Fermion field quantization.

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi \quad , \quad \text{note that } [\psi] = [E]^{3/2} \text{ dimension.}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi} i \gamma^\mu \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \partial_\mu \bar{\psi}\, i\gamma^\mu$$

$$\Rightarrow \quad \partial_\mu \bar{\psi}\, i\gamma^\mu + m\bar{\psi} = 0 \Rightarrow (\partial_\mu \bar{\psi}\, i\gamma^\mu + m\bar{\psi})^\dagger = 0$$

$$\Rightarrow -i\gamma^{\mu\dagger} \partial_\mu \bar{\psi}^\dagger + m\bar{\psi}^\dagger = 0$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi \ , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \qquad \Downarrow$$

$$\Rightarrow \quad i\gamma^\mu \partial_\mu \psi - m\psi = 0 \qquad -i\gamma^{\mu\dagger} \partial_\mu (\psi^\dagger \gamma^0)^\dagger + m(\psi^\dagger\gamma^0)^\dagger = 0$$

$$\Downarrow$$

$$-i\gamma^0\gamma^\mu\gamma^0\gamma^0 \partial_\mu \psi + m\gamma^0\psi = 0$$

Since $\begin{cases} \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \\ (\gamma^0)^2 = 1 \end{cases}$ $\Downarrow$ ← left times $\gamma^0$

$$-i\gamma^\mu \partial_\mu \psi + m\psi = 0$$

$$\Downarrow$$

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

canonical momentum field $\pi$ is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial t)} = \bar{\psi} i \gamma^0 = i\psi^\dagger$$

In fact, $\psi$ and $\bar{\psi}$ mutually conjugate to each other (up to $i$ and $\gamma^0$), so we only have one set of commutation relations.

$$\text{So,} \ H = \int d^3\vec{x}\, \mathcal{H} = \int d^3\vec{x} \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \partial_0 \psi + \partial_0 \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} - \mathcal{L} \right)$$

$$= \int d^3\vec{x} \left[ i\psi^\dagger \dot{\psi} - (\bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi) \right]$$

However, since $\psi$ in $H$ satisfies the Dirac equation (i.e., $\psi$ is on-shell), then $i\gamma^\mu \partial_\mu \psi - m\psi = 0 \Rightarrow \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi = 0$.

$$\Rightarrow \quad H = \int d^3\vec{x}\, i\psi^\dagger \dot{\psi} = \int d^3\vec{x}\, i\bar{\psi}\gamma^0 \dot{\psi}$$

$$\vec{P} = -\int d^3\vec{x} \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \vec{\nabla}\psi + \vec{\nabla}\bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} \right) = -\int d^3\vec{x} \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \vec{\nabla}\psi \right) = -\int d^3\vec{x}\, i\psi^\dagger \vec{\nabla}\psi$$

(note: $T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi + \partial_\nu \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \delta^\mu{}_\nu \mathcal{L}$, $T^0{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \partial_\nu \psi + \partial_\nu \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} - \delta^0_\nu \mathcal{L}$)

Do an internal global transformation of the fields,

$$\psi(x) \to \psi'(x) = e^{-i\alpha}\psi \simeq \psi(x) - i\alpha\psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = e^{i\alpha}\bar{\psi}(x) \simeq \bar{\psi}(x) + i\alpha\bar{\psi}(x)$$

$$\Rightarrow \quad \bar{\psi}'(x)\, i\gamma^\mu \partial_\mu \psi'(x) = \bar{\psi}(x) e^{i\alpha} i\gamma^\mu \partial_\mu \psi\, e^{-i\alpha}$$
$$= \bar{\psi}(x)\, i\gamma^\mu \partial_\mu \psi(x)$$

$$\bar{\psi}'(x)\psi'(x) = \bar{\psi}(x) e^{i\alpha} e^{-i\alpha} \psi(x) = \bar{\psi}(x)\psi(x)$$

So the Lagrangian $\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$ is invariant.

$$\Rightarrow \mathcal{L}' = \mathcal{L}.$$

Similar to what we did for complex scalar field,

let $\delta\omega = \delta\alpha$,
$$\psi \to \psi' = e^{-i\delta\alpha}\psi \simeq \psi - i(\delta\alpha)\psi$$
$$\bar{\psi} \to \bar{\psi}' = e^{i\delta\alpha}\bar{\psi} \simeq \bar{\psi} + i(\delta\alpha)\bar{\psi}$$

$$\Rightarrow \quad \frac{\delta\psi}{\delta\omega} = -i\psi, \quad \frac{\delta\bar{\psi}}{\delta\omega} = i\bar{\psi} \quad \text{and} \quad \frac{\delta x^\mu}{\delta\omega} = 0.$$

$$\Rightarrow \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (-i\psi) + i\bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}$$
$$= \bar{\psi}\gamma^\mu \psi$$

$$\Rightarrow \quad Q = \int d^3\vec{x}\, j^0 = \int d^3\vec{x}\, \bar{\psi}\gamma^0\psi = \int d^3\vec{x}\, \psi^\dagger\psi$$

From the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

times $-i\gamma^\nu \partial_\nu$ from the left

$$\Rightarrow \quad -i\gamma^\nu \partial_\nu ( i\gamma^\mu \partial_\mu \psi - m\psi ) = 0$$

$$\Rightarrow \quad \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi + i m \gamma^\nu \partial_\nu \psi = 0$$

$$\Rightarrow \quad \frac{1}{2} \underbrace{(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu)}_{2g^{\mu\nu}} \partial_\nu \partial_\mu \psi + m^2 \psi = 0$$

$$\Rightarrow \quad \partial^\mu \partial_\mu \psi + m^2 \psi = 0$$

$$\Rightarrow \quad (\Box + m^2)\psi = 0$$

$\Rightarrow$ we can expect that $e^{\pm i p\cdot x}$ are still (part of) solutions of Dirac equation, and they should be included in the decomposition of $\psi$.

If we write a solution as $\psi(x) = u(\vec{p})\, e^{-ip\cdot x}$, where $u(\vec{p})$ is an $x$-independent spinor, then

$$i\gamma^\mu \partial_\mu ( u(\vec{p})\, e^{-ip\cdot x}) - m\, u(\vec{p})\, e^{-ip\cdot x} = 0$$

$$\Rightarrow \quad i\gamma^\mu u(\vec{p}) (-i) \underbrace{\partial_\mu (p_\nu x^\nu)}_{p_\nu \delta_\mu^\nu = p_\mu} e^{-ip\cdot x} - m\, u(\vec{p})\, e^{-ip\cdot x} = 0$$

$$\Rightarrow \quad [\not{p}\, u(\vec{p}) - m\, u(\vec{p})]\, e^{-ip\cdot x} = 0$$

$$\Rightarrow \quad \boxed{(\not{p} - m)\, u(\vec{p}) = 0}$$

For the other solution $\psi(x) = v(\vec{p})\, e^{ip\cdot x}$, where $v(\vec{p})$ is also an $x$-independent spinor, then

$$i\gamma^\mu \partial_\mu ( v(\vec{p})\, e^{ip\cdot x}) - m\, v(\vec{p})\, e^{ip\cdot x} = 0$$

$$\Rightarrow \quad [-\gamma^\mu p_\mu\, v(\vec{p}) - m\, v(\vec{p})]\, e^{ip\cdot x} = 0$$

$$\Rightarrow \quad (\not{p} + m)\, v(\vec{p})\, e^{ip\cdot x} = 0$$

$$\Rightarrow \quad \boxed{(\not{p} + m)\, v(\vec{p}) = 0}$$

Now we look at the solutions of

$$(\not{p} - m)\, u(\vec{p}) = 0$$

let's use standard representation of $\gamma$-matrices, $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}$$

$$\Rightarrow \quad (\not{p} - m) = \begin{pmatrix} E - m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E - m \end{pmatrix}$$

the above matrix is a block matrix, so we write $u(\vec{p})$ as a block column, $u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, where $\xi$ & $\eta$ are $2\times1$ columns

$$\Rightarrow \begin{cases} (E-m)\,\xi - \vec{p}\cdot\vec{\sigma}\,\eta = 0 & ① \\ \vec{p}\cdot\vec{\sigma}\,\xi - (E+m)\,\eta = 0 & ② \end{cases}$$

using $\sigma^i \sigma^j = \delta^{ij} \mathbb{1}_{2\times2} + i\varepsilon^{ijk}\sigma^k$

left times $\sum_{j=1}^{3} p^j \sigma^j$ on ①

$$\Rightarrow \quad \vec{p}\cdot\vec{\sigma}\,(E-m)\,\xi - \underbrace{p^j\sigma^j p^i \sigma^i}_{\vec{p}^{\,2} + \cancel{i\varepsilon^{ijk}\sigma^k p^i p^j} \to 0}\,\eta = 0$$

$$\vec{p}^{\,2} = E^2 - m^2 = (E+m)(E-m)$$

$$\Rightarrow \quad \eta = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\,\xi \qquad (\text{for } E \neq m).$$

② gives the same result.

$$\Rightarrow \quad u(\vec{p}) = \begin{pmatrix} \xi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\xi \end{pmatrix}$$

So there're two independent solutions for $u(\vec{p})$.

For example, we can choose the two independent solutions as

$$u(\vec{p}) = \underbrace{N}_{\text{normalization factor}} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad u(\vec{p}) = N \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

Any $2\times1$ column $\begin{pmatrix} a \\ b \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

i.e., $\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similarly, for

$$(\not{p} + m)\, v(\vec{p}) = 0$$

we have

$$\begin{pmatrix} E+m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E+m \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} (E+m)\,\xi' - \vec{p}\cdot\vec{\sigma}\,\eta' = 0 & ① \\ \vec{p}\cdot\vec{\sigma}\,\xi' - (E-m)\,\eta' = 0 & ② \end{cases}$$

left times $\sum_{j=1}^{3} p^j \sigma^j$ on ②

$$\Rightarrow \underbrace{\vec{p}^{\,2}}_{(E+m)(E-m)}\,\xi' - (E-m)\,\vec{p}\cdot\vec{\sigma}\,\eta' = 0$$

$$\Rightarrow \xi' = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\,\eta' \qquad (\text{for } E \neq m)$$

① gives the same result.

So, there're two independent solutions for $v(\vec{p})$.

For example, we can choose the two independent solution as

$$v(\vec{p}) = \underbrace{N'}_{\text{normalization factor}} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad v(\vec{p}) = N' \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix}$$

Again, any $2\times1$ column $\begin{pmatrix} a \\ b \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

i.e., $\begin{pmatrix} a \\ b \end{pmatrix} = -b\begin{pmatrix} -1 \\ 0 \end{pmatrix} + a\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In the $E = m$ case, that is, in $\vec{p} = 0$ reference frame.

$$\cancel{p} - m = m(\gamma^0 - 1) = m\begin{pmatrix} 1-1 & 0 \\ 0 & -1-1 \end{pmatrix} = -2m\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then write $u = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \Rightarrow \quad \eta = 0$, and we can write the two independent solutions as

$$u(\vec{p}=0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u(\vec{p}=0) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

normalization factor

Similarly, for

$$0 = (\cancel{p} + m)\, v(\vec{p}=0) = m(\gamma^0 + 1)\, v(\vec{p}=0)$$

$$= m\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$$

$$\Rightarrow \quad \xi' = 0$$

and we can write the two independent solutions as

$$v(\vec{p}=0) = N'\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v(\vec{p}=0) = N'\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

normalization factor

In the $\vec{p} = 0$ reference frame, we can construct the spin operator as $\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$, then

$$\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

The charge conjugation operator in the Standard representation is

$$C = i\gamma^2\gamma^0$$

$$= i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$$

In the $\vec{p}=0$ frame, let's denote

$$u(\vec{p}=0, 1) = u(\vec{0}, 1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$u(\vec{0}, 2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v(\vec{0}, 1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v(\vec{0}, 2) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

then $$C\bar{u}^T(\vec{0}, 1) = C(u^+\gamma^0)^T = C\left[(1\ 0\ 0\ 0)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right]^T$$

$$= C\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = v(\vec{0}, 1)$$

$$C\bar{u}^T(\vec{0}, 2) = C\left[(0\ 1\ 0\ 0)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right]^T$$

$$= C\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = v(\vec{0}, 2)$$

$$C\bar{v}^T(\vec{0}, 1) = C\left[(0\ 0\ 0\ 1)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right]^T$$

$$= C\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = u(\vec{0}, 1)$$

$$C\bar{v}^T(\vec{0}, 2) = C\left[(0\ 0\ -1\ 0)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right]^T$$

$$= C\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = u(\vec{0}, 2)$$

In fact, for any $\vec{p}$ and any representation, if we want.

$$C\bar{u}^T(\vec{p},s) = v(\vec{p},s)$$

and $C\bar{v}^T(\vec{p},s) = u(\vec{p},s)$

where s is for spin

This is the reason we choose the solutions of $v(\vec{p})$, when $u(\vec{p})$ solutions are given.

