Solution

1.

$$\text{From } \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \Rightarrow \begin{cases} 2\gamma_i^2 = -2 \Rightarrow \gamma_i^2 = -1 \\ 2\gamma_0^2 = 2 \Rightarrow \gamma_0^2 = 1 \end{cases}$$

$$\gamma^\mu = g^{\mu\nu}\gamma_\nu \Rightarrow \gamma^0 = \gamma_0 , \ \gamma^i = -\gamma_i$$

$$\Rightarrow \quad (\gamma^0)^2 = (\gamma_0)^2 = 1 ,$$

$$(\gamma^i)^2 = (-\gamma_i)^2 = (\gamma_i)^2 = -1$$

Done 1) & 2) ✓

$$(\gamma^5)^2 = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^0\gamma^1\gamma^2\gamma^3)$$

$$\text{From } \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \Rightarrow \begin{cases} \gamma_0\gamma_i = -\gamma_i\gamma_0 \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0 \\ \gamma_i\gamma_j = -\gamma_j\gamma_i \ (\text{for } i\neq j), \Rightarrow \gamma^i\gamma^j = -\gamma^j\gamma^i \\ \qquad (\text{for } i\neq j) \end{cases}$$

$$\begin{aligned} \Rightarrow \ & (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^0\gamma^1\gamma^2\gamma^3) \\ &= (-1) \ \gamma^0\underbrace{\gamma^1\gamma^2\gamma^3}\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= (-1)\cdot(-1)^3(\gamma^0)^2 \ \gamma^1\underbrace{\gamma^2\gamma^3}\gamma^1\gamma^2\gamma^3 \\ &= (-1)^2(\gamma^1)^2 \ \gamma^2\underbrace{\gamma^3}\gamma^2\gamma^3 \\ &= (-1)\cdot(-1)^1(\gamma^2)^2\gamma^3\gamma^3 \\ &= 1 \end{aligned}$$

Done 3) ✓

$$(\gamma^5)^+ = (i\gamma^0\gamma^1\gamma^2\gamma^3)^+ = (-i)\cdot(\gamma^3)^+(\gamma^2)^+(\gamma^1)^+(\gamma^0)^+$$

$$\text{From } \gamma_\mu^+ = \gamma_0\gamma_\mu\gamma_0 \Rightarrow \begin{cases} \gamma_0^+ = (\gamma_0)^3 = \gamma_0 \Rightarrow (\gamma^0)^+ = \gamma^0 \\ \gamma_i^+ = \gamma_0\gamma_i\gamma_0 = -\gamma_0\gamma_0\gamma_i = -\gamma_i \Rightarrow (\gamma^i)^+ = -\gamma^i \end{cases}$$

$$\begin{aligned} \Rightarrow (\gamma^5)^+ &= (-i)(-1)^3\cdot\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^3\underbrace{\gamma^2\gamma^1\gamma^0} \\ &= i\underbrace{\gamma^2\gamma^1}\gamma^0\gamma^3(-1)^3 = -i(-1)^2\underbrace{\gamma^1}\gamma^0\gamma^2\gamma^3 = -i(-1)^1\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \end{aligned}$$

Done 4)

$$\gamma_5\gamma_\mu = i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\mu = \begin{cases} (\text{for } \mu=0) \ i\gamma_0\underbrace{\gamma_1\gamma_2\gamma_3}\gamma_0 = -i\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_0\gamma_5 \\ (\text{for } \mu=1) \ i\gamma_0\gamma_1\underbrace{\gamma_2\gamma_3}\gamma_1 = i\gamma_0\gamma_1\gamma_1\gamma_2\gamma_3 = -i\gamma_1\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_1\gamma \\ (\text{for } \mu=2) \ i\gamma_0\gamma_1\gamma_2\underbrace{\gamma_3}\gamma_2 = -i\underbrace{\gamma_0\gamma_1}\gamma_2\gamma_2\gamma_3 = -i\gamma_2\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_2\gamma \\ (\text{for } \mu=3) \ i\underbrace{\gamma_0\gamma_1\gamma_2}\gamma_3\gamma_3 = -i\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_3\gamma_5 \end{cases}$$

$$\Rightarrow \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$$

Done 5)

