Lorentz Symmetry of spin $\frac{1}{2}$ fields

Do a Lorentz transformation

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \text{ where } \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}$$

then $\psi'_a(x') = S_{ab}(\Lambda)\psi_b(x)$, note that $\Lambda^\mu{}_\nu$ are constants.

(recall that for scalar field $\phi'(x') = \phi(x)$,
for vector field $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$)

Since the theory should not change after a Lorentz transformation, then $\psi'(x')$ should also satisfies Dirac equation.

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0.$$

times S from the left to $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

$$S(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

and insert $S^{-1}S = 1$

$$\Rightarrow S(i\gamma^\mu S^{-1}S\partial_\mu - m)\psi(x) = 0$$

$$\Rightarrow i S\gamma^\mu S^{-1}\partial_\mu(S\psi(x)) - mS\psi(x) = 0$$

(note that $S(\Lambda)\partial_\mu = \partial_\mu S(\Lambda)$, since S is a 4×4 constant matrix)

$$\Rightarrow (iS\gamma^\mu S^{-1}\partial_\mu - m)\psi'(x') = 0$$

So we have

$$S\gamma^\mu S^{-1}\partial_\mu = \gamma^\mu\partial'_\mu.$$

Since $\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \partial'_\nu$

then $S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu\partial'_\nu = \gamma^\mu\partial'_\mu$

$$\Rightarrow \quad S\gamma^\mu S^{-1} \Lambda^\nu{}_\mu = \gamma^\nu$$

$$\Rightarrow \quad S^{-1} S \gamma^\mu S^{-1} S \Lambda^\nu{}_\mu = S^{-1}\gamma^\nu S$$

$$\Rightarrow \quad S^{-1}\gamma^\nu S = \Lambda^\nu{}_\mu \gamma^\mu .$$

This is the meaning of γ^μ behaves as a Lorentz vector.

Now consider an infinitesimal Lorentz transformation,

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon^\mu{}_\nu$, and $\varepsilon_{\mu\nu} \equiv g_{\mu\lambda}\varepsilon^\lambda{}_\nu$

and we have shown before that $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$.

To first order in $\varepsilon_{\mu\nu}$, $S(\Lambda)$ must have the form that

$$S(\Lambda) \simeq 1 - \frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}$$

where $\sigma_{\mu\nu}$ are 4X4 matrices, $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$.

the factor $\left(-\frac{i}{4}\right)$ is introduced by convention.

Now let's find $\sigma_{\mu\nu}$.

Then from $S^{-1}\gamma^\nu S = \Lambda^\nu{}_\mu \gamma^\mu$, to first order in ε

$$\Rightarrow \quad \left(1 + \frac{i}{4}\varepsilon^{\alpha\beta}\sigma_{\alpha\beta}\right)\gamma^\nu\left(1 - \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma}\right) \simeq \left(\delta^\nu_\mu + \varepsilon^\nu{}_\mu\right)\gamma^\mu$$

$$\Rightarrow \quad \left(\gamma^\nu + \frac{i}{4}\varepsilon^{\alpha\beta}\sigma_{\alpha\beta}\gamma^\nu\right)\left(1 - \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma}\right) = \gamma^\nu + \varepsilon^\nu{}_\mu\gamma^\mu$$

$$\Rightarrow \quad \frac{i}{4}\varepsilon^{\alpha\beta}\sigma_{\alpha\beta}\gamma^\nu - \frac{i}{4}\gamma^\nu\varepsilon^{\rho\sigma}\sigma_{\rho\sigma} \simeq \varepsilon^\nu{}_\mu\gamma^\mu$$

$$\Rightarrow \quad -\frac{i}{4}\varepsilon^{\kappa\lambda}\left(\gamma^\nu\sigma_{\kappa\lambda} - \sigma_{\kappa\lambda}\gamma^\nu\right) = \varepsilon^\nu{}_\mu\gamma^\mu$$

$$\text{the RHS} = \varepsilon^\nu{}_\mu\gamma^\mu = \varepsilon^{\nu\mu}\gamma_\mu = \varepsilon^{\kappa\lambda}\left(\delta^\nu_\kappa\gamma_\lambda\right)$$

$$= \frac{1}{2}\varepsilon^{\kappa\lambda}\delta^\nu_\kappa\gamma_\lambda + \underbrace{\frac{1}{2}\varepsilon^{\kappa\lambda}\delta^\nu_\kappa\gamma_\lambda}_{-\frac{1}{2}\varepsilon^{\kappa\lambda}\delta^\nu_\lambda\gamma_\kappa}$$

$$\Rightarrow -\frac{i}{4}\varepsilon^{\kappa\lambda}\left(\gamma^\nu\sigma_{\kappa\lambda} - \sigma_{\kappa\lambda}\gamma^\nu\right) = \frac{1}{2}\varepsilon^{\kappa\lambda}\left(\delta^\nu_\kappa\gamma_\lambda - \delta^\nu_\lambda\gamma_\kappa\right)$$

$$\Rightarrow \quad 2i\left(\delta^\nu_\kappa\gamma_\lambda - \delta^\nu_\lambda\gamma_\kappa\right) = \left[\gamma^\nu, \sigma_{\kappa\lambda}\right]$$

we can check that $\sigma_{\kappa\lambda} = \frac{i}{2}[\gamma_\kappa, \gamma_\lambda]$ is the solution to the above equation.