We need to emphasis that we do not have to choose

$$u(\vec{p}) = N\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}, \quad u(\vec{p}) = N\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}$$

$$v(\vec{p}) = N'\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1\end{pmatrix}, \quad v(\vec{p}) = N'\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\begin{pmatrix}-1\\0\end{pmatrix} \\ \begin{pmatrix}-1\\0\end{pmatrix}\end{pmatrix}$$

We can choose any 2x1 columns $\chi_s$ & $\eta_s$ ($s=1,2$, or call it $s=\pm\frac{1}{2}$)

$$\chi_s^+\chi_r = \eta_s^+\eta_r = \delta_{sr} \quad (\text{for } s,r=1,2) \longrightarrow \text{normal orthagonal relation}$$

and $$\sum_{s=1}^{2}\chi_s\chi_s^+ = \sum_{s=1}^{2}\eta_s\eta_s^+ = \mathbb{1}_{2\times2} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \rightarrow \text{complete relation}$$

then $$u(\vec{p},s) = N\begin{pmatrix}\chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\end{pmatrix} \quad \text{and} \quad v(\vec{p},s) = N'\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s \\ \eta_s\end{pmatrix}$$

can be used to form a complete set to decomposite $\psi(x)$.

The important point of the analysis of the solutions of $u(\vec{p})$ and $v(\vec{p})$ is that they also include the spin degrees of freedom, so the Dirac field operator should have the following decompositions,

$$\psi(\vec{x},t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left( u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, d^{\dagger}_{\vec{p},s}\, e^{ip\cdot x} \right)$$

Note that it does not matter you label $s$ as $\pm\frac{1}{2}$ or 1 & 2; all you need to know is that both $u(\vec{p})$ and $v(\vec{p})$ have two independent solutions, corresponding to the spin degrees of freedom.

$$\psi^{\dagger}(\vec{x},t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left( u^{\dagger}(\vec{p},s)\, b^{\dagger}_{\vec{p},s}\, e^{ip\cdot x} + v^{\dagger}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x} \right)$$

$$\bar{\psi}(\vec{x},t) = \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \sum_{s=\pm\frac{1}{2}} \left( \bar{u}(\vec{p},s)\, b^{\dagger}_{\vec{p},s}\, e^{ip\cdot x} + \bar{v}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x} \right)$$

Let's look at the normalization of $u$ and $v$ spinors.

For $u(\vec{p},s) = N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix}$, where $\chi_s$ is a $2\times 1$ column and satisfies $\chi_s^+ \chi_{s'} = \delta_{ss'}$

( e.g., if choose the two solutions as $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\chi_{\frac{1}{2}}^+ = (1 \ \ 0)$, $\chi_{-\frac{1}{2}}^+ = (0 \ \ 1) \Rightarrow \chi_s^+ \chi_{s'} = \delta_{ss'}$ )

then

$$u^+(\vec{p},s)\, u(\vec{p},s') = |N|^2 \left( \chi_s^+ \quad \chi_s^+ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right) \begin{pmatrix} \chi_{s'} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_{s'} \end{pmatrix}$$

$$= |N|^2 \left( \chi_s^+\chi_{s'} + \chi_s^+ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi_{s'} \right)$$

$$= |N|^2 \left( \chi_s^+\chi_{s'} + \chi_s^+ \frac{(E+m)(E-m)}{(E+m)^2} \mathbb{I} \chi_{s'} \right)$$

$$= |N|^2 \frac{E+m+E-m}{E+m} \chi_s^+ \chi_{s'}$$

$$= |N|^2 \frac{2E}{E+m} \delta_{ss'}$$

Since we know that $\bar{\psi}(x)\gamma^\mu\psi(x)$ transform as a Lorentz vector and for $\psi(x) = u(\vec{p},s)e^{-ip\cdot x}$, we have

$$\bar{\psi}(x)\gamma^\mu\psi(x) = \bar{u}(\vec{p},s)\, e^{ip\cdot x} \gamma^\mu u(\vec{p},s)\, e^{-ip\cdot x}$$

$$= \bar{u}(\vec{p},s)\gamma^\mu u(\vec{p},s)$$

$$= u^+(\vec{p},s)\gamma^0\gamma^\mu u(\vec{p},s)$$

So, we expect $u^+(\vec{p},s)\,u(\vec{p},s) = u^+(\vec{p},s)\gamma^0\gamma^0 u(\vec{p},s)$ behaves as the time component of a Lorentz vector, so we choose $N = (E+m)^{\frac{1}{2}}$ then $u^+(\vec{p},s)\,u(\vec{p},s') = 2E\delta_{ss'}$.

Similarly, for $v(\vec{p},s) = N'\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s \\ \eta_s \end{pmatrix}$, and $\eta_s^+\eta_{s'} = \delta_{ss'}$.

(e.g., $\eta_{-\frac{1}{2}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\eta_{+\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \eta^+_{-\frac{1}{2}} = (-1 \;\; 0)$, $\eta^+_{+\frac{1}{2}} = (0 \;\; 1)$

$\Rightarrow \eta_s^+\eta_{s'} = \delta_{ss'}$ )

$$\Rightarrow \quad v^+(\vec{p},s)\, v(\vec{p},s') = |N'|^2 \left(\eta_s^+ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \quad \eta_s^+\right) \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_{s'} \\ \eta_{s'} \end{pmatrix}$$

$$= |N'|^2 \cdot \left(\eta_s^+ \frac{|\vec{p}|^2}{(E+m)^2} \mathbb{I}\, \eta_{s'} + \eta_s^+\eta_{s'}\right)$$

$$= |N'|^2 \frac{E-m+E+m}{E+m} \eta_s \eta_{s'}$$

$$= |N'|^2 \frac{2E}{E+m} \delta_{ss'}$$

again, let's choose $N' = \sqrt{E+m}$

$$\Rightarrow \quad v^+(\vec{p},s)\; v(\vec{p},s') = 2E\,\delta_{ss'}$$

Also, $u^+(\vec{p},s)\, v(-\vec{p},s')$

$$= (E+m)\cdot \left(\chi_s^+ \quad \chi_s^+\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) \begin{pmatrix} \frac{-\vec{p}\cdot\vec{\sigma}}{E+m}\eta_{s'} \\ \eta_{s'} \end{pmatrix}$$

$$= (E+m)\cdot \left(-\chi_s^+ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_{s'} + \chi_s^+ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\eta_{s'}\right)$$

$$= 0$$

$v^+(\vec{p},s)\; u(-\vec{p},s')$

$$= (E+m)\cdot \left(\eta_s^+ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \quad \eta_s^+\right) \begin{pmatrix} \chi_{s'} \\ -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_{s'} \end{pmatrix}$$

$$= (E+m) \left(\eta_s^+ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_{s'} - \eta_s^+ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_{s'}\right)$$

$$= 0$$

In the $\vec{p}=0$ refrence frame, $u(0,\frac{1}{2})=N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $u(0,-\frac{1}{2})=N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$v(0,-\frac{1}{2})=N'\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $v(0,+\frac{1}{2})=N'\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

$\Rightarrow u^{+}(0,s)\,u(0,s')=|N|^2\delta_{ss'}$

and $N$ is choosen to be $(2m)^{\frac{1}{2}}$

$\Rightarrow u^{+}(0,s)\,u(0,s')=2m\delta_{ss'}$.

also $v^{+}(0,s)\,v(0,s')=|N'|^2\delta_{ss'}$

and $N'$ is choosen to be $(2m)^{\frac{1}{2}}$

$\Rightarrow v^{+}(0,s)\,v(0,s')=2m\delta_{ss'}$

or, we just use $u(0,s)=N\begin{pmatrix}\chi_s\\0\end{pmatrix}$

$v(0,s)=N'\begin{pmatrix}0\\\eta_s\end{pmatrix}$

then $u^{+}(0,s)u(0,s')=|N|^2\chi_s^{+}\chi_{s'}$

$=|N|^2\delta_{ss'}$

$v^{+}(0,s)\,v(0,s')=|N'|^2\eta_s^{+}\eta_{s'}$

$=|N'|^2\delta_{ss'}$

$u^{+}(0,s)\,v(0,s')=0$

$v^{+}(0,s)\,u(0,s')=0$

also we have

$$u^{+}(0,s)\,v(0,s')=0$$

$$v^{+}(0,s)\,u(0,s')=0$$

Since $E\big|_{\vec{p}=0}=m$

then in general, no matter whether $\vec{p}=0$, we have the formulae

$$\boxed{\begin{aligned}&u^{+}(\vec{p},s)\,u(\vec{p},s')=v^{+}(\vec{p},s)\,v(\vec{p},s')=2E\delta_{ss'}\\&u^{+}(\vec{p},s)\,v(-\vec{p},s')=v^{+}(\vec{p},s)\,u(-\vec{p},s')=0\end{aligned}}$$

let's further look at $\bar{u}u$, $\bar{v}v$, $\bar{u}v$ and $\bar{v}u$

left times $u^{+}(\vec{p},s)$ on $(\not{p}-m)\,u(\vec{p},s')=0$

$\Rightarrow$ $u^{+}(\vec{p},s)(\not{p}-m)\,u(\vec{p},s')=0$

$\Rightarrow$ $u^{+}(\vec{p},s)(E\gamma^0-p^i\gamma^i-m)\,u(\vec{p},s')=0$ $\quad\cdots(1)$

take hermitian conjugate and switch $s\leftrightarrow s'$

$\Rightarrow$ $u^{+}(\vec{p},s)(E\gamma^{0+}-p^i\gamma^{i+}-m)\,u(\vec{p},s')=0$

using $\gamma^{0+} = \gamma^0$, $\gamma^{i+} = \gamma^i$

$\Rightarrow$

$$u^+(\vec{p}, s)(E\gamma^0 + p^i\gamma^i - m)\,u(\vec{p}, s') = 0 \quad \cdots [2]$$

[1] + [2]

$$\Rightarrow \quad E u^+(\vec{p}, s)\gamma^0 u(\vec{p}, s') = m\, u^+(\vec{p}, s)\, u(\vec{p}, s')$$

using $u^+(\vec{p}, s)\, u(\vec{p}, s') = 2E\,\delta_{ss'}$

$\Rightarrow$

$$\underbrace{E u^+(\vec{p}, s)\gamma^0 u(\vec{p}, s')}_{\bar{u}(\vec{p}, s) u(\vec{p}, s')} = m \cdot 2E\,\delta_{ss'}$$

$$\Rightarrow \quad \bar{u}(\vec{p}, s)\, u(\vec{p}, s') = 2m\,\delta_{ss'}$$

Similarly, left times $v^+(\vec{p}, s)$ on $(\not{p} + m)\,v(\vec{p}, s') = 0$

$$\Rightarrow \quad v^+(\vec{p}, s)(E\gamma^0 - p^i\gamma^i + m)\,v(\vec{p}, s') = 0$$

take hermitian conjugate and $s \leftrightarrow s'$, and use $\gamma^{0+} = \gamma^0$, $\gamma^{i+} = -\gamma^i$

$\Rightarrow$

$$v^+(\vec{p}, s)(E\gamma^0 + p^i\gamma^i + m)\,v(\vec{p}, s') = 0$$

$$\Rightarrow \quad E v^+(\vec{p}, s)\gamma^0 v(\vec{p}, s') = -m\, v^+(\vec{p}, s)\, v(\vec{p}, s') = -2mE\,\delta_{ss'}$$

$$\Rightarrow \quad \bar{v}(\vec{p}, s)\, v(\vec{p}, s') = -2m\,\delta_{ss'}$$

Also, since $(\not{p} - m)\,u(\vec{p}, s') = 0$, then $\not{p}\, u(\vec{p}, s') = m\, u(\vec{p}, s')$

left times $\bar{v}(\vec{p}, s)$ $\Rightarrow$

$$\bar{v}(\vec{p}, s)\not{p}\, u(\vec{p}, s') = m\bar{v}(\vec{p}, s)\, u(\vec{p}, s') \quad \cdots [3]$$

Since $\bar{v}(\vec{p}, s)(\not{p} + m) = 0 \Rightarrow \bar{v}(\vec{p}, s)\not{p} = -m\bar{v}(\vec{p}, s)$

right times $u(\vec{p}, s')$ $\Rightarrow$

$$\bar{v}(\vec{p}, s)\not{p}\, u(\vec{p}, s') = -m\bar{v}(\vec{p}, s)\, u(\vec{p}, s') \quad \cdots [4]$$

[3]−[4] $\Rightarrow$

$$\bar{v}(\vec{p}, s)\, u(\vec{p}, s') = 0.$$

take hermitian conjugate and $s \leftrightarrow s'$

$$\Rightarrow \quad u^+(\vec{p}, s)\gamma^{0+} v(\vec{p}, s') = 0 \Rightarrow \bar{u}(\vec{p}, s)\, v(\vec{p}, s') = 0$$

So, we have

$$\boxed{\begin{aligned}\bar{u}(\vec{p},s)\,u(\vec{p},s') &= -\bar{v}(\vec{p},s)\,v(\vec{p},s') = 2m\,\delta_{ss'}\\ \bar{u}(\vec{p},s)\,v(\vec{p},s') &= \bar{v}(\vec{p},s)\,u(\vec{p},s') = 0\end{aligned}}$$

