Forgetting in CTL to Compute Necessary and Sufficient Conditions

Abstract

Computation Tree Logic (CTL) is one of the central formalisms in formal verification. As a specification language, it is used to express a property that the system at hand is expected to satisfy. From both the verification and the system design points of view, some information content of such property might become irrelevant for the system due to various reasons e.g., it might become obsolete by time, or perhaps infeasible due to practical difficulties. Then, the problems arises on how to subtract such piece of information without altering the relevant system behaviour or violating the existing specifications. Moreover, in such a scenario, two crucial notions are informative: the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC) of a given property.

To address such a scenario in a principled way, we introduce a *forgetting*-based approach in CTL and show that it can be used to compute SNC and WSC of a property under a given model. We study its theoretical properties and also show that our notion of forgetting satisfies existing essential postulates. Furthermore, we analyse the computational complexity of basic tasks, including various results for the relevant fragment CTL_{AF}.

1 Introduction

Weakest precondition, we also call *weakest sufficient condition* (WSC), is introduced by Dijkstra in [Dijkstra, 1978]. *Strongest postcondition* (we also call *strongest necessary condition* (SNC)), a dual concept, was introduced subsequently. WSC was widely used in program *verification*, especially in generating counterexamples [**?**] and refinement of system [**?**]. *Computation Tree Logic* (CTL) [Clarke and Emerson, 1981] Model modification, which has been developed in [**?**; **?**; **?**], is an extension of refinement of system. This paper explore a method to compute the WSC of a property (a CTL formula) under a given model system that may be modified for guiding CTL Model modification. It is known that the computing of WSC for code fragment S with respect to assertion Q requires S must terminate [**?**] due to it just concerns relation among input values and output values. However, in the case of model checking, it concerns properties about execution runs, which may not be terminate. It can be shown by the following example.

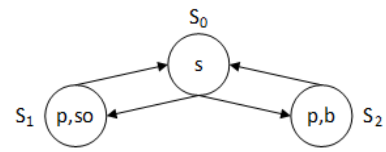

Figure 1: A Beverage Vending Machine

Example 1 A Beverage Vending Machine can be described as a Kripke structure $\mathcal{M} = (S, R, L, s_0)$ in Figure 1 on $V_a = \{select, pay, beer, soda\}$. Which means that when we in s_0 if we select soda and pay for it then we change to the s_1, else if we select beer and pay for it then we change to the s_2, after takeing out the drink we transform to s_0. This is somewhat different from that in [Baier and Katoen, 2008] for simply. For convenience, we use s for $select$, p for pay, b for $beer$, so for $soda$ and r for orange juice. Let $\varphi = \text{AGAF}(p \wedge r)$, which means $p \wedge r$ will be satisfied infinite times in the structure, be a CTL formula.

We can decide $(\mathcal{M}, s_0) \not\models \varphi$ easily due to this structure do not contain the atom r. In order for $(\mathcal{M}, s_0)$ satisfy φ, we should find a condition ψ such that $(\mathcal{M}, s_0) \models \psi \supset \varphi$. As we know that if this condition exists, there are many conditions that satisfy the need. In this case, if we are clever enough to judge in advance the set of possible atomic propositions that make up the condition, then we can find this condition in the set only, and the smaller the set, the easier it is to work out the condition. In this paper, we always assume that the condition is a property defined on the specified atomic proposition set V, for our example $V = \{p, r\}$, and find the weakest property (that is, the weakest sufficient condition) satisfying the condition on the set. Finding this property is called discovering theorem by Lin in [**?**]. Inspired by the forgetting-based method to compute SNC (WSC) [Lin, 2001], in this paper, we tackle this problem by proposing a semantic forgetting for CTL.

However, as we have said that $\mathcal{M}$ is a Kripke structure, which needs to be converted into a logical formula (theory), that is the characteristic formula, which is a CTL formula proposed in [Browne *et al.*, 1988]. Thanks to we find the

WSC in a set V of atoms, hence a set-based bisimulation between two K-structures (a Kripke structure with a state in it), V-bisimulation, and characteristic formula on V will be proposed in this paper. Our V-bisimulation is a more general bisimulation relation than others. On the one hand, the above set-based bisimulation is an extension of the bisimulation-equivalence of Definition 7.1 in [Baier and Katoen, 2008] in the sense that if $V = \mathcal{A}$ then our bisimulation is almost same to the latter. On the other hand, the above set-based bisimulation notion is similar to the state equivalence in [Browne *et al.*, 1988]. But it is different in the sense that ours is defined on K-structures, while it is defined on states in [Browne *et al.*, 1988]. What's more, the set-based bisimulation notion is also different from the state-based bisimulation notion of Definition 7.7 in [Baier and Katoen, 2008], which is defined for states of a given K-structure.

