Solution

1) $c \equiv 1 = 2.998 \times 10^{10}\ cm/sec \Rightarrow 1\ sec = 2.998 \times 10^{10}\ cm$

$$\Rightarrow 3 \times 10^{-26}\ cm^3/sec = 3 \times 10^{-26}\ cm^3/(2.998 \times 10^{10}\ cm)$$

$$\simeq \boxed{1 \times 10^{-36}\ cm^2}$$

2) $1\ eV = 1.602 \times 10^{-19}\ J,$

$c \equiv 1 = 2.998 \times 10^{10}\ cm \cdot sec^{-1}$

$\hbar \equiv 1 = 1.055 \times 10^{-34}\ J \cdot sec$

$$\Rightarrow \hbar c = 1 = 1.055 \times 10^{-34} \times 2.998 \times 10^{10}\ J \cdot cm$$

$$= 1.055 \times 10^{-34} \times 2.998 \times 10^{10} \cdot \frac{1\ eV}{1.602 \times 10^{-19}} \cdot cm$$

$$\Rightarrow 1\ cm = 1.6022 \times 10^{-19} \times (1.055 \times 10^{-34} \times 2.998 \times 10^{10})^{-1}\ eV^{-1}$$

$$1\ eV^{-1} = \frac{1}{1\ eV} = \frac{10^9}{10^9\ eV} = 10^9\ GeV^{-1}$$

$$\Rightarrow 3 \times 10^{-26}\ cm^3/sec = 3 \times 10^{-26}\ cm^3/(2.998 \times 10^{10}\ cm)$$

$$= 3 \times 10^{-26} \times (2.998 \times 10^{10})^{-1}\ cm^2$$

$$= 3 \times 10^{-26} \times (2.998 \times 10^{10})^{-1} \times [1.6022 \times 10^{-19} \times (1.055 \times 10^{-34} \times 2.998 \times 10^{10})^{-1} \times 10^9]\ GeV^{-2}$$

$$= 3 \times 10^{-26} \times (2.998 \times 10^{10})^{-3} \times (1.6022 \times 10^{-19})^2 \times (1.055 \times 10^{-34})^{-2} \times 10^{18}\ GeV^{-2}$$

$$\simeq \boxed{3 \times 10^{-9}\ GeV^{-2}}$$

solution first term

$$= \bar{u}(\vec{p}_2, s_2)\left(\frac{-\not{p}_1+\not{p}_2+m}{(p_1-p_2)^2-m^2}\right)\not{p}_4\, v(\vec{p}_3, s_3)$$

$p_1^\mu = p_2^\mu + p_3^\mu + p_4^\mu$

$$= \bar{u}(\vec{p}_2, s_2)\left(-\not{p}_2-\not{p}_3-\not{p}_4+\not{p}_2+m\right)\not{p}_4\, v(\vec{p}_3, s_3)\frac{1}{(p_1-p_2)^2-m^2}$$

$\not{p}_3\not{p}_4+\not{p}_4\not{p}_3 = 2p_3\cdot p_4$

$$= \bar{u}(\vec{p}_2, s_2)\left(+\not{p}_4\not{p}_3-2p_3\cdot p_4-\not{p}_4\not{p}_4+m\not{p}_4\right)v(\vec{p}_3, s_3)\frac{1}{(p_1-p_2)^2-m^2}$$

$\not{p}_4\not{p}_4 = p_4^2$

$\not{p}_3\, v(\vec{p}_3, s_3) = -m\, v(\vec{p}_3, s_3)$

$$= \bar{u}(\vec{p}_2, s_2)\left(\not{p}_4(-m)-2p_3\cdot p_4-p_4^2+m\not{p}_4\right)v(\vec{p}_3, s_3)\frac{1}{(p_1-p_2)^2-m^2}$$

$p_4^2 = 0$

$$= \bar{u}(\vec{p}_2, s_2)\, v(\vec{p}_3, s_3)\frac{-2p_3\cdot p_4}{(p_1-p_2)^2-m^2}$$

where $\dfrac{-2p_3\cdot p_4}{(p_1-p_2)^2-m^2} = \dfrac{-2p_3\cdot p_4}{(p_3+p_4)^2-m^2} = \dfrac{-2p_3\cdot p_4}{p_3^2+p_4^2+2p_3\cdot p_4-m^2}$

$$= -1$$

$p_3^2 = m^2,\ p_4^2 = 0$

$\Rightarrow$ first term $= -\bar{u}(\vec{p}_2, s_2)\, v(\vec{p}_3, s_3)$

Similarly,

second term

$$= \bar{u}(\vec{p}_2, s_2)\not{p}_4\left(\frac{\not{p}_1-\not{p}_3+m}{(p_1-p_3)^2-m^2}\right)v(\vec{p}_3, s_3)$$

$$= \bar{u}(\vec{p}_2, s_2)\not{p}_4\left(\not{p}_2+\not{p}_4+m\right)v(\vec{p}_3, s_3)\frac{1}{(p_2+p_4)^2-m^2}$$

$$= \bar{u}(\vec{p}_2, s_2)\left(2p_2\cdot p_4-\not{p}_2\not{p}_4+m\not{p}_4\right)v(\vec{p}_3, s_3)\frac{1}{2p_2\cdot p_4}$$

$$= \bar{u}(\vec{p}_2, s_2)\left(2p_2\cdot p_4-m\not{p}_4+m\not{p}_4\right)v(\vec{p}_3, s_3)\frac{1}{2p_2\cdot p_4}$$

