Vector field quantization

For massive spin-1 complex vector field,

$$\mathcal{L} = -\frac{1}{2} F^*_{\mu\nu} F^{\mu\nu} + m^2 A^*_\mu A^\mu$$

$$= -\frac{1}{2} (\partial_\mu A^*_\nu - \partial_\nu A^*_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^*_\mu A^\mu$$

Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial A^*_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^*_\nu)} = 0$$

using $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^{*\nu}$, $\dfrac{\partial \mathcal{L}}{\partial A^*_\nu} = m^2 A^\nu$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial\left[-\frac{1}{2}(\partial_\rho A^*_\sigma - \partial_\sigma A^*_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)\right]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{2}(\partial^\rho A^{*\sigma} - \partial^\sigma A^{*\rho})(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$$

$$= -\frac{1}{2}\left[(\partial^\mu A^{*\nu} - \partial^\nu A^{*\mu}) - (\partial^\nu A^{*\mu} - \partial^\mu A^{*\nu})\right]$$

$$= -(\partial^\mu A^{*\nu} - \partial^\nu A^{*\mu})$$

$$= -F^{*\mu\nu}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial_\mu \partial^\mu A^{*\nu} + \partial_\mu \partial^\nu A^{*\mu} = -\partial_\mu F^{*\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^*_\nu)} = \frac{\partial\left[-\frac{1}{2}(\partial_\rho A^*_\sigma - \partial_\sigma A^*_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)\right]}{\partial(\partial_\mu A^*_\nu)}$$

$$= \frac{\partial\left[-\partial_\rho A^*_\sigma \partial^\rho A^\sigma + \partial_\rho A^*_\sigma \partial^\sigma A^\rho\right]}{\partial(\partial_\mu A^*_\nu)}$$

$$= -\delta^\mu_\rho \delta^\nu_\sigma \partial^\rho A^\sigma + \delta^\mu_\rho \delta^\nu_\sigma \partial^\sigma A^\rho$$

$$= -\partial^\mu A^\nu + \partial^\nu A^\mu$$

$$= -F^{\mu\nu}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^*_\nu)} = -\partial_\mu \partial^\mu A^\nu + \partial_\mu \partial^\nu A^\mu = -\partial_\mu F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu F^{*\mu\nu} + m^2 A^{*\nu} = 0 = \partial_\mu \partial^\mu A^{*\nu} - \partial_\mu \partial^\nu A^{*\mu} + m^2 A^{*\nu} = 0$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + m^2 A^\nu = 0$$

] Proca equation.

$$\Rightarrow \quad \partial_\nu (\partial_\mu F^{\mu\nu} + m^2 A^\nu) = 0 \Rightarrow \partial_\nu A^\nu = 0 \quad , \text{since } m \neq 0$$

Similarly, $\partial_\nu A^{*\nu} = 0$

$$\Rightarrow \quad \partial_\mu \partial^\mu A^\nu + m^2 A^\nu = 0 \text{ and } \partial_\mu \partial^\mu A^{*\nu} + m^2 A^{*\nu} = 0$$

that is $(\Box + m^2) A^\nu = 0$ and $(\Box + m^2) A^{*\nu} = 0$.

For massive spin-1 real vector field,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

$$= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2} m^2 A_\mu A^\mu$$

Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

using $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu$,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4} \frac{\partial[(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{4} \frac{\partial[2 \partial_\rho A_\sigma \partial^\rho A^\sigma - 2 \partial_\rho A_\sigma \partial^\sigma A^\rho]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{2} (\delta^\mu_\rho \delta^\nu_\sigma \partial^\rho A^\sigma + \delta^\mu_\rho \delta^\nu_\sigma \partial^\rho A^\sigma - \delta^\mu_\rho \delta^\nu_\sigma \partial^\sigma A^\rho - \delta^\mu_\sigma \delta^\nu_\rho \partial^\rho A^\sigma)$$

$$= -\frac{1}{2} (\partial^\mu A^\nu + \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu)$$

$$= -\partial^\mu A^\nu + \partial^\nu A^\mu$$

$$= -F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu + m^2 A^\nu = 0$$

$$\Rightarrow \quad \partial_\nu(\partial_\mu F^{\mu\nu} + m^2 A^\nu) = 0 \Rightarrow \quad \partial_\nu A^\nu = 0 \quad \text{since } m \neq 0$$

$$\Rightarrow \quad \partial_\mu \partial^\mu A^\nu + m^2 A^\nu = 0$$

that is, $(\Box + m^2) A^\nu = 0$

For massless spin-1 complex vector field,

$$\mathcal{L} = -\frac{1}{2} F^*_{\mu\nu} F^{\mu\nu}$$

Euler-Lagrange equation is

$$\partial_\mu F^{*\mu\nu} = 0 = \partial_\mu \partial^\mu A^{*\nu} - \partial_\mu \partial^\nu A^{*\mu} = 0$$

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

For massless spin-1 real vector field.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Euler-Lagrange equation is

$$\underline{\partial_\mu F^{\mu\nu} = 0} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

give two Maxwell equations without sources.

Note that since $F^{\mu\nu}$ is defined as $\partial^\mu A^\nu - \partial^\nu A^\mu$, we have

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}$$

$$= \underline{\partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu)} + \partial_\mu(\partial_\nu A_\lambda - \underline{\partial_\lambda A_\nu}) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda)$$

$$= 0$$

— this give the other two Maxwell equations for the photon field (a massless spin-1 real vector field).

In sum, for massive spin-1 field, no matter real or complex, the equation of motion is

$$(\Box + m^2) A^\mu = 0$$

where A^μ satisfies $\partial_\mu A^\mu = 0$

(and the complex conjugate equation $(\Box + m^2) A^{*\mu} = 0$ and $\partial_\mu A^{*\mu} = 0$ for complex field)

While for massless spin-1 field, no matter real or complex, the equation of motion is

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

(and the complex conjugate equation $\partial_\mu F^{*\mu\nu} = \partial_\mu \partial^\mu A^{*\nu} - \partial_\mu \partial^\nu A^{*\mu} = 0$ for complex scaler field).

Massive vector field case

For massive spin-1 particle, spin degrees of freedom is $2\times1+1=3$, corresponding to the spin polarization ±1 & 0. Therefore, the condition $\partial_\mu A^\mu=0$ is exactly the one we need to reduce the number of independent equations, $(\Box+m^2)A^\mu=0$, from four to three.

The solution to the Klein-Gordon equation is $e^{\pm ik\cdot x}$,

then since A^μ is a vector, the solution is written as $A^\mu(x)\propto\varepsilon^\mu(\vec{k})e^{\pm ik\cdot x}$.

The condition $\partial_\mu A^\mu=0 \Rightarrow k_\mu\varepsilon^\mu(\vec{k})=0$

We can find three independent ε^μ to satisfy $k_\mu\varepsilon^\mu(\vec{k})=0$.

(note that in 4-dimensional space-time, we can use 4 independent basis vectors to span the space, and they can be chosen to be $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. However, the condition $k_\mu\varepsilon^\mu=0$ reduce the number of independent basis vectors from 4 to 3.)

For example, placing the z-axis in the direction of the momentum $\vec{k}$, so that $k^\mu=(E_{\vec{k}},0,0,|\vec{k}|)$, where $E_{\vec{k}}=(|\vec{k}|^2+m^2)^{\frac{1}{2}}$, then a possible choice of three orthonormal vectors are

$$\varepsilon^\mu(\vec{k},1)=(0,1,0,0)$$

$$\varepsilon^\mu(\vec{k},2)=(0,0,1,0)$$

$$\varepsilon^\mu(\vec{k},3)=\frac{1}{m}(|\vec{k}|,0,0,E_{\vec{k}})$$

(check: $\varepsilon^\mu(\vec{k},\lambda)\varepsilon_\mu(\vec{k},\lambda')=-\delta_{\lambda\lambda'}$, $\varepsilon^\mu(\vec{k},\lambda)k_\mu=0$ for $\lambda=1,2,3$)

(note that if $\vec{k}=0$, then $k^\mu=(m,0,0,0)$, $\varepsilon^\mu(\vec{0},1)=(0,1,0,0)$, $\varepsilon^\mu(\vec{0},2)=(0,0,1,0)$, $\varepsilon^\mu(\vec{0},3)=(0,0,0,1)$)

In this choice, $\varepsilon^\mu(\vec{k},1)$ & $\varepsilon^\mu(\vec{k},2)$ describe the transverse polarizations of the field, while $\varepsilon^\mu(\vec{k},3)$ describes the longitudinal polarization.