As a logical notion, *forgetting* was first formally defined in propostional and first order logics by Lin and Reiter [Lin and Reiter, 1994]. Over the last twenty years, researchers have developed forgetting notions and theories not only in classical logic but also in other non-classical logic systems [**?**], such as forgetting in logic programs under answer set/stable model semantics [Zhang and Foo, 2006; Eiter and Wang, 2008; Wong, 2009; Wang *et al.*, 2012; Wang *et al.*, 2013], forgetting in description logic [Wang *et al.*, 2010; Lutz and Wolter, 2011; Zhao and Schmidt, 2017] and knowledge forgetting in modal logic [Zhang and Zhou, 2009; Su *et al.*, 2009; Liu and Wen, 2011; Fang *et al.*, 2019]. In application, forgetting has been used in planning [Lin, 2003], conflict solving [Lang and Marquis, 2010; Zhang *et al.*, 2005], createing restricted views of ontologies [Zhao and Schmidt, 2017], strongest and weakest definitions [Lang and Marquis, 2008], SNC (WSC) [Lin, 2001] and so on.

Though forgetting has been extensively investigated from various aspects of different logical systems. However, the existing forgetting method in propositional logic, answer set programming, description logic and modal logic are not directly applicable in CTL. For instance, in propositional forgetting theory, forgetting atom q from φ is equivalent to a formula $\varphi[q/\top] \vee \varphi[q/\perp]$, where $\varphi[q/X]$ is a formula obtained from φ by replacing each q with X ($X \in \{\top, \perp\}$). However, this method cannot be extended to a CTL formula. Consider a CTL formula $\psi = \mathrm{AG}p \wedge \neg \mathrm{AG}q \wedge \neg \mathrm{AG}\neg q$. If we want to forget atom q from ψ by using the above method, we would have $\psi[q/\top] \vee \psi[q/\perp] \equiv \perp$. This is obviously not correct because after forgetting q this specification should not become inconsistent. Similar with that in [Zhang and Zhou, 2009], we research forgetting in CTL from the semantic forgetting point of view. And it is shown that our definition of forgetting satisfies those four postulates of forgetting.

The rest of the paper is organised as follows. Section 2 introduces the related notions for forgetting in CTL, including the syntax and semantics of CTL, the language we aimed for. A formal definition of concept forgetting and its properties for CTL follows in Section 3. Section 4 explores the relation between forgetting and SNC (WSC). From the point of view of model, we propose an algorithm for computing forgetting on CTL in Section 5. Finally, we conclude this paper.

2 Preliminaries

We start with some technical and notational preliminaries. Throughout this paper, we fix a finite set $\mathcal{A}$ of propositional variables (or atoms), and use V, V' for subsets of $\mathcal{A}$. In this part, we will introduce the structure we will use for CTL and syntax and semantic of CTL.

2.1 Model structure in CTL

In general, a transition system [1] is described as a *model structure* (or *Kripke structure*), and a model structure is a triple $\mathcal{M} = (S, R, L)$, where

- S is a finite nonempty set of states,
- $R \subseteq S \times S$ and, for each $s \in S$, there is $s' \in S$ such that $(s, s') \in R$,
- L is a labeling function $S \to 2^{\mathcal{A}}$.

We call a model structure $\mathcal{M}$ on a set V of atoms if $L : S \to 2^V$, *i.e.* , the labeling function L map every state to V (not the $\mathcal{A}$). A *path* π_{s_i} start from s_i of $\mathcal{M}$ is a infinite sequence of states $\pi_{s_i} = (s_i, s_{i+1}s_{i+2}, \dots)$, where for each j ($0 \leq i \leq j$), $(s_j, s_{j+1}) \in R$. By $s' \in \pi_{s_i}$ we mean that s' is a state in the path π_{s_i}. A sate $s \in S$ is *initial* if for any state $s' \in S$, there is a path π_s s.t $s' \in \pi_s$. We denote this model structure as (S, R, L, s_0), where s_0 is initial.

