

复习) Lorentz transformations.

① $x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha$, where a^α and Λ^α_β are constants

Λ^α_β satisfy $\Lambda^\alpha_\gamma \Lambda^\beta_\delta \gamma_{\alpha\beta} = \delta_{\gamma\delta}$, $\gamma_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, note that $\Lambda^\alpha_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}$

② pure Lorentz boost, pure rotation, translations

$$3 \quad + \quad 3 \quad + \quad 4 \quad = \quad 10$$

e.g. $x'^\alpha = \Lambda^\alpha_\beta x^\beta$

where

$$\Lambda^\alpha_\beta = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta$$

where

$$\Lambda^\alpha_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x'^\alpha = x^\alpha + a^\alpha$$

where

$$a^\alpha = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}$$

③

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \vec{\nabla}) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i} \right)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_0, -\vec{\nabla})$$

d'Alembert $\square \equiv \partial^2 \equiv \partial_\mu \partial^\mu = \left(\frac{\partial}{\partial x^0} \right)^2 - \vec{\nabla}^2$

④

Four-momentum $P^\mu = (P^0, \vec{P}) = (E, \vec{p})$, $P_\mu = (E, -\vec{p})$

$$P^2 \equiv P_\mu P^\mu = P^\mu P_\mu = E^2 - |\vec{p}|^2 = (P^0)^2 - p_i p_i = E^2 - \vec{p} \cdot \vec{p} = P \cdot P = g_{\mu\nu} P^\mu P^\nu$$

In general, four-vector inner product

$$a \cdot b = a^\mu b_\mu = g^{\mu\nu} a_\mu b_\nu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \vec{a} \cdot \vec{b}$$

⑤ Consider two observers O and O' moving in two different inertial reference frames related by a Lorentz transformation. If observer O describes a function $\varphi(x) = \varphi(t, \vec{x})$ using the coordinates of her own frame, then observer O' will describe the same field by another function $\varphi'(x') = \varphi'(t', \vec{x}')$ in terms of the transformed coordinates $x'^\mu = \Lambda^\mu_\nu x^\nu$.

Then how are $\phi(x)$ and $\phi'(x')$ related?

Statement: a theory consistent with relativistic principles can only contain fields that have well-defined transformation properties. They include

(a) scalar fields,

$$\phi'(x') = \phi(x) \quad (\text{e.g., } \pi, K, \text{ Higgs}, \text{ (spin 0)})$$

(b) vector fields,

$$A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x) \quad (\text{e.g., photon, } W, Z, \text{ (spin 1)})$$

(c) tensor fields,

$$F'^{\mu\nu}(x') = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}(x) \quad (\text{e.g., gravitational field (spin 2)})$$

(d) Other fields, e.g., spinor field, describing spin $\frac{1}{2}$ particles (e.g. e^\pm, p)

Now we study scalar fields first.

Why we talk about Lorentz transformations in the previous classes?

① 引出标量、矢量、张量 这些概念.

② 熟悉四矢量的计算 (内积)

③ 熟悉四维时空的概念、度规、上下标变换.

The ultimate goal in this course is to calculate tree-level diagram and get cross section & decay rate. So, we just need to introduce the elements we need for this goal.

Space-time translation of a scalar field

$$x^\mu \rightarrow x'^\mu = x^\mu - \underbrace{a^\mu}_{\text{constant displacement parameter}}$$

↪ constant displacement parameter

Consider an infinitesimal transformation

$$x^\mu \rightarrow x'^\mu = x^\mu - \underbrace{\delta a^\mu}_{\text{an infinitesimal constant}}$$

↪ an infinitesimal constant

The total variation of the field is defined as

$$\delta\phi(x) \equiv \phi'(x') - \phi(x)$$

Since the field has the same value at the same spacetime point, which is labeled as x in one coordinate and x' in the other, and the field variation $\delta\phi(x) = 0$

Refine $\delta_0\phi(x) \equiv \phi'(x) - \phi(x)$, which is the variation of the field alone, keeping the argument fixed.

↪ called the intrinsic variation of the field.

$$\phi'(x - \delta a^\mu) = \phi(x)$$

Substitute $x^\mu - \delta a^\mu \rightarrow x^\mu$, then $x^\mu \rightarrow x^\mu + \delta a^\mu$

$$\Rightarrow \phi'(x) = \phi(x + \delta a^\mu) = (1 + \delta a^\mu \partial_\mu) \phi(x)$$

$$\Rightarrow \delta_0\phi(x) = \delta a^\mu \partial_\mu \phi(x) = i \delta a^\mu (-i \partial_\mu \phi(x))$$

A finite transformation is obtained by repeating the process

$$\phi(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{\delta a^\mu}{n} \partial_\mu\right)^n \phi(x) = \exp(\delta a^\mu \partial_\mu) \phi(x) \equiv U(a) \phi(x)$$

- $i \partial_\mu$ is called
the generator of infinitesimal translations.