In general, $\varepsilon^\mu(\vec{k},\lambda)$ $(\lambda=1,2,3)$ can be taken as complex vectors, so we can write

$$A^\mu(x)=\int d^3\vec{k}\,C(E_{\vec{k}})\sum_{\lambda=1}^{3}\left(\varepsilon^\mu(\vec{k},\lambda)a_{\vec{k},\lambda}e^{-ik\cdot x}+\varepsilon^{*\mu}(\vec{k},\lambda)b^\dagger_{\vec{k},\lambda}e^{ik\cdot x}\right),$$

5. $b^\dagger_{\vec{k},\lambda}$ should be changed to $a^\dagger_{\vec{k},\lambda}$ for real massive vector field.

Massless photon field. case

Let's now focus on a massless vector field — the photon field, or say, the electromagnetic field.

We have the EoM as

$$\partial_\mu F^{\mu\nu} = 0 = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu)$$

However, we do not automatically have $\partial_\mu A^\mu = 0$ as in the massive vector fields case.

On the other hand, the EoM has a property: it is invariant to transformations defined by

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \Lambda(x),$$

where $\Lambda(x)$ is an arbitrary function of x.

(check: $\Box(A^\mu + \partial^\mu \Lambda) - \partial^\mu(\partial_\nu A^\nu + \partial_\nu \partial^\nu \Lambda) = \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$)

Also, $F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu A^\nu - \partial^\mu \partial^\nu \Lambda - \partial^\nu A^\mu - \partial^\nu \partial^\mu \Lambda = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$

$\Rightarrow \quad -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

(the massive vector field does not have this property, since

$(\Box + m^2) A'^\mu = (\Box + m^2) A^\mu + (\Box + m^2) \partial^\mu \Lambda = (\Box + m^2) \partial^\mu \Lambda \neq 0$, and also,

$A'_\mu A'^\mu = (A_\mu + \partial_\mu \Lambda)(A^\mu + \partial^\mu \Lambda) = A_\mu A^\mu + \partial^\mu \Lambda \partial_\mu \Lambda + 2A^\mu \partial_\mu \Lambda \neq A_\mu A^\mu$)

This property for the electromagnetic field is called a local gauge symmetry, and the transformation is called the local gauge transformation. "Local" means that the transformation depends on x.

Since the Lagrangian is invariant to gauge transformation, the physics described by the Lagrangian does not depend on the $\Lambda(x)$. That is, although the intermediate steps in calculations may differ for different choices of $\Lambda(x)$, the final expression of any observable, if correctly calculated, must be independent of $\Lambda(x)$.

We know that for electromagnetic field, the spin polarizations can take values of ± 1 only, since electromagnetic wave is transversal.
Therefore, we should reduce the number of independent solutions from four to three.

First, we can choose A^μ to satisfy $\partial_\mu A^\mu = 0$ (this is called the Lorentz condition).

This choice is always possible since we can always make a gauge transformation.

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \Lambda$$

so that $\partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^\mu \partial_\mu \Lambda = 0$ is satisfied by choosing $\Lambda(x)$ such that $\Box \Lambda = -\partial_\mu A^\mu$

So, the equations to solve can be always written as

$$\begin{cases} \Box A^\mu = 0 \\ \partial_\mu A^\mu = 0 \end{cases}$$

However, we still have a residual gauge transformation:

as long as $\Box \Lambda = 0$, then any $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ give

$$\begin{cases} \Box A'^\mu = \Box A^\mu + \Box \partial^\mu \Lambda = \Box A^\mu + \partial^\mu \Box \Lambda = 0 \\ \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda = \partial_\mu A^\mu + \Box \Lambda = 0 \end{cases}$$

So, to further reduce the independent solutions from three to two, we should choose a specific $\Lambda(x)$ which satisfies $\Box \Lambda(x) = 0$.

A plane-wave solution to $\Box A^\mu = 0$ takes the form

$$A_\mu(x) = \varepsilon_\mu(\vec{k})\, e^{-ik\cdot x}, \quad k^2 = 0 = E_{\vec{k}}^2 - |\vec{k}|^2 \Rightarrow E_{\vec{k}} = |\vec{k}|$$

$$\partial_\mu A^\mu = 0 \Rightarrow k_\mu \varepsilon^\mu = 0 \Rightarrow E_{\vec{k}} \varepsilon^0 = \vec{k}\cdot\vec{\varepsilon} \Rightarrow \varepsilon^0 = \frac{\vec{k}}{|\vec{k}|}\cdot\vec{\varepsilon}$$

Now consider a particular solution to $\Box \Lambda = 0$,

$$\Lambda(x) = \lambda(\vec{k})\, e^{-ik\cdot x},$$

then $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \Rightarrow \varepsilon'_\mu = \varepsilon_\mu - i k_\mu \lambda$

If we choose λ such that $\lambda = \frac{\varepsilon_0}{iE_{\vec{k}}}$, then $\varepsilon_0' = 0$, and

$$\vec{\varepsilon}' = \vec{\varepsilon} - i\vec{k}\lambda = \vec{\varepsilon} - i\vec{k}\frac{\varepsilon_0}{iE_{\vec{k}}} \Rightarrow \vec{\varepsilon}'\cdot\vec{k} = \vec{\varepsilon}\cdot\vec{k} - i|\vec{k}|^2\frac{\varepsilon_0}{iE_{\vec{k}}} = \vec{\varepsilon}\cdot\vec{k} - |\vec{k}|\varepsilon_0 = 0$$

(note that $\varepsilon_\mu^*(\vec{k})e^{ik\cdot x}$ is the other plane-wave solution to $\Box A^\mu = 0$,
then $\varepsilon_\mu' = \varepsilon_\mu - ik_\mu\lambda \Rightarrow \varepsilon_\mu'^* = \varepsilon_\mu^* + ik_\mu\lambda^*$ and $\lambda^* = \frac{\varepsilon_0^*}{-ik_0} = \frac{\varepsilon_0^*}{-iE_{\vec{k}}}$ makes $\varepsilon_0'^* = 0$,
and $\vec{\varepsilon}'^* = \vec{\varepsilon}^* + i\vec{k}\lambda^* = \vec{\varepsilon}^* + i\vec{k}\cdot\frac{\varepsilon_0^*}{-iE_{\vec{k}}} = \vec{\varepsilon}^* - \frac{\vec{k}\cdot\varepsilon_0^*}{E_{\vec{k}}} \Rightarrow \vec{\varepsilon}'^*\cdot\vec{k} = \vec{\varepsilon}^*\cdot\vec{k} - \frac{|\vec{k}|^2\varepsilon_0^*}{E_{\vec{k}}}$

$$= \vec{\varepsilon}^*\cdot\vec{k} - E_{\vec{k}}\varepsilon_0^*$$

$$= 0$$

Since $k_\mu\varepsilon^{*\mu} = 0$)

Therefore, this choice of $\Lambda(x)$ makes $\varepsilon_\mu'(\vec{k})$ become a spacelike vector lying in a plane perpendicular to $\vec{k}$. Since we can choose two basis vectors to span that plane, we reduce the number of independent solutions from four to two, satisfying (drop the "′", and write $\varepsilon_\mu(\vec{k})$ in term of the two basis vectors)

$$\varepsilon^\mu(\vec{k},1) = (0, \vec{e}(\vec{k},1)), \quad \varepsilon^\mu(\vec{k},2) = (0, \vec{e}(\vec{k},2))$$

where $\vec{e}(\vec{k},1)\cdot\vec{k} = 0$ and $\vec{e}(\vec{k},2)\cdot\vec{k} = 0$.