For a given model structure (S, R, L, s_0) and $s \in S$, the *computation tree* $\mathrm{Tr}_n^{\mathcal{M}}(s)$ of $\mathcal{M}$(or simply $\mathrm{Tr}_n(s)$), that has depth n and is rooted at s, is recursively defined as [Browne *et al.*, 1988], for $n \geq 0$,

- $\mathrm{Tr}_0(s)$ consists of a single node s with label s.
- $\mathrm{Tr}_{n+1}(s)$ has as its root a node m with label s, and if $(s, s') \in R$ then the node m has a subtree $\mathrm{Tr}_n(s')$.

By s_n we mean a nth level node of tree $\mathrm{Tr}_m(s)$ ($m \geq n$).

A K-*structure* (or K-*interpretation*) is a model structure $\mathcal{M} = (S, R, L, s_0)$ associating with a state $s \in S$, which is written as $(\mathcal{M}, s)$ for convenience in the following. In the case $s = s_0$ is an initial state of $\mathcal{M}$, the K-structure is *initial*.

2.2 Syntax and semantics of CTL

In the following we briefly review the basic syntax and semantics of the CTL [Clarke *et al.*, 1986]. The *signature* of the language $\mathcal{L}$ of CTL includes:

- a finite set of Boolean variables, called *atoms* of $\mathcal{L}$: $\mathcal{A}$;
- constant symbols $\perp$ and $\top$
- the classical connectives: $\vee$ and $\neg$;
- the path quantifiers: A and E;
- the temporal operators: X, F, G U and W, that means 'neXt state', 'some Future state', 'all future states (Globally)', 'Until' and 'Unless', respectively;
- parentheses: (and).

[1] According to [Baier and Katoen, 2008], a *transition system* TS is a tuple $(S, Act, \to, I, AP, L)$ where (1) S is a set of states, (2) Act is a set of actions, (3) $\to \subseteq S \times \mathrm{Act} \times S$ is a transition relation, (4) $I \subseteq S$ is a set of initial states, (5) AP is a set of atomic propositions, and (6) $L : S \to 2^{\mathrm{AP}}$ is a labeling function.

The *(existential normal form or ENF in short) formulas* of $\mathcal{L}$ are inductively defined via a Backus Naur form:

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi \vee \phi \mid \mathrm{EX}\phi \mid \mathrm{EG}\phi \mid \mathrm{E}[\phi \,\mathrm{U}\, \phi] \quad (1)$$

where $p \in \mathcal{A}$. The formulas $\phi \wedge \psi$ and $\phi \to \psi$ are defined in a standard manner of propositional logic. The other form formulas of $\mathcal{L}$ are abbreviated using the forms of (1). The priorities for the CTL connectives are assumed to be (from the highest to the lowest):

$$\neg, \mathrm{EX}, \mathrm{EF}, \mathrm{EG}, \mathrm{AX}, \mathrm{AF}, \mathrm{AG} \prec \wedge \prec \vee \prec \mathrm{EU}, \mathrm{AU}, \mathrm{EW}, \mathrm{AW}, \to .$$

We are now in the position to define the semantics of $\mathcal{L}$. Let $\mathcal{M} = (S, R, L, s_0)$ be a model structure, $s \in S$ and ϕ a formula of $\mathcal{L}$. The *satisfiability* relationship between $(\mathcal{M}, s)$ and ϕ, written $(\mathcal{M}, s) \models \phi$, is inductively defined on the structure of ϕ as follows:

- $(\mathcal{M}, s) \not\models \bot$ and $(\mathcal{M}, s) \models \top$;
- $(\mathcal{M}, s) \models p$ iff $p \in L(s)$;
- $(\mathcal{M}, s) \models \phi_1 \vee \phi_2$ iff $(\mathcal{M}, s) \models \phi_1$ or $(\mathcal{M}, s) \models \phi_2$;
- $(\mathcal{M}, s) \models \neg\phi$ iff $(\mathcal{M}, s) \not\models \phi$;
- $(\mathcal{M}, s) \models \mathrm{EX}\phi$ iff $(\mathcal{M}, s_1) \models \phi$ for some $s_1 \in S$ and $(s, s_1) \in R$;
- $(\mathcal{M}, s) \models \mathrm{EG}\phi$ iff $\mathcal{M}$ has a path $(s_1 = s, s_2, \ldots)$ such that $(\mathcal{M}, s_i) \models \phi$ for each $i \geq 1$;
- $(\mathcal{M}, s) \models \mathrm{E}[\phi_1 \mathrm{U} \phi_2]$ iff $\mathcal{M}$ has a path $(s_1 = s, s_2, \ldots)$ such that, for some $i \geq 1$, $(\mathcal{M}, s_i) \models \phi_2$ and $(\mathcal{M}, s_j) \models \phi_1$ for each $1 \leq j < i$.