(checked)

$$\gamma_\lambda\gamma^\lambda = \gamma_0\gamma^0 + \sum_{i=1}^{3}\gamma_i\gamma^i = (\gamma_0)^2-(\gamma_1)^2-(\gamma_2)^2-(\gamma_3)^2 = 1+1+1+1 = 4$$

(or, just from $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} = \gamma_\mu\gamma^\mu + \gamma_\mu\gamma^\mu = 2\times\delta^\mu_\mu = 8 \Rightarrow \gamma_\lambda\gamma^\lambda = 4$)

Done 6)

From $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ and $\gamma^\mu = g^{\mu\nu}\gamma_\nu \Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

$$\Rightarrow \gamma_\lambda\gamma^\alpha\gamma^\lambda = \gamma_\lambda(2g^{\alpha\lambda} - \gamma^\lambda\gamma^\alpha) = 2\gamma^\alpha - 4\gamma^\alpha = -2\gamma^\alpha$$

Done 7)

$$\begin{aligned}\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda &= \gamma_\lambda\gamma^\alpha(2g^{\beta\lambda} - \gamma^\lambda\gamma^\beta) = 2\gamma^\beta\gamma^\alpha - \gamma_\lambda\gamma^\alpha\gamma^\lambda\gamma^\beta \\ &= 2\gamma^\beta\gamma^\alpha + 2\gamma^\alpha\gamma^\beta \\ &= 4g^{\alpha\beta}\end{aligned}$$

Done 8)

$$\begin{aligned}\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\lambda &= \gamma_\lambda\gamma^\alpha\gamma^\beta(2g^{\sigma\lambda} - \gamma^\lambda\gamma^\sigma) = 2\gamma^\sigma\gamma^\alpha\gamma^\beta - \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\sigma \\ &= 2\gamma^\sigma\gamma^\alpha\gamma^\beta - 4g^{\alpha\beta}\gamma^\sigma \\ &= 2\gamma^\sigma(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha) - 4g^{\alpha\beta}\gamma^\sigma \\ &= -2\gamma^\sigma\gamma^\beta\gamma^\alpha\end{aligned}$$

Done 9)

$$\begin{aligned}\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\eta\gamma^\lambda &= \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma(2g^{\eta\lambda} - \gamma^\lambda\gamma^\eta) = 2\gamma^\eta\gamma^\alpha\gamma^\beta\gamma^\sigma + 2\gamma^\sigma\gamma^\beta\gamma^\alpha\gamma^\eta \\ &= 2(\gamma^\eta\gamma^\alpha\gamma^\beta\gamma^\sigma + 2\gamma^\sigma\gamma^\beta\gamma^\alpha\gamma^\eta)\end{aligned}$$

Done 10)

11) $$\begin{aligned}\not{A}\not{A} &= A^\alpha A^\beta\gamma_\alpha\gamma_\beta = \tfrac{1}{2}(A^\alpha A^\beta\gamma_\alpha\gamma_\beta + A^\alpha A^\beta\gamma_\alpha\gamma_\beta) \\ &= \tfrac{1}{2}A^\alpha A^\beta(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha) = \tfrac{1}{2}A^\alpha A^\beta\, 2g_{\alpha\beta} = A^\alpha A_\alpha = A\cdot A \quad ✓\end{aligned}$$

$\not{A}\not{B} + \not{B}\not{A} = A^\alpha B^\beta(\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha) = A^\alpha B^\beta\, 2g_{\alpha\beta} = 2A\cdot B.$ ✓

$\gamma_\lambda\not{A}\gamma^\lambda = A_\alpha\gamma_\lambda\gamma^\alpha\gamma^\lambda = -2A_\alpha\gamma^\alpha = -2\not{A}$ ✓

$\gamma_\lambda\not{A}\not{B}\gamma^\lambda = A_\alpha B_\beta\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = 4A_\alpha B_\beta g^{\alpha\beta} = 4A\cdot B$ ✓

$\gamma_\lambda\not{A}\not{B}\not{C}\gamma^\lambda = A_\alpha B_\beta C_\sigma\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\lambda = -2A_\alpha B_\beta C_\sigma\gamma^\sigma\gamma^\beta\gamma^\alpha = -2\not{C}\not{B}\not{A}$ ✓

$$\begin{aligned}\gamma_\lambda\not{A}\not{B}\not{C}\not{D}\gamma^\lambda &= A_\alpha B_\beta C_\sigma D_\eta\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\eta\gamma^\lambda = A_\alpha B_\beta C_\sigma D_\eta\, 2(\gamma^\eta\gamma^\alpha\gamma^\beta\gamma^\sigma + \gamma^\sigma\gamma^\beta\gamma^\alpha\gamma^\eta) \\ &= 2(\not{D}\not{A}\not{B}\not{C} + \not{C}\not{B}\not{A}\not{D}) \quad ✓\end{aligned}$$

(checked)

2.

1) use $(\gamma_5)^2 = 1$ and $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$

and $T_r(AB) = T_r(BA)$.

(proof: $T_r(AB) = \sum_{i=1}^{N}\sum_{k=1}^{M} A_{ik}B_{ki}$ (for $N\times M$ matrix A and $M\times N$ matrix B)

if all $A_{ik}B_{ki} = B_{ki}A_{ik}$, then

$T_r(AB) = \sum_{i=1}^{N}\sum_{k=1}^{M} B_{ki}A_{ik} = T_r(BA)$)

$\Rightarrow T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu) = T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu\gamma_5^2) = T_r(\gamma_5\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu\gamma_5)$

Do permutation to move the γ_5 in front γ^α to the position after γ^ν. Since $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$ and there're odd number of permutations due to there're odd number of γ's, then

$T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu) = T_r(\gamma_5\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu\gamma_5) = (-1)^n\, T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu\gamma_5^2)$

$= (-1)^n\, T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu)$, where n is an odd number,

$\Rightarrow T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu) = -T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu)$

$\Rightarrow T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu) = 0$

2) Since $T_r(A) = T_r(A^T)$

(proof: $T_r(A) = \sum_{i=1}^{N} A_{ii}$, $T_r(A^T) = \sum_{i=1}^{N}(A^T)_{ii} = \sum_{i=1}^{N} A_{ii}$, for $N\times N$ matrix

then $T_r(A) = T_r(A^T)$.)

$\Rightarrow T_r(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu) = T_r[(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu)^T]$

Since $A^\dagger = (A^T)^*$, then $A^T = (A^\dagger)^*$

Since $T_r(A^*) = \sum_{i=1}^{N}(A^*)_{ii} = \left(\sum_{i=1}^{N} A_{ii}\right)^* = (T_r(A))^*$

then $T_r[(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu)^T] = \{T_r[(\gamma^\alpha\gamma^\beta\dots\gamma^\mu\gamma^\nu)^\dagger]\}^*$

$= \{T_r(\gamma^{\dagger\nu}\gamma^{\dagger\mu}\dots\gamma^{\dagger\beta}\gamma^{\dagger\alpha})\}^*$

use $\gamma^{\dagger\nu} = \gamma^0\gamma^\nu\gamma^0$

$\Rightarrow \{T_r(\gamma^{\dagger\nu}\gamma^{\dagger\mu}\dots\gamma^{\dagger\beta}\gamma^{\dagger\alpha})\}^* = \{T_r(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0\dots\gamma^0\gamma^\beta\gamma^0\gamma^0\gamma^\alpha\gamma^0)\}^*$

(checked)

Since $(\gamma^0)^2 = 1$, then

$$\{Tr(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0\cdots\gamma^0\gamma^\beta\gamma^0\gamma^0\gamma^\alpha\gamma^0)\}^* = \{Tr(\gamma^0\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha\gamma^0)\}^*$$

use $Tr(AB) = Tr(BA)$ to put the γ^0 in the front to the end position, then

$$\{Tr(\gamma^0\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha\gamma^0)\}^* = \{Tr(\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha\gamma^0\gamma^0)\}^*$$

$$= \{Tr(\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha)\}^* \quad (\gamma^0\gamma^0 = 1 \text{ is used })$$

For the trace of even number of γ-matrices,

$$Tr(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$

use $\gamma^{\mu_1}\gamma^{\mu_2} + \gamma^{\mu_2}\gamma^{\mu_1} = 2g^{\mu_1\mu_2}$

$$\Rightarrow Tr(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) = 2g^{\mu_1\mu_2}Tr(\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$
$$- Tr(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$