Together with $\frac{\vec{k}}{|\vec{k}|} = \hat{k}$, the three spacelike vectors form a complete, orthonormal basis in three-dimensional space, so that

$$\vec{e}^*(\vec{k},\lambda)\cdot\vec{e}(\vec{k},\lambda') = \delta_{\lambda\lambda'}, \quad (\lambda,\lambda' = 1,2)$$

$$\hat{k}\cdot\vec{e}(\vec{k},\lambda) = 0, \quad (\lambda = 1,2)$$

$$\hat{k}\cdot\hat{k} = 1$$

$$(\vec{e}^*(\vec{k},1))^i(\vec{e}(\vec{k},1))^j + (\vec{e}^*(\vec{k},2))^i(\vec{e}(\vec{k},2))^j + \hat{k}^i\hat{k}^j = \delta^{ij}$$

$$\Rightarrow \sum_{\lambda=1}^{2}(\vec{e}^*(\vec{k},\lambda))^i(\vec{e}(\vec{k},\lambda))^j = \delta^{ij} - \frac{(\vec{k})^i(\vec{k})^j}{|\vec{k}|^2}$$

Correspondingly, $\varepsilon_\mu^*(\vec{k},\lambda)\varepsilon^\mu(\vec{k},\lambda') = -\vec{e}^*(\vec{k},\lambda)\cdot\vec{e}(\vec{k},\lambda') = -\delta_{\lambda\lambda'}$

$$\varepsilon_\mu(\vec{k},\lambda)k^\mu = -\vec{e}(\vec{k},\lambda)\cdot\vec{k} = 0$$

This choice of $\Lambda(x)$ (called the choice of gauge) is the Coulomb gauge, since

$$A^0(x) = 0 \qquad (\text{since } \varepsilon^0(\vec{k},\lambda) = 0 \text{ for } \lambda = 1,2)$$

and $\vec{\nabla}\cdot\vec{A}(x) = 0 \qquad (\text{since } \vec{\nabla}\cdot(\vec{e}(\vec{k},\lambda)e^{-ik\cdot x}) = i\cdot\vec{k}\cdot\vec{e}(\vec{k},\lambda)e^{-ik\cdot x} = 0)$

The decomposition of $A^\mu(x)$ is then

$$A^\mu(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\varepsilon^\mu(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \varepsilon^{*\mu}(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{ik\cdot x} \right)$$

$$= (0, \vec{A}(x))$$

$$= \left(0, \int d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^{\,*}(\vec{k},\lambda) a^+_{\vec{k},\lambda} e^{ik\cdot x} \right)\right)$$

Now let's look at the commutation relations in the Coulomb gauge.

Since $A^0(x) = 0$, we might guess that

$$i\delta_{ij}\delta^3(\vec{x}-\vec{y}) = [A_i(\vec{x},t), \pi^j(\vec{y},t)] = [A_i(\vec{x},t), \frac{\partial \mathcal{L}}{\partial(\partial_0 A_j)}] = [A_i(\vec{x},t), -F^{0j}(\vec{y},t)]$$

$$= [A_i(\vec{x},t), -\dot{A}^j(\vec{y},t)] = [A_i(\vec{x},t), \dot{A}_j(\vec{y},t)]$$

However, since $\vec{\nabla}\cdot\vec{A} = 0$, then $\underbrace{\vec{\nabla}}_{\text{act on } \vec{x} \text{ only}}\cdot[\vec{A}(\vec{x},t), \pi^j(\vec{y},t)] = [\vec{\nabla}\cdot\vec{A}(\vec{x},t), \pi^j(\vec{y},t)] = 0$

but $i\underbrace{\vec{\nabla}}_{\text{act on } \vec{x} \text{ only}}\cdot\delta^3(\vec{x}-\vec{y}) \neq 0$

So, the above guess can not be correct.

Then let's write $[A_i(\vec{x},t), \dot{A}_j(\vec{y},t)] = i\underbrace{\int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \Delta_{ij}(\vec{k})}$

(if $\Delta_{ij} = \delta_{ij}$, then it is $i\delta^3(\vec{x}-\vec{y})\delta_{ij}$)

$$\Rightarrow 0 = \underbrace{\vec{\nabla}}_{\text{act on } \vec{x} \text{ only}}\cdot[\vec{A}(\vec{x},t), \dot{A}_j(\vec{y},t)] = i\int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^3} \sum_{l=1}^{3} i k^l \Delta_{lj}(\vec{k}) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= -\int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^3} \sum_{l=1}^{3} k^l \Delta_{lj}(\vec{k}) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

It is clear that if $\Delta_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}$, then $\sum_{l=1}^{3} k^l \Delta_{lj}(\vec{k}) = \underbrace{k^j - \frac{k^l k_l k_j}{|\vec{k}|^2}}_{=0} = k^j - \frac{k^j|\vec{k}|^2}{|\vec{k}|^2}$

So, let's impose the commutation relations as

$$[A_i(\vec{x},t), \dot{A}_j(\vec{y},t)] = i\int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^3} \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$[A_i(\vec{x},t), A_j(\vec{y},t)] = [\dot{A}_i(\vec{x},t), \dot{A}_j(\vec{y},t)] = 0$$

Then from $\vec{A}(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^{\,*}(\vec{k},\lambda) a^{+}_{\vec{k},\lambda} e^{ik\cdot x}\right)$

$$\dot{\vec{A}}(x) = \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}})(-iE_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} - \vec{e}^{\,*}(\vec{k},\lambda) a^{+}_{\vec{k},\lambda} e^{ik\cdot x}\right)$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, \vec{A}(x)\cdot\vec{e}(\vec{p},r)\, e^{-ip\cdot x}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} + \vec{e}^{\,*}(\vec{k},\lambda) a^{+}_{\vec{k},\lambda} e^{ik\cdot x}\right)\cdot \vec{e}(\vec{p},r)\, e^{-ip\cdot x}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}} \delta^3(\vec{p}+\vec{k}) + e^{*}(\vec{k},\lambda) a^{+}_{\vec{k},\lambda} e^{iE_{\vec{k}}t - i\vec{k}\cdot\vec{x}} \delta^3(\vec{p}-\vec{k})\right)\cdot \vec{e}(\vec{p},r)$$

$$= C(E_{\vec{p}})\left(\sum_{\lambda=1}^{2} \vec{e}(-\vec{p},\lambda)\cdot\vec{e}(\vec{p},r)\, a_{-\vec{p},\lambda} e^{-2iE_{\vec{p}}t}\right) + C(E_{\vec{p}})\, a^{+}_{\vec{p},r}$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, \dot{\vec{A}}(x)\cdot\vec{e}(\vec{p},r)\, e^{-ip\cdot x}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{k}\, C(E_{\vec{k}})(-iE_{\vec{k}}) \sum_{\lambda=1}^{2} \left(\vec{e}(\vec{k},\lambda) a_{\vec{k},\lambda} e^{-ik\cdot x} - \vec{e}^{\,*}(\vec{k},\lambda) a^{+}_{\vec{k},\lambda} e^{ik\cdot x}\right)\cdot \vec{e}(\vec{p},r)\, e^{-ip\cdot x}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}})(-iE_{\vec{k}}) \sum_{\lambda=1}^{2} \Big(\vec{e}(\vec{k},\lambda)\cdot\vec{e}(\vec{p},r)\, a_{\vec{k},\lambda} \delta^3(\vec{p}+\vec{k})\, e^{-iE_{\vec{k}}t - iE_{\vec{p}}t}$$
$$- \vec{e}^{\,*}(\vec{k},\lambda)\cdot\vec{e}(\vec{p},r)\, a^{+}_{\vec{k},\lambda} e^{iE_{\vec{k}}t - iE_{\vec{p}}t} \delta^3(\vec{p}-\vec{k})\Big)$$

