$\hbar = c\,(= k_B) = 1$ unit. SI unit. m, kg, sec.

$1\,eV = 1.6\times10^{-19}\,J$ 电子伏特 (eV)

$W = UQ$

$[J] = [V][C]$

$1\,MeV = 10^6\,eV$

$1\,keV = 10^3\,eV$

$1\,GeV = 10^9\,eV$

$1\,TeV = 10^{12}\,eV$

$c = 3\times10^8\,m\cdot sec^{-1}$

$$\hbar = \frac{h}{2\pi} = 1.05\times10^{-34}\,J\cdot sec = 1.05\times10^{-34}\times 1eV\,\frac{1J}{1eV}\cdot sec$$

$$= 1.05\times10^{-34}\,eV\,\frac{J}{1.6\times10^{-19}J}\cdot sec = 6.6\times10^{-16}\,eV\cdot sec$$

$\Rightarrow \hbar c = 20\times10^{-8}\,eV\cdot m = 200\,MeV\cdot fm$

$k_B = 1.38\times10^{-23}\,J\cdot K^{-1}$

If set $c=1$, then $1\,sec = 3\times10^8\,m$, so $[L]=[T]$, i.e., length and time are equivalent dimensions. (so velocity becomes dimensionless)

If set $\hbar=1$, then $1\,sec = \dfrac{1}{1.05\times10^{-34}J} = 1.5\times10^{21}\,MeV^{-1}$, so $[T]=[E]^{-1}$ (so action becomes dimensionless)

In Newtonian physics $P = mv$, $KE = \frac{1}{2}mv^2$, then $[P]=[M]=[E]$.

In special relativity $E^2 = P^2c^2 + m^2c^4$, so $c=1$ makes $[E]=[P]=[M]$

$$1\,m = \frac{1\,sec}{3\times10^8} = \frac{1.5\times10^{21}}{3\times10^8}\,MeV^{-1} = 5\times10^{12}\,MeV^{-1}$$

$$1\,kg = \frac{1\,J}{1\cdot m^2\cdot sec^{-2}} = \frac{\frac{1}{1.6\times10^{-19}}\,eV\cdot(1.5\times10^{21}MeV^{-1})^2}{(5\times10^{12}MeV^{-1})^2} = 5.6\times10^{29}\,MeV$$

mass of electron: $9.1\times10^{-31}kg = 9.1\times10^{-31}\times5.6\times10^{29}\,MeV = 0.51\,MeV$

G: (from $F = \dfrac{GmM_2}{r^2} \Rightarrow [F] = [J]\cdot[m]^{-1} \Rightarrow [G] = [J][m]^{-1}\cdot[m]^2[kg]^{-2}$)

$G = 6.67\times10^{-11}\,J\cdot m\cdot kg^{-2} = 6.67\times10^{-11}\,kg\cdot m^2\cdot sec^{-2}\cdot m\cdot kg^{-2} = 6.67\times10^{-11}\,m^3\cdot kg^{-1}\cdot sec^{-2}$

$\Rightarrow G = 6.67\times10^{-11}\,(5\times10^{12}\,\mathrm{MeV}^{-1})^3\cdot(5.6\times10^{-9}\,\mathrm{MeV})^{-1}\,(1.5\times10^{21}\,\mathrm{MeV}^{-1})^{-2} = 6.7\times10^{-39}\,\mathrm{GeV}^{-2}$

usually

So we use ω as energy (since $\hbar\omega = E$), $\vec{k}$ (波数) as momentum (since $\vec{p} = \hbar\vec{k}$)

If set $k_B = 1$, then $1\,K = 1.38\times10^{-23}\,J = 1.38\times10^{-23}\,\frac{1}{1.6\times10^{-19}}\,eV$