Do the same thing to move γ^{μ_1} to the end of the γ-matrices,

$$Tr(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$
$$= 2g^{\mu_1\mu_2}Tr(\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) - 2g^{\mu_1\mu_3}Tr(\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$
$$+ Tr(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_1}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$

$$= 2g^{\mu_1\mu_2}Tr(\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) - 2g^{\mu_1\mu_3}Tr(\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$
$$+ 2g^{\mu_1\mu_4}Tr(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_5}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$
$$- 2g^{\mu_1\mu_5}Tr(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_6}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$
$$+ \cdots$$
$$- 2g^{\mu_1\mu_{2N-1}}Tr(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N}})$$
$$+ Tr(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}}\gamma^{\mu_1}\gamma^{\mu_{2N}})$$

Note that the last term is

$$Tr(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}}\gamma^{\mu_1}\gamma^{\mu_{2N}}) = 2g^{\mu_1\mu_{2N}}Tr(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}})$$
$$- Tr(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}\gamma^{\mu_1})$$

(checked)

use $T_r(AB) = T_r(BA)$ to put γ^{μ_1} at the end position in the second term to the front position, then

$$T_r(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}}\gamma^{\mu_1}\gamma^{\mu_{2N}})$$

$$= 2g^{\mu_1\mu_{2N}} T_r(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}}) - T_r(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$

$$\Rightarrow T_r(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}})$$

$$\begin{aligned}
= \; & g^{\mu_1\mu_2} T_r(\gamma^{\mu_3}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) \\
& - g^{\mu_1\mu_3} T_r(\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) \\
& + g^{\mu_1\mu_4} T_r(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_5}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) \\
& - g^{\mu_1\mu_5} T_r(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_6}\cdots\gamma^{\mu_{2N-1}}\gamma^{\mu_{2N}}) \\
& + \cdots \\
& - 2g^{\mu_1\mu_{2N-1}} T_r(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N}}) \\
& + g^{\mu_1\mu_{2N}} T_r(\gamma^{\mu_2}\gamma^{\mu_3}\cdots\gamma^{\mu_{2N-2}}\gamma^{\mu_{2N-1}})
\end{aligned}$$

that is, the trace of $(2N)$ γ-matrices can be written as the sum of $(2N-1)$ traces of $(2N-2)$ γ-matrices.

We can repeat the process so that the trace of $(2N)$ γ-matrices can be written as the sum of $T_r(\gamma^{\mu_i}\gamma^{\mu_j})$.

Since $T_r(\gamma^{\mu_i}\gamma^{\mu_j}) = T_r(2g^{\mu_i\mu_j} - \gamma^{\mu_j}\gamma^{\mu_i})$

$$= 2g^{\mu_i\mu_j} T_r(1) - T_r(\gamma^{\mu_j}\gamma^{\mu_i})$$

$$= 8g^{\mu_i\mu_j} - T_r(\gamma^{\mu_i}\gamma^{\mu_j})$$

then $T_r(\gamma^{\mu_i}\gamma^{\mu_j}) = 4g^{\mu_i\mu_j}$, which is a real number.

Therefore, the trace of a product of even number of γ-matrices is real, that is

$$[T_r(\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha)]^* = T_r(\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha)$$

$$\Rightarrow T_r(\gamma^\alpha\gamma^\beta\cdots\gamma^\mu\gamma^\nu) = T_r(\gamma^\nu\gamma^\mu\cdots\gamma^\beta\gamma^\alpha) \quad ✓$$

(checked)

3) As already shown in 2)

$$Tr(\gamma^\alpha\gamma^\beta) = Tr(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha)$$

$$= 2g^{\alpha\beta}\, Tr(1) - Tr(\gamma^\beta\gamma^\alpha) = 2g^{\alpha\beta}\times 4 - Tr(\gamma^\alpha\gamma^\beta)$$

$$\Rightarrow Tr(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}$$

4) As already shown in 2)

$$Tr(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = g^{\alpha\beta}\, Tr(\gamma^\gamma\gamma^\delta) - g^{\alpha\gamma}\, Tr(\gamma^\beta\gamma^\delta) + g^{\alpha\delta}\, Tr(\gamma^\beta\gamma$$