In later calculations, we also need $\sum_s u(\vec{p},s)\,\bar{u}(\vec{p},s)$ and $\sum_s v(\vec{p},s)\,\bar{v}(\vec{p},s)$

$$\sum_s u(\vec{p},s)\,\bar{u}(\vec{p},s) = (E+m)\sum_s \begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\end{pmatrix}\left(\chi_s^+ \quad \chi_s^+\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\gamma^0$$

$$= (E+m)\sum_s \begin{pmatrix}\chi_s\chi_s^+ & \chi_s\chi_s^+\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\chi_s^+ & \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s\chi_s^+\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\gamma^0$$

using $\sum_s \chi_s\chi_s^+ = \mathbb{1}_{2\times2}$

$$\Rightarrow \sum_s u(\vec{p},s)\,\bar{u}(\vec{p},s) = \begin{pmatrix}E+m & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & E-m\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -E+m\end{pmatrix} = \not{p}+m$$

while $\sum_s v(\vec{p},s)\,\bar{v}(\vec{p},s) = (E+m)\sum_s \begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s\\ \eta_s\end{pmatrix}\left(\eta_s^+\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \quad \eta_s^+\right)\gamma^0$

$$= (E+m)\sum_s \begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s\eta_s^+\frac{\vec{p}\cdot\vec{\sigma}}{E+m} & \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\eta_s\eta_s^+\\ \eta_s\eta_s^+\frac{\vec{p}\cdot\vec{\sigma}}{E+m} & \eta_s\eta_s^+\end{pmatrix}\gamma^0$$

using $\sum_s \eta_s\eta_s^+ = \mathbb{1}_{2\times2}$

$$\Rightarrow \sum_s v(\vec{p},s)\,\bar{v}(\vec{p},s) = (E+m)\begin{pmatrix}\frac{|\vec{p}|^2}{(E+m)^2} & \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & \mathbb{1}_{2\times2}\end{pmatrix}\begin{pmatrix}\mathbb{1}_{2\times2} & 0\\ 0 & -\mathbb{1}_{2\times2}\end{pmatrix}$$

$$= \begin{pmatrix} E-m & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & E+m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E-m \end{pmatrix}$$

$$= \not{p} - m$$

So,

$$\boxed{\begin{aligned} \sum_s u(\vec{p},s)\,\bar{u}(\vec{p},s) &= \not{p} + m \\ \sum_s v(\vec{p},s)\,\bar{v}(\vec{p},s) &= \not{p} - m. \end{aligned}}$$

In the $\vec{p}=0$ refrence frame.

$$\bar{u}(0,\tfrac{1}{2}) = (2m)^{\frac{1}{2}}(1\ 0\ 0\ 0)\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix} = (2m)^{\frac{1}{2}}(1\ 0\ 0\ 0) = u^{+}(0,\tfrac{1}{2})$$

$$\bar{u}(0,-\tfrac{1}{2}) = (2m)^{\frac{1}{2}}(0\ 1\ 0\ 0) = u^{+}(0,-\tfrac{1}{2})$$

$$\bar{v}(0,-\tfrac{1}{2}) = (2m)^{\frac{1}{2}}(0\ 0\ -1\ 0)\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix} = (2m)^{\frac{1}{2}}(0\ 0\ 1\ 0) = -v^{+}(0,-\tfrac{1}{2})$$

$$\bar{v}(0,+\tfrac{1}{2}) = (2m)^{\frac{1}{2}}(0\ 0\ 0\ -1) = -v^{+}(0,+\tfrac{1}{2})$$

$$\Rightarrow \quad \bar{u}(0,s)\,u(0,s') = 2m\,\delta_{ss'},$$

$$\bar{v}(0,s)\,v(0,s') = -2m\,\delta_{ss'}$$

$$\bar{u}(0,s)\,v(0,s') = \bar{v}(0,s)\,u(0,s') = 0.$$

and

$$\sum_s u(0,s)\,\bar{u}(0,s) = 2m\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} = \begin{pmatrix} 2m & 0 \\ 0 & 0 \end{pmatrix} = m(\gamma^0+1)$$

$$\sum_s v(0,s)\,\bar{v}(0,s) = 2m\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2m \end{pmatrix} = m(\gamma^0-1)$$

So, the formulae for $\vec{p}=0$ are consistent with the one for $\vec{p}\neq 0$.

Or, just write $u(0,s) = (2m)^{\frac{1}{2}}\begin{pmatrix}\chi_s\\0\end{pmatrix}$, $v(0,s) = (2m)^{\frac{1}{2}}\begin{pmatrix}0\\\eta_s\end{pmatrix}$,

$$\Rightarrow \quad \bar{u}(0,s) = (2m)^{\frac{1}{2}}(\chi_s^{+}\ \ 0)\begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2}\end{pmatrix} = (2m)^{\frac{1}{2}}(\chi_s^{+}\ \ 0)$$

$$\bar{v}(0,s) = (2m)^{\frac{1}{2}}(0\ \ \eta_s^{+})\begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2}\end{pmatrix} = (2m)^{\frac{1}{2}}(0\ \ -\eta_s^{+})$$

$$\Rightarrow \bar{u}(0,s)\,u(0,s') = 2m\,\chi_s^{+}\chi_{s'} = 2m\,\delta_{ss'}, \quad \bar{v}(0,s)\,v(0,s') = -2m\,\eta_s^{+}\eta_{s'} = -2m\,\delta_{ss'}$$

$$\bar{u}(0,s)\,v(0,s') = 0, \quad \bar{v}(0,s)\,u(0,s') = 0,$$

$$\sum_s u(0,s)\,\bar{u}(0,s) = 2m\begin{pmatrix} \sum_s \chi_s\chi_s^{+} & 0 \\ 0 & 0 \end{pmatrix} = m(\gamma^0+1), \quad \sum_s v(0,s)\,\bar{v}(0,s) = 2m\begin{pmatrix} 0 & 0 \\ 0 & -\sum_s \eta_s\eta_s^{+}\end{pmatrix} = m(\gamma^0-1)$$

Note: the above normalization for the spinors are used in Peskin & Schroeder and Ho Kim & Yem.

However, in Ryder, the normalization differs by 2m, i.e.,

$$u^+(\vec{p},s)\,u(\vec{p},s') = v^+(\vec{p},s)\,v(\vec{p},s') = \frac{2E}{2m} = \frac{E}{m}\,\delta_{ss'}$$

$$\bar{u}(\vec{p},s)\,u(\vec{p},s') = -\bar{v}(\vec{p},s)\,v(\vec{p},s') = \delta_{ss'}$$

$$\sum_s u(\vec{p},s)\,\bar{u}(\vec{p},s) = (\not{p}+m)/(2m), \quad \sum_s v(\vec{p},s)\,\bar{v}(\vec{p},s) = (\not{p}-m)/(2m)$$

Let's look at $H$.

$$H = \int_{-\infty}^{+\infty} d^3\vec{x}\ i\psi^{+}\dot{\psi}$$

$$= i\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k} \sum_{s,r} (-iE_{\vec{k}})\, C(E_{\vec{p}})\, C(E_{\vec{k}})$$

$$\times \left( u^{+}(\vec{p},s)\, b^{+}_{\vec{p},s}\, e^{ip\cdot x} + v^{+}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x} \right)$$

$$\times \left( u(\vec{k},r)\, b_{\vec{k},r}\, e^{-ik\cdot x} - v(\vec{k},r)\, d^{+}_{\vec{k},r}\, e^{ik\cdot x} \right)$$

$$= i\int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\, (-iE_{\vec{k}})\, C(E_{\vec{p}})\, C(E_{\vec{k}})\, (2\pi)^3$$

$$\times \sum_{s,r} \Big[ u^{+}(\vec{p},s)\, u(\vec{k},r)\, b^{+}_{\vec{p},s} b_{\vec{k},r}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$- v^{+}(\vec{p},s)\, v(\vec{k},r)\, d_{\vec{p},s} d^{+}_{\vec{k},r}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$- u^{+}(\vec{p},s)\, v(\vec{k},r)\, b^{+}_{\vec{p},s} d^{+}_{\vec{k},r}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$

$$+ v^{+}(\vec{p},s)\, u(\vec{k},r)\, d_{\vec{p},s} b_{\vec{k},r}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k}) \Big]$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, E_{\vec{p}} \left(C(E_{\vec{p}})\right)^2 (2\pi)^3$$

$$\times \sum_{s,r} \Big[ u^{+}(\vec{p},s)\, u(\vec{p},r)\, b^{+}_{\vec{p},s} b_{\vec{p},r} - v^{+}(\vec{p},s)\, v(\vec{p},r)\, d_{\vec{p},s} d^{+}_{\vec{p},r}$$

$$- u^{+}(\vec{p},s)\, v(-\vec{p},r)\, b^{+}_{\vec{p},s}\, d_{-\vec{p},r}\, e^{2iE_{\vec{p}}t}$$

$$+ v^{+}(\vec{p},s)\, u(-\vec{p},r)\, d_{\vec{p},s}\, b_{-\vec{p},r}\, e^{-2iE_{\vec{p}}t} \Big]$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, E_{\vec{p}} \left(C(E_{\vec{p}})\right)^2 (2\pi)^3 \sum_{s,r} \left( 2E_{\vec{p}}\delta_{sr}\, b^{+}_{\vec{p},s} b_{\vec{p},r} - 2E_{\vec{p}}\delta_{sr}\, d_{\vec{p},s} d^{+}_{\vec{p},r} \right)$$

$$= \int_{-\infty}^{+\infty} \left[ (2\pi)^3 \left(C(E_{\vec{p}})\right)^2 (2E_{\vec{p}}) \right] E_{\vec{p}} \sum_{s} \left( b^{+}_{\vec{p},s} b_{\vec{p},s} - d_{\vec{p},s} d^{+}_{\vec{p},s} \right)$$

Now let's pause here and look at the commutation relations.

If we impose commutation rules, then

$$[\psi_a(t,\vec{x}), \psi_b^+(t,\vec{y})] = \delta_{ab}\,\delta^3(\vec{x}-\vec{y})$$, where $a, b$ are spinor indices.

If we impose anticommutation rules, then

$$\{\psi_a(t,\vec{x}), \psi_b^+(t,\vec{y})\} = \delta_{ab}\,\delta^3(\vec{x}-\vec{y}).$$

Let's look at both possibilities and see what will happen.

$$\psi_a(t,\vec{x})\psi_b^+(t,\vec{y}) + g\,\psi_b^+(t,\vec{y})\psi_a(t,\vec{x}) = \delta_{ab}\,\delta^3(\vec{x}-\vec{y})$$

$$\psi_a(t,\vec{x})\psi_b(t,\vec{y}) + g\,\psi_b(t,\vec{y})\psi_a(t,\vec{x}) = 0$$

Hermitian conjugate

$$\psi_b^+(t,\vec{y})\psi_a^+(t,\vec{x}) + g\,\psi_a^+(t,\vec{x})\psi_b^+(t,\vec{y}) = 0$$

$a \leftrightarrow b$, $x \leftrightarrow y$

$$\psi_a^+(t,\vec{x})\psi_b^+(t,\vec{y}) + g\,\psi_b^+(t,\vec{y})\psi_a^+(t,\vec{x}) = 0$$

$g = -1$ for commutator, $g = +1$ for anticommutator.

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\; u^+(\vec{k},r)\, e^{ik\cdot x}\, \psi(x)$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\; u^+(\vec{k},r)\, e^{ik\cdot x}\, C(E_{\vec{p}}) \sum_s \left( u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, d^+_{\vec{p},s}\, e^{ip\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\; u^+(\vec{k},r)\, C(E_{\vec{p}}) \sum_s \Big( u(\vec{p},s)\, b_{\vec{p},s}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$
$$+ v(\vec{p},s)\, d^+_{\vec{p},s}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})\Big)$$

$$= C(E_{\vec{k}}) \sum_s \left( u^+(\vec{k},r)\, u(\vec{k},s)\, b_{\vec{k},s} + u^+(\vec{k},r)\, v(-\vec{k},s)\, d^+_{-\vec{k},s}\, e^{2iE_{\vec{k}}t}\right)$$

$$= C(E_{\vec{k}})\, b_{\vec{k},r}\, 2E_{\vec{k}}$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\; v^+(\vec{k},r)\, e^{-ik\cdot x}\, \psi(x)$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\;\; v^+(\vec{k},r)\, e^{-ik\cdot x}\, C(E_{\vec{p}})\sum_s \left(u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, d^+_{\vec{p},s}\, e^{ip\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\;\; v^+(\vec{k},r)\, C(E_{\vec{p}})\sum_s \Big( u(\vec{p},s)\, b_{\vec{p},s}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$
$$+ v(\vec{p},s)\, d^+_{\vec{p},s}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) \Big)$$

$$= C(E_{\vec{k}}) \sum_s \Big( v^+(\vec{k},r)\, u(-\vec{k},s)\, b_{-\vec{k},s}\, e^{-2iE_{\vec{k}}t} + v^+(\vec{k},r)\, v(\vec{k},s)\, d^+_{\vec{k},s} \Big)$$

$$= C(E_{\vec{k}})\, 2E_{\vec{k}}\, d^+_{\vec{k},r}$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\; \psi^+(x)\, u(\vec{k},r)\, e^{-ik\cdot x}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\; C(E_{\vec{p}})\sum_s \left(u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} + v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right) u(\vec{k},r)\, e^{-ik\cdot x}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \sum_s \Big( u^+(\vec{p},s)\, u(\vec{k},r)\, b^+_{\vec{p},s}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{k}-\vec{p})$$
$$+ v^+(\vec{p},s)\, u(\vec{k},r)\, d_{\vec{p},s}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{k}+\vec{p}) \Big)$$

$$= C(E_{\vec{k}}) \sum_s \Big( u^+(\vec{k},s)\, u(\vec{k},r)\, b^+_{\vec{k},s} + v^+(-\vec{k},s)\, u(\vec{k},r)\, d_{-\vec{k},s}\, e^{-2iE_{\vec{k}}t} \Big)$$

