k}_2,r_2)\frac{i}{(p_2+k_2)^2-m^2+i\varepsilon}$

$$\Rightarrow i\mathcal{M}_{fi} = i(-i)^2 e^2 \bar{u}(\vec{p}_2, s_2)\left[\gamma^i \frac{(\not{p}_2 - \not{k}_1 + m)}{(p_2-k_1)^2 - m^2 + i\varepsilon}\gamma^j + \gamma^j \frac{(\not{p}_2 + \not{k}_2 + m)}{(p_2+k_2)^2 - m^2 + i\varepsilon}\gamma^i\right]$$

$$\cdot u(\vec{p}_1, s_1) \quad e^i(\vec{k}_1, r_1) e^{*j}(\vec{k}_2, r_2)$$

Since we are using Coulomb gauge, then

$$\varepsilon^\mu(\vec{k}_1, r_1) = (0, \vec{e}(\vec{k}_1, r_1))$$

$$\varepsilon^{*\mu}(\vec{k}_2, r_2) = (0, \vec{e}^{\,*}(\vec{k}_2, r_2))$$

$$\Rightarrow \quad \gamma^i e^i(\vec{k}_1, r_1) = -\gamma^\mu \varepsilon_{1\mu}(\vec{k}_1, r_1) \equiv -\not{\varepsilon}_1$$

$$\gamma^j e^{*j}(\vec{k}_2, r_2) = -\gamma^\mu \varepsilon^*_{2\mu}(\vec{k}_2, r_2) \equiv -\not{\varepsilon}_2^*$$

$$\Rightarrow i\mathcal{M}_{fi} = -ie^2 \bar{u}(\vec{p}_2, s_2)\left[\frac{\not{\varepsilon}_1(\not{p}_2 - \not{k}_1 + m)\not{\varepsilon}_2^*}{(p_2-k_1)^2 - m^2 + i\varepsilon} + \frac{\not{\varepsilon}_2^*(\not{p}_2 + \not{k}_2 + m)\not{\varepsilon}_1}{(p_2+k_2)^2 - m^2 + i\varepsilon}\right]u(\vec{p}_1, s_1)$$

using. $\vec{k}_1 \cdot \vec{e}(\vec{k}_1, r_1) = 0 \Rightarrow k_1 \cdot \varepsilon_1 = 0$, $\quad (\not{p}_1 - m)\, u(\vec{p}_1, s_1) = 0$

$\vec{k}_2 \cdot \vec{e}(\vec{k}_2 \cdot r_2) = 0 \Rightarrow k_2 \cdot \varepsilon_2 = 0$, $\quad \bar{u}(\vec{p}_2, s_2)(\not{p}_2 - m) = 0$

$k_1^2 = k_2^2 = 0$, $\; p_1^2 = p_2^2 = m^2$, $\quad (p_1+k_1)^\mu = (p_2+k_2)^\mu$

$$\Rightarrow \bar{u}(\vec{p}_2, s_2)\not{\varepsilon}_1(\not{p}_2 - \not{k}_1 + m)\not{\varepsilon}_2^* u(\vec{p}_1, s_1) = \bar{u}(\vec{p}_2, s_2)\not{\varepsilon}_1(\not{p}_1 - \not{k}_2 + m)\not{\varepsilon}_2^* u(\vec{p}_1, s_1)$$

$$= \bar{u}(\vec{p}_2, s_2)\not{\varepsilon}_1(2p_1\cdot\varepsilon_2^* - \not{k}_2\not{\varepsilon}_2^*) \quad u(\vec{p}_1, s_1)$$

$$\bar{u}(\vec{p}_2, s_2)\not{\varepsilon}_2^*(\not{p}_2 + \not{k}_2 + m)\not{\varepsilon}_1 u(\vec{p}_1, s_1) = \bar{u}(\vec{p}_2, s_2)\not{\varepsilon}_2^*(\not{p}_1 + \not{k}_1 + m)\not{\varepsilon}_1 u(\vec{p}_1, s_1)$$

$$= \bar{u}(\vec{p}_2, s_2)\not{\varepsilon}_2^*(2p_1\cdot\varepsilon_1 + \not{k}_1\not{\varepsilon}_1)u(\vec{p}_1, s_1)$$

$$(p_2-k_1)^2 - m^2 = (p_1-k_2)^2 - m^2 = -2p_1 k_2, \quad (p_2+k_2)^2 - m^2 = (p_1+k_1)^2 - m^2 = 2p_1\cdot k_1$$

$$\Rightarrow i\mathcal{M}_{fi} = -ie^2 \bar{u}(\vec{p}_2, s_2)\left[\frac{\not{\varepsilon}_1(2p_1\cdot\varepsilon_2^* - \not{k}_2\not{\varepsilon}_2^*)}{-2p_1\cdot k_2} + \frac{\not{\varepsilon}_2^*(2p_1\cdot\varepsilon_1 + \not{k}_1\not{\varepsilon}_1)}{2p_1\cdot k_1}\right]u(\vec{p}_1, s_1)$$

Now let's sum over and averaged over the electrons' polorizations

$$\frac{1}{2}\sum_{s_1,s_2}|\mathcal{M}|^2 = \frac{1}{2}e^4\,\mathrm{Tr}\left[(\not{p}_1+m)\left(\frac{2(p_1\cdot\varepsilon_2)\not{\varepsilon}_1^* - \not{\varepsilon}_2\not{k}_2\not{\varepsilon}_1^*}{-2p_1\cdot k_2} + \frac{2(p_1\cdot\varepsilon_1^*)\not{\varepsilon}_2 + \not{\varepsilon}_1^*\not{k}_1\not{\varepsilon}_2}{2p_1\cdot k_1}\right)\right.$$

$$\left.\cdot(\not{p}_2+m)\left(\frac{2(p_1\cdot\varepsilon_2^*)\not{\varepsilon}_1 - \not{\varepsilon}_1\not{k}_2\not{\varepsilon}_2^*}{-2p_1\cdot k_2} + \frac{2(p_1\cdot\varepsilon_1)\not{\varepsilon}_2^* + \not{\varepsilon}_2^*\not{k}_1\not{\varepsilon}_1}{2p_1\cdot k_1}\right)\right]$$

