复习 Lorentz transformations

① $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$, where a^α and $\Lambda^\alpha{}_\beta$ are constants

$\Lambda^\alpha{}_\beta$ satisfy $\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta}$, $\eta_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, note that $\Lambda^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$

($\eta_{\alpha\beta} = g_{\alpha\beta}$ in special relativity)

② pure Lorentz boost, pure rotation, translations

$3 + 3 + 4 = 10$

e.g. $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$ where $\Lambda^\alpha{}_\beta = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$ where $\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$x'^\alpha = x^\alpha + a^\alpha$ where $a^\alpha = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}$

③ $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_0, \vec{\nabla}) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right)$

$\partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_0, -\vec{\nabla})$

d'Alembert $\Box \equiv \partial^2 \equiv \partial_\mu \partial^\mu = \left(\frac{\partial}{\partial x^0}\right)^2 - \vec{\nabla}^2$

④ Four-momentum $p^\mu = (p^0, \vec{p}) = (E, \vec{p})$, $p_\mu = (E, -\vec{p})$

$p^2 \equiv p_\mu p^\mu = p^\mu p_\mu = E^2 - |\vec{p}|^2 = (p^0)^2 - p^i p^i = E^2 - \vec{p}\cdot\vec{p} = p\cdot p = g_{\mu\nu} p^\mu p^\nu$

$= g^{\mu\nu} p_\mu p_\nu$

In general, four-vector inner product $a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3)$, $b^\mu = (b^0, \vec{b}) = (b^0, b^1, b^2, b^3)$

$a\cdot b = a^\mu b_\mu = g^{\mu\nu} a_\mu b_\nu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \vec{a}\cdot\vec{b}$

⑤ Consider two observers O and O' moving in two different inertial reference frames related by a Lorentz transformation. If observer O describes a field by a certain function $\varphi(x) = \varphi(t, \vec{x})$ using the coordinates of her own frame, then observer O' will describe the same field by another function $\varphi'(x') = \varphi'(t', \vec{x}')$ in terms of the transformed coordinates $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

note here it is an arbitrary function, not scalar field

Then how are $\varphi(x)$ and $\varphi'(x')$ related?

Statement: a theory consistent with relativistic principles can only contain fields that have well-defined transformation properties. They include

(a) scalar fields, $\phi'(x') = \phi(x)$ (e.g., π, K, Higgs, (spin 0))

(b) vector fields, $A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x)$ (e.g., photon, W, Z, (spin 1))

(c) tensor fields, $F'^{\mu\nu}(x') = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} F^{\rho\sigma}(x)$ (e.g., gravitational field (spin 2)
electromagnetic field tensor
(constructed from spin 1))

(d) Other fields, e.g., spinor field, describing spin $\frac{1}{2}$ particles (e.g. $e^{\pm}$, p

Now we study scalar fields first.

Why we talk about Lorentz transformations in the previous classes?

① 引出标量、矢量、张量 这些 概念

② 熟悉四矢量的计算（内积）

③ 熟悉四维时空的概念、度规、上下标变换

The ultimate goal in this course is to calculate tree-level diagram and get cross section & decay rate. So, we just need to introduce the elements we need for this goal.

Space-time translation of a scalar field

$$x^\mu \to x'^\mu = x^\mu - \underbrace{a^\mu}$$

↳ constant displacement parameter

Consider an infinitesimal transformation

$$x^\mu \to x'^\mu = x^\mu - \underbrace{\delta a^\mu}$$

↳ an infinitesmal constant

The total variation of the field is defined as

$$\delta\phi(x) \equiv \phi'(x') - \phi(x)$$

scalar field means $\delta\phi(x) = 0$

Refine $\delta_0\phi(x) \equiv \phi'(x) - \phi(x)$, which is the variation of the field alone, keeping the argument fixed.

↳ called the intrinsic variation of the field.

$$\phi'(x - \delta a) = \phi(x)$$

Substitute $x^\mu - \delta a^\mu \to x^\mu$, then $x^\mu \to x^\mu + \delta a^\mu$

$$\Rightarrow \phi'(x) = \phi(x^\mu + \delta a^\mu) \simeq (1 + \delta a^\mu \partial_\mu)\phi(x)$$

$$\Rightarrow \delta_0\phi(x) = \delta a^\mu \partial_\mu \phi(x) = i\delta a^\mu(-i\partial_\mu \phi(x))$$

A finite transformation is obtained by repeating the process

$$\phi'(x) = \lim_{n\to\infty}\left(1 + \frac{a^\mu}{n}\partial_\mu\right)^n \phi(x) = \exp(a^\mu\partial_\mu)\phi(x) \equiv U(a)\phi(x)$$

$-i\partial_\mu$ is called the generator of infinitesimal translations.