For two successive translations $U(a)$ and $U(b)$, we have

$$U(a) U(b) \phi(x) = U(b) U(a) \phi(x) \Rightarrow [U(a), U(b)] = 0$$

$$\exp((a^\mu + b^\mu) \partial_\mu) \phi(x) \quad \exp((b^\mu + a^\mu) \partial_\mu) \phi(x)$$

That is to say, translations do not depend on the order in which they are applied.

Lorentz transformation of a scalar field
i.e., boost and rotation

Consider infinitesimal Lorentz transformation,

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \approx (\delta^\mu_\nu + \varepsilon^\mu_\nu) x^\nu$$

where $\varepsilon_{\mu\nu} \equiv g_{\mu\nu} \varepsilon^\lambda_\nu$ are infinitesimal constants. ($4 \times 4 = 16$)
 \uparrow
definition of $\varepsilon_{\mu\nu}$.

Recall Lorentz transformations satisfy $\Lambda^\alpha_\beta \Lambda^\beta_\gamma g_{\alpha\beta} = g_{\alpha\gamma}$

$$\Rightarrow (\delta^\alpha_\beta + \varepsilon^\alpha_\beta)(\delta^\beta_\gamma + \varepsilon^\beta_\gamma) g_{\alpha\beta} \approx g_{\alpha\gamma}$$

up to first order of ε

$$\Rightarrow \delta^\alpha_\beta \delta^\beta_\gamma g_{\alpha\gamma} + \varepsilon^\alpha_\beta \delta^\beta_\gamma g_{\alpha\beta} + \delta^\alpha_\beta \varepsilon^\beta_\gamma g_{\alpha\beta} = g_{\alpha\gamma}$$

$$\Rightarrow \varepsilon_{\alpha\gamma} + \varepsilon_{\gamma\alpha} = 0$$

That is, $\varepsilon_{\mu\nu}$ is anti-symmetric.

Recall that the infinitesimal Lorentz boost (\hat{x}, \hat{s}_V) is

$$\Lambda^\alpha_\beta = \begin{pmatrix} 1 & -v\gamma & 0 & 0 \\ -v\gamma & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{\uparrow}{\approx} \begin{pmatrix} 1 & -v & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for small v (call it SU), then

$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & -v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \delta^\alpha_\beta + \varepsilon^\alpha_\beta$$

$$\Rightarrow \varepsilon_1^0 = -sv = \varepsilon'_0 \Rightarrow \varepsilon_{01} = g_{01} \varepsilon_1^0 = g_{00} \varepsilon_1^0 = -sv,$$

$$\varepsilon_{10} = g_{10} \varepsilon_1^0 = g_{11} \varepsilon'_0 = sv$$

Note that if we rapidity w , $\Lambda^\alpha_\beta = \begin{pmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where $\tanh w = v$
 $\cosh w = \gamma$

$$\text{then } \varepsilon_{01} = -sw, \quad \varepsilon_{10} = sw$$

For infinitesimal rotation $(\hat{\theta}, \delta\theta)$

$$\Lambda^\alpha_\beta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\theta & 0 \\ 0 & -\delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon'_2 = \delta\theta, \quad \varepsilon^2_1 = -\delta\theta$$

$$\Rightarrow \varepsilon_{12} = g_{12} \varepsilon'_2 = -\delta\theta, \quad \varepsilon_{21} = g_{21} \varepsilon^2_1 = \delta\theta$$

For scalar field,

$$\phi'(x') = \phi(x) = \phi(\Lambda^{-1}x')$$

Substitute $x' \rightarrow x$, then

$$\begin{aligned} \phi'(x) &= \phi(\Lambda^{-1}x) = \phi((\delta^\mu_\nu - \varepsilon^\mu_\nu)x^\nu) = \phi(x^\mu - \varepsilon^\mu_\nu x^\nu) \\ &\quad \text{check } (\delta^\mu_\lambda - \varepsilon^\mu_\lambda)(\delta^\lambda_\nu + \varepsilon^\lambda_\nu) \\ &= \delta^\mu_\nu - \varepsilon^\mu_\nu + \varepsilon^\mu_\nu + O(\varepsilon^2) \\ &= \delta^\mu_\nu + O(\varepsilon^2) \quad \checkmark \\ &\approx \phi(x) - \varepsilon^\mu_\nu x^\nu \partial_\mu \phi(x) \end{aligned}$$