$$= \frac{e^4}{2} \mathrm{Tr}\Big[(\not{P}_2 + m) \left(\frac{2P_1^\mu \gamma^\nu + \gamma^\nu \not{k}_1 \gamma^\mu}{2P_1\cdot k_1} - \frac{2P_1^\nu \gamma^\mu - \gamma^\mu \not{k}_2 \gamma^\nu}{2P_1\cdot k_2} \right) (\not{P}_1 + m)$$

$$\cdot \left(\frac{2P_1^\alpha \gamma^\beta + \gamma^\alpha \not{k}_1 \gamma^\beta}{2P_1\cdot k_1} - \frac{2P_1^\beta \gamma^\alpha - \gamma^\beta \not{k}_2 \gamma^\alpha}{2P_1\cdot k_2} \right) \Big]$$

$$\times \; \varepsilon_{1\mu}(\vec{k}_1, r_1)\, \varepsilon^*_{2\nu}(\vec{k}_2, r_2)\, \varepsilon^*_{1\alpha}(\vec{k}_1, r_1)\, \varepsilon_{2\beta}(k_2, r_2)$$

$\boxed{\text{Let's evaluate the above expression in the rest frame of } P_1.}$

use $P_1^\mu \varepsilon_{1\mu}(\vec{k}_1, r_1) = m\cdot\varepsilon_{10}(\vec{k}_1, r_1) - \vec{0}\cdot\vec{e}_1(\vec{k}_1, \vec{r}_1) = 0 - 0 = 0$

and similarly $P_1^\nu \varepsilon^*_{2\nu}(\vec{k}_2, r_2) = 0$

$P_1^\alpha \varepsilon^*_{1\alpha}(\vec{k}_1, r_1) = 0$

$P_1^\beta \varepsilon_{2\beta}(k_2, r_2) = 0$

$$\Rightarrow \frac{1}{2}\sum_{s_1, s_2} |\mathcal{M}|^2 = \frac{e^4}{2}\mathrm{Tr}\Big[(\not{P}_2 + m) \left(\frac{\gamma^\nu \not{k}_1 \gamma^\mu}{2P_1\cdot k_1} + \frac{\gamma^\mu \not{k}_2 \gamma^\nu}{2P_1\cdot k_2} \right) (\not{P}_1 + m)$$

$$\cdot \left(\frac{\gamma^\alpha \not{k}_1 \gamma^\beta}{2P_1\cdot k_1} + \frac{\gamma^\beta \not{k}_2 \gamma^\alpha}{2P_1\cdot k_2} \right) \Big] \; \varepsilon_{1\mu}(\vec{k}_1, r_1)\, \varepsilon^*_{2\nu}(\vec{k}_2, r_2)\, \varepsilon^*_{1\alpha}(\vec{k}_1, r_1)\, \varepsilon_{2\beta}(\vec{k}_2, r_2)$$

$$= \frac{e^4}{2}\left[\frac{T_1}{(2P_1\cdot k_1)^2} + \frac{T_2}{(2P_1\cdot k_2)^2} + \frac{T_3 + T_4}{2(P_1\cdot k_1)2(P_1\cdot k_2)} \right]$$

where
$$T_1 \equiv \mathrm{Tr}[(\not{P}_2 + m)\not{\varepsilon}_2^* \not{k}_1 \not{\varepsilon}_1 (\not{P}_1 + m) \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$
$$T_2 \equiv \mathrm{Tr}[(\not{P}_2 + m)\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* (\not{P}_1 + m) \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*]$$
$$T_3 \equiv \mathrm{Tr}[(\not{P}_2 + m)\not{\varepsilon}_2^* \not{k}_1 \not{\varepsilon}_1 (\not{P}_1 + m) \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*]$$
$$T_4 \equiv \mathrm{Tr}[(\not{P}_2 + m)\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* (\not{P}_1 + m) \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$

Now let's do the polarization sum for the initial and final state photons.

$$\Rightarrow \sum_{r_1} \not{\varepsilon}_1 \not{\varepsilon}_1^* = \sum_{r_1} e^i(\vec{k}_1, r_1)\, e^{j*}(\vec{k}_1, r_1)\, \gamma^i\gamma^j = \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}_1|^2}\right)\gamma^i\gamma^j = \gamma^i\gamma^i - \frac{\frac{1}{2}k_1^i k_1^j 2g^{ij}}{|\vec{k}_1|^2}$$

$$= -3 + 1 = -2 = \sum_{r_1} \not{\varepsilon}_1^* \not{\varepsilon}_1$$

and $\sum_{r_2} \not{\varepsilon}_2 \not{\varepsilon}_2^* = -2 = \sum_{r_2} \not{\varepsilon}_2^* \not{\varepsilon}_2$

$$\Rightarrow \sum_{r_1,r_2} T_1 = \sum_{r_1 r_2}\Big(\mathrm{Tr}[\not{p}_2 \not{\varepsilon}_2^* \not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2] + m^2 \underbrace{\mathrm{Tr}[\not{\varepsilon}_2^* \not{k}_1 \not{\varepsilon}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]}_{\underbrace{\mathrm{Tr}[\not{\varepsilon}_2 \not{\varepsilon}_2^* \not{k}_1 (-2) \not{k}_1]}_{\underbrace{\mathrm{Tr}[4 \not{k}_1 \not{k}_1]}_{\mathrm{Tr}[4 \cdot k_1^2] = 0}}} \Big)$$

$$= \sum_{r_1,r_2} \mathrm{Tr}[\not{p}_2 \not{\varepsilon}_2^* \not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$

$$= \sum_{r_1,r_2} \mathrm{Tr}\Big[(2 p_2\cdot\varepsilon_2^* - \not{\varepsilon}_2^* \not{p}_2) \not{k}_1 (\underbrace{2 p_1\cdot\varepsilon_1}_{0} - \not{p}_1 \not{\varepsilon}_1) \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2\Big]$$

using
$$p_2\cdot\varepsilon_2^* = (p_1+k_1-k_2)\cdot\varepsilon_2^* = k_1\cdot\varepsilon_2^*$$
$$p_1\cdot\varepsilon_2^* = 0 \text{ and } k_2\cdot\varepsilon_2^* = -k_2^i e_2^{*i}(\vec{k}_2, r_2) = 0$$

$$\Rightarrow \sum_{r_1,r_2} T_1 = \sum_{r_1,r_2} \mathrm{Tr}\Big[\big(2 \underbrace{k_1\cdot\varepsilon_2^*}_{-\vec{k}_1\cdot\vec{e}_2^*} \not{\varepsilon}_2 - \not{\varepsilon}_2 \not{\varepsilon}_2^* \not{p}_2\big) \not{k}_1 (-\not{p}_1) \not{\varepsilon}_1 \not{\varepsilon}_1^* \not{k}_1\Big]$$

using
$$\sum_{r_2} e_2^{*i} \not{\varepsilon}_2 = \sum_{r_2} e_2^{*i}(-e_2^j \gamma^j) = -\Big(\delta^{ij} - \frac{k_2^i k_2^j}{|\vec{k}_2|^2}\Big)\gamma^j$$
$$= -\Big(\gamma^i - \frac{k_2^i \vec{\gamma}\cdot\vec{k}_2}{|\vec{k}_2|^2}\Big)$$

$$\Rightarrow \sum_{r_1,r_2} T_1 = \mathrm{Tr}\Big[\Big(+2 k_1^i\Big(\gamma^i - \frac{k_2^i \vec{\gamma}\cdot\vec{k}_2}{|\vec{k}_2|^2}\Big) + 2\not{p}_2\Big) \not{k}_1 (-\not{p}_1)(-2)\not{k}_1\Big]$$

$$= \mathrm{Tr}\Big[\Big(2\vec{k}_1\cdot\gamma - \frac{2(\vec{k}_1\cdot\vec{k}_2)(\vec{\gamma}\cdot\vec{k}_2)}{|\vec{k}_2|^2} + 2\not{p}_2\Big) 2 (-\not{p}_1 \not{k}_1 + 2p_1\cdot k_1)\not{k}_1\Big]$$

$$= 8\,\mathrm{Tr}\Big\{\Big[(\vec{k}_1\cdot\gamma) - \frac{(\vec{k}_1\cdot\vec{k}_2)(\vec{\gamma}\cdot\vec{k}_2)}{|\vec{k}_2|^2} + \not{p}_2\Big] \not{k}_1 (p_1\cdot k_1)\Big\}$$

$$= 8(p_1\cdot k_1)\Big[k_1^i k_{1\mu} 4 g^{i\mu} - \frac{(\vec{k}_1\cdot\vec{k}_2) k_2^i k_{1\mu} 4 g^{i\mu}}{|\vec{k}_2|^2} + 4 p_2\cdot k_1\Big]$$

$$= 32(p_1\cdot k_1)\Big(|\vec{k}_1|^2 - \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_2|^2} + \underbrace{(p_2\cdot k_1)}_{p_1\cdot k_2}\Big)$$

$$\sum_{r_1,r_2} T_2 = \sum_{r_1,r_2}\Big(\mathrm{Tr}[\not{p}_2 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*] + m^2 \mathrm{Tr}[\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*]\Big)$$

$$= \sum_{r_1,r_2} \mathrm{Tr}[\not{p}_2 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*]$$

$$= \sum_{r_1,r_2} \mathrm{Tr}\Big[(2\underbrace{p_2\cdot\varepsilon_1}_{-\vec{p}_2\cdot\vec{e}_1} - \not{\varepsilon}_1 \not{p}_2) \not{k}_2 (2\underbrace{p_1\cdot\varepsilon_2^*}_{0} - \not{p}_1 \not{\varepsilon}_2^*) \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*\Big]$$

using
$$\sum_{r_1} e_1^i \not{\varepsilon}_1^* = \sum_{r_1} e_1^i(-e_1^{*j}\gamma^j) = -\Big(\delta^{ij} - \frac{k_1^i k_1^j}{|\vec{k}_1|^2}\Big)\gamma^j = -\Big(\gamma^i - \frac{k_1^i(\vec{k}_1\cdot\vec{\gamma})}{|\vec{k}_1|^2}\Big)$$

$$\Rightarrow \sum_{r_1,r_2} T_2 = T_r\Big[\Big(+2p_2^i\big(\gamma^i - \frac{k_1^i(\vec{k}_1\cdot\vec{\gamma})}{|\vec{k}_1|^2}\big) + 2\not{p}_2\Big)\underbrace{\not{k}_2(-\not{p}_1)(-2)\not{k}_2}_{2(2p_1\cdot k_2 - \not{k}_2\not{p}_1)\not{k}_2}\Big]$$

$$= 8(p_1\cdot k_2)\, T_r\Big[\Big(\vec{p}_2\cdot\vec{\gamma} - \frac{(\vec{p}_2\cdot\vec{k}_1)(\vec{k}_1\cdot\vec{\gamma})}{|\vec{k}_1|^2} + \not{p}_2\Big)\not{k}_2\Big]$$

$$= 32(p_1\cdot k_2)\Big[p_2^i k_{2\mu} g^{i\mu} - \frac{(\vec{p}_2\cdot\vec{k}_1)}{|\vec{k}_1|^2} k_1^i k_{2\mu} g^{i\mu} + p_2\cdot k_2\Big]$$

$$= 32(p_1\cdot k_2)\Big((\vec{p}_2\cdot\vec{k}_2) - \frac{(\vec{p}_2\cdot\vec{k}_1)(\vec{k}_1\cdot\vec{k}_2)}{|\vec{k}_1|^2} + p_2\cdot k_2\Big]$$

using $\vec{p}_2\cdot\vec{k}_2 = (\vec{p}_1+\vec{k}_1-\vec{k}_2)\cdot\vec{k}_2 = \underset{0}{\underset{\parallel}{\vec{p}_1\cdot\vec{k}_2}} + \vec{k}_1\cdot\vec{k}_2 - |\vec{k}_2|^2$

$\vec{p}_2\cdot\vec{k}_1 = (\vec{p}_1+\vec{k}_1-\vec{k}_2)\cdot\vec{k}_1 = \underset{0}{\underset{\parallel}{\vec{p}_1\cdot\vec{k}_1}} + |\vec{k}_1|^2 - \vec{k}_1\cdot\vec{k}_2$

$$\Rightarrow \sum_{r_1,r_2} T_2 = 32(p_1\cdot k_2)\Big[(\vec{k}_1\cdot\vec{k}_2) - |\vec{k}_2|^2 - \frac{(|\vec{k}_1|^2 - (\vec{k}_1\cdot\vec{k}_2))(\vec{k}_1\cdot\vec{k}_2)}{|\vec{k}_1|^2} + p_2\cdot k_2\Big]$$

$$= 32(p_1\cdot k_2)\Big(-|\vec{k}_2|^2 + \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2} + \underbrace{p_2\cdot k_2}_{p_1\cdot k_1}\Big)$$

$$\sum_{r_1,r_2} T_3 = \sum_{r_1,r_2}\Big(T_r(\not{p}_2\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{p}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*) + m^2 T_r(\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*)\Big)$$

$$= \sum_{r_1,r_2}\Big(T_r\big[(\not{p}_1+\not{k}_1-\not{k}_2)\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{p}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*\big] + m^2 T_r(\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*)\Big)$$

$$= \sum_{r_1,r_2}\Big(T_r\big[\not{p}_1\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{p}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*\big] + 2(k_1\cdot\epsilon_2^*)T_r\big[\not{k}_1\not{\epsilon}_1\not{p}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*\big]$$
$$- 2(k_2\cdot\epsilon_1^*)T_r\big[\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{p}_1\not{\epsilon}_2\not{k}_2\big]$$
$$+ m^2 T_r\big[\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*\big]\Big)$$

where $\sum_{r_1,r_2} T_r\big[\not{p}_1\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{p}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*\big]$

$$= \sum_{r_1,r_2} T_r\big[(-\not{\epsilon}_2^*\not{p}_1)\not{k}_1(-\not{p}_1\not{\epsilon}_1)\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*\big]$$

$$= m^2\Big(\sum_{r_1,r_2}\Big[-T_r\big(\underbrace{\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*}_{\not{\epsilon}_2^*\not{\epsilon}_1\not{k}_1\not{k}_2\not{\epsilon}_2\not{\epsilon}_1^*}\big)\Big]\Big) + \sum_{r_1 r_2} 2(p_1\cdot k_1)T_r\big(\underbrace{\not{\epsilon}_2^*\not{p}_1\not{\epsilon}_1\not{\epsilon}_2\not{k}_2\not{\epsilon}_1^*}_{\not{\epsilon}_2^*\not{\epsilon}_1\not{\epsilon}_2\not{p}_1\not{k}_2\not{\epsilon}_1^*}\big)$$

$$= m^2 \left(\sum_{r_1, r_2} \left[-\mathrm{Tr}\left(\not{\varepsilon}_2 \not{\varepsilon}_1^* \not{\varepsilon}_2^* \not{\varepsilon}_1 \not{k}_1 \not{k}_2 \right) \right] \right) + 2 \sum_{r_1 r_2} \left((p_1 \cdot k_1)\, \mathrm{Tr}\left(\not{\varepsilon}_1^* \not{\varepsilon}_2^* \not{\varepsilon}_1 \not{\varepsilon}_2 \not{p}_1 \not{k}_2 \right) \right)$$

$$= m^2 \left(\sum_{r_1, r_2} \left[-2 (\varepsilon_1^* \cdot \varepsilon_2^*)\, \mathrm{Tr}\left(\not{\varepsilon}_2 \not{\varepsilon}_1 \not{k}_1 \not{k}_2 \right) + \mathrm{Tr}\left(\not{\varepsilon}_2 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{\varepsilon}_1 \not{k}_1 \not{k}_2 \right) \right] \right)$$

$$+ \sum_{r_1, r_2} \left[2 (p_1 \cdot k_1) \left(2 (\varepsilon_1 \cdot \varepsilon_2^*)\, \mathrm{Tr}\left(\not{\varepsilon}_1^* \not{\varepsilon}_2 \not{p}_1 \not{k}_2 \right) - \mathrm{Tr}\left(\not{\varepsilon}_1^* \not{\varepsilon}_1 \not{\varepsilon}_2^* \not{\varepsilon}_2 \not{p}_1 \not{k}_2 \right) \right) \right]$$

$$= m^2 \left\{ -\sum_{r_1 r_2} 8 (\varepsilon_1^* \cdot \varepsilon_2^*) \left[(\varepsilon_1 \cdot \varepsilon_2)(k_1 \cdot k_2) - (k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_1) \right] \right\}$$

$$+ 16 m^2 (k_1 \cdot k_2) - 32 (p_1 \cdot k_1)(p_1 \cdot k_2)$$

$$+ 16 \left\{ \sum_{r_1 r_2} (p_1 \cdot k_1)(\varepsilon_1 \cdot \varepsilon_2^*) \left[(\varepsilon_1^* \cdot \varepsilon_2)(p_1 \cdot k_2) \right] \right\}$$

using $\sum_{r_1 r_2} (\varepsilon_1^* \cdot \varepsilon_2^*)(\varepsilon_1 \cdot \varepsilon_2)$

$$= \sum_{r_1 r_2} e_1^{*i} e_2^{i*} e_1^j e_2^j = \left(\delta^{ij} - \frac{k_1^i k_1^j}{|\vec{k}_1|^2} \right) \left(\delta^{ij} - \frac{k_2^i k_2^j}{|\vec{k}_2|^2} \right)$$

$$= 3 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2} - 1 - 1$$

$$= 1 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}$$

$$\sum_{r_1 r_2} (\varepsilon_1^* \cdot \varepsilon_2^*)(k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_1)$$

$$= -\sum_{r_1 r_2} e_1^{*i} e_2^{*i} k_1^j e_2^j k_2^a e_1^a$$

$$= -\left(\delta^{ia} - \frac{k_1^i k_1^a}{|\vec{k}_1|^2} \right) \left(\delta^{ij} - \frac{k_2^i k_2^j}{|\vec{k}_2|^2} \right) k_1^j k_2^a$$

$$= -\left(\delta^{aj} + \frac{(\vec{k}_1 \cdot \vec{k}_2)\, k_1^a k_2^j}{|\vec{k}_1|^2 |\vec{k}_2|^2} - \frac{k_1^j k_1^a}{|\vec{k}_1|^2} - \frac{k_2^a k_2^j}{|\vec{k}_2|^2} \right) k_1^j k_2^a$$

$$= -\left(\vec{k}_1 \cdot \vec{k}_2 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^3}{|\vec{k}_1|^2 |\vec{k}_2|^2} - \vec{k}_1 \cdot \vec{k}_2 - \vec{k}_1 \cdot \vec{k}_2 \right)$$

$$= (\vec{k}_1 \cdot \vec{k}_2) \left(1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2} \right)$$

$$\sum_{r_1 r_2} (\varepsilon_1 \cdot \varepsilon_2^*)(\varepsilon_1^* \cdot \varepsilon_2)$$

$$= \sum_{r_1 r_2} e_1^i e_2^{i*} e_1^{*a} e_2^a$$

$$= \left(\delta^{ia} - \frac{k_1^i k_1^a}{|\vec{k}_1|^2} \right) \left(\delta^{ia} - \frac{k_2^i k_2^a}{|\vec{k}_2|^2} \right)$$

$$= 3 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2} - 1 - 1 = 1 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}$$

$$\Rightarrow \sum_{r_1, r_2} \mathrm{Tr}\left[\not{p}_1 \not{\varepsilon}_2^{*} \not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^{*}\right]$$

$$= 8m^2 \left\{ -\left(1 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}\right)(k_1 \cdot k_2) + (\vec{k}_1 \cdot \vec{k}_2)\left(1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}\right) \right\}$$

$$+ 16m^2 (k_1 \cdot k_2) - 32 (p_1 \cdot k_1)(p_1 \cdot k_2)$$

$$+ 16 (p_1 \cdot k_1)(p_1 \cdot k_2)\left(1 + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}\right)$$

$$\sum_{r_1, r_2} 2(k_1 \cdot \varepsilon_2^{*}) \mathrm{Tr}(\not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^{*})$$

$$= \sum_{r_1, r_2} 2(k_1 \cdot \varepsilon_2^{*}) \mathrm{Tr}(-\not{\varepsilon}_1 \not{k}_1 \not{p}_1 \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^{*})$$

$$= \sum_{r_2} 4(k_1 \cdot \varepsilon_2^{*}) \mathrm{Tr}(\not{k}_1 \not{p}_1 \not{\varepsilon}_2 \not{k}_2)$$

$$= \sum_{r_2} 16(k_1 \cdot \varepsilon_2^{*})\left[-(k_1 \cdot \varepsilon_2)(p_1 \cdot k_2)\right]$$

$$= -16(p_1 \cdot k_2) \sum_{r_2} k_1^i e_2^{i*} k_1^j e_2^j$$

$$= -16(p_1 \cdot k_2) k_1^i k_1^j \left(\delta^{ij} - \frac{k_2^i k_2^j}{|\vec{k}_2|^2}\right)$$

$$= -16(p_1 \cdot k_2)\left(|\vec{k}_1|^2 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_2|^2}\right)$$

$$= -16(p_1 \cdot k_2)|\vec{k}_1|^2\left(1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}\right)$$

$$\sum_{r_1, r_2} (-2)(k_2 \cdot \varepsilon_1^{*}) \mathrm{Tr}\left[\not{\varepsilon}_2^{*} \not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_2 \not{k}_2\right]$$

$$= \sum_{r_1, r_2} (-2)(k_2 \cdot \varepsilon_1^{*}) \mathrm{Tr}\left[-\not{\varepsilon}_2^{*} \not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{k}_2 \not{\varepsilon}_2\right]$$

$$= \sum_{r_1} (-4)(k_2 \cdot \varepsilon_1^{*}) \mathrm{Tr}\left[\not{k}_1 \not{\varepsilon}_1 \not{p}_1 \not{k}_2\right]$$

$$= 16 \sum_{r_1} (k_2 \cdot \varepsilon_1^{*})(k_1 \cdot p_1)(\varepsilon_1 \cdot k_2)$$

$$= 16(k_1 \cdot p_1) \sum_{r_1} k_2^i e_1^{i*} e_1^j k_2^j$$

$$= 16(k_1 \cdot p_1) k_2^i k_2^j \left(\delta^{ij} - \frac{k_1^i k_1^j}{|\vec{k}_1|^2}\right)$$

$$= 16(p_1 \cdot k_1)|\vec{k}_2|^2\left(1 - \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{|\vec{k}_1|^2 |\vec{k}_2|^2}\right)$$

$$\sum_{r_1 r_2} m^2 \,\mathrm{Tr}[\not{\varepsilon}_2^* \not{k}_1 \not{\varepsilon}_1 \not{\varepsilon}_2 \not{k}_2 \not{\varepsilon}_1^*]$$

$$= m^2 \sum_{r_1 r_2} \mathrm{Tr}[\not{\varepsilon}_2^* \not{\varepsilon}_1 \not{k}_1 \not{k}_2 \not{\varepsilon}_2 \not{\varepsilon}_1^*]$$

$$= 8m^2 \left\{ -\left(1+\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)(k_1\cdot k_2) - (\vec{k}_1\cdot\vec{k}_2)\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right\} - 16m^2(k_1\cdot k_2)$$

$$\Rightarrow \sum_{r_1 r_2} T_3 = -32 (p_1\cdot k_1)(p_1\cdot k_2)$$
$$+16 (p_1\cdot k_1)(p_1\cdot k_2)\left(1+\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$
$$-16 (p_1\cdot k_2)|\vec{k}_1|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$
$$+16 (p_1\cdot k_1)|\vec{k}_2|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$$\sum_{r_1 r_2} T_4 = \sum_{r_1 r_2} \mathrm{Tr}[(\not{p}_1 + \not{k}_1 - \not{k}_2 + m)\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* (\not{p}_1 + m)\not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$
$$= \sum_{r_1 r_2} \Big\{ \mathrm{Tr}[\not{p}_1 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2] + \mathrm{Tr}[\not{k}_1 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$
$$- \mathrm{Tr}[\not{k}_2 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2] + m^2 \mathrm{Tr}[\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]\Big\}$$
$$= \sum_{r_1 r_2} \Big\{ \mathrm{Tr}[-\not{\varepsilon}_1 \not{p}_1 \not{k}_2 \not{p}_1 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2] + 2(k_1\cdot\varepsilon_2)\,\mathrm{Tr}[\not{k}_1 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^*]$$
$$- 2(k_2\cdot\varepsilon_1)\,\mathrm{Tr}[\not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2] + m^2 \mathrm{Tr}[\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]\Big\}$$

where $\sum_{r_1 r_2} \mathrm{Tr}[\not{\varepsilon}_1 \not{p}_1 \not{k}_2 \not{p}_1 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$

$$= \sum_{r_1 r_2}\left(-m^2 \mathrm{Tr}[\not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2] + 2(p_1\cdot k_2)\,\mathrm{Tr}[\not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]\right)$$

$$\Rightarrow \sum_{r_1 r_2} T_4 = 2(p_1\cdot k_2)\sum_{r_1 r_2} \mathrm{Tr}(\not{\varepsilon}_1 \not{p}_1 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$
$$+ 2\sum_{r_1 r_2} (k_1\cdot\varepsilon_2)\,\mathrm{Tr}[\not{k}_1 \not{\varepsilon}_1 \not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^*]$$
$$- 2\sum_{r_1 r_2} (k_2\cdot\varepsilon_1)\,\mathrm{Tr}[\not{k}_2 \not{\varepsilon}_2^* \not{p}_1 \not{\varepsilon}_1^* \not{k}_1 \not{\varepsilon}_2]$$
$$= 2(p_1\cdot k_2)\sum_{r_1 r_2} \mathrm{Tr}(\not{\varepsilon}_2 \not{\varepsilon}_1 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{p}_1 \not{k}_1)$$
$$+ 2\sum_{r_1 r_2} (k_1\cdot\varepsilon_2)\,\mathrm{Tr}(\not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{\varepsilon}_1 \not{k}_1 \not{k}_2 \not{p}_1)$$
$$- 2\sum_{r_1 r_2} (k_2\cdot\varepsilon_1)\,\mathrm{Tr}(\not{\varepsilon}_2 \not{\varepsilon}_2^* \not{\varepsilon}_1^* \not{p}_1 \not{k}_1 \not{k}_2)$$

$$= 2(p_1\cdot k_2)\sum_{r_1 r_2}\left[2(\varepsilon_1\cdot\varepsilon_2^*)\,\mathrm{Tr}(\not\varepsilon_2\not\varepsilon_1^*\not p_1\not k_1) - \mathrm{Tr}(\not\varepsilon_2\not\varepsilon_2^*\not\varepsilon_1\not\varepsilon_1^*\not p_1\not k_1)\right]$$

$$-4\sum_{r_2}(k_1\cdot\varepsilon_2)\,\mathrm{Tr}(\not\varepsilon_2^*\not k_1\not k_2\not p_1)$$

$$+4\sum_{r_1}(k_2\cdot\varepsilon_1)\,\mathrm{Tr}(\not\varepsilon_1^*\not p_1\not k_1\not k_2)$$

$$= 2(p_1\cdot k_2)\left(\sum_{r_1 r_2} 8(\varepsilon_1\cdot\varepsilon_2^*)(\varepsilon_2\cdot\varepsilon_1^*)(p_1\cdot k_1)\right) - 32(p_1\cdot k_2)(p_1\cdot k_1)$$

$$-16\sum_{r_2}(k_1\cdot\varepsilon_2)(\varepsilon_2^*\cdot k_1)(k_2\cdot p_1)$$

$$+16\sum_{r_1}(k_2\cdot\varepsilon_1)(\varepsilon_1^*\cdot k_2)(p_1\cdot k_1)$$

using

$$\sum_{r_1 r_2}(\varepsilon_1\cdot\varepsilon_2^*)(\varepsilon_2\cdot\varepsilon_1^*)$$

$$= \sum_{r_1 r_2}(e_1^i e_2^{i*})(e_2^j e_1^{j*})$$

$$= \left(\delta^{ij} - \frac{k_1^i k_1^j}{|\vec{k}_1|^2}\right)\left(\delta^{ij} - \frac{k_2^i k_2^j}{|\vec{k}_2|^2}\right)$$

$$= 3 + \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2} - 2$$

$$= 1 + \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}$$

$$\sum_{r_2}(k_1\cdot\varepsilon_2)(\varepsilon_2^*\cdot k_1)$$

$$= \sum_{r_2} k_1^i e_2^i e_2^{j*} k_1^j$$

$$= \left(\delta^{ij} - \frac{k_2^i k_2^j}{|\vec{k}_2|^2}\right) k_1^i k_1^j$$

$$= |\vec{k}_1|^2 - \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_2|^2}$$

$$= |\vec{k}_1|^2\left(1 - \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$$\sum_{r_1}(k_2\cdot\varepsilon_1)(\varepsilon_1^*\cdot k_2)$$

$$= \sum_{r_2} k_2^i e_1^i e_1^{*j} k_2^j = \left(\delta^{ij} - \frac{k_1^i k_1^j}{|\vec{k}_1|^2}\right) k_2^i k_2^j = |\vec{k}_2|^2 - \frac{(\vec{k}_1\cdot\vec{k}_2)}{|\vec{k}_1|^2}$$

$$= |\vec{k}_2|^2\left(1 - \frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$$\Rightarrow \sum_{r_1 r_2} T_4 = 16 (p_1\cdot k_2)(p_1\cdot k_1)\left(1+\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$
$$-32(p_1\cdot k_2)(p_1\cdot k_1)$$
$$-16(p_1\cdot k_2)|\vec{k}_1|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$
$$+16(p_1\cdot k_1)|\vec{k}_2|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$$\Rightarrow \sum_{r_1 r_2} T_3 = \sum_{r_1 r_2} T_4$$

$$\Rightarrow \frac{1}{4}\sum_{\substack{s_1,s_2\\ r_1,r_2}}|M|^2 = \frac{e^4}{4}\left\{\frac{1}{4(p_1\cdot k_1)^2}32\left[(p_1\cdot k_1)(p_1\cdot k_2)+(p_1\cdot k_1)|\vec{k}_1|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right]\right.$$
$$+\frac{1}{4(p_1\cdot k_2)^2}32\left[(p_1\cdot k_1)(p_1\cdot k_2)-(p_1\cdot k_2)|\vec{k}_2|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right]$$
$$+\frac{1}{4(p_1\cdot k_1)(p_1\cdot k_2)}\left[32(p_1\cdot k_1)(p_1\cdot k_2)\left(1+\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right.$$
$$-64(p_1\cdot k_1)(p_1\cdot k_2)-32(p_1\cdot k_2)|\vec{k}_1|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$
$$\left.\left.+32(p_1\cdot k_1)|\vec{k}_2|^2\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right]\right\}$$

$$=2e^4\left\{\frac{p_1\cdot k_2}{p_1\cdot k_1}+\frac{|\vec{k}_1|^2}{p_1\cdot k_1}\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)+\frac{p_1\cdot k_1}{p_1\cdot k_2}-\frac{|\vec{k}_2|^2}{p_1\cdot k_2}\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right.$$
$$+\left(1+\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)-2-\frac{|\vec{k}_1|^2}{p_1\cdot k_1}\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$
$$\left.+\frac{|\vec{k}_2|^2}{p_1\cdot k_2}\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right\}$$

$$=2e^4\left\{\frac{p_1\cdot k_1}{p_1\cdot k_2}+\frac{p_1\cdot k_2}{p_1\cdot k_1}-\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right\}$$