$= 8.6\times10^{-11}\,\mathrm{MeV}$

Fine structure constant

$$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \quad \text{where } \varepsilon_0 = 8.85\times10^{-12}\,F\cdot m^{-1} = 8.85\times10^{-12}\,C^2 J^{-1}\cdot m^{-1}$$

$$\left(1F = \frac{1\,\mathrm{Coulomb}}{1V} = \frac{1\,\mathrm{Coulomb}^2}{1J}\right)$$

$$\equiv \frac{k_e e^2}{\hbar c}$$

(note: ε_0 is vacuum permittivity, also called electric constant)

and $k_e \simeq 9\times10^9\,J\cdot m\cdot C^{-2}$

$$= \frac{9\times10^9\,J\cdot m\cdot C_{\mathrm{Coulomb}}^{-2}\times(1.6\times10^{-19}\,C_{\mathrm{Coulomb}})^2}{\hbar c}$$

$$= 9\times10^9\times(1.6\times10^{-19})^2\,J\cdot m = 9\times10^9\times(1.6\times10^{-19})^2\times\frac{1\,eV}{1.6\times10^{-19}}\,5\times10^{12}\,\mathrm{MeV}^{-1}$$

$$= \frac{1}{137}$$

If we take $\varepsilon_0 = 1$, $\hbar = 1$, $c = 1$, then $\alpha = \frac{e^2}{4\pi}$

then $1\,\mathrm{coulomb} = (8.85\times10^{-12})^{-\frac{1}{2}}(J\cdot m)^{\frac{1}{2}}$

$$= (8.85\times10^{-12})^{-\frac{1}{2}}\left(\frac{1\,eV}{1.6\times10^{-19}}\,5\times10^{12}\,\mathrm{MeV}^{-1}\right)^{\frac{1}{2}}$$

$$= 1.9\times10^{18}$$

$$\Rightarrow \quad e = 1.6\times10^{-19}\,\mathrm{coulomb} = 0.3$$

this dimensionless number "e" is what appeared in interaction Lagrangian, e.g., $\mathcal{L}_{int} = e\bar{\psi}\gamma^\mu\psi A_\mu$, or just write it as

$$\mathcal{L}_{int} = \sqrt{4\pi\alpha}\,\bar{\psi}\gamma^\mu\psi A_\mu$$

Note: α is dimensionless by definition, while e is dimensionless by taking $\underbrace{\varepsilon_0\hbar c}_{\text{the product}}$ dimensionless.

计算静电力和引力的比值

以两个电子为例.

$$F_{电} = \frac{K_e e^2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \quad (\text{i.e., } F = K_e q^2/r^2)$$

$$F_{引} = \frac{G m_e^2}{r^2}$$

$$\Rightarrow \frac{F_{电}}{F_{引}} = \frac{K_e e^2}{G m_e^2} = \frac{\overset{J\cdot m\cdot C^{-2}}{9\times10^9} \times (1.6\times10^{-19}C)^2}{\underset{J\cdot m\cdot kg^{-2}}{6.67\times10^{-11}} \times (9.1\times10^{-31}kg)^2} \simeq 4\times10^{42}$$

or directly use natrul unit ($\varepsilon_0 = \hbar = c = 1$)

$$F_{电} = \frac{e^2}{4\pi r^2} = \frac{\alpha\hbar c}{r^2} = \frac{\alpha}{r^2}$$

$$\Rightarrow \frac{F_{电}}{F_{引}} = \frac{\frac{1}{137}}{6.7\times10^{-39}GeV^{-2}\cdot(0.51MeV)^2} \simeq 4\times10^{42}$$

or $$\frac{F_{电}}{F_{引}} = \frac{\alpha\hbar c}{G m_e^2} = \frac{\frac{1}{137}\times 1.05\times10^{-34}J\cdot sec\cdot 3\times10^8 m\cdot sec^{-1}}{6.67\times10^{-11}J\cdot m\cdot kg^{-2}\times(9.1\times10^{-31}kg)^2}$$

$$= 9\times10^9 J\cdot m\cdot C^{-2}(1.6\times10^{-19}C)^2$$

$$\simeq 4\times10^{42}$$

or $$\frac{F_{电}}{F_{引}} = \frac{9\times10^9\cdot J\cdot m\, C^{-2}\times(1.6\times10^{-19}C)^2}{6.7\times10^{-39}GeV^{-2}\,(0.51MeV)^2}$$

use $1J = \frac{1}{1.6\times10^{-19}}eV$,

$1m = 5\times10^{12}MeV^{-1}$

$$\Rightarrow \frac{F_{电}}{F_{引}} = \frac{9\times10^9\frac{1}{1.6\times10^{-19}}eV\cdot 5\times10^{12}MeV^{-1}\cdot(1.6\times10^{-19})^2}{6.7\times10^{-39}GeV^{-2}\,(0.51MeV)^2}$$

$$= \frac{\frac{1}{137}}{6.7\times10^{-39}GeV^{-2}\cdot(0.51MeV)^2} \simeq 4\times10^{42}$$