The intrinsic variation of the field is then

$$\delta_0 \phi = \phi'(x) - \phi(x) = -\varepsilon^\mu_\nu x^\nu \partial_\mu \phi(x) = \frac{1}{2} \varepsilon^{\mu\nu} (x_\mu \partial_\nu - x_\nu \partial_\mu) \phi(x)$$

$$\text{where } \varepsilon^{\mu\nu} \equiv \varepsilon^\mu_\lambda g^{\lambda\nu} = \varepsilon_{\alpha\lambda} g^{\lambda\nu} g^{\mu\alpha} = -\frac{i}{2} \varepsilon^{\mu\nu} L_{\mu\nu} \phi(x)$$

definition of $\varepsilon^{\mu\nu}$

$L_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu)$, is called the generator of infinitesimal transformations.
 $L_{\mu\nu}$ is antisymmetric.

It can be shown that $[L_{\mu\nu}, L_{\rho\sigma}] = -ig_{\mu\rho} L_{\nu\sigma} - ig_{\nu\sigma} L_{\mu\rho} + ig_{\mu\sigma} L_{\nu\rho} + ig_{\nu\rho} L_{\mu\sigma}$

$$\text{proof: } [L_{\mu\nu}, L_{\rho\sigma}] = -[(x_\mu \partial_\nu - x_\nu \partial_\mu)(x_\rho \partial_\sigma - x_\sigma \partial_\rho) - (x_\rho \partial_\sigma - x_\sigma \partial_\rho)(x_\mu \partial_\nu - x_\nu \partial_\mu)]$$

$$= [x_\mu x_\rho \partial_\nu \partial_\sigma + x_\mu g_{\rho\mu} \partial_\nu \partial_\sigma - x_\rho x_\nu \partial_\mu \partial_\sigma - x_\rho g_{\nu\mu} \partial_\mu \partial_\sigma]$$

$$\begin{aligned} &- x_\sigma x_\nu \partial_\mu \partial_\rho - x_\sigma g_{\nu\mu} \partial_\mu \partial_\rho + x_\sigma x_\mu \partial_\nu \partial_\rho + x_\sigma g_{\mu\nu} \partial_\nu \partial_\rho \\ &- x_\mu x_\sigma \partial_\nu \partial_\rho - x_\mu g_{\nu\mu} \partial_\nu \partial_\rho + x_\mu x_\rho \partial_\nu \partial_\sigma + x_\mu g_{\nu\rho} \partial_\nu \partial_\sigma \\ &+ x_\nu x_\sigma \partial_\mu \partial_\rho + x_\nu g_{\mu\sigma} \partial_\mu \partial_\rho - x_\nu x_\rho \partial_\mu \partial_\sigma - x_\nu g_{\mu\rho} \partial_\mu \partial_\sigma \end{aligned}$$

Note that

$$\partial_\mu x_\nu = \partial_\mu g_{\nu\lambda} x^\lambda$$

$$= g_{\nu\lambda} \delta^\lambda_\mu = g_{\nu\mu}$$

$$= g_{\mu\nu}$$

$$= -ig_{ue} L_{v0} - ig_{ve} L_{u0} + ig_{vu} L_{u0} + ig_{vu} L_{v0}$$

While for the generator of infinitesimal translation, $-i\partial_\mu$,
 $[-i\partial_\mu, -i\partial_\nu] = 0$.

That is, the order of successive applications of two arbitrary Lorentz transformations matters, while the order of two arbitrary Lorentz boosts and rotations translations does not matter.

In other words, the Lorentz translation group is Abelian,
the Lorentz transformation group is non-Abelian
boost and rotation

Space rotations are generated by the operators

$$\begin{aligned} L^1 &\equiv L_{23} = -i\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right), & \text{i.e., } L^i &= \frac{1}{2}\epsilon^{ijk}L_{jk} \\ L^2 &\equiv L_{31} = -i\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right), \\ L^3 &\equiv L_{12} = -i\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right). \end{aligned}$$

They satisfy $[L^i, L^j] = i\epsilon^{ijk}L^k$.

where ϵ^{ijk} is the completely antisymmetric Levi-Civita tensor,
such that $\epsilon^{123} = +1$.

