Solution 1

$m_e = 0.511\ \text{MeV}, \quad \alpha = \frac{1}{137}$

(1) $\tau = \frac{2}{m_e\alpha^5} = \frac{2}{0.511\times10^{-3}\ \text{GeV}\left(\frac{1}{137}\right)^5} \simeq \boxed{1.9\times10^{14}\ \text{GeV}^{-1}}$

(2)

$[\hbar] = \text{kg}\cdot\text{m}^2\cdot\text{sec}^{-1}, \quad [c] = \text{m}\cdot\text{sec}^{-1}$

To balance the dimension in SI units, we need, from $\tau = \frac{2}{m_e\alpha^5}\hbar^{n_1}c^{n_2}$

$$\text{sec} = \text{kg}^{-1}(\text{kg}\cdot\text{m}^2\,\text{sec}^{-1})^{n_1}(\text{m}\cdot\text{sec}^{-1})^{n_2}$$

$$\Rightarrow \begin{array}{ll} \text{for sec}: & 1 = -n_1 - n_2 \\ \text{for kg}: & 0 = -1 + n_1 \\ \text{for m}: & 0 = 2n_1 + n_2 \end{array} \Rightarrow \boxed{\begin{cases} n_1 = 1 \\ n_2 = -2 \end{cases}}$$

(3) use $\hbar = 1.0546\times10^{-34}\ \text{kg}\cdot\text{m}^2\,\text{sec}^{-1}$, $c = 2.9979\times10^{8}\ \text{m}\cdot\text{sec}^{-1}$,

$m_e = 9.1094\times10^{-31}\ \text{kg}$

$$\Rightarrow \tau = \frac{2}{m_e\alpha^5}\hbar c^{-2} = \frac{2}{(9.1094\times10^{-31})\left(\frac{1}{137}\right)^5}\times1.0546\times10^{-34}\times(2.9979\times10^8)^{-2}$$

$$\simeq \boxed{1.2\times10^{-10}\ \text{sec.}}$$

Alternatively, from $1\ \text{eV} = 1.6022\times10^{-19}\ \text{J} \Rightarrow 1\ \text{J} = (1.6022\times10^{-19})^{-1}\ \text{eV}$

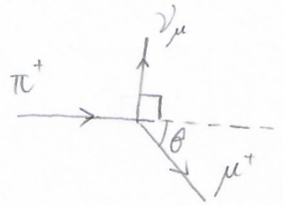
from $\hbar = 1 = 1.0546\times10^{-34}\ \text{J}\cdot\text{sec} \Rightarrow 1\ \text{sec} = (1.0546\times10^{-34}\ \text{J})^{-1}$
$= (1.0546\times10^{-34})^{-1}\times1.6022\times10^{-19}\ \text{eV}^{-1}$

$$\Rightarrow 1\ \text{GeV}^{-1} = (1\ \text{GeV})^{-1} = (10^9\ \text{eV})^{-1} = 10^{-9}\frac{1\ \text{sec}}{(1.0546\times10^{-34})^{-1}\times1.6022\times10^{-19}}$$

$$\Rightarrow \tau = \frac{2}{m_e\alpha^5} = \frac{2}{0.511\times10^{-3}\ \text{GeV}\left(\frac{1}{137}\right)^5} = \frac{2}{0.511\times10^{-3}\left(\frac{1}{137}\right)^5}\times10^{-9}\frac{1\ \text{sec}}{(1.0546\times10^{-34})^{-1}\times1.6022\times10^{-19}}$$

$$\simeq 1.2\times10^{-10}\ \text{sec}$$

solution 2



$$0 = P_{\nu_\mu}^2 = (P_\pi - P_\mu)^2 = m_\pi^2 + m_\mu^2 - 2P_\pi \cdot P_\mu$$

$$= m_\pi^2 + m_\mu^2 - 2E_\pi E_\mu + 2\vec{P}_\pi \cdot \vec{P}_\mu$$

$$= m_\pi^2 + m_\mu^2 - 2E_\pi E_\mu + 2|\vec{P}_\pi||\vec{P}_\mu| \cos\theta. \quad \cdots \text{①}$$

$$m_\mu^2 = P_\mu^2 = (P_\pi - P_{\nu_\mu})^2 = m_\pi^2 + 0 - 2P_\pi \cdot P_{\nu_\mu}$$