氢原子电离能为 13.6 eV,

then calculate wave length:

$$\lambda = \frac{c}{\nu} = \frac{c}{E/h} = \frac{2\pi \hbar c}{E} = \frac{2\pi}{13.6\,eV}$$

$$= \frac{2\pi}{13.6\times10^{-6}\,MeV} = \frac{2\pi}{13.6\times10^{-6}}\,\frac{1}{MeV}$$

$$= \frac{2\pi}{13.6\times10^{-6}}\,\frac{1\,m}{5\times10^{12}}$$

$$\simeq 9\times10^{-8}\,m$$

<u>Construct Planck mass from $\hbar$, c, G</u>

start by writing the dimension of $\hbar$, c and G in terms of m, kg, sec,

$$[\hbar] = kg\cdot m^2\cdot sec^{-1}$$
$$[c] = m\cdot sec^{-1}$$
$$[G] = m^3\cdot kg^{-1}\cdot sec^{-2}$$

then $[\hbar]^{n_1}[c]^{n_2}[G]^{n_3} = kg^{n_1}\cdot m^{2n_1}\,sec^{-n_1}\,m^{n_2}\,sec^{-n_2}\cdot m^{3n_3}\,kg^{-n_3}\,sec^{-2n_3}$

need

$$n_1 - n_3 = 1 \quad ①$$
$$2n_1 + n_2 + 3n_3 = 0 \quad ②$$
$$-n_1 - n_2 - 2n_3 = 0 \quad ③$$
$$\Rightarrow \quad -n_2 - 3n_3 = 1 \quad (① + ③)$$
$$\Rightarrow \quad 2(1+n_3) + (-1-3n_3) + 3n_3 = 0 \quad (②)$$

$$n_3 = -\frac{1}{2}$$
$$n_1 = \frac{1}{2}$$
$$n_2 = \frac{1}{2}$$

$$\Rightarrow \quad m_{pl} = \left(\frac{\hbar c}{G}\right)^{\frac{1}{2}} = 1.2\times10^{19}\,GeV$$
$$\simeq 2\times10^{-8}\,kg$$

Planck length:

$$n_1 - n_3 = 0 \quad ①$$
$$2n_1 + n_2 + 3n_3 = 1 \quad ②$$
$$-n_1 - n_2 - 2n_3 = 0 \quad ③$$

$\Rightarrow$ $n_1 = n_3$ (①)

$n_1 + n_3 = 1$ (②+③)

$\Rightarrow$ $n_1 = n_3 = \frac{1}{2}$

$n_2 = -\frac{3}{2}$

$$\Rightarrow l_{pl} = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} = 8 \times 10^{-20}\ \text{GeV}^{-1}$$
$$= 1.6 \times 10^{-35}\ \text{m}$$

Planck energy.

$$E_{pl} = m_{pl} c^2 = 1.2 \times 10^{18}\ \text{GeV} = 2 \times 10^{9}\ \text{J}$$
$$= \left(\frac{\hbar c^5}{G}\right)^{\frac{1}{2}}$$

Planck time (or directly $t_{pl} = \frac{l_{pl}}{c} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}}$

$$n_1 - n_3 = 0 \quad ①$$
$$2n_1 + n_2 + 3n_3 = 0 \quad ②$$
$$-n_1 - n_2 - 2n_3 = 1 \quad ③$$

$\Rightarrow$ $-n_2 - 3n_3 = 1$ (①+②)

$\Rightarrow$ $2n_3 + (-3n_3 - 1) + 3n_3 = 0$

$\Rightarrow$ $n_3 = \frac{1}{2}$

$n_1 = \frac{1}{2}$

$n_2 = -\frac{5}{2}$

$$\Rightarrow t_{pl} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} = 8 \times 10^{-20}\ \text{GeV}^{-1} = 5 \times 10^{-44}\ \text{sec}$$

Note that sometimes reduced planck mass is used

$$\sqrt{\frac{\hbar c}{8\pi G}} \simeq 2.4 \times 10^{18}\ \text{GeV} = 4 \times 10^{-9}\ \text{kg}$$

this 8π simplifies equations in general relativity

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Reconstruct the correct power of c, $\hbar$.