$$= C(E_{\vec{k}})\, 2E_{\vec{k}}\, b^+_{\vec{k},r}$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\; \psi^+(x)\, v(\vec{k},r)\, e^{ik\cdot x}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\; C(E_{\vec{p}})\sum_s \left(u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} + v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right) v(\vec{k},r)\, e^{ik\cdot x}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \sum_s \Big( u^+(\vec{p},s)\, v(\vec{k},r)\, b^+_{\vec{p},s}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$
$$+ v^+(\vec{p},s)\, v(\vec{k},r)\, d_{\vec{p},s}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) \Big)$$

$$= C(E_{\vec{k}}) \sum_s \Big( u^+(-\vec{k},s)\, v(\vec{k},r)\, b^+_{-\vec{k},s}\, e^{2iE_{\vec{k}}t} + v^+(\vec{k},s)\, v(\vec{k},r)\, d_{\vec{k},s} \Big)$$

$$= C(E_{\vec{k}})\, 2E_{\vec{k}}\, d_{\vec{k},r}$$

$$\Rightarrow \quad b_{\vec{p},s} = \frac{1}{(2\pi)^3 C(E_{\vec{p}})\, 2E_{\vec{p}}} \int_{-\infty}^{+\infty} d^3\vec{x}\; u^+(\vec{p},s)\, \psi(x)\, e^{ip\cdot x}$$

$$b^+_{\vec{p},s} = \frac{1}{(2\pi)^3 C(E_{\vec{p}})\, 2E_{\vec{p}}} \int_{-\infty}^{+\infty} d^3\vec{x}\; \psi^+(x)\, u(\vec{p},s)\, e^{-ip\cdot x}$$

$$d_{\vec{p},s} = \frac{1}{(2\pi)^3 C(E_{\vec{p}})\, 2E_{\vec{p}}} \int_{-\infty}^{+\infty} d^3\vec{x}\; \psi^+(x)\, v(\vec{p},s)\, e^{ip\cdot x}$$

$$d^+_{\vec{p},s} = \frac{1}{(2\pi)^3 C(E_{\vec{p}})\, 2E_{\vec{p}}} \int_{-\infty}^{+\infty} d^3\vec{x}\; v^+(\vec{p},s)\, \psi(x)\, e^{-ip\cdot x}$$

$$\Rightarrow \quad b_{\vec{p},s}\, b^+_{\vec{k},r} + g\, b^+_{\vec{k},r}\, b_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{2E_{\vec{p}}}\frac{1}{2E_{\vec{k}}}\frac{1}{C(E_{\vec{p}})}\frac{1}{C(E_{\vec{k}})} \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\; e^{ip\cdot x} e^{-ik\cdot y}$$

$$\times \left( u^+(\vec{p},s)\,\psi(x)\,\psi^+(y)\,u(\vec{k},r) + g\,\psi^+(y)\,u(\vec{k},r)\,u^+(\vec{p},s)\,\psi(x) \right)$$

$$(\qquad) = \sum_{a,b=1}^{4} \left( u^+_a(\vec{p},s)\,\psi_a(x)\,\psi^+_b(y)\,u_b(\vec{k},r) + g\,\psi^+_b(y)\,u_b(\vec{k},r)\,u^+_a(\vec{p},s)\,\psi_a(x) \right)$$

$$= \sum_{a,b=1}^{4} u^+_a(\vec{p},s)\,u_b(\vec{k},r)\,\delta_{ab}\,\delta^3(\vec{x}-\vec{y}) = u^+(\vec{p},s)\,u(\vec{k},r)\,\delta^3(\vec{x}-\vec{y})$$

$$\Rightarrow \quad b_{\vec{p},s}\, b^+_{\vec{k},r} + g\, b^+_{\vec{k},r}\, b_{\vec{p},s} = \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{2E_{\vec{p}}}\frac{1}{2E_{\vec{k}}}\frac{1}{C(E_{\vec{p}})}\frac{1}{C(E_{\vec{k}})} \int_{-\infty}^{+\infty} d^3\vec{x}\; e^{ip\cdot x - ik\cdot x}$$

$$\times\, u^+(\vec{p},s)\,u(\vec{k},r)$$

$$= \frac{1}{(2\pi)^3}\left(\frac{1}{2E_{\vec{p}}}\right)^2 \left(\frac{1}{C(E_{\vec{p}})}\right)^2 u^+(\vec{p},s)\,u(\vec{p},r)\,\delta^3(\vec{p}-\vec{k})$$

$$= \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr}\,\delta^3(\vec{p}-\vec{k})$$

$$d_{\vec{p},s}\, d^{\dagger}_{\vec{k},r} + g\, d^{\dagger}_{\vec{k},r}\, d_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{2E_{\vec{p}}}\,\frac{1}{2E_{\vec{k}}}\,\frac{1}{C(E_{\vec{p}})}\,\frac{1}{C(E_{\vec{k}})} \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\; e^{ip\cdot x}\, e^{-ik\cdot y}$$

$$\times \sum_{a,b=1}^{4} \left( \psi^{\dagger}_a(x)\, v_a(\vec{p},s)\, v^{\dagger}_b(\vec{k},r)\, \psi_b(y) + g\, v^{\dagger}_b(\vec{k},r)\, \psi_b(y)\, \psi^{\dagger}_a(x)\, v_a(\vec{p},s) \right)$$

$$\sum_{a,b=1}^{4} \left( \qquad \right) = \sum_{a,b=1}^{4} g \left( g\, \psi^{\dagger}_a(x)\, \psi_b(y) + \psi_b(y)\, \psi^{\dagger}_a(x) \right) v_a(\vec{p},s)\, v^{\dagger}_b(\vec{k},r)$$

↑ use $g^2 = 1$ for $g = \pm 1$

$$= \sum_{a,b=1}^{4} g\, \delta_{ab}\, \delta^3(\vec{x}-\vec{y})\, v_a(\vec{p},s)\, v^{\dagger}_b(\vec{k},r)$$

$$= g\, \delta^3(\vec{x}-\vec{y})\, v^{\dagger}(\vec{k},r)\, v(\vec{p},s)$$

$$\Rightarrow d_{\vec{p},s}\, d^{\dagger}_{\vec{k},r} + g\, d^{\dagger}_{\vec{k},r}\, d_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{2E_{\vec{p}}}\,\frac{1}{2E_{\vec{k}}}\,\frac{1}{C(E_{\vec{p}})}\,\frac{1}{C(E_{\vec{k}})} \int_{-\infty}^{+\infty} d^3\vec{x}\; e^{ip\cdot x - ik\cdot x}\; g\; v^{\dagger}(\vec{k},r)\, v(\vec{p},s)$$

$$= \frac{1}{(2\pi)^3}\,\frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 g\, \delta_{rs}\, \delta^3(\vec{p}-\vec{k})$$

$$b_{\vec{p},s}\, b_{\vec{k},r} + g\, b_{\vec{k},r}\, b_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E_{\vec{p}})}\,\frac{1}{C(E_{\vec{k}})}\,\frac{1}{2E_{\vec{p}}}\,\frac{1}{2E_{\vec{k}}} \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\; e^{ip\cdot x}\, e^{ik\cdot y}$$

$$\times \sum_{a,b=1}^{4} \left( u^{\dagger}_a(\vec{p},s)\, \underline{\psi_a(x)}\, u^{\dagger}_b(\vec{k},r)\, \underline{\psi_b(y)} + g\, u^{\dagger}_b(\vec{k},r)\, \underline{\psi_b(y)}\, u^{\dagger}_a(\vec{p},s)\, \underline{\psi_a(x)} \right)$$

$= 0$ , since $\psi_a(t,\vec{x})\, \psi_b(t,\vec{y}) + g\, \psi_b(t,\vec{y})\, \psi_a(t,\vec{x}) = 0$

$$d_{\vec{p},s}\, d_{\vec{k},r} + g\, d_{\vec{k},r}\, d_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E_{\vec{p}})}\,\frac{1}{C(E_{\vec{k}})}\,\frac{1}{2E_{\vec{p}}}\,\frac{1}{2E_{\vec{k}}} \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\; e^{ip\cdot x}\, e^{ik\cdot y}$$

$$\times \sum_{a,b=1}^{4} \left( \underline{\psi^{\dagger}_a(x)}\, v_a(\vec{p},s)\, \underline{\psi^{\dagger}_b(y)}\, v_b(\vec{k},r) + g\, \underline{\psi^{\dagger}_b(y)}\, v_b(\vec{k},r)\, \underline{\psi^{\dagger}_a(x)}\, v_a(\vec{p},s) \right)$$

$= 0$ , since $\psi^{\dagger}_a(t,\vec{x})\, \psi^{\dagger}_b(t,\vec{y}) + g\, \psi^{\dagger}_b(t,\vec{y})\, \psi^{\dagger}_a(t,\vec{x}) = 0$

$$b_{\vec{p},s} d_{\vec{k},r} + g\, d_{\vec{k},r} b_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E_{\vec{p}})} \frac{1}{C(E_{\vec{k}})} \frac{1}{2E_{\vec{p}}} \frac{1}{2E_{\vec{k}}} \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\; e^{ip\cdot x} e^{ik\cdot y}$$

$$\times \sum_{a,b=1}^{4} \left( u_a^\dagger(\vec{p},s) \psi_a(x) \psi_b^\dagger(y) v_b(\vec{k},r) + g \psi_b^\dagger(y) v_b(\vec{k},r) u_a^\dagger(\vec{p},s) \psi_a(x) \right)$$

where $\sum_{a,b=1}^{4} (\qquad) = \sum_{a,b=1}^{4} \delta_{ab} \delta^3(\vec{x}-\vec{y})\, u_a^\dagger(\vec{p},s) v_b(\vec{k},r) = \delta^3(x-y)\, u^\dagger(\vec{p},s) v(\vec{k},r)$

$$\Rightarrow b_{\vec{p},s} d_{\vec{k},r} + g\, d_{\vec{k},r} b_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E_{\vec{p}})} \frac{1}{C(E_{\vec{k}})} \frac{1}{2E_{\vec{p}}} \frac{1}{2E_{\vec{k}}} \int_{-\infty}^{+\infty} d^3\vec{x}\; e^{ip\cdot x + ik\cdot x}\; u^\dagger(\vec{p},s) v(\vec{k},r)$$

$$= \frac{1}{(2\pi)^3} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \left(\frac{1}{2E_{\vec{p}}}\right)^2 \delta^3(\vec{p}+\vec{k})\, u^\dagger(-\vec{k},s) v(\vec{k},r)$$

$$= 0$$

$$b_{\vec{p},s} d_{\vec{k},r}^\dagger + g\, d_{\vec{k},r}^\dagger b_{\vec{p},s}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E_{\vec{p}})} \frac{1}{C(E_{\vec{k}})} \frac{1}{2E_{\vec{p}}} \frac{1}{2E_{\vec{k}}} \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\; e^{ip\cdot x} e^{-ik\cdot y}$$

$$\times \sum_{a,b=1}^{4} \left( u_a^\dagger(\vec{p},s) \psi_a(x) v_b^\dagger(\vec{k},r) \psi_b(y) + g\, v_b^\dagger(\vec{k},r) \psi_b(y) u_a^\dagger(\vec{p},s \psi_a(x) \right)$$

$$= 0$$

$$\Rightarrow g\, b_{\vec{p},s}^\dagger b_{\vec{k},r}^\dagger + b_{\vec{k},r}^\dagger b_{\vec{p},s}^\dagger = 0 \Rightarrow b_{\vec{p},s}^\dagger b_{\vec{k},r}^\dagger + g\, b_{\vec{k},r}^\dagger b_{\vec{p},s}^\dagger = 0$$

$$d_{\vec{p},s}^\dagger d_{\vec{k},r}^\dagger + g\, d_{\vec{k},r}^\dagger d_{\vec{p},s}^\dagger = 0$$

$$g\, b_{\vec{p},s}^\dagger d_{\vec{k},r}^\dagger + d_{\vec{k},r}^\dagger b_{\vec{p},s}^\dagger = 0 \Rightarrow b_{\vec{p},s}^\dagger d_{\vec{k},r}^\dagger + g\, d_{\vec{k},r}^\dagger b_{\vec{p},s}^\dagger = 0$$

$$g\, b_{\vec{p},s}^\dagger d_{\vec{k},r} + d_{\vec{k},r} b_{\vec{p},s}^\dagger = 0 \Rightarrow b_{\vec{p},s}^\dagger d_{\vec{k},r} + g\, d_{\vec{k},r} b_{\vec{p},s}^\dagger = 0$$

Let's go back the decomposition of $H$.

We expect that $:H:$ is positive definite, and we want the effect of $: \ :$ is dropping infinite zero-point energy.

If we take $g=-1$ (as in the scalar field quantization)

then $d_{\vec{p},s} d^{\dagger}_{\vec{p},s} = d^{\dagger}_{\vec{p},s} d_{\vec{p},s} - \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)$

$$\Rightarrow H = \int_{-\infty}^{+\infty} \left[(2\pi)^3 (C(E_{\vec{p}}))^2 2E_{\vec{p}}\right] E_{\vec{p}} \sum_s \left( b^{\dagger}_{\vec{p},s} b_{\vec{p},s} - d^{\dagger}_{\vec{p},s} d_{\vec{p},s} \right)$$
$$+ \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} (2\pi)^3 \delta^3(0) \times \underset{\substack{\uparrow \\ \text{from } \sum_s = 2}}{2}$$

then $:H: = \int_{-\infty}^{+\infty} \left[(2\pi)^3 (C(E_{\vec{p}}))^2 2E_{\vec{p}}\right] E_{\vec{p}} \sum_s \left( b^{\dagger}_{\vec{p},s} b_{\vec{p},s} - d^{\dagger}_{\vec{p},s} d_{\vec{p},s} \right)$

This means that the anti-particle have negative energy.

