On Sufficient and Necessary Conditions in Bounded CTL

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Abstract

Computation Tree Logic (CTL) is one of the central formalisms in formal verification. As a specification language, it is used to express a property that the system at hand is expected to satisfy. From both the verification and the system design points of view, some information content of such property might become irrelevant for the system due to various reasons e.g., it might become obsolete by time, or perhaps infeasible due to practical difficulties. Then, the problem arises on how to subtract such piece of information without altering the relevant system behaviour or violating the existing specifications. Moreover, in such a scenario, two crucial notions are informative: the strongest necessary condition (SNC) and the weakest sufficient condition (WSC) of a given property.

To address such a scenario in a principled way, we introduce a forgetting-based approach in CTL and show that it can be used to compute SNC and WSC of a property under a given model. We study its theoretical properties and also show that our notion of forgetting satisfies existing essential postulates. Furthermore, we analyse the computational complexity of basic tasks, including various results for the relevant fragment CTL_{AF}.

1 Introduction

Consider a car-manufacturing company which produces two types of automobiles: a sedan car (basically a four-doored classical passenger car) and a sports car. No matter a sedan or a sports car, both production lines are subject to a single standard criterion which is indispensable: safety restrictions. This shared feature is also complemented several major differences aligned with these types in general. That is, a sedan car is produced with a small engine, while a sports car is produced with a large one. Moreover, due to its large amount of production, the sedan car is subject to some very restrictive low-carbon emission regulations, while a sports car is not. On the verge of shifting to an upcoming new engine technology, the company aims to adapt the sedan production to electrical engines. Such major shift in production also comes with one in regulations; electric sedans are not subject to low-carbon emission restrictions any more. In fact, due to the difference in its underlying technology, a sedan car drastically emits much less carbon, hence such standard is obsolete, and can be dropped. Yet dropping some restrictions in a large and complex production system in automative industry, without affecting the working system components or violating dependent specifications is a non-trivial task.

Similar scenarios may arise in many different domains such as business-process modelling, software development, concurrent systems and more (Baier and Katoen 2008). In general, some information content of such property might become irrelevant for the system due to various reasons e.g., it might become obsolete by time like in the above example, or perhaps becomes infeasible due to practical difficulties. Then, the problem arises on how to subtract such piece of information without altering the behaviour of the relevant system components or violating the existing specifications. Moreover, in such a scenario, two logical notions introduced by E. Dijkstra in (Dijkstra 1978) are very informative: the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC) of a given specification. These correspond to *most general consequence* and the *most specific abduction* of such specification, respectively.

To address such a scenario in a principled way, we employ a method based on formal verification.[1] In particular, we introduce a *forgetting*-based approach in Computation Tree Logic (CTL) (Clarke and Emerson 1981) a central formalism in formal verification, and show that it can be used to compute SNC and WSC, in the same spirit of (Lin 2001).

The scenario we mentioned concerning car-engine manufacturing can be easily represented as a small example by the following Kripke structure $\mathcal{M} = (S, R, L, s_0)$ in Figure 1 on $V = \{sl, sr, se, le, lc\}$ whose elements correspond to *select*, *safety restrictions*, *small engine*, *large engine* and *low-carbon emission requirements*, respectively. Moreover, s_0 is the initial state where we select either choose producing an engine for *sedan* which corresponds to the state s_1 or a *sports car* which corresponds to the state s_2. Since only a single-type of production is possible at a time, after each production state (s_1 or s_2), we turn back to the initial state (s_0) to start over.

The notions of SNC and WSC were considered in the scope of formal verification among others, in generating counterexamples (Dailler et al. 2018) and refinement of system (Woodcock and Morgan 1990). On the *forgetting* side, it was first formally defined in propositional and first order

[1] This is especially useful for abstracting away the domain-dependent problems, and focusing on conceptual ones.

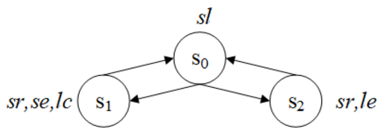

Figure 1: Car Engine Manufacturing Scenario

logics by Lin and Reiter (Lin and Reiter 1994). Over the last decades, researchers have developed forgetting notions and theories not only in classical logic but also in non-classical logic systems (Eiter and Kern-Isberner 2019), such as forgetting in logic programs under answer-set semantics (Zhang and Foo 2006; Eiter and Wang 2008; Wong 2009; Wang et al. 2012; Wang, Wang, and Zhang 2013), description logics (Wang et al. 2010; Lutz and Wolter 2011; Zhao and Schmidt 2017) and knowledge forgetting in modal logic (Zhang and Zhou 2009; Su et al. 2009; Liu and Wen 2011; Fang, Liu, and Van Ditmarsch 2019). It has also been considered in planning (Lin 2003) and conflict solving (Lang and Marquis 2010; Zhang, Foo, and Wang 2005), creating restricted views of ontologies (Zhao and Schmidt 2017), strongest and weakest definitions (Lang and Marquis 2008), SNC (WSC) (Lin 2001), among others.

Although forgetting has been extensively investigated from various aspects of different logical systems, the existing forgetting techniques are not directly applicable in CTL. For instance, in propositional forgetting theory, forgetting atom q from φ is equivalent to a formula $\varphi[q/\top] \vee \varphi[q/\perp]$, where $\varphi[q/X]$ is a formula obtained from φ by replacing each q with X ($X \in \{\top, \perp\}$). This method cannot be extended to a CTL formula. Consider a CTL formula $\psi = \mathrm{AG}p \wedge \neg\mathrm{AG}q \wedge \neg\mathrm{AG}\neg q$. If we want to forget atom q from ψ by using the above method, we would have $\psi[q/\top] \vee \psi[q/\perp] \equiv \perp$. This is obviously not correct since after forgetting q this specification should not become inconsistent. Similar to (Zhang and Zhou 2009), we research forgetting in CTL from the semantic forgetting point of view. And it is shown that our definition of forgetting satisfies those four postulates of forgetting presented in (Zhang and Zhou 2009).

The rest of the paper is organised as follows. Section 2 introduces the notation and technical preliminaries. As key contributions, Section 3, introduces the notion of forgetting in bounded CTL, via developing the notion of V-bisimulation. Such bisimulation is constructed through a set-based bisimulation and more general than the classical bisimulation. Moreover, it provides a CTL characterization for model structures (with the initial state), and studies the semantic properties of forgetting. In addition, a complexity analysis, including a relevant fragment $\mathrm{CTL}_{\mathrm{AF}}$, is carried out. Section 4 explores the relation between forgetting and SNC (WSC). Section 5 gives a model-based algorithm for computing forgetting in CTL and outline its complexity. Conclusion closes the paper.

Due to space restrictions and to avoid hindering the flow of content, some of the proofs are moved to the supplementary material [2].

2 Preliminaries

We start with some technical and notational preliminaries. Throughout this paper, we fix a finite set $\mathcal{A}$ of propositional variables (or atoms), and use V, V' for subsets of $\mathcal{A}$. Throughout this paper, we fix a finite set $\mathcal{A}$ of propositional variables (or atoms), use V, V' for subsets of $\mathcal{A}$ and $\overline{V} = \mathcal{A} - V$.

2.1 Model structures in CTL

In general, a transition system can be described by a *model structure* (or *Kripke structure*) (see (Baier and Katoen 2008) for details). A model structure is a triple $\mathcal{M} = (S, R, L)$, where

- S is a finite nonempty set of states[3],
- $R \subseteq S \times S$ and, for each $s \in S$, there is $s' \in S$ such that $(s, s') \in R$,
- L is a labeling function $S \to 2^{\mathcal{A}}$.

Given a model structure $\mathcal{M} = (S, R, L)$, a *path* π_{s_i} starting from s_i of $\mathcal{M}$ is an infinite sequence of states $\pi_{s_i} = (s_i, s_{i+1}s_{i+2}, \ldots)$, where for each j ($0 \leq i \leq j$), $(s_j, s_{j+1}) \in R$. By $s' \in \pi_{s_i}$ we mean that s' is a state in the path π_{s_i}. A state $s \in S$ is *initial* if for any state $s' \in S$, there is a path π_s s.t $s' \in \pi_s$. If s_0 is an initial state of $\mathcal{M}$, then we denote this model structure $\mathcal{M}$ as (S, R, L, s_0).

For a given model structure $\mathcal{M} = (S, R, L, s_0)$ and $s \in S$, the *computation tree* $\mathrm{Tr}_n^{\mathcal{M}}(s)$ of $\mathcal{M}$ (or simply $\mathrm{Tr}_n(s)$), that has depth n and is rooted at s, is recursively defined as (Browne, Clarke, and Grumberg 1988), for $n \geq 0$,

- $\mathrm{Tr}_0(s)$ consists of a single node s with label s.
- $\mathrm{Tr}_{n+1}(s)$ has as its root a node m with label s, and if $(s, s') \in R$ then the node m has a subtree $\mathrm{Tr}_n(s')$.

A K-*structure* (or K-*interpretation*) is a model structure $\mathcal{M} = (S, R, L, s_0)$ associating with a state $s \in S$, which is written as $(\mathcal{M}, s)$ for convenience in the following. In the case $s = s_0$ is an initial state of $\mathcal{M}$, the K-structure is *initial*.

2.2 Syntax and semantics of CTL

In the following we briefly review the basic syntax and semantics of the CTL (Clarke, Emerson, and Sistla 1986). The *signature* of the language $\mathcal{L}$ of CTL includes:

- a finite set of Boolean variables, called *atoms* of $\mathcal{L}$: $\mathcal{A}$;
- constant symbols: $\perp$ and $\top$;
- the classical connectives: $\vee$ and $\neg$;
- the path quantifiers: A and E;

[2]https://github.com/fengrenyan/proof-of-CTL.git

[3]Since CTL has finite model property (Emerson and Halpern 1985), we assume that the signature of states is fixed and finite, *i.e.* $S \subseteq \mathcal{S}$ with $\mathcal{S} = \{b_1, \ldots, b_m\}$, such that any CTL formula with bounded length is satisfiable if and only if it is satisfiable in a such model structure. Thus, there are only finite number of model structures.

- the temporal operators: X, F, G U and W, that means 'neXt state', 'some Future state', 'all future states (Globally)', 'Until' and 'Unless', respectively;
- parentheses: (and).

The *(existential normal form or ENF in short) formulas* of $\mathcal{L}$ are inductively defined via a Backus Naur form:

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi \vee \phi \mid \text{EX}\phi \mid \text{EG}\phi \mid \text{E}[\phi \text{ U } \phi] \quad (1)$$

where $p \in \mathcal{A}$. The formulas $\phi \wedge \psi$ and $\phi \rightarrow \psi$ are defined in a standard manner of propositional logic. The other form formulas of $\mathcal{L}$ are abbreviated using the forms of (1). In the following we assume every formula of $\mathcal{L}$ has bounded size, where the size $|\varphi|$ of formula φ is its length over the alphbet of $\mathcal{L}$ (Emerson and Halpern 1985).

We are now in the position to recall the semantics of $\mathcal{L}$. Let $\mathcal{M} = (S, R, L, s_0)$ be a model structure, $s \in S$ and ϕ a formula of $\mathcal{L}$. The *satisfiability* relationship between $(\mathcal{M}, s)$ and ϕ, written $(\mathcal{M}, s) \models \phi$, is inductively defined on the structure of ϕ as follows:

- $(\mathcal{M}, s) \not\models \bot$ and $(\mathcal{M}, s) \models \top$;
- $(\mathcal{M}, s) \models p$ iff $p \in L(s)$;
- $(\mathcal{M}, s) \models \phi_1 \vee \phi_2$ iff $(\mathcal{M}, s) \models \phi_1$ or $(\mathcal{M}, s) \models \phi_2$;
- $(\mathcal{M}, s) \models \neg\phi$ iff $(\mathcal{M}, s) \not\models \phi$;
- $(\mathcal{M}, s) \models \text{EX}\phi$ iff $(\mathcal{M}, s_1) \models \phi$ for some $s_1 \in S$ and $(s, s_1) \in R$;
- $(\mathcal{M}, s) \models \text{EG}\phi$ iff $\mathcal{M}$ has a path $(s_1 = s, s_2, \ldots)$ such that $(\mathcal{M}, s_i) \models \phi$ for each $i \geq 1$;
- $(\mathcal{M}, s) \models \text{E}[\phi_1 \text{U} \phi_2]$ iff $\mathcal{M}$ has a path $(s_1 = s, s_2, \ldots)$ such that, for some $i \geq 1$, $(\mathcal{M}, s_i) \models \phi_2$ and $(\mathcal{M}, s_j) \models \phi_1$ for each $1 \leq j < i$.