Bohr radius of Hydrogen.

$$a = \frac{1}{\alpha\mu} \qquad \text{where } \mu = \frac{m_e m_p}{m_e + m_p} \simeq m_e$$

$$= \frac{1}{\frac{1}{137}\, 0.511\,\text{MeV}} = \frac{1}{\frac{1}{137} \times 0.511} \frac{1\,\text{m}}{5 \times 10^{12}} = 0.53 \times 10^{-10}\ \text{m}.$$

or start from

SI unit. $$a = \frac{\hbar^2 4\pi\varepsilon_0}{\mu e^2} = \frac{(1.05 \times 10^{-34}\ \text{J}\cdot\text{sec})^2\, 4\pi\, 8.85 \times 10^{-12}\, \text{C}^2\, \text{J}^{-1}\, \text{m}^{-1}}{9.1 \times 10^{-31}\ \text{kg} \cdot (1.6 \times 10^{-19}\ \text{C})^2}$$

$$= 0.53 \times 10^{-10}\ \underbrace{\underbrace{\frac{\text{J}}{\text{kg}}\, \text{sec}^2}_{\frac{\text{kg}\, \text{m}^2 \cdot \text{sec}^{-2}\ \text{sec}^2 \cdot \text{m}^{-1}}{\text{kg}}}}_{\text{m}}$$

Question. if we are given $a = \frac{1}{\alpha\mu}$, how to go back $a = \frac{\hbar^2 4\pi\varepsilon_0}{\mu e^2}$?

From $a = \frac{\hbar^2 \cdot 4\pi\varepsilon_0}{\mu e^2}$ to $a = \frac{1}{\alpha\mu}$ is easy.

(Just use $\frac{e^2}{4\pi\varepsilon_0 \hbar c} = \alpha$, and then set $\hbar = c = 1$)

we get $a = \frac{\hbar^2}{\mu \alpha \hbar c} = \frac{1}{\mu\alpha}$.)

From $a = \frac{1}{\alpha\mu}$, first put ε_0 in by using

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

So $$a = \frac{4\pi\varepsilon_0 \hbar c}{e^2 \mu}$$

then write $a = \frac{4\pi\varepsilon_0 \hbar c}{e^2\mu} \hbar^{n_1} c^{n_2}$

and let the dimension by $[m]$ to get $n_1 \& n_2$

using $[\varepsilon_0] = [\text{Coulomb}]^2 [J]^{-1} [m]^{-1}$

$[e] = [\text{Coulomb}]$

$[\mu] = [kg]$

$[\hbar] = kg \cdot m^2 \, sec^{-1}$

$[c] = m \cdot sec^{-1}$

$\Rightarrow$ $[kg]^{n_1+1} [m]^{2(n_1+1)} [sec]^{-(n_1+1)} [m]^{n_2+1} [sec]^{-(n_2+1)} [J]^{-1} [m]^{-1}$

$[kg]^{-1} \equiv [m]$

and using $[J] = [kg][m]^2 [sec]^{-2}$

$\Rightarrow$ $n_1 + 1 - 1 - 1 = 0$

$2(n_1+1) + (n_2+1) - 1 - 2 = 1$

$-(n_1+1) - (n_2+1) + 2 = 0$

$\Rightarrow$ $n_1 = 1$

$n_2 = -1$

$\Rightarrow$ $$a = \frac{4\pi\varepsilon_0 \hbar^2}{e^2 \mu}$$

In $8\pi G = \hbar = c = k_B = 1$ unit, the black hole temperature is $T = \frac{1}{M}$, get back the version in SI unit.