$\not{p}_2\, u(\vec{p}_2, s_2) = m\, u(\vec{p}_2, s_2)$

$\Rightarrow u^\dagger(\vec{p}_2, s_2)\not{p}_2^\dagger = m\, u^\dagger(\vec{p}_2, s_2)$

$\Rightarrow \bar{u}(\vec{p}_2, s_2)\not{p}_2 = m\,\bar{u}(\vec{p}_2, s_2)$

$$= \bar{u}(\vec{p}_2, s_2)\, v(\vec{p}_3, s_3)$$

$\Rightarrow$ first term + second term $= \boxed{0}$

Solution 1). $\partial_\mu \phi(x) \underset{\uparrow}{=} \int_{-\infty}^{+\infty} d^3\vec{P}\, C(E_{\vec{P}})(-iP_\mu)(b_{\vec{P}} e^{-iP\cdot x} - b^+_{\vec{P}} e^{iP\cdot x})$

use $\partial_\mu(P\cdot x) = \partial_\mu(P_\nu x^\nu) = P_\nu \delta^\nu_\mu = P_\mu$

$$\Rightarrow \langle q_1, q_2 | \int_{-\infty}^{+\infty} d^4x : \sigma(x)\phi(x)\partial_\mu\phi(x) : | k_1 \rangle$$
$$= C(E_{\vec{q}_1})(2\pi)^3 2E_{\vec{q}_1}\, C(E_{\vec{q}_2})(2\pi)^3 2E_{\vec{q}_2}\, C(E_{\vec{k}_1})(2\pi)^3 2E_{\vec{k}_1}$$
$$\times \langle 0 | b_{\vec{q}_1} b_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x : \sigma(x)\phi(x)\partial_\mu\phi(x) : a^+_{\vec{k}_1} | 0 \rangle$$

where $: \quad : = \int_{-\infty}^{+\infty} d^3\vec{P}_1 d^3\vec{P}_2 d^3\vec{P}_3\, C(E_{\vec{P}_1}) C(E_{\vec{P}_2}) C(E_{\vec{P}_3})$
$$\cdot : (a_{\vec{P}_1} e^{-iP_1\cdot x} + a^+_{\vec{P}_1} e^{iP_1\cdot x})(b_{\vec{P}_2} e^{-iP_2\cdot x} + b^+_{\vec{P}_2} e^{iP_2\cdot x})$$
$$\cdot (-iP_{3\mu})(b_{\vec{P}_3} e^{-iP_3\cdot x} - b^+_{\vec{P}_3} e^{iP_3\cdot x}) :$$

Since $\langle 0 | a^+_{\vec{P}_1} = 0$,

$b_{\vec{P}_2} |0\rangle = 0$ and $b_{\vec{P}_3} |0\rangle = 0$.

then the only term that contributes is

$$\langle 0 | b_{\vec{q}_1} b_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x : \sigma(x)\phi(x)\partial_\mu\phi(x) : a^+_{\vec{k}_1} | 0 \rangle$$
$$= \langle 0 | b_{\vec{q}_1} b_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x\, d^3\vec{P}_1 d^3\vec{P}_2 d^3\vec{P}_3\, C(E_{\vec{P}_1}) C(E_{\vec{P}_2}) C(E_{\vec{P}_3})\, b^+_{\vec{P}_2} b^+_{\vec{P}_3} a_{\vec{P}_1} e^{iP_2\cdot x} e^{iP_3\cdot x}$$
$$\times e^{-iP_1\cdot x} (+iP_{3\mu})\, a^+_{\vec{k}_1} | 0 \rangle$$

using $\langle 0 | b_{\vec{q}_1} b_{\vec{q}_2} b^+_{\vec{P}_2} b^+_{\vec{P}_3}$

$$= \langle 0 | b_{\vec{q}_1} ([b_{\vec{q}_2}, b^+_{\vec{P}_2}] + b^+_{\vec{P}_2} b_{\vec{q}_2}) b^+_{\vec{P}_3}$$
$$= [b_{\vec{q}_2}, b^+_{\vec{P}_2}] \langle 0 | b_{\vec{q}_1} b^+_{\vec{P}_3} + \langle 0 | b_{\vec{q}_1} b^+_{\vec{P}_2} b_{\vec{q}_2} b^+_{\vec{P}_3}$$
$$= [b_{\vec{q}_2}, b^+_{\vec{P}_2}] \langle 0 | ([b_{\vec{q}_1}, b^+_{\vec{P}_3}] + b^+_{\vec{P}_3} b_{\vec{q}_1})$$
$$+ \langle 0 | ([b_{\vec{q}_1}, b^+_{\vec{P}_2}] + b^+_{\vec{P}_2} b_{\vec{q}_1})([b_{\vec{q}_2}, b^+_{\vec{P}_3}] + b^+_{\vec{P}_3} b_{\vec{q}_2})$$
$$= \left([b_{\vec{q}_2}, b^+_{\vec{P}_2}][b_{\vec{q}_1}, b^+_{\vec{P}_3}] + [b_{\vec{q}_1}, b^+_{\vec{P}_2}][b_{\vec{q}_2}, b^+_{\vec{P}_3}]\right) \langle 0 |$$
$$= \left[\frac{1}{(2\pi)^3}\right]^2 \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 [\delta^3(\vec{P}_2 - \vec{q}_2)\delta^3(\vec{P}_3 - \vec{q}_1)$$
$$+ \delta^3(\vec{P}_2 - \vec{q}_1)\delta^3(\vec{P}_3 - \vec{q}_2)] \langle 0 |$$