Similar to the work in [Browne *et al.*, 1988; Bolotov, 1999], only initial K-structures are considered to be candidate models in the following, unless explicitly stated. Formally, an initial K-structure $\mathcal{K}$ is a *model* of a formula ϕ whenever $\mathcal{K} \models \phi$. We denote *Mod*(ϕ) the set of models of ϕ. The formula ϕ is *satisfiable* if *Mod*$(\phi) \neq \emptyset$. Since the states in model structure is finite, *Mod*(ϕ) is finite for any formula ϕ.

Let ϕ_1 and ϕ_2 be two formulas. By $\phi_1 \models \phi_2$ we denote *Mod*$(\phi_1) \subseteq$ *Mod*(ϕ_2). By $\phi_1 \equiv \phi_2$ we mean $\phi_1 \models \phi_2$ and $\phi_2 \models \phi_1$. In this case ϕ_1 is *equivalent* to ϕ_2. By *Var*(ϕ_1) we mean the set of atoms occurring in ϕ_1. ϕ_1 is V*-irrelevant*, written IR(ϕ_1, V), if there is a formula ψ with *Var*$(\psi) \cap V = \emptyset$ such that $\phi_1 \equiv \psi$.

3 Forgetting in CTL

In this section, we will define the forgetting in CTL by V-bisimulation, set-based bisimulations. Besides, some properties of forgetting are also explored. For convenience, let $\mathcal{M} = (S, R, L, s_0)$, $\mathcal{M}' = (S', R', L', s_0')$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ with $\mathcal{M}_i = (S_i, R_i, L_i, s_0^i)$, $s_i \in S_i$ and i is an integer.

3.1 Set-based bisimulation

To present a formal definition of forgetting, we need the concept of V-bisimulation. Inspired by the notion of bisimulation in [Browne *et al.*, 1988], we define the relations $\mathcal{B}_0, \mathcal{B}_1, \ldots$ between K-structures on V as follows: let $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ with $i \in \{1, 2\}$,

- $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0$ if $L_1(s_1) \setminus V = L_2(s_2) \setminus V$;
- for $n \geq 0$, $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}$ if
  - $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0$,
  - for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_n$, and
  - for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_n$,

  where $\mathcal{K}_i' = (\mathcal{M}_i, s_i')$ with $i \in \{1, 2\}$.

Now, we define the notion of V-bisimulation between K-structures:

Definition 1 (V-bisimulation) *Let $V \subseteq \mathcal{A}$. Given two* K*-structures $\mathcal{K}_1$ and $\mathcal{K}_2$ are V-*bisimilar, *denoted $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ if and only if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$. Moreover, two paths $\pi_i = (s_{i,1}, s_{i,2}, \ldots)$ of $\mathcal{M}_i$ with $i \in \{1, 2\}$ are V-*bisimilar *if $\mathcal{K}_{1,j} \leftrightarrow_V \mathcal{K}_{2,j}$ for every $j \geq 0$ where $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$.*

It's apparent that $\leftrightarrow_V$ is a binary relation. In the sequel, we abbreviate $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ by $s_1 \leftrightarrow_V s_2$ whenever the underlying model structures of states s_1 and s_2 are clear from the context.

Lemma 1 *The relation $\leftrightarrow_V$ is an equivalence relation.*

Besides, we have the following properties:

Proposition 1 *Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s_i's be two states and π_i's be two pathes, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be* K*-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:*

(i) $s_1' \leftrightarrow_{V_i} s_2'$ $(i = 1, 2)$ implies $s_1' \leftrightarrow_{V_1 \cup V_2} s_2'$;

(ii) $\pi_1' \leftrightarrow_{V_i} \pi_2'$ $(i = 1, 2)$ implies $\pi_1' \leftrightarrow_{V_1 \cup V_2} \pi_2'$;

(iii) for each path π_{s_1} of $\mathcal{M}_1$ there is a path π_{s_2} of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Intuitively, if two K-structures are V-bisimilar, then they satisfy the same formula φ that dose not contain any atoms in V, *i.e.* IR(φ, V).