$$T = \frac{1}{M} (8\pi G)^{n_1} (\hbar)^{n_2} (c)^{n_3} (k_B)^{n_4}.$$

$$[G] = m^3 \cdot kg^{-1} \cdot sec^{-2}$$
$$[\hbar] = kg \cdot m^2 \cdot sec^{-1}$$
$$[c] = m \cdot sec^{-1}$$
$$[k_B] = J \cdot K^{-1} = kg \cdot m^2 \cdot sec^{-2} \cdot K^{-1}.$$
$$[T] = K$$
$$[M] = kg$$

$$m^{3n_1} kg^{-n_1} sec^{-2n_1} \cdot kg^{n_2} m^{2n_2} sec^{-n_2} \cdot m^{n_3} sec^{-n_3} \cdot kg^{n_4} m^{2n_4} sec^{-2n_4} K^{-n_4} kg^{-1} = K$$

$\Rightarrow$

$$n_4 = -1$$

$$3n_1 + 2n_2 + n_3 + 2n_4 = 0$$
$$-n_1 + n_2 + n_4 - 1 = 0$$
$$-2n_1 - n_2 - n_3 - 2n_4 = 0$$

$\Rightarrow$

$$3n_1 + 2n_2 + n_3 = 2 \quad ①$$
$$-n_1 + n_2 = 2 \quad ②$$
$$-2n_1 - n_2 - n_3 = -2 \quad ③$$

$\Rightarrow$

$$5n_1 + n_3 = -2 \quad (① - 2②)$$
$$-3n_1 - n_3 = 0 \quad (② + ③)$$

$\Rightarrow$

$$5n_1 + (-3n_1) = -2 \Rightarrow n_1 = -1$$
$$n_3 = 3$$
$$n_2 = 1$$

$$\Rightarrow T = \frac{\hbar c^3}{M 8\pi G k_B}$$

WIMP miracle

$$\langle \sigma v \rangle = 3 \times 10^{-26}\ cm^3 \cdot sec^{-1}.$$

Now convert it to GeV^{-2}

$$cm^3 = (10^{-2}\ m)^3 = (10^{-2} \times 5 \times 10^{12}\ MeV^{-1})^3.$$

$$sec^{-1} = (1.5 \times 10^{21}\ MeV^{-1})^{-1}$$

$$\Rightarrow \quad cm^3 \cdot sec^{-1} = (10^{-2} \times 5 \times 10^{12})^3 \times (1.5 \times 10^{21})^{-1} \cdot \underbrace{MeV^{-2}}_{\underbrace{(10^{-3} GeV)^{-2}}_{10^6\ GeV^{-2}}}$$

$$\Rightarrow \langle \sigma v \rangle$$

$$= 2.5 \times 10^{-9}\ GeV^{-2}$$

$$= \frac{1}{(100\ GeV)^2}\ 100^2 \times 2.5 \times 10^{-9}$$

$$= \frac{2.5 \times 10^{-5}}{(100\ GeV)^2}$$

$$= \frac{\alpha^2 \cdot (2.5 \times 10^{-5}/\alpha^2)}{(100\ GeV)^2}$$

$$= 0.5\ \frac{\alpha^2}{(100\ GeV)^2}.$$

Estimate the number of CMB photons in $1cm^3$.

From blackbody radiation of CMB., we know

energy density $\rho \propto T^4$

the average photon energy is $\sim T$

so the number density is T^3

Since it is 2.73K temperature, we have

$$n \sim (2.73K)^3 = (2.73 \times 8.6 \times 10^{-11} MeV)^3$$

$$= (2.73 \times 8.6 \times 10^{-11})^3 \cdot (5 \times 10^{12})^3 \cdot \underbrace{m^{-3}}_{\frac{1}{1m^3} = \frac{1}{(10^2 cm)^3} = 10^{-6} cm^{-3}}$$

$$= (2.73 \times 8.6 \times 10^{-11} \times 5 \times 10^{12} \times 10^{-2})^3 \ cm^{-3}$$

$$= 1.6 \times 10^3 \ cm^{-3}$$

The order one coefficient is $\frac{2 \times \zeta(3)}{\pi^2} = 0.25$

and the result is $n \sim 400 \ cm^{-3}$.

the SI unit formula is

$$n = \frac{2 \times \zeta(3)}{\pi^2} \frac{k^3}{\hbar^3 c^3} T^3$$

check unit:

$[k] = J \cdot K^{-1}$, $[\hbar] = J \cdot sec$ $[c] = m \cdot sec^{-1}$

$$\Rightarrow \left[\frac{k^3}{\hbar^3 c^3} T^3\right] = \left(\frac{J \cdot K^{-1} \cdot K}{J \cdot sec \cdot m \cdot sec^{-1}}\right)^3 = m^{-3} \quad \checkmark$$

but the point is that in natural unit, you don't need to worry about $\hbar$, c & k_B.