and $\quad a_{\vec{p}_1} a^{+}_{\vec{k}_1} |0\rangle$

$$= \left([a_{\vec{p}_1}, a^{+}_{\vec{k}_1}] + a^{+}_{\vec{k}_1} a_{\vec{p}_1} \right)|0\rangle$$

$$= \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{k}_1}}\left(\frac{1}{C(E_{\vec{k}_1})}\right)^2 \delta^3(\vec{p}_1 - \vec{k}_1)|0\rangle$$

$$\Rightarrow \langle 0| b_{\vec{q}_1} b_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x \, :\sigma(x)\phi(x)\partial_\mu\phi(x): a^{+}_{\vec{k}_1}|0\rangle$$

$$= \left[\frac{1}{(2\pi)^3}\right]^3 \frac{1}{2E_{\vec{q}_1}}\frac{1}{2E_{\vec{q}_2}}\frac{1}{2E_{\vec{k}_1}}\left(\frac{1}{C(E_{\vec{k}_1})}\right)^2\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2$$

$$\times \int_{-\infty}^{+\infty} d^4x\, d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3 \cdot C(E_{\vec{p}_1})\, C(E_{\vec{p}_2})\, C(E_{\vec{p}_3})\, e^{-i(p_1-p_2-p_3)\cdot x}(i p_{3\mu})$$

$$\times \left[\delta^3(\vec{p}_2-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_1)+\delta^3(\vec{p}_2-\vec{q}_1)\delta^3(\vec{p}_3-\vec{q}_2)\right]\delta^3(\vec{p}_1-\vec{k}_1)\langle 0|0\rangle$$

$$= \left[\frac{1}{(2\pi)^3}\right]^3 \frac{1}{2E_{\vec{q}_1}}\frac{1}{2E_{\vec{q}_2}}\frac{1}{2E_{\vec{k}_1}}\left(\frac{1}{C(E_{\vec{k}_1})}\right)^2\left(\frac{1}{C(E_{\vec{q}_1})}\right)^2\left(\frac{1}{C(E_{\vec{q}_2})}\right)^2$$

$$\times \int (2\pi)^4 \delta^4(p_1-p_2-p_3)\, d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3 \; C(E_{\vec{p}_1})\, C(E_{\vec{p}_2})\, C(E_{\vec{p}_3}) \cdot (i p_{3\mu})$$

$$\left[\delta^3(\vec{p}_2-\vec{q}_2)\delta^3(\vec{p}_3-\vec{q}_1)+\delta^3(\vec{p}_2-\vec{q}_1)\delta^3(\vec{p}_3-\vec{q}_2)\right]\delta^3(\vec{p}_1-\vec{k}_1)$$

$$= \left[\frac{1}{(2\pi)^3}\right]^3 \frac{1}{2E_{\vec{q}_1}}\frac{1}{2E_{\vec{q}_2}}\frac{1}{2E_{\vec{k}_1}}\frac{1}{C(E_{\vec{k}_1})}\frac{1}{C(E_{\vec{q}_1})}\frac{1}{C(E_{\vec{q}_2})}$$

$$(2\pi)^4\left[\delta^4(k_1-q_1-q_2)\, i q_{1\mu} + \delta^4(k_1-q_1-q_2)\, i q_{2\mu}\right]$$

$$= \left[\frac{1}{(2\pi)^3}\right]^3 \frac{1}{2E_{\vec{q}_1}}\frac{1}{2E_{\vec{q}_2}}\frac{1}{2E_{\vec{k}_1}}\frac{1}{C(E_{\vec{k}_1})}\frac{1}{C(E_{\vec{q}_1})}\frac{1}{C(E_{\vec{q}_2})}(2\pi)^4\delta^4(k_1-q_1-q_2)\, i(q_1+q_2$$

$$\Rightarrow \langle q_1, q_2| \int_{-\infty}^{+\infty} d^4x \, :\sigma(x)\phi(x)\partial_\mu\phi(x): |k_1\rangle$$

$$\boxed{\begin{aligned} &= (2\pi)^4\delta^4(k_1-q_1-q_2)\; i(q_1+q_2)_\mu \\ &= (2\pi)^4\delta^4(k_1-q_1-q_2)\; i k_{1\mu} \end{aligned}}$$