Theorem 1 *Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two* K*-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.*

Let $V \subseteq \mathcal{A}$, $\mathcal{M}_i$ $(i = 1, 2)$ be model structures. A computation tree $\mathrm{Tr}_n(s_1)$ of $\mathcal{M}_1$ is V*-bisimilar* to a computation tree $\mathrm{Tr}_n(s_2)$ of $\mathcal{M}_2$, written $(\mathcal{M}_1, \mathrm{Tr}_n(s_1)) \leftrightarrow_V (\mathcal{M}_2, \mathrm{Tr}_n(s_2))$ (or simply $\mathrm{Tr}_n(s_1) \leftrightarrow_V \mathrm{Tr}_n(s_2)$), if

- $L_1(s_1) \setminus V = L_2(s_2) \setminus V$,
- for every subtree $\mathrm{Tr}_{n-1}(s_1')$ of $\mathrm{Tr}_n(s_1)$, $\mathrm{Tr}_n(s_2)$ has a subtree $\mathrm{Tr}_{n-1}(s_2')$ such that $\mathrm{Tr}_{n-1}(s_1') \leftrightarrow_V \mathrm{Tr}_{n-1}(s_2')$, and

Please note that the last condition in the above definition hold trivially for $n = 0$.

Proposition 2 *Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two* K*-structures. Then*

$$(s_1, s_2) \in \mathcal{B}_n \textit{ iff } Tr_j(s_1) \leftrightarrow_V Tr_j(s_2) \textit{ for every } 0 \leq j \leq n.$$

This means that $\mathrm{Tr}_j(s_1) \leftrightarrow_V \mathrm{Tr}_j(s_2)$ for all $j \geq 0$ if $s_1 \leftrightarrow_V s_2$, otherwise there is some k such that $\mathrm{Tr}_k(s_1)$ and $\mathrm{Tr}_k(s_2)$ are not V-bisimilar.

Proposition 3 *Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least k such that $Tr_k(s)$ and $Tr_k(s')$ are not V-bisimilar.*

In this case the model structure $\mathcal{M}$ is called V-*distinguishable* (by states s and s' at the least depth k), which is denoted by $\text{dis}_V(\mathcal{M}, s, s', k)$. It is evident that $\text{dis}_V(\mathcal{M}, s, s', k)$ implies $\text{dis}_V(\mathcal{M}, s, s', k')$ whenever $k' \geq k$. The V-*characterization number* of $\mathcal{M}$, written $ch(\mathcal{M}, V)$, is defined as

$$ch(\mathcal{M}, V) = \begin{cases} \max\{k \mid s, s' \in S \;\&\; \text{dis}_V(\mathcal{M}, s, s', k)\}, & \\ \qquad\qquad \mathcal{M} \text{ is } V\text{-distinguishable;} & \\ \min\{k \mid \mathcal{B}_k = \mathcal{B}_{k+1}\}, & \text{otherwise.} \end{cases}$$

3.2 Characterization of Initial K-structure

In order to introduce our notion of forgetting, and to compute strongest necessary and weakest sufficient conditions, we need a formula that captures the initial K-structure on V syntactically. We call such formula as characterizing formula. In the following, we present such characterization.

Given a set $V \subseteq \mathcal{A}$, we define a formula φ of V (that is $Var(\varphi) \subseteq V$) in CTL that describes a computation tree.

Definition 2 *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ be a model structure and $s \in S$. The* characterizing formula *of the computation tree $Tr_n(s)$ on V, written $\mathcal{F}_V(Tr_n(s))$, is defined recursively as:*

$$\mathcal{F}_V(Tr_0(s)) = \bigwedge_{p \in V \cap L(s)} p \wedge \bigwedge_{q \in V - L(s)} \neg q,$$

$$\mathcal{F}_V(Tr_{k+1}(s)) = \bigwedge_{(s,s') \in R} \text{EX}\mathcal{F}_V(Tr_k(s'))$$

$$\wedge\, \text{AX}\left(\bigvee_{(s,s') \in R} \mathcal{F}_V(Tr_k(s'))\right) \wedge \mathcal{F}_V(Tr_0(s))$$

for $k \geq 0$.