In $\hbar = c = 1$ unit

$v = 0.3$ means v is 0.3 speed of light

In $\hbar = c = G = 1$ unit,

$[L]^3 [M]^{-1} [T]^{-2}$ is the SI unit of G means

$[M] = [L]$ since $c = 1$ gives $[L] = [T]$

(However $\hbar = 1$ means $[E] = [T^{-1}]$, then together with $[E] = [M][L]^2[T]^{-2} \Rightarrow [E] = [M]$)

or from $\hbar = 1.05 \times 10^{-34}$ J·sec $= 1.05 \times 10^{-34}$ kg·m²·sec⁻¹

$\Rightarrow$ $[M][L]^2[T]^{-1}$ is dimensionless when let $\hbar = 1$

$\Rightarrow$ $[M] = [L]^{-1}$

Therefore in $\hbar = c = G = 1$ unit,

$[M] = [L]$

$[M] = [L]^{-1}$

$\Rightarrow$ $[M]$ is dimensionless.

$[L]$ is dimensionless

$[T]$ is dimensionless

By solving
$$\begin{cases} \hbar = 1.05 \times 10^{-34}\ \text{kg}\cdot\text{m}^2\cdot\text{sec}^{-1} = 1 & ① \\ c = 3 \times 10^8\ \text{m}\cdot\text{sec}^{-1} = 1 & ② \\ G = 6.67 \times 10^{-11}\ \text{m}^3\cdot\text{kg}^{-1}\ \text{sec}^{-2} = 1 & ③ \end{cases}$$

$\Rightarrow$ $1\ \text{sec} = 3 \times 10^8\ \text{m}$ (from ②)

$1.05 \times 10^{-34}\ \text{kg}\cdot\text{m}^2 \cdot \dfrac{1}{3 \times 10^8\ \text{m}} = 1$ (from ①)

$\Rightarrow$ $1\ \text{kg} = \dfrac{3 \times 10^8}{1.05 \times 10^{-34}}\ \text{m}^{-1}$

$\Rightarrow$ $6.67 \times 10^{-11}\ \text{m}^3 \cdot \left(\dfrac{1.05 \times 10^{-34}}{3 \times 10^8}\ \text{m}\right)(3 \times 10^8\ \text{m})^{-2} = 1$ (from ③)

$\Rightarrow$ $1\ \text{m} = 6.2 \times 10^{34}$

Recall planck length $l_{pl} = \left(\dfrac{\hbar G}{c^3}\right)^{\frac{1}{2}} = 1.6 \times 10^{-35}$ m ($= 1$ in $c = \hbar = G = 1$ unit)

So $1\ \text{m} = 6.2 \times 10^{34}$ means $1\ \text{m} = 6.2 \times 10^{34}\ l_{pl}$

$c = 1 \Rightarrow [L] = [T]$

$\hbar = 1 \Rightarrow [E] = [T]^{-1}$

Since $[E] = [M]\cdot[L]^2[T]^{-2} \Rightarrow [E] = [M]$

That is, $[L] = [E]^{-1}$, $[T] = [E]^{-1}$, $[M] = [E]^{-1}$.

so all length, time and energy can be given unit of GeV^{-1} and GeV.

$c = 1 \Rightarrow 1\,sec = 3\times10^8\,m$

$\hbar = 1 \Rightarrow 1.05\times10^{-34}\,kg\cdot m^2\cdot sec^{-1} = 1 \Rightarrow 1.05\times10^{-34}\,kg\cdot m\cdot\frac{1}{3\times10^8} = 1$

$\Rightarrow 1\,kg = 3\times10^8\,\frac{1}{1.05\times10^{-34}}\,m^{-1}$

$8\pi G = 1 \Rightarrow 6.67\times10^{-11}\,kg^{-1}\cdot m^3\cdot sec^{-2} = \frac{1}{8\pi}$

