Unit 3

From Proposition to Predicate

Outline of Unit 3

- ❑ 3.1 Inference Rules for Propositional Logic

- ❑ 3.2 Beginning Predicate Logic
- ❑ 3.3 Negation of Quantification
- ❑ 3.4 Nested Quantification

Predicate Logic

Unit 3.1

Inference Rules for Propositional Logic

Validity of an Argument

- An argument is valid if its conclusion is a logical consequence of the premises.
  - This was defined in Unit 1.
- Consider an argument in this form:

$$\frac{p_1, p_2, \ldots, p_n}{c}$$

The argument is valid if and only if
$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow c$ is a tautology.

Checking Validity by Truth Table

- To check the validity of an argument, we can always
  1. tabulate the truth values of premises and conclusion,
  2. check whether there is a row in which all premises are true (i.e. critical row) while the conclusion is false.
  3. The argument is valid iff no such rows exist.

- Drawback: When there are n proposition variables, there are 2^n rows in the truth table, which grows exponentially in n.

Inference Rules

- A rule of inference is a valid argument form which takes premises, analyses their syntax, and returns a conclusion.
- By repeatedly applying inference rules, we can demonstrate the validity of an argument by
  - starting with its premises,
  - taking one tiny valid step at a time, and finally
  - reaching its conclusion.
- We will consider nine elementary inference rules.

Argument Form

Argument 1:

If you study hard, then you will get a good grade in this course.

You study hard.

You will get a good grade in this course.

Argument 1 has the form:

$$p \rightarrow q$$

$$p$$

$$q$$

- This argument form is called Modus Ponens (MP).

Inference Rule #1

Modus Ponens
(method of affirming)

$$\begin{array}{c} p \to q \\ p \\ \hline q \end{array}$$

	variables		premises	conclusion
p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

- There is only one critical row (i.e., with all premises true).
- In that row, the conclusion is true.
- Therefore, MP is a valid argument form.

Inference Rule #2 – 3

Modus Tollens
(method of denying)

$$p \rightarrow q$$
$$\sim q$$

$$\sim p$$

If it is Saturday morning, then Vincent is sleeping in.

Vincent is not sleeping in.

It is not Saturday morning.

Hypothetical Syllogism
(transitivity)

$$p \rightarrow q$$
$$q \rightarrow r$$

$$p \rightarrow r$$

If you love logic, then you will study hard.
If you study hard, then you will get an A in EE1001.

If you love logic, then you will get an A in EE1001.

Valid but may not be sound!

Inference Rules #4 – 6

Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{p \wedge q}$$

Simplification

$$\frac{p \wedge q}{p}$$

Absorption

$$\frac{p \rightarrow q}{p \rightarrow (p \wedge q)}$$

If Tiffany visits Japan, then she will eat sushi.

If Tiffany visits Japan, then Tiffany visits Japan and she will eat sushi.

Why Absorption?

- The main use for Absorption will be in cases where you need to have $p \wedge q$ in order to take further step in the argument.
- Example:
  1) $\boldsymbol{p \rightarrow q}$ (Premise)
  2) $\boldsymbol{(p \wedge q) \rightarrow r}$ (Premise)
  3) $\boldsymbol{p \rightarrow (p \wedge q)}$ (Abs 1)
  4) $\boldsymbol{p \rightarrow r}$ (HS 3,2)

$$\begin{array}{c} \boldsymbol{p \rightarrow q} \\ \boldsymbol{(p \wedge q) \rightarrow r} \\ \hline \boldsymbol{p \rightarrow r} \end{array}$$

Inference Rules #7 – 9

Addition

$$\frac{p}{p \vee q}$$

Disjunctive Syllogism

$$\frac{\begin{array}{c} p \vee q \\ \sim p \end{array}}{q}$$

Constructive Dilemma

$$\frac{\begin{array}{c} (p \to q) \wedge (r \to s) \\ p \vee r \end{array}}{q \vee s}$$

Handy for deducing the consequences of "either-or" situations.

If Tiffany comes along on the trip, then Vincent will be happy; and if Vincent goes without Tiffany, then he will be lonely.

Either Tiffany comes along on the trip or Vincent goes without Tiffany.

Either Vincent will be happy or Vincent will be lonely.

Example

- What conclusion can be drawn?

