Conserved current and Noether theorem

Noether theorem: every continuous global symmetry of the action leads to a conserved current and thus a conserved charge, for solutions of the equations of motion. condition.

Start from $S = \int d^4x \, \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)]$, $\mathcal{L} \equiv \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)]$.

then $\delta S = \int d^4x \, \delta_0 \mathcal{L} = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta_0 \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 (\partial_\mu \varphi) \right]$ --- ①

here I write $\delta_0 \mathcal{L}$ instead of $\delta \mathcal{L}$ to emphasis the fact that S is not a function of x, and $\delta_0 \mathcal{L} = \mathcal{L}[\varphi'(x), \partial_\mu \varphi'(x)] - \mathcal{L}[\varphi(x), \partial_\mu \varphi]$

just continue with ①, we get.

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta_0 \varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta_0 \varphi \right] \quad \text{--- ②}$$

note that here the relation $\partial_\mu (\delta_0 \varphi) = \partial_\mu (\varphi'(x) - \varphi(x))$
$= \partial_\mu \varphi'(x) - \partial_\mu \varphi(x)$
$= \delta_0 (\partial_\mu \varphi(x))$ is used.

the first and third term can be combined as

$$\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta_0 \varphi.$$

Since we require that φ is solution of the equation of motion, this combination vanishes.

$$\Rightarrow \delta S = \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi \right) \quad \text{--- ③}$$

"A global symmety of the action" means that $\delta_0 \mathcal{L}$ is a total derivative (if $\delta_0 \mathcal{L}$ is not zero) that is,

$$\delta S = \int d^4x \, \delta_0 \mathcal{L} = \underset{\substack{\uparrow \\ \text{does not depend on spacetime}}}{\delta w} \int d^4x \, \underbrace{\partial_\mu K^\mu}_{\text{total derivative}} \quad \text{--- ④}$$

Equating 3 and 4 gives $\delta\omega \int d^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta_0 \varphi}{\delta\omega} - K^\mu \right) = 0.$

Since $\delta\omega$ is an arbitrary constant, then

$$\partial_\mu j^\mu = 0 \text{ , where } j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta_0 \varphi}{\delta\omega} - K^\mu$$

Here, of course, by writing $\frac{\delta_0 \varphi}{\delta\omega}$ we mean that the variation of φ is induced by $\delta\omega$.

If $\mathcal{L} = \mathcal{L}[\varphi_1, \varphi_2, \cdots, \varphi_N, \partial_\mu\varphi_1, \partial_\mu\varphi_2, \cdots, \partial_\mu\varphi_N]$,

then $$j^\mu = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \frac{\delta_0 \varphi_i}{\delta\omega} - K^\mu$$

We can define $Q \equiv \int d^3\vec{x}\, j^0(t, \vec{x})$, then

$$\frac{dQ}{dt} = \int d^3\vec{x}\, \partial_0 j^0(t, \vec{x}) = \int d^3\vec{x} \left(\cancelto{0}{\partial_\mu j^\mu} - \vec{\nabla} \cdot \vec{j} \right)$$

$$\underset{\uparrow}{=} - \underbrace{\int d\vec{s} \cdot \vec{j}}$$

Gauss's theorem $\hookrightarrow$ this is a surface integral.

If we <u>assume</u> that $\vec{j}$ vanishes at the spatial boundary (actually, not neccesserily be space infinity in practice, as long as the current $\vec{j}$ does not leave our experimental apparatus) then $\frac{dQ}{dt} = 0$, so we get a conserved charge Q (since it does not change with time).

Example (a) Internal transformation of a complex scalar field.

$$\mathcal{L}(x) = \partial_\mu \phi^*_{(x)} \partial^\mu \phi(x) - m^2 \phi^*_{(x)} \phi(x) - \lambda (\phi^*_{(x)} \phi_{(x)})^2$$

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\delta\alpha} \phi(x) \simeq \phi(x) - i(\delta\alpha)\phi(x)$$

$$\phi^*(x) \longrightarrow \phi'^*(x) = e^{i\delta\alpha} \phi^*(x) \simeq \phi^*(x) + i(\delta\alpha)\phi^*(x)$$

where $\delta\alpha$ is an infinitesimal real parameter, independent of x.

