The Euler-Lagrangian Equation

具有N个自由度的系统，其运动状态完全由
N个广义坐标及广义速度决定。系统的运动状态由
它们的函数描述（即拉格朗日函数或称拉氏量

In classical mechanics, the equations of motion (EoM) can be derived from the action, S,

$$S = \int_{t_1}^{t_2} dt \; L(q(t), \dot{q}(t), t)$$

where L is the Lagrange function, $q(t)$ are the generalized coordinates, $\dot{q}(t)$ the generalized velocities.

The minimum action principle says that among all possible paths joining any two fixed points at time t_1 and t_2 ($t_2 > t_1$), the path for which S is minimum corresponds to the physical path that determines the actual motion of the particles.

[Note that the path may be actually a maximum. Therefore, more accurately, it should be called the principle of stationary action.]

[Note that the degrees of freedom can be more than 1. Therefore, $q(t)$ and $\dot{q}(t)$ should be understood as $q = (q_1, q_2, \cdots, q_N)$ and $\dot{q} = (\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_N)$

[Mathematically, the principle is

$$\delta S = 0$$

It means "the path taken by the system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the action is stationary (no change) to first order]

Thus, by varying $q \longrightarrow q + \delta q$, subject to the constraint $\delta q(t_1) = \delta q(t_2) = 0$, one gets

$$S \longrightarrow S + \delta S$$

where

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right)$$

Integration by part for the second term, then

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \delta q \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q + \underbrace{\cancel{\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\Bigg|_{t_1}^{t_2}}}_{0 \text{ due to } \delta q(t_1) = \delta(q(t_2)) = 0}$$

Because δq is arbitrary,

$$\delta S = 0 \Rightarrow \underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}_{\text{Euler-Lagrangian equation}}$$

For N degrees of freedom.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad , \quad i = 1, \cdots, N.$$

Introduce

$$P_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad \text{and get Hamiltonian}$$

$$H(q_i, P_i, t) = \sum_i \dot{q}_i P_i - L(q, \dot{q}, t)$$

where $\dot{q}_i = \frac{\partial H}{\partial P_i}$

(From $dH = \sum_i \left[\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial P_i} dP_i\right] + \frac{\partial H}{\partial t} dt$

$$= \sum_i \left[\dot{q}_i dP_i + P_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right] - \frac{\partial L}{\partial t} dt$$

"~" cancel due to $P_i = \frac{\partial L}{\partial \dot{q}_i}$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial P_i} \quad , \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \underset{\uparrow}{=} -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\dot{P}_i$$

Euler-Lagrange equation

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$)

$\boxed{\text{In field theory}}$: $q(t) \to \varphi(t, \vec{x})$, $\dot{q}(t) \to \partial_\mu \varphi(t, \vec{x})$

$$L = \int d^3x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

$\mathcal{L}$ is called the Lagrangian density.

and the action
$$S = \int_{\tau_1}^{\tau_2} d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

where τ_1 and τ_2 are the integration boundary (i.e., the limiting surfaces of integration).

note: ① $[\mathcal{L}] = [E]^4$, $[S] = [E]^0 = 1$

② We usually directly call $\mathcal{L}$ the "Lagrangian", rather than use the words "Lagrangian density", for simplicity.

③ $\mathcal{L}$ is usually the starting point when studying a theory (or model) in particle physics, since it includes the dynamical information and symmetries of the theory (or model).

Now follow the same story as in classical mechanics to find the Euler-Lagrangian equation in field theory.

Do a variation for a generic field $\varphi(x)$,
$$\varphi(x) \to \varphi'(x) = \varphi(x) + \delta_0 \varphi(x)$$

such that $\delta_0 \varphi = 0$ at the integration limits, i.e., $\delta_0 \varphi(\tau_1) = \delta_0 \varphi(\tau_2) = 0$

then
$$\delta S = \int_{\tau_1}^{\tau_2} d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta_0 \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 (\partial_\mu \varphi) \right)$$
$$= \int_{\tau_1}^{\tau_2} d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta_0 \varphi + \int_{\tau_1}^{\tau_2} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi \right)$$

the second term $= \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi \right] \Big|_{\tau_1}^{\tau_2} = 0$.