$$= C(E_{\vec{p}})(-iE_{\vec{p}}) \sum_{\lambda=1}^{2} \left(\vec{e}(-\vec{p},\lambda)\cdot\vec{e}(\vec{p},r)\, a_{-\vec{p},\lambda} e^{-2iE_{\vec{p}}t}\right) - C(E_{\vec{p}})(-iE_{\vec{p}})\, a^{+}_{\vec{p},r}$$

$$\Rightarrow a^{+}_{\vec{p},r} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x} \left(\vec{A}(x) - \frac{\dot{\vec{A}}(x)}{(-iE_{\vec{p}})}\right)\cdot \vec{e}(\vec{p},r)\, e^{-ip\cdot x} \frac{1}{2C(E_{\vec{p}})}$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\ \vec{A}(x)\cdot\vec{e}^{\,*}(\vec{p},r)\,e^{ip\cdot x}$$

$$=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{k}\,C(E_{\vec{k}})\sum_{\lambda=1}^{2}\left(\vec{e}(\vec{k},\lambda)\,a_{\vec{k},\lambda}\,e^{-ik\cdot x}+\vec{e}^{\,*}(\vec{k},\lambda)\,a^{+}_{\vec{k},\lambda}\,e^{ik\cdot x}\right)\cdot\vec{e}^{\,*}(\vec{p},r)\,e^{ip\cdot x}$$

$$=\int_{-\infty}^{+\infty} d^3\vec{k}\,C(E_{\vec{k}})\sum_{\lambda=1}^{2}\Big(\vec{e}(\vec{k},\lambda)\cdot\vec{e}^{\,*}(\vec{p},r)\,a_{\vec{k},\lambda}\,\delta^3(\vec{p}-\vec{k})\,e^{+iE_{\vec{p}}t-iE_{\vec{k}}t}$$
$$+\vec{e}^{\,*}(\vec{k},\lambda)\cdot\vec{e}^{\,*}(\vec{p},r)\,a^{+}_{\vec{k},\lambda}\,\delta^3(\vec{p}+\vec{k})\,e^{iE_{\vec{p}}t+iE_{\vec{k}}t}\Big)$$

$$=C(E_{\vec{p}})\,a_{\vec{p},r}+C(E_{\vec{p}})\,\vec{e}^{\,*}(-\vec{p},\lambda)\cdot\vec{e}^{\,*}(\vec{p},r)\,a^{+}_{-\vec{p},\lambda}\,e^{2iE_{\vec{p}}t}$$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\ \dot{\vec{A}}(x)\cdot\vec{e}^{\,*}(\vec{p},r)\,e^{ip\cdot x}$$

$$=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{k}\,C(E_{\vec{k}})(-iE_{\vec{k}})\sum_{\lambda=1}^{2}\left(\vec{e}(\vec{k},\lambda)\,a_{\vec{k},\lambda}\,e^{-ik\cdot x}-\vec{e}^{\,*}(\vec{k},\lambda)\,a^{+}_{\vec{k},\lambda}\,e^{ik\cdot x}\right)\cdot\vec{e}^{\,*}(\vec{p},r)\,e^{ip\cdot r}$$

$$=\int_{-\infty}^{+\infty} d^3\vec{k}\,C(E_{\vec{k}})\cdot(-iE_{\vec{k}})\sum_{\lambda=1}^{2}\Big(\vec{e}(\vec{k},\lambda)\cdot\vec{e}^{\,*}(\vec{p},r)\,a_{\vec{k},\lambda}\,\delta^3(\vec{p}-\vec{k})\,e^{-iE_{\vec{k}}t+iE_{\vec{p}}t}$$
$$-\vec{e}^{\,*}(\vec{k},\lambda)\cdot\vec{e}^{\,*}(\vec{p},r)\,a^{+}_{\vec{k},\lambda}\,e^{iE_{\vec{p}}t+iE_{\vec{k}}t}\,\delta^3(\vec{p}+\vec{k})\Big)$$

$$=C(E_{\vec{p}})(-iE_{\vec{p}})\,a_{\vec{p},r}-C(E_{\vec{p}})(-iE_{\vec{p}})\sum_{\lambda=1}^{2}\vec{e}^{\,*}(-\vec{p},\lambda)\cdot\vec{e}^{\,*}(\vec{p},r)\,a^{+}_{-\vec{p},r}\,e^{2iE_{\vec{p}}t}$$

$$\Rightarrow\quad a_{\vec{p},r}=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\left(\vec{A}(x)+\frac{\dot{\vec{A}}(x)}{(-iE_{\vec{p}})}\right)\cdot\vec{e}^{\,*}(\vec{p},r)\,e^{ip\cdot x}\,\frac{1}{2C(E_{\vec{p}})}\ \checkmark$$

$$\Rightarrow\ [a_{\vec{p},r},\,a^{+}_{\vec{k},s}]$$

$$=\left(\frac{1}{(2\pi)^3}\right)^2\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{y}\left[A^i(x)+\frac{\dot{A}^i(x)}{(-iE_{\vec{p}})},\ A^j(y)-\frac{\dot{A}^j(y)}{(-iE_{\vec{k}})}\right]e^{*i}(\vec{p},r)$$
$$e^{j}(\vec{k},s)\,e^{ip\cdot x-ik\cdot y}\,\frac{1}{2C(E_{\vec{p}})}\,\frac{1}{2C(E_{\vec{k}})}$$

$$=\left(\frac{1}{(2\pi)^3}\right)^2\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{y}\Big\{\,i\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\left(\delta^{ij}-\frac{q^iq^j}{|\vec{q}|^2}\right)e^{i\vec{q}\cdot(\vec{x}-\vec{y})}\,\frac{1}{iE_{\vec{k}}}$$
$$+\,i\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\left(\delta^{ji}-\frac{q^jq^i}{|\vec{q}|^2}\right)e^{i\vec{q}\cdot(\vec{y}-\vec{x})}\,\frac{1}{iE_{\vec{p}}}\Big\}$$
$$e^{i*}(\vec{p},r)\,e^{j}(\vec{k},s)\,e^{ip\cdot x-ik\cdot y}\,\frac{1}{2C(E_{\vec{p}})}\,\frac{1}{2C(E_{\vec{k}})}$$

$$=\left(i\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\left(\delta^{ij}-\frac{q^iq^j}{|\vec{q}|^2}\right)\frac{1}{iE_{\vec{k}}}\,\delta^3(\vec{q}-\vec{p})\,\delta^3(\vec{q}-\vec{k})\,e^{iE_{\vec{p}}t-iE_{\vec{k}}t}\right.$$
$$\left.+\,i\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\left(\delta^{ij}-\frac{q^iq^j}{|\vec{q}|^2}\right)\frac{1}{iE_{\vec{p}}}\,\delta^3(\vec{q}+\vec{k})\,\delta^3(\vec{q}+\vec{p})\,e^{iE_{\vec{p}}t-iE_{\vec{k}}t}\right)\begin{matrix}e^{i*}(\vec{p},r)\\ e^{j}(\vec{k},s)\end{matrix}\,\frac{1}{2C(E_{\vec{p}})}\,\frac{1}{2C(E_{\vec{k}})}$$

$$= \frac{2}{E_{\vec{p}}}\left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}\right) e^{i*}(\vec{p},r)\, e^{j}(\vec{p},s)\, \delta^3(\vec{p}-\vec{k}) \left(\frac{1}{2C(E_{\vec{p}})}\right)^2 \frac{1}{(2\pi)^3}$$

$$= \frac{1}{2E_{\vec{p}}} \cdot \vec{e}^{\,*}(\vec{p},r)\cdot \vec{e}(\vec{p},s)\, \delta^3(\vec{p}-\vec{k}) \cdot \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \frac{1}{(2\pi)^3}$$

$$= \delta_{rs}\, \delta^3(\vec{p}-\vec{k})\, \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \frac{1}{(2\pi)^3}$$

$$[a_{\vec{p},r}, a_{\vec{k},s}]$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y} \left[A^i(x) + \frac{\dot{A}^i(x)}{(-iE_{\vec{p}})},\ A^j(y) + \frac{\dot{A}^j(y)}{(-iE_{\vec{k}})} \right] e^{i*}(\vec{p},r)\, e^{*}_{j}(\vec{k},s)$$
$$e^{ip\cdot x + ik\cdot y} \frac{1}{2C(E_{\vec{p}})} \frac{1}{2C(E_{\vec{k}})}$$

$$= \left(\frac{1}{(2\pi)^3}\right)^2 \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y} \left\{ \frac{i}{(-iE_{\vec{k}})} \int_{-\infty}^{+\infty} \frac{d^3\vec{q}}{(2\pi)^3} \left(\delta^{ij} - \frac{q^i q^j}{|\vec{q}|^2}\right) e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \right.$$
$$\left. + \frac{i}{iE_{\vec{p}}} \int_{-\infty}^{+\infty} \frac{d^3\vec{q}}{(2\pi)^3} \left(\delta^{ji} - \frac{q^j q^i}{|\vec{q}|^2}\right) e^{i\vec{q}\cdot(\vec{y}-\vec{x})} \right\}$$
$$e^{i*}(\vec{p},r)\, e^{*j}(\vec{k},s)\, e^{ip\cdot x + ik\cdot y} \frac{1}{2C(E_{\vec{p}})} \frac{1}{2C(E_{\vec{k}})}$$

$$= \int_{-\infty}^{+\infty} \frac{d^3\vec{q}}{(2\pi)^3} \left\{ -\frac{1}{E_{\vec{k}}} \delta^3(\vec{q}-\vec{p})\, \delta^3(\vec{q}+\vec{k})\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t} \left(\delta^{ij} - \frac{q^i q^j}{|\vec{q}|^2}\right) \right.$$
$$\left. + \frac{1}{E_{\vec{p}}} \delta^3(\vec{q}-\vec{k})\, \delta^3(\vec{q}+\vec{p})\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t} \left(\delta^{ij} - \frac{q^i q^j}{|\vec{q}|^2}\right) \right\}$$
$$e^{i*}(\vec{p},r)\, e^{*j}(\vec{k},s) \frac{1}{2C(E_{\vec{p}})} \frac{1}{2C(E_{\vec{k}})}$$

$$= \frac{1}{(2\pi)^3} \left\{ -\frac{1}{E_{\vec{p}}} e^{2iE_{\vec{p}}t}\, \delta^3(\vec{p}+\vec{k}) \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}\right) + \frac{1}{E_{\vec{p}}} e^{2iE_{\vec{p}}t}\, \delta^3(\vec{p}+\vec{k}) \left(\delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}\right) \right\}$$
$$e^{i*}(\vec{p},r)\, e^{*j}(-\vec{p},s) \left(\frac{1}{2C(E_{\vec{p}})}\right)^2$$

$$= 0$$

$$[a^{+}_{\vec{p},r}, a^{+}_{\vec{k},s}] = a^{+}_{\vec{p},r} a^{+}_{\vec{k},s} - a^{+}_{\vec{k},s} a^{+}_{\vec{p},r} = -[a_{\vec{p},r}, a_{\vec{k},s}]^{+} = 0$$

Now calculate the Hamiltonian & momentum operators.

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} \partial_0 A_i - \mathcal{L}$$

(since in the Coulomb gauge, $A_0 = 0$)

$$= -F^{0i}\dot{A}_i + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$= -(\partial^0 A^i - \underbrace{\cancel{\partial^i A^0}}_{0})\dot{A}_i + \frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \dot{A}^i\dot{A}^i + \frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu)$$

$$= \dot{A}^i\dot{A}^i + \frac{1}{2}(\partial^0 A^i \partial_0 A_i + \partial^i A^j \partial_i A_j - \partial^i A^j \partial_j A_i)$$

$$= \frac{1}{2}\dot{A}^i\dot{A}^i + \frac{1}{2}\partial^i A^j \partial^i A^j - \frac{1}{2}\left[\partial^i(A^j\partial_j A_i) - A^j\partial^i\partial_j A_i\right]$$

since $A^j\partial^i\partial_j A_i = -A^j\partial_j\,\vec{\nabla}\cdot\vec{A} = 0$, and

$$\int d^3\vec{x}\,\partial^i(A^j\partial_j A_i) = \int d^3\vec{x}\,\vec{\nabla}\cdot(A^j\partial_j\vec{A}) = \iint_{\text{surface}}(A^j\partial_j\vec{A})\,ds$$

and we take the surface term $=0$ by assuming the field vanishes at infinity.

Also, since $E^i = -F^{0i} = -(\partial^0 A^i - \underbrace{\cancel{\partial^i A^0}}_{0}) = -\dot{A}^i$

and $F^{ij} = -\varepsilon^{ijk}B^k \Rightarrow \varepsilon^{ijm}F^{ij} = -\varepsilon^{ijm}\varepsilon^{ijk}B^k = -2B^m$

$$\Rightarrow B^k = -\frac{1}{2}\varepsilon^{ijk}F^{ij}$$

$$\Rightarrow (\vec{B})^2 = B^kB^k = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{mnk}F^{ij}F^{mn}$$

$$= \frac{1}{4}(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})F^{ij}F^{mn}$$

$$= \frac{1}{4}(F^{ij}F^{ij} - F^{ij}F^{ji})$$

$$= \frac{1}{2}F^{ij}F^{ij}$$

$$= \frac{1}{2}(\partial^i A^j - \partial^j A^i)(\partial^i A^j - \partial^j A^i)$$

$$= \partial^i A^j \partial^i A^j - \partial^i A^j \partial^j A^i$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}\vec{B}\cdot\vec{B} + \frac{1}{2}\vec{E}\cdot\vec{E}.$$

$$\Rightarrow \quad H = \frac{1}{2}\int_{-\infty}^{+\infty} d^3\vec{x}\,(\vec{E}^2 + \vec{B}^2)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} d^3\vec{x}\,(\dot{\vec{A}}^2 + \partial^i A^j \partial^i A^j)$$

where $\int_{-\infty}^{+\infty} d^3\vec{x}\,(\dot{\vec{A}})^2$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\; C(E_{\vec{p}})\, C(E_{\vec{k}})\,(-iE_{\vec{p}})(-iE_{\vec{k}}).$$

$$\times \sum_{s,r} \left(\vec{e}(\vec{p},s)\, a_{\vec{p},s}\, e^{-ip\cdot x} - \vec{e}^{\,*}(\vec{p},s)\, a^{+}_{\vec{p},s}\, e^{ip\cdot x} \right)$$

$$\cdot \left(\vec{e}(\vec{k},r)\, a_{\vec{k},r}\, e^{-ik\cdot x} - \vec{e}^{\,*}(\vec{k},r)\, a^{+}_{\vec{k},r}\, e^{ik\cdot x} \right)$$

$$= (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\; C(E_{\vec{p}})\, C(E_{\vec{k}})\,(-iE_{\vec{p}})(-iE_{\vec{k}}).$$

$$\times \sum_{s,r} \Big(\vec{e}(\vec{p},s)\cdot\vec{e}(\vec{k},r)\, a_{\vec{p},s}\, a_{\vec{k},r}\, \delta^3(\vec{p}+\vec{k})\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}$$

$$+ \vec{e}^{\,*}(\vec{p},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a^{+}_{\vec{p},s}\, a^{+}_{\vec{k},r}\, \delta^3(\vec{p}+\vec{k})\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}$$

$$- \vec{e}(\vec{p},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a_{\vec{p},s}\, a^{+}_{\vec{k},r}\, \delta^3(\vec{p}-\vec{k})\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}$$

$$- \vec{e}^{\,*}(\vec{p},s)\cdot\vec{e}(\vec{k},r)\, a^{+}_{\vec{p},s}\, a_{\vec{k},r}\, \delta^3(\vec{p}-\vec{k})\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t} \Big)$$

$$= (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, \left(C(E_{\vec{p}})\right)^2 (-E_{\vec{p}}^2)$$

$$\Big\{ \sum_{s,r} \Big(\vec{e}(\vec{p},s)\cdot\vec{e}(-\vec{p},r)\, a_{\vec{p},s}\, a_{-\vec{p},r}\, e^{-2iE_{\vec{p}}t}$$