Similar to the work in (Browne, Clarke, and Grumberg 1988; Bolotov 1999), only initial K-structures are considered to be candidate models in the following, unless otherwise noted. Formally, an initial K-structure $\mathcal{K}$ is a *model* of a formula ϕ whenever $\mathcal{K} \models \phi$. We denote *Mod*(ϕ) the set of models of ϕ. The formula ϕ is *satisfiable* if *Mod*$(\phi) \neq \emptyset$. Given two formulas ϕ_1 and ϕ_2, $\phi_1 \models \phi_2$ we mean *Mod*$(\phi_1) \subseteq$ *Mod*(ϕ_2), and by $\phi_1 \equiv \phi_2$, we mean $\phi_1 \models \phi_2$ and $\phi_2 \models \phi_1$. In this case ϕ_1 is *equivalent* to ϕ_2. The set of atoms occurring in ϕ_1 is denoted by *Var*(ϕ_1). The formula ϕ_1 is *irrelevant to* the atoms in a set V (or simply V*-irrelevant*), written IR(ϕ_1, V), if there is a formula ψ with *Var*$(\psi) \cap V = \emptyset$ such that $\phi_1 \equiv \psi$.

3 Forgetting in CTL

In this section, we present the notion of forgetting in CTL and report its properties. For convenience, in the following we denote $\mathcal{M} = (S, R, L, s_0)$, $\mathcal{M}' = (S', R', L', s'_0)$, $\mathcal{M}_i = (S_i, R_i, L_i, s^i_0)$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ with $s_i \in S_i$ and $i \in \mathbb{N}$.

3.1 Set-based bisimulation

In this section we will introduce the notion of Set-based bisimulation. Intuitively, bisimulation relates states that mutually mimic all individual transitions (Baier and Katoen 2008). While the Set-based bisimulation relates states that mutually mimic all individual transitions on some set of atoms, this is close with forgetting some set of atoms.

Let $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ with $i \in \{1, 2\}$,

- $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0$ if $L_1(s_1) - V = L_2(s_2) - V$;
- for $n \geq 0$, $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}$ if:
  - $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0$,
  - for every $(s_1, s'_1) \in R_1$, there is a $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_n$, and
  - for every $(s_2, s'_2) \in R_2$, there is a $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_n$,

  where $\mathcal{K}'_i = (\mathcal{M}_i, s'_i)$ with $i \in \{1, 2\}$.

Now, we define the notion of V-bisimulation between K-structures:

Definition 1 (V-bisimulation)**.** *Let $V \subseteq \mathcal{A}$. Given two* K*-structures $\mathcal{K}_1$ and $\mathcal{K}_2$ are V*-bisimilar, *denoted $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ if and only if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$. Moreover, two paths $\pi_i = (s_{i,1}, s_{i,2}, \ldots)$ of $\mathcal{M}_i$ with $i \in \{1, 2\}$ are V-*bisimilar *if $\mathcal{K}_{1,j} \leftrightarrow_V \mathcal{K}_{2,j}$ for every $j \geq 1$ where $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$.*

On the one hand, the above set-based bisimulation is an extension of the bisimulation-equivalence of Definition 7.1 in (Baier and Katoen 2008) in the sense that if $V = \mathcal{A}$ then our bisimulation is almost same to the latter. On the other hand, the above set-based bisimulation notion is similar to the state equivalence in (Browne, Clarke, and Grumberg 1988). But it is different in the sense that ours is defined on K-structures, while it is defined on states in (Browne, Clarke, and Grumberg 1988). What's more, the set-based bisimulation notion is also different from the state-based bisimulation notion of Definition 7.7 in (Baier and Katoen 2008), which is defined for states of a given K-structure.

Example 1. *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be two initial* K*-structures described in Figure 2. It is easy to check $\mathcal{K}_1 \leftrightarrow_{\{y\}} \mathcal{K}_2$.*

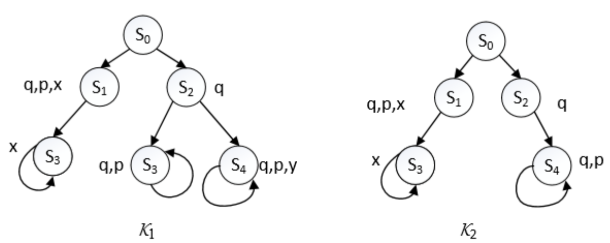

Figure 2: Two $\{y\}$-bisimular initial K-structures

It is apparent that $\leftrightarrow_V$ is a binary relation. In the sequel, we abbreviate $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ by $s_1 \leftrightarrow_V s_2$ whenever the underlying model structures of states s_1 and s_2 are clear from the context.

Lemma 1. *The relation $\leftrightarrow_V$ is an equivalence relation.*

Besides, we have the following properties:

Proposition 1. *Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s'_is be two states and π'_is be two paths, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be* K*-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:*

(i) $s_1' \leftrightarrow_{V_i} s_2'$ $(i = 1, 2)$ *implies* $s_1' \leftrightarrow_{V_1 \cup V_2} s_2'$*;*

(ii) $\pi_1' \leftrightarrow_{V_i} \pi_2'$ $(i = 1, 2)$ *implies* $\pi_1' \leftrightarrow_{V_1 \cup V_2} \pi_2'$*;*

(iii) for each path π_{s_1} *of* $\mathcal{M}_1$ *there is a path* π_{s_2} *of* $\mathcal{M}_2$ *such that* $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$*, and vice versa;*

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$*;*

(v) If $V_1 \subseteq V_2$ *then* $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$*.*

Proof. We give proofs of (iii) and (iv) here. Proofs of other propositions can be found in the appendix. For convenience, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$.

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$, and

(c) for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$,

where $\mathcal{K}_i' = (\mathcal{M}_i, s_i')$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

$(\Rightarrow)$ (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s_1') \in R_1$, there is a $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s_1') - V = L_2(s_2') - V$. Therefore, $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$. (c) This is similar with (b).

$(\Leftarrow)$ (a) $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) Condition (ii) implies that for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$; (c) Condition (iii) implies that for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B} \text{ and } (w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing (s_1, s_3) from the (a), (b) and (c) of the previous step (iii) of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

□

The (iv) in this proposition shows that if a K-structure is V_1 and V_2-bisimulation with the other two K-structures respectively, then those two K-structures are $V_1 \cup V_2$-bisimulation. This is important for forgetting since this laid the foundation of forgetting atoms in V one by one. Besides, (v) means that if two K-structures are V_1-bisimulation then they are V_2-bisimulation for each V_2 with $V_1 \subseteq V_2 \subseteq \mathcal{A}$.

Intuitively, if two K-structures are V-bisimilar, then they satisfy the same formula φ that dose not contain any atoms in V, *i.e.* IR(φ, V).

Theorem 1. *Let* $V \subseteq \mathcal{A}$*,* $\mathcal{K}_i$ $(i = 1, 2)$ *be two* K*-structures such that* $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ *and* ϕ *a formula with IR*(ϕ, V)*. Then* $\mathcal{K}_1 \models \phi$ *if and only if* $\mathcal{K}_2 \models \phi$*.*

Proof. (sketch) This can be proved by induction on the structures of ϕ. For instance, let $\phi = \psi_1 \vee \psi_2$, the induction hypothesis is $\mathcal{K}_1 \models \psi_i$ iff $\mathcal{K}_2 \models \psi_i$ with $i \in \{1, 2\}$. Then we can see that $\mathcal{K}_1 \models \phi$ iff $\mathcal{K}_1 \models \psi_1$ or $\mathcal{K}_1 \models \psi_2$ iff $\mathcal{K}_2 \models \psi_1$ or $\mathcal{K}_2 \models \psi_2$ by induction hypothesis. □

Example 2. *Let* $\varphi_1 = \neg p \wedge \text{AX}q \wedge \text{EX}(x \rightarrow \text{EX}x)$ *and* $\varphi_2 = q \wedge \text{AX}q$ *be two CTL formulae. They are* $\{y\}$*-irrelevant. One can check that* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *in Figure 2 satisfy* φ_1*, but they do not satisfy* φ_2*.*

Let $V \subseteq \mathcal{A}$, $\mathcal{M}_i$ $(i = 1, 2)$ be model structures. A computation tree $\text{Tr}_n(s_1)$ of $\mathcal{M}_1$ is V*-bisimilar* to a computation tree $\text{Tr}_n(s_2)$ of $\mathcal{M}_2$, written $(\mathcal{M}_1, \text{Tr}_n(s_1)) \leftrightarrow_V (\mathcal{M}_2, \text{Tr}_n(s_2))$ (or simply $\text{Tr}_n(s_1) \leftrightarrow_V \text{Tr}_n(s_2)$), if

- $L_1(s_1) - V = L_2(s_2) - V$,
- for every subtree $\text{Tr}_{n-1}(s_1')$ of $\text{Tr}_n(s_1)$, $\text{Tr}_n(s_2)$ has a subtree $\text{Tr}_{n-1}(s_2')$ such that $\text{Tr}_{n-1}(s_1') \leftrightarrow_V \text{Tr}_{n-1}(s_2')$, and vice
- for every subtree $\text{Tr}_{n-1}(s_2')$ of $\text{Tr}_n(s_2)$, $\text{Tr}_n(s_1)$ has a subtree $\text{Tr}_{n-1}(s_1')$ such that $\text{Tr}_{n-1}(s_1') \leftrightarrow_V \text{Tr}_{n-1}(s_2')$.

The last condition in the above definition holds trivially for $n = 0$.

Proposition 2. *Let* $V \subseteq \mathcal{A}$ *and* $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ *be two* K*-structures. Then*

$$(s_1, s_2) \in \mathcal{B}_n \textit{ iff } Tr_j(s_1) \leftrightarrow_V Tr_j(s_2) \textit{ for every } 0 \leq j \leq n.$$

Proof. (sketch) From the left to the right is apparent since $(s_1, s_2) \in \mathcal{B}_n$ implies that $(s_1, s_2) \in \mathcal{B}_m$ for every $0 \leq m \leq n$.

For the other direction, in order to show $(s_1, s_2) \in \mathcal{B}_n$ we need only to prove for any s_1' with $(s_1, s_1') \in R_1$ there is s_2' with $(s_2, s_2') \in R_2$ s.t. $(s_2, s_2') \in \mathcal{B}_{n-1}$ and vice versa. $\text{Tr}_0(s_1) \leftrightarrow_V \text{Tr}_0(s_2)$ implies $(s_1, s_2) \in B_0$, $\text{Tr}_1(s_1) \leftrightarrow_V \text{Tr}_1(s_2)$ implies for any s_1' with $(s_1, s_1') \in R_1$ there is s_2' with $(s_2, s_2') \in R_2$ s.t. $(s_2, s_2') \in B_0$ and vice versa, hence $(s_1, s_2) \in B_1$. Therefore, we can prove $(s_1, s_2) \in \mathcal{B}_n$ recursively. □

This means that $\text{Tr}_j(s_1) \leftrightarrow_V \text{Tr}_j(s_2)$ for all $j \geq 0$ if $s_1 \leftrightarrow_V s_2$, otherwise there is some k such that $\text{Tr}_k(s_1)$ and $\text{Tr}_k(s_2)$ are not V-bisimilar.

Proposition 3. *Let* $V \subseteq \mathcal{A}$*,* $\mathcal{M}$ *be a model structure and* $s, s' \in S$ *such that* $s \not\leftrightarrow_V s'$*. There exists a least* k *such that* $Tr_k(s)$ *and* $Tr_k(s')$ *are not* V*-bisimilar.*

Proof. If $s \not\leftrightarrow_V s'$, then there exists a least constant c such that $(s_i, s_j) \notin \mathcal{B}_c$, and then there is a least constant m ($m \leq c$) such that $\text{Tr}_m(s_i)$ and $\text{Tr}_m(s_j)$ are not V-bisimilar by Proposition 2. Let $k = m$, the lemma is proved. □

In this case the model structure $\mathcal{M}$ is called V-*distinguishable* (by states s and s' at the least depth k), which is denoted by $\text{dis}_V(\mathcal{M}, s, s', k)$. The V-*characterization number* of $\mathcal{M}$, written $ch(\mathcal{M}, V)$, is defined as

$$ch(\mathcal{M}, V) = \begin{cases} \max\{k \mid s, s' \in S \text{ and } \text{dis}_V(\mathcal{M}, s, s', k)\}, \\ \qquad\qquad \mathcal{M} \text{ is } V\text{-distinguishable;} \\ \min\{k \mid \mathcal{B}_k = \mathcal{B}_{k+1}\}, \qquad \text{otherwise.} \end{cases}$$

3.2 Characterization of initial K-structure

In the following we present characterizing formulas of initial K-structures over a signature. As a basis, we first give the definition of characterizing formulas of a computation tree.

Definition 2. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ be a model structure and $s \in S$. The* characterizing formula *of the computation tree $Tr_n(s)$ on V, written $\mathcal{F}_V(Tr_n(s))$, is defined recursively as:*

$$\mathcal{F}_V(Tr_0(s)) = \bigwedge_{p \in V \cap L(s)} p \wedge \bigwedge_{q \in V - L(s)} \neg q,$$

$$\mathcal{F}_V(Tr_{k+1}(s)) = \bigwedge_{(s,s') \in R} \text{EX}\mathcal{F}_V(Tr_k(s'))$$

$$\wedge\, \text{AX}\left(\bigvee_{(s,s') \in R} \mathcal{F}_V(Tr_k(s'))\right) \wedge \mathcal{F}_V(Tr_0(s))$$

for $k \geq 0$.

The characterizing formula of a computation tree formally exhibits the content of each node on V (i.e., atoms that are *true* at this node if they are in V, *false* otherwise) and the temporal relation between states recursively. The following result shows that the V-bisimulation between two computation trees implies the semantic equivalence of the corresponding characterizing formulas.

Lemma 2. *Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ and $\mathcal{M}'$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$, then $\mathcal{F}_V(Tr_n(s)) \equiv \mathcal{F}_V(Tr_n(s'))$.*

Proof. (sketch) This result can be proved by inducting on n.

For the base. It is apparent that for any $s \in S$ and $s' \in S'$, if $\text{Tr}_0(s) \leftrightarrow_{\overline{V}} \text{Tr}_0(s')$ then $\mathcal{F}_V(\text{Tr}_0(s)) \equiv \mathcal{F}_V(\text{Tr}_0(s'))$ due to $L(s) - \overline{V} = L'(s') - \overline{V}$ by known.

For the induction step. If $\text{Tr}_n(s) \leftrightarrow_{\overline{V}} \text{Tr}_n(s')$ we can prove for each state s_1 with $(s, s_1) \in R$ there is s_1' with $(s', s_1') \in R'$ such that $\mathcal{F}_V(\text{Tr}_n(s_1)) \equiv \mathcal{F}_V(\text{Tr}_n(s_1'))$ and versa vice. Then it is easy check $\mathcal{F}_V(\text{Tr}_n(s)) \equiv \mathcal{F}_V(\text{Tr}_n(s'))$. □

Let $s' = s$. It is clear that, for any formula φ of V, if φ is a characterizing formula of $\text{Tr}_n(s)$ then $\varphi \equiv \mathcal{F}_V(\text{Tr}_n(s))$.

Let $V \subseteq \mathcal{A}$, $\mathcal{K} = (\mathcal{M}, s_0)$ be an initial K-structure, $c = ch(\mathcal{M}, V)$ and $T(s') = \mathcal{F}_V(\text{Tr}_c(s'))$ for each state s' in $\mathcal{M}$. The *characterizing formula* of $\mathcal{K}$ on V, written $\mathcal{F}_V(\mathcal{M}, s_0)$ (or $\mathcal{F}_V(\mathcal{K})$ in short), is the following formula:

$$\mathcal{F}_V(\text{Tr}_c(s_0)) \wedge$$

$$\bigwedge_{s \in S} \text{AG}\left(T(s) \rightarrow \bigwedge_{(s,s') \in R} \text{EX}T(s') \wedge \text{AX}(\bigvee_{(s,s') \in R} T(s'))\right)$$

It is apparent that $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. Besides, given a set of atomic propositions V, it is easy to check that any initial K-structure has its own characterizing formula on V. As we will see that the characterizing formula is critical to show some important properties of forgetting and compute the S-NC and WSC of some property under the initial K-structure.

The following example illustrates how one can compute a characterizing formula:

Example 3. *Let $V = \{sr\}$ and $\overline{V} = \{sl, se, lc, le\}$ ($\mathcal{A} = V \cup \{sr\}$), and $\mathcal{M}$ is as illustrated in Figure 1. We have $Tr_0(s_0) \not\leftrightarrow_{\overline{V}} Tr_0(s_1)$ and $Tr_0(s_0) \not\leftrightarrow_{\overline{V}} Tr_0(s_2)$, then $dis_{\overline{V}}(\mathcal{M}, s_0, s_1, 0)$ and $dis_{\overline{V}}(\mathcal{M}, s_0, s_2, 0)$. Besides, it is easy checking that $s_1 \leftrightarrow_{\overline{V}} s_2$ since they have the same direct successor s_0. Hence, $ch(\mathcal{M}, \overline{V}) = 0$. Therefore,*

$\mathcal{F}_V(Tr_0(s_0)) = \neg sr$

$\mathcal{F}_V(Tr_0(s_1)) = \mathcal{F}_V(Tr_0(s_2)) = sr$, *and*

$\mathcal{F}_V(\mathcal{M}, s_0) = \neg sr \wedge \text{AG}(\neg sr \rightarrow \text{AX}sr) \wedge \text{AG}(sr \rightarrow \text{AX}\neg sr)$.

The following theorem shows that the characterizing formulas of an initial K-structure are equivalent. Given a set of atomic propositions V, two kripke structures are bisimilar w.r.t. V's complement iff one of them models the others characterizing formula. Moreover, the V-bisimulation between two initial K-structures implies the semantic equivalence of the corresponding characterizing formulas. Formally:

Theorem 2. *Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s_0')$ be two model structures. Then,*

(i) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s_0')$;

(ii) $s_0 \leftrightarrow_{\overline{V}} s_0'$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s_0')$.

Proof. (sketch) Let $k = ch(\mathcal{M}, V)$, On the one hand, one can verify that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s))$ holds for $n \geq 0$, which implies $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_k(s))$. On the other hand, $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s'))$ implies $\text{Tr}_n(s) \leftrightarrow_{\overline{V}} \text{Tr}_n(s')$. In this case, It is not difficult to see that, for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(\text{Tr}_k(s))$.

(i) From the left to the right is apparent due to $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$, hence $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

For the other direction. We can prove this by showing that $\forall n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s_0')$ by Proposition 2. The base case, *i.e.* $n = 0$, is easy. A key point for the induction step is that $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$ implies $\forall s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\text{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1) \in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s,s_1) \in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ for any $s \in S$. For further details, please see Appendix A.

(ii) This is implied by Lemma 2. □

Recall that the number of model structures are finite. The next lemma is evident in terms of the above theorem.

Lemma 3. *Let φ be a formula. We have*

$$\varphi \equiv \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M},s_0). \tag{2}$$

It shows that any CTL formula can be equivalently transformed into a disjunction of the characterizing formulas for its models.

3.3 Semantic properties of forgetting in CTL

In this subsection we present the definition of forgetting in CTL and investigate its semantic properties. Based on our notion of V-bisimulation, we can now introduce forgetting in CTL from the semantic forgetting point view. Which means the result of forgetting the atoms in set V of atoms from CTL formula φ is a formula which shares the same models as φ and models that are V-bisimulation with one of the models of φ.

Definition 3 (Forgetting)**.** *Let $V \subseteq \mathcal{A}$ and ϕ be a formula. A formula ψ with $Var(\psi)\cap V = \emptyset$ is a* result of forgetting V from ϕ*, if*

$$Mod(\psi) = \{\mathcal{K} \textit{ is initial} \mid \exists \mathcal{K}' \in Mod(\phi) \,\&\, \mathcal{K}' \leftrightarrow_V \mathcal{K}\}.$$

Note that if both ψ and ψ' are results of forgetting V from ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the forgetting result is unique (up to equivalence). By Lemma 3, such a formula always exists, which is equivalent to

$$\bigvee_{\mathcal{K}\in\{\mathcal{K}'\mid\exists\mathcal{K}''\in Mod(\phi) \text{ and } \mathcal{K}''\leftrightarrow_V\mathcal{K}'\}} \mathcal{F}_{\overline{V}}(\mathcal{K}).$$

The forgetting result is denoted by $\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

Following from the knowledge forgetting point of view (Zhang and Zhou 2009), we show that the above forgetting repectes the four forgetting postulates in CTL:

- Weakening (**W**): $\varphi \models \varphi'$;
- Positive Persistence (**PP**): for any formula η, if $\mathrm{IR}(\eta, V)$ and $\varphi \models \eta$ then $\varphi' \models \eta$;
- Negative Persistence (**NP**): for any formual η, if $\mathrm{IR}(\eta, V)$ and $\varphi \not\models \eta$ then $\varphi' \not\models \eta$;
- Irrelevance (**IR**): $\mathrm{IR}(\varphi', V)$

where $V \subseteq \mathcal{A}$, φ is a formula and φ' is a result of forgetting V from φ. Intuitivly speaking, the postulate (**W**) says, forgetting weakens the original formula; the postulates (**PP**) and (**NP**) say that forgetting results have no effect on formulas that are irrelevant to fogotten atoms; the postulate (**IR**) states that forgetting result is irrelevant to forgotten atoms.

Theorem 3 (Representation Theorem)**.** *Let φ, φ' and ϕ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

(i) $\varphi' \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$,

(ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi$ *and* $IR(\phi, V)\}$,

*(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.*

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\mathrm{F}_{\mathrm{CTL}}(\varphi, V)) = Mod(\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\})$$
$$= Mod(\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)).$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. Then there exists an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of φ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all ϕ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s_0')$ is a models of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$ due to $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$ is irrelevant to V.

Finally, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$. Then there exists $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$. Hence, $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ by Theorem 2. Thus $(\mathcal{M}', s_0')$ is also a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. By Positive Persistence, we have $\varphi' \models \{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Now we show that $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \varphi'$. Otherwise, there exists formula ϕ' such that $\varphi' \models \phi'$ but $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\} \not\models \phi'$. There are three cases:

- ϕ' is relevant to V. Thus, φ' is also relevant to V, a contradiction to Irrelevance.
- ϕ' is irrelevant to V and $\varphi \models \phi'$. This contradicts to our assumption.
- ϕ' is irrelevant to V and $\varphi \not\models \phi'$. By Negative Persistence, $\varphi' \not\models \phi'$, a contradiction.

Thus, φ' is equivalent to $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. □

The next lemma is obvious.

Lemma 4. *Let φ and α be two CTL formulae and $q \in Var(\varphi) \cup Var(\alpha)$. Then $\mathrm{F}_{CTL}(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.*

The following proposition shows that the forgetting a set of atoms can be obtained by forgetting atoms in the set one by one.

Proposition 4. *Given a formula $\varphi \in CTL$, V a set of atoms and p an atom such that $p \notin V$. Then,*

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V).$$

Proof. Let $(\mathcal{M}_1, s_1)$ with $\mathcal{M}_1 = (S_1, R_1, L_1, s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V)$. By the definition, there exists a model $(\mathcal{M}, s)$ with $\mathcal{M} = (S, R, L, s)$ of φ, such that $(\mathcal{M}_1, s_1) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}, s)$. We construct an initial K-structure $(\mathcal{M}_2, s_2)$ with $\mathcal{M}_2 = (S_2, R_2, L_2, s_2)$ as follows:

(1) for s_2: let s_2 be the state such that:
  - $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
  - for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
  - for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:

(i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\}\cup V} w_1$, let $w_2 \in S_2$ and

- $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
- for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
- for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

(ii) if $(w_1', w_1) \in R_1$, w_2 is constructed based on w_1 and $w_2' \in S_2$ is constructed based on w_1', then $(w_2', w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$. Thus, $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\text{CTL}}(\mathrm{F}_{\text{CTL}}(\varphi, p), V)$.

On the other hand, suppose that $(\mathcal{M}_1, s_1)$ is a model of $\mathrm{F}_{\text{CTL}}(\mathrm{F}_{\text{CTL}}(\varphi, p), V)$, then there exists an initial K-structure $(\mathcal{M}_2, s_2)$ such that $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\varphi, p)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$, and there exists $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \varphi$ and $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$. Therefore, $(\mathcal{M}, s) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}_1, s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\text{CTL}}(\varphi, \{p\} \cup V)$. $\square$

The next corollary follows from the above proposition.

Corollary 4 (Commutativity)**.** *Let φ be a formula and $V_i \subseteq \mathcal{A}$ $(i = 1, 2)$. Then:*

$$\mathrm{F}_{\text{CTL}}(\varphi, V_1 \cup V_2) \equiv \mathrm{F}_{\text{CTL}}(\mathrm{F}_{\text{CTL}}(\varphi, V_1), V_2).$$

The following results, that hold in both classical propositional logic and modal logic **S5** (Zhang and Zhou 2009), further illustrate other important properties of CTL forgetting.

Proposition 5. *Let φ, φ_i, ψ_i $(i = 1, 2)$ be formulas and $V \subseteq \mathcal{A}$. We have*

(i) $\mathrm{F}_{\text{CTL}}(\varphi, V)$ *is satisfiable iff φ is;*
(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\text{CTL}}(\varphi_1, V) \equiv \mathrm{F}_{\text{CTL}}(\varphi_2, V)$*;*
(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\text{CTL}}(\varphi_1, V) \models \mathrm{F}_{\text{CTL}}(\varphi_2, V)$*;*
(iv) $\mathrm{F}_{\text{CTL}}(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_{\text{CTL}}(\psi_1, V) \vee \mathrm{F}_{\text{CTL}}(\psi_2, V)$*;*
(v) $\mathrm{F}_{\text{CTL}}(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_{\text{CTL}}(\psi_1, V) \wedge \mathrm{F}_{\text{CTL}}(\psi_2, V)$*;*

The next proposition shows that forgetting a set $V \subseteq \mathcal{A}$ from a formula with path quantifiers is equivalent to quantify the result of forgetting V from the formula with the same path quantifiers.

Proposition 6 (Homogeneity)**.** *Let $V \subseteq \mathcal{A}$ and $\phi \in$ CTL,*

(i) $\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V) \equiv \text{AX}\mathrm{F}_{\text{CTL}}(\phi, V)$.
(ii) $\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V) \equiv \text{EX}\mathrm{F}_{\text{CTL}}(\phi, V)$.
(iii) $\mathrm{F}_{\text{CTL}}(\text{AF}\phi, V) \equiv \text{AF}\mathrm{F}_{\text{CTL}}(\phi, V)$.
(iv) $\mathrm{F}_{\text{CTL}}(\text{EF}\phi, V) \equiv \text{EF}\mathrm{F}_{\text{CTL}}(\phi, V)$.

Proof. (stretch) We given the proof of (i), others can be proved similarly.

For one thing, for all model $\mathcal{K} = (\mathcal{M}, s_0)$ of the left side there is a model $\mathcal{K}' = (\mathcal{M}', s_0')$ of $\text{AX}\phi$ such that $\mathcal{K} \leftrightarrow_V \mathcal{K}'$, i.e. $\forall s_1$ with $(s_0, s_1) \in R$ there is s_1' with $(s_0', s_1') \in R'$ such that $(\mathcal{M}, s_1) \leftrightarrow_V (\mathcal{M}', s_1')$ and then $(\mathcal{M}, s_1) \models \mathrm{F}_{\text{CTL}}(\phi, V)$ due to $\mathrm{IR}(\mathrm{F}_{\text{CTL}}(\phi, V), V)$ and $(\mathcal{M}', s_1') \models \phi$ and versa vice. Therefore, $\mathcal{K} \models \text{AX}\mathrm{F}_{\text{CTL}}(\phi, V)$.

For another, for all model $\mathcal{K} = (\mathcal{M}, s_0)$ of $\text{AX}\mathrm{F}_{\text{CTL}}(\phi, V)$, we can easily construct an initial K-structure $\mathcal{K}' = (\mathcal{M}', s_0')$ such that $\mathcal{K} \leftrightarrow_V \mathcal{K}'$ and $\mathcal{K}' \models \text{AX}\phi$ since for each model $\mathcal{K}_1$ of $\mathrm{F}_{\text{CTL}}(\phi, V)$ there is a model $\mathcal{K}_2$ of ϕ s.t. $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$. Therefore, $\mathcal{K} \models \mathrm{F}_{\text{CTL}}(\text{AX}\phi, V)$ by the definition of forgetting. $\square$

3.4 Complexity Results

In the following, we analyze the computational complexity of the various tasks regarding the forgetting in CTL and the fragment CTL_{AF}.

Proposition 7 (Model Checking on Forgetting)**.** *Let $(\mathcal{M}, s_0)$ be an initial* K*-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{CTL}(\varphi, V)$ is NP-complete.*

Proof. One can (1) guess an initial K-structure $(\mathcal{M}', s_0')$ satisfying φ; and (2) check if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Both guessing and checking can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard (Zhang and Zhou 2008). $\square$

The fragment CTL_{AF} of CTL, in which each formula contains only AF temporal connective, corresponds to specifications that are desired to hold in all branches eventually. Such properties are of special interest in concurrent systems e.g., mutual exclusion and waiting events (Baier and Katoen 2008). In the following, we investigate some complexity results concerning forgetting and the logical entailment in this fragment.

Theorem 5 (Entailment on Forgetting)**.** *Let φ and ψ be two CTL_{AF} formulas and V a set of atoms. Then, results:*

(i) deciding $\mathrm{F}_{CTL}(\varphi, V) \models^? \psi$ is co-NP-complete,
(ii) deciding $\psi \models^? \mathrm{F}_{CTL}(\varphi, V)$ is Π_2^P-complete,
(iii) deciding $\mathrm{F}_{CTL}(\varphi, V) \models^? \mathrm{F}_{CTL}(\psi, V)$ is Π_2^P-complete.

Proof. (i) It is known that deciding whether ψ is satisfiable is NP-Complete (Meier et al. 2015). The hardness is easy to see by setting $\mathrm{F}_{CTL}(\varphi, Var(\varphi)) \equiv \top$, i.e., deciding whether ψ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{CTL}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $\mathrm{IR}(\psi, V)$. Clearly, in CTL_{AF}, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $\mathrm{IR}(\psi, V)$ is also in co-NP. Without loss of generality, we assume that ψ is satisfiable. We consider the complement of the problem: deciding whether ψ is not irrelevant to V. It is easy to see that ψ is not irrelevant to V iff there exist a model $(\mathcal{M}, s_0)$ of ψ and an initial K-structure $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ and $(\mathcal{M}', s_0') \not\models \psi$. So checking whether ψ is not irrelevant to V can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s_0')$, (2) check if $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s_0') \not\models \psi$, and (3) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Obviously (1) can be done in polynomial time and also (2) and (3) can be done in polynomial time.

(ii) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ and check whether $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}, s_0) \not\models \mathrm{F_{CTL}}(\varphi, V)$. From Proposition 7, we know that this is in Σ_2^P. So the original problem is in Π_2^P. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F_{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case temporal forgetting, the hardness is directly followed from the proof of Proposition 24 in (Lang, Liberatore, and Marquis 2003).

(iii) Membership. If $\mathrm{F_{CTL}}(\varphi, V) \not\models \mathrm{F_{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F_{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F_{CTL}}(\psi, V)$, i.e.,, there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M}, s)$, $(\mathcal{M}_1, s_1)$ with $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ and checking $(\mathcal{M}_1, s_1) \models \varphi$ are feasible while checking $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ can be done in polynomial time. Thus the problem is in Π_2^P.

Hardness. It follows from (2) due to the fact that $\mathrm{F_{CTL}}(\varphi, V) \models \mathrm{F_{CTL}}(\psi, V)$ iff $\varphi \models \mathrm{F_{CTL}}(\psi, V)$ by $\mathrm{IR}(\mathrm{F_{CTL}}(\psi, V), V)$.

□

The following corollary follow from Theorem 5.

Corollary 6. *Let φ and ψ be two CTL$_{\mathrm{AF}}$ formulas and V a set of atoms. Then*

(i) deciding $\psi \equiv^? \mathrm{F}_{CTL}(\varphi, V)$ is Π_2^P-complete,

(ii) deciding $\mathrm{F}_{CTL}(\varphi, V) \equiv^? \varphi$ is co-NP-complete,

(iii) deciding $\mathrm{F}_{CTL}(\varphi, V) \equiv^? \mathrm{F}_{CTL}(\psi, V)$ is Π_2^P-complete.

4 Necessary and Sufficient Conditions

In this section, we present the definitions of strongest necessary and weakest sufficient conditions (SNC and WSC, respectively) of a specification in CTL and show that they can be obtained by forgetting under a given initial K-structure and a set V of atoms.

Definition 4 (sufficient and necessary condition). *Let ϕ be a formula (or an initial* K*-structure), ψ be a formula, $V \subseteq Var(\phi)$, $q \in Var(\phi) - V$ and $Var(\psi) \subseteq V$.*

- *ψ is a* necessary condition *(NC in short) of q on V under ϕ if $\phi \models q \rightarrow \psi$.*
- *ψ is a* sufficient condition *(SC in short) of q on V under ϕ if $\phi \models \psi \rightarrow q$.*
- *ψ is a* strongest necessary condition *(SNC in short) of q on V under ϕ if it is a NC of q on V under ϕ and $\phi \models \psi \rightarrow \psi'$ for any NC ψ' of q on V under ϕ.*
- *ψ is a* weakest sufficient condition *(SNC in short) of q on V under ϕ if it is a SC of q on V under ϕ and $\phi \models \psi' \rightarrow \psi$ for any SC ψ' of q on V under ϕ.*

Note that if both ψ and ψ' are SNC (WSC) of q on V under ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In the sense of equivalence the SNC (WSC) is unique (up to equivalence).

Example 4. *Let $\psi = sl \rightarrow (\mathrm{AX}sr \vee \mathrm{EX}le \vee \mathrm{EX}(se \vee lc))$, $\varphi = sl \rightarrow \mathrm{AX}sr$, $\mathcal{A} = \{sl, sr, se, lc, le\}$ and $V = \{sl, sr\}$, then we can check that the WSC of ψ on V under the initial* K*-structure $(\mathcal{M}, s_0)$ in Figure 1 is φ.*

We verify this result by the following two steps:

(i) It is apparent that $\varphi \models \psi$ and $Var(\varphi) \subseteq V$. Besides, $(\mathcal{M}, s_0) \models \varphi \wedge \psi$, hence $(\mathcal{M}, s_0) \models \varphi \rightarrow \psi$, which means φ is a SC of ψ on V under $(\mathcal{M}, s_0)$,

(ii) We will show that for any SC φ' of ψ on V under $(\mathcal{M}, s_0)$, there is $(\mathcal{M}, s_0) \models \varphi' \rightarrow \varphi$. Following Figure 3, we can see that $(\mathcal{M}', s_0) \leftrightarrow_{\mathcal{A} \setminus V} (\mathcal{M}, s_0)$. By Theorem 2 we can easily obtain the characterizing formula of $(\mathcal{M}, s_0)$ on V, i.e. $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s_0) = sl \wedge \neg sr \wedge \mathrm{AG}((sl \wedge \neg sr) \rightarrow \mathrm{AX}(\neg sl \wedge sr)) \wedge \mathrm{AG}((\neg sl \wedge sr) \rightarrow \mathrm{AX}(sl \wedge \neg sr))$, due to $ch(\mathcal{M}', \mathcal{A} \setminus V) = 0$.

If $\mathcal{F}_V(\mathcal{M}', s_0) \not\models \varphi'$ or $\varphi' \models \neg sl$ then we have $\mathcal{F}_V(\mathcal{M}', s_0) \models \varphi' \rightarrow \varphi$ i.e. $(\mathcal{M}, s_0) \models \varphi' \rightarrow \varphi$. Therefore, we suppose $\mathcal{F}_V(\mathcal{M}', s_0) \models \varphi'$. In this case we can construct φ' as follows: (1) sl is a sub-formula of φ' due to $\mathcal{F}_V(\mathcal{M}', s_0) \models \varphi'$; (2) $sl \rightarrow \mathrm{AX}sr$ is also a sub-formula by (1) and we need $(\mathcal{M}, s_0) \models \varphi' \rightarrow \psi$ and $Var(\varphi') \subseteq V$. This means that each SC φ' should be the following form with β be a CTL formula of V:

$$(sl \wedge (sl \rightarrow \mathrm{AX}sr)) \wedge \beta.$$

In this case, we have $(\mathcal{M}, s_0) \models \varphi' \rightarrow \varphi$ for all SC φ' of ψ on V under $(\mathcal{M}, s_0)$ since for any α with $Var(\alpha) \subseteq V$ $(\mathcal{M}, s_0) \models \alpha$ iff $(\mathcal{M}', s_0) \models \alpha$ by Theorem 1.

Figure 3: A model structure $\mathcal{M}'$ with initial state s_0

Proposition 8 (Dual). *Let V, q, φ and ψ are the ones in Definition 4. The formula ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.*

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \rightarrow \psi$. Thus $\varphi \models \neg\psi \rightarrow \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \rightarrow \neg q$. Then $\varphi \models q \rightarrow \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \rightarrow \neg\psi'$ by assumption. So $\varphi \models \psi' \rightarrow \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case.

□

This shows that the SNC and WSC are in fact dual conditions.

In order to generalise Definition 4 to arbitrary formulas, one can replace q (in the definition) by any formula α, and redefine V as a subset of $Var(\alpha) \cup Var(\phi)$. It turns out that the previous notion of SNC and WSC for an atomic proposition can be lifted to any formula, or conversely the SNC and WSC of any formula can be reduced to that of a proposition.

Proposition 9. *Let Γ and α be two formulas, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q is a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.*

Proof. We prove this for SNC. The case for WSC is similar. Let $SNC(\varphi, \alpha, V, \Gamma)$ denote that φ is the SNC of α on V under Γ, and $NC(\varphi, \alpha, V, \Gamma)$ denote that φ is the NC of α on V under Γ.

