Homework 4
Due date: 2019.12.18

Problem 1. [10 points] Consider a real scalar field, decomposed as

$$\phi(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}})(a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x}) d^3\vec{p},$$

for which the commutation relations are

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(\vec{p} - \vec{q}) \ ,$$
$$[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0 \, .$$

Use the above field decomposition and commutation relations, calculate

$$\langle q_1, q_2| \int_{-\infty}^{+\infty} d^4x : \phi^4(x) : |k_1, k_2\rangle \ ,$$

where

$$\begin{aligned} \langle q_1, q_2| &= C(E_{\vec{q}_1})(2\pi)^3(2E_{\vec{q}_1})C(E_{\vec{q}_2})(2\pi)^3(2E_{\vec{q}_2}) \langle 0| a_{\vec{q}_1} a_{\vec{q}_2} \ , \\ |k_1, k_2\rangle &= C(E_{\vec{k}_1})(2\pi)^3(2E_{\vec{k}_1})C(E_{\vec{k}_2})(2\pi)^3(2E_{\vec{k}_2}) a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle \ . \end{aligned}$$

The vacuum state is normalized as $\langle 0|0\rangle = 1$. The function $C(E_{\vec{p}})$ is a real function of $E_{\vec{p}}$.
Note: Please show all your steps. Please do not use the results for the one-particle state given in class. Also, please do not use the Feynman rule (if you know it), since the purpose of this problem is to let you derive the Feynman rule.

Problem 2. [5 points] Show that

$$2m\bar{v}(\vec{p}_2, s_2)\gamma^\mu u(\vec{p}_1, s_1) = \bar{v}(\vec{p}_2, s_2) \left[(p_1 - p_2)^\mu - i\sigma^{\mu\nu}(p_1 + p_2)_\nu\right] u(\vec{p}_1, s_1) \, ,$$

where $m > 0$, $(\not{p}_1 - m)u(\vec{p}_1, s_1) = 0$, $(\not{p}_2 + m)v(\vec{p}_2, s_2) = 0$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.
Note: If you need, you can use the results of problem 1 and 2 of Homework 3.