$$= 4g^{\alpha\beta}g^{\gamma\delta} - 4g^{\alpha\gamma}g^{\beta\delta} + 4g^{\alpha\delta}g^{\beta\gamma}$$

$$= 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$$

✓

5) use $(\gamma^0)^2 = 1$

$$Tr(\gamma^5) = Tr(\gamma^5\gamma^0\gamma^0) = Tr(-\gamma^0\gamma^5\gamma^0) = -Tr(\gamma^0\gamma^0\gamma^5) = -Tr(\gamma^5$$

use $(\gamma^0)^2 = 1$ use $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ use $Tr(AB) = Tr(BA)$

$$\Rightarrow Tr(\gamma^5) = 0$$

✓

use $\gamma^5\gamma^\alpha = -\gamma^\alpha\gamma^5$

$$Tr(\gamma^5\gamma^\alpha) = Tr(\gamma^\alpha\gamma^5) = Tr(-\gamma^5\gamma^\alpha) = -Tr(\gamma^5\gamma^\alpha)$$

use $Tr(AB) = Tr(BA)$

$$\Rightarrow Tr(\gamma^5\gamma^\alpha) = 0$$

✓

For $Tr(\gamma^5\gamma^\alpha\gamma^\beta)$,

① if $\alpha = \beta = 0$, then $Tr(\gamma^5\gamma^\alpha\gamma^\beta) = Tr(\gamma^5(\gamma^0)^2) = Tr(\gamma^5) = 0$

② if $\alpha = \beta = i$, then $Tr(\gamma^5\gamma^\alpha\gamma^\beta) = Tr(\gamma^5(\gamma^i)^2) = -Tr(\gamma^5) = 0$

③ if $\alpha \neq \beta$ and $\alpha \neq 0$ and $\beta \neq 0$, then

$$Tr(\gamma^5\gamma^\alpha\gamma^\beta) = Tr(\gamma^5\gamma^\alpha\gamma^\beta\gamma^0\gamma^0) = -Tr(\gamma^0\gamma^5\gamma^\alpha\gamma^\beta\gamma^0)$$

since $\gamma^0\gamma^i = -\gamma^i\gamma^0$, $\gamma^0\gamma^5 = -\gamma^5\gamma^0$

$$= -Tr(\gamma^0\gamma^0\gamma^5\gamma^\alpha\gamma^\beta) = -Tr(\gamma^5\gamma^\alpha\gamma^\beta)$$

use $Tr(AB) = Tr(BA)$

$$\Rightarrow Tr(\gamma^5\gamma^\alpha\gamma^\beta) = 0$$

(checked)

④ if $\alpha \neq \beta$ and either $\alpha = 0$ or $\beta = 0$, then choose $i \neq \alpha$ & $i \neq \beta$,

$$\Rightarrow Tr(\gamma^5 \gamma^\alpha \gamma^\beta) \underset{\uparrow}{=} -Tr(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^i \gamma^i) \underset{\uparrow}{=} Tr(\gamma^i \gamma^5 \gamma^\alpha \gamma^\beta \gamma^i)$$

use $(\gamma^i)^2 = -1$

use $\gamma^\alpha \gamma^i = -\gamma^i \gamma^\alpha$, $\gamma^i \gamma^\beta = -\gamma^\beta \gamma^i$ when $i \neq \alpha$ & $i \neq \beta$ and $\gamma^i \gamma^5 = -\gamma^5 \gamma^i$

$$\underset{\uparrow}{=} Tr(\gamma^i \gamma^i \gamma^5 \gamma^\alpha \gamma^\beta) = -Tr(\gamma^5 \gamma^\alpha \gamma^\beta)$$

use $Tr(AB) = Tr(BA)$

$$\Rightarrow Tr(\gamma^5 \gamma^\alpha \gamma^\beta) = 0$$

Therefore, for all cases, $Tr(\gamma^5 \gamma^\alpha \gamma^\beta) = 0$

$$Tr(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu) = i\, Tr(\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\alpha \gamma^\beta \gamma^\mu)$$

there're 7 γ-matrices (odd number of γ-matrices), so

$Tr(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu) = 0$ ✓

So, $Tr(\gamma^5) = Tr(\gamma^5 \gamma^\alpha) = Tr(\gamma^5 \gamma^\alpha \gamma^\beta) = Tr(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu) = 0$