2) $\langle q_2, k_2 | \int_{-\infty}^{+\infty} d^4x : \sigma^2(x)\phi^2(x) : | q_1, k_1 \rangle$

$$= C(E_{\vec{k}_1}) C(E_{\vec{k}_2}) C(E_{\vec{q}_1}) C(E_{\vec{q}_2}) [(2\pi)^3]^4 \, 2E_{\vec{k}_1} \, 2E_{\vec{k}_2} \, 2E_{\vec{q}_1} \, 2E_{\vec{q}_2}$$

$$\times \langle 0 | b_{\vec{q}_2} a_{\vec{k}_2} \int_{-\infty}^{+\infty} d^4x \, d^3\vec{p}_1 \, d^3\vec{p}_2 \, d^3\vec{p}_3 \, d^3\vec{p}_4 \, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) C(E_{\vec{p}_4})$$

$$: (a_{\vec{p}_1} e^{-ip_1\cdot x} + a^+_{\vec{p}_1} e^{ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + a^+_{\vec{p}_2} e^{ip_2\cdot x})$$

$$(b_{\vec{p}_3} e^{-ip_3\cdot x} + b^+_{\vec{p}_3} e^{ip_3\cdot x})(b_{\vec{p}_4} e^{-ip_4\cdot x} + b^+_{\vec{p}_4} e^{ip_4\cdot x}) : \; b^+_{\vec{q}_1} a^+_{\vec{k}_1} | 0 \rangle$$

For $: \; :$, there is 1 term taking the form $a^+a^+b^+b^+$;
there are 2 terms taking the form $a^+a^+b^+b$;
there are 2 terms taking the form $b^+b^+a^+a$;
there is 1 term taking the form a^+a^+bb;
there is 1 term taking the form b^+b^+aa;
there are 4 terms taking the form a^+b^+ab;
there are 2 terms taking the form a^+abb;
there are 2 terms taking the form b^+baa;
there is 1 term taking the form $aabb$.

For arbitrary $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5$ and $\vec{p}_6$,

$$a_{\vec{p}_1} a_{\vec{p}_2} b_{\vec{p}_3} b_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} | 0 \rangle$$

$$= a_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_6} b_{\vec{p}_3} b_{\vec{p}_4} b^+_{\vec{p}_5} | 0 \rangle = a_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_6} b_{\vec{p}_3} ([b_{\vec{p}_4}, b^+_{\vec{p}_5}] + \underbrace{b^+_{\vec{p}_5} b_{\vec{p}_4}) | 0 \rangle}_{=0}$$

since $b_{\vec{p}} | 0 \rangle = 0$ for any $\vec{p}$

$$\Rightarrow \langle 0 | b_{\vec{p}_5} a_{\vec{p}_6} b^+_{\vec{p}_4} b^+_{\vec{p}_3} a^+_{\vec{p}_2} a^+_{\vec{p}_1} = (a_{\vec{p}_1} a_{\vec{p}_2} b_{\vec{p}_3} b_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} | 0 \rangle)^* = 0$$

$$b^+_{\vec{p}_1} b_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} | 0 \rangle$$

$$= b^+_{\vec{p}_1} b_{\vec{p}_2} b^+_{\vec{p}_5} a_{\vec{p}_3} a_{\vec{p}_4} a^+_{\vec{p}_6} | 0 \rangle = b^+_{\vec{p}_1} b_{\vec{p}_2} b^+_{\vec{p}_5} a_{\vec{p}_3} ([a_{\vec{p}_4}, a^+_{\vec{p}_6}] + \underbrace{a^+_{\vec{p}_6} a_{\vec{p}_4}) | 0 \rangle}_{=0}$$

since $a_{\vec{p}} | 0 \rangle = 0$ for any $\vec{p}$.

$$\Rightarrow \langle 0| b_{\vec{p}_5} a_{\vec{p}_6} a^+_{\vec{p}_4} a^+_{\vec{p}_3} b^+_{\vec{p}_2} b_{\vec{p}_1} = \left(b^+_{\vec{p}_1} b_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} |0\rangle \right)^* = 0$$

$$a^+_{\vec{p}_1} a_{\vec{p}_2} b_{\vec{p}_3} b_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} |0\rangle$$

$$= a^+_{\vec{p}_1} a_{\vec{p}_2} a^+_{\vec{p}_6} b_{\vec{p}_3} \left([b_{\vec{p}_4}, b^+_{\vec{p}_5}] + b^+_{\vec{p}_5} b_{\vec{p}_4} \right) |0\rangle$$

$$\underset{\uparrow}{=} 0$$

since $b_{\vec{p}}|0\rangle = 0$ for any $\vec{p}$

$$\Rightarrow \langle 0| b_{\vec{p}_5} a_{\vec{p}_6} b^+_{\vec{p}_4} b^+_{\vec{p}_3} a^+_{\vec{p}_2} a_{\vec{p}_1} = \left(a^+_{\vec{p}_1} a_{\vec{p}_2} b_{\vec{p}_3} b_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} |0\rangle \right)^* = 0$$

$$b^+_{\vec{p}_1} b^+_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} |0\rangle$$

$$= b^+_{\vec{p}_1} b^+_{\vec{p}_2} b^+_{\vec{p}_5} a_{\vec{p}_3} \left([a_{\vec{p}_4}, a^+_{\vec{p}_6}] + a^+_{\vec{p}_6} a_{\vec{p}_4} \right) |0\rangle$$

$$\underset{\uparrow}{=} 0$$

since $a_{\vec{p}}|0\rangle = 0$ for any $\vec{p}$

$$\Rightarrow \langle 0| b_{\vec{p}_5} a_{\vec{p}_6} a^+_{\vec{p}_4} a^+_{\vec{p}_3} b_{\vec{p}_2} b_{\vec{p}_1} = \left(b^+_{\vec{p}_1} b^+_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} b^+_{\vec{p}_5} a^+_{\vec{p}_6} |0\rangle \right)^* = 0$$