The characterizing formula of a computation tree formally exhibit the context of each node on V (atoms are true at this node if they are in V, else false) and the temporal relation between states recursively. In this way, we know:

Lemma 2 *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s_0')$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$, then $\mathcal{F}_V(Tr_n(s)) \equiv \mathcal{F}_V(Tr_n(s'))$.*

Let $s' = s$, it shows that for any formula φ of V, if φ is a characterizing formula of $\text{Tr}_n(s)$ then $\varphi \equiv \mathcal{F}_V(\text{Tr}_n(s))$.

Let $V \subseteq \mathcal{A}$, $\mathcal{K} = (\mathcal{M}, s_0)$ be an initial K-structure and $T(s') = \mathcal{F}_V(\text{Tr}_c(s'))$. The *characterizing formula* of $\mathcal{K}$ on V, written $\mathcal{F}_V(\mathcal{M}, s_0)$ (or $\mathcal{F}_V(\mathcal{K})$), is defined as the conjunction of the following formulas:

$\mathcal{F}_V(\text{Tr}_c(s_0))$, and

$$\bigwedge_{s \in S} \text{AG}\left(\mathcal{F}_V(\text{Tr}_c(s)) \rightarrow \bigwedge_{(s,s') \in R} \text{EX}T(s') \wedge \text{AX} \bigvee_{(s,s') \in R} T(s')\right)$$

where $c = ch(\mathcal{M}, V)$. It is apparent that $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$.

The following example show how to compute characterizing formula:

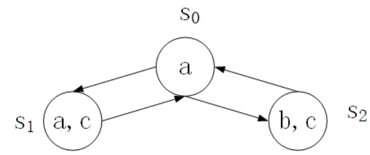

Figure 2: A simple Kripke structure

Example 2 Let $\mathcal{K} = (\mathcal{M}, s_0)$ in Figure 2 be an initial K-structure and $V = \{a, b\}$, then compute the characterizing formula of $\mathcal{K}$ on V.

It is apparent that $\text{Tr}_0(s_0) \leftrightarrow_{\overline{V}} \text{Tr}_0(s_1)$ due to $L(s_0) \setminus \overline{V} = L(s_1) \setminus \overline{V}$, $\text{Tr}_1(s_0) \not\leftrightarrow_{\overline{V}} \text{Tr}_1(s_1)$ due to there is $(s_0, s_2) \in R$ such that for any $(s_1, s') \in R$ (there is only one immediate successor $s' = s_0$) there is $L(s_2) \setminus \overline{V} \neq L(s') \setminus \overline{V}$. Hence, we have that $\mathcal{M}$ is $\overline{V}$-distinguished by state s_0 and s_1 at the least depth 1, *i.e.* $\text{dis}_{\overline{V}}(\mathcal{M}, s_0, s_1, 1)$. Similarly, we have $\text{dis}_{\overline{V}}(\mathcal{M}, s_0, s_2, 0)$ and $\text{dis}_{\overline{V}}(\mathcal{M}, s_1, s_2, 0)$. Therefore, $ch(\mathcal{M}, \overline{V}) = \max\{k \mid s, s' \in S \;\&\; \text{dis}_{\overline{V}}(\mathcal{M}, s, s', k)\} = 1$. Then we have:

$$\mathcal{F}_V(\text{Tr}_0(s_0)) = a \wedge \neg b,$$
$$\mathcal{F}_V(\text{Tr}_0(s_1)) = a \wedge \neg b,$$
$$\mathcal{F}_V(\text{Tr}_0(s_2)) = b \wedge \neg a,$$
$$\mathcal{F}_V(\text{Tr}_1(s_0)) = \text{EX}(a \wedge \neg b) \wedge \text{EX}(b \wedge \neg a) \wedge \text{AX}((a \wedge \neg b) \vee (b \wedge \neg a)) \wedge (a \wedge \neg b),$$
$$\mathcal{F}_V(\text{Tr}_1(s_1)) = \text{EX}(a \wedge \neg b) \wedge \text{AX}(a \wedge \neg b) \wedge (a \wedge \neg b),$$
$$\mathcal{F}_V(\text{Tr}_1(s_2)) = \text{EX}(a \wedge \neg b) \wedge \text{AX}(a \wedge \neg b) \wedge (b \wedge \neg a).$$

Then it is easy to obtain $\mathcal{F}_V(\mathcal{M}, s_0)$.

Lemma 3 *Let φ be a formula. We have*

$$\varphi \equiv \bigvee_{(\mathcal{M}, s_0) \in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \tag{2}$$

It follows that any CTL formula can be described by the disjunction of the characterizing formulas of all the models of itself due to the number of models of a CTL formula is finite.