(checked)

3. 1) $\sigma^1\sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma^3$

$\sigma^2\sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma^3$

$\sigma^2\sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma^1$

$\sigma^3\sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma^1$

$\sigma^3\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2$

$\sigma^1\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2$

$\sigma^1\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma^2\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma^3\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

that is $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$ where $\varepsilon^{123} = 1$

$\Rightarrow \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}$

also $(\sigma^1)^\dagger = \sigma^1$

$(\sigma^2)^\dagger = \sigma^2$

$(\sigma^3)^\dagger = \sigma^3$

For $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$.

① $\mu=\nu=0$, $2(\gamma^0)^2 = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2g^{00}\mathbb{I}_{4\times4}$

② $(\mu=0 \,\&\, \nu=i)$ or $(\mu=i \,\&\, \nu=0)$, $\gamma^0\gamma^i+\gamma^i\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0 = 2g^{0i}$

③ $\mu=i \,\&\, \nu=j$, $\gamma^i\gamma^j+\gamma^j\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + (i \leftrightarrow j)$

$= \begin{pmatrix} -\sigma^i\sigma^j - \sigma^j\sigma^i & 0 \\ 0 & -\sigma^i\sigma^j - \sigma^j\sigma^i \end{pmatrix} = -2\begin{pmatrix} \delta^{ij} & 0 \\ 0 & \delta^{ij} \end{pmatrix} = 2g^{ij}$

(checked)

So. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

For $\gamma^{+\mu}$

① $\mu=0$, $\gamma^{0+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^+ = \begin{pmatrix} 1^+ & 0 \\ 0 & -1^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0$

$\gamma^0\gamma^0\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0$

② $\mu=i$, $\gamma^{i+} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -\sigma^{i+} \\ \sigma^{i+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = -\gamma^i$

$\gamma^0\gamma^i\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$

$= -\gamma^i$

So, $\gamma^{+\mu} = \gamma^0\gamma^\mu\gamma^0$

2) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$

$= i\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$

$= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} = \begin{pmatrix} 0 & (\sigma^1)^2 \\ (\sigma^1)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(checked)

4. $$\gamma^0\gamma^i+\gamma^i\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}+\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}-\sigma^i&0\\0&\sigma^i\end{pmatrix}+\begin{pmatrix}\sigma^i&0\\0&-\sigma^i\end{pmatrix}$$

$$=0=2g^{0i}$$

✓

$$\gamma^0\gamma^0+\gamma^0\gamma^0=2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=2\begin{pmatrix}1&0\\0&1\end{pmatrix}=2g^{00}$$ ✓

Since the spatial part $\gamma^i\gamma^j+\gamma^j\gamma^i=2g^{ij}$ has been checked in problem 3.

then done the check of $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$

$$\gamma^{+0}=\begin{pmatrix}0&1\\1&0\end{pmatrix}^+=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\gamma^0\gamma^0\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\Rightarrow \gamma^{+0}=\gamma^0\gamma^0\gamma^0$$

Since $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$.

then $\gamma^0\gamma^i+\gamma^i\gamma^0=0 \Rightarrow \gamma^0\gamma^i=-\gamma^i\gamma^0$

also $(\gamma^0)^2=1$

$\Rightarrow \gamma^0\gamma^i\gamma^0=-\gamma^i(\gamma^0)^2=-\gamma^i$

and. $\gamma^{+i}=-\gamma^i$ has been checked in problem 3.

So, done the check of $\gamma^{\mu +}=\gamma^0\gamma^\mu\gamma^0$

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=i\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma^1\\-\sigma^1&0\end{pmatrix}\begin{pmatrix}0&\sigma^2\\-\sigma^2&0\end{pmatrix}\begin{pmatrix}0&\sigma^3\\-\sigma^3&0\end{pmatrix}$$

$$=i\begin{pmatrix}-\sigma^1&0\\0&\sigma^1\end{pmatrix}\begin{pmatrix}-i\sigma^1&0\\0&-i\sigma^1\end{pmatrix}=\begin{pmatrix}-(\sigma^1)^2&0\\0&(\sigma^1)^2\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

For problem 3&4, the Mathematica code is also attached.

