Solution

$$\langle q_1, q_2 | \int_{-\infty}^{+\infty} d^4x :\phi^4(x): | k_1, k_2 \rangle$$

$$= C(E_{\vec{q}_1}) C(E_{\vec{q}_2}) C(E_{\vec{k}_1}) C(E_{\vec{k}_2}) [(2\pi)^3]^4 2E_{\vec{q}_1}\, 2E_{\vec{q}_2}\, 2E_{\vec{k}_1}\, 2E_{\vec{k}_2} \langle 0 | a_{\vec{q}_1} a_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x :\phi^4(x): a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2}$$

where $\langle 0 | a_{\vec{q}_1} a_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x :\phi^4(x): a^\dagger_{\vec{k}_1} a^\dagger_{k_2} | 0 \rangle$

$$= \langle 0 | a_{\vec{q}_1} a_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x\, d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p}_3 d^3\vec{p}_4\, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) C(E_{\vec{p}_4})$$

$$\times :(a_{\vec{p}_1} e^{-ip_1\cdot x} + a^\dagger_{\vec{p}_1} e^{ip_1\cdot x})(a_{\vec{p}_2} e^{-ip_2\cdot x} + a^\dagger_{\vec{p}_2} e^{ip_2\cdot x})$$

$$\times (a_{\vec{p}_3} e^{-ip_3\cdot x} + a^\dagger_{\vec{p}_3} e^{ip_3\cdot x})(a_{\vec{p}_4} e^{-ip_4\cdot x} + a^\dagger_{\vec{p}_4} e^{ip_4\cdot x}): a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} | 0 \rangle$$

where $: \; : = a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} e^{i(p_1+p_2+p_3+p_4)\cdot x} + a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4} e^{i(p_1+p_2+p_3-p_4)\cdot x}$

$$+ \cdots + a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a_{\vec{p}_3} a_{\vec{p}_4} e^{i(p_1+p_2-p_3-p_4)\cdot x}$$

$$+ \cdots + a^\dagger_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_2} a_{\vec{p}_3} e^{i(p_1-p_2-p_3-p_4)\cdot x}$$

$$+ \cdots + a_{\vec{p}_1} a_{p_2} a_{p_3} a_{4} e^{i(-p_1-p_2-p_3-p_4)\cdot x}$$

that is, there is 1 term with four $a^\dagger$'s, in the form $a^\dagger a^\dagger a^\dagger a^\dagger$;
there are 4 terms with three $a^\dagger$'s and one $a$, in the form $a^\dagger a^\dagger a^\dagger a$;
there are 6 terms with two $a^\dagger$'s and two $a$'s, in the form $a^\dagger a^\dagger a a$;
there are 4 terms with one $a^\dagger$ and three $a$'s, in the form $a^\dagger a a a$;
there is 1 term with four $a$'s, in the form $a a a a$.

For the term with four $a^\dagger$'s, since for arbitrary $\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2, \vec{p}_3 \& \vec{p}_4$

we have $\langle 0 | a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$

$$= \langle 0 | a_{\vec{q}_1} \left( [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] + a^\dagger_{\vec{p}_1} a_{\vec{q}_2} \right) a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

$$= [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \langle 0 | a_{\vec{q}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} + \langle 0 | a_{\vec{q}_1} a^\dagger_{\vec{p}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

$$= [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \langle 0 | \left( [a_{\vec{q}_1}, a^\dagger_{\vec{p}_2}] + a^\dagger_{\vec{p}_2} a_{\vec{q}_1} \right) a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

$$+ \langle 0 | \left( [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] + a^\dagger_{\vec{p}_1} a_{\vec{q}_1} \right) a_{\vec{q}_2} a^\dagger_{p_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

$$= [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \left( [a_{\vec{q}_1}, a^\dagger_{\vec{p}_2}] \langle 0 | a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} + \langle 0 | a^\dagger_{\vec{p}_2} a_{\vec{q}_1} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} \right)$$

$$[a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \langle 0 | a_{\vec{q}_2} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} + \langle 0 | a^\dagger_{\vec{p}_1} a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

use $\langle 0| a^\dagger = 0$

$$\stackrel{\downarrow}{=} [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \langle 0| a_{\vec{q}_2} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

$$= [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \langle 0| \left( [a_{\vec{q}_2}, a^\dagger_{\vec{p}_2}] + a^\dagger_{\vec{p}_2} a_{\vec{q}_2} \right) a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4}$$

$$= [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \left( [a_{\vec{q}_2}, a^\dagger_{\vec{p}_2}] \langle 0| a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} + \langle 0| a^\dagger_{\vec{p}_2} a_{\vec{q}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} \right)$$

$$= 0$$

$$\Rightarrow 0 = \left( \langle 0| a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} \right)^* = a_{\vec{p}_4} a_{\vec{p}_3} a_{\vec{p}_2} a_{\vec{p}_1} a^\dagger_{\vec{q}_2} a^\dagger_{\vec{q}_1} |0\rangle$$