1) If Aunt Mary comes to visit, then Vincent will escape to his bedroom.
2) If Vincent escapes to his bedroom, then his Mom will be displeased.
3) His Mom is not displeased.

The inference steps may not be unique.

1) $p \to q$
2) $q \to r$
3) $\sim r$
4) $p \to r$ (HS 1,2)
5) $\sim p$ (MT 3,4)

Conclusion: Aunt Mary did not come to visit.

Summary

1. MODUS PONENS
2. MODUS TOLLENS
3. HYPOTHETICAL SYLLOGISM
4. CONJUNCTION
5. SIMPLIFICATION
6. ABSORPTION
7. ADDITION
8. DISJUNCTIVE SYLLOGISM
9. CONSTRUCTIVE DILEMMA

Unit 3.2

Beginning Predicate Logic

Knights and Knaves on the Island

All of us here are of the same type.

Each one said the same thing.

- You asked each person living on the island to tell you something about the types of all the people living there.

Are they really all of the same type? If so, which type are they?

Limitation of Propositional Logic

All men are mortal.

Socrates was a man.

Socrates was mortal.

- This argument can't be expressed with propositional logic.

Why?

Predicates

- In ordinary language, *predicate* refers to the part of a sentence that gives information about the subject.

Issac Newton is a physicist.

subject predicate

- We use $P(x)$ to represent "x is a physicist," where x is a predicate variable.

Predicates

- In logic, a predicate is a statement that contains variables and that may be true or false depending on the values of these variables.

- Example:
  - $P(x)$ represents "x is a physicist."
  - P(Issac Newton) is true.
  - P(William Shakespeare) is false.

Predicate Instantiated

- A *predicate instantiated* (where variables are evaluated in specific values) is a proposition.
  - P(Issac Newton) = "Issac Newton is a physicist."

- The *domain* of a predicate variable is the set of all possible values that the variable may take.
  - The domain of x may be

    {Issac Newton, William Shakespeare, Albert Einstein}

The Universal Quantifier ∀

- A quantifier tells the amount or quantity.
- The symbol ∀ denotes the *universal quantifier*, which means "given any" or "for all".
- "$\forall x \in D, Q(x)$" is a universal statement.
  - It asserts that all elements in D have the property Q.
  - e.g. "$\forall x \in D, x \geq 0$" means all $x \in D$ are non-negative.
  - The domain D can be omitted if no ambiguity.
- "$\forall x \in D, Q(x)$" is true iff

  $Q(x)$ is true for every x in D.

Universal Statements

- Example 1:
  - Let $P(x, y) =$ "$\forall x, y \in D,\ x > y$", where D is the set of integers.
  - $P(6, 2)$ is true, but it doesn't mean that $P(x, y)$ is true.
  - $P(3, 5)$ is false, a counter-example which shows that $P(x, y)$ is false.

- Example 2:
  - Let $Q(x) =$ "$\forall x \in D, x^2 \geq x$“, where $D = \{1, 2, 3, 4, 5\}$.
  - Check that

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

  - Hence, $Q(x)$ is true, (by the method of exhaustion).

True or False?

1. $\forall x \in \boldsymbol{R}, x^2 \geq x$, where $\boldsymbol{R}$ is the set of real numbers.

2. $\forall x \in \emptyset, x^2 \geq x$, where $\emptyset$ is the empty set.

The Existential Quantifier ∃

- The symbol ∃ denotes the *existential quantifier*, which means "there exists".
- "$\exists x \in D, Q(x)$" is an existential statement.
  - It asserts that at least one element in D has the property Q.
  - e.g. "$\exists x \in \boldsymbol{Z}, 1.5 < x < 2.5$" means that there is an integer between 1.5 and 2.5.
- "$\exists x \in D, Q(x)$" is true iff

$$Q(x) \text{ is true for some } x \text{ in } D.$$

Existential Statements

- Example 3:
  - $\exists m \in \boldsymbol{Z}$ such that $m^2 = m$.
  - It can be shown to be true by the method of case (i.e., giving an example):

    1 is an integer and $1^2 = 1$.
- Example 4:
  - $\exists m \in \{5, 6, 7, 8\}$ such that $m^2 = m$.
  - It can be shown to be false by the method of exhaustion,

$$5^2 = 25 \neq 5, \quad 6^2 = 36 \neq 6, \quad 7^2 = 49 \neq 7, \quad 8^2 = 64 \neq 8.$$

Truth Values

Statement	When True	When False
$\forall x \in D, P(x)$	$P(x)$ is true for every x.	There is one x for which $P(x)$ is false.
$\exists x \in D, P(x)$	There is one x for which $P(x)$ is true.	$P(x)$ is false for every x.