$$= m_\pi^2 - 2E_\pi E_{\nu_\mu}$$

use $E_{\nu_\mu} = E_\pi - E_\mu$

$\Rightarrow$

$$m_\mu^2 = m_\pi^2 - 2E_\pi (E_\pi - E_\mu) = m_\pi^2 - 2E_\pi^2 + 2E_\pi E_\mu \quad \cdots \text{②}$$

$$\Rightarrow \quad E_\mu = \frac{2E_\pi^2 + m_\mu^2 - m_\pi^2}{2E_\pi}$$

Also, $|\vec{P}_\mu| = (E_\mu^2 - m_\mu^2)^{\frac{1}{2}}$

$E_\pi = m_\pi \gamma$, where $\gamma = (1-v^2)^{-\frac{1}{2}}$,

$|\vec{P}_\pi| = v E_\pi = m_\pi \gamma v$

① + ② $\Rightarrow$

$$0 + m_\mu^2 = 2m_\pi^2 + m_\mu^2 - 2E_\pi^2 + 2|\vec{P}_\pi||\vec{P}_\mu| \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{E_\pi^2 - m_\pi^2}{|\vec{P}_\pi||\vec{P}_\mu|} = \frac{|\vec{P}_\pi|^2}{|\vec{P}_\pi||\vec{P}_\mu|} = \frac{|\vec{P}_\pi|}{|\vec{P}_\mu|}$$

$$= \frac{|\vec{P}_\pi|}{\left[\left(\frac{2E_\pi^2 + m_\mu^2 - m_\pi^2}{2E_\pi}\right)^2 - m_\mu^2\right]^{\frac{1}{2}}} = \frac{|\vec{P}_\pi| \, 2E_\pi}{\left[4E_\pi^4 + m_\mu^4 + m_\pi^4 - 4E_\pi^2 m_\pi^2 - 2m_\mu^2 m_\pi^2\right]^{\frac{1}{2}}}$$

$$= \frac{2|\vec{P}_\pi| E_\pi}{\left[4|\vec{P}_\pi|^2 E_\pi^2 + (m_\pi^2 - m_\mu^2)^2\right]^{\frac{1}{2}}}$$

$$\Rightarrow \boxed{\tan\theta =} \frac{m_\pi^2 - m_\mu^2}{2|\vec{P}_\pi| E_\pi} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi^2 \gamma^2 v} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi^2} \frac{1-v^2}{v} = \boxed{\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \frac{1-v^2}{2v}}$$

Alternatively,

from conservation of momentum,

$$\vec{P}_\pi = \vec{P}_\mu + \vec{P}_{\nu_\mu}$$

do dot product $\vec{P}_\pi$ on both sides,

$$\Rightarrow \quad \vec{P}_\pi \cdot \vec{P}_\pi = \vec{P}_\pi \cdot (\vec{P}_\mu + \vec{P}_{\nu_\mu})$$

$$\Rightarrow \quad |\vec{P}_\pi|^2 = |\vec{P}_\pi||\vec{P}_\mu| \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{|\vec{P}_\pi|}{|\vec{P}_\mu|}$$

$$\Rightarrow \quad \tan\theta = \frac{(|\vec{P}_\mu|^2 - |\vec{P}_\pi|^2)^{\frac{1}{2}}}{|\vec{P}_\pi|}$$

From $\vec{P}_\pi - \vec{P}_{\nu_\mu} = \vec{P}_\mu$

$$\Rightarrow \quad (\vec{P}_\pi - \vec{P}_{\nu_\mu}) \cdot (\vec{P}_\pi - \vec{P}_{\nu_\mu}) = \vec{P}_\mu \cdot \vec{P}_\mu$$

$$\Rightarrow \quad |\vec{P}_\pi|^2 + |\vec{P}_{\nu_\mu}|^2 - 2\vec{P}_\pi \cdot \vec{P}_{\nu_\mu} = |\vec{P}_\mu|^2$$

$$\Rightarrow \quad |\vec{P}_\pi|^2 + |\vec{P}_{\nu_\mu}|^2 = |\vec{P}_\mu|^2$$

Therefore, $\tan\theta = \dfrac{|\vec{P}_{\nu_\mu}|}{|\vec{P}_\pi|} = \dfrac{E_{\nu_\mu}}{|\vec{P}_\pi|}$