$$+ \vec{e}^{\,*}(\vec{p},s)\cdot e^{*}(-\vec{p},r)\, a^{+}_{\vec{p},s}\, a^{+}_{-\vec{p},r}\, e^{2iE_{\vec{p}}t} \Big)$$

$$- \sum_{s} \left(a_{\vec{p},s}\, a^{+}_{\vec{p},s} + a^{+}_{\vec{p},s}\, a_{\vec{p},s} \right) \Big\}$$

and. $\int_{-\infty}^{+\infty} d^3\vec{x}\ \partial^i A^j \partial^i A^j$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})\, C(E_{\vec{k}})\, (ip^i)(ik^i)$$

$$\times \sum_{s,r} \Big(e^j(\vec{p},s)\, a_{\vec{p},s}\, e^{-ip\cdot x} - e^{*j}(\vec{p},s)\, a^+_{\vec{p},s}\, e^{ip\cdot x} \Big)$$

$$\Big(e^j(\vec{k},r)\, a_{\vec{k},r}\, e^{-ik\cdot x} - e^{*j}(\vec{k},r)\, a^+_{\vec{k},r}\, e^{ik\cdot x} \Big)$$

$$= (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})\, C(E_{\vec{k}})\, (-\vec{p}\cdot\vec{k})$$

$$\times \sum_{s,r} \Big(\vec{e}(\vec{p},s)\cdot\vec{e}(\vec{k},r)\, a_{\vec{p},s}\, a_{\vec{k},r}\, \delta^3(\vec{p}+\vec{k})\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}$$

$$+ \vec{e}^{\,*}(\vec{p},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a^+_{\vec{p},s}\, a^+_{\vec{k},r}\, \delta^3(\vec{p}+\vec{k})\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}$$

$$- \vec{e}(\vec{p},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a_{\vec{p},s}\, a^+_{\vec{k},r}\, \delta^3(\vec{p}-\vec{k})\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}$$

$$- e^*(\vec{p},s)\cdot \vec{e}(\vec{k},r)\, a^+_{\vec{p},s}\, a_{\vec{k},r}\, \delta^3(\vec{p}-\vec{k})\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t} \Big)$$

$$= (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, \big(C(E_{\vec{p}})\big)^2$$

$$\times \Big\{ \sum_{s,r} \Big(|\vec{p}|^2\, \vec{e}(\vec{p},s)\cdot\vec{e}(-\vec{p},r)\, a_{\vec{p},s}\, a_{-\vec{p},r}\, e^{-2iE_{\vec{p}}t}$$

$$+ |\vec{p}|^2\, \vec{e}^{\,*}(\vec{p},s)\cdot\vec{e}^{\,*}(-\vec{p},r)\, a^+_{\vec{p},s}\, a^+_{-\vec{p},r}\, e^{2iE_{\vec{p}}t}$$

$$+ |\vec{p}|^2 \sum_s \big(a_{\vec{p},s}\, a^+_{\vec{p},s} + a^+_{\vec{p},s}\, a_{\vec{p},s} \big)$$

using $E_{\vec{p}}^2 = |\vec{p}|^2$

$$\Rightarrow \quad H = (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, \big(C(E_{\vec{p}})\big)^2\, E_{\vec{p}}^2 \sum_s \big(a_{\vec{p},s}\, a^+_{\vec{p},s} + a^+_{\vec{p},s}\, a_{\vec{p},s} \big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \Big[(2\pi)^3\, 2E_{\vec{p}}\, \big(C(E_{\vec{p}})\big)^2 \Big] \frac{E_{\vec{p}}}{2} \sum_s \big(a_{\vec{p},s}\, a^+_{\vec{p},s} + a^+_{\vec{p},s}\, a_{\vec{p},s} \big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \Big[(2\pi)^3\, 2E_{\vec{p}}\, \big(C(E_{\vec{p}})\big)^2 \Big] E_{\vec{p}} \sum_s \big(a^+_{\vec{p},s}\, a_{\vec{p},s} \big)$$

$$+ \int_{-\infty}^{+\infty} d^3\vec{p} \Big[(2\pi)^3\, 2E_{\vec{p}}\, \big(C(E_{\vec{p}})\big)^2 \Big] \frac{E_{\vec{p}}}{2} \times 2\, \delta^3(0)\, \frac{1}{2E_{\vec{p}}} \Big(\frac{1}{C(E_{\vec{p}})} \Big)^2 \frac{1}{(2\pi)^3}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \Big[(2\pi)^3\, 2E_{\vec{p}}\, \big(C(E_{\vec{p}})\big)^2 \Big] E_{\vec{p}} \sum_s \big(a^+_{\vec{p},s}\, a_{\vec{p},s} \big)$$

$$+ \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{E_{\vec{p}}}{2} \cdot 2\, (2\pi)^3\, \delta^3(0)$$

This result is the expected one: the degrees of freedom is 2, the vacuum energy is positive.

$$:H: = \int_{-\infty}^{+\infty} d^3\vec{p}\,[(2\pi)^3 2E_{\vec{p}}(C(E_{\vec{p}}))^2]\,E_{\vec{p}} \sum_s (a^\dagger_{\vec{p},s} a_{\vec{p},s})$$

$$(\hat{\vec{P}})^i = \int_{-\infty}^{+\infty} d^3\vec{x}\,\frac{\partial f}{\partial(\partial_0 A_\mu)}\,\partial^i A_\mu \underset{\substack{\uparrow \\ \text{note that } A^0=0}}{=} \int_{-\infty}^{+\infty} d^3\vec{x}\,\frac{\partial f}{\partial(\partial_0 A_j)}\,\partial^i A_j$$

where $\dfrac{\partial f}{\partial(\partial_0 A_j)} = -F^{0j} = -(\partial^0 A^j - \cancel{\partial^j A^0}^{\,0}) = -\dot{A}^j = E^j$

$$\Rightarrow \quad (\vec{E}\times\vec{B})^i = \varepsilon^{ijk}E^jB^k = \varepsilon^{ijk}E^j\frac{1}{(-2)}\varepsilon^{mnk}F^{mn}$$

$$= -\frac{1}{2}(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})E^jF^{mn}$$

$$= -\frac{1}{2}(E^jF^{ij} - E^jF^{ji})$$

$$= -E^jF^{ij}$$

$$= \dot{A}^j(\partial^iA^j - \partial^jA^i)$$

where $\dot{A}^j\partial^jA^i = (\partial^0A^j)(\partial^jA^i) = \partial^j(A^i\partial^0A^j) - A^i\partial^0\cancel{\partial^jA^j}$ 0 due to $\vec{\nabla}\cdot\vec{A}=0$

and $\displaystyle\int_{-\infty}^{+\infty} d^3\vec{x}\,\partial^j(A^i\partial^0A^j) = -\int_{-\infty}^{+\infty} d^3\vec{x}\,\vec{\nabla}\cdot(A^i\partial^0\vec{A}) = -\iint_{\text{surface}}(A^i\partial^0\vec{A})\,ds$

and we take the surface term be zero due to the assumption that $\vec{A}$ vanishes at infinity.