Theorem 2 *Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s_0')$ be two model structures. Then,*

(a) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$ *iff* $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s_0')$.

(b) $s_0 \leftrightarrow_{\overline{V}} s_0'$ *implies* $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s_0')$.

By the following theorem we also have that given a set $V \subseteq \mathcal{A}$, the characterizing formula of an initial K-structure is equivalent uniquely describe this initial K-structure on V.

3.3 Semantic Properties of Forgetting in CTL

In this subsection we will give the definition of forgetting in CTL and study it's semantic properties. We will first show via a representation theorem that our definition of forgetting correspond to the readily existing notion of forgetting which is characterised by several desirable properties (also called postulates) suggested in [Zhang and Zhou, 2009]. Next, we discuss various additional semantic properties of forgetting.

Now, we give the formal definition of forgetting in CTL from the semantic point view.

Definition 3 (Forgetting) *Let $V \subseteq \mathcal{A}$ and ϕ a formula. A formula ψ with $Var(\psi) \cap V = \emptyset$ is a* result of forgetting V from ϕ*, if*

$$Mod(\psi) = \{\mathcal{K} \text{ is initial} \mid \exists \mathcal{K}' \in Mod(\phi) \,\&\, \mathcal{K}' \leftrightarrow_V \mathcal{K}\}. \quad (3)$$

Note that if both ψ and ψ' are results of forgetting V from ϕ then $Mod(\psi) = Mod(\psi')$, *i.e.* , ψ and ψ' have the same models. In the sense of equivalence the forgetting result is unique (up to equivalence). By Lemma 3,such a formula always exists, which is equivalent to

$$\bigvee_{\mathcal{K} \in \{\mathcal{K}' \mid \exists \mathcal{K}'' \in Mod(\phi) \text{ and } \mathcal{K}'' \leftrightarrow_V \mathcal{K}'\}} \mathcal{F}_{\overline{V}}(\mathcal{K}).$$

For this reason, the forgetting result is denoted by $\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

Assume you are given a formula φ, and φ' is the formula after forgetting V, then we have the following desired properties, also called *postulates* of forgetting [Zhang and Zhou, 2009].

- Weakening (**W**): $\varphi \models \varphi'$;
- Positive Persistence (**PP**): Given $\eta \in$ CTL if IR(η, V) and $\varphi \models \eta$, then $\varphi' \models \eta$;
- Negative Persistence (**NP**): Given $\eta \in$ CTL if IR(η, V) and $\varphi \not\models \eta$, then $\varphi' \not\models \eta$;
- Irrelevance (**IR**): IR(φ', V).

Intuitive enough, the postulate (**W**) says, forgetting weakens the original formula. (**PP**) and (**NP**) correspond to the fact that so long as forgotten atoms V are irrelevant to the remaining positive and the negative information, respectively, they do not affect them. (**IR**) states that forgotten atoms V are not relevant for the final formula anymore (i.e., φ' is V-irrelevant).

Theorem 3 *(Representation theorem). Let φ, φ' and ψ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

(i) $\varphi' \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$*,*

(ii) $\varphi' \equiv \{\psi | \varphi \models \psi$ *and IR*$(\psi, V)\}$*,*

*(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.*

The above theorem says that CTL is closed under our definition of forgetting, *i.e.* , for any CTL formula the result of forgetting is also a CTL formula, and captures and entailed by the four postulates that forgetting should satisfy.

Lemma 4 *Let φ and α be two CTL formulae and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then* $\mathrm{F}_{CTL}(\varphi \cup \{q \leftrightarrow \alpha\}, q) \equiv \varphi$*.*

Proposition 4 *Let φ be a formula, V a set of atoms and p an atom such that $p \notin V$. Then,*

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V).$$

This means that the result of forgetting V from φ can be obtained by forgetting atoms in V one by one. Moreover, the order of atoms does not matter (commutativity), which follows from Proposition 4.

Corollary 4 *Let φ be a formula and $V_i \subseteq \mathcal{A}$ $(i = 1, 2)$. Then:*

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, V_1 \cup V_2) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, V_1), V_2).$$

The following results, which are satisfied in both classical proposition logic and modal logic **S5** [Zhang and Zhou, 2009], further illustrate other essential semantic properties of forgetting.

Proