# Unit 2

## Beginning Propositional Logic

# Outline of Unit 2

- 2.1 True or False?
- 2.2 Logical Equivalence
- 2.3 Conditionals

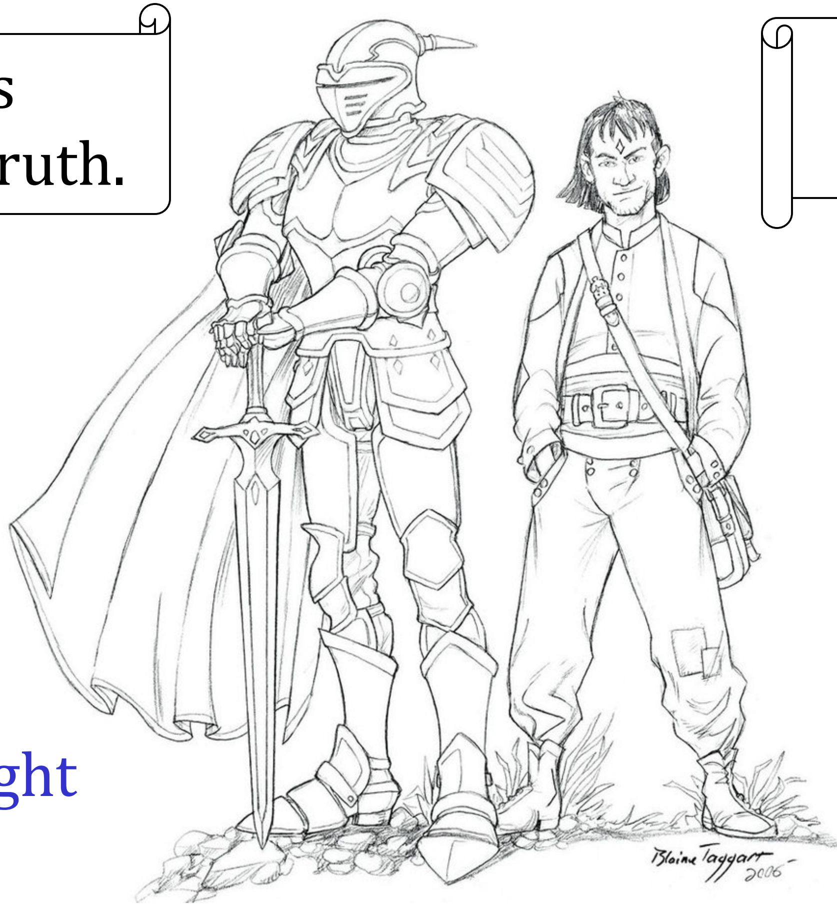
# Unit 2.1

True or False?

# On an island, there are two types of inhabitants…

He always tells the truth.

He always lies.




knight

knave

# You met two islanders…

I’m a knave
but he isn’t.

Are they knights
or knaves?

A

B

1) Both are knights.
2) Both are knaves.
3) A is knight and B is knave.
4) A is knave and B is knight.

# Propositions

- A proposition is a statement that is either true or false, but not both.
- Each proposition has a truth value, either T or F.

- Are they propositions?

1) "1 + 1 = 3."
2) "What's your name?"
3) "God exists."
4) "Study logic more often, please."

# Logical Connectives

- Propositions can be combined to form compound propositions by using logical connectives.

- Three basic logical connectives are
  - negation, NOT
  - conjunction, AND
  - disjunction, OR

# Negation - NOT

- Given a proposition $p$, the compound proposition $\sim p$ is called the negation of $p$.
  - It is read as "not $p$" or "It is not the case that $p$".
  - Sometimes, it is written as $\neg p$.
- Truth table:

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

# Conjunction - AND

- Given $p$ and $q$, the compound proposition $p \wedge q$ is called the conjunction of $p$ and $q$.
  - It is read as "$p$ and $q$".
- Truth table:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

$2^2 = 4$ rows

It is true only when both $p$ and $q$ are true.

# Disjunction - OR

- Given $p$ and $q$, the compound proposition $p \vee q$ is called the disjunction of $p$ and $q$.
  - It is read as "$p$ or $q$".
- Truth table:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

It is false only when both $p$ and $q$ are false.