$$\Rightarrow \quad \int_{-\infty}^{+\infty}(\vec{E}\times\vec{B})^i d^3\vec{x} = \int_{-\infty}^{+\infty} d^3\vec{x}\,\dot{A}^j\partial^iA^j = (\hat{\vec{P}})^i$$

that is $\displaystyle\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{x}\,(\vec{E}\times\vec{B}) = -\int_{-\infty}^{+\infty}(\dot{\vec{A}}\cdot\nabla\vec{A})\,d^3\vec{x}$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{p}\,d^3\vec{k}\,C(E_{\vec{p}})(iE_{\vec{p}})(i\vec{k})C(E_{\vec{k}})$$

$$\times \sum_{s,r}(\vec{e}(\vec{p},s)a_{\vec{p},s}e^{-ip\cdot x} - \vec{e}^*(\vec{p},s)a^\dagger_{\vec{p},s}e^{ip\cdot x})$$

$$\cdot(\vec{e}(\vec{k},r)a_{\vec{k},r}e^{-ik\cdot x} - \vec{e}^*(\vec{k},r)a^\dagger_{\vec{k},r}e^{ik\cdot x})$$

$$= (2\pi)^3\int_{-\infty}^{+\infty} d^3\vec{p}\,d^3\vec{k}\,C(E_{\vec{p}})C(E_{\vec{k}})(-E_{\vec{p}}\vec{k})$$

$$\times \sum_{s,r}\Big(\vec{e}(\vec{p},s)\cdot\vec{e}(\vec{k},r)\,a_{\vec{p},s}a_{\vec{k},r}\,\delta^3(\vec{p}+\vec{k})\,e^{-iE_{\vec{p}}t-iE_{\vec{k}}t}$$

$$+\vec{e}^*(\vec{p},s)\cdot\vec{e}^*(\vec{k},r)\,a^\dagger_{\vec{p},s}a^\dagger_{\vec{k},r}\,\delta^3(\vec{p}+\vec{k})\,e^{iE_{\vec{p}}t+iE_{\vec{k}}t}$$

$$-\vec{e}(\vec{p},s)\cdot\vec{e}^*(\vec{k},r)\,a_{\vec{p},s}a^\dagger_{\vec{k},r}\,\delta^3(\vec{p}-\vec{k})\,e^{-iE_{\vec{p}}t+iE_{\vec{k}}t}$$

$$-\vec{e}^*(\vec{p},s)\cdot\vec{e}(\vec{k},r)\,a^\dagger_{\vec{p},s}a_{\vec{k},r}\,\delta^3(\vec{p}-\vec{k})\,e^{iE_{\vec{p}}t-iE_{\vec{k}}t}\Big)$$

$$= (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{k} \left(C(E_{\vec{k}})\right)^2 (-E_{\vec{k}}\vec{k})$$

$$\times \Big\{ \sum_{s,r} \big(\vec{e}(-\vec{k},s)\cdot\vec{e}(\vec{k},r)\, a_{-\vec{k},s}\, a_{\vec{k},r}\, e^{-2iE_{\vec{k}}t} + \vec{e}^{\,*}(-\vec{k},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a^{+}_{-\vec{k},s} a^{+}_{\vec{k},r}\, e^{2iE_{\vec{k}}t} \big)$$

$$- \sum_{s} \big(a_{\vec{k},s} a^{+}_{\vec{k},s} + a^{+}_{\vec{k},s} a_{\vec{k},s} \big) \Big\}$$

Since $(2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{k} \left(C(E_{\vec{k}})\right)^2 (-E_{\vec{k}}\vec{k}) \sum_{s,r} \big(\vec{e}(-\vec{k},s)\cdot\vec{e}(\vec{k},r)\, a_{-\vec{k},s}\, a_{\vec{k},r}\, e^{-2iE_{\vec{k}}t}$

$$+ \vec{e}^{\,*}(-\vec{k},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a^{+}_{-\vec{k},s} a^{+}_{\vec{k},r}\, e^{2iE_{\vec{k}}t} \big)$$

$\vec{k}\to-\vec{p},\ s\leftrightarrow r$

$$= (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p} \left(C(E_{\vec{p}})\right)^2 (+E_{\vec{p}}\vec{p}) \sum_{s,r} \big(\vec{e}(\vec{p},r)\cdot\vec{e}(-\vec{p},s)\, a_{\vec{p},r}\, a_{-\vec{p},s}\, e^{-2iE_{\vec{p}}t}$$

$$+ \vec{e}^{\,*}(\vec{p},r)\cdot\vec{e}^{\,*}(-\vec{p},s)\, a^{+}_{\vec{p},r} a^{+}_{-\vec{p},s}\, e^{2iE_{\vec{p}}t} \big)$$

$\vec{p}\to\vec{k}$

$$= -(2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{k} \left(C(E_{\vec{k}})\right)^2 (-E_{\vec{k}}\vec{k}) \sum_{s,r} \big(\vec{e}(-\vec{k},s)\cdot\vec{e}(\vec{k},r)\, a_{-\vec{k},s}\, a_{\vec{k},r}\, e^{-2iE_{\vec{k}}t}$$

$\Rightarrow$ this term is zero. $(C=-C \Rightarrow C=0)$

$$+ \vec{e}^{\,*}(-\vec{k},s)\cdot\vec{e}^{\,*}(\vec{k},r)\, a^{+}_{-\vec{k},s} a^{+}_{\vec{k},r}\, e^{2iE_{\vec{k}}t} \big)$$

$$\Rightarrow \hat{\vec{P}} = (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p} \left(C(E_{\vec{p}})\right)^2 E_{\vec{p}}\,\vec{p} \sum_{s} \big(a_{\vec{p},s} a^{+}_{\vec{p},s} + a^{+}_{\vec{p},s} a_{\vec{p},s} \big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 \left(C(E_{\vec{p}})\right)^2 2E_{\vec{p}} \right] \vec{p} \sum_{s} \big(a^{+}_{\vec{p},s} a_{\vec{p},s} \big)$$

$$+ (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p} \left(C(E_{\vec{p}})\right)^2 E_{\vec{p}}\,\vec{p}\, \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \frac{1}{(2\pi)^3}\, 2\delta^3(0)$$

Since the second term $= \int_{-\infty}^{+\infty} d^3\vec{p}\ \vec{p}\cdot\delta^3(0) = 0$

then $$\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 \left(C(E_{\vec{p}})\right)^2 2E_{\vec{p}} \right] \vec{p} \sum_{s} \big(a^{+}_{\vec{p},s} a_{\vec{p},s} \big)$$

Again, we can define the particle number operator

$$\hat{N} = \int_{-\infty}^{+\infty} d^3\vec{p}\, [(2\pi)^3\, 2E_{\vec{p}}\, (C(E_{\vec{p}}))^2] \sum_s (a^+_{\vec{p},s}\, a_{\vec{p},s})$$

$$\Rightarrow [\hat{N}, a_{\vec{k},r}] = [\int_{-\infty}^{+\infty} d^3\vec{p}\, [(2\pi)^3\, 2E_{\vec{p}}\, (C(E_{\vec{p}}))^2] \sum_s (a^+_{\vec{p},s}\, a_{\vec{p},s}),\ a_{\vec{k},r}]$$

$$= -a_{\vec{k},r}$$

$$[\hat{N}, a^+_{\vec{k},r}] = a^+_{\vec{k},r}$$

The rest of the discussions for the number operator $\hat{N}$ acting on $(a^+_{\vec{k}_1,r_1})^{n_1} (a^+_{\vec{k}_2,r_2})^{n_2} \cdots (a^+_{\vec{k}_\ell,r_\ell})^{n_\ell} |s\rangle$ ect. are the same as the ones for real scaler field. For example,

For $\hat{N}|s\rangle = c|s\rangle$

$$\hat{N} (a^+_{\vec{k}_1,r_1})^{n_1} (a^+_{\vec{k}_2,r_2})^{n_2} \cdots (a^+_{\vec{k}_\ell,r_\ell})^{n_\ell} |s\rangle = (c + n_1 + \cdots + n_\ell)(a^+_{\vec{k}_1,r_1})^{n_1} \cdots (a^+_{\vec{k}_\ell,r_\ell})^{n_\ell} |s\rangle$$

$$\hat{N} (a_{\vec{k}_1,r_1})^{n_1} \cdots (a_{\vec{k}_\ell,r_\ell})^{n_\ell} |s\rangle = (c - n_1 - \cdots - n_\ell)(a_{\vec{k}_1,r_1})^{n_1} \cdots (a_{\vec{k}_\ell,r_\ell})^{n_\ell} |s\rangle$$

The vaccum state $|0\rangle$ satisfies $a_{\vec{k},r}|0\rangle = 0$ for any $\vec{k}$ and r.