Since

$$\begin{aligned}
\delta_0 \mathcal{L} &= [\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - m^2 \phi'^*_{(x)} \phi'(x) - \lambda (\phi'^*(x) \phi'(x))^2] \\
&\quad - [\partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x) - \lambda (\phi^*(x) \phi(x))^2] \\
&= [e^{i\delta\alpha} \partial_\mu \phi^*(x) e^{-i\delta\alpha} \partial^\mu \phi(x) - m^2 e^{i\delta\alpha} \phi^*(x) e^{-i\delta\alpha} \phi(x) \\
&\qquad - \lambda (e^{i\delta\alpha} \phi^*(x) e^{-i\delta\alpha} \phi(x))^2] \\
&\quad - [\partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x) - \lambda (\phi^*(x) \phi(x))^2] \\
&= [\partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x) - \lambda (\phi^*(x) \phi(x))^2] \\
&\quad - [\partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x) - \lambda (\phi^*(x) \phi(x))^2] \\
&= 0
\end{aligned}$$

$$\Rightarrow K^\mu = 0$$

Since

$$\delta_0 \phi = \phi'(x) - \phi(x) = -i(\delta\alpha)\phi(x)$$

$$\delta_0 \phi^* = \phi'^*(x) - \phi^*(x) = +i(\delta\alpha)\phi^*(x)$$

Then let $\delta\omega = \delta\alpha \Rightarrow \dfrac{\delta_0 \phi}{\delta\omega} = -i\phi(x)$ and $\dfrac{\delta_0 \phi^*}{\delta\omega} = i\phi^*(x)$

$$\begin{aligned}
\Rightarrow \quad j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta_0 \phi}{\delta\omega} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta_0 \phi^*}{\delta\omega} \\
&= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-i\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} (i\phi^*) \\
&= (\partial^\mu \phi^*)(-i\phi) + (\partial^\mu \phi)(i\phi^*) = i[(\partial^\mu \phi)\phi^* - (\partial^\mu \phi^*)\phi]
\end{aligned}$$

$\Rightarrow \quad Q = \int d^3\vec{x}\, j^0 = i\int d^3\vec{x}\,(\dot{\phi}\phi^* - \dot{\phi}^*\phi)$

Example (b). Translation of a generic field.

Consider infinitesimal translation $x^\mu \longrightarrow x'^\mu = x^\mu - \delta a^\mu$, so that $\delta x^\mu = x'^\mu - x^\mu = -\delta a^\mu$, note that δa^μ is independent of x.

For a scalar field, we have shown that

$$\delta_0 \phi(x) = (\delta a^\mu)\partial_\mu \phi(x)$$

In fact, for a generic field $\varphi(x)$, we have
scalar, vector, spinor etc

$$\varphi'(x') = \varphi'(x-a) = \varphi(x)$$

This relation expresses the fact that the field has the same value at the same spacetime point, which is labeled as x in one coordinate and x' in the other coordinate, and the difference of the two coordinates is merely a translation. (or, each of the field components, for $A'^\mu(x') = A^\mu(x)$, $\psi'_a(x') = \psi_a(x)$ etc.)

$$\Rightarrow \quad \delta_0 \varphi(x) = \varphi'(x) - \varphi(x) = \varphi(x+a) - \varphi(x)$$
$$= (\delta a^\mu)\partial_\mu \varphi(x)$$

Furthermore, the Lagrangian density itself is a scalar field, that is, $\mathcal{L}'[\varphi'(x'), \partial'_\mu\varphi'(x')] = \mathcal{L}[\varphi(x), \partial_\mu\varphi(x)]$, where x' and x are related by a translation, and therefore

$$\delta_0 \mathcal{L}[\varphi(x), \partial_\mu\varphi(x)] = \delta a^\mu \partial_\mu \mathcal{L}[\varphi(x), \partial_\mu\varphi(x)] \equiv \delta a^\mu \partial_\mu \mathcal{L},$$

note that the dependence of $\mathcal{L}$ on x is achieved through the dependence of φ on x.

Let $\delta\omega = \delta a^\nu$,

$$\Rightarrow \quad K^{\mu}{}_{\nu} = \frac{\delta a^{\mu}}{\delta a^{\nu}} \mathcal{L} = \delta^{\mu}{}_{\nu} \mathcal{L}$$

since each $\delta a^0, \delta a^1, \delta a^2$ and δa^3 are independent then $\frac{\delta a^{\mu}}{\delta a^{\nu}}$ can be understood as $\frac{\partial a^{\mu}}{\partial a^{\nu}} = \delta^{\mu}{}_{\nu}$

$$\frac{\delta_0 \varphi(x)}{\delta \omega} = \frac{\delta a^{\mu}}{\delta a^{\nu}} \partial_{\mu} \varphi(x) = \delta^{\mu}{}_{\nu} \partial_{\mu} \varphi(x) = \partial_{\nu} \varphi(x)$$

$$\Rightarrow \quad j^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)} \partial_{\nu} \varphi(x) - \delta^{\mu}{}_{\nu} \mathcal{L}$$

Here we notice that the additional index ν for the Noether current is introduced by $\delta \omega$.