In the rest frame of the initial electron

$$2m^2\left(\frac{1}{p_1\cdot k_1}-\frac{1}{p_1\cdot k_2}\right)+m^4\left(\frac{1}{p_1\cdot k_1}-\frac{1}{p_1\cdot k_2}\right)^2$$

$$=2m^2\left(\frac{1}{m|\vec{k}_1|}-\frac{1}{m|\vec{k}_2|}\right)+m^4\left(\frac{1}{m|\vec{k}_1|}-\frac{1}{m|\vec{k}_2|}\right)^2$$

$$=2m\frac{|\vec{k}_2|-|\vec{k}_1|}{|\vec{k}_1||\vec{k}_2|}+m^2\left(\frac{|\vec{k}_2|-|\vec{k}_1|}{|\vec{k}_1||\vec{k}_2|}\right)^2\equiv **$$

while $-\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$

$$=-\left(1-\frac{\left(-(k_1\cdot k_2-k_1^0k_2^0)\right)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$$=-\left(1-\frac{(k_1\cdot k_2-|\vec{k}_1||\vec{k}_2|)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$k_1\cdot k_2=-\frac{(k_1-k_2)^2}{2}=-\frac{(p_2-p_1)^2}{2}=p_1\cdot p_2-m^2=p_1\cdot(p_1+k_1-k_2)-m^2=m(|\vec{k}_1|-|\vec{k}_2|)$

$$=-\left(1-\frac{\left[m(|\vec{k}_1|-|\vec{k}_2|)-|\vec{k}_1||\vec{k}_2|\right]^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)$$

$$=-1+\frac{1}{|\vec{k}_1|^2|\vec{k}_2|^2}\left[m^2(|\vec{k}_1|-|\vec{k}_2|)^2+|\vec{k}_1|^2|\vec{k}_2|^2-2|\vec{k}_1||\vec{k}_2|m(|\vec{k}_1|-|\vec{k}_2|)\right]$$

$$=2m\frac{|\vec{k}_2|-|\vec{k}_1|}{|\vec{k}_1||\vec{k}_2|}+m^2\left(\frac{|\vec{k}_2|-|\vec{k}_1|}{|\vec{k}_1||\vec{k}_2|}\right)^2=**$$

Since $|M|^2$ is Lorentz invariant, then $\frac{1}{4}\sum_{\substack{s_1,s_2\\r_1,r_2}}|M|^2$ is Lorentz invariant, then we can write it in terms of Lorentz invariant quantities.

$$\Rightarrow \frac{1}{4}\sum_{\substack{s_1,s_2\\r_1,r_2}}|M|^2=2e^4\left[\frac{p_1\cdot k_1}{p_1\cdot k_2}+\frac{p_1\cdot k_2}{p_1\cdot k_1}+2m^2\left(\frac{1}{p_1\cdot k_1}-\frac{1}{p_1\cdot k_2}\right)+m^4\left(\frac{1}{p_1\cdot k_1}-\frac{1}{p_1\cdot k_2}\right)^2\right]$$

In the rest frame of the initial electron,

$$\frac{1}{4}\sum_{\substack{r_1,r_2\\ s_1,s_2}}|M|^2 \equiv |\overline{M}|^2 = 2e^4\left\{\frac{|\vec{k}_1|}{|\vec{k}_2|}+\frac{|\vec{k}_2|}{|\vec{k}_1|}-\left(1-\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{|\vec{k}_1|^2|\vec{k}_2|^2}\right)\right\}$$

let $|\vec{k}_1| \equiv \omega$, $|\vec{k}_2| \equiv \omega'$

$\Rightarrow \vec{k}_1\cdot\vec{k}_2 = \omega\omega'\cos\theta_{lab}$, where θ_{lab} is the angle between the incoming and outgoing photons.

$$\Rightarrow |\overline{M}|^2 = 2e^4\left\{\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\left(1-\frac{\omega^2\omega'^2\cos^2\theta_{lab}}{\omega^2\omega'^2}\right)\right\}$$

$$= 2e^4\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta_{lab}\right]$$

$$\Rightarrow \left(\frac{d\overline{\sigma}}{d\Omega}\right)_{lab} = \frac{|\overline{M}|^2}{64\pi^2 m}\,\frac{\omega'}{\omega}\,\frac{1}{\left[(\omega+m)-\frac{\omega}{\omega'}\omega'\cos\theta_{lab}\right]}$$

$$= \frac{|\overline{M}|^2}{64\pi^2 m^2}\,\frac{\omega'}{\omega}\,\frac{1}{1+\frac{\omega}{m}(1-\cos\theta_{lab})}$$

using $k_1\cdot k_2 = P_1\cdot P_2 - m^2 = P_1\cdot(P_1+k_1-k_2)-m^2 = m(|\vec{k}_1|-|\vec{k}_2|) = m(\omega-\omega')$

and $k_1\cdot k_2 = \omega\omega' - \omega\omega'\cos\theta_{lab}$

$$\Rightarrow \omega\omega'(1-\cos\theta_{lab}) = m(\omega-\omega')$$

$$\Rightarrow \frac{\omega}{m}(1-\cos\theta_{lab}) = \frac{\omega}{\omega'}-1$$

$$\Rightarrow 1+\frac{\omega}{m}(1-\cos\theta_{lab}) = \frac{\omega}{\omega'}$$

$$\Rightarrow \left(\frac{d\overline{\sigma}}{d\Omega}\right)_{lab} = \frac{|\overline{M}|^2}{64\pi^2 m^2}\left(\frac{\omega'}{\omega}\right)^2 \overset{e^2=4\pi\alpha}{=} \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta_{lab}\right]$$

$$\Rightarrow \boxed{\frac{d\overline{\sigma}_{lab}}{d\cos\theta_{lab}} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta_{lab}\right]}$$

, this is the Klein-Nishina formula.