Still, let's define the normalization of one particle state as

$$|p,s\rangle \equiv f(p)\, a^+_{\vec{p},s}\, |0\rangle \quad \text{by requiring}$$

$$\langle \ell, r | p, s\rangle = (2\pi)^3\, 2E_{\vec{p}}\, \delta_{rs}\, \delta^3(\vec{p} - \vec{\ell}) \text{ and } \langle 0|0\rangle = 1 \text{ and } f(p) \text{ is real.}$$

$$\Rightarrow \quad f(p) = C(E_{\vec{p}})(2\pi)^3\, 2E_{\vec{p}}$$

Therefore $\langle 0| A_\mu(x) |p,s\rangle$

$$= \langle 0| \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec{k}}) \sum_{\lambda=1}^{2} (\varepsilon_\mu(\vec{k},\lambda)\, a_{\vec{k},\lambda}\, e^{-ik\cdot x} + \varepsilon^*_\mu(\vec{k},\lambda)\, a^+_{\vec{k},\lambda}\, e^{ik\cdot x})$$

$$C(E_{\vec{p}})(2\pi)^3\, 2E_{\vec{p}}\, a^+_{\vec{p},s}\, |0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{k}\, \underline{C(E_{\vec{k}})} \sum_{\lambda=1}^{2} \varepsilon_\mu(\vec{k},\lambda)\, \underline{\frac{1}{(2\pi)^3}}\, \underline{\frac{1}{2E_{\vec{k}}}} \underline{\left(\frac{1}{C(E_{\vec{k}})}\right)^2} \delta^3(\vec{k} - \vec{p})\, \delta_{\lambda s}\, e^{-ik\cdot x}$$

$$\cdot \underline{C(E_{\vec{p}})}\, \underline{(2\pi)^3}\, \underline{2E_{\vec{p}}}$$

$$= \varepsilon_\mu(\vec{p},s)\, e^{-ip\cdot x}$$

$$\langle p,s| A_\mu(x)|0\rangle = \varepsilon^*_\mu(\vec{p},s)\, e^{ip\cdot x}$$

For electromagnetic field $A_\mu(x)$ satisfying the EoM

$$\Box A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

we can make a gauge transformation $A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$ in which $\Box \Lambda(x) = -\partial_\mu A^\mu$

$$\Rightarrow \begin{cases} \partial_\mu A'^\mu = \partial_\mu (A^\mu + \partial^\mu \Lambda) = \partial_\mu A^\mu + \Box \Lambda = 0 \\ \Box A'^\mu = \Box (A^\mu + \partial^\mu \Lambda) = \Box A^\mu + \partial^\mu \Box \Lambda = \Box A^\mu - \partial^\mu \partial_\nu A^\nu = 0 \end{cases}$$

In the above two equations, the equation

$\Box A'^\mu = 0$ ensures $A'^\mu = C \varepsilon'^\mu(\vec{k}) e^{-ik\cdot x} + d\, \varepsilon'^{*\mu}(\vec{k}) e^{ik\cdot x}$, where $k^2 = 0$

C & d are coefficients not depends on x.

That is, it only ensures that $e^{\pm ik\cdot x}$ should be part of the solutions, where $k^2 = 0$

(Since $\Box e^{\pm ik\cdot x} = (\pm ik_\mu)(\pm ik^\mu) \cdot e^{\pm ik\cdot x} = -k^2 e^{\pm ik\cdot x} = 0 \Rightarrow k^2 = 0$)

(This is similar to the equation for massive vector field.

$\Box A^\mu + m^2 A^\mu = 0$ ensures $A^\mu = C \varepsilon^\mu(\vec{k}) e^{-ik\cdot x} + d\, \varepsilon^{*\mu}(\vec{k}) e^{ik\cdot x}$, since

$\Box e^{\pm ik\cdot x} + m^2 e^{\pm ik\cdot x} = 0 \Rightarrow -k^2 + m^2 = 0 \Rightarrow k^2 = m^2$.

That is, it only ensures that $e^{\pm ik\cdot x}$ should be part of the solutions, where $k^2 = m^2$)

The equation $\partial_\mu A'^\mu = 0$ reduce the number of independent $\varepsilon'^\mu(\vec{k})$ from four to three, since $\partial_\mu A'^\mu = \underset{=0}{C}(-ik_\mu)\varepsilon'^\mu(\vec{k}) e^{-ik\cdot x} + d(ik_\mu)\varepsilon'^{*\mu}(\vec{k}) e^{ik\cdot x}$

$\Rightarrow k_\mu \varepsilon'^\mu(\vec{k}) = 0$

Because of this constraint, $\varepsilon'^\mu(k)$ should be lying on the three-surface perpendicular to k_μ. Since a three-surface is spanned by three basis vectors, we have three independent $\varepsilon'^\mu(\vec{k})$, namely, $\varepsilon'^\mu(\vec{k}, 1)$, $\varepsilon'^\mu(\vec{k}, 2)$ and $\varepsilon'^\mu(\vec{k}, 3)$

(clarify 1).

Now, start from $\begin{cases} \partial_\mu A'^\mu = 0 \\ \Box A'^\mu = 0 \end{cases}$, we can make another gauge transformation

$A'^{(x)}_\mu \longrightarrow A''_\mu(x) = A'_\mu(x) + \partial_\mu \Lambda'(x)$ in which Λ' satisfy $\Box \Lambda' = 0$.

then we notice that

$$\begin{cases} \partial_\mu A''^\mu = \partial_\mu (A'^\mu + \partial^\mu \Lambda') = \partial_\mu A'^\mu + \Box \Lambda' = \partial_\mu A'^\mu = 0 \\ \Box A''^\mu = \Box (A'^\mu + \partial^\mu \Lambda') = \Box A'^\mu + \partial^\mu \Box \Lambda' = \Box A'^\mu = 0 \end{cases}$$

This means that A'^μ is not uniquely determined by $\begin{cases} \partial_\mu A'^\mu = 0 \\ \Box A'^\mu = 0 \end{cases}$,

since once we find a solution to $A'^\mu(x)$, say S'^μ, such that $\begin{cases} \partial_\mu S'^\mu = 0 \\ \Box S'^\mu = 0 \end{cases}$, then

$S''^\mu = (S'^\mu + \partial^\mu \Lambda')$ is also a solution as long as $\Box \Lambda' = 0$, since

$\begin{cases} \partial_\mu (S'^\mu + \partial^\mu \Lambda') = \partial_\mu S'^\mu = 0 \\ \Box (S'^\mu + \partial^\mu \Lambda') = \Box S'^\mu = 0 \end{cases}$.

So, by choosing a specific $\Lambda'(x)$ satisfying $\Box \Lambda'(x) = 0$, say,

$$\Lambda'(x) = \lambda(\vec{k}) c e^{-ik\cdot x} + \lambda^*(\vec{k}) d e^{ik\cdot x}$$

then $\partial_\mu \Lambda'(x) = \lambda(\vec{k}) c(-ik_\mu) e^{-ik\cdot x} + \lambda^*(\vec{k}) d i k_\mu e^{ik\cdot x}$

$$\Rightarrow \quad \varepsilon''_\mu = \varepsilon'_\mu - i k_\mu \lambda$$
$$\varepsilon''^*_\mu = \varepsilon'^*_\mu + i k_\mu \lambda^*$$

we can choose $\lambda = \dfrac{\varepsilon'_0}{i E_{\vec{k}}}$, then $\varepsilon''_0 = 0$

and $\vec{\varepsilon}'' \cdot \vec{k} = \vec{\varepsilon}' \cdot \vec{k} - i \vec{k} \cdot \vec{k} \cdot \lambda = \vec{\varepsilon}' \cdot \vec{k} - i \vec{k} \cdot \vec{k} \dfrac{\varepsilon_0}{i E_{\vec{k}}} = \vec{\varepsilon}' \cdot \vec{k} - \varepsilon_0 E_k$

$= -\varepsilon'_\mu k^\mu = 0$

So we can make $\varepsilon''_\mu(\vec{k})$ become a space like vector lying in a plane perpendicular to $\vec{k}$, and this plane can be spanned by two basis vectors.

(clarify 2)