Let's look at the gauge transformation further.

$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$.

We know that in classical field theory, the Lagrangian for electromagnetic field with source is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = -j^\nu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu(-F^{\mu\nu}) + j^\nu = 0$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = j^\nu.$$

This gives the two Maxwell equations $\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \end{cases}$

The other two Maxwell equations are given by $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

$$\Downarrow$$

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E} = 0. \end{cases}$$

where $j^\mu = (\rho, \vec{j})$

We also know that for Dirac field, the Noether current from the internal phase transformation, $\psi \to \psi' = e^{-i\delta\alpha}\psi$, is

$$j^\mu_{Dirac} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}(-i\psi) + i\bar{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = \bar{\psi}\gamma^\mu\psi, \text{ where } \mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi.$$

and we know that $\int d^3\vec{x}\, :j^0_{Dirac}: = :\hat{Q}: = \int d^3\vec{p}\, [(C(E_{\vec{p}}))^2(2\pi)^3 2E_{\vec{p}}] \sum_s (b^\dagger_{\vec{p},s}b_{\vec{p},s} - d^\dagger_{\vec{p},s}d_{\vec{p},s})$, after quantization.

We note the above $:\hat{Q}:$ is nothing but $(\hat{N} - \hat{\bar{N}})$, that is, we have not specified what the charge is (electric charge, baryon number, lepton number ect.), and it only means that the charges for particle and anti-particle are opposite.

Therefore, by multipling the electric charge to $\bar{\psi}\gamma^\mu\psi$ for the particle described by the ψ field, we get the electromagnetic current.

Since historically we call the electron as particle, and positron as anti-particle, then when ψ describes the electron-positron field, the multiplication factor to $\bar{\psi}\gamma^\mu\psi$ is $-|e|$, where $|e|$ is the absolute value of the electron or positron charge, with value 1.6×10^{-19} Coulomb $\doteq 0.3$ in $\hbar = c = \varepsilon_0 = 1$ units (recell that the fine structure constant, which is dimensionless in any unit system, is defined as $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \doteq \frac{1}{137}$).

That is, the multiplication factor is always $\varepsilon|e|$, where ε is the charge of the particle (not anti-particle) described by the field ψ. For example, $\varepsilon = -1$ when ψ describes the electron-positron field, $\varepsilon = +1$ when ψ describes the proton-antiproton field, $\varepsilon = +\frac{2}{3}$ when ψ describes the up-quark anti-up-quark field, $\varepsilon = -\frac{1}{3}$ when ψ describes the down-quark anti-down-quark field.

Therefore, the j^ν appears in $\partial_\mu F^{\mu\nu} = j^\nu$ is

$$j^\nu = \varepsilon|e|\,\bar{\psi}\gamma^\nu\psi = \varepsilon|e|\,j^\nu_{Dirac}$$

and the Lagrangian for the Dirac fermion + electromagnetic fields system is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \varepsilon|e|\,\bar{\psi}\gamma^\mu\psi A_\mu + \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

However, we immediately encounter a problem for this Lagrangian: for gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda(x)$, the term $-\varepsilon|e|\bar{\psi}\gamma^\mu\psi A_\mu \to -\varepsilon|e|\bar{\psi}\gamma^\mu\psi A_\mu - \varepsilon|e|\bar{\psi}\gamma^\mu\psi\,\partial_\mu\Lambda(x)$.

Therefore, if the field ψ do nothing when making a gauge transformation for A_μ, the Lagrangian $\mathcal{L}$ changes, so that gauge symmetry would not be a symmetry of the ψ & A_μ system, so that we would lose the benefit to use gauge symmetry to reduce the polarization degrees of freedom for the electromagnetic field from four to two.

Clearly, a global phase transformation for ψ won't help, since for $\psi \to \psi' = e^{-ic}\psi$ where c is a constant, $\bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{+ic}$, so that $\bar{\psi}\gamma^\mu\psi \to \bar{\psi}'\gamma^\mu\psi' = \bar{\psi}e^{ic}\gamma^\mu e^{-ic}\psi = \bar{\psi}\gamma^\mu\psi$

and $\bar{\psi}\psi \to \bar{\psi}'\psi' = \psi\psi$, $\bar{\psi}i\gamma^\mu\partial_\mu\psi \to \bar{\psi}'i\gamma^\mu\partial_\mu\bar{\psi}' = \bar{\psi}e^{ic}i\gamma^\mu\partial_\mu e^{-ic}\psi$

$$= \bar{\psi}e^{ic}i\gamma^\mu e^{-ic}\partial_\mu\psi$$
$$= \bar{\psi}i\gamma^\mu\partial_\mu\psi.$$

However, if c depends on space-time coordinates, then

$$\partial_\mu(e^{-ic(x)}\psi) = -i(\partial_\mu c(x))\,e^{-ic(x)}\psi + e^{-ic(x)}\partial_\mu\psi$$

and therefore, by choosing $c(x) = \varepsilon|e|\Lambda(x)$, we get.