So, we only need to consider the form $a^+ b^+ a b$ from : :

$$a_{\vec{p}_1} b_{\vec{p}_2} b^+_{\vec{p}_3} a^+_{\vec{p}_4} |0\rangle$$

$$= [a_{\vec{p}_1}, a^+_{\vec{p}_4}] [b_{\vec{p}_2}, b^+_{\vec{p}_3}] |0\rangle$$

while $\langle 0| b_{\vec{p}_3} a_{\vec{p}_4} a^+_{\vec{p}_1} b^+_{\vec{p}_2}$

$$\underset{\uparrow}{=} \langle 0| \left([b_{\vec{p}_3}, b^+_{\vec{p}_2}] + b^+_{\vec{p}_2} b_{\vec{p}_3} \right) \left([a_{\vec{p}_4}, a^+_{\vec{p}_1}] + a^+_{\vec{p}_1} a_{\vec{p}_4} \right)$$

since $\langle 0| a^+_{\vec{p}} = 0$, $\langle 0| b^+_{\vec{p}} = 0$ for any $\vec{p}$

$$= [b_{\vec{p}_3}, b^+_{\vec{p}_2}] [a_{\vec{p}_4}, a^+_{\vec{p}_1}] \langle 0|$$

$$\Rightarrow \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} : (a_{\vec{p}_1} e^{-ip_1\cdot x} + a^+_{\vec{p}_1} e^{ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + a^+_{\vec{p}_2} e^{ip_2\cdot x})$$
$$(b_{\vec{p}_3} e^{-ip_3\cdot x} + b^+_{\vec{p}_3} e^{ip_3\cdot x})(b_{\vec{p}_4} e^{-ip_4\cdot x} + b^+_{\vec{p}_4} e^{ip_4\cdot x}) :$$
$$b^+_{\vec{q}_1} a^+_{\vec{k}_1} |0\rangle$$

$$= e^{i(p_1+p_3-p_2-p_4)\cdot x} \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} a^+_{\vec{p}_1} b^+_{\vec{p}_3} a_{\vec{p}_2} b_{\vec{p}_4} b^+_{\vec{q}_1} a^+_{\vec{k}_1} |0\rangle$$
$$+ e^{i(p_1+p_4-p_2-p_3)\cdot x} \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} a^+_{\vec{p}_1} b^+_{\vec{p}_4} a_{\vec{p}_2} b_{\vec{p}_3} b^+_{\vec{q}_1} a^+_{\vec{k}_1} |0\rangle$$
$$+ e^{i(p_2+p_3-p_1-p_4)} \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} a^+_{\vec{p}_2} b^+_{\vec{p}_3} a_{\vec{p}_1} b_{\vec{p}_4} b^+_{\vec{q}_1} a^+_{\vec{k}_1} |0\rangle$$
$$+ e^{i(p_2+p_4-p_1-p_3)} \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} a^+_{\vec{p}_2} b^+_{\vec{p}_4} a_{\vec{p}_1} b_{\vec{p}_3} b^+_{\vec{q}_1} a^+_{\vec{k}_1} |0\rangle$$

$$= e^{i(p_1+p_3-p_2-p_4)\cdot x} [a_{\vec{p}_2}, a^+_{\vec{k}_1}][b_{\vec{p}_4}, b^+_{\vec{q}_1}][a_{\vec{k}_2}, a^+_{\vec{p}_1}][b_{\vec{q}_2}, b^+_{\vec{p}_3}]\langle 0|0\rangle$$
$$+ (p_3 \leftrightarrow p_4) + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4)$$

$$\underset{\substack{\uparrow \\ \text{use } \langle 0|0\rangle = 1}}{=} e^{i(p_1+p_3-p_2-p_4)\cdot x} \left[\frac{1}{(2\pi)^3}\right]^4 \frac{1}{2E_{\vec{k}_1}} \frac{1}{2E_{\vec{k}_2}} \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{k}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{k}_2})}\right)^2$$
$$\cdot \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{p}_1 - \vec{k}_2)\delta^3(\vec{p}_3 - \vec{q}_2)\delta^3(\vec{p}_2 - \vec{k}_1)\delta^3(\vec{p}_4 - \vec{q}_1)$$
$$+ (p_3 \leftrightarrow p_4) + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4)$$

$$\Rightarrow \langle 0| b_{\vec{q}_2} a_{\vec{k}_2} \int_{-\infty}^{+\infty} d^4x\, d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3\, d^3\vec{p}_4\, C(E_{\vec{p}_1})C(E_{\vec{p}_2})C(E_{\vec{p}_3})C(E_{\vec{p}_4})$$
$$:(a_{\vec{p}_1} e^{-ip_1\cdot x} + a^+_{\vec{p}_1} e^{ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + a^+_{\vec{p}_2} e^{ip_2\cdot x})$$
$$(b_{\vec{p}_3} e^{-ip_3\cdot x} + b^+_{\vec{p}_3} e^{ip_3\cdot x})(b_{\vec{p}_4} e^{-ip_4\cdot x} + b^+_{\vec{p}_4} e^{ip_4\cdot x}): b^+_{\vec{q}_1} a^+_{\vec{k}_1} |0\rangle$$

$$= \int (2\pi)^4 \delta^4(p_1+p_3-p_2-p_4)\, d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3\, d^3\vec{p}_4\, C(E_{\vec{p}_1})C(E_{\vec{p}_2})C(E_{\vec{p}_3})C(E_{\vec{p}_4})$$
$$\cdot \left[\frac{1}{(2\pi)^3}\right]^4 \frac{1}{2E_{\vec{k}_1}} \frac{1}{2E_{\vec{k}_2}} \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{k}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{k}_2})}\right)^2 \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2$$
$$\cdot \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \delta^3(\vec{p}_1 - \vec{k}_2)\delta^3(\vec{p}_3 - \vec{q}_2)\delta^3(\vec{p}_2 - \vec{k}_1)\delta^3(\vec{p}_4 - \vec{q}_1)$$
$$+ (p_3 \leftrightarrow p_4) + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4)$$

$$= 4(2\pi)^4 \delta^4(q_1+k_1-q_2-k_2) \left[\frac{1}{(2\pi)^3}\right]^4 \frac{1}{2E_{\vec{k}_1}} \frac{1}{2E_{\vec{k}_2}} \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}}$$
$$\cdot \left(\frac{1}{C(E_{\vec{k}_1})}\right)\left(\frac{1}{C(E_{\vec{k}_2})}\right)\left(\frac{1}{C(E_{\vec{q}_1})}\right)\left(\frac{1}{C(E_{\vec{q}_2})}\right)$$

$$\Rightarrow \langle q_2, k_2 | \int_{-\infty}^{+\infty} d^4x : \sigma^2(x)\phi^2(x) : | q_1, k_1\rangle$$
$$\boxed{= 4(2\pi)^4 \delta^4(q_1+k_1-q_2-k_2)}$$

solution

Assign momentum as

$$P(P_{in}) + P(P_{r_1}) \longrightarrow P(P_{f_1}) + P(P_{f_2}) + K^+(P_{f_3}) + K^-(P_{K^-})$$

$$K^-(P_{K^-}) + P(P_{r_2}) \longrightarrow \Omega^-(P_\Omega) + K^\circ(P_{K^\circ}) + K^+(P_{K^+})$$

Since in the second reaction the mass sum of the final state ($m_{\Omega^-} + m_{K^\circ} + m_{K^\pm}$) is larger than the mass sum of the initial state ($m_{K^\pm} + m_p$), and considering that the proton in the second reaction is at rest, then K^- needs to have some kinetic energy to make this reaction happen (i.e., to satisfy the requirement of energy-momentum conservation in the second reaction).

For the second reaction,

$$\Rightarrow (P_{K^-} + P_{r_2})^2 = (P_\Omega + P_{K^\circ} + P_{K^+})^2$$

Evaluate the LHS in the rest frame of proton (i.e., lab frame)

$$\Rightarrow LHS = m_{K^\pm}^2 + m_p^2 + 2m_p E_{K^-}^{lab},$$

The minimum $E_{K^-}^{lab}$ for this reaction to happen means that the RHS takes the minimum, that is

$$E_{K^-_{min}}^{lab} = \frac{(P_\Omega + P_{K^\circ} + P_{K^+})^2_{min} - m_{K^\pm}^2 - m_p^2}{2m_p}$$

It's easy to get $(P_\Omega + P_{K^\circ} + P_{K^+})^2_{min}$ when evaluate it in the center of momentum frame of $\Omega^-, K^\circ \& K^+$ (label it as cm'), which is $(E_\Omega^{cm'} + E_{K^\circ}^{cm'} + E_{K^+}^{cm'})$ and its minimum is $(m_\Omega + m_{K^\circ} + m_{K^\pm})^2$

Therefore, $E_{K^-_{min}}^{lab} = \left[(m_\Omega + m_{K^\circ} + m_{K^\pm})^2 - m_{K^\pm}^2 - m_p^2\right] / (2m_p)$

Now, let's look for the minimum kinetic energy of the incident proton that can give $E_{K^-_{min}}^{lab}$, which is the <u>maximum</u> energy of the produced K^- in the first reaction for the minimum E_{in}^{lab}.

For a given E_{in}^{lab} of the incident proton in the lab frame, the center of energy of the LHS of the first reaction is

$$\sqrt{s} = [(E_{in}^{lab} + m_p)^2 - |\vec{P}_{in}^{lab}|^2]^{\frac{1}{2}} = [2m_p^2 + 2m_p E_{in}^{lab}]^{\frac{1}{2}}$$

and $s = (P_{in} + P_{r_1})^2$

Using $(P_{in} + P_{r_1} - P_{K^-})^2 = (P_{f_1} + P_{f_2} + P_{f_3})^2$

and Evaluate this expression in the center of momentum frame of the initial state (and final state) of the first reaction, (label it as cm)

$$LHS = (P_{in} + P_{r_1})^2 + P_{K^-}^2 - 2P_{K^-}\cdot(P_{in} + P_{r_1})$$

$$= s + m_{K^\pm}^2 - 2\sqrt{s}\, E_{K^-}^{cm}$$

For the produced K^- in the first reaction, let's look at its energy & momentum relation between the lab and the cm frame

$E_{K^-}^{lab} = \gamma(E_{K^-}^{cm} + \vec{v}\cdot\vec{P}_{K^-}^{cm})$, where $|\vec{v}|$ is the relative speed between the two frames, and its direction is along the direction of the incident proton.

For a given E_{in}^{lab}, γ and $|\vec{v}|$ are fixed, as

$$\frac{\sqrt{s}}{2} = m_p \gamma \quad (\text{since } P_{r_1}^{0\,lab} = m_p, \text{ while } P_{r_1}^{0\,cm} = \frac{\sqrt{s}}{2}, \vec{P}_{r_1}^{lab} = 0$$

$$\Rightarrow \gamma = \frac{\sqrt{s}}{2m_p} = \frac{(2m_p^2 + 2m_p E_{in}^{lab})^{\frac{1}{2}}}{2m_p}$$

and $|\vec{v}| = (1 - \frac{1}{\gamma^2})^{\frac{1}{2}}$

Therefore, for a given E_{in}^{lab}, $|\vec{v}|$ and γ is fixed.

Also, for a given $E_{K^-}^{cm}$, $|\vec{P}_{K^-}^{cm}|$ is fixed as $[(E_{K^-}^{cm})^2 - m_{K^-}^2]^{\frac{1}{2}}$, then $E_{K^-}^{lab}$ achieves its maximum when $\vec{P}_{K^-}^{cm}$ is along the direction of $\vec{v}$.

Since $s + m_{K^\pm}^2 - 2\sqrt{s}\, E_{K^-}^{cm} = (P_{f_1} + P_{f_2} + P_{f_3})^2$

where $s = 2m_p^2 + 2m_p E_{in}^{lab}$.

then the minimum E_{in}^{lab} is achieved when $E_{K^-}^{cm}$ is just big enough to give $E_{K^-_{min}}^{lab}$ needed in the second reaction, and

$(P_{f_1}+P_{f_2}+P_{f_3})^2$ takes minimum value, which is $(m_p+m_p+m_{K^\pm})^2$ (this is easy to see in the center of momentum frame of the three particles)

Therefore, the configuration corresponding to the minimum E_{in}^{lab} is that in the first reaction the produced two protons and K^+ move together without relative velocities among them, and the produced K^- moves along the direction of the incident proton; and then in the second reaction, the produced Ω^-, K^0 and K^+ move together also along the direction of the incident proton of the first reaction.

In the lab frame, by three momentum conservation,

$$(\vec{P}_{f_1}^{lab}+\vec{P}_{f_2}^{lab}+\vec{P}_{f_3}^{lab}) = \vec{P}_{in}^{lab} - \vec{P}_{K^-}^{lab}$$

Since we know that the direction of $\vec{P}_{in}^{lab}$ and $\vec{P}_{K^-}^{lab}$ is the same, then if $(|\vec{P}_{in}^{lab}|-|\vec{P}_{K^-}^{lab}|)>0$ (or <0), then in the lab frame the produced two protons and K^+ in the first reaction move together along (or opposite) the direction of the incident proton.

In this configuration, in the lab frame,

$$E_{in}^{lab}+m_p = E_{K^-_{min}}^{lab} + \left\{\left[(E_{in}^{2\,lab}-m_p^2)^{\frac{1}{2}} - (E_{K^-_{min}}^{2\,lab}-m_{K^\pm}^2)^{\frac{1}{2}}\right]^2 + (m_p+m_p+m_{K^\pm})^2\right\}^{\frac{1}{2}}$$

$$\Rightarrow (E_{in}^{lab}+m_p-E_{K^-_{min}}^{lab})^2 = \left([(E_{in}^{lab})^2-m_p^2]^{\frac{1}{2}} - [(E_{K^-_{min}}^{lab})^2-m_{K^\pm}^2]^{\frac{1}{2}}\right)^2 + (2m_p+m_{K^\pm})^2$$

$$\Rightarrow (E_{in}^{lab})^2+m_p^2+(E_{K^-_{min}}^{lab})^2+2m_pE_{in}^{lab}-2E_{in}^{lab}E_{K^-_{min}}^{lab}-2m_pE_{K^-_{min}}^{lab}$$
$$= (E_{in}^{lab})^2-m_p^2+(E_{K^-_{min}}^{lab})^2-m_{K^\pm}^2-2[(E_{in}^{lab})^2-m_p^2]^{\frac{1}{2}}[(E_{K^-_{min}}^{lab})^2-m_{K^\pm}^2]^{\frac{1}{2}}$$
$$+4m_p^2+m_{K^\pm}^2+4m_pm_{K^\pm}$$

$$\Rightarrow \quad 2\left[(E_{in}^{lab})^2 - m_p^2\right]^{\frac{1}{2}}\left[(E_{K^-_{min}}^{lab})^2 - m_{K^\pm}^2\right]^{\frac{1}{2}}$$

$$= 2m_p^2 + 4m_p m_{K^\pm} - 2m_p E_{in}^{lab} + 2E_{in}^{lab}E_{K^-_{min}}^{lab} + 2m_p E_{K^-_{min}}^{lab}$$

$$\Rightarrow \quad \underline{(E_{in}^{lab})^2(E_{K^-_{min}}^{lab})^2} + m_p^2 m_{K^\pm}^2 - (E_{in}^{lab})^2 m_{K^\pm}^2 - (E_{K^-_{min}}^{lab})^2 m_p^2$$

$$= m_p^4 + 4m_p^2 m_{K^\pm}^2 + m_p^2(E_{in}^{lab})^2 + \underline{(E_{in}^{lab})^2(E_{K^-_{min}}^{lab})^2} + m_p^2(E_{K^-_{min}}^{lab})^2$$