For two successive translations $U(a)$ and $U(b)$, we have

$$U(a)U(b)\phi(x) = U(b)U(a)\phi(x) \Rightarrow [U(a), U(b)] = 0$$

$$U(a)U(b)\phi(x) = \exp((a^\mu + b^\mu)\partial_\mu)\phi(x), \quad U(b)U(a)\phi(x) = \exp((b^\mu + a^\mu)\partial_\mu)\phi(x)$$

That is to say, translations do not depend on the order in which they are applied.

Lorentz transformation of a scalar field

i.e., boost and rotation

Consider infinitesimal Lorentz transformation,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \simeq (\delta^\mu{}_\nu + \varepsilon^\mu{}_\nu) x^\nu$$

where $\underset{\substack{\uparrow \\ \text{definition of } \varepsilon_{\mu\nu}}}{\varepsilon_{\mu\nu} \equiv g_{\mu\lambda}\varepsilon^\lambda{}_\nu}$ are infinitesimal constants. $(4\times4=16)$

Recall Lorentz transformations satisfy $\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta g_{\alpha\beta} = g_{\gamma\delta}$

$$\Rightarrow (\delta^\alpha{}_\gamma + \varepsilon^\alpha{}_\gamma)(\delta^\beta{}_\delta + \varepsilon^\beta{}_\delta) g_{\alpha\beta} \simeq g_{\gamma\delta}$$

up to first order of ε

$$\Rightarrow \delta^\alpha{}_\gamma \delta^\beta{}_\delta g_{\gamma\delta} + \varepsilon^\alpha{}_\gamma \delta^\beta{}_\delta g_{\alpha\beta} + \delta^\alpha{}_\gamma \varepsilon^\beta{}_\delta g_{\alpha\beta} = g_{\gamma\delta}$$

$$\Rightarrow \varepsilon_{\delta\gamma} + \varepsilon_{\gamma\delta} = 0$$

That is, $\varepsilon_{\mu\nu}$ is antisymmetric.

Recall that the infinitesimal Lorentz boost $(\hat{x}, \delta v)$ is

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{\uparrow}{\simeq} \begin{pmatrix} 1 & -\delta v & 0 & 0 \\ -\delta v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for small v (call it δv), then

$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\delta v & 0 & 0 \\ -\delta v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \delta^\alpha{}_\beta + \varepsilon^\alpha{}_\beta$$

$$\Rightarrow \varepsilon^0{}_1 = -\delta v = \varepsilon^1{}_0 \Rightarrow \varepsilon_{01} = g_{0\lambda}\varepsilon^\lambda{}_1 = g_{00}\varepsilon^0{}_1 = -\delta v,$$

$$\varepsilon_{10} = g_{1\lambda}\varepsilon^\lambda{}_0 = g_{11}\varepsilon^1{}_0 = \delta v$$

Note that if use rapidity ω, $\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where $\tanh\omega = v$, $\cosh\omega = \gamma$

then $\varepsilon_{01} = -\delta\omega$, $\varepsilon_{10} = \delta\omega$

For infinitesimal rotation $(\hat{z}, \delta\theta)$

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\theta & 0 \\ 0 & -\delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \varepsilon^{1}{}_{2} = \delta\theta , \quad \varepsilon^{2}{}_{1} = -\delta\theta$$

$$\Rightarrow \quad \varepsilon_{12} = g_{1\lambda}\varepsilon^{\lambda}{}_{2} = -\delta\theta , \quad \varepsilon_{21} = g_{2\lambda}\varepsilon^{\lambda}{}_{1} = \delta\theta$$

For scalar field,

$$\phi'(x') = \phi(x) = \phi(\Lambda^{-1}x')$$

Substitute $x' \to x$, then

$$\phi'(x) = \phi(\Lambda^{-1}x) \underset{\uparrow}{=} \phi((\delta^{\mu}{}_{\nu} - \varepsilon^{\mu}{}_{\nu})x^{\nu}) = \phi(x^{\mu} - \varepsilon^{\mu}{}_{\nu}x^{\nu})$$

check $(\delta^{\mu}{}_{\lambda} - \varepsilon^{\mu}{}_{\lambda})(\delta^{\lambda}{}_{\nu} + \varepsilon^{\lambda}{}_{\nu})$

$$= \delta^{\mu}{}_{\nu} - \varepsilon^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu} + O(\varepsilon^2)$$

$$= \delta^{\mu}{}_{\nu} + O(\varepsilon^2) \quad ✓$$

$$\simeq \phi(x) - \varepsilon^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi(x)$$