(checked)

This code checks the defining properties of gamma matrices in the standard and Weyl representations

This code is checked on Nov. 1, 2018

check γ-matrices properties in the standard representation

define γ-matrices and $g^{\mu\nu}$

In[1]:=
```
gamma[0] = ArrayFlatten[{{IdentityMatrix[2], ConstantArray[0, {2, 2}]},
    {ConstantArray[0, {2, 2}], -IdentityMatrix[2]}}];
MatrixForm[
 %]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In[3]:=
```
gamma[1] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[1]},
    {-PauliMatrix[1], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

In[5]:=
```
gamma[2] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[2]},
    {-PauliMatrix[2], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[6]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

In[7]:=
```
gamma[3] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[3]},
    {-PauliMatrix[3], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[9]:=
```
gmunu = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
MatrixForm[%]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

check $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$ (note that in Mathematica the column and row start from 1, rather than

0)

```
In[11]:= Do[If[(gamma[mdo].gamma[ndo] + gamma[ndo].gamma[mdo] -
      2 * gmunu[[mdo + 1, ndo + 1]] * IdentityMatrix[4]) ≠
    ConstantArray[0, {4, 4}], Print["error"], Print["ok"]],
  {mdo, 0, 3, 1}, {ndo, 0, 3, 1}]
```

ok
ok
ok
ok
ok
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ok
ok
ok
ok
ok
ok
ok

check $(\gamma^{\mu})^{+} = \gamma^{0}\, \gamma^{\mu}\, \gamma^{0}$

```
In[12]:= Do[If[(ConjugateTranspose[gamma[mdo]] - gamma[0].gamma[mdo].gamma[0]) ≠
    ConstantArray[0, {4, 4}], Print["error"], Print["ok"]], {mdo, 0, 3, 1}]
```

ok
ok
ok
ok

find gamma[5]

```
In[13]:= MatrixForm[I * gamma[0].gamma[1].gamma[2].gamma[3]]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[14]:=

check γ-matrices properties in the Weyl representation (note that the only difference compared to the standard representation is γ^{0})

define γ-matrices and $g^{\mu\nu}$

In[15]:=
```
gamma[0] = ArrayFlatten[{{ConstantArray[0, {2, 2}], IdentityMatrix[2]},
    {IdentityMatrix[2], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[17]:=
```
gamma[1] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[1]},
    {-PauliMatrix[1], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[18]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