$$+ 4m_p^3 m_{K^\pm} - 2m_p^3 E_{in}^{lab} + \underline{2m_p^2 E_{in}^{lab}E_{K^-_{min}}^{lab}} + 2m_p^3 E_{K^-_{min}}^{lab}$$

$$- 4m_p^2 m_{K^\pm}E_{in}^{lab} + 4m_p m_{K^\pm}E_{in}^{lab}E_{K^-_{min}}^{lab} + 4m_p^2 m_{K^\pm}E_{K^-_{min}}^{lab}$$

$$- 2m_p E_{K^-_{min}}^{lab}(E_{in}^{lab})^2 - \underline{2m_p^2 E_{in}^{lab}E_{K^-_{min}}^{lab}} + 2m_p E_{in}^{lab}(E_{K^-_{min}}^{lab})^2$$

$$\Rightarrow \quad (m_{K^\pm}^2 + m_p^2 - 2m_p E_{K^-_{min}}^{lab})(E_{in}^{lab})^2$$

$$+ \left(-2m_p^3 - 4m_p^2 m_{K^\pm} + 4m_p m_{K^\pm}E_{K^-_{min}}^{lab} + 2m_p(E_{K^-_{min}}^{lab})^2\right)E_{in}^{lab}$$

$$+ \left(3m_p^2 m_{K^\pm}^2 + 2m_p^2(E_{K^-_{min}}^{lab})^2 + m_p^4 + 4m_p^3 m_{K^\pm} + 2m_p^3 E_{K^-_{min}}^{lab} + 4m_p^2 m_{K^\pm}E_{K^-_{min}}^{lab}\right) = 0$$

Plug in $E_{K^-_{min}}^{lab} = \left[(m_\Omega + m_{K^0} + m_{K^\pm})^2 - m_{K^\pm}^2 - m_p^2\right]/(2m_p)$

and $m_p = 938.27\,\text{MeV}$, $m_{K^\pm} = 493.7\,\text{MeV}$, $m_{K^0} = 497.6\,\text{MeV}$,
$m_\Omega = 1672\,\text{MeV}$

$\Rightarrow \quad E_{in}^{lab} \simeq -1.166\times10^3\,\text{MeV}$ or $E_{in}^{lab} \simeq 5.602\times10^3\,\text{MeV}$

drop the negative solution since the energy is positive.

Therefore, the minimum kinetic energy required is

$$(T_{in}^{lab})_{min} = E_{in}^{lab} - m_p \simeq \boxed{4.66\times10^3\,\text{MeV}}$$

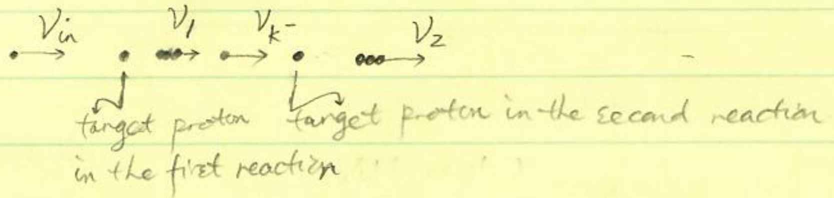
Also, we can find $|\vec{P}_{in}^{lab}| = \left[(E_{in}^{lab})^2 - m_p^2\right]^{\frac{1}{2}} \simeq 5.52\times10^3\,\text{MeV}$

and $|\vec{P}_{K^-}^{lab}| = \left[(E_{K^-_{min}}^{lab})^2 - m_{K^\pm}^2\right]^{\frac{1}{2}} \simeq 3.14\times10^3\,\text{MeV}$

$\Rightarrow \quad |\vec{P}_{in}^{lab}| - |\vec{P}_{K^-}^{lab}| > 0$

So the produced two protons and K^+ in the first reaction move together

along the direction of the incident proton.
The configuration is, in the lab frame:



$\vec{v}_{in}$, $\vec{v}_1$, $\vec{v}_{K^-}$ and $\vec{v}_2$ are all along the same line connecting the two target protons, which are at rest in the lab frame; All the four velocities are in the same direction; $\vec{v}_1$ is the common velocity of the produced two protons and the K^+ in the first reaction (they move with the same velocity), $\vec{v}_2$ is the common velocity of the produced Ω^-, K^0 & K^+ in the second reaction (they also move with the same velocity).

$$|\vec{v}_{in}| = \frac{|\vec{P}_{in}^{\,lab}|}{E_{in}^{lab}} \simeq 0.986,$$

$$|\vec{v}_1| = \frac{|\vec{P}_{in}^{\,lab}| - |\vec{P}_{K^-}^{\,lab}|}{E_{in}^{lab} + m_p - E_{K^-_{min}}^{lab}} \simeq 0.709$$

$$|\vec{v}_{K^-}| = \frac{|\vec{P}_{K^-}^{\,lab}|}{E_{K^-_{min}}^{lab}} \simeq 0.988$$

$$|\vec{v}_2| = \frac{\left[(E_{K^-_{min}}^{lab} + m_p)^2 - (m_{\Omega} + m_{K^0} + m_{K^\pm})^2\right]^{\frac{1}{2}}}{E_{K^-_{min}}^{lab} + m_p} \simeq 0.763$$