($\Rightarrow$) We will show that if $SNC(\varphi, \alpha, V, \Gamma)$ holds, then $SNC(\varphi, q, V, \Gamma')$ will be true. According to $SNC(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \to \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F}_{\text{CTL}}(\Gamma', q) \models \alpha \to \varphi'$ due to $\alpha \equiv q$, $IR(\alpha \to \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \to \varphi'$ by Lemma 4, this means $NC(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \to \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \to \varphi'$. Hence, $SNC(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $SNC(\varphi, q, V, \Gamma')$ holds, then $SNC(\varphi, \alpha, V, \Gamma)$ will be true. According to $SNC(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F}_{\text{CTL}}(\Gamma', \{q\}) \models \alpha \to \varphi$ due to $\alpha \equiv q$, $IR(\alpha \to \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \to \varphi$ by Lemma 4, this means $NC(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \to \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $NC(\varphi', q, V, \Gamma')$. According to $SNC(\varphi, q, V, \Gamma')$, $IR(\varphi \to \varphi', \{q\})$ and (**PP**), we have $\mathrm{F}_{\text{CTL}}(\Gamma', \{q\}) \models \varphi \to \varphi'$, and $\Gamma \models \varphi \to \varphi'$ by Lemma 4. Hence, $SNC(\varphi, \alpha, V, \Gamma)$ holds. □

The following result establishes the bridge between these two notions which are central to the paper.

Theorem 7. *Let φ be a formula, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.*

(i) $\mathrm{F}_{\text{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ *is a SNC of q on V under φ.*

(ii) $\neg\mathrm{F}_{\text{CTL}}(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ *is a WSC of q on V under φ.*

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F}_{\text{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \to \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \to \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \to \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $IR(\psi, (Var(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ. □

As aforementioned, since any initial K-structure can be characterized by a CTL formula, one can obtain the SNC (and its dual WSC) of a target property (a formula) under an initial K-structure by forgetting.

Theorem 8. *Let $\mathcal{K} = (\mathcal{M}, s)$ be an initial* K*-structure with $\mathcal{M} = (S, R, L, s_0)$ on the set $\mathcal{A}$ of atoms, $V \subseteq \mathcal{A}$ and $q \in V' = \mathcal{A} - V$. Then:*

(i) the SNC of q on V under $\mathcal{K}$ is $\mathrm{F}_{\text{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q on V under $\mathcal{K}$ is $\neg\mathrm{F}_{\text{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge \neg q, V')$.

Input: A CTL formula φ and a set V of atoms
Output: $\mathrm{F}_{\text{CTL}}(\varphi, V)$
$\psi \leftarrow \bot$;
foreach *initial structure $\mathcal{K}$ (over $\mathcal{A}$ and $\mathcal{S}$)* **do**
  if $\mathcal{K} \not\models \varphi$ **then continue foreach** *initial structure $\mathcal{K}'$ with $\mathcal{K} \leftrightarrow_V \mathcal{K}'$* **do**
    $\psi \leftarrow \psi \vee \mathcal{F}_{\overline{V}}(\mathcal{K}')$;
  end
end
return ψ;

Algorithm 1: A model-based CTL forgetting procedure

5 An Algorithm Computing CTL Forgetting

To compute the forgetting in CTL, we propose a model-based method. Intuitively, the model-based method means that we can compute the forgetting applied to a formula simply by considering all the possible models of that formula.

Next, we give a trivial algorithm computing CTL forgetting result, Algorithm 1.Its correctness is guaranteed by Lemma 3 and Theorem 2.

Example 5. *Consider the model given in Figure 1. Assume that we are given a property $\alpha = \mathrm{AGEF}(lc \wedge sr)$ and $\{lc\}$. Then, it is easy to check that $\mathrm{F}_{\text{CTL}}(\alpha, \{lc\}) \equiv \mathrm{AGEF}sr$.*

Proposition 10. *Let φ be a CTL formula and $V \subseteq \mathcal{A}$ with $|\mathcal{S}| = m$, $|\mathcal{A}| = n$ and $|V| = x$. The the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity of Algorithm 1 is at least the same as the space.*

Proof. Supposing each state or atom occupy one byte, then a state pair (s, s') occupy two bytes. For any $B \subseteq \mathcal{S}$ with $B \neq \emptyset$ and $s_0 \in B$, we can construct an initial K-structure $(\mathcal{M}, s_0)$ with $\mathcal{M} = (B, R, L, s_0)$, in which there is at most $\frac{|B|^2}{2}$ state pairs in R and $|B| * n$ (s, A) $(A \subseteq \mathcal{A})$ in L. Hence, the $(\mathcal{M}, s_0)$ occupy at most $|B| + |B|^2 + |B| * n$ bytes. Besides, there is at most $|B|^{|B|+1} * 2^{nm}$ number of initial K-structures. Therefore, there is at most $m^{m+2} * 2^{nm}$ number of initial K-structures, hence it will at most cost $m^{m+2} * 2^{nm} * (m + m^2 + nm)$ bytes.

Let $k = n - x$, for any initial K-structure $\mathcal{K} = (\mathcal{M}, s_0)$ with $i \geq 1$ nodes, in the worst, i.e., $ch(\mathcal{M}, V) = i$, we will spend $N(i)$ space to store the characterizing formula.

$$
\begin{aligned}
N(i) &= (k + (\cdots + (k + 2ik) * (2i)) \cdots * (2i)) \\
&= (2i)^0 k + 2ik + (2i)^2 k + \cdots + (2i)^{(i-1)} k \\
&= \frac{(2i)^i - 1}{2i - 1} k.
\end{aligned}
$$

In the worst case, i.e., there is $m^{m+2} * 2^{nm}$ initial K-structures with m nodes, we will spent $m^{m+2} * 2^{nm} * N(m)$ bytes to store the result of forgetting.

Therefore, the space complexity is $O((n - x)m^{2(m+1)}2^{nm})$ and the time complexity is at least the same as the space. □

6 Concluding Remarks

Summary In this article, we generalized the notion of bisimulation from model structures of Computation Tree Logic (CTL) to model structures with initial states over a given signature V, named V*-bisimulation*. Based on this new bisimulation, we presented the notion of forgetting for CTL, which enables computing weakest sufficient and strongest necessary conditions of specifications. Further properties and complexity issues in this context were also explored. In particular, we have shown that our notion of forgetting satisfies the existing postulates of forgetting, which means it faithfully extends the notion of forgetting from classical propositional logic and modal logic S5 to CTL. On the complexity theory side, we investigated the model checking of forgetting, which turned out to be NP-complete, and the relevant fragment of $\mathrm{CTL}_{\mathrm{AF}}$ which ranged from co-NP to Π_2^p-completeness. And finally, we proposed a model-based algorithm which computes the forgetting of a given formula and a set of variables, and outlined its complexity.

Future work Note that, when a transition system $\mathcal{M}$ does not satisfy a specification ϕ, one can evaluate the weakest sufficient condition ψ over a signature V under which $\mathcal{M}$ satisfies ϕ, viz., $\mathcal{M} \models \psi \rightarrow \phi$ and ψ mentions only atoms from V. It is worthwhile to explore how the condition ψ can guide the design of a new transition system $\mathcal{M}'$ satisfying ψ.

Moreover, a further study regarding the computational complexity for other general fragments is required and part of the future research agenda. Such investigation can be coupled with fine-grained parameterized analysis.

A Supplementary Material: Proof Appendix

Lemma 5. *Let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be the ones in the definition of section 3.1. Then, for each $i \geq 0$,*

(i) $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$;

(ii) there is a (smallest) $k \geq 0$ such that $\mathcal{B}_{k+1} = \mathcal{B}_k$;

(iii) $\mathcal{B}_i$ is reflexive, symmetric and transitive.

Proof. (i) Base: it is clear for $i = 0$ by the above definition.

Step: suppose it holds for $i = n$, i.e., $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.

$(s, s') \in \mathcal{B}_{n+2}$

$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s_1') \in R'$ such that $(s_1, s_1') \in \mathcal{B}_{n+1}$, and (c) for every $(s', s_1') \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s_1') \in \mathcal{B}_{n+1}$

$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s_1') \in R'$ such that $(s_1, s_1') \in \mathcal{B}_n$ by inductive assumption, and (c) for every $(s', s_1') \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s_1') \in \mathcal{B}_n$ by inductive assumption

$\Rightarrow (s, s') \in \mathcal{B}_{n+1}$.

(ii) and (iii) are evident from (i) and the definition of $\mathcal{B}_i$. $\square$

Lemma 1 The relation $\leftrightarrow_V$ is an equivalence relation.

Proof. It is clear from Lemma 5 (ii) such that there is a $k \geq 0$ where $\mathcal{B}_k = \mathcal{B}_{k+1}$ which is $\leftrightarrow_V$, and it is reflexive, symmetric and transitive by (iii). $\square$

Proposition 1 Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s_i's be two states and π_i's be two paths, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:

(i) $s_1' \leftrightarrow_{V_i} s_2'$ $(i = 1, 2)$ implies $s_1' \leftrightarrow_{V_1 \cup V_2} s_2'$;

(ii) $\pi_1' \leftrightarrow_{V_i} \pi_2'$ $(i = 1, 2)$ implies $\pi_1' \leftrightarrow_{V_1 \cup V_2} \pi_2'$;

(iii) for each path π_{s_1} of $\mathcal{M}_1$ there is a path π_{s_2} of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Proof. In order to distinguish the relations $\mathcal{B}_0, \mathcal{B}_1, \dots$ for different set $V \subseteq \mathcal{A}$, by $\mathcal{B}_i^V$ we mean the relation $\mathcal{B}_1, \mathcal{B}_2, \dots$ for $V \subseteq \mathcal{A}$. Denote as $\mathcal{B}_0, \mathcal{B}_1, \dots$ when the underlying set V is clear from the context. Moreover, for the ease of notation, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$ (i.e., without subindex).

(i) Base: it is clear for $n = 0$.

Step: For $n > 0$, supposing if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1 \cup V_2}$ for all $0 \leq i \leq n$. We will show that if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1 \cup V_2}$.

(a) It is evident that $L_1(s_1) - (V_1 \cup V_2) = L_2(s_2) - (V_1 \cup V_2)$.

(b) We will show that for each $(s_1, s_1^1) \in R_1$ there is a $(s_2, s_2^1) \in R_2$ such that $(s_1^1, s_2^1) \in \mathcal{B}_n^{V_1 \cup V_2}$. There is $(\mathcal{K}_1^1, \mathcal{K}_2^1) \in \mathcal{B}_{n-1}^{V_1 \cup V_2}$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_1 \cup V_2}$ by inductive assumption. Then we only need to prove for each $(s_1^1, s_1^2) \in R_1$ there is a $(s_2^1, s_2^2) \in R_2$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$ and for each $(s_2^1, s_2^2) \in R_2$ there is a $(s_1^1, s_1^2) \in R_1$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$. Therefore, we only need to prove that for each $(s_1^n, s_1^{n+1}) \in R_1$ there is a $(s_2^n, s_2^{n+1}) \in R_2$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$ and for each $(s_2^n, s_2^{n+1}) \in R_2$ there is a $(s_1^n, s_1^{n+1}) \in R_1$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$. It is apparent that $L_1(s_1^{n+1}) - (V_1 \cup V_2) = L_1(s_2^{n+1}) - (V_1 \cup V_2)$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$. Where $\mathcal{K}_i^j = (\mathcal{M}_i, s_i^j)$ with $i \in \{1, 2\}$ and $0 < j \leq n + 1$.

(c) It is similar with (b).

(ii) It is clear from (i).

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$, and

(c) for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$,

where $\mathcal{K}_i' = (\mathcal{M}_i, s_i')$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

($\Rightarrow$) (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s_1') \in R_1$, there is a $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s_1') - V = L_2(s_2') - V$. Therefore, $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$. (c) This is similar with (b).

($\Leftarrow$) (a) $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) Condition (ii) implies that for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$; (c) Condition (iii) implies that for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$

$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$

$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing (s_1, s_3) from the (a), (b) and (c) of the previous step (iii) of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(v) Let $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$ and $(s_{i,k}, s_{i,k+1}) \in R_i$ mean that $s_{i,k+1}$ is the $(k + 2)$-th node in the path

$(s_i, s_{i,1}, s_{i,2}, \ldots, s_{i,k+1}, \ldots)$ $(i = 1, 2)$. We will show that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_2}$ for all $n \geq 0$ inductively.

Base: $L_1(s_1) - V_1 = L_2(s_2) - V_1$
$\Rightarrow \forall q \in \mathcal{A} - V_1$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$
$\Rightarrow \forall q \in \mathcal{A} - V_2$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$ due to $V_1 \subseteq V_2$
$\Rightarrow L_1(s_1) - V_2 = L_2(s_2) - V_2$, i.e., $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0^{V_2}$.