# Well-Formed Formulas

- A well-formed formula (or simply formula) is an expression constructed according to the following rules:

1) Each propositional variable is a formula.
   - e.g. $p$, $q$, and $r$
2) Formulas combined by logical connectives (with parenthesis if involving two formulas) is a formula.
   - e.g. $\sim p$, $(p \vee q)$, $(p \wedge q)$
   - Outer parenthesis is often omitted, e.g., $p \vee q$.
   - More logical connectives will be introduced in the next section.

# Classwork

- Are they (well-formed) formulas?

1) $(p \wedge \sim\sim q)$

2) $(p \wedge q \vee r)$

3) $(p \vee q) \wedge \sim(p \wedge q)$

# Compound Truth Tables

- The compound truth table for $(p \vee q) \wedge \sim(p \wedge q)$:

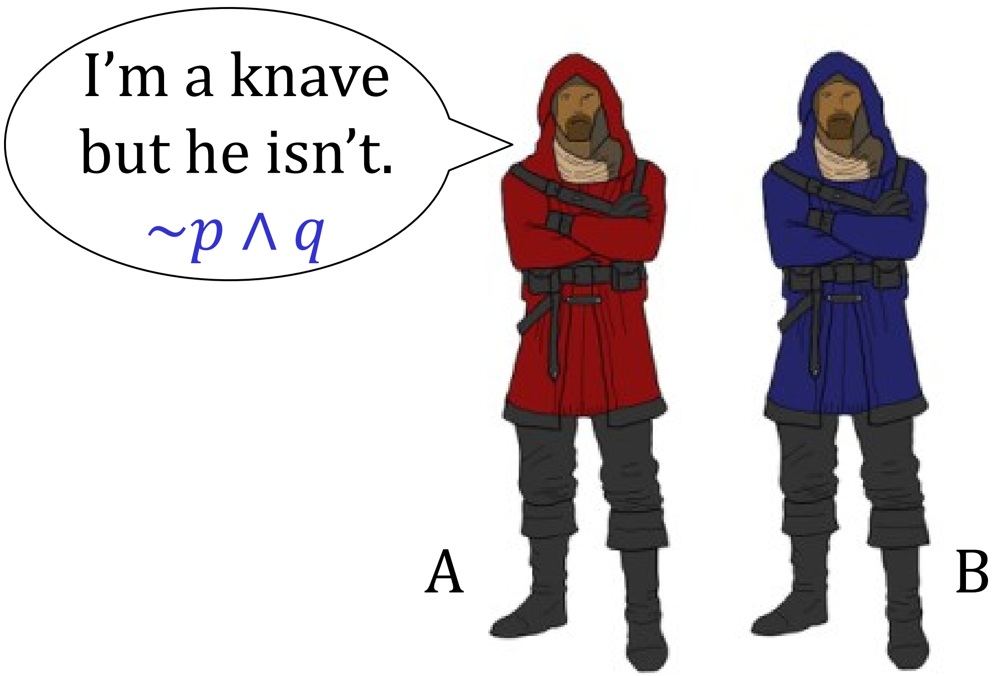
| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $(p \vee q) \wedge \sim(p \wedge q)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | F | T | F | T | T |
| F | T | T | F | T | T |
| F | F | F | F | T | F |

Intermediate result (columns $p \vee q$, $p \wedge q$, $\sim(p \wedge q)$)

Final result (column $(p \vee q) \wedge \sim(p \wedge q)$)

The formula means "($p$ or $q$) and not (both $p$ and $q$)."
It is called Exclusive-OR (XOR), and denoted by $p \oplus q$.

# Knights and Knaves (Revisited)




- $p$ = "A is a knight"
- $q$ = "B is a knight"

| $p$ | $q$ | $\sim p \vee q$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | T |
| F | F | F |

This must be the case.

- $p$ and $\sim p \wedge q$ must have identical truth values.
  - If A is a knight (i.e. $p$ is true) , he tells the truth, so $\sim p \wedge q$ is true.
  - If A is a knave (i.e. $p$ is false), he lies, so $\sim p \wedge q$ is false.