This is not desirable.

So, we need to choose $g=+1$,

then $d_{\vec{p},s} d^{\dagger}_{\vec{p},s} = - d^{\dagger}_{\vec{p},s} d_{\vec{p},s} + \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)$

$$\Rightarrow H = \int_{-\infty}^{+\infty} \left[(2\pi)^3 (C(E_{\vec{p}}))^2 2E_{\vec{p}}\right] E_{\vec{p}} \sum_s \left( b^{\dagger}_{\vec{p},s} b_{\vec{p},s} + d^{\dagger}_{\vec{p},s} d_{\vec{p},s} \right)$$
$$- \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} (2\pi)^3 \delta^3(0) \times 2$$

$$\Rightarrow :H: = \int_{-\infty}^{+\infty} \left[(2\pi)^3 (C(E_{\vec{p}}))^2 2E_{\vec{p}}\right] E_{\vec{p}} \sum_s \left( b^{\dagger}_{\vec{p},s} b_{\vec{p},s} + d^{\dagger}_{\vec{p},s} d_{\vec{p},s} \right)$$

We also see the sign for the zero-point energy of the Dirac field is opposite of the one for scalar field.

So, in sum, the anticommutation relations are

$$\{\psi_a(t,\vec{x}),\ \psi_b^{\dagger}(t,\vec{y})\} = \delta_{ab}\,\delta^3(\vec{x}-\vec{y})$$

$$\{\psi_a(t,\vec{x}),\ \psi_b(t,\vec{y})\} = \{\psi_a^{\dagger}(t,\vec{x}),\ \psi_b^{\dagger}(t,\vec{y})\} = 0$$

$$\Rightarrow\ \{b_{\vec{p},s},\ b^{\dagger}_{\vec{k},r}\} = \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr}\,\delta^3(\vec{p}-\vec{k}) = \{d_{\vec{p},s},\ d^{\dagger}_{\vec{k},r}\}$$

$$0 = \{b_{\vec{p},s},\ b_{\vec{k},r}\} = \{b_{\vec{p},s},\ d_{\vec{k},r}\} = \{b_{\vec{p},s},\ d^{\dagger}_{\vec{k},r}\} = \{b^{\dagger}_{\vec{p},s},\ b^{\dagger}_{\vec{k},r}\}$$

$$= \{d_{\vec{p},s},\ d_{\vec{k},r}\} = \{d^{\dagger}_{\vec{p},s},\ d^{\dagger}_{\vec{k},r}\} = \{d_{\vec{p},s},\ b^{\dagger}_{\vec{k},r}\} = \{b^{\dagger}_{\vec{p},s},\ d^{\dagger}_{\vec{k},r}\}$$

For $\hat{\vec{P}}$, if write it as $\hat{\vec{P}} = -\int_{-\infty}^{+\infty} d^3\vec{x}\, i\psi^+\vec{\nabla}\psi$, then

$$\hat{\vec{P}} = -\int_{-\infty}^{+\infty} d^3\vec{x}\, i\psi^+\vec{\nabla}\psi$$

$$= (-i)\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}})\cdot(i\vec{k})$$

$$\times \sum_{s,r} \left(u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} + v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)\left(u(\vec{k},r)\, b_{\vec{k},r}\, e^{-ik\cdot x} - v(\vec{k},r)\, d^+_{\vec{k},r}\, e^{ik\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}})\, \vec{k}\, (2\pi)^3$$

$$\times \sum_{s,r}\Big( u^+(\vec{p},s)\, u(\vec{k},r)\, b^+_{\vec{p},s}\, b_{\vec{k},r}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) - v^+(\vec{p},s)\, v(\vec{k},r)\, d_{\vec{p},s}\, d^+_{\vec{k},r}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$- u^+(\vec{p},s)\, v(\vec{k},r)\, b^+_{\vec{p},s}\, d^+_{\vec{k},r}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k}) + v^+(\vec{p},s)\, u(\vec{k},r)\, d_{\vec{p},s}\, b_{\vec{k},r}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})\Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\, C(E_{\vec{p}})\, \vec{p}\, (2\pi)^3$$

$$\times \sum_{s,r}\left( 2E_{\vec{p}}\,\delta_{sr}\, b^+_{\vec{p},s}\, b_{\vec{p},r} - 2E_{\vec{p}}\,\delta_{sr}\, d_{\vec{p},s}\, d^+_{\vec{k},r}\right)$$

$$= \int_{-\infty}^{+\infty} \left[(C(E_{\vec{p}}))^2 (2\pi)^3\, 2E_{\vec{p}}\right] \sum_{s}\left( b_{\vec{p},s}\, b_{\vec{p},s} - d_{\vec{p},s}\, d^+_{\vec{p},s}\right)\vec{p}$$

If write it as. $\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{x}\, i(\vec{\nabla}\psi^+)\psi$, then

$$\hat{\vec{P}} = i\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}}).\ (-i\vec{p})$$

$$\times \sum_{s,r}\left( u^+(\vec{p},s)\, b^+_{\vec{p},s}\, e^{ip\cdot x} - v^+(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)\left(u(\vec{k},r)\, b_{\vec{k},r}\, e^{-ik\cdot x} + v(\vec{k},r)\, d^+_{\vec{k},r}\, e^{ik\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}})\, \vec{p}\, (2\pi)^3$$

$$\times \sum_{s,r}\Big( u^+(\vec{p},s)\, u(\vec{k},r)\, b^+_{\vec{p},s}\, b_{\vec{k},r}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) - v^+(\vec{p},s)\, v(\vec{k},r)\, d_{\vec{p},s}\, d^+_{\vec{k},r}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$+ u^+(\vec{p},s)\, v(\vec{k},r)\, b^+_{\vec{p},s}\, d^+_{\vec{k},r}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k}) - v^+(\vec{p},s)\, u(\vec{k},r)\, d_{\vec{p},s}\, b_{\vec{k},r}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, \vec{p}\, (2\pi)^3 \times \sum_{s,r}\left( 2E_{\vec{p}}\,\delta_{sr}\, b^+_{\vec{p},s}\, b_{\vec{p},r} - 2E_{\vec{p}}\,\delta_{sr}\, d_{\vec{p},s}\, d^+_{\vec{p},r}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2 (2\pi)^3\, 2E_{\vec{p}}\right]\sum_{s}\left( b^+_{\vec{p},s}\, b_{\vec{p},s} - d_{\vec{p},s}\, d^+_{\vec{p},s}\right)\vec{p}.$$

So $-\int_{-\infty}^{+\infty} d^3\vec{x}\, i\psi^\dagger \nabla\psi = \int_{-\infty}^{+\infty} d^3\vec{x}\, i(\vec{\nabla}\psi^\dagger)\psi = \int_{-\infty}^{+\infty} d^3\vec{x}\left(-\frac{i}{2}\psi^\dagger\vec{\nabla}\psi + \frac{i}{2}(\vec{\nabla}\psi^\dagger)\psi\right)$

So it does not matter which form of the classical $\vec{P}$ to use for the decomposition of $\hat{\vec{P}}$.

For $\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right]\sum_s \left(b^\dagger_{\vec{p},s} b_{\vec{p},s} - d_{\vec{p},s} d^\dagger_{\vec{p},s}\right)\vec{p}$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right]\sum_s \vec{p}\left(b^\dagger_{\vec{p},s} b_{\vec{p},s} + d^\dagger_{\vec{p},s} d_{\vec{p},s} - \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right]\vec{p}\sum_s \left(b^\dagger_{\vec{p},s} b_{\vec{p},s} + d^\dagger_{\vec{p},s} d_{\vec{p},s}\right)$$

$$- \underbrace{\int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}}_{=0}\, (2\pi)^3 \delta^3(0) \times 2$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right]\vec{p}\sum_s \left(b^\dagger_{\vec{p},s} b_{\vec{p},s} + d^\dagger_{\vec{p},s} d_{\vec{p},s}\right)$$

Therefore, again, there is no need to do normal ordering for $\hat{\vec{P}}$.

For $\hat{Q}$,

$$\hat{Q} = \int_{-\infty}^{+\infty} d^3\vec{x}\, \psi^\dagger\psi$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}}) \sum_{s,r}\left(u^\dagger(\vec{p},s) b^\dagger_{\vec{p},s} e^{ip\cdot x} + v^\dagger(\vec{p},s) d_{\vec{p},s} e^{-ip\cdot x}\right) \times \left(u(\vec{k},r) b_{\vec{k},r} e^{-ik\cdot x} + v(\vec{k},r) d^\dagger_{\vec{k},r} e^{ik\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})\, C(E_{\vec{k}})\, (2\pi)^3 \sum_{s,r}\Big(u^\dagger(\vec{p},s) u(\vec{k},r) b^\dagger_{\vec{p},s} b_{\vec{k},r} e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\delta^3(\vec{p}-\vec{k})$$
$$+ v^\dagger(\vec{p},s) v(\vec{k},r) d_{\vec{p},s} d^\dagger_{\vec{k},r} e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\delta^3(\vec{p}-\vec{k}) + u^\dagger(\vec{p},s) v(\vec{k},r) b^\dagger_{\vec{p},s} d^\dagger_{\vec{k},r} e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\delta^3(\vec{p}+\vec{k})$$
$$+ v^\dagger(\vec{p},s) u(\vec{k},r) d_{\vec{p},s} b_{\vec{k},r} e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\delta^3(\vec{p}+\vec{k})\Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,(C(E_{\vec{p}}))^2 (2\pi)^3 \sum_{s,r} \Big( u^\dagger(\vec{p},s)\, u(\vec{p},r)\, b^\dagger_{\vec{p},s} b_{\vec{p},r} + v^\dagger(\vec{p},s)\, v(\vec{p},r)\, d_{\vec{p},s} d^\dagger_{\vec{p},r}$$

$$+ u^\dagger(\vec{p},s)\, v(-\vec{p},r)\, b^\dagger_{\vec{p},s} d^\dagger_{-\vec{p},r} e^{2iE_{\vec{p}}t} + v^\dagger(\vec{p},s)\, u(-\vec{p},r)\, d_{\vec{p},s} b_{-\vec{p},r} e^{-2iE_{\vec{p}}t} \Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left[ (C(E_{\vec{p}}))^2 (2\pi)^3\, 2E_{\vec{p}} \right] \sum_s \left( b^\dagger_{\vec{p},s} b_{\vec{p},s} + d_{\vec{p},s} d^\dagger_{\vec{p},s} \right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left[ (C(E_{\vec{p}}))^2 (2\pi)^3\, 2E_{\vec{p}} \right] \sum_s \left( b^\dagger_{\vec{p},s} b_{\vec{p},s} - d^\dagger_{\vec{p},s} d_{\vec{p},s} \right)$$

$$+ \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} (2\pi)^3 \delta^3(0) \times 2$$

So, $$:\hat{Q}: = \int_{-\infty}^{+\infty} d^3\vec{p} \left[ (C(E_{\vec{p}}))^2 (2\pi)^3\, 2E_{\vec{p}} \right] \sum_s \left( b^\dagger_{\vec{p},s} b_{\vec{p},s} - d^\dagger_{\vec{p},s} d_{\vec{p},s} \right)$$

Note that $$: d_{\vec{p},s} d^\dagger_{\vec{p},s} : = - d^\dagger_{\vec{p},s} d_{\vec{p},s}$$

Define $\hat N \equiv \int_{-\infty}^{+\infty} d^3\vec p\, [(2\pi)^3 (C(E_{\vec p}))^2\, 2E_{\vec p}] \sum_s b^\dagger_{\vec p,s} b_{\vec p,s}$

$\Rightarrow \hat{\bar N} \equiv \int_{-\infty}^{+\infty} - - - - - - - - - - - - - - - d^\dagger_{\vec p,s} d_{\vec p,s}$

$$\Rightarrow [\hat N, b_{\vec k,r}] = \int_{-\infty}^{+\infty} d^3\vec p\,[\quad] \sum_s \big(\underbrace{b^\dagger_{\vec p,s} b_{\vec p,s} b_{\vec k,r} - b_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}}\big)$$

$$b^\dagger_{\vec p,s} b_{\vec p,s} b_{\vec k,r} - \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec p}}\left(\frac{1}{C(E_{\vec p})}\right)^2 \delta_{rs}\, \delta^3(\vec p - \vec k)\, b_{\vec p,s}$$

$$+ b^\dagger_{\vec p,s} b_{\vec k,r} b_{\vec p,s}$$

use $\{b_{\vec p,s}, b_{\vec k,r}\} = 0$

$\downarrow$

$= \quad - b_{\vec k,r}$

$$[\hat N, b^\dagger_{\vec k,r}] = \int_{-\infty}^{+\infty} d^3\vec p\,[\quad] \sum_s \big(\underbrace{b^\dagger_{\vec p,s} b_{\vec p,s} b^\dagger_{\vec k,r} - b^\dagger_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}}\big)$$

$$b^\dagger_{\vec p,s} \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec p}}\left(\frac{1}{C(E_{\vec p})}\right)^2 \delta_{sr}\, \delta^3(\vec p - \vec k)$$

$$- b^\dagger_{\vec p,s} b^\dagger_{\vec k,r} b_{\vec p,s} - b^\dagger_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}$$

use $\{b^\dagger_{\vec p,s}, b^\dagger_{\vec k,r}\} = 0$

$\downarrow$

$= \quad b^\dagger_{\vec k,r}$

$$[\hat N, d_{\vec k,r}] = \int_{-\infty}^{+\infty} d^3\vec p\,[\quad] \sum_s \big(\underbrace{b^\dagger_{\vec p,s} b_{\vec p,s} d_{\vec k,r}} - d_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}\big) = 0$$

$$\underbrace{- b^\dagger_{\vec p,s} d_{\vec k,r} b_{\vec p,s}}$$

$$+ d_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}$$

$$[\hat N, d^\dagger_{\vec k,r}] = \int_{-\infty}^{+\infty} d^3\vec p\,[\quad] \sum_s \big(\underbrace{b^\dagger_{\vec p,s} b_{\vec p,s} d^\dagger_{\vec k,r}}_{d^\dagger_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}} - d^\dagger_{\vec k,r} b^\dagger_{\vec p,s} b_{\vec p,s}\big) = 0$$

Similarly,

$$[\hat{\bar N}, d_{\vec k,r}] = - d_{\vec k,r}$$

$$[\hat{\bar N}, d^\dagger_{\vec k,r}] = d^\dagger_{\vec k,r}$$

$$[\hat{\bar N}, b_{\vec k,r}] = 0$$

$$[\hat{\bar N}, b^\dagger_{\vec k,r}] = 0$$

Also, $[\hat{\bar N}, \hat N] = 0$.