For $\mathcal{L} = \mathcal{L}[\varphi_1, \varphi_2, \cdots, \varphi_N, \partial_{\mu}\varphi_1, \partial_{\mu}\varphi_2, \cdots, \partial_{\mu}\varphi_N]$, we get

$$j^{\mu}{}_{\nu} = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi_i)} \partial_{\nu} \varphi_i - \delta^{\mu}{}_{\nu} \mathcal{L}$$

$$\Rightarrow \quad Q_{\nu} = \int d^3\vec{x}\, j^0{}_{\nu} = \int d^3\vec{x} \left(\sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_i)} \partial_{\nu} \varphi_i - \delta^0{}_{\nu} \mathcal{L} \right)$$

recall that $\pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_i)}$

then $Q_{\nu} = \int d^3\vec{x} \left[\sum_{i=1}^{N} \pi_i \partial_{\nu} \varphi_i - \delta^0{}_{\nu} \mathcal{L} \right]$

where $Q_0 = \int d^3\vec{x} \left[\sum_{i=1}^{N} \pi_i \partial_0 \varphi_i - \mathcal{L} \right]$

$$= \int d^3\vec{x}\, \mathcal{H} = H$$

recall that $\mathcal{H}[\pi_1, \pi_2, \cdots \pi_N, \varphi_1, \varphi_2, \cdots, \varphi_N] = \sum_{i=1}^{N} \pi_i \partial_0 \varphi_i - \mathcal{L}$

$$Q_i = \int d^3\vec{x} \sum_{j=1}^{N} \pi_j \partial_i \varphi_j$$

Since Q_0 is the Hamiltonian H, and (Q^0, Q^1, Q^2, Q^3) form a four-vector, it is natural to call $Q_i \equiv P_i$, and $j^{\mu}{}_{\nu}$ is actually just the energy-momentum tensor, usually denoted as $T^{\mu}{}_{\nu}$.

$\frac{dQ_{\mu}}{dt} = 0$ means we get $\underbrace{\text{energy}}_{Q_0}$ and $\underbrace{\text{momentum}}_{Q_i}$ conservation from the translation symmetry of the action.

In the case of a free real scalar field,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

the energy-momentum tensor $T^\mu{}_\nu$ is

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L}$$

$$= \partial^\mu\phi\,\partial_\nu\phi - \delta^\mu{}_\nu\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}m^2\phi^2\right)$$

$$\Rightarrow \quad T^{\mu\nu} = g^{\nu\beta}T^\mu{}_\beta = \partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}m^2\phi^2\right)$$

Apparently, $T^{\mu\nu} = T^{\nu\mu}$

However, it's not always automatically that $T^{\mu\nu} = T^{\nu\mu}$, which is a property we need in the case for example in General Relativity ($R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G\,T^{\mu\nu}$). Nevertheless, there is typically a way to massage the energy-momentum tensor of any theory into a symmetric form by adding an extra term

$$T^{\mu\nu} \longrightarrow T^{\mu\nu}_{new} = T^{\mu\nu} + \partial_\rho N^{\rho\mu\nu},$$

where $N^{\rho\mu\nu}$ is some function of the fields that is anti-symmetric in the first two indices so $N^{\rho\mu\nu} = -N^{\mu\rho\nu}$, and therefore

$$\partial_\mu\partial_\rho N^{\rho\mu\nu} = -\partial_\mu\partial_\rho N^{\mu\rho\nu} = -\partial_\rho\partial_\mu N^{\mu\rho\nu} = -\partial_\mu\partial_\rho N^{\rho\mu\nu} = 0$$

$$\Rightarrow \quad \partial_\mu T^{\mu\nu}_{new} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\rho N^{\rho\mu\nu} = 0 + 0 = 0$$

So, $T^{\mu\nu}_{new}$ is also conserved, i.e., $\frac{d}{dt}\int d^3\vec{x}\,T^{0\nu}_{new} = 0$

Example (c). Lorentz transformation of a free real scalar field

As shown before, $x'^\mu = \Lambda^\mu{}_\nu x^\nu = (\delta^\mu{}_\nu + \varepsilon^\mu{}_\nu) x^\nu$

$$\Rightarrow \delta x^\mu = \varepsilon^\mu{}_\nu x^\nu = \varepsilon^{\mu\nu} x_\nu$$

$$\phi'(x') = \phi(x) \Rightarrow \delta_0 \phi(x) = \phi'(x) - \phi(x) = -\varepsilon^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$
$$= -\varepsilon^{\mu\nu} x_\nu \partial_\mu \phi(x)$$

Since the Lagrangian density itself is a scalar field, then

$$\delta_0 \mathcal{L}(x) = -\varepsilon^{\mu\nu} x_\nu \partial_\mu \mathcal{L}(x)$$

where the dependence of $\mathcal{L}$ on x is achieved through the dependence of ϕ on x. In fact, the relation $\delta_0 \mathcal{L}(x) = -\varepsilon^{\mu\nu} x_\nu \partial_\mu \mathcal{L}(x)$ is valid for any Lagrangian as a function of generic fields (scalar, vector, spinor etc.) for Lorentz transformation, because Lagrangian density as a whole behaves as a scalar field, which is just a single number for a spacetime point.