The requirement that S is stationary for any arbitrary variation $\delta_a \varphi$ means

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0$$

this is the Euler - Lagrangian equation (i.e., the EoM)

For N fields, $\mathcal{L}(\varphi_1, \varphi_2, \cdots \varphi_N, \partial_\mu \varphi_1, \partial_\mu \varphi_2, \cdots, \partial_\mu \varphi_N)$

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} = 0, \quad i = 1, \cdots, N.$$

Again, introduce conjugate momentum density

$$\pi_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial \varphi_i / \partial t)}$$

and then the Hamiltonian density is

$$\mathcal{H}(\pi_i, \varphi_i) = \sum_i (\pi_i \frac{\partial \varphi_i}{\partial t}) - \mathcal{L}$$

Examples (we will study them later in this course.)

(a) Real scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 ,$$ m and λ are parameters.

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} = - m^2 \phi - \frac{\lambda}{3!} \phi^3 .$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\Rightarrow \quad (\Box + m^2) \phi = - \frac{\lambda}{3!} \phi^3 .$$ $\leftarrow$ the EoM. (when $\lambda = 0$, it is just the Klein-Gordon equation for free field.)

$$[\mathcal{L}] = [E]^4 \Rightarrow [\phi] = [E]^1 \Rightarrow [\lambda] = [E]^0 .$$

also $$\pi = \frac{\partial \mathcal{L}}{\partial(\partial \phi / \partial t)} = \frac{\partial \phi}{\partial t} .$$

(b) Complex scalar field.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2,$$ note that ϕ^* and ϕ are independent field component.

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^* - 2\lambda(\phi^*\phi)\phi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^* \Rightarrow \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \Box \phi^*$$

$$\Rightarrow (\Box + m^2)\phi^* = -2\lambda(\phi^*\phi)\phi^*. \quad , \quad \pi^* = \frac{\partial \mathcal{L}}{\partial(\partial \phi^*/\partial t)} = \frac{\partial \phi}{\partial t}$$

and $$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2\phi - 2\lambda(\phi^*\phi)\phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi \Rightarrow \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\right) = \Box\phi$$

$$\Rightarrow (\Box + m^2)\phi = -2\lambda(\phi^*\phi)\phi \quad , \quad \pi = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} = \frac{\partial \phi^*}{\partial t}$$

If want to consider real scalar field, then can write

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \Rightarrow \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2$$
$$- \frac{1}{4}\lambda(\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2\phi_1 - \lambda\phi_1^3 - \lambda\phi_1\phi_2^2 \quad , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_1)} = \partial^\mu\phi_1$$

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2\phi_2 - \lambda\phi_2^3 - \lambda\phi_1^2\phi_2 \quad , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_2)} = \partial^\mu\phi_2$$

$$\Rightarrow (\Box + m^2)\phi_1 = -\lambda(\phi_1^2 + \phi_2^2)\phi_1 ,$$
$$(\Box + m^2)\phi_2 = -\lambda(\phi_1^2 + \phi_2^2)\phi_2$$

Directly from $(\Box + m^2)\phi = -2\lambda(\phi^*\phi)\phi \Rightarrow (\Box + m^2)\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = -\frac{1}{\sqrt{2}}\lambda(\phi_1^2 + \phi_2^2)(\phi_1 + i\phi_2)$

$$\Rightarrow (\Box + m^2)\phi_1 = -\lambda(\phi_1^2 + \phi_2^2)\phi_1 , \quad (\Box + m^2)\phi_2 = -\lambda(\phi_1^2 + \phi_2^2)\phi_2$$

Note: for $\lambda = 0$, EoM is just the Klein-Gordon Equation for free field. ✓

c) Real vector field.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu$$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Each component of A_μ can be considered as an independent component,