From $m_\mu^2 = P_\mu^2 = (P_\pi - P_{\nu_\mu})^2 = m_\pi^2 + 0 - 2P_\pi \cdot P_{\nu_\mu} = m_\pi^2 - 2E_\pi E_{\nu_\mu}$

$$\Rightarrow \quad E_{\nu_\mu} = \frac{m_\pi^2 - m_\mu^2}{2E_\pi}$$

Finally, use $E_\pi = m\gamma$, $|\vec{P}_\pi| = vE_\pi$

$$\Rightarrow \quad \boxed{\tan\theta} = \frac{\frac{m_\pi^2 - m_\mu^2}{2E_\pi}}{vE_\pi} = \frac{m_\pi^2 - m_\mu^2}{2v\, m_\pi^2 \gamma^2} = \boxed{\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\frac{1-v^2}{2v}}$$

solution 3

$$a^\mu = (2, 0, -1, 9), \quad b^\mu = (0, -9, 2, 5)$$

$$\Rightarrow \quad a_\mu = g_{\mu\nu} a^\nu = (2, 0, 1, -9),$$

$$b_\mu = g_{\mu\nu} b^\nu = (0, 9, -2, -5),$$

$$\vec{a}\cdot\vec{a} = 0^2 + (-1)^2 + 9^2 = 82$$

$$\vec{a}\cdot\vec{b} = 0\times(-9) + (-1)\times 2 + 9\times 5 = 43.$$

$$a^2 = (a^0)^2 - \vec{a}\cdot\vec{a} = 2^2 - 82 = -78$$

$$a\cdot b = a^0 b^0 - \vec{a}\cdot\vec{b} = 2\times 0 - 43 = -43$$

$$(a+b)^2 = a^2 + b^2 + 2a\cdot b = -78 + \left(0^2 - (-9)^2 - 2^2 - 5^2\right) + 2\times(-43)$$

$$= -78 - 110 - 86$$

$$= -274$$

Alternatively, $(a+b)^\mu = (2+0, 0-9, -1+2, 9+5)$

$$= (2, -9, 1, 14)$$

$$\Rightarrow \quad (a+b)^2 = 2^2 - [(-9)^2 + 1^2 + 14^2]$$

$$= 4 - (81 + 1 + 196)$$

$$= -274$$

Solution 4

1)
$$s+t+u = (P_A+P_B)^2 + (P_A-P_C)^2 + (P_A-P_D)^2$$

$$= P_A^2 + P_B^2 + P_A^2 + P_C^2 + P_A^2 + P_D^2 + 2P_A\cdot P_B - 2P_A\cdot P_C - 2P_A\cdot P_D$$

$$= P_A^2 + P_B^2 + P_C^2 + P_D^2 + 2P_A\cdot \underbrace{(P_A+P_B-P_C-P_D)}_{0}$$

$$= m_A^2 + m_B^2 + m_C^2 + m_D^2$$

2)

use $\vec{P}_A = -\vec{P}_B$ in CM frame

$$S = (P_A+P_B)^2 = m_A^2 + m_B^2 + 2P_A\cdot P_B = m_A^2 + m_B^2 + 2E_AE_B + 2|\vec{P}_A|^2$$

$$= m_A^2 + m_B^2 + 2E_AE_B + 2(E_A^2 - m_A^2) = 2E_A(E_A+E_B) + m_B^2 - m_A^2$$

Also, $S = (P_A+P_B)^2 = (E_A+E_B)^2 - |\vec{P}_A+\vec{P}_B|^2 = (E_A+E_B)^2$

$$\Rightarrow \quad E_A + E_B = \sqrt{S}$$

$$\Rightarrow \quad \boxed{E_A = \frac{S+m_A^2-m_B^2}{2(E_A+E_B)} = \frac{S+m_A^2-m_B^2}{2\sqrt{S}}}$$

3)

$$S = (P_A+P_B)^2 = m_A^2 + m_B^2 + 2P_A\cdot P_B = m_A^2 + m_B^2 + 2E_A m_B$$

$$\Rightarrow \quad \boxed{E_A = \frac{S-m_A^2-m_B^2}{2m_B}}$$

Solution 5

$$e^+ + e^- \rightarrow \gamma$$

For the initial state, a reference frame can be found in which the three-momenta sum of the e^+ and e^- is zero, i.e., the CM frame. However, in this CM frame, the single photon in the final state cannot have zero three-momentum. Therefore, the conservation of momentum is violated.

Another way to see this is that the final state photon has $P_\gamma^2 = 0$, while in the initial state $(P_{e^+} + P_{e^-})^2 \geq (2m_e)^2 > 0$

$\Rightarrow \quad P_\gamma^2 \neq (P_{e^+} + P_{e^-})^2$, and therefore the process cannot happen.