$\Rightarrow 6.67\times10^{-11}\,\frac{1.05\times10^{-34}}{3\times10^8}\,m\;m^3\cdot\left(\frac{1}{3\times10^8\,m}\right)^2 = \frac{1}{8\pi}$

$\Rightarrow 1\,m = 1.2\times10^{34}$

(Recall $l_{pl} = 1.6\times10^{-35}\,m = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}}$

If define reduced planck length as $\left(\frac{\hbar\cdot 8\pi G}{c^3}\right)^{\frac{1}{2}}$

then $l_{pl,reduced} = 8.0\times10^{-35}\,m$)

Therefore, $1\,m = 1.2\times10^{34}$ means $1\,m = 1.2\times10^{34}\,l_{pl,reduced}$.

Similarly, $c = \hbar = 8\pi G = 1$

$\Rightarrow 1\,kg = 3\times10^8\,\frac{1}{1.05\times10^{-34}}\,\frac{1}{1.2\times10^{34}} = 2.4\times10^8$

(Recall the reduced planck mass is defined as $\left(\frac{\hbar c}{8\pi G}\right)^{\frac{1}{2}} = 4\times10^{-9}\,kg$)

Therefore, $1\,kg = 2.4\times10^8$ means $1\,kg = 2.4\times10^8\,m_{pl,reduced}$.

$1\,sec = 3\times10^8\,m = 3\times10^8\times1.2\times10^{34} = 3.6\times10^{42}$.

(recall that the Planck time is $\left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} = 5\times10^{-44}\,sec$, then define reduced Planck time as $\left(\frac{\hbar\cdot 8\pi G}{c^5}\right)^{\frac{1}{2}} = 2.5\times10^{-43}\,sec$)

Therefore $1\,sec = 3.6\times10^{42}$ means $1\,sec = 3.6\times10^{42}\,t_{pl,reduced}$

Similarly

$$1\ \text{sec} = 3\times10^{8}\ \text{m} = 3\times10^{8} \times 6.2\times10^{34} = 2\times10^{43}$$

recall $t_{pl} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} = 5\times10^{-44}\ \text{sec}$ ($= 1$ in $c=\hbar=G=1$ unit.)

So $1\ \text{sec} = 2\times10^{43}$ means 1 sec is $2\times10^{43}\ t_{pl}$.

$$1\ \text{kg} = \frac{3\times10^{8}}{1.05\times10^{-34}}\ \frac{1}{6.2\times10^{34}} = 5\times10^{7}$$

recall $m_{pl} = \left(\frac{\hbar c}{G}\right)^{\frac{1}{2}} = 2\times10^{-8}$

So $1\ \text{kg} = 5\times10^{7}\ m_{pl}$

Therefore, in $G=\hbar=c=1$ unit, the length is measured in terms of Planck length, mass is measured in terms of Planck mass, and time is measured in terms of Planck time.

However, choose natural unit need to avoid inconsistency.

For example, we can not simultaneously choose both the electron and proton mass to be 1, because the proton/electron mass ratio is a fixed dimensionless number which should not depend on the choice of unit. (i.e., $\frac{m_p}{m_e} = 1836$, $\frac{m_p}{m_e}$ never equals 1, so cannot choose $m_p = 1$ & $m_e = 1$ simultaneously)

Atomic units

$$\boxed{\begin{aligned} m_e &\equiv 1 \\ e &\equiv 1 \\ \hbar &\equiv 1 \\ k_e = \frac{1}{4\pi\varepsilon_0} &\equiv 1 \end{aligned}}$$

Therefore, since $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \dfrac{1}{137}$, then $c \simeq 137$

NOTE that dimensionless physical constants retain their values in any system of units.