$$-\varepsilon|e|\bar{\psi}\gamma^\mu\psi A_\mu + \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

$$\to -\varepsilon|e|\bar{\psi}'\gamma^\mu\psi' A'_\mu + \bar{\psi}'i\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi'$$

$$= -\varepsilon|e|\bar{\psi}e^{i\varepsilon|e|\Lambda(x)}\gamma^\mu e^{-i\varepsilon|e|\Lambda(x)}\psi(A_\mu + \partial_\mu\Lambda(x))$$
$$+ \bar{\psi}e^{i\varepsilon|e|\Lambda(x)}i\gamma^\mu\partial_\mu(e^{-i\varepsilon|e|\Lambda(x)}\psi) - m\bar{\psi}e^{i\varepsilon|e|\Lambda(x)}e^{-i\varepsilon|e|\Lambda(x)}\psi$$

$$= -\varepsilon|e|\bar{\psi}\gamma^\mu\psi A_\mu - \varepsilon|e|\bar{\psi}\gamma^\mu\psi(\partial_\mu\Lambda(x))$$
$$+ \bar{\psi}i\gamma^\mu\partial_\mu\psi + \varepsilon|e|\partial_\mu\Lambda(x)\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi$$

$$= -\varepsilon|e|\bar{\psi}\gamma^\mu\psi A_\mu + \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi.$$

So the Lagrangian $\mathcal{L}$ is unchanged under the simultaneous transformation

$$\begin{cases} A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \\ \psi(x) \to \psi'(x) = e^{-iq|e|\Lambda(x)} \psi(x) \end{cases}$$

We can then introduce the covariant derivative

$$D_\mu \equiv \partial_\mu + i q|e| A_\mu$$

$$\Rightarrow \quad \bar{\psi} i\gamma^\mu D_\mu \psi = \bar{\psi} i\gamma^\mu (\partial_\mu + iq|e|A_\mu)\psi = \bar{\psi} i\gamma^\mu \psi - q|e|\bar{\psi}\gamma^\mu \psi A_\mu$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i\gamma^\mu D_\mu \psi - m\bar{\psi}\psi.$$

This is the QED Lagrangian, which describes the electromagnetic field and a Dirac field with the Dirac particle (not anti-particle) charge q.

To include, for example, both electron-positron and proton-antiproton, we just write

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_e i\gamma^\mu D_\mu \psi_e - m\bar{\psi}_e \psi_e + \bar{\psi}_p i\gamma^\mu D_\mu \psi_p - m\bar{\psi}_p \psi_p$$

where
$$D_\mu \psi_e = \partial_\mu \psi_e - i|e| A_\mu \psi_e$$
$$D_\mu \psi_p = \partial_\mu \psi_p + i|e| A_\mu \psi_p.$$

We can similarly write the Lagrangian describes the electromagnetic field and a complex scalar field with the particle charge q.

Again, the free field Lagrangian for a complex scalar field

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^*$$

gives

$$j^\mu_{\text{scalar}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-i\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} i\phi^*$$

$$= \partial^\mu \phi^* (-i\phi) + \partial^\mu \phi (i\phi^*)$$

$$= i[\phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi]$$

and

$$:\hat{Q}: = \int d^3\vec{x} :j^0_{\text{scalar}}: = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 \, 2E_{\vec{p}} \left(C(E_{\vec{p}}) \right)^2 \right] (a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}})$$

$$= \hat{N} - \hat{\bar{N}}$$

So the electromagnetic current in the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

is

$$j^\mu = q|e| \, i[\phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi] = q|e| \, j^\mu_{\text{scalar}}$$

where q is the electric charge of the particle (not anti-particle) described by the field ϕ & ϕ^*.

We expect the same covariant derivative,

$$D_\mu \phi = \partial_\mu \phi + i q |e| A_\mu \phi$$

Should make the Lagrangian of the eletromagnetic field + complex scalar field unchanged under the gauge transformation

$$\begin{cases} A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \\ \phi(x) \longrightarrow \phi'(x) = e^{-iq|e|\Lambda(x)} \phi(x). \end{cases}$$

check:

$$\mathcal{L}' = \partial_\mu \phi' \partial^\mu \phi^{*\prime} - m^2 \phi' \phi^{*\prime} - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu}$$
$$- i q |e| [\phi^{*\prime} \partial^\mu \phi' - (\partial^\mu \phi'^*) \phi'] A'_\mu$$

$$= \partial_\mu (e^{-iq|e|\Lambda(x)} \phi(x)) \, \partial^\mu (e^{iq|e|\Lambda(x)} \phi^*(x))$$
$$- m^2 \phi \phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
$$- i q |e| [\phi^* e^{iq|e|\Lambda(x)} \partial^\mu (e^{-iq|e|\Lambda(x)} \phi(x))$$
$$- (\partial^\mu (\phi^*(x) e^{iq|e|\Lambda(x)})) (e^{-iq|e|\Lambda(x)} \phi(x))]$$
$$(A_\mu(x) + \partial_\mu \Lambda(x))$$