Again, let's define a common eigenstate of $\hat{N}$ and $\hat{\bar{N}}$ as $|t\rangle$ such that

$$\hat{N}|t\rangle = c|t\rangle$$
$$\hat{\bar{N}}|t\rangle = \bar{c}|t\rangle$$

where $c$ & $\bar{c}$ are real numbers (since $\hat{N}$ and $\hat{\bar{N}}$ are Hermitian operators)

$$\Rightarrow \quad \hat{N} b^{+}_{\vec{k},r}|t\rangle = (b^{+}_{\vec{k},r}\hat{N} + b^{+}_{\vec{k},r})|t\rangle = (c+1)\, b^{+}_{\vec{k},r}|t\rangle$$
$$\hat{N} b_{\vec{k},r}|t\rangle = (b_{\vec{k},r}\hat{N} - b_{\vec{k},r})|t\rangle = (c-1)\, b_{\vec{k},r}|t\rangle$$
$$\hat{\bar{N}} d^{+}_{\vec{k},r}|t\rangle = (\bar{c}+1)\, d^{+}_{\vec{k},r}|t\rangle$$
$$\hat{\bar{N}} d_{\vec{k},r}|t\rangle = (\bar{c}-1)\, d_{\vec{k},r}|t\rangle$$
$$\hat{N} d_{\vec{k},r}|t\rangle = c\, d_{\vec{k},r}|t\rangle$$
$$\hat{N} d^{+}_{\vec{k},r}|t\rangle = c\, d^{+}_{\vec{k},r}|t\rangle$$
$$\hat{\bar{N}} b_{\vec{k},r}|t\rangle = \bar{c}\, b_{\vec{k},r}|t\rangle$$
$$\hat{\bar{N}} b^{+}_{\vec{k},r}|t\rangle = \bar{c}\, b^{+}_{\vec{k},r}|t\rangle$$

Now, what about $(b^{+}_{\vec{k},r})^2|t\rangle$ ?

$$(b^{+}_{\vec{k},r})^2|t\rangle = b^{+}_{\vec{k},r} b^{+}_{\vec{k},r}|t\rangle \underset{\uparrow}{=} - b^{+}_{\vec{k},r} b^{+}_{\vec{k},r}|t\rangle$$

$\Rightarrow (b^{+}_{\vec{k},r})^2|t\rangle = 0$ since $\{b^{+}_{\vec{k},r}, b^{+}_{\vec{k},r}\} = 0$.

Similarly, $(d^{+}_{\vec{k},r})^2|t\rangle = 0$, since $\{d^{+}_{\vec{k},r}, d^{+}_{\vec{k},r}\} = 0$

that is, since $(b^{+}_{\vec{k},r})^2 = \frac{1}{2}\{b^{+}_{\vec{k},r}, b^{+}_{\vec{k},r}\} = \frac{1}{2}\times 0 = 0$, we have

$$(b^{+}_{\vec{k},r})^n = 0 \quad \text{for } n \geq 2$$

Similarly, we have $(d^{+}_{\vec{k},r})^n = 0$ for $n \geq 2$

Also, $(b_{\vec{k},r})^n = 0$ for $n \geq 2$,

$(d_{\vec{k},r})^n = 0$ for $n \geq 2$.

Define vaccum state $|0\rangle$, satisfying $b_{\vec{k},r}|0\rangle=0$, $d_{\vec{k},r}|0\rangle=0$ for any $\vec{k}$ and $r$.

$$\Rightarrow \quad \hat{N}|0\rangle = 0$$

$$\hat{\bar{N}}|0\rangle = 0$$

$$\hat{N} b^{+}_{\vec{k},r}|0\rangle = (b^{+}_{\vec{k},r}\hat{N} + b^{+}_{\vec{k},r})|0\rangle = b^{+}_{\vec{k},r}|0\rangle$$

$$\hat{\bar{N}} d^{+}_{\vec{k},r}|0\rangle = (d^{+}_{\vec{k},r}\hat{\bar{N}} + d^{+}_{\vec{k},r})|0\rangle = d^{+}_{\vec{k},r}|0\rangle$$

$$\hat{N} d^{+}_{\vec{k},r}|0\rangle = d^{+}_{\vec{k},r}\hat{N}|0\rangle = 0$$

$$\hat{\bar{N}} b^{+}_{\vec{k},r}|0\rangle = b^{+}_{\vec{k},r}\hat{\bar{N}}|0\rangle = 0$$

Since $\{b^{+}_{\vec{k}_1,r_1}, b^{+}_{\vec{k}_2,r_2}\} = 0$

then
$$b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}|0\rangle = -b^{+}_{\vec{k}_2,r_2} b^{+}_{\vec{k}_1,r_1}|0\rangle$$

while $\hat{N} b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}|0\rangle = (b^{+}_{\vec{k}_1}\hat{N} + b^{+}_{\vec{k}_1,r_1}) b^{+}_{\vec{k}_2,r_2}|0\rangle$

$$= [b^{+}_{\vec{k}_1,r_1}(b^{+}_{\vec{k}_2,r_2}\hat{N} + b^{+}_{\vec{k}_2,r_2}) + b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}]|0\rangle$$

$$= 2 b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}|0\rangle$$

and $\hat{N} b^{+}_{\vec{k}_2,r_2} b^{+}_{\vec{k}_1,r_1}|0\rangle = 2 b^{+}_{\vec{k}_2,r_2} b^{+}_{\vec{k}_1,r_1}|0\rangle$.

$$:H: b^{+}_{\vec{k},r}|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\,[(2\pi)^3 2E_{\vec{p}}(C(E_{\vec{p}}))^2] E_{\vec{p}} \sum_s (b^{+}_{\vec{p},s} b_{\vec{p},s} + d^{+}_{\vec{p},s} d_{\vec{p},s}) b^{+}_{\vec{k},r}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] \sum_s b^{+}_{\vec{p},s} \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr}\delta^3(\vec{p}-\vec{k})|0\rangle$$

$$= E_{\vec{k}} b^{+}_{\vec{k},r}|0\rangle$$

$$:H: b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}} \sum_s (b^{+}_{\vec{p},s} b_{\vec{p},s} + d^{+}_{\vec{p},s} d_{\vec{p},s}) b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}} \sum_s (b^{+}_{\vec{p},s} b^{+}_{\vec{k}_2,r_2} \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr_1}\delta^3(\vec{p}-\vec{k}_1) - b^{+}_{\vec{p},s} b^{+}_{\vec{k}_1,r_1} b_{\vec{p},s} b^{+}_{\vec{k}_2,r_2})|0\rangle$$

$$= E_{\vec{k}_1} b^{+}_{\vec{k}_1,r_1} b^{+}_{\vec{k}_2,r_2}|0\rangle - \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}} b^{+}_{\vec{p},s} b^{+}_{\vec{k}_1,r_1} \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr_2}\delta^3(\vec{p}-\vec{k}_2)|0\rangle$$

$$= E_{\vec{k}_1} b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}|0\rangle - E_{\vec{k}_2} b^\dagger_{\vec{k}_2,r_2} b^\dagger_{\vec{k}_1,r_1}|0\rangle$$

$$= (E_{\vec{k}_1}+E_{\vec{k}_2}) b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}|0\rangle$$

$$:H: b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}} \sum_s (b^\dagger_{\vec{p},s} b_{\vec{p},s}) b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}} \sum_s b^\dagger_{\vec{p},s}\left(\frac{1}{[\quad]}\delta_{sr_1}\delta^3(\vec{p}-\vec{k}_1) - b^\dagger_{\vec{k}_1,r_1} b_{\vec{p},s}\right) b^\dagger_{\vec{k}_2,r_2}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$= E_{\vec{k}_1} b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle - \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}}\sum_s b^\dagger_{\vec{p},s} b^\dagger_{\vec{k}_1,r_1}\Big(\frac{1}{[\quad]}\delta_{sr_2}\delta^3(\vec{p}-\vec{k}_2)$$
$$- b^\dagger_{\vec{k}_2,r_2} b_{\vec{p},s}\Big) b^\dagger_{\vec{k}_3,r_3}\cdots$$
$$b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$= (E_{\vec{k}_1}+E_{\vec{k}_2}) b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle + \int_{-\infty}^{+\infty} d^3\vec{p}\,[\quad] E_{\vec{p}}\sum_s b^\dagger_{\vec{p},s} b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}$$
$$\times\left(\frac{1}{[\quad]}\delta^3(\vec{p}-\vec{k}_3)\delta_{sr_3} - b^\dagger_{\vec{k}_3,r_3} b_{\vec{p},s}\right) b^\dagger_{\vec{k}_4,r_4}$$
$$\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$= (E_{\vec{k}_1}+E_{k_2}+\cdots+E_{\vec{k}_q}) b^\dagger_{\vec{k}_1,r_1} b^\dagger_{\vec{k}_2,r_2}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

Similarly,

$$:H: d^\dagger_{\vec{k}_1,r_1} d^\dagger_{\vec{k}_2,r_2}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle = (E_{\vec{k}_1}+E_{\vec{k}_2}+\cdots+E_{\vec{k}_q}) d^\dagger_{\vec{k}_1,r_1}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$\hat{\vec{P}} b^\dagger_{\vec{k}_1,r_1}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle = (\vec{k}_1+\vec{k}_2+\cdots+\vec{k}_q) b^\dagger_{\vec{k}_1,r_1}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$\hat{\vec{P}} d^\dagger_{\vec{k}_1,r_1}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle = (\vec{k}_1+\vec{k}_2+\cdots+\vec{k}_q) d^\dagger_{\vec{k}_1,r_1}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$\hat{N} b^\dagger_{\vec{k}_1,r_1}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle = q\, b^\dagger_{\vec{k}_1,r_1}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$\hat{\bar{N}} d^\dagger_{\vec{k}_1,r_1}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle = q\, d^\dagger_{\vec{k}_1,r_1}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle$$

$$\hat{N} d^\dagger_{\vec{k}_1,r_1}\cdots d^\dagger_{\vec{k}_q,r_q}|0\rangle = 0$$



$$\hat{\bar{N}} b^\dagger_{\vec{k}_1,r_1}\cdots b^\dagger_{\vec{k}_q,r_q}|0\rangle = 0$$

$$\hat{Q}\, b^{\dagger}_{\vec{k}_1,r_1}\cdots b^{\dagger}_{\vec{k}_\ell,r_\ell}|0\rangle = \ell\, b^{\dagger}_{\vec{k}_1,r_1}\cdots b^{\dagger}_{\vec{k}_\ell,r_\ell}|0\rangle$$

$$\hat{Q}\, d^{\dagger}_{\vec{k}_1,r_1}\cdots d^{\dagger}_{\vec{k}_\ell,r_\ell}|0\rangle = -\ell\, d^{\dagger}_{\vec{k}_1,r_1}\cdots d^{\dagger}_{\vec{k}_\ell,r_\ell}|0\rangle$$

$$:H:\ b^{\dagger}_{\vec{k}_1,r_1}\cdots b^{\dagger}_{\vec{k}_\ell,r_\ell} d^{\dagger}_{\vec{p}_1,s_1}\cdots d^{\dagger}_{\vec{p}_h,s_h}|0\rangle = (E_{\vec{k}_1}+E_{\vec{k}_2}+\cdots+E_{\vec{k}_\ell}+E_{\vec{p}_1}+E_{\vec{p}_2}+\cdots+E_{\vec{p}_h})$$
$$\times\, b^{\dagger}_{\vec{k}_1,r_1}\cdots b^{\dagger}_{\vec{k}_\ell,r_\ell} d^{\dagger}_{\vec{p}_1,s_1}\cdots d^{\dagger}_{\vec{p}_h,s_h}|0\rangle$$

$$\hat{\vec{P}}\ \text{- - - - - - - -}\ |0\rangle = (\vec{k}_1+\vec{k}_2+\cdots+\vec{k}_\ell+\vec{p}_1+\vec{p}_2+\cdots+\vec{p}_h)$$
$$\times\ \text{- - - - - - - -}\ |0\rangle$$

$$\hat{Q}\ \text{- - - - - - - -}\ |0\rangle = (\ell-h)\times\ \text{- - - - - - - -}\ |0\rangle$$

① Therefore, $b^{\dagger}_{\vec{k},r}|0\rangle$ is a one particle state with energy $E_{\vec{k}}$, momentum $\vec{k}$ and spin label $r$. $d^{\dagger}_{\vec{k},r}|0\rangle$ is a one anti-particle state with energy $E_{\vec{k}}$, momentum $\vec{k}$ and spin label $r$.