The intrinsic variation of the field is then

$$\delta_0\phi = \phi'(x) - \phi(x) = -\varepsilon^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi(x) = \tfrac{1}{2}\varepsilon^{\mu\nu}(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\phi(x)$$

where $\varepsilon^{\mu\nu} \underset{\uparrow}{\equiv} \varepsilon^{\mu}{}_{\lambda}g^{\lambda\nu} = \varepsilon_{\alpha\lambda}g^{\lambda\nu}g^{\alpha\mu} \quad \equiv -\tfrac{i}{2}\varepsilon^{\mu\nu}L_{\mu\nu}\phi(x)$

definition of $\varepsilon^{\mu\nu}$

$L_{\mu\nu} \equiv i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$, is called the generator of infinitesmal transformations.

$L_{\mu\nu}$ is antisymmetric.

It can be shown that $[L_{\mu\nu}, L_{\rho\sigma}] = -ig_{\mu\rho}L_{\nu\sigma} - ig_{\nu\sigma}L_{\mu\rho} + ig_{\mu\sigma}L_{\nu\rho} + ig_{\nu\rho}L_{\mu\sigma}$

proof: $[L_{\mu\nu}, L_{\rho\sigma}] = -[(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}) - (x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho})(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})]$

$$= [\cancel{x_{\rho}x_{\mu}\partial_{\sigma\nu}} + x_{\rho}g_{\sigma\mu}\partial_{\nu} - \cancel{x_{\rho}x_{\nu}\partial_{\sigma\mu}} - x_{\rho}g_{\sigma\nu}\partial_{\mu}$$

$$- \cancel{x_{\sigma}x_{\mu}\partial_{\rho\nu}} - x_{\sigma}g_{\rho\mu}\partial_{\nu} + \cancel{x_{\sigma}x_{\nu}\partial_{\rho\mu}} + x_{\sigma}g_{\rho\nu}\partial_{\mu}$$

$$- \cancel{x_{\mu}x_{\rho}\partial_{\nu\sigma}} - x_{\mu}g_{\nu\rho}\partial_{\sigma} + \cancel{x_{\mu}x_{\sigma}\partial_{\nu\rho}} + x_{\mu}g_{\nu\sigma}\partial_{\rho}$$

$$+ \cancel{x_{\nu}x_{\rho}\partial_{\mu\sigma}} + x_{\nu}g_{\mu\rho}\partial_{\sigma} - \cancel{x_{\nu}x_{\sigma}\partial_{\mu\rho}} - x_{\nu}g_{\mu\sigma}\partial_{\rho}]$$

note that

$$\partial_{\mu}x_{\nu} = \partial_{\mu}g_{\nu\lambda}x^{\lambda} = g_{\nu\lambda}\delta^{\lambda}{}_{\mu} = g_{\nu\mu} = g_{\mu\nu}$$

$$= -ig_{\mu\rho} L_{\nu\sigma} - ig_{\nu\sigma} L_{\mu\rho} + ig_{\mu\sigma} L_{\nu\rho} + ig_{\nu\rho} L_{\mu\sigma} \quad \checkmark$$

While for the generator of infinitesmal translation, $-i\partial_\mu$,

$$[-i\partial_\mu, -i\partial_\nu] = 0.$$

That is, the order of successive applications of two arbitrary Lorentz transformations matters, while the order of two arbitrary Lorentz

boosts and rotations

translations does not matter.

In other words, the Lorentz translation group is Abelian;
the Lorentz transformation group is non-Abelian.

boost and rotation

Space rotations are generated by the operators

$$L^1 \equiv L_{23} = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$

$$L^2 \equiv L_{31} = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \qquad \text{i.e., } L^i \equiv \frac{1}{2}\varepsilon^{ijk} L_{jk}$$

$$L^3 = L_{12} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

They satisfy $[L^i, L^j] = i\varepsilon^{ijk} L^k$

where ε^{ijk} is the completely antisymmetric Levi-Civitá tensor, such that $\varepsilon^{123} \equiv +1$.