$$= [(-iq|e| \partial_\mu \Lambda) e^{-iq|e|\Lambda} \phi + e^{-iq|e|\Lambda} \partial_\mu \phi]$$
$$[(iq|e| \partial^\mu \Lambda) e^{iq|e|\Lambda} \phi^* + e^{iq|e|\Lambda} \partial^\mu \phi^*]$$

$$- m^2 \phi \phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
$$- i q |e| \{ \phi^* e^{iq|e|\Lambda} e^{-iq|e|\Lambda} [(-iq|e| \partial^\mu \Lambda) \phi + \partial^\mu \phi]$$
$$- [\partial^\mu \phi^* + (iq|e| \partial^\mu \Lambda) \phi^*] e^{iq|e|\Lambda} e^{-iq|e|\Lambda} \phi \}$$
$$(A_\mu + \partial_\mu \Lambda)$$

$$= \{ \underline{(-iq|e| \partial_\mu \Lambda)(iq|e| \partial^\mu \Lambda) \phi \phi^*} - (iq|e| \partial_\mu \Lambda) \phi \partial^\mu \phi^*$$
$$+ (\partial_\mu \phi) \phi^* (iq|e| \partial^\mu \Lambda) + (\partial_\mu \phi)(\partial^\mu \phi^*) \}$$

$$- m^2 \phi \phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
$$- i q |e| \{ (-iq|e| \partial^\mu \Lambda) \phi^* \phi + \phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi - (iq|e| \partial^\mu \Lambda) \phi^* \phi \}$$
$$A_\mu$$
$$- i q |e| (\partial_\mu \Lambda) \{ \underline{(-iq|e| \partial^\mu \Lambda) \phi^* \phi} + \phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi - (iq|e| \partial^\mu \Lambda) \phi^* \phi \}$$

$$= (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$- i\varepsilon|e|\left[\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi\right]A_\mu$$
$$- i\varepsilon|e|(-2i\varepsilon|e|\,\partial^\mu\Lambda)A_\mu\,\phi^*\phi$$
$$+(i\varepsilon|e|)^2(\partial_\mu\Lambda)(\partial^\mu\Lambda)\,\phi^*\phi$$

$$= \mathcal{L} - i\varepsilon|e|(-2i\varepsilon|e|\,\partial^\mu\Lambda)A_\mu\phi^*\phi + (i\varepsilon|e|)^2(\partial_\mu\Lambda)(\partial^\mu\Lambda)\phi^*\phi$$
$$= \mathcal{L} - 2(\varepsilon|e|)^2(\partial^\mu\Lambda)A_\mu\phi^*\phi - (\varepsilon|e|)^2(\partial_\mu\Lambda)(\partial^\mu\Lambda)\phi^*\phi$$

We actually miss a term $(\varepsilon|e|)^2 A_\mu A^\mu\phi^*\phi$ in the Lagrangian.

This term gives

$$(\varepsilon|e|)^2 A'_\mu A'^\mu\,\phi'^*\phi'$$
$$= (\varepsilon|e|)^2(A_\mu + \partial_\mu\Lambda)(A^\mu + \partial^\mu\Lambda)\,\phi^*\phi$$
$$= (\varepsilon|e|)^2 A_\mu A^\mu\phi^*\phi + 2(\varepsilon|e|)^2 A^\mu(\partial_\mu\Lambda)\phi^*\phi$$
$$+ (\varepsilon|e|)^2(\partial^\mu\Lambda)(\partial_\mu\Lambda)\,\phi^*\phi$$

Therefore, the Lagrangian

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$- i\varepsilon|e|\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right]A_\mu$$
$$+ (\varepsilon|e|)^2 A_\mu A^\mu\phi^*\phi$$

is unchanged under gauge transformation

$$\begin{cases} A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \\ \phi(x) \to \phi'(x) = e^{-i\varepsilon|e|\Lambda(x)}\phi(x) \\ \phi^*(x) \to \phi'^*(x) = e^{i\varepsilon|e|\Lambda(x)}\phi^*(x) \end{cases}$$

In fact, we can write it as

$$\mathcal{L} = D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where
$$D_\mu\phi^* = \partial_\mu\phi^* - i\varepsilon|e|A_\mu\phi^*$$
$$D^\mu\phi = \partial^\mu\phi + i\varepsilon|e|A^\mu\phi$$

(check: $D_\mu\phi^* D^\mu\phi = (\partial_\mu\phi^* - i\varepsilon|e|A_\mu\phi^*)(\partial^\mu\phi + i\varepsilon|e|A^\mu\phi) = (\partial_\mu\phi^*)(\partial^\mu\phi) - i\varepsilon|e|A_\mu\phi^*\partial^\mu\phi + i\varepsilon|e|A^\mu(\partial_\mu\phi^*)\phi + (\varepsilon|e|)^2A_\mu A^\mu\phi^*\phi$.)