$$\begin{aligned} \text{check: } [L^1, L^2] &= [L_{23}, L_{31}] = -\left\{ \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \right. \\ &\quad \left. - \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \right\} \\ &= \left[\cancel{y \frac{\partial}{\partial z}} - \cancel{z \frac{\partial}{\partial y}} - \cancel{xz \frac{\partial}{\partial z}} + \cancel{xy \frac{\partial}{\partial y}} \right] \\ &\quad - \left[\cancel{y \frac{\partial}{\partial x}} + \cancel{zx \frac{\partial}{\partial x}} - \cancel{yx \frac{\partial}{\partial z}} + \cancel{zx \frac{\partial}{\partial y}} \right] \\ &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} = i\epsilon^{123}L^3 \end{aligned}$$

$$\begin{aligned}
 [L^3, L^3] &= [L_{31}, L_{12}] = - \left\{ \left(\frac{\partial}{\partial x} - \frac{x \partial}{\partial z} \right) \left(\frac{x \partial}{\partial y} - \frac{y \partial}{\partial x} \right) - \left(\frac{x \partial}{\partial y} - \frac{y \partial}{\partial x} \right) \left(\frac{\partial}{\partial z} - \frac{x \partial}{\partial y} \right) \right\} \\
 &= \left[\cancel{xy \frac{\partial}{\partial y}} - \cancel{x^2 \frac{\partial}{\partial z}} \quad - \cancel{yz \frac{\partial}{\partial x}} + \cancel{ye \frac{\partial}{\partial z}} + \cancel{yx \frac{\partial}{\partial x}} \right] \\
 &\quad - \left[\cancel{z \frac{\partial}{\partial y}} + \cancel{zx \frac{\partial}{\partial z}} - \cancel{zy \frac{\partial}{\partial x}} - \cancel{x^2 \frac{\partial}{\partial y}} + \cancel{xy \frac{\partial}{\partial z}} \right] \\
 &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} = i \epsilon^{231} L^1
 \end{aligned}$$

$$\begin{aligned}
 [L^3, L^1] &= [L_{12}, L_{23}] = - \left\{ \left(\frac{x \partial}{\partial y} - \frac{y \partial}{\partial x} \right) \left(\frac{y \partial}{\partial z} - \frac{z \partial}{\partial y} \right) - \left(\frac{y \partial}{\partial z} - \frac{z \partial}{\partial y} \right) \left(\frac{x \partial}{\partial y} - \frac{y \partial}{\partial x} \right) \right\} \\
 &= \left[\cancel{yx \frac{\partial}{\partial z}} - \cancel{y^2 \frac{\partial}{\partial x}} - \cancel{zx \frac{\partial}{\partial y}} + \cancel{ze \frac{\partial}{\partial z}} + \cancel{zy \frac{\partial}{\partial x}} \right] \\
 &\quad - \left[\cancel{xy \frac{\partial}{\partial z}} + \cancel{x^2 \frac{\partial}{\partial y}} - \cancel{yz \frac{\partial}{\partial x}} + \cancel{ye \frac{\partial}{\partial z}} + \cancel{yx \frac{\partial}{\partial x}} \right] \\
 &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} = i \epsilon^{312} L^2
 \end{aligned}$$

other: $[L^2, L^1] = -[L^1, L^2] = -i \epsilon^{123} L^3 = i \epsilon^{213} L^3$

$$[L^3, L^2] = -[L^2, L^3] = -i \epsilon^{231} L^1 = i \epsilon^{321} L^1$$

$$[L^1, L^3] = -[L^3, L^1] = -i \epsilon^{312} L^2 = i \epsilon^{132} L^2$$

when $i = j$, $[L^i, L^i] = 0$, and $\epsilon^{ijk} = 0$.

Done the check.

Lorentz transformation of a vector field

boost and rotation.

$$A'^\alpha(x') = \Lambda^\alpha_\beta A^\beta(x) = \Lambda^\alpha_\beta A^\beta(\Lambda^{-1}x')$$

For an infinitesimal Lorentz transformation, $\Lambda^\alpha_\beta = \delta^\alpha_\beta + \varepsilon^\alpha_\beta$,
Substitute $x' \rightarrow x$, then

$$\begin{aligned} A'^\alpha(x) &= \Lambda^\alpha_\beta A^\beta(\Lambda^{-1}x) \simeq (\delta^\alpha_\beta + \varepsilon^\alpha_\beta)[A^\beta(x) - \varepsilon^{\mu\nu} x^\nu \partial_\mu A^\beta(x)] \\ &= [\delta^\alpha_\beta - \frac{i}{2} \varepsilon^{\mu\nu} (\sum_{\mu\nu})^\alpha_\beta] [A^\beta(x) - \frac{i}{2} \varepsilon^{\mu\nu} L_{\mu\nu} A^\beta(x)] \end{aligned}$$

where $(\sum_{\mu\nu})^\alpha_\beta \equiv i(\delta^\alpha_\mu g_{\nu\rho} - \delta^\alpha_\nu g_{\mu\rho})$ ↳ defined before
in scalar field
transformation.