Step: Supposing that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ for all $0 \leq i \leq k$ $(k > 0)$, we will show $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{k+1}^{V_2}$.

(a) It is apparent that $L_1(s_1) - V_2 = L_2(s_2) - V_2$ by base.

(b) $\forall (s_1, s_{1,1}) \in R_1$, we will show that there is a $(s_2, s_{2,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_{k-1}^{V_2}$ by inductive assumption, we need only to prove the following points:
(a) $\forall (s_{1,k}, s_{1,k+1}) \in R_1$ there is a $(s_{2,k}, s_{2,k+1}) \in R_2$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. It is easy to see that $L_1(s_{1,k+1}) - V_1 = L_1(s_{2,k+1}) - V_1$, then there is $L_1(s_{1,k+1}) - V_2 = L_1(s_{2,k+1}) - V_2$. Therefore, $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$.
(b) $\forall (s_{2,k}, s_{2,k+1}) \in R_1$ there is a $(s_{1,k}, s_{1,k+1}) \in R_1$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. This can be proved as (a).

(c) $\forall (s_2, s_{2,1}) \in R_1$, we will show that there is a $(s_1, s_{1,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. This can be proved as (ii).

□

Theorem1 Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.

Proof. This theorem can be proved by inducting on the formula ϕ and supposing *Var*$(\phi) \cap V = \emptyset$. Let $\mathcal{K}_1 = (\mathcal{M}, s)$ and $\mathcal{K}_2 = (\mathcal{M}', s')$.

Case $\phi = p$ where $p \in \mathcal{A} - V$:
$(\mathcal{M}, s) \models \phi$ iff $p \in L(s)$ (by the definition of satisfiability)
$\Leftrightarrow p \in L'(s')$ $\quad (s \leftrightarrow_V s')$
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \neg\psi$:
$(\mathcal{M}, s) \models \phi$ iff $(\mathcal{M}, s) \not\models \psi$
$\Leftrightarrow (\mathcal{M}', s') \not\models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \psi_1 \vee \psi_2$:
$(\mathcal{M}, s) \models \phi$
$\Leftrightarrow (\mathcal{M}, s) \models \psi_1$ or $(\mathcal{M}, s) \models \psi_2$
$\Leftrightarrow (\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{EX}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s, s_1, \ldots)$ such that $\mathcal{M}, s_1 \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s', s_1', \ldots)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_1 \leftrightarrow_V s_1'$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s_1') \models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{EG}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, \ldots)$ such that for each $i \geq 0$ there is $(\mathcal{M}, s_i) \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s' = s_0', s_1', \ldots)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_i \leftrightarrow_V s_i'$ for each $i \geq 0$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s_i') \models \psi$ for each $i \geq 0$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{E}[\psi_1 \mathrm{U} \psi_2]$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, \ldots)$ such that there is $i \geq 0$ such that $(\mathcal{M}, s_i) \models \psi_2$, and for all $0 \leq j < i$, $(\mathcal{M}, s_j) \models \psi_1$
$\Leftrightarrow$ There is a path $\pi' = (s = s_0', s_1', \ldots)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow (\mathcal{M}', s_i') \models \psi_2$, and for all $0 \leq j < i$ $(\mathcal{M}', s_j') \models \psi_1$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$ □

Proposition 2 Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two K-structures. Then

$(s_1, s_2) \in \mathcal{B}_n$ iff $\mathrm{Tr}_j(s_1) \leftrightarrow_V \mathrm{Tr}_j(s_2)$ for every $0 \leq j \leq n$.

Proof. We will prove this from two aspects:

($\Rightarrow$) If $(s_1, s_2) \in \mathcal{B}_n$, then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$. $(s, s') \in \mathcal{B}_n$ implies both roots of $Tr_n(s_1)$ and $Tr_n(s_2)$ have the same atoms except those atoms in V. Besides, for any $s_{1,1}$ with $(s_1, s_{1,1}) \in R_1$, there is a $s_{2,1}$ with $(s_2, s_{2,1}) \in R_2$ s.t. $(s_{1,1}, s_{2,1}) \in \mathcal{B}_{n-1}$ and vice versa. Then we have $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$. Therefore, $Tr_n(s_1) \leftrightarrow_V Tr_n(s_2)$ by use such method recursively, and then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$.

($\Leftarrow$) If $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$, then $(s_1, s_2) \in \mathcal{B}_n$. $Tr_0(s_1) \leftrightarrow_V Tr_0(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and then $(s, s') \in \mathcal{B}_0$. $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and for every successors s of the root of one, it is possible to find a successor of the root of the other s' such that $(s, s') \in \mathcal{B}_0$. Therefore $(s_1, s_2) \in \mathcal{B}_1$, and then we will have $(s_1, s_2) \in \mathcal{B}_n$ by use such method recursively. □

Proposition 3 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least k such that $\mathrm{Tr}_k(s)$ and $\mathrm{Tr}_k(s')$ are not V-bisimilar.

Proof. If $s \not\leftrightarrow_V s'$, then there exists a least constant c such that $(s_i, s_j) \notin \mathcal{B}_c$, and then there is a least constant m $(m \leq c)$ such that $\mathrm{Tr}_m(s_i)$ and $\mathrm{Tr}_m(s_j)$ are not V-bisimilar by Proposition 2. Let $k = m$, the lemma is proved. □

Lemma2 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ and $\mathcal{M}'$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $\mathrm{Tr}_n(s) \leftrightarrow_{\overline{V}} \mathrm{Tr}_n(s')$, then $\mathcal{F}_V(\mathrm{Tr}_n(s)) \equiv \mathcal{F}_V(\mathrm{Tr}_n(s'))$.

Proof. This result can be proved by inducting on n.

Base. It is apparent that for any $s_n \in S$ and $s_n' \in S'$, if $\mathrm{Tr}_0(s_n) \leftrightarrow_{\overline{V}} \mathrm{Tr}_0(s_n')$ then $\mathcal{F}_V(\mathrm{Tr}_0(s_n)) \equiv \mathcal{F}_V(\mathrm{Tr}_0(s_n'))$ due to $L(s_n) - \overline{V} = L'(s_n') - \overline{V}$ by known.

Step. Supposing that for $k = m$ $(0 < m \leq n)$ there is if $\text{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k}(s'_k)$ then $\mathcal{F}_V(\text{Tr}_{n-k}(s_k)) \equiv \mathcal{F}_V(\text{Tr}_{n-k}(s'_k))$, then we will show if $\text{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k+1}(s'_{k-1})$ then $\mathcal{F}_V(\text{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\text{Tr}_{n-k+1}(s'_{k-1}))$. Apparent that:
$\mathcal{F}_V(\text{Tr}_{n-k+1}(s_{k-1})) = \left(\bigwedge_{(s_{k-1},s_k)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_{n-k}(s_k))\right) \wedge$
$\text{AX}\left(\bigvee_{(s_{k-1},s_k)\in R} \mathcal{F}_V(\text{Tr}_{n-k}(s_k))\right) \wedge \mathcal{F}_V(\text{Tr}_0(s_{k-1}))$
$\mathcal{F}_V(\text{Tr}_{n-k+1}(s'_{k-1})) = \left(\bigwedge_{(s'_{k-1},s'_k)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_{n-k}(s'_k))\right) \wedge$
$\text{AX}\left(\bigvee_{(s'_{k-1},s'_k)\in R} \mathcal{F}_V(\text{Tr}_{n-k}(s'_k))\right) \wedge \mathcal{F}_V(\text{Tr}_0(s'_{k-1}))$
by the definition of characterizing formula of the computation tree. Then we have for any $(s_{k-1}, s_k) \in R$ there is $(s'_{k-1}, s'_k) \in R'$ such that $\text{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k}(s'_k)$ by $\text{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k+1}(s'_{k-1})$. Besides, for any $(s'_{k-1}, s'_k) \in R'$ there is $(s_{k-1}, s_k) \in R$ such that $\text{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k}(s'_k)$ by $\text{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k+1}(s'_{k-1})$. Therefore, we have $\mathcal{F}_V(\text{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\text{Tr}_{n-k+1}(s'_{k-1}))$ by induction hypothesis. $\square$

Theorem 2 Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures. Then,

(i) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s'_0)$;

(ii) $s_0 \leftrightarrow_{\overline{V}} s'_0$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s'_0)$.

In order to prove Theorem 2, we prove the following two lemmas at first.

Lemma 6. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s))$.

(ii) If $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s'))$ *then* $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$.

Proof. (i) It is apparent from the definition of $\mathcal{F}_V(\text{Tr}_n(s))$.
Base. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_0(s))$.
Step. For $k \geq 0$, supposing the result talked in (i) is correct in $k-1$, we will show that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_{k+1}(s))$, i.e.,:

$$(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \text{EX}T(s')\right) \wedge \text{AX}\left(\bigvee_{(s,s')\in R} T(s')\right) \wedge \mathcal{F}_V(\text{Tr}_0(s)).$$

Where $T(s') = \mathcal{F}_V(\text{Tr}_k(s'))$. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_0(s))$ by Base. It is apparent that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \mathcal{F}_V(\text{Tr}_k(s'))$ by inductive assumption. Then we have $(\mathcal{M}, s) \models \text{EX}\mathcal{F}_V(\text{Tr}_k(s')$, and then $(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \text{EX}\mathcal{F}_V(\text{Tr}_k(s'))\right)$. Similarly, we have that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \bigvee_{(s,s'')\in R} \mathcal{F}_V(\text{Tr}_k(s''))$. Therefore, $(\mathcal{M}, s) \models \text{AX}\left(\bigvee_{(s,s'')\in R} \mathcal{F}_V(\text{Tr}_k(s''))\right)$.

(ii) **Base**. If $n = 0$, then $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_0(s'))$ implies $L(s) - \overline{V} = L'(s') - \overline{V}$. Hence, $\text{Tr}_0(s) \leftrightarrow_{\overline{V}} \text{Tr}_0(s')$.
Step. Supposing $n > 0$ and the result talked in (ii) is correct in $n-1$.
(a) It is easy to see that $L(s) - \overline{V} = L'(s') - \overline{V}$.
(b) We will show that for each $(s, s_1) \in R$, there is a $(s', s'_1) \in R'$ such that $\text{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s'_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \text{AX}\left(\bigvee_{(s',s'_1)\in R} \mathcal{F}_V(\text{Tr}_{n-1}(s'_1))\right)$. Therefore, for each $(s, s_1) \in R$ there is a $(s', s'_1) \in R'$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\text{Tr}_{n-1}(s'_1))$. Hence, $\text{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s'_1)$ by inductive hypothesis.
(c) We will show that for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $\text{Tr}_{n-1}(s'_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \bigwedge_{(s',s'_1)\in R'} \text{EX}\mathcal{F}_V(\text{Tr}_{n-1}(s'_1))$. Therefore, for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\text{Tr}_{n-1}(s'_1)$. Hence, $\text{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s'_1)$ by inductive hypothesis. $\square$

A consequence of the previous lemma is:

Lemma 7. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ a model structure, $k = ch(\mathcal{M}, V)$ and $s \in S$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_k(s))$*, and*

(ii) for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(Tr_k(s))$.

Proof. (i) It is apparent from the (i) of Lemma 6.

(ii) Let $\phi = \mathcal{F}_V(\text{Tr}_k(s))$, where k is the V-characteristic number of $\mathcal{M}$. $(\mathcal{M}, s) \models \phi$ by the definition of $\mathcal{F}$, and then $\forall s' \in S$, if $s \leftrightarrow_{\overline{V}} s'$ there is $(\mathcal{M}, s') \models \phi$ by Theorem 1 due to $\text{IR}(\phi, \mathcal{A} - V)$. Supposing $(\mathcal{M}, s') \models \phi$, if $s \not\leftrightarrow_{\overline{V}} s'$, then $\text{Tr}_k(s) \not\leftrightarrow_{\overline{V}} \text{Tr}_k(s')$, and then $(\mathcal{M}, s') \not\models \phi$ by Lemma 6, a contradiction. $\square$

Now we are in the position of proving Theorem 2.

Proof. (i) Let $\mathcal{F}_V(\mathcal{M}, s_0)$ be the characterizing formula of $(\mathcal{M}, s_0)$ on V. It is apparent that $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. We will show that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ at first.