- Assume that $D = \{x_1, x_2, \dots, x_n\}$.

$$\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \cdots \wedge P(x_n)$$

$$\exists x \in D, P(x) \equiv P(x_1) \vee P(x_2) \cdots \vee P(x_n)$$

Universal Conditional Statements

- A universal conditional statement takes the form

$$\forall x \in D, \quad \text{if } P(x), \text{then } Q(x).$$

- Example:

$$\forall x \in \boldsymbol{R}, \quad \text{if } x > 2, \text{then } x^2 > 4.$$

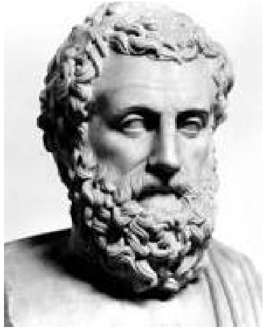
  - "For all real numbers *x*, if *x* is greater than 2, then its square is greater than 4."
  - "If a real number is greater than 2, then its square is greater than 4."
    - An implicitly quantified statement, which occurs commonly in mathematics writing.

Unit 3.3

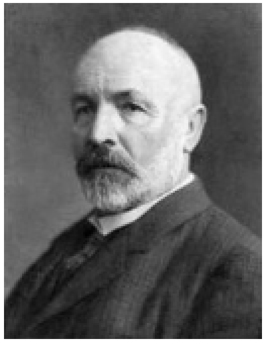
Negation of Quantification

“All logicians are male.”

What is the negation of this universal statement?

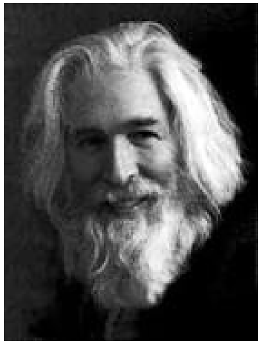


Aristotle (384-322), the father of logic.



Gottlob Frege (1848-1925), one of the founders of modern logic.

Kurt Gödel (1906-1978), who published his two Incompleteness Theorems in 1931, which showed the limit of logic.



Raymond Merrill Smullyan (1919-2017), author of many books on recreational logic.

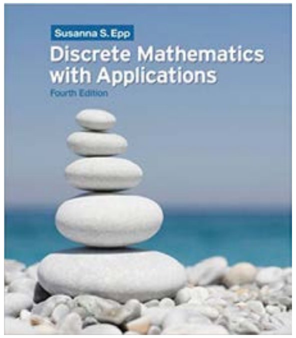
“Not all logicians are male.”

- “Not all logicians are male.”

  is equivalent to

- “There exists a logician who are not male.”

A true statement.



Susanna S. Epp, the author of our major reference.

Negation of Universal Statements

- The negation of a universal statement is logically equivalent to an existential statement.

The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

Negation of Existential Statements

- The negation of an existential statement is logically equivalent to a universal statement.

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

Negation of Universal Conditional Statements

- The negation of a universal statement is

$$\sim(\forall x,\ P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)).$$

- And we know that $p \rightarrow q \equiv \sim p \vee q$, so

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x).$$ De Morgan's Law

- Hence, the negation of a universal conditional statement is

$$\sim(\forall x,\ P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Classwork

- Consider "$\forall x \in \boldsymbol{R}$, if $x^2 > 4$, then $x > 2$".
  a) What is its negation?

  b) Is its negation true?

Unit 3.4

Nested Quantification

Nested Quantification ($\forall, \exists$)

- Two quantifiers are nested if one is within the scope of the other.

"Every girl has a boy that she'll never lose feelings for."

- Domains: G = {girls}, B = {boys}
- Predicate $L(x, y)$: The girl x never loses feelings for the boy y.
- In symbols,

$$\forall x \in G, \exists y \in B, L(x, y).$$

A Closer Look

$$\forall x \in G, \underbrace{\boxed{\exists y \in B, F(x, y)}}_{P(x)}$$

- The statement is equivalent to

$$\forall x \in G, P(x),$$

where

$$P(x) \equiv \exists y \in B, F(x, y).$$

Nested Quantification ($\exists, \forall$)

"There is a book which every Christian reads."