At low energies, as $\omega \longrightarrow 0$,

$$\Rightarrow \quad \frac{\omega}{\omega'} - 1 = \frac{\omega}{m}(1 - \cos\theta_{lab}) \longrightarrow 0.$$

$$\Rightarrow \quad \frac{\omega}{\omega'} \rightarrow 1$$

$$\Rightarrow \frac{d\bar{\sigma}_{lab}}{d\cos\theta_{lab}} \longrightarrow \frac{\pi\alpha^2}{m^2}\left[2 - \sin^2\theta_{lab}\right] = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta_{lab})$$

$$\Rightarrow \quad \bar{\sigma}_{lab} \longrightarrow \int_{-1}^{1} d(\cos\theta_{lab}) \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta_{lab})$$

$$= \frac{\pi\alpha^2}{m^2}\left(2 + \frac{1}{3}\times 2\right)$$

$$= \left(\frac{8\pi}{3}\right)\frac{\alpha^2}{m^2}$$

This is the Thomson cross section formula.

If we use Feynman rules directly.

$$i\mathcal{M}_{fi} = \bar{u}(\vec{p}_2, s_2)\left(-i|e|(-1)\right)\gamma^\nu \frac{i(\not{p}_1 + \not{k}_1 + m)}{(p_1+k_1)^2 - m^2 + i\varepsilon}\left(-i|e|(-1)\right)\gamma^\mu$$
$$\cdot u(\vec{p}_1, s_1)\,\varepsilon_\mu(\vec{k}_1, r_1)\,\varepsilon^*_\nu(\vec{k}_2, r_2)$$
$$+ \bar{u}(\vec{p}_2, s_2)\left(-i|e|(-1)\right)\gamma^\mu \frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1-k_2)^2 - m^2 + i\varepsilon}\left(-i|e|(-1)\right)\gamma^\nu$$
$$\cdot u(p_1 \cdot s_1)\,\varepsilon_\mu(\vec{k}_1, r_1)\,\varepsilon^*_\nu(\vec{k}_2, r_2)$$

$$= -ie^2\,\bar{u}(\vec{p}_2, s_2)\left[\frac{\not{\varepsilon}_2^*(\not{p}_2 + \not{k}_2 + m)\not{\varepsilon}_1}{(p_2+k_2)^2 - m^2 + i\varepsilon} + \frac{\not{\varepsilon}_1(\not{p}_2 - \not{k}_1 + m)\not{\varepsilon}_2^*}{(p_2-k_1)^2 - m^2 + i\varepsilon}\right]u(\vec{p}_1, s_1)$$

$$= -ie^2\,\bar{u}(\vec{p}_2, s_2)\left[\frac{\gamma^\mu(\not{p}_1 + \not{k}_1 + m)\gamma^\nu}{2p_1\cdot k_1} + \frac{\gamma^\nu(\not{p}_1 - \not{k}_2 + m)\gamma^\mu}{-2p_1\cdot k_2}\right]u(\vec{p}_1 \cdot s_1)$$
$$\cdot\,\varepsilon^*_\mu(\vec{k}_2, r_2)\,\varepsilon_\nu(\vec{k}_1, r_1)$$

$$= -ie^2\, \bar{u}(\vec{p}_2, s_2)\left[\frac{\gamma^\mu(2p_1^\nu + \not{k}_1\gamma^\nu)}{2p_1\cdot k_1} - \frac{\gamma^\nu(2p_1^\mu - \not{k}_2\gamma^\mu)}{2p_1\cdot k_2}\right] u(\vec{p}_1, s_1)$$

$$\cdot\, \varepsilon^*_\mu(\vec{k}_2, r_2)\, \varepsilon_\nu(\vec{k}_1, r_1)$$

$\uparrow$

$(\not{p}_1 - m)\, u(\vec{p}_1, s_1) = 0$

We can use $\sum_\lambda \varepsilon^*_\mu(\vec{k},\lambda)\,\varepsilon_\nu(\vec{k},\lambda) \longrightarrow -g_{\mu\nu}$ for photon polarization sum.

$$\Rightarrow\ \overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\substack{s_1, s_2\\ r_1, r_2}} |\mathcal{M}|^2$$

$$= \frac{e^4}{4}\,\mathrm{Tr}\left[(\not{p}_2+m)\left(\frac{2\gamma^\mu p_1^\nu + \gamma^\mu\not{k}_1\gamma^\nu}{2p_1\cdot k_1} - \frac{2\gamma^\nu p_1^\mu - \gamma^\nu\not{k}_2\gamma^\mu}{2p_1\cdot k_2}\right)\right.$$

$$\left.\cdot\,(\not{p}_1+m)\left(\frac{2\gamma^\alpha p_1^\beta + \gamma^\beta\not{k}_1\gamma^\alpha}{2p_1\cdot k_1} - \frac{2\gamma^\beta p_1^\alpha - \gamma^\alpha\not{k}_2\gamma^\beta}{2p_1\cdot k_2}\right)\right]$$

$$(-g_{\mu\alpha})(-g_{\nu\beta})$$

$$= \frac{e^4}{4}\,\mathrm{Tr}\left[(\not{p}_2+m)\left(\frac{2\gamma^\mu p_1^\nu + \gamma^\mu\not{k}_1\gamma^\nu}{2p_1\cdot k_1} - \frac{2\gamma^\nu p_1^\mu - \gamma^\nu\not{k}_2\gamma^\mu}{2p_1\cdot k_2}\right)\right.$$

$$\left.\cdot\,(\not{p}_1+m)\left(\frac{2\gamma_\mu p_{1\nu} + \gamma_\nu\not{k}_1\gamma_\mu}{2p_1\cdot k_1} - \frac{2\gamma_\nu p_{1\mu} - \gamma_\mu\not{k}_2\gamma_\nu}{2p_1\cdot k_2}\right)\right]$$

The same as the starting point of Homework 4