It is apparent that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\text{Tr}_c(s_0))$ by Lemma 6. We must show that $(\mathcal{M}, s_0) \models \bigwedge_{s\in S} G(\mathcal{M}, s)$. Let $\mathcal{X} = \mathcal{F}_V(\text{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right)$ $\wedge \text{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$, we will show $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$. Where $G(\mathcal{M}, s) = \text{AG}\mathcal{X}$. There are two cases we should consider:

- If $(\mathcal{M}, s_0) \not\models \mathcal{F}_V(\text{Tr}_c(s))$, it is apparent that $(\mathcal{M}, s_0) \models \mathcal{X}$;
- If $(\mathcal{M}, s_0) \models \mathcal{F}_V(\text{Tr}_c(s))$:
  $(\mathcal{M}, s_0) \models \mathcal{F}_V(\text{Tr}_c(s))$
  $\Rightarrow s_0 \leftrightarrow_{\overline{V}} s$ by the definition of characteristic number and Lemma 7.
  For each $(s, s_1) \in R$ there is:
  $(\mathcal{M}, s_1) \models \mathcal{F}_V(\text{Tr}_c(s_1))$ $\quad (s_1 \leftrightarrow_{\overline{V}} s_1)$
  $\Rightarrow (\mathcal{M}, s) \models \bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))$
  $\Rightarrow (\mathcal{M}, s_0) \models \bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))$ (by $\text{IR}(\bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1)), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$).
  For each (s, s_1) there is:
  $\mathcal{M}, s_1 \models \bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))$

$\Rightarrow (\mathcal{M}, s) \models \text{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$
$\Rightarrow (\mathcal{M}, s_0) \models \text{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$ (by IR($\text{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right), \overline{V}$), $s_0 \leftrightarrow_{\overline{V}} s$)
$\Rightarrow (\mathcal{M}, s_0) \models \mathcal{X}$.

For any other states s' which can reach from s_0 can be proved similarly, i.e.,, $(\mathcal{M}, s') \models \mathcal{X}$. Therefore, $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$, and then $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$.

We will prove this theorem from the following two aspects:

($\Leftarrow$) If $s_0 \leftrightarrow_{\overline{V}} s_0'$, then $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$. Since $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and IR($\mathcal{F}_V(\mathcal{M}, s_0), \overline{V}$), hence $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

($\Rightarrow$) If $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$, then $s_0 \leftrightarrow_{\overline{V}} s_0'$. We will prove this by showing that $\forall n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s_0')$.

Base. It is apparent that $Tr_0(s_0) \equiv Tr_0(s_0')$.

Step. Supposing $\text{Tr}_k(s_0) \leftrightarrow_{\overline{V}} \text{Tr}_k(s_0')$ $(k > 0)$, we will prove $\text{Tr}_{k+1}(s_0) \leftrightarrow_{\overline{V}} \text{Tr}_{k+1}(s_0')$. We should only show that $\text{Tr}_1(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_1(s_k')$. Where $(s_0, s_1), (s_1, s_2), \ldots, (s_{k-1}, s_k) \in R$ and $(s_0', s_1'), (s_1', s_2'), \ldots, (s_{k-1}', s_k') \in R'$, i.e., s_{i+1} (s_{i+1}') is an immediate successor of s_i (s_i') for all $0 \leq i \leq k-1$.

(a) It is apparent that $L(s_k) - \overline{V} = L'(s_k') - \overline{V}$ by inductive assumption.

Before talking about the other points, note the following fact that:
$(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$
$\Rightarrow \forall s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\text{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ for any $s \in S$. **(fact)**
(I) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\text{Tr}_c(s_0)) \rightarrow \left(\bigwedge_{(s_0,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ **(fact)**
(II) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\text{Tr}_c(s_0)))$ (known)
(III) $(\mathcal{M}', s_0') \models \left(\bigwedge_{(s_0,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ ((I),(II))

(b) We will show that for each $(s_k, s_{k+1}) \in R$ there is a $(s_k', s_{k+1}') \in R'$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.
(1) $(\mathcal{M}', s_0') \models \bigwedge_{(s_0,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))$ (III)
(2) $\forall (s_0, s_1) \in R$, $\exists (s_0', s_1') \in R'$ s.t. $(\mathcal{M}', s_1') \models \mathcal{F}_V(\text{Tr}_c(s_1))$ (2)
(3) $\text{Tr}_c(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_c(s_1')$ ((2), Lemma 6)
(4) $L(s_1) - \overline{V} = L'(s_1') - \overline{V}$ ((3), $c \geq 0$)
(5) $(\mathcal{M}', s_1') \models \mathcal{F}_V(\text{Tr}_c(s_1)) \rightarrow \left(\bigwedge_{(s_1,s_2)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_2))\right) \wedge \text{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$ **(fact)**
(6) $(\mathcal{M}', s_1') \models \left(\bigwedge_{(s_1,s_2)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_2))\right) \wedge \text{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$ ((2), (5))
(7) ......
(8) $(\mathcal{M}', s_k') \models \left(\bigwedge_{(s_k,s_{k+1})\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_{k+1}))\right) \wedge \text{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\text{Tr}_c(s_{k+1}))\right)$ (similar with (6))
(9) $\forall (s_k, s_{k+1}) \in R$, $\exists (s_k', s_{k+1}') \in R'$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\text{Tr}_c(s_{k+1}))$ (8)
(10) $\text{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \text{Tr}_c(s_{k+1}')$ ((9), Lemma 6)
(11) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((10), $c \geq 0$)

(c) We will show that for each $(s_k', s_{k+1}') \in R'$ there is a $(s_k, s_{k+1}) \in R$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.
(1) $(\mathcal{M}', s_k') \models \text{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\text{Tr}_c(s_{k+1}))\right)$ (by (8) talked above)
(2) $\forall (s_k', s_{k+1}') \in R'$, $\exists (s_k, s_{k+1}) \in R$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\text{Tr}_c(s_{k+1}'))$ (1)
(3) $\text{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \text{Tr}_c(s_{k+1}')$ ((2), Lemma 6)
(4) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((3), $c \geq 0$)

(ii) This is following Lemma 2 and the definition of the characterizing formula of initial K-structure $\mathcal{K}$ on V.

□

Lemma 3 Let φ be a formula. We have

$$\varphi \equiv \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \quad (3)$$

Proof. Let $(\mathcal{M}', s_0')$ be a model of φ. Then $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$ due to $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}', s_0')$. On the other hand, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. Then there is a $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. And then $(\mathcal{M}, s_0) \leftrightarrow_{\emptyset} (\mathcal{M}', s_0')$ by Theorem 2. Therefore, $(\mathcal{M}, s_0)$ is also a model of φ by Theorem 1. □

Theorem 3 **(Representation theorem)** Let φ, φ' and ϕ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

(i) $\varphi' \equiv \text{F}_{\text{CTL}}(\varphi, V)$,
(ii) $\varphi' \equiv \{\phi | \varphi \models \phi \text{ and IR}(\phi, V)\}$,
(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$\begin{aligned} &Mod(\text{F}_{\text{CTL}}(\varphi, V)) = Mod(\{\phi | \varphi \models \phi, \text{IR}(\phi, V)\}) \\ &= Mod(\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)). \end{aligned}$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\text{F}_{\text{CTL}}(\varphi, V)$. Then there exists an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of φ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all ϕ that $\varphi \models \phi$ and IR(ϕ, V). Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi | \varphi \models \phi, \text{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s_0')$ is a models of $\{\phi | \varphi \models \phi, \text{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s_0')$

$\models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$ due to $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$ is irrelevant to V.

Finally, suppose that $(\mathcal{M}',s_0')$ is a model of $\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$. Then there exists $(\mathcal{M},s_0) \in Mod(\varphi)$ such that $(\mathcal{M}',s_0') \models \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$. Hence, $(\mathcal{M},s_0) \leftrightarrow_V (\mathcal{M}',s_0')$ by Theorem 2. Thus $(\mathcal{M}',s_0')$ is also a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. Suppose that all postulates hold. By Positive Persistence, we have $\varphi' \models \{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\}$. Now we show that $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\} \models \varphi'$. Otherwise, there exists formula ϕ' such that $\varphi' \models \phi'$ but $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\} \not\models \phi'$. There are three cases:

- ϕ' is relevant to V. Thus, φ' is also relevant to V, a contradiction to Irrelevance.
- ϕ' is irrelevant to V and $\varphi \models \phi'$. This contradicts to our assumption.
- ϕ' is irrelevant to V and $\varphi \not\models \phi'$. By Negative Persistence, $\varphi' \not\models \phi'$, a contradiction.

Thus, φ' is equivalent to $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\}$. □

Lemma 4 Let φ and α be two CTL formulae and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then $\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $(\mathcal{M},s)$ of $\mathrm{F}_{\mathrm{CTL}}(\varphi',q)$ there is an initial K-structure $(\mathcal{M}',s')$ s.t. $(\mathcal{M},s) \leftrightarrow_{\{q\}} (\mathcal{M}',s')$ and $(\mathcal{M}',s') \models \varphi'$. It's apparent that $(\mathcal{M}',s') \models \varphi$, and then $(\mathcal{M},s) \models \varphi$ since $\mathrm{IR}(\varphi,\{q\})$ and $(\mathcal{M},s) \leftrightarrow_{\{q\}} (\mathcal{M}',s')$ by Theorem 1.

Let $(\mathcal{M},s) \in Mod(\varphi)$ with $\mathcal{M} = (S,R,L,s)$. We construct $(\mathcal{M}',s)$ with $\mathcal{M}' = (S,R,L',s)$ as follows:

$L' : S \to \mathcal{A}$ *and* $\forall s^* \in S, L'(s^*) = L(s^*)$ *if* $(\mathcal{M},s^*) \not\models \alpha$, *else* $L'(s^*) = L(s^*) \cup \{q\}$,

$L'(s) = L(s) \cup \{q\}$ *if* $(\mathcal{M},s) \models \alpha$, *and* $L'(s) = L(s)$ *otherwise*.

It is clear that $(\mathcal{M}',s) \models \varphi$, $(\mathcal{M}',s) \models q \leftrightarrow \alpha$ and $(\mathcal{M}',s) \leftrightarrow_{\{q\}} (\mathcal{M},s)$. Therefore $(\mathcal{M}',s) \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $(\mathcal{M}',s) \leftrightarrow_{\{q\}} (\mathcal{M},s)$. □

Proposition 4 Given a formula $\varphi \in$ CTL, V a set of atoms and p an atom such that $p \notin V$. Then,

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V).$$

Proof. Let $(\mathcal{M}_1,s_1)$ with $\mathcal{M}_1 = (S_1,R_1,L_1,s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,\{p\} \cup V)$. By the definition, there exists a model $(\mathcal{M},s)$ with $\mathcal{M} = (S,R,L,s)$ of φ, such that $(\mathcal{M}_1,s_1) \leftrightarrow_{\{p\}\cup V} (\mathcal{M},s)$. We construct an initial K-structure $(\mathcal{M}_2,s_2)$ with $\mathcal{M}_2 = (S_2,R_2,L_2,s_2)$ as follows:

(1) for s_2: let s_2 be the state such that:
  - $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
  - for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
  - for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:
  - (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\}\cup V} w_1$, let $w_2 \in S_2$ and
    - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
    - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
    - for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.
  - (ii) if $(w_1',w_1) \in R_1$, w_2 is constructed based on w_1 and $w_2' \in S_2$ is constructed based on w_1', then $(w_2',w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $(\mathcal{M},s) \leftrightarrow_{\{p\}} (\mathcal{M}_2,s_2)$ and $(\mathcal{M}_2,s_2) \leftrightarrow_V (\mathcal{M}_1,s_1)$. Thus, $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,p)$. And therefore $(\mathcal{M}_1,s_1) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V)$.

On the other hand, suppose that $(\mathcal{M}_1,s_1)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V)$, then there exists an initial K-structure $(\mathcal{M}_2,s_2)$ such that $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,p)$ and $(\mathcal{M}_2,s_2) \leftrightarrow_V (\mathcal{M}_1,s_1)$, and there exists $(\mathcal{M},s)$ such that $(\mathcal{M},s) \models \varphi$ and $(\mathcal{M},s) \leftrightarrow_{\{p\}} (\mathcal{M}_2,s_2)$. Therefore, $(\mathcal{M},s) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}_1,s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1,s_1) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,\{p\} \cup V)$. □

Proposition 5 Let φ, φ_i, ψ_i $(i = 1,2)$ be formulas and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$ is satisfiable iff φ is;

(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1,V) \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi_2,V)$;

(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1,V) \models \mathrm{F}_{\mathrm{CTL}}(\varphi_2,V)$;

(iv) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V) \equiv \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$;

(v) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \wedge \psi_2,V) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \wedge \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$;

Proof. (i) ($\Rightarrow$) Supposing $(\mathcal{M},s)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$, then there is a model $(\mathcal{M}',s')$ of φ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

($\Leftarrow$) Supposing $(\mathcal{M},s)$ is a model of φ, then there is an initial Kripke structure $(\mathcal{M}',s')$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$, and then $(\mathcal{M}',s') \models \mathrm{F}_{\mathrm{CTL}}(\varphi,V)$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

The (ii) and (iii) can be proved similarly.