- Domains: B = {books}, C = {Christians}
- Predicate $R(x, y)$: The Christian x reads the book y.
- In symbols,

$$\exists y \in B, \forall x \in C, R(x, y).$$

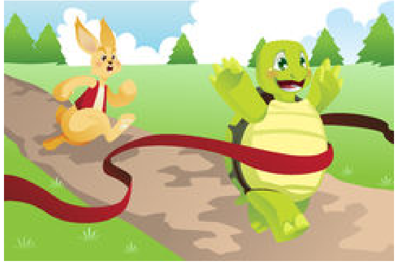
Nested Quantification (∀, ∀) (∃, ∃)

- Domains: R = {rabbits}, T = {tortoises}
- Predicate $F(x, y)$: x is faster than y.

"All rabbits are faster than all tortoises."

$$\forall x \in R\ \forall y \in T, F(x, y).$$

"There is a tortoise which is faster than some rabbit."



$$\exists y \in T, \exists x \in R, F(y, x).$$

Negation of Nested Quantification

- To find $\sim\left(\forall x \in G, \exists y \in B, L(x,y)\right)$.

$$
\begin{aligned}
&\sim\left(\forall x \in G, \exists y \in B, L(x,y)\right) \\
&\qquad \equiv \exists x \in G, \sim\left(\exists y \in B, L(x,y)\right) \\
&\qquad \equiv \exists x \in G, \forall y \in B, \sim L(x,y)
\end{aligned}
$$

- Similarly, we can show that

$$
\begin{aligned}
&\sim(\exists y \in B, \forall x \in C, R(x,y)) \\
&\qquad \equiv \forall y \in B, \exists x \in C, \sim R(x,y)
\end{aligned}
$$

Classwork

- Express the negation of the statement
$$\forall x \in \boldsymbol{R}, \exists y \in \boldsymbol{R}, xy = 1,$$
where $\boldsymbol{R}$ is the set of real numbers.

- Answer:

Order of Nesting

- Equivalent or not?

$$\forall x, \exists y, P(x, y) \overset{?}{\equiv} \exists y, \forall x, P(x, y)$$

Every girl is loved by some boy. $\neq$ There is some boy who loves all girls.

How about $\forall x, \forall y$?
And $\exists x, \exists y$?

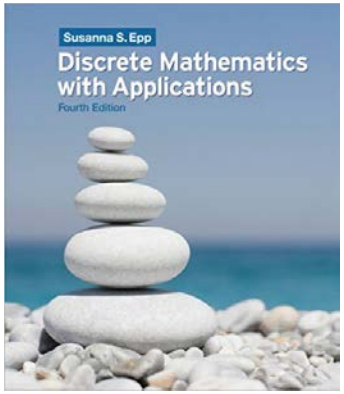
Order of Nesting

- If two (or more) quantifiers are the same, their order *doesn't* matter.
- Example 1:
  - Let $P(x, y)$ be $(x + y)^2 = x^2 + 2xy + y^2$.
  - $\forall x \in \boldsymbol{R}\ \forall y \in \boldsymbol{R}\ P(x, y)$ means that given any x, no matter how we choose y, $P(x, y)$ is true.
  - $\forall y \in \boldsymbol{R}\ \forall x \in \boldsymbol{R}\ P(x, y)$ means that given any y, no matter how we choose x, $P(x, y)$ is true.
- Example 2:
  - $\exists x \in \boldsymbol{R}\ \exists y \in \boldsymbol{Z}\ x > y$ means that there exists a real number x, such that we can find an integer y that is less than x.
  - $\exists y \in \boldsymbol{Z}\ \exists x \in \boldsymbol{R}\ x > y$ means that there exists an integer y, such that we can find a real number x that is greater than y.

Order of Nesting

- If two quantifiers are different, their order *does* matter.
- Example 3:
  - $\forall x \in \boldsymbol{R}\ \exists y \in \boldsymbol{R}\ \ x > y$ is true.
    - Reason: Given any real number x, we can always find a real number y which is less than x.
  - $\exists y \in \boldsymbol{R}\ \forall x \in \boldsymbol{R}\ \ x > y$ is false.
    - Meaning of the statement: There exists a real number y which is less than all real numbers.
    - It is false because there is no smallest real number.

Recommended Reading



- Sections 2.3, 3.1-3.3, Susanna S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.