# Unit 2.2

Logical Equivalence

# Logical Equivalence

Two formulas $X$ and $Y$ are called logically equivalent (or simply equivalent), denoted by

$$X \equiv Y,$$

if they have identical truth values in all cases.

- Example: $p \land q \equiv q \land p$
  - It can be verified by truth table:

| $p$ | $q$ | $p \land q$ | $q \land p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | F | F |
| F | F | F | F |

$p \land q$ and $q \land p$ always have the same truth values, so they are logically equivalent

# Classwork

- Is it true that $\sim(p \wedge q) \equiv \sim p \wedge \sim q$?

# De Morgan’s Law

- The negation of a conjunction is the disjunction of the negations:

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

- The negation of a disjunction is the conjunction of the negations.

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q.}$$

Augustus De Morgan (1806-1871), a British mathematician and logician.

# Applying De Morgan's Law

- Write the negations for

1. The bus was late or Tom's watch was slow.

2. $-1 < x \leq 4$

# Tautologies

- A statement that is always true is called a tautology.

Example: $p \vee {\sim}p$

| $p$ | ${\sim}p$ | $p \vee {\sim}p$ |
|---|---|---|
| T | F | T |
| F | T | T |

# Tautology – Stating the Obvious?!


*Remark:* Tautologies reveal hidden truths. For example, all theorems in mathematics are tautologies.

# Contradictions

- A statement that is always false is called a contradiction.

Example: $p \wedge \sim p$

- This course is interesting and this course is not interesting.
- You can use truth table to verify that it's always false.

# A fable from an ancient Chinese book 《韓非子 - Hanfeizi》

What if you used your lances to pierce your shields?


My shields are so firm, that there is nothing that can pierce them.

My lances are so sharp, that there is nothing that they cannot pierce.

# Equivalence involving **t** and **c**

- If **t** is a tautology and **c** is a contradiction, then

$$p \wedge \mathbf{t} \equiv p \text{ and } p \wedge \mathbf{c} \equiv \mathbf{c}.$$

| $p$ | $\mathbf{t}$ | $p \wedge \mathbf{t}$ | $p$ |
|---|---|---|---|
| T | T | T | T |
| F | T | F | F |

same truth values, so $p \wedge \mathbf{t} \equiv p$

| $p$ | $\mathbf{c}$ | $p \wedge \mathbf{c}$ |
|---|---|---|
| T | F | F |
| F | F | F |

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

# Theorem 2.1 (Logical Equivalences)

Given any statement variables $p$, $q$, and $r$, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

| | | |
|---|---|---|
| 1. *Commutative laws:* | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. *Associative laws:* | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. *Distributive laws:* | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. *Identity laws:* | $p \wedge \mathbf{t} \equiv p$ | $p \vee \mathbf{c} \equiv p$ |
| 5. *Negation laws:* | $p \vee {\sim}p \equiv \mathbf{t}$ | $p \wedge {\sim}p \equiv \mathbf{c}$ |
| 6. *Double negative law:* | ${\sim}({\sim}p) \equiv p$ | |
| 7. *Idempotent laws:* | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. *Universal bound laws:* | $p \vee \mathbf{t} \equiv \mathbf{t}$ | $p \wedge \mathbf{c} \equiv \mathbf{c}$ |
| 9. *De Morgan's laws:* | ${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$ | ${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$ |
| 10. *Absorption laws:* | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. *Negations of* **t** *and* **c**: | ${\sim}\mathbf{t} \equiv \mathbf{c}$ | ${\sim}\mathbf{c} \equiv \mathbf{t}$ |

# Applying Theorem 2.1

- Prove that $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$.

$$
\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law} \\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law} \\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge \\
&\equiv p \vee \mathbf{c} && \text{by the negation law} \\
&\equiv p && \text{by the identity law.}
\end{aligned}
$$

*Remark:* Truth table can be used to prove both equivalence and non-equivalence.