(iv) ($\Rightarrow$) $\forall(\mathcal{M},s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V))$, $\exists(\mathcal{M}',s') \in Mod(\psi_1 \vee \psi_2)$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$ and $(\mathcal{M}',s') \models \psi_1$ or $(\mathcal{M}',s') \models \psi_2$
$\Rightarrow \exists(\mathcal{M}_1,s_1) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1,V))$ s.t. $(\mathcal{M}',s') \leftrightarrow_V (\mathcal{M}_1,s_1)$ or $\exists(\mathcal{M}_2,s_2) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_2,V))$ s.t. $(\mathcal{M}',s') \leftrightarrow_V (\mathcal{M}_2,s_2)$
$\Rightarrow (\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$ by Theorem 1.

($\Leftarrow$) $\forall(\mathcal{M},s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V))$
$\Rightarrow (\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V)$ or $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1,s_1)$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}_1,s_1)$ and $(\mathcal{M}_1,s_1) \models \psi_1$ or $(\mathcal{M}_1,s_1) \models \psi_2$
$\Rightarrow (\mathcal{M}_1,s_1) \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2,s_2)$ s.t. $(\mathcal{M}_1,s_1) \leftrightarrow_V (\mathcal{M}_2,s_2)$ and $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V)$
$\Rightarrow (\mathcal{M},s) \leftrightarrow_V (\mathcal{M}_2,s_2)$ and $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V)$.

The (v) can be proved as (iv). □

Proposition 6 **(Homogeneity)** Let $V \subseteq \mathcal{A}$ and $\phi \in$ CTL,

(i) $\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V) \equiv \text{AX}\mathrm{F}_{\text{CTL}}(\phi, V)$.

(ii) $\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V) \equiv \text{EX}\mathrm{F}_{\text{CTL}}(\phi, V)$.

(iii) $\mathrm{F}_{\text{CTL}}(\text{AF}\phi, V) \equiv \text{AF}\mathrm{F}_{\text{CTL}}(\phi, V)$.

(iv) $\mathrm{F}_{\text{CTL}}(\text{EF}\phi, V) \equiv \text{EF}\mathrm{F}_{\text{CTL}}(\phi, V)$.

Proof. Let $\mathcal{M} = (S, R, L, s_0)$ with initial state s_0 and $\mathcal{M}' = (S', R', L', s_0')$ with initial state s_0', then we call $\mathcal{M}', s_0'$ be a sub-structure of $\mathcal{M}, s_0$ if:

- $S' \subseteq S$ and $S' = \{s' | s'$ is reachable from $s_0'\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to 2^{\mathcal{A}}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- s_0' is s_0 or a state reachable from s_0.

(i) In order to prove $\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V) \equiv \text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V))$, we only need to prove $Mod(\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V)) = Mod(\text{AX}\mathrm{F}_{\text{CTL}}(\phi, V))$:

($\Rightarrow$) $\forall(\mathcal{M}', s') \in Mod(\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \text{AX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ for any sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ there is $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ with s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow$ $(\mathcal{M}_3, s_3) \models \text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V))$ and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow (\mathcal{M}', s') \models \text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V))$.

($\Leftarrow$) $\forall$ $(\mathcal{M}_3, s_3) \in Mod(\text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V)))$, then for any sub-structure $(\mathcal{M}_2, s_2)$ with s_2 is a directed successor of s_3 there is $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$
$\Rightarrow$ for any $(\mathcal{M}_2, s_2)$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ with s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \text{AX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\text{CTL}}(\text{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V) \equiv \text{EX}\mathrm{F}_{\text{CTL}}(\phi, V)$, we only need to prove Mod $(\mathrm{F}_{\text{CTL}}(\text{EX}\phi, \ V)) = Mod(\text{EX}\mathrm{F}_{\text{CTL}}(\phi, V))$:

($\Rightarrow$) $\forall \mathcal{M}', s' \in Mod(\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \text{EX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ there is a sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ that s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow (\mathcal{M}_3, s_3) \models \text{EX}(\mathrm{F}_{\text{CTL}}(\phi, V))$
$\Rightarrow (\mathcal{M}', s') \models \text{EX}(\mathrm{F}_{\text{CTL}}(\phi, V))$.

($\Leftarrow$) $\forall$ $(\mathcal{M}_3, s_3) \in Mod(\text{EX}(\mathrm{F}_{\text{CTL}}(\phi, V)))$, then there exists a sub-structure $(\mathcal{M}_2, s_2)$ of $(\mathcal{M}_3, s_3)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ that s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \text{EX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\text{CTL}}(\text{EX}\phi, V)$.

(iii) and (iV) can be proved as (i) and (ii) respectively. $\square$

Proposition7 **(Model Checking on Forgetting)** Let $(\mathcal{M}, s_0)$ be an initial K-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{\text{CTL}}(\varphi, V)$ is NP-complete.

Proof. The problem can be determined by the following two things: (1) guessing an initial K-structure $(\mathcal{M}', s_0')$ satisfying φ; and (2) checking if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Both two steps can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard (Zhang and Zhou 2008). $\square$

Theorem 5 **(Entailment on Forgetting)** Let φ and ψ be two CTL_{AF} formulas and V a set of atoms. Then, results:

(i) deciding $\mathrm{F}_{\text{CTL}}(\varphi, V) \models^? \psi$ is co-NP-complete,

(ii) deciding $\psi \models^? \mathrm{F}_{\text{CTL}}(\varphi, V)$ is Π_2^P-complete,

(iii) deciding $\mathrm{F}_{\text{CTL}}(\varphi, V) \models^? \mathrm{F}_{\text{CTL}}(\psi, V)$ is Π_2^P-complete.

Proof. (1) It is proved that deciding whether ψ is satisfiable is NP-Complete (Meier et al. 2015). The hardness is easy to see by setting $\mathrm{F}_{\text{CTL}}(\varphi, Var(\varphi)) \equiv \top$, i.e., deciding whether ψ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{\text{CTL}}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $IR(\psi, V)$. Clearly, in CTL_{AF}, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $IR(\psi, V)$ is also in co-NP. Without loss of generality, we assume that ψ is satisfiable. We consider the complement of the problem: deciding whether ψ is not irrelevant to V. It is easy to see that ψ is not irrelevant to V iff there exist a model $(\mathcal{M}, s_0)$ of ψ and an initial K-structure $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ and $(\mathcal{M}', s_0') \not\models \psi$. So checking whether ψ is not irrelevant to V can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s_0')$, (2) check if $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s_0') \not\models \psi$, and (3) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Obviously (1) can be done in polynomial time and also (2) and (3) can be done in polynomial time.

(2) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ and check whether $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}, s_0) \not\models \mathrm{F}_{\text{CTL}}(\ \varphi, V)$. From Proposition 7, we know that this is in Σ_2^P. So the original problem is in Π_2^P. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F}_{\text{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case temporal forgetting, the hardness is directly followed from the proof of Proposition 24 in (Lang, Liberatore, and Marquis 2003).

(3) Membership. If $\mathrm{F}_{\text{CTL}}(\varphi, V) \not\models \mathrm{F}_{\text{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F}_{\text{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F}_{\text{CTL}}(\psi, V)$, i.e.,, there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models$

ψ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M}, s)$, $(\mathcal{M}_1, s_1)$ with $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ and checking $(\mathcal{M}_1, s_1) \models \varphi$ are feasible while checking $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ can be done in polynomial time. Thus the problem is in Π_2^P.

Hardness. It follows from (2) due to the fact that $\mathrm{F_{CTL}}(\varphi, V) \models \mathrm{F_{CTL}}(\psi, V)$ iff $\varphi \models \mathrm{F_{CTL}}(\psi, V)$ thanks to $IR(\mathrm{F_{CTL}}(\psi, V), V)$. $\square$

Proposition 8 **(dual)** Let V, q, φ and ψ are like in Definition 4. The ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \rightarrow \psi$. Thus $\varphi \models \neg\psi \rightarrow \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \rightarrow \neg q$. Then $\varphi \models q \rightarrow \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \rightarrow \neg\psi'$ by assumption. So $\varphi \models \psi' \rightarrow \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case. $\square$

Proposition 9 Let Γ and α be two formulas, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q is a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

Proof. We prove this for SNC. The case for WSC is similar. Let $SNC(\varphi, \alpha, V, \Gamma)$ denote that φ is the SNC of α on V under Γ, and $NC(\varphi, \alpha, V, \Gamma)$ denote that φ is the NC of α on V under Γ.

($\Rightarrow$) We will show that if $SNC(\varphi, \alpha, V, \Gamma)$ holds, then $SNC(\varphi, q, V, \Gamma')$ will be true. According to $SNC(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \rightarrow \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F_{CTL}}(\Gamma', q) \models \alpha \rightarrow \varphi'$ due to $\alpha \equiv q$, $IR(\alpha \rightarrow \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi'$ by Lemma 4, this means $NC(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \rightarrow \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \rightarrow \varphi'$. Hence, $SNC(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $SNC(\varphi, q, V, \Gamma')$ holds, then $SNC(\varphi, \alpha, V, \Gamma)$ will be true. According to $SNC(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F_{CTL}}(\Gamma', \{q\}) \models \alpha \rightarrow \varphi$ due to $\alpha \equiv q$, $IR(\alpha \rightarrow \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi$ by Lemma 4, this means $NC(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \rightarrow \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $NC(\varphi', q, V, \Gamma')$. According to $SNC(\varphi, q, V, \Gamma')$, $IR(\varphi \rightarrow \varphi', \{q\})$ and (**PP**), we have $\mathrm{F_{CTL}}(\Gamma', \{q\}) \models \varphi \rightarrow \varphi'$, and $\Gamma \models \varphi \rightarrow \varphi'$ by Lemma 4. Hence, $SNC(\varphi, \alpha, V, \Gamma)$ holds. $\square$

Theorem 7 Let φ be a formula, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.

(i) $\mathrm{F_{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ.

(ii) $\neg\mathrm{F_{CTL}}(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ.

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F_{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \rightarrow \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \rightarrow \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \rightarrow \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $IR(\psi, (Var(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ. $\square$

Theorem 8 Let $\mathcal{K} = (\mathcal{M}, s)$ be an initial K-structure with $\mathcal{M} = (S, R, L, s_0)$ on the set $\mathcal{A}$ of atoms, $V \subseteq \mathcal{A}$ and $q \in V' = \mathcal{A} - V$. Then:

(i) the SNC of q on V under $\mathcal{K}$ is $\mathrm{F_{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q on V under $\mathcal{K}$ is $\neg\mathrm{F_{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge \neg q, V')$.

Proof. (i) As we know that any initial K-structure $\mathcal{K}$ can be described as a characterizing formula $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$, then the SNC of q on V under $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$ is $\mathrm{F_{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, \mathcal{A} - V)$.

(ii) This is proved by the dual property. $\square$

Proposition10 Let φ be a CTL formula and $V \subseteq \mathcal{A}$ with $|\mathcal{S}| = m$, $|\mathcal{A}| = n$ and $|V| = x$. The the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity of Algorithm 1 is at least the same as the space.

Proof. Supposing each state or atom occupy one byte, then a state pair (s, s') occupy two bytes. For any $B \subseteq \mathcal{S}$ with $B \neq \emptyset$ and $s_0 \in B$, we can construct an initial K-structure $(\mathcal{M}, s_0)$ with $\mathcal{M} = (B, R, L, s_0)$, in which there is at most $\frac{|B|^2}{2}$ state pairs in R and $|B| * n$ (s, A) $(A \subseteq \mathcal{A})$ in L. Hence, the $(\mathcal{M}, s_0)$ occupy at most $|B| + |B|^2 + |B| * n$ bytes. Besides, there is at most $|B|^{|B|+1} * 2^{nm}$ number of initial K-structures. Therefore, there is at most $m^{m+2} * 2^{nm}$ number of initial K-structures, hence it will at most cost $m^{m+2} * 2^{nm} * (m + m^2 + nm)$ bytes.

Let $k = n - x$, for any initial K-structure $\mathcal{K} = (\mathcal{M}, s_0)$ with $i \geq 1$ nodes, in the worst, i.e., $ch(\mathcal{M}, V) = i$, we will spend $N(i)$ space to store the characterizing formula.

$$\begin{aligned} N(i) &= (k + (\cdots + (k + 2ik) * (2i)) \cdots * (2i)) \\ &= (2i)^0 k + 2ik + (2i)^2 k + \cdots + (2i)^{(i-1)} k \\ &= \frac{(2i)^i - 1}{2i - 1} k. \end{aligned}$$

In the worst case, i.e., there is $m^{m+2} * 2^{nm}$ initial K-structures with m nodes, we will spent $m^{m+2} * 2^{nm} * N(m)$ bytes to store the result of forgetting.

Therefore, the space complexity is $O((n - x)m^{2(m+1)}2^{nm})$ and the time complexity is at least the same as the space. $\square$

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