Since $\partial_\mu x_\nu = g_{\mu\nu}$, we have

$$\delta_0 \mathcal{L}(x) = -\left[\varepsilon^{\mu\nu} \partial_\mu (x_\nu \mathcal{L}(x)) - (\varepsilon^{\mu\nu} \partial_\mu x_\nu) \mathcal{L}(x)\right]$$
$$= -\varepsilon^{\mu\nu} \partial_\mu (x_\nu \mathcal{L}(x)) + \underbrace{\varepsilon^{\mu\nu} g_{\mu\nu} \mathcal{L}(x)}_{0}$$
$$= -\varepsilon^{\mu\nu} \partial_\mu (x_\nu \mathcal{L}(x))$$

Let $\delta\omega = \varepsilon^{\rho\sigma}$, then

$$K^\mu{}_{\rho\sigma} = \frac{-\varepsilon^{\mu\nu} x_\nu \mathcal{L}}{\varepsilon^{\rho\sigma}} = -(\delta^\mu{}_\rho \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \delta^\nu{}_\rho) x_\nu \mathcal{L}$$

again, $\frac{\varepsilon^{\mu\nu}}{\varepsilon^{\rho\nu}}$ is understood as $\frac{\partial \varepsilon^{\mu\nu}}{\partial \varepsilon^{\rho\nu}}$

$$= -(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho) \mathcal{L}$$

$$\frac{\delta_0 \phi}{\delta\omega} = -(\delta^\mu{}_\rho \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \delta^\nu{}_\rho) x_\nu \partial_\mu \phi = -(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho) \partial_\mu \phi$$
$$= x_\rho \partial_\sigma \phi - x_\sigma \partial_\rho \phi$$

$$\Rightarrow j^\mu{}_{\rho\sigma} = (\partial^\mu \phi)(x_\rho \partial_\sigma \phi - x_\sigma \partial_\rho \phi) + (\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho)\left(\tfrac{1}{2} \partial_\lambda \phi \partial^\lambda \phi - \tfrac{1}{2} m^2 \phi^2\right)$$

$$= x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho = -j^\mu{}_{\sigma\rho}$$

$$\Rightarrow \quad Q_{\rho\sigma} = \int d^3\vec{x}\, j^0{}_{\rho\sigma} = \int d^3\vec{x}\,(x_\rho T^0{}_\sigma - x_\sigma T^0{}_\rho) = -Q_{\sigma\rho}$$

$$\Rightarrow \quad Q^{ij} = \int d^3\vec{x}\,(x^i T^{0j} - x^j T^{0i}) = -Q^{ji}$$

Since $\int d^3\vec{x}\, T^{0i} = p^i$,

then $\frac{dQ^{ij}}{dt} = 0$ means orbital angular momentum conservation.

Also, $Q^{0i} = \int d^3\vec{x}\,(x^0 T^{0i} - x^i T^{00}) = t\int d^3\vec{x}\, T^{0i} - \int d^3\vec{x}\, x^i T^{00}$
$= t p^i - \int d^3\vec{x}\, x^i T^{00} = -Q^{i0}$

$$\Rightarrow 0 = \frac{dQ^{0i}}{dt} = p^i + t\underbrace{\cancel{\frac{dp^i}{dt}}}_{0} - \frac{d}{dt}\int d^3\vec{x}\, x^i T^{00}$$

$$\Rightarrow \quad \text{constant} = p^i = \frac{d}{dt}\int d^3\vec{x}\, x^i T^{00}$$

Since $\int d^3\vec{x}\, T^{00} = H$, then the above expression means that the center of energy of the field travels with a constant velocity, i.e., we can define the center of energy as

$$x^i_{cm} = \frac{\int d^3\vec{x}\, x^i T^{00}}{\int d^3\vec{x}\, T^{00}} = \frac{\int d^3\vec{x}\, x^i T^{00}}{H}$$

$$\Rightarrow \quad \frac{dx^i_{cm}}{dt} = v^i_{cm} = \frac{\frac{d}{dt}\int d^3\vec{x}\, x^i T^{00}}{H} = \frac{p^i}{H} = \text{constant}$$

This is kind of like a field theoretic version of Newton's first law.

In sum, for translation, we get for generic fields,

$$T^\mu{}_\nu = j^\mu{}_\nu = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial_\nu \varphi_i - \delta^\mu{}_\nu \mathcal{L}$$

for Lorentz transformation, we get for generic fields,

$$j^\mu{}_{\rho\sigma} = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \frac{\delta_0 \varphi_i}{\varepsilon^{\rho\sigma}} + (\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho)\mathcal{L}$$, where $\delta x^\mu = \varepsilon^{\mu\nu} x_\nu$.