check:

$$RHS = [\gamma^\nu, \frac{i}{2}[\gamma_\kappa, \gamma_\lambda]]$$

$$= \frac{i}{2}[\gamma^\nu, \gamma_\kappa\gamma_\lambda - \gamma_\lambda\gamma_\kappa]$$

$$= \frac{i}{2}(\gamma^\nu\gamma_\kappa\gamma_\lambda - \gamma^\nu\gamma_\lambda\gamma_\kappa - \gamma_\kappa\gamma_\lambda\gamma^\nu + \gamma_\lambda\gamma_\kappa\gamma^\nu)$$

use $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \Rightarrow \gamma_\mu\gamma^\nu + \gamma^\nu\gamma_\mu = 2\delta^\nu_\mu$

$$\Rightarrow RHS = \frac{i}{2}\Big[(2\delta^\nu_\kappa - \gamma_\kappa\gamma^\nu)\gamma_\lambda - (2\delta^\nu_\lambda - \gamma_\lambda\gamma^\nu)\gamma_\kappa - \gamma_\kappa(2\delta^\nu_\lambda - \gamma^\nu\gamma_\lambda) + \gamma_\lambda(2\delta^\nu_\kappa - \gamma^\nu\gamma_\kappa)\Big]$$

$$= \frac{i}{2}[4\delta^\nu_\kappa\gamma_\lambda - 4\delta^\nu_\lambda\gamma_\kappa - \cancel{\gamma_\kappa\gamma^\nu\gamma_\lambda} + \cancel{\gamma_\lambda\gamma^\nu\gamma_\kappa} + \cancel{\gamma_\kappa\gamma^\nu\gamma_\lambda} - \cancel{\gamma_\lambda\gamma^\nu\gamma_\kappa}]$$

$$= 2i(\delta^\nu_\kappa\gamma_\lambda - \delta^\nu_\lambda\gamma_\kappa)$$

$$= LHS \quad \checkmark$$

From $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \Rightarrow \sigma^+_{\mu\nu} = -\frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)^+ = -\frac{i}{2}(\gamma_\nu^+\gamma_\mu^+ - \gamma_\mu^+\gamma_\nu^+)$

$= \gamma_0 \sigma_{\mu\nu}\gamma_0$

$\Rightarrow \quad S^+ = 1 + \frac{i}{4}\varepsilon^{\mu\nu}\sigma^+_{\mu\nu} = 1 + \frac{i}{4}\varepsilon^{\mu\nu}\gamma_0\sigma_{\mu\nu}\gamma_0$

while $S^{-1} = 1 + \frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}$, $\quad \gamma_0^2 = 1$

$\Rightarrow \quad S^+ = \gamma_0 S^{-1}\gamma_0$

Since $\psi'(x') = S\psi(x)$

then $\psi'^+(x') = \psi^+(x) S^+$

$\bar{\psi}'(x') = \psi'^+(x')\gamma^0 = \psi^+(x)S^+\gamma^0 = \psi^+(x)\gamma_0 S^{-1} = \bar{\psi}(x)S^{-1}$

So, under Lorentz transformation.

$\bar{\psi}(x)\psi(x) \longrightarrow \bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}S\psi(x) = \bar{\psi}(x)\psi(x)$

$i\bar{\psi}\gamma^\mu\partial_\mu\psi \longrightarrow i\bar{\psi}'(x')\gamma^\mu\partial'_\mu\psi'(x') = i\bar{\psi}(x)S^{-1}\gamma^\mu\partial'_\mu S\psi(x)$

$= i\bar{\psi}(x)\gamma^\nu\Lambda^\mu{}_\nu\partial'_\mu\psi(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$

Therefore, the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

is invariant under Lorentz transformation.