Since in SI unit,

$$k_e = 9\times10^{9}\ \text{Coulomb}^{-2}\cdot kg\, m^3 \cdot sec^{-2}$$
$$\hbar = 1.05\times10^{-34}\ kg\cdot m^2\cdot sec^{-1}$$
$$e = 1.6\times10^{-19}\ \text{Coulomb}$$
$$m_e = 9.1\times10^{-31}\ kg$$

$\Rightarrow$ In Atomic Units,

$$\boxed{\begin{aligned} 1\ kg &= \frac{1}{9.1\times10^{-31}} = 1.1\times10^{30} \\ 1\ \text{Coulomb} &= \frac{1}{1.6\times10^{-19}} = 6.3\times10^{18} \end{aligned}}$$

$$\hbar = 1.05\times10^{-34}\cdot\frac{1}{9.1\times10^{-31}}\ m^2\cdot sec^{-1} \equiv 1 \qquad ①$$

$$k_e = 9\times10^{9}\,\frac{(1.6\times10^{-19})^2}{9.1\times10^{-31}}\ m^3\cdot sec^{-2} \equiv 1 \qquad ②$$

$$\Rightarrow \boxed{\begin{aligned} 1\ m &= 1.9\times10^{10} \\ 1\ sec &= 4.2\times10^{16} \end{aligned}} \qquad \begin{aligned} &(①^2/②) \\ &(①^3/②^2) \end{aligned}$$

$$\Rightarrow \text{check}: 3\times10^{8}\ m\cdot sec^{-1} = 3\times10^{8}\cdot\frac{1.9\times10^{10}}{4.2\times10^{16}} \simeq 137 \quad ✓$$

Note that Bohr radius is $a_0 = \dfrac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \simeq 5.3\times10^{-11}\ m$ in SI unit

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1$ in Atomic unit

So $1\ m = 1.9\times10^{10}\ a_0$.

Check dimension:

$$\left[\frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}\right] = \frac{\text{Coulomb}^2\ \text{kg}^{-1}\ \text{m}^{-3}\ \text{sec}^2 \cdot \text{kg}^2\ \text{m}^4\ \text{sec}^{-2}}{\text{kg} \cdot \text{Coulomb}^2} = \text{m} \ \checkmark$$

Now try to find a combination of $1/(4\pi\varepsilon_0)$, $\hbar$, m_e and e with dimension second, so that we can understand what is 1 sec in terms of.

$$\left[\frac{1}{4\pi\varepsilon_0}\right]^{n_1} [\hbar]^{n_2} [m_e]^{n_3} [e]^{n_4} \equiv \text{sec}$$

$$\text{Coulomb}^{-2n_1} \cdot \text{kg}^{+n_1}\ \text{m}^{+3n_1}\ \text{sec}^{-2n_1} \cdot \text{kg}^{n_2}\ \text{m}^{2n_2}\ \text{sec}^{-n_2} \cdot \text{kg}^{n_3}\ \text{Coulomb}^{n_4} \equiv \text{sec}$$

$\Rightarrow$

$$-2n_1 + n_4 = 0 \quad ①$$

$$+n_1 + n_2 + n_3 = 0 \quad ②$$

$$+3n_1 + 2n_2 = 0 \quad ③$$

$$-2n_1 - n_2 = 1 \quad ④$$

$\Rightarrow$ put ④ in ③ $\Rightarrow +3n_1 + 2(-2n_1 - 1) = 0 \Rightarrow n_1 = -2$

$\Rightarrow n_2 = 3,\ n_3 = -1,\ n_4 = -4$

$\Rightarrow$ 1 unit of time is $\left(\frac{1}{4\pi\varepsilon_0}\right)^{-2} \hbar^3 m_e^{-1} e^{-4}$

$$= \frac{(4\pi\varepsilon_0)^2 \cdot \hbar^2 \cdot \hbar}{e^4 \cdot m_e}$$

what is this ?

In Bohr model, the potential energy of the electron in Bohr radius is

$$\frac{e^2}{4\pi\varepsilon_0 \cdot a_0} = \frac{e^2\, m_e e^2}{4\pi\varepsilon_0\, 4\pi\varepsilon_0 \hbar^2} \equiv E_h$$ (called Hartree energy $= 2 \times 13.6$ eV)

So 1 unit of time is $\frac{\hbar}{E_h}$

What about the meaning of $c \simeq 137$?

It is in unit of electron speed in Bohr orbit.

Check: $\frac{e^2}{4\pi\varepsilon_0 a_0^2} = \frac{m_e v^2}{a_0} \Rightarrow v = \left(\frac{e^2}{4\pi\varepsilon_0 a_0 m_e}\right)^{\frac{1}{2}} = 2.2 \times 10^6$ m/s in SI unit

$\parallel$

1 in Atomic unit