For the term with three $a^\dagger$'s and one $a$, since for arbitrary $\vec{q}_1, \vec{q}_2$, $\vec{p}_1, \vec{p}_2, \vec{p}_3$ and $\vec{p}_4$, we have

$$\langle 0| a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= \langle 0| a_{\vec{q}_1} \left( [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] + a^\dagger_{\vec{p}_1} a_{\vec{q}_2} \right) a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \langle 0| a_{\vec{q}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4} + \langle 0| a_{\vec{q}_1} a^\dagger_{\vec{p}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \langle 0| \left( [a_{\vec{q}_1}, a^\dagger_{\vec{p}_2}] + a^\dagger_{\vec{p}_2} a_{\vec{q}_1} \right) a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$
$$+ \langle 0| \left( [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] + a^\dagger_{\vec{p}_1} a_{\vec{q}_1} \right) a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \left( [a_{\vec{q}_1}, a^\dagger_{\vec{p}_2}] \langle 0| a^\dagger_{\vec{p}_3} a_{\vec{p}_4} + \langle 0| a^\dagger_{\vec{p}_2} a_{\vec{q}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_4} \right)$$
$$+ [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \langle 0| a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_4} + \langle 0| a^\dagger_{\vec{p}_1} a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \langle 0| a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \langle 0| \left( [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] + a^\dagger_{\vec{p}_1} a_{\vec{q}_2} \right) a^\dagger_{\vec{p}_3} a_{\vec{p}_4}$$

$$= [a_{\vec{q}_1}, a^\dagger_{\vec{p}_1}] \left( [a_{\vec{q}_2}, a^\dagger_{\vec{p}_1}] \langle 0| a^\dagger_{\vec{p}_3} a_{\vec{p}_4} + \langle 0| a^\dagger_{\vec{p}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4} \right)$$

$$= 0$$

$$\Rightarrow 0 = \left( \langle 0| a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_3} a_{\vec{p}_4} \right)^* = a^\dagger_{\vec{p}_4} a_{\vec{p}_3} a_{\vec{p}_2} a_{\vec{p}_1} a^\dagger_{\vec{q}_2} a^\dagger_{\vec{q}_1} |0\rangle$$

Therefore, only the 6 terms with two $a^\dagger$'s and two $a$'s in the form $a^\dagger a^\dagger a a$ contribute to $\langle 0| a_{\vec{q}_1} a_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x : \phi(x) : a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle$.

For arbitrary $\vec{q}_1, \vec{q}_2, \vec{p}_1$ and $\vec{p}_2$, we have

$$< 0 | a_{\vec{q}_1} a_{\vec{q}_2} a^+_{\vec{p}_1} a^+_{\vec{p}_2}$$

$$= <0| a_{\vec{q}_1} \left( [ a_{\vec{q}_2}, a^+_{\vec{p}_1} ] + a^+_{\vec{p}_1} a_{\vec{q}_2} \right) a^+_{\vec{p}_2}$$

$$= [ a_{\vec{q}_2}, a^+_{\vec{p}_1} ] <0| a_{\vec{q}_1} a^+_{\vec{p}_2} + <0| a_{\vec{q}_1} a^+_{\vec{p}_1} a_{\vec{q}_2} a^+_{\vec{p}_2}$$

$$= [ a_{\vec{q}_2}, a^+_{\vec{p}_1} ] <0| \left( [ a_{\vec{q}_1}, a^+_{\vec{p}_2} ] + a^+_{\vec{p}_2} a_{\vec{q}_1} \right)$$
$$+ <0| \left( [ a_{\vec{q}_1}, a^+_{\vec{p}_1} ] + a^+_{\vec{p}_1} a_{\vec{q}_1} \right) \left( [ a_{\vec{q}_2}, a^+_{\vec{p}_2} ] + a^+_{\vec{p}_2} a_{\vec{q}_2} \right)$$