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}m\psi$$

$$= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \partial_\mu(\bar{\psi}i\gamma^\mu) = i\gamma^\mu\partial_\mu\bar{\psi}$$

$$\Rightarrow i\gamma^\mu\partial_\mu\bar{\psi} + m\bar{\psi} = 0$$

$$\Rightarrow \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \quad ✓$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = \partial_\mu 0 = 0$$

$$\Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0 \quad ✓$$

if use ψ^+ as independent field, then

$$\mathcal{L} = i\psi^+\gamma^0\gamma^\mu\partial_\mu\psi - m\psi^+\gamma^0\psi$$

$$\frac{\partial\mathcal{L}}{\partial\psi^+} = i\gamma^0\gamma^\mu\partial_\mu\psi - m\gamma^0\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^+)} = 0$$

$$\Rightarrow \gamma^0(i\gamma^\mu\partial_\mu\psi - m\psi) = 0$$

times γ^0 from the left and use $(\gamma^0)^2 = 1$

$$\Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$$

spin of the Dirac field

Do a Lorentz transformation

$$\psi'(x') = S(\Lambda)\psi(x)$$

$$\Rightarrow \psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x)$$

(Recall for scalar field $\phi'(x') = \phi(x) \Rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$

for vector field $A'^{\alpha}(x') = \Lambda^{\alpha}{}_{\beta}A^{\beta}(x) \Rightarrow A'^{\alpha}(x) = \Lambda^{\alpha}{}_{\beta}A^{\beta}(\Lambda^{-1}x)$)

For an infinitesimal transformation, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}$

where $\varepsilon_{\mu\nu} \equiv g_{\mu\lambda}\varepsilon^{\lambda}{}_{\nu}$ and $\varepsilon_{\delta\gamma} + \varepsilon_{\gamma\delta} = 0$

$$\Rightarrow \psi'(x) = \left(1 - \frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right)\psi(x^{\rho} - \varepsilon^{\rho}{}_{\sigma}x^{\sigma})$$

$$= \left(1 - \frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right)\left[\psi(x) - \frac{i}{2}\varepsilon^{\mu\nu}L_{\mu\nu}\psi(x)\right]$$

$$= \psi(x) - \frac{i}{2}\varepsilon^{\mu\nu}\left(L_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}\right)\psi(x)$$

(recall for scalar field $\phi'(x) = \phi(\Lambda^{-1}x) = \phi((\delta^{\mu}{}_{\nu} - \varepsilon^{\mu}{}_{\nu})x^{\nu}) = \phi(x^{\mu} - \varepsilon^{\mu}{}_{\nu}x^{\nu})$

$$= \phi(x) - \varepsilon^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi(x)$$

$$= \phi(x) - \frac{i}{2}\varepsilon^{\mu\nu}L_{\mu\nu}\phi(x)$$

where $L_{\mu\nu} \equiv i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$

$L_{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$.

for vector field. $A'^{\alpha}(x) = \Lambda^{\alpha}{}_{\beta}A^{\beta}(\Lambda^{-1}x) \simeq (\delta^{\alpha}{}_{\beta} + \varepsilon^{\alpha}{}_{\beta})(A^{\beta}(x) - \frac{i}{2}\varepsilon^{\mu\nu}L_{\mu\nu}A^{\beta}(x))$

$$= \left[\delta^{\alpha}{}_{\beta} - \frac{i}{2}\varepsilon^{\mu\nu}(\Sigma_{\mu\nu})^{\alpha}{}_{\beta}\right](A^{\beta}(x) - \frac{i}{2}\varepsilon^{\mu\nu}L_{\mu\nu}A^{\beta}(x))$$

$$= A^{\alpha}(x) - \frac{i}{2}\varepsilon^{\mu\nu}\left[L_{\mu\nu}\delta^{\alpha}{}_{\beta} + (\Sigma_{\mu\nu})^{\alpha}{}_{\beta}\right]A^{\beta}(x)$$

where $(\Sigma_{\mu\nu})^{\alpha}{}_{\beta} = i(\delta^{\alpha}{}_{\mu}g_{\nu\beta} - \delta^{\alpha}{}_{\nu}g_{\beta\mu})$

$\Sigma_{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$.)

$$\Rightarrow \delta_0\psi(x) = -\frac{i}{2}\varepsilon^{\mu\nu}\left(L_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}\right)\psi(x)$$

write the spinor indices explicitly,

$$\delta_0\psi_a(x) = -\frac{i}{2}\varepsilon^{\mu\nu}\left[L_{\mu\nu}\delta_{ab} + \frac{1}{2}(\sigma_{\mu\nu})_{ab}\right]\psi_b(x)$$

$\left(L_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}\right)$ are the generators for infinitesimal Lorentz transformation of a Dirac field.

(note that $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}]$ is antisymmetric in $\mu \leftrightarrow \nu$)

recall for scalar field, $\delta_0 \phi(x) = \phi'(x) - \phi(x) = -\frac{i}{2}\varepsilon^{\mu\nu} L_{\mu\nu} \phi(x)$

for vector field, $\delta_0 A^\alpha(x) = -\frac{i}{2}\varepsilon^{\mu\nu}(L_{\mu\nu} + \Sigma_{\mu\nu}) A^\alpha(x)$

where $\Sigma_{\mu\nu} A^\alpha \equiv (\Sigma_{\mu\nu})^\alpha{}_\beta A^\beta$, $L_{\mu\nu} A^\alpha \equiv L_{\mu\nu}\delta^\alpha_\beta A^\beta$

$L_{\mu\nu}$ are the generators of infinitesimal Lorentz transformation of a scalar field.

