Homework 3
Due date: 2019.11.18

Problem 1. Start with the following defining properties of the γ-matrices,

$$
\begin{aligned}
\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= 2g_{\mu\nu}\,, \\
\gamma^\mu &= g^{\mu\nu}\gamma_\nu\,, \\
\gamma_\mu &= g_{\mu\nu}\gamma^\nu\,, \\
\gamma_\mu^\dagger &= \gamma_0\gamma_\mu\gamma_0\,, \\
\gamma^5 \equiv \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3\,,
\end{aligned}
$$

prove the following identities: [1 point for each, 2 points in total]
1) $(\gamma_0)^2 = (\gamma^0)^2 = 1$
2) $(\gamma_i)^2 = (\gamma^i)^2 = -1$, for i = 1,2,3
3) $(\gamma^5)^2 = 1$
4) $(\gamma^5)^\dagger = \gamma^5$
5) $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$
6) $\gamma_\lambda\gamma^\lambda = 4$
7) $\gamma_\lambda\gamma^\alpha\gamma^\lambda = -2\gamma^\alpha$
8) $\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = 4g^{\alpha\beta}$
9) $\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\lambda = -2\gamma^\sigma\gamma^\beta\gamma^\alpha$
10) $\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\eta\gamma^\lambda = 2(\gamma^\eta\gamma^\alpha\gamma^\beta\gamma^\sigma + \gamma^\sigma\gamma^\beta\gamma^\alpha\gamma^\eta)$
11) A, B, ... denote four-vectors, and they commute with each other (i.e., for any two vectors, $A^\mu B^\nu = B^\nu A^\mu$ for all their indices μ and ν). Use the definition of the 'slashed' vector (i.e, $\not{A} \equiv \gamma^\alpha A_\alpha$), show that
$\not{A}\not{A} = A\cdot A$
$\not{A}\not{B} + \not{B}\not{A} = 2A\cdot B$
$\gamma_\lambda\not{A}\gamma^\lambda = -2\not{A}$
$\gamma_\lambda\not{A}\not{B}\gamma^\lambda = 4A\cdot B$
$\gamma_\lambda\not{A}\not{B}\not{C}\gamma^\lambda = -2\not{C}\not{B}\not{A}$
$\gamma_\lambda\not{A}\not{B}\not{C}\not{D}\gamma^\lambda = 2(\not{D}\not{A}\not{B}\not{C} + \not{C}\not{B}\not{A}\not{D})$
Note: If you want to use the identities derived in class, please re-derive them in your work.
Please use the convention $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$.

Problem 2. Prove the following trace identities for γ-matrices: [1 point for each, 2 points in total]
1) If $(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu)$ contains an odd number of γ-matrices, then $\mathrm{Tr}(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu) = 0$.
2) If $(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu)$ contains an even number of γ-matrices, then $\mathrm{Tr}(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu...\gamma^\beta\gamma^\alpha)$.
3) $\mathrm{Tr}(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}$.
4) $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$.
5) $\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\alpha) = \mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta) = \mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu) = 0$.
Note: You can use the results in problem 1 if you need.

Problem 3. The γ-matrices in the standard representation are

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where $\mathbb{1}$ is the 2×2 identity matrix, 0 is the 2×2 zero matrix, and σ^i $(i = 1, 2, 3)$ are the 2×2 Pauli matrices, namely,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1) Show that the γ-matrices in this representation satisfy the two defining properties, (a) $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and (b) $\gamma^{\dagger\mu} = \gamma^0\gamma^\mu\gamma^0$. [3 points]
2) Calculate γ^5. [1 point]
Note: If you want to use the properties of the Pauli matrices to do this problem (for example, the commutation and/or the anticommutation relations of the Pauli matrices, the Hermitian property of them), please prove these properties first. If you would rather check (a) and (b) for all the μ and ν indices explicitly, you can either do it by hand or print your computer program.

Problem 4. Repeat the calculations of problem 3 (namely, check the two defining properties [1 point] and calculate γ^5 [1 point]), but this time for the γ-matrices in the Weyl representation, in which

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Note: The spatial components of the γ-matrices in the standard representation and in the Weyl representation are the same. Therefore, you don't need to check the parts which you have already done in problem 3.

Problem 5. Prove the trace identity

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma},$$

where $\epsilon^{\mu\nu\lambda\sigma} = -1$ if $\mu\nu\lambda\sigma$ is an even permutation of 0123, $+1$ for an odd permutation, 0 if any two indices are the same. [2 point]
Note: You can use the results in problems 1 and 2 if you need.

Problem 6. The 2×2 Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that often we use numerical indices: $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$; $\vec{\sigma}$ is not part of a four-vector, and we do not distinguish upper and lower indices: $\sigma^1 = \sigma_1$, $\sigma^2 = \sigma_2$, $\sigma^3 = \sigma_3$. Show that
1)

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad [1 \text{ point}]$$

where a 2×2 unit matrix is implied in the first term, and summation over k in the second. Note that ϵ_{ijk} is the Levi-Civita symbol: $\epsilon_{ijk} = 1$ if ijk is an even permutation of 123, -1 for an odd permutation, 0 if any two indices are the same.

2) For any three-vectors $\vec{a}$ and $\vec{b}$,

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b}) + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})\,. \text{ [1 point]}$$

3)

$$e^{i\vec{\theta}\cdot\vec{\sigma}} = \cos\theta + i\hat{\theta} \cdot \vec{\sigma} \sin\theta\,, \text{ [1 point]}$$

where $\theta \equiv |\vec{\theta}|$ and $\hat{\theta} \equiv \vec{\theta}/|\vec{\theta}|$.