The state $b^{\dagger}_{\vec{k}_1,r_1}\cdots b^{\dagger}_{\vec{k}_\ell,r_\ell} d^{\dagger}_{\vec{p}_1,s_1}\cdots d^{\dagger}_{\vec{p}_h,s_h}|0\rangle$ is a multi-particle state.

② However,

$$b^{\dagger}_{\vec{k}_1,r_1} b^{\dagger}_{\vec{k}_2,r_2}|0\rangle = -b^{\dagger}_{\vec{k}_2,r_2} b^{\dagger}_{\vec{k}_1,r_1}|0\rangle,$$
$$d^{\dagger}_{\vec{k}_1,r_1} d^{\dagger}_{\vec{k}_2,r_2}|0\rangle = -d^{\dagger}_{\vec{k}_2,r_2} d^{\dagger}_{\vec{k}_1,r_1}|0\rangle,$$
$$b^{\dagger}_{\vec{k}_1,r_1} d^{\dagger}_{\vec{k}_2,r_2}|0\rangle = -d^{\dagger}_{\vec{k}_2,r_2} b^{\dagger}_{\vec{k}_1,r_1}|0\rangle,$$

in contrast to the scalar case, where the sign on the RHS is "+". This reflect the difference of Bose statistics and Fermi statistics.

Again, let's normalize the one-particle state

$|p,s\rangle = f(\vec{p})\, b^{+}_{\vec{p},s} |0\rangle$ by requiring

$\langle q,r|p,s\rangle = (2\pi)^3\, 2E_{\vec{p}}\, \delta_{rs}\, \delta^3(\vec{p}-\vec{q})$ ,

$\langle 0|0\rangle = 1$ ,

and $f(\vec{p})$ is real.

Since $\langle q,r| = f(\vec{q})\langle 0|\, b_{\vec{q},r}$

$$\Rightarrow \langle q,r|p,s\rangle = f(\vec{q}) f(\vec{p}) \langle 0| b_{\vec{q},r}\, b^{+}_{\vec{p},s}|0\rangle$$

$$= f(\vec{q}) f(\vec{p}) \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr}\,\delta^3(\vec{p}-\vec{q})$$

$$= \left(f(\vec{p})\right)^2 \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr}\,\delta^3(\vec{p}-\vec{q})$$

$$\overset{\text{require}}{=} (2\pi)^3\, 2E_{\vec{p}}\,\delta_{sr}\,\delta^3(\vec{p}-\vec{q})$$

$$\Rightarrow \quad f(\vec{p}) = C(E_{\vec{p}})\,(2\pi)^3\, 2E_{\vec{p}}$$

Similarly, one anti-particle state is normalized as

$|\bar{p},\bar{s}\rangle = \bar{f}(\vec{p})\, d^{+}_{\vec{p},\bar{s}}|0\rangle$

satisfying $\langle \bar{q},\bar{r}|\bar{p},\bar{s}\rangle = (2\pi)^3\, 2E_{\vec{p}}\, \delta_{\bar{s}\bar{r}}\,\delta^3(\vec{p}-\vec{q})$

$$\Rightarrow \quad \bar{f}(\vec{p}) = C(E_{\vec{p}})(2\pi)^3\, 2E_{\vec{p}}$$

$$\Rightarrow \langle 0|\psi(x)|k,r\rangle = \langle 0|\int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}}) \sum_s \left( u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x} + v(\vec{p},s)\, d^{+}_{\vec{p},s}\, e^{ip\cdot x}\right)$$
$$C(E_{\vec{k}})(2\pi)^3\, 2E_{\vec{k}}\, b^{+}_{\vec{k},r}|0\rangle$$

$$= \langle 0|\int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}})\, C(E_{\vec{k}})(2\pi)^3\, 2E_{\vec{k}}\, u(\vec{p},s)\, e^{-ip\cdot x}$$
$$\times \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta_{sr}\,\delta^3(\vec{p}-\vec{k})|0\rangle$$

$$= u(\vec{k},r)\, e^{-ik\cdot x}$$

$$\langle 0|\bar{\psi}(x)|\bar{k},\bar{r}\rangle = \langle 0|\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\sum_s \left(\bar{u}(\vec{p},s)\, b^{\dagger}_{\vec{p},s}\, e^{ip\cdot x} + \bar{v}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)$$

$$\times (2\pi)^3\, 2E_{\vec{k}}\, C(E_{\vec{k}})\, d^{\dagger}_{\vec{k},\bar{r}}|0\rangle$$

$$= \bar{v}(\vec{k},\bar{r})\, e^{-ik\cdot x}$$

$$\langle 0|\psi(x)|\bar{k},\bar{r}\rangle = 0$$

$$\langle 0|\bar{\psi}(x)|k,r\rangle = 0$$

$$\langle \bar{k},\bar{r}|\psi(x)|0\rangle = C(E_{\vec{k}})(2\pi)^3\, 2E_{\vec{k}}\, \langle 0|\, d_{\vec{k},\bar{r}} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\sum_s \left(u(\vec{p},s)\, b_{\vec{p},s}\, e^{-ip\cdot x}\right.$$

$$\left. + v(\vec{p},s)\, d^{\dagger}_{\vec{p},s}\, e^{ip\cdot x}\right)|0\rangle$$

$$= v(\vec{k},\bar{r})\, e^{ik\cdot x}$$

$$\langle k,r|\bar{\psi}(x)|0\rangle = C(E_{\vec{k}})(2\pi)^3\, 2E_{\vec{k}}\, \langle 0|\, b_{\vec{k},r} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}})\sum_s \left(\bar{u}(\vec{p},s)\, b^{\dagger}_{\vec{p},s}\, e^{ip\cdot x}\right.$$

$$\left. + \bar{v}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)|0\rangle$$

$$= \bar{u}(\vec{k},r)\, e^{ik\cdot x}$$

$$\langle k,r|\psi(x)|0\rangle = 0$$

$$\langle \bar{k},\bar{r}|\bar{\psi}(x)|0\rangle = 0$$

What about the canonical momentum of $\bar{\psi}$ ?

we can rewrite the Lagrangian as.

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

$$= \partial_\mu (\bar{\psi} i\gamma^\mu \psi) - (\partial_\mu \bar{\psi}) i\gamma^\mu \psi - m\bar{\psi}\psi$$

The first term is a total derivative. Adding a total derivative in the Lagrangian does not change the equation of motion.

check: for the term $\partial_\mu (\bar{\psi} i\gamma^\mu \psi) = (\partial_\mu \bar{\psi}) i\gamma^\mu \psi + \bar{\psi} i\gamma^\mu \partial_\mu \psi$

$$\frac{\partial \mathcal{L}}{\partial \psi} = (\partial_\mu \bar{\psi}) i\gamma^\mu, \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \partial_\mu \bar{\psi} i\gamma^\mu$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi, \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = i\gamma^\mu \partial_\mu \psi$$

So, we can drop the total derivative in the Lagrangian, provided $\psi$ and $\bar{\psi}$ are vanishing in the boundary, that is, $\int d^4x\, \partial_\mu (\bar{\psi} i\gamma^\mu \psi)$

$$= \int d^3s\, \bar{\psi} i\gamma^\mu \psi = 0$$

surface of integration boundary

$$\Rightarrow \quad \mathcal{L} = -(\partial_\mu \bar{\psi}) i\gamma^\mu \psi - m\bar{\psi}\psi$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial(\partial\bar{\psi}/\partial t)} = -i\gamma^0 \psi.$$

$$\Rightarrow \quad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \partial_0 \psi + \partial_0 \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} - \mathcal{L}$$

However, the equation of motion is derived as

$$\frac{\partial \mathcal{L}}{\partial \psi} = -(\partial_\mu \bar{\psi}) i\gamma^\mu - m\bar{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0$$

$$\Rightarrow i\, \partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = -m\psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -i\gamma^\mu \psi \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -i\gamma^\mu \partial_\mu \psi$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

extra 1.

So the equation of motion obtained is the same as obtained from the Lagrangian $\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$.

$$\Rightarrow \quad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \partial_0 \psi + \partial_0 \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} - \cancel{\mathcal{L}}$$

0 when $\psi$ satisfies the equation of motion.

$$= \quad 0 + \partial_0 \bar{\psi} (-i\gamma^0 \psi)$$

$$= \quad -i \dot{\psi}^\dagger \psi$$

$$= \quad (i\psi^\dagger \dot{\psi})^\dagger$$

$$\vec{P} = -\int d^3\vec{x} \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \vec{\nabla}\psi + \vec{\nabla}\bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} \right)$$

$$= -\int d^3\vec{x} \; \vec{\nabla}\bar{\psi} (-i\gamma^0 \psi)$$

$$= \int d^3\vec{x} \; i(\vec{\nabla}\bar{\psi})\gamma^0 \psi$$

$$= \int d^3\vec{x} \; i(\vec{\nabla}\psi^\dagger)\psi$$

$$= \int d^3\vec{x} \; (-i\psi^\dagger \vec{\nabla}\psi)^\dagger$$

$$Q = \int d^3\vec{x} \left[ \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}(-i\psi) + i\bar{\psi}\left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})}\right) \right]$$

$$= \int d^3\vec{x} \left[ i\bar{\psi}(-i\gamma^0 \psi) \right]$$

$$= \int d^3\vec{x} \; \psi^\dagger \psi$$

$$= \int d^3\vec{x} \; (\psi^\dagger \psi)^\dagger$$

$$= Q^\dagger$$

So if we want $\mathcal{L}$, $H$, $\vec{P}$ are Hermitian at this state, we can write

$$\mathcal{L} = \frac{1}{2}\bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{2}(\partial_\mu \bar{\psi}) i\gamma^\mu \psi - m\bar{\psi}\psi$$

So equation of motion is

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \psi} = -\frac{1}{2}(\partial_\mu \bar{\psi}) i\gamma^\mu - m\bar{\psi},$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{1}{2}\bar{\psi} i\gamma^\mu \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{1}{2}(\partial_\mu \bar{\psi}) i\gamma^\mu$$

$$\Rightarrow \quad (\partial_\mu \bar{\psi}) i\gamma^\mu + m\bar{\psi} = 0.$$ ✓ same as before.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{1}{2} i\gamma^\mu \partial_\mu \psi - m\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -\frac{1}{2} i\gamma^\mu \psi \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -\frac{1}{2} i\gamma^\mu \partial_\mu \psi$$

$$\Rightarrow \quad i\gamma^\mu \partial_\mu \psi - m\psi = 0,$$ ✓ same as before.

$$\mathcal{L}^\dagger = -\frac{i}{2}(\partial_\mu \psi^\dagger)\gamma^{\mu\dagger}\gamma^{0\dagger}\psi + \frac{i}{2}\psi^\dagger \gamma^{\mu\dagger}\gamma^{0\dagger}\partial_\mu \psi - m\psi^\dagger \gamma^{0\dagger}\psi$$

$$\underset{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0,\ (\gamma^0)^2 = 1}{=} -\frac{i}{2}(\partial_\mu \bar{\psi})\gamma^\mu \psi + \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

$$= \mathcal{L}$$ ✓

$$\frac{\partial \mathcal{L}}{\partial(\partial \psi/\partial t)} = \frac{i}{2}\bar{\psi}\gamma^0 = \frac{i}{2}\psi^\dagger, \quad \frac{\partial \mathcal{L}}{\partial(\partial \bar{\psi}/\partial t)} = -\frac{i}{2}\gamma^0 \psi$$

$$\Rightarrow \quad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}\partial_0 \psi + \partial_0 \bar{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} - \cancel{\mathcal{L}}$$ $\nearrow 0$ due to equation of motion

$$= \frac{i}{2}\psi^\dagger \dot{\psi} - \frac{i}{2}\dot{\psi}^\dagger \psi = \left(\frac{i}{2}\psi^\dagger \dot{\psi} - \frac{i}{2}\dot{\psi}^\dagger \psi\right)^\dagger = \mathcal{H}^\dagger$$ ✓

extra 3.