$(L_{\mu\nu} + \Sigma_{\mu\nu})$ are the generators of infinitesimal Lorentz transformation of a vector field.

For the pure spatial part (i.e., for pure Lorentz rotation, no boost)

Define $L^k \equiv \frac{1}{2}\varepsilon^{ijk} L_{ij}$

$S^k \equiv \frac{1}{2}\varepsilon^{ijk} \frac{1}{2}\sigma_{ij}$

$\Rightarrow$ $L^1 = L_{23}$, $L^2 = L_{31}$, $L^3 = L_{12}$

$S^1 = \frac{1}{2}\sigma_{23}$, $S^2 = \frac{1}{2}\sigma_{31}$, $S^3 = \frac{1}{2}\sigma_{12}$

we have shown before (when we do scalar field)

$$[L^i, L^j] = i\varepsilon^{ijk} L^k$$

For $[S^i, S^j]$,

$[S^1, S^1] = [S^2, S^2] = [S^3, S^3] = 0$

use $\sigma_{ij} = \frac{i}{2}[\gamma_i, \gamma_j] = i\gamma_i\gamma_j$ for $i \neq j$.

$$\begin{aligned}
\Rightarrow [S^1, S^2] &= \frac{1}{4}[\sigma_{23}, \sigma_{31}] = \frac{1}{4} i^2 [\gamma_2\gamma_3, \gamma_3\gamma_1] \\
&= \frac{1}{4} i^2 (\gamma_2\gamma_3\gamma_3\gamma_1 - \gamma_3\gamma_1\gamma_2\gamma_3) \\
&= \frac{1}{4} i^2 (-\gamma_2\gamma_1 + \gamma_1\gamma_2) \\
&= \frac{1}{2} i^2 \gamma_1\gamma_2 \\
&= \frac{i}{2}\sigma_{12} = i\varepsilon^{123} S^3
\end{aligned}$$

$$[S^2, S^3] = \frac{1}{4}[\sigma_{31}, \sigma_{12}] = \frac{1}{4} i^2 [\gamma_3\gamma_1, \gamma_1\gamma_2]$$

$$= \frac{1}{4} i^2 (\gamma_3\gamma_1\gamma_1\gamma_2 - \gamma_1\gamma_2\gamma_3\gamma_1)$$

$$= \frac{1}{4} i^2 (-\gamma_3\gamma_2 + \gamma_2\gamma_3)$$

$$= \frac{i^2}{2} \gamma_2\gamma_3$$

$$= \frac{i}{2} \sigma_{23}$$

$$= i \varepsilon^{231} S^1$$

$$[S^3, S^1] = \frac{1}{4}[\sigma_{12}, \sigma_{23}] = \frac{1}{4} i^2 [\gamma_1\gamma_2, \gamma_2\gamma_3]$$

$$= \frac{1}{4} i^2 (\gamma_1\gamma_2\gamma_2\gamma_3 - \gamma_2\gamma_3\gamma_1\gamma_2)$$

$$= \frac{1}{4} i^2 (-\gamma_1\gamma_3 + \gamma_3\gamma_1)$$

$$= \frac{1}{4} i^2 \, 2 \gamma_3\gamma_1$$

$$= \frac{i}{2} \sigma_{31}$$

$$= i \varepsilon^{312} S^2$$

$$\Rightarrow \quad [S^i, S^j] = i \varepsilon^{ijk} S^k$$

Since $[\mathbb{1}, S^i] = 0$, then $[L^i, S^j] = 0$

Define $J^i \equiv L^i + S^i \Rightarrow [J^i, J^j] = i\varepsilon^{ijk} J^k$

So J^i is total angular momentum, L^i is orbital angular momentum, S^i is spin angular momentum.

Since $$\sum_{i=1}^{3} S^i S^i = S^1S^1 + S^2S^2 + S^3S^3 = \frac{1}{4}(\sigma_{23}\sigma_{23} + \sigma_{31}\sigma_{31} + \sigma_{12}\sigma_{12})$$

$$= \frac{1}{4} i^2 (\gamma_2\gamma_3\gamma_2\gamma_3 + \gamma_3\gamma_1\gamma_3\gamma_1 + \gamma_1\gamma_2\gamma_1\gamma_2)$$

$$= \frac{1}{4} i^2 (-1 - 1 - 1)$$

$$= \frac{3}{4}$$

$$= \frac{1}{2}\left(\frac{1}{2} + 1\right) \Rightarrow \text{spin is } \frac{1}{2}$$