# Unit 2.3

Conditionals

# The Conditional "$\rightarrow$"

- "If $p$ then $q$," or equivalently, "$p$ implies $q$", is represented by "$p \rightarrow q$".
  - $p$ is called the antecedent.
  - $q$ is called the consequent.
  - "$p \rightarrow q$" is sometimes written as "$p \supset q$".

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

If $p$ is true, then $p \rightarrow q$ takes the truth value of $q$.

If $p$ is false, then $p \rightarrow q$ is regarded to be true.

# Counter-Intuitive?

True or False?

"If the sun is made of cheese,
then life exists on the moon."

- For $p \rightarrow q$ to be true, there is not necessarily any relevance between $p$ and $q$,
  - It depends only on the truth values of $p$ and $q$.
- $p \rightarrow q$ is called material implication.
  - Its meaning is *not* exactly the same as "if … then…" in daily life.

# Promise: to keep or to break?

- $p \rightarrow q$ *does* conform to some of our ordinary intuitions about implication.
- "If I am healthy, then I will come to class."

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

I keep the promise.

I break the promise.

I do not violate the promise.

# “→” defined by NOT and OR

- → can be defined by ~ and ∨.

$$p \to q \equiv {\sim}p \vee q$$

| $p$ | $q$ | ${\sim}p$ | ${\sim}p \vee q$ | $p \to q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

# Contrapositive: $\sim q \rightarrow \sim p$

"If it is raining, then the grass is wet."

$\equiv ?$

"If the grass is not wet, then it is not raining"

# Converse and Inverse

- Consider $p \to q$.
- Its converse is $q \to p$.
- Its inverse is $\sim p \to \sim q$.
- Converse is equivalent to Inverse.

$$\boxed{q \to p \equiv \sim p \to \sim q}$$

# Only If

- “*p* only if *q*” means
  - “If not *q* then not *p*”, or equivalently,
  - “If *p* then *q*”.

- If it is raining, then the grass is wet.
- It is raining only if the grass is wet.

# Necessary & Sufficient Conditions

- When $p \rightarrow q$,
  - $p$ is called a sufficient condition for $q$.
  - $q$ is called a necessary condition for $p$.
- "If you are below the lion rock, you are in Hong Kong."
  - Being below the Lion Rock is a sufficient condition of being in Hong Kong.
  - Being in Hong Kong is a necessary condition of being below the Lion Rock.

# The Biconditional "↔"

- Biconditional:

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- Also called *If and only If* *(iff)*

| $p$ | $q$ | $p \rightarrow q$ | $q \rightarrow p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

- True only when $p$ and $q$ have identical truth values.

# Operator Precedence

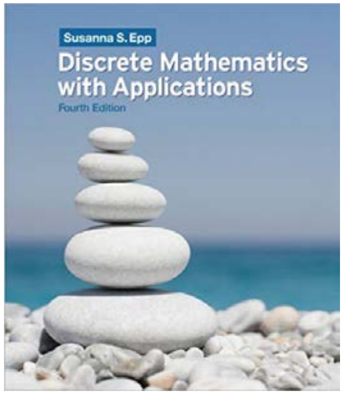
- It is more convenient if some parenthesis ( ) can be omitted.
- To avoid ambiguity, the operators have different priorities:

  from high to low: ( ), $\sim$, $\wedge\vee$, $\rightarrow$, $\leftrightarrow$.
- Example:

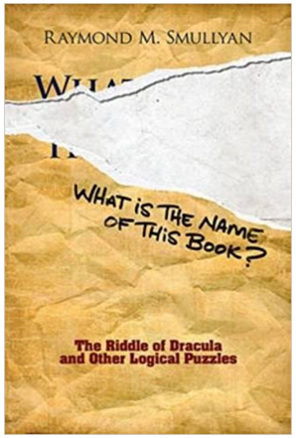
  $p \vee \sim q \rightarrow \sim p$ means $(p \vee (\sim q)) \rightarrow (\sim p)$.

# Knights and Knaves


# Recommended Readings




- Sections 2.1-2.2, Susanna S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.




- Raymond Smullyan, *What Is the Name of This Book?: The Riddle of Dracula and Other Logical Puzzles*, Dover, 2011.
  - Many logical puzzles.
  - Just for fun; ***not*** required in this course.