Final Exam
Due date: 2019.12.30

Problem 1. The "magic" number that can give the correct WIMP dark matter relic abundance is $3 \times 10^{-26}\text{cm}^3/\text{sec}$, which is the product of the annihilation cross section and the relative speed between two annihilating dark matter particles, i.e, σv.
1) In $c = 1$ units, what is this number in cm^2? [1 point]
2) In $\hbar = c = 1$ units, what is this number in GeV^{-2}? [1 point]
NOTE: Please round your results to one significant figure, and don't worry about uncertainties.

Problem 2. [4 points] Calculate

$$\bar{u}(\vec{p}_2, s_2)\left(\frac{-\not{p}_1 + \not{p}_2 + m}{(p_1 - p_2)^2 - m^2}\right)\not{p}_4 v(\vec{p}_3, s_3) + \bar{u}(\vec{p}_2, s_2)\not{p}_4\left(\frac{\not{p}_1 - \not{p}_3 + m}{(p_1 - p_3)^2 - m^2}\right)v(\vec{p}_3, s_3)\,,$$

where

$$(\not{p}_2 - m)u(\vec{p}_2, s_2) = 0\,, \quad (\not{p}_3 + m)v(\vec{p}_3, s_3) = 0\,,$$
$$p_1^\mu = p_2^\mu + p_3^\mu + p_4^\mu\,, \quad p_2^2 = p_3^2 = m^2\,, \quad p_4^2 = 0\,.$$

NOTE: You can use all the properties of the γ-matrices that you have proved in the problem 1 in homework 3, and you don't have to re-do the proofs here.

Problem 3. For two real scalar fields, σ and ϕ, decomposed as

$$\begin{aligned}
\sigma(x) &= \int_{-\infty}^{+\infty} C(E_{\vec{p}})(a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x})d^3\vec{p}, \\
\phi(x) &= \int_{-\infty}^{+\infty} C(E_{\vec{p}})(b_{\vec{p}}e^{-ip\cdot x} + b_{\vec{p}}^\dagger e^{ip\cdot x})d^3\vec{p},
\end{aligned}$$

for which the commutation relations are

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p} - \vec{p}')\ ,$$
$$[b(\vec{p}), b^\dagger(\vec{p}')] = \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p} - \vec{p}')\ ,$$
$$[a(\vec{p}), a(\vec{p}')] = [b(\vec{p}), b(\vec{p}')] = [a(\vec{p}), b^\dagger(\vec{p}')] = [b(\vec{p}), a^\dagger(\vec{p}')] = 0\ ,$$
$$[a^\dagger(\vec{p}), a^\dagger(\vec{p}')] = [b^\dagger(\vec{p}), b^\dagger(\vec{p}')] = [a^\dagger(\vec{p}), b^\dagger(\vec{p}')] = [a(\vec{p}), b(\vec{p}')] = 0\,.$$

Note that in fact, the two sets of annihilation and creation operators for the fields σ and ϕ are independent, so they commute. Use the above field decompositions and commutation relations, calculate

1) [3 points]

$$\langle q_1, q_2| \int_{-\infty}^{+\infty} d^4x : \sigma(x)\phi(x)\partial_\mu\phi(x) : |k_1\rangle \ ,$$

where

$$\begin{aligned} \langle q_1, q_2| &= C(E_{\vec{q}_1})(2\pi)^3(2E_{\vec{q}_1})C(E_{\vec{q}_2})(2\pi)^3(2E_{\vec{q}_2}) \langle 0| b_{\vec{q}_1} b_{\vec{q}_2} \ , \\ |k_1\rangle &= C(E_{\vec{k}_1})(2\pi)^3(2E_{\vec{k}_1}) a^\dagger_{\vec{k}_1} |0\rangle \ . \end{aligned}$$

2) [3 points] Calculate

$$\langle q_2, k_2| \int_{-\infty}^{+\infty} d^4x : \sigma^2(x)\phi^2(x) : |q_1, k_1\rangle \ ,$$

where

$$\begin{aligned} \langle q_2, k_2| &= C(E_{\vec{q}_2})(2\pi)^3(2E_{\vec{q}_2})C(E_{\vec{k}_2})(2\pi)^3(2E_{\vec{k}_2}) \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} \ , \\ |q_1, k_1\rangle &= C(E_{\vec{q}_1})(2\pi)^3(2E_{\vec{q}_1})C(E_{\vec{k}_1})(2\pi)^3(2E_{\vec{k}_1}) b^\dagger_{\vec{q}_1} a^\dagger_{\vec{k}_1} |0\rangle \ . \end{aligned}$$

NOTE: Please note that there is a ∂_μ in question 1).
The vacuum state is normalized as $\langle 0|0\rangle = 1$. $C(E_{\vec{k}})$ is a real function of $E_{\vec{k}}$.
Please show all your steps. Please don't use directly the Feynman rule (if you know it), since the purpose is to let you derive it.

Problem 4. The first man-made Ω^- particle was created by firing a high energy proton at a stationary proton to produce a K^+/K^- pair: $p + p \rightarrow p + p + K^+ + K^-$; the K^- in turn hit another stationary proton, $K^- + p \rightarrow \Omega^- + K^0 + K^+$. The masses of the particles involved in these two reactions are 938.27 MeV for a proton, 493.7 MeV for a charged kaon, 497.6 MeV for a neutral kaon, and 1672 MeV for an Ω^-. What minimum kinetic energy is required for the incident proton, to make an Ω^- in this way? [6 points]
NOTE: Please give your result in the unit of MeV. Round this number to three significant figures, and don't worry about uncertainties.