$$\begin{aligned} (\text{check: } -\frac{i}{2} \varepsilon^{\mu\nu} (\sum_{\mu\nu})^\alpha_\beta) &= -\frac{i}{2} \varepsilon^{\mu\nu} i(\delta^\alpha_\mu g_{\nu\rho} - \delta^\alpha_\nu g_{\mu\rho}) \\ &= \frac{1}{2} (\varepsilon^{\mu\nu} \delta^\alpha_\mu g_{\nu\rho} + \varepsilon^{\nu\mu} \delta^\alpha_\nu g_{\rho\mu}) \\ &= \frac{1}{2} (\varepsilon^\alpha_\rho + \varepsilon^\alpha_\beta) = \varepsilon^\alpha_\beta \end{aligned}$$

then use notation $\sum_{\mu\nu} A^\alpha \equiv (\sum_{\mu\nu})^\alpha_\beta A^\beta$

$$\Rightarrow A'^\alpha(x) \stackrel{\text{to first order of } \varepsilon}{=} A^\alpha(x) - \frac{i}{2} \varepsilon^{\mu\nu} (L_{\mu\nu} + \sum_{\mu\nu}) A^\alpha(x)$$

Note that here $L_{\mu\nu}$ and $\sum_{\mu\nu}$ are 4×4 matrices,
while the $L_{\mu\nu}$ for scalar field Lorentz transformation is just
 1×1 matrix. (because a vector field has 4 components,
while a scalar field only has 1 component).

Also note that $\sum_{\mu\nu}$ acts on the vector label i.e., generates the
variations stemming from the vectorial character of the field, while

$(L_{\mu\nu})^\alpha_\beta = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \delta^\alpha_\beta$ 物理量特征
generates the variations due to
the functional field dependence.

↑
物理量特征

The antisymmetric operator $J_{\mu\nu} \equiv L_{\mu\nu} + \sum_{\mu\nu}$ is the generator of
infinitesimal Lorentz transformations of vector fields: $S_\alpha A^\alpha(x) = -\frac{i}{2} \varepsilon^{\alpha\beta\gamma} J_{\mu\nu} A^\mu(x)$

Again, let's look at the pure spatial components, J_{ij} , $i,j=1,2,3$.

Refine $J^i = L^i + S^i \equiv \frac{1}{2} \epsilon^{ijk} J_{jk}$, i.e., $L^i = \frac{1}{2} \epsilon^{ijk} J_{jk}$,
 with $(L^i)_a^b = \frac{1}{2} \epsilon^{ijk} (x_j \partial_k - x_k \partial_j) \delta_a^b$, $a,b=1,2,3$.

$$(S^i)_b^a = -i \epsilon^{iab}$$

→ consider J act
on the space components
of the field.

check: $(S^i)_b^a = \frac{1}{2} \epsilon^{ijk} (\sum_{jk})_b^a = \frac{i}{2} \epsilon^{ijk} (\delta_j^a g_{kb} - \delta_k^a g_{jb})$
 $= \frac{i}{2} (-\epsilon^{iab} + \epsilon^{iba}) = -i \epsilon^{iab}$

Still we have $[L^i, L^j] = i \epsilon^{ijk} L^k$

and we have

$$[S^i, S^j]_b^a = -(\epsilon^{iac} \epsilon^{jcb} - \epsilon^{iac} \epsilon^{icb})$$

$$= -(-\delta_{ij} \delta^{ab} + \delta^{ib} \delta^{aj} + \delta^{ia} \delta^{bj} - \delta^{ib} \delta^{ai})$$

$$= \delta^{ib} \delta^{ai} - \delta^{ib} \delta^{aj}$$

$$= i \epsilon^{ijk} \epsilon^{iab} = i \epsilon^{ijk} (S^k)_b^a$$

(Note that $\epsilon^{ijk} \epsilon^{abk} = \delta^{ia} \delta^{jb} - \delta^{ib} \delta^{ja}$)

while $[L^i, S^j] = 0$

$$\Rightarrow [J^i, J^j] = i \epsilon^{ijk} J^k$$

The square of the operator S is

$$(S^2)_b^a = \sum_{k,c} (S^k)_c^a (S^k)_b^c = \epsilon^{ka} \epsilon^{kc} = 2 \delta_b^a$$

which means the eigenvalue of S^2 is $s(s+1) = 1(1+1) = 2$.

That is, the intrinsic spin of a vector field is 1.