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu - j^\nu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4} \frac{\partial[(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{4} \left\{ 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \frac{\partial(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)} \right\}$$

$$= -\frac{1}{4} \left\{ 2(\partial^\alpha A^\beta - \partial^\beta A^\alpha)(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) \right\}$$

$$= -\frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu + \partial^\mu A^\nu)$$

$$= -(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -F^{\mu\nu}$$

(or, do the second equal sign as

$$-\frac{1}{4} \frac{\partial[2\partial_\alpha A_\beta \partial^\alpha A^\beta - 2\partial_\alpha A_\beta \partial^\beta A^\alpha]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{2} [2\partial^\alpha A^\beta \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\alpha \delta^\nu_\beta \partial^\beta A^\alpha - \partial^\alpha A^\beta \delta^\mu_\beta \delta^\nu_\alpha]$$

$$= -\frac{1}{2}[2\partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu] = -F^{\mu\nu}$$)

$$\Rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\partial_\mu F^{\mu\nu}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

note $[A_\mu] = [E]^1$, $[F_{\mu\nu}] = [E]^2$

Note that for $m=0$, the EoM are just the Maxwell equations (2 of the 4)

Let's check that we can get the familiar Maxwell equations from $F_{\mu\nu}$

First, $F_{\mu\nu}$ is 4×4 matrix and antisymmetric, so it has 6 independent components ($E^{1,2,3}$ & $B^{1,2,3}$, also 6, Great!)

Construct $F^{i0} = -F^{0i} = E^i$, $F^{ij} = -\varepsilon^{ijk}B^k$ (note $\varepsilon^{123}=1$.)

that is
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1, & -E^2, & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^2 & -B^2 & B^1 & 0 \end{pmatrix}$$

and construct the four-current $j^\mu = (\rho, \vec{j})$

Then $\partial_\mu F^{\mu\nu} = j^\nu$

$\Rightarrow$ for $\nu=0$, LHS $= \partial_0 F^{00} + \partial_i F^{i0} = \dfrac{\partial E^1}{\partial x} + \dfrac{\partial E^2}{\partial y} + \dfrac{\partial E^3}{\partial z} = \vec{\nabla}\cdot\vec{E}$

RHS $= \rho$

$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = \rho$ ✓

for $\nu = i$, $i=1,2,3$, LHS $= \partial_0 F^{0i} + \partial_j F^{ji} = -\dfrac{\partial E^i}{\partial t} + \partial_j \varepsilon^{ijk} B_k$

RHS $= j^i$

$$\Rightarrow \quad -\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \vec{j} \quad ✓$$

The current conservation is abtained by do ∂_ν on both sides.

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu$$
$$\|$$
0 due to $F^{\mu\nu}$ is antisymmetric while $\partial_\nu\partial_\mu = \partial_\mu\partial_\nu$.

(i.e, $\partial_\nu\partial_\mu F^{\mu\nu} = -\partial_\nu\partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu} \Rightarrow \partial_\nu\partial_\mu F^{\mu\nu} = 0$.)

Since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, then

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = \partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda) = 0$$

For $(\lambda, \mu, \nu) = (i, j, k)$, $i \neq j \neq k$,

$$\partial^i F^{jk} = -\partial_i(\varepsilon^{jkm} B^m) = \partial_i \varepsilon^{jkm} B^m$$

$$\partial^j F^{ki} = \partial_j \varepsilon^{kim} B^m$$

$$\partial^k F^{ij} = \partial_k \varepsilon^{ijm} B^m$$

$\Rightarrow$ for $i=1, j=2, k=3$, (other combination give the same thing)

$$\Rightarrow \quad \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0,$$

that is $\vec{\nabla} \cdot \vec{B} = 0$ ✓

For $(\lambda, \mu, \nu) = (0, i, j)$

$$\partial^0 F^{ij} = -\partial_0 \varepsilon^{ijm} B^m$$

$$\partial^i F^{j0} = -\partial_i E^j$$

$$\partial^j F^{0i} = -\partial_j (-E^i) = \partial_j E^i$$

$$\Rightarrow \quad \partial_i E^j - \partial_j E^i + \varepsilon^{ijk} \frac{\partial B^k}{\partial t} = 0$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$ ✓

Done the check!

From $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

$\Rightarrow$ for $\mu = 0$, $\nu = i$

$$-E^i = \frac{\partial A^i}{\partial t} + \frac{\partial A^0}{\partial x^i} \Rightarrow \underline{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A^0}$$

for $\mu = i$, $\nu = j$

$$-\varepsilon^{ijk} B^k = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i$$

$$\Rightarrow \quad \underline{\vec{B} = \vec{\nabla} \times \vec{A}}$$

— the familiar ones.