In[19]:=
```
gamma[2] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[2]},
    {-PauliMatrix[2], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

In[21]:=
```
gamma[3] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[3]},
    {-PauliMatrix[3], ConstantArray[0, {2, 2}]}}];
MatrixForm[
 %]
```

Out[22]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[23]:=
```
gmunu = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
MatrixForm[%]
```

Out[24]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

check $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$ (note that in Mathematica the column and row start from 1, rather than 0)

In[25]:=
```
Do[If[(gamma[mdo].gamma[ndo] + gamma[ndo].gamma[mdo] -
      2 * gmunu[[mdo + 1, ndo + 1]] * IdentityMatrix[4]) ≠
    ConstantArray[0, {4, 4}], Print["error"], Print["ok"]],
  {mdo, 0, 3, 1}, {ndo, 0, 3, 1}]
```

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

ok

check $(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0$

In[26]:=
```
Do[If[(ConjugateTranspose[gamma[mdo]] - gamma[0].gamma[mdo].gamma[0]) ≠
    ConstantArray[0, {4, 4}], Print["error"], Print["ok"]], {mdo, 0, 3, 1}]
```

ok

ok

ok

ok

find gamma[5]

In[27]:=
```
MatrixForm[I * gamma[0].gamma[1].gamma[2].gamma[3]]
```

Out[27]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[28]:=
```
Date[]
```

Out[28]= {2018, 11, 1, 16, 15, 44.082165}

5. If $\mu=\nu$, then $\gamma^\mu\gamma^\nu = \begin{cases} +1, & \text{for } \mu=\nu=0 \\ -1, & \text{for } \mu=\nu=1,2,3 \end{cases}$

$\Rightarrow \quad LHS \equiv Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = \pm\, Tr(\gamma^5\gamma^\lambda\gamma^\sigma) = 0$

If $\mu\neq\nu$, but $\mu=\lambda$, then $\gamma^\mu\gamma^\nu\gamma^\lambda = \pm\gamma^\nu \Rightarrow LHS = \pm Tr(\gamma^5\gamma^\nu\gamma^\sigma) = 0$

If $\mu\neq\nu$, $\mu\neq\lambda$, but $\mu=\sigma$, then $\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma = \pm\gamma^\nu\gamma^\lambda \Rightarrow LHS = \pm Tr(\gamma^5\gamma^\nu\gamma^\lambda) = 0$

If $\nu=\lambda$, then $LHS = \pm Tr(\gamma^5\gamma^\mu\gamma^\sigma) = 0$

If $\nu\neq\lambda$, but $\nu=\sigma$, then $\gamma^\nu\gamma^\lambda\gamma^\sigma = \pm\gamma^\lambda \Rightarrow LHS = \pm Tr(\gamma^5\gamma^\mu\gamma^\lambda) = 0$

If $\lambda=\sigma$, then $LHS = \pm Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$.

That is, if any two indices are the same, $LHS=0$, since $\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma$ can be always reduced to a product of only two gamma matrices.

For $\mu=0$, $\nu=1$, $\lambda=2$, $\sigma=3$, we get

$$LHS = Tr(i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3) = -i\,Tr(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3)$$

$$= -i\,Tr(\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3) = -i\,Tr(\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3)$$

$$= i\,Tr(\gamma^2\gamma^3\gamma^2\gamma^3) = -i\,Tr(\gamma^2\gamma^2\gamma^3\gamma^3) = -4i$$

$$= 4i\,\varepsilon^{0123}, \quad \text{since } \varepsilon^{0123} = -1$$

Since for any $\mu\neq\nu$, we have $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$, then define

$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma \equiv (\mu\nu\lambda\sigma)$

$$\Rightarrow (0123) = (1203) = (2013) = (0231) = (0312)$$
$$= (1032) = (2130)$$
$$= (1320) = (2301)$$
$$= (3210) = (3021)$$
$$= (3102)$$

$(0\,1\,2\,3) = -(1\,0\,2\,3) = -(0\,2\,1\,3) = -(0\,1\,3\,2)$

$= -(2\,1\,0\,3) = -(0\,3\,2\,1) = -(3\,0\,1\,2)$

$= -(1\,2\,3\,0) = -(2\,0\,3\,1) = -(3\,1\,2\,0)$

$= -(1\,3\,0\,2) = -(2\,3\,1\,0) = -(3\,2\,0\,1)$

Therefore, it is the same as the permutation property of $\varepsilon^{\mu\nu\lambda\sigma}$

$\Rightarrow$ LHS = RHS

Done the proof.

6. 1)

$$\sigma_1\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2$$

$$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3$$

$$\sigma_2\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

$$\sigma_3\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1$$

$$\sigma_3\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

Therefore, we have shown explicitly that

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$$

2)

$$\begin{aligned} LHS &= (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (a_i\sigma_i)(b_j\sigma_j) = a_i b_j(\delta_{ij} + i\varepsilon_{ijk}\sigma_k) \\ &= \vec{a}\cdot\vec{b} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \\ &= RHS \end{aligned}$$

Done!

3)

$$LHS = 1 + i\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2!}(i\vec{\theta}\cdot\vec{\sigma})^2 + \frac{1}{3!}(i\vec{\theta}\cdot\vec{\sigma})^3 + \dots$$

where $(\vec{\theta}\cdot\vec{\sigma})^2 = |\vec{\theta}|^2 = \theta^2$ according to $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b}) + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

$$\begin{aligned} \Rightarrow LHS &= 1 + i(\hat{\theta}\cdot\vec{\sigma})\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}(\hat{\theta}\cdot\vec{\sigma})\theta^3 + \frac{1}{4!}\theta^4 \\ &\quad + i\frac{1}{5!}(\hat{\theta}\cdot\vec{\sigma})\theta^5 - \frac{1}{6!}\theta^6 - i\frac{1}{7!}(\hat{\theta}\cdot\vec{\sigma})\theta^7 + \dots \\ &= \cos\theta + i\hat{\theta}\cdot\vec{\sigma}\sin\theta = RHS \end{aligned}$$

Done!