check: $[L^1, L^2] = [L_{23}, L_{31}] = -\left\{\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right\}$

$$= \left[\cancel{zy\frac{\partial}{\partial x\partial z}} - \cancel{z^2\frac{\partial}{\partial x\partial y}} - \cancel{xy\frac{\partial}{\partial z^2}} + x\frac{\partial}{\partial y} + \cancel{xz\frac{\partial}{\partial z\partial y}}\right]$$

$$- \left[y\frac{\partial}{\partial x} + \cancel{yz\frac{\partial}{\partial z\partial x}} - \cancel{yx\frac{\partial}{\partial z^2}} - \cancel{z^2\frac{\partial}{\partial y\partial x}} + \cancel{zx\frac{\partial}{\partial y\partial z}}\right]$$

$$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = i\varepsilon^{123} L^3$$

$$[L^2, L^3] = [L_{31}, L_{12}] = -\left\{\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\}$$

$$= \left[\cancel{xz\frac{\partial}{\partial y\partial x}} - \cancel{x^2\frac{\partial}{\partial y\partial z}} - \cancel{yz\frac{\partial}{\partial x^2}} + y\frac{\partial}{\partial z} + \cancel{yx\frac{\partial}{\partial x\partial z}}\right]$$

$$-\left[z\frac{\partial}{\partial y} + \cancel{zx\frac{\partial}{\partial x\partial y}} - \cancel{zy\frac{\partial}{\partial x^2}} - \cancel{x^2\frac{\partial}{\partial z\partial y}} + \cancel{xy\frac{\partial}{\partial z\partial x}}\right]$$

$$= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = i\varepsilon^{231}L^1$$

$$[L^3, L^1] = [L_{12}, L_{23}] = -\left\{\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) - \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right\}$$

$$= \left[\cancel{yx\frac{\partial}{\partial z\partial y}} - \cancel{y^2\frac{\partial}{\partial z\partial x}} - \cancel{zx\frac{\partial}{\partial y^2}} + z\frac{\partial}{\partial x} + \cancel{zy\frac{\partial}{\partial x\partial y}}\right]$$

$$-\left[\cancel{xy\frac{\partial}{\partial y\partial z}} + x\frac{\partial}{\partial z} - \cancel{xz\frac{\partial}{\partial y^2}} - \cancel{y^2\frac{\partial}{\partial x\partial z}} + \cancel{yz\frac{\partial}{\partial x\partial y}}\right]$$

$$= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = i\varepsilon^{312}L^2$$

other: $[L^2, L^1] = -[L^1, L^2] = -i\varepsilon^{123}L^3 = i\varepsilon^{213}L^3$

$[L^3, L^2] = -[L^2, L^3] = -i\varepsilon^{231}L^1 = i\varepsilon^{321}L^1$

$[L^1, L^3] = -[L^3, L^1] = -i\varepsilon^{312}L^2 = i\varepsilon^{132}L^2$

when $i = j$, $[L^i, L^i] = 0$, and $\varepsilon^{iik} = 0$.

Done the check.

Lorentz transformation of a vector field

boost and rotation.

$$A'^{\alpha}(x') = \Lambda^{\alpha}{}_{\beta} A^{\beta}(x) = \Lambda^{\alpha}{}_{\beta} A^{\beta}(\Lambda^{-1}x')$$

For an infinitesimal Lorentz transformation, $\Lambda^{\alpha}{}_{\beta} \simeq \delta^{\alpha}{}_{\beta} + \varepsilon^{\alpha}{}_{\beta}$,

Substitute $x' \to x$, then

$$A'^{\alpha}(x) = \Lambda^{\alpha}{}_{\beta} A^{\beta}(\Lambda^{-1}x) \simeq (\delta^{\alpha}{}_{\beta} + \varepsilon^{\alpha}{}_{\beta})[A^{\beta}(x) - \varepsilon^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} A^{\beta}(x)]$$

$$= [\delta^{\alpha}{}_{\beta} - \frac{i}{2}\varepsilon^{\mu\nu}(\Sigma_{\mu\nu})^{\alpha}{}_{\beta}][A^{\beta}(x) - \frac{i}{2}\varepsilon^{\mu\nu} \underline{L_{\mu\nu}} A^{\beta}(x)]$$

where $(\Sigma_{\mu\nu})^{\alpha}{}_{\beta} \equiv i(\delta^{\alpha}{}_{\mu} g_{\nu\beta} - \delta^{\alpha}{}_{\nu} g_{\beta\mu})$ 　↳ defined before in scalar field transformation.