$$= \left( [ a_{\vec{q}_2}, a^+_{\vec{p}_1} ] [ a_{\vec{q}_1}, a^+_{\vec{p}_2} ] + [ a_{\vec{q}_1}, a^+_{\vec{p}_1} ] [ a_{\vec{q}_2}, a^+_{\vec{p}_2} ] \right) <0|$$

$$= \left[\frac{1}{(2\pi)^3}\right]^2 \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \Big( \delta^3(\vec{p}_1 - \vec{q}_1) \delta^3(\vec{p}_2 - \vec{q}_2)$$
$$+ \delta^3(\vec{p}_1 - \vec{q}_2) \delta^3(\vec{p}_2 - \vec{q}_1) \Big) <0|$$

$$\Rightarrow a_{\vec{p}_2} a_{\vec{p}_1} a^+_{\vec{q}_2} a^+_{\vec{q}_1} |0> = \left( <0| a_{\vec{q}_1} a_{\vec{q}_2} a^+_{\vec{p}_1} a^+_{\vec{p}_2} \right)^*$$

$$= \left[\frac{1}{(2\pi)^3}\right]^2 \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \Big( \delta^3(\vec{p}_1 - \vec{q}_1) \delta^3(\vec{p}_2 - \vec{q}_2)$$
$$\delta^3(\vec{p}_1 - \vec{q}_2) \delta^3(\vec{p}_2 - \vec{q}_1) \Big) |0>$$

$$\Rightarrow <0| a_{\vec{q}_1} a_{\vec{q}_2} \int_{-\infty}^{+\infty} d^4x : \phi^4(x) : a^+_{\vec{k}_1} a^+_{\vec{k}_2} |0>$$

$$\overset{\text{use } <0|0>=1}{=} \int_{-\infty}^{+\infty} d^4x\, d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3\, d^3\vec{p}_4\, C(E_{\vec{p}_1}) C(E_{\vec{p}_2}) C(E_{\vec{p}_3}) C(E_{\vec{p}_4})$$
$$\times \left[\frac{1}{(2\pi)^3}\right]^4 \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \frac{1}{2E_{\vec{k}_1}} \frac{1}{2E_{\vec{k}_2}} \left(\frac{1}{C(E_{\vec{q}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{q}_2})}\right)^2 \left(\frac{1}{C(E_{\vec{k}_1})}\right)^2 \left(\frac{1}{C(E_{\vec{k}_2})}\right)^2$$
$$\times \Big( [\delta^3(\vec{p}_1 - \vec{q}_1) \delta^3(\vec{p}_2 - \vec{q}_2) + \delta^3(\vec{p}_1 - \vec{q}_2) \delta^3(\vec{p}_2 - \vec{q}_1)]$$
$$\times [\delta^3(\vec{p}_3 - \vec{k}_1) \delta^3(\vec{p}_4 - \vec{k}_2) + \delta^3(\vec{p}_3 - \vec{k}_2) \delta^3(\vec{p}_4 - \vec{k}_1)]\, e^{i(p_1+p_2-p_3-p_4)\cdot x}$$
$$+ (p_2 \leftrightarrow p_3) + (p_2 \leftrightarrow p_4) + (p_1 \leftrightarrow p_3) + (p_1 \leftrightarrow p_4)$$
$$+ (p_1 \leftrightarrow p_3,\ p_2 \leftrightarrow p_4) \Big)$$

$$= \left[\frac{1}{(2\pi)^3}\right]^4 \frac{1}{2E_{\vec{q}_1}} \frac{1}{2E_{\vec{q}_2}} \frac{1}{2E_{\vec{k}_1}} \frac{1}{2E_{\vec{k}_2}} \left(\frac{1}{C(E_{\vec{q}_1})}\right) \left(\frac{1}{C(E_{\vec{q}_2})}\right) \left(\frac{1}{C(E_{\vec{k}_1})}\right) \left(\frac{1}{C(E_{\vec{k}_2})}\right)$$
$$\times \int d^4x \left( 4 e^{i(q_1+q_2-k_1-k_2)\cdot x} \times 6 \right)$$