$$\vec{\mathcal{P}} = -\left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \vec{\nabla}\psi + \vec{\nabla}\bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})}\right)$$

$$= -\left(\frac{i}{2}\psi^\dagger \vec{\nabla}\psi - \frac{i}{2}(\vec{\nabla}\bar{\psi})\gamma^0 \psi\right)$$

$$= -\frac{i}{2}\psi^\dagger \vec{\nabla}\psi + \frac{i}{2}(\vec{\nabla}\psi^\dagger)\psi$$

$$= \left[-\frac{i}{2}\psi^\dagger \vec{\nabla}\psi + \frac{i}{2}(\vec{\nabla}\psi^\dagger)\psi\right]^\dagger$$

$$= \vec{\mathcal{P}}^\dagger \quad ✓$$

$$Q = \int d^3\vec{x}\, j^0 = \int d^3\vec{x} \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}(-i\psi) + i\bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})}\right)$$

$$= \int d^3\vec{x} \cdot \left(\frac{i}{2}\psi^\dagger(-i\psi) + i\bar{\psi}\left(-\frac{i}{2}\right)\gamma^0\psi\right)$$

$$= \int d^3\vec{x} \cdot (\psi^\dagger \psi)$$

since $(\psi^\dagger\psi)^\dagger = \psi^\dagger\psi$

$Q^\dagger = Q$ ✓

If we use $H = \int d^3\vec{x}\,(-i\dot{\psi}^{+}\psi)$, then

$$H = -i\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\; C(E_{\vec{p}})\, C(E_{\vec{k}})\, (iE_{\vec{p}})\,.$$

$$\sum_{s,r} \left(u^{+}(\vec{p},s)\, b^{+}_{\vec{p},s}\, e^{ip\cdot x} - v^{+}(\vec{p},s)\, d_{\vec{p},s}\, e^{-ip\cdot x}\right)$$

$$\left(u(\vec{k},r)\, b_{\vec{k},r}\, e^{-ik\cdot x} + v(\vec{k},r)\, d^{+}_{\vec{k},r}\, e^{ik\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\; C(E_{\vec{p}})\, C(E_{\vec{k}})\, E_{\vec{p}}\, (2\pi)^3$$

$$\times \sum_{s,r} \Big( u^{+}(\vec{p},s)\, u(\vec{k},r)\, b^{+}_{\vec{p},s} b_{\vec{k},r}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$- v^{+}(\vec{p},s)\, v(\vec{k},r)\, d_{\vec{p},s} d^{+}_{\vec{k},r}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$+ u^{+}(\vec{p},s)\, v(\vec{k},r)\, b^{+}_{\vec{p},s} d^{+}_{\vec{k},r}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$

$$- v^{+}(\vec{p},s)\, u(\vec{k},r)\, d_{\vec{p},s}\, b_{\vec{k},r}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k}) \Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, \left(C(E_{\vec{p}})\right)^2 E_{\vec{p}}\, (2\pi)^3 \sum_{s,r} \left(2E_{\vec{p}}\, \delta_{sr}\, b^{+}_{\vec{p},s} b_{\vec{p},r} - 2E_{\vec{p}}\, \delta_{sr}\, d_{\vec{p},s} d^{+}_{\vec{p},r}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left[\left(C(E_{\vec{p}})\right)^2 2E_{\vec{p}}\, (2\pi)^3\right] E_{\vec{p}} \sum_{s} \left(b^{+}_{\vec{p},s} b_{\vec{p},s} - d_{\vec{p},s} d^{+}_{\vec{p},s}\right)$$

the same as $\int d^3\vec{x}\; i\psi^{+}\dot{\psi}$

$$\Rightarrow \int d^3\vec{x}\,(i\psi^{+}\dot{\psi}) = \int d^3\vec{x}\,(-i\dot{\psi}^{+}\psi) = \int d^3\vec{x}\left(\frac{i}{2}\psi^{+}\dot{\psi} - \frac{i}{2}\dot{\psi}^{+}\psi\right)$$

So it does not matter which form of the classical $H$ to use for the decomposition of $H$.

Can we switch the order of $\psi^\dagger$ and $\psi$ in building $H$, $\hat{\vec{P}}$ and $\hat{Q}$?

If start $\mathcal{L} = \bar{\psi}\, i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi = \sum_{a,b=1}^{4} \bar{\psi}_a\, i\gamma^\mu_{ab}\, \partial_\mu \psi_b - \sum_{a=1}^{4} m\,\bar{\psi}_a \psi_a$

then we have

$$\frac{\partial \mathcal{L}}{\partial \psi_a} = -m\bar{\psi}_a, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} = \bar{\psi}_b\, i\gamma^\mu_{ba} \Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} = \bar{\psi}_b\, i\gamma^0_{ba}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a)} = 0 \qquad\qquad = i\psi^\dagger_a$$

So we could have written Hamiltonian, momentum and charge operators as

$$H = \int d^3\vec{x} \sum_{a=1}^{4} \left( \dot{\psi}_a\, i\psi^\dagger_a \right)$$

$$\hat{\vec{P}} = -\int d^3\vec{x} \sum_{a=1}^{4} \left( \vec{\nabla}\psi_a\, i\psi^\dagger_a \right)$$

$$\hat{Q} = \int d^3\vec{x} \sum_{a=1}^{4} (-i\psi_a)\, i\psi^\dagger_a = \int d^3\vec{x} \sum_{a=1}^{4} \left( \psi_a \psi^\dagger_a \right)$$

If start from $\mathcal{L} = -(\partial_\mu \bar{\psi})\, i\gamma^\mu \psi - m\bar{\psi}\psi$

$$= -\sum_{a,b=1}^{4} (\partial_\mu \bar{\psi}_a)\, i\gamma^\mu_{ab}\, \psi_b - \sum_{a=1}^{4} m\,\bar{\psi}_a \psi_a$$

then we have

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_a} = -m\psi_a, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a)} = -i\gamma^\mu_{ab}\psi_b \Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi}_a)} = -i\gamma^0_{ab}\psi_b$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} = 0$$

So we could have written

$$H = \int d^3\vec{x} \sum_{a,b=1}^{4} \left( -i\gamma^0_{ab}\psi_b\, \dot{\bar{\psi}}_a \right) = \int d^3\vec{x}\, (-i) \sum_{a=1}^{4} \left( \psi_a \dot{\psi}^\dagger_a \right)$$

$$\hat{\vec{P}} = -\int d^3\vec{x} \sum_{a,b=1}^{4} \left( -i\gamma^0_{ab}\psi_b\, \vec{\nabla}\bar{\psi}_a \right) = i\int d^3\vec{x} \sum_{a=1}^{4} \left( \psi_a \vec{\nabla}\psi^\dagger_a \right)$$

$$\hat{Q} = \int d^3\vec{x} \sum_{a,b=1}^{4} \left( -i\gamma^0_{ab}\psi_b\, i\bar{\psi}_a \right) = \int d^3\vec{x} \sum_{a=1}^{4} \left( \psi_a \psi^\dagger_a \right)$$

If start from $\mathcal{L} = \frac{1}{2}\bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{2}\partial_\mu \bar{\psi} i\gamma^\mu \psi - m\bar{\psi}\psi$,

$$= \frac{1}{2}\sum_{a,b=1}^{4}\left(\bar{\psi}_a i\gamma^\mu_{ab}\partial_\mu \psi_b - \partial_\mu \bar{\psi}_a i\gamma^\mu_{ab}\psi_b\right) - m\sum_{a=1}^{4}\bar{\psi}_a\psi_a$$

then we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} = \frac{i}{2}\psi_a^\dagger \,, \quad \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi}_a)} = -\frac{i}{2}\gamma^0_{ab}\psi_b$$

$$\Rightarrow \quad H = \int d^3\vec{x}\sum_{a=1}^{4}\left(\frac{i}{2}\psi_a^\dagger\dot{\psi}_a - \frac{i}{2}\dot{\psi}_a^\dagger\psi_a\right) = \int d^3\vec{x}\,\frac{i}{2}\left(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi\right)$$

or write it as

$$H = \int d^3\vec{x}\sum_{a=1}^{4}\left(\frac{i}{2}\dot{\psi}_a\psi_a^\dagger - \frac{i}{2}\dot{\psi}_a^\dagger\psi_a\right)$$

or write it as

$$H = \int d^3\vec{x}\sum_{a=1}^{4}\left(\frac{i}{2}\psi_a^\dagger\dot{\psi}_a - \frac{i}{2}\psi_a\dot{\psi}_a^\dagger\right)$$

or write it as

$$H = \int d^3\vec{x}\sum_{a=1}^{4}\left(\frac{i}{2}\dot{\psi}_a\psi_a^\dagger - \frac{i}{2}\psi_a\dot{\psi}_a^\dagger\right)$$

$$\hat{\vec{P}} = -\int d^3\vec{x}\,\frac{i}{2}\sum_{a=1}^{4}\left[(\vec{\nabla}\psi_a)\psi_a^\dagger - \psi_a\vec{\nabla}\psi_a^\dagger\right]$$

or write it as

$$\hat{\vec{P}} = -\int d^3\vec{x}\,\frac{i}{2}\sum_{a=1}^{4}\left[(\vec{\nabla}\psi_a)\psi_a^\dagger - (\vec{\nabla}\psi_a^\dagger)\psi_a\right]$$

or write it as

$$\hat{\vec{P}} = -\int d^3\vec{x}\,\frac{i}{2}\sum_{a=1}^{4}\left[\psi_a^\dagger\vec{\nabla}\psi_a - \psi_a\vec{\nabla}\psi_a^\dagger\right]$$

or write it as

$$\hat{\vec{P}} = -\int d^3\vec{x}\,\frac{i}{2}\sum_{a=1}^{4}\left[\psi_a^\dagger\vec{\nabla}\psi_a - (\vec{\nabla}\psi_a^\dagger)\psi_a\right]$$

$$\hat{Q} = \int d^3\vec{x}\sum_{a=1}^{4}\left(\frac{1}{2}\psi_a\psi_a^\dagger + \frac{1}{2}\psi_a^\dagger\psi_a\right)$$

or write it as

$$\hat{Q} = \int d^3\vec{x}\sum_{a=1}^{4}\left(\psi_a\psi_a^\dagger\right)$$

or write it as

$$\hat{Q} = \int d^3\vec{x}\sum_{a=1}^{4}(\psi_a^\dagger\psi_a) = \int d^3\vec{x}\,\psi^\dagger\psi$$

Let's calculate $i\int d^3\vec{x} \sum_{a=1}^{4} (\dot{\psi}_a \psi_a^+) \equiv **$

$$** = i\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})C(E_{\vec{k}})(-iE_{\vec{p}}).$$

$$\times \sum_{a=1}^{4}\sum_{s,r}\left(u_a(\vec{p},s)\, b_{\vec{p},s} e^{-ip\cdot x} - v_a(\vec{p},s)\, d^+_{\vec{p},s} e^{ip x}\right)\left(u_a^+(\vec{k},r)\, b^+_{\vec{k},r} e^{ik\cdot x} + v_a^+(\vec{k},r)\, d_{\vec{k},r} e^{-ik\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\, C(E_{\vec{p}})C(E_{\vec{k}})E_{\vec{p}}(2\pi)^3 .$$

$$\times \sum_{a=1}^{4}\sum_{s,r}\Big(u_a(\vec{p},s)u_a^+(\vec{k},r)\, b_{\vec{p},s} b^+_{\vec{k},r} e^{-iE_{\vec{p}}t+iE_{\vec{k}}t}\delta^3(\vec{p}-\vec{k}) - v_a(\vec{p},s)v_a^+(\vec{k},r)\, d^+_{\vec{p},s} d_{\vec{k},r} e^{iE_{\vec{p}}t-iE_{\vec{k}}t}\delta^3(\vec{p}-\vec{k})$$

$$+ u_a(\vec{p},s)v_a^+(\vec{k},r)\, b_{\vec{p},s} d_{\vec{k},r} e^{-iE_{\vec{p}}t-iE_{\vec{k}}t}\delta^3(\vec{p}+\vec{k}) - v_a(\vec{p},s)u_a^+(\vec{k},r)\, d^+_{\vec{p},s} b^+_{\vec{k},r} e^{iE_{\vec{p}}t+iE_{\vec{k}}t}\delta^3(\vec{p}+\vec{k})\Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2 E_{\vec{p}}(2\pi)^3 \times \sum_{a=1}^{4}\sum_{s,r}\Big(u_a(\vec{p},s)u_a^+(\vec{p},r)\, b_{\vec{p},s} b^+_{\vec{p},r} - v_a(\vec{p},s)v_a^+(\vec{p},r)\, d^+_{\vec{p},s} d_{\vec{p},r}$$

$$+ u_a(\vec{p},s)v_a^+(-\vec{p},r)\, b_{\vec{p},s} d_{-\vec{p},r} e^{-2iE_{\vec{p}}t} - v_a(\vec{p},s)u_a^+(-\vec{p},r)\, d^+_{\vec{p},s} b^+_{-\vec{p},r} e^{2iE_{\vec{p}}t}\Big)$$

Note that
$$\sum_{a=1}^{4} u_a(\vec{p},s)u_a^+(\vec{p},r) = u^+(\vec{p},r)u(\vec{p},s) = 2E_{\vec{p}}\delta_{rs}$$
$$\sum_{a=1}^{4} v_a(\vec{p},s)v_a^+(\vec{p},r) = v^+(\vec{p},r)v(\vec{p},s) = 2E_{\vec{p}}\delta_{rs}$$
$$\sum_{a=1}^{4} u_a(\vec{p},s)v_a^+(-\vec{p},r) = v^+(-\vec{p},r)u(\vec{p},s) = 0$$
$$\sum_{a=1}^{4} v_a(\vec{p},s)u_a^+(-\vec{p},r) = u^+(-\vec{p},r)v(\vec{p},s) = 0$$

$$\Rightarrow ** = \int_{-\infty}^{+\infty} d^3\vec{p}\left[(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}\right]E_{\vec{p}}\sum_s\left(b_{\vec{p},s}b^+_{\vec{p},s} - d^+_{\vec{p},s}d_{\vec{p},s}\right)$$

This raises a problem: even if we do $:**:$, we still cannot get positive definite energy.

The reason is that for the observables, e.g., $H$, $\hat{\vec{P}}$ and $\hat{Q}$, they should be constructed from bilinear terms $\psi^+ \mathcal{O} \psi$, where $\mathcal{O}$ are 4x4 matrices, and the derivative of $\psi^+$ and $\psi$ can appear as well.

So, we can not switch the order of $\psi$ and $\psi^+$ in building $H$, $\hat{\vec{P}}$ and $\hat{Q}$.