(check: $-\frac{i}{2}\varepsilon^{\mu\nu}(\Sigma_{\mu\nu})^{\alpha}{}_{\beta} = -\frac{i}{2}\varepsilon^{\mu\nu} i(\delta^{\alpha}{}_{\mu} g_{\nu\beta} - \delta^{\alpha}{}_{\nu} g_{\beta\mu})$

$$= \frac{1}{2}(\varepsilon^{\mu\nu}\delta^{\alpha}{}_{\mu} g_{\nu\beta} + \varepsilon^{\nu\mu}\delta^{\alpha}{}_{\nu} g_{\beta\mu})$$

$$= \frac{1}{2}(\varepsilon^{\alpha}{}_{\beta} + \varepsilon^{\alpha}{}_{\beta}) = \varepsilon^{\alpha}{}_{\beta}$$

then use notation $\Sigma_{\mu\nu} A^{\alpha} \equiv (\Sigma_{\mu\nu})^{\alpha}{}_{\beta} A^{\beta}$

$$\Rightarrow A'^{\alpha}(x) \overset{\text{to first order of } \varepsilon}{=} A^{\alpha}(x) - \frac{i}{2}\varepsilon^{\mu\nu}(L_{\mu\nu} + \Sigma_{\mu\nu})A^{\alpha}(x)$$

Note that here $L_{\mu\nu}$ and $\Sigma_{\mu\nu}$ are 4x4 matrices. while the $L_{\mu\nu}$ for scalar field Lorentz transformation is just 1x1 matrix. (because a vector field has 4 components, while a scalar field only has 1 component)

Also note that $\Sigma_{\mu\nu}$ acts on the vector label, i.e., generates the variations stemming from the <u>vectorial character of the field</u>, while
场的矢量特征
$(L_{\mu\nu})^{\alpha}{}_{\beta} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\delta^{\alpha}{}_{\beta}$ generates the variations due to the <u>functional field dependence</u>.
场的函数形式.

The antisymmetric operator $J_{\mu\nu} \equiv L_{\mu\nu} + \Sigma_{\mu\nu}$ is the generator of infinitesimal Lorentz transformations of vector fields: $\delta_0 A^{\alpha}(x) = -\frac{i}{2}\varepsilon^{\mu\nu} J_{\mu\nu} A^{\alpha}(x)$

Again, let's look at the pure spatial components, J_{ij}, $i,j=1,2,3$.

Define $J^i \equiv L^i + S^i \equiv \frac{1}{2}\varepsilon^{ijk} J_{jk}$, i.e., $L^i = \frac{1}{2}\varepsilon^{ijk} L_{jk}$, $S^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk}$

with $(L^i)^a{}_b = \frac{i}{2}\varepsilon^{ijk}(x_j\partial_k - x_k\partial_j)\delta^a{}_b$, $\underbrace{a,b=1,2,3}$

↳ consider J act on the space components of the field.

$$\underbrace{(S^i)^a{}_b = -i\varepsilon^{iab}}$$

check: $(S^i)^a{}_b = \frac{1}{2}\varepsilon^{ijk}(\Sigma_{jk})^a{}_b = \frac{i}{2}\varepsilon^{ijk}(\delta^a{}_j g_{kb} - \delta^a{}_k g_{jb})$

$$= \frac{i}{2}(-\varepsilon^{iab} + \varepsilon^{iba}) = -i\varepsilon^{iab}$$

Still we have $[L^i, L^j] = i\varepsilon^{ijk}L^k$

and we have
$$[S^i, S^j]^a{}_b = -(\varepsilon^{iac}\varepsilon^{jcb} - \varepsilon^{jac}\varepsilon^{icb})$$
$$= -(\cancel{-\delta^{ij}\delta^{ab}} + \delta^{ib}\delta^{aj} + \cancel{\delta^{ji}\delta^{ba}} - \delta^{jb}\delta^{ai})$$
$$= \delta^{jb}\delta^{ai} - \delta^{ib}\delta^{aj}$$
$$= i\varepsilon^{ijk}(-i)\varepsilon^{kab} = i\varepsilon^{ijk}(S^k)^a{}_b$$

(Note that $\varepsilon^{ijk}\varepsilon^{abk} = \delta^{ia}\delta^{jb} - \delta^{ib}\delta^{ja}$)

while $[L^i, S^j] = 0$

$\Rightarrow [J^i, J^j] = i\varepsilon^{ijk}J^k$

The square of the operator S is

$$(S^2)^a{}_b = \sum_{k,c}(S^k)^a{}_c (S^k)^c{}_b = \varepsilon^{kca}\varepsilon^{kcb} = 2\delta^a{}_b$$

which means the eigenvalue of S^2 is $s(s+1) = 1\times(1+1) = 2$.

That is, the intrinsic spin of a vector field is 1.