$$= 24\,\delta^4(q_1+q_2-k_1-k_2)\left[\frac{1}{(2\pi)^3}\right]^4\left(\frac{1}{2E_{\vec{q}_1}}\right)\left(\frac{1}{2E_{\vec{q}_2}}\right)\left(\frac{1}{2E_{\vec{k}_1}}\right)\left(\frac{1}{2E_{\vec{k}_2}}\right)\left(\frac{1}{C(E_{\vec{k}_1})}\right)\left(\frac{1}{C(E_{\vec{k}_2})}\right)$$

$$\times\left(\frac{1}{C(E_{\vec{q}_1})}\right)\left(\frac{1}{C(E_{\vec{q}_2})}\right)$$

$$\Rightarrow \langle q_1, q_2 | \int_{-\infty}^{+\infty} d^4x :\phi^4(x): | k_1, k_2 \rangle$$

$$= \boxed{24\,\delta^4(q_1+q_2-k_1-k_2)}$$

Solution

From $(\not{p}_2 + m)\, v(\vec{p}_2, s_2) = 0$

$\Rightarrow \quad v^{+}(\vec{p}_2, s_2)\,(\not{p}_2^{+} + m) = 0 \Rightarrow v^{+}\gamma^0\gamma^0(\not{p}_2^{+} + m)\gamma^0 = 0$

use $\gamma^0\gamma^{+\mu}\gamma^0 = \gamma^\mu$ $\Rightarrow$ $\bar{v}(\vec{p}_2, s_2)\,(\not{p}_2 + m) = 0 \Rightarrow \bar{v}(\vec{p}_2, s_2)\not{p}_2 = -m\,\bar{v}(\vec{p}_2,$

Therefore, $\text{RHS} = \bar{v}(\vec{p}_2, s_2)\left[(p_1 - p_2)^\mu - i\cdot\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_1+p_2)_\nu\right] u(\vec{p}_1, s_1)$

$= \bar{v}(\vec{p}_2, s_2)\left[(p_1-p_2)^\mu + \frac{1}{2}\left(\gamma^\mu(\not{p}_1 + \not{p}_2) - (\not{p}_1 + \not{p}_2)\gamma^\mu\right)\right] u(\vec{p}_1, s_1)$

$= \bar{v}(\vec{p}_2, s_2)\left[(p_1-p_2)^\mu + \frac{1}{2}\gamma^\mu(m + \not{p}_2) - \frac{1}{2}(\not{p}_1 - m)\gamma^\mu\right] u(\vec{p}_1, s$

use $\not{p}_1\, u(\vec{p}_1, s_1) = m\, u(\vec{p}_1, s_1)$

$\bar{v}(\vec{p}_2, s_2)\not{p}_2 = -m\,\bar{v}(\vec{p}_2, s_2)$

$= \bar{v}(\vec{p}_2, s_2)\left[m\gamma^\mu + (p_1 - p_2)^\mu + \frac{1}{2}\gamma^\mu\not{p}_2 - \frac{1}{2}\not{p}_1\gamma^\mu\right] u(\vec{p}_1, s_1)$

$= \bar{v}(\vec{p}_2, s_2)\left[m\gamma^\mu + (p_1-p_2)^\mu + \frac{1}{2}(2p_2^\mu - \not{p}_2\gamma^\mu) - \frac{1}{2}(2p_1^\mu - \gamma^\mu\not{p}_1)\right.$ $\cdot u(\vec{p}_1, s_1)$

use $\not{p}\gamma^\mu + \gamma^\mu\not{p} = p_\nu(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu) = p_\nu 2g^{\mu\nu} = 2p^\mu$

$= \bar{v}(\vec{p}_2, s_2)\left[m\gamma^\mu + (p_1-p_2)^\mu + (p_2 - p_1)^\mu - \frac{1}{2}\not{p}_2\gamma^\mu + \frac{1}{2}\gamma^\mu\not{p}_1\right] \cdot u(\vec{p}_1, s_1)$

use $\not{p}_1\, u(\vec{p}_1, s_1) = m\, u(\vec{p}_1, s_1)$, $\bar{v}(\vec{p}_2, s_2)\not{p}_2 = -m\,\bar{v}(\vec{p}_2, s_2)$

$= \bar{v}(\vec{p}_2, s_2)\left[m\gamma^\mu + \frac{1}{2}m\gamma^\mu + \frac{1}{2}\gamma^\mu m\right] u(\vec{p}_1, s_1)$

$= 2m\,\bar{v}(\vec{p}_2, s_2)\gamma^\mu u(\vec{p}_1, s_1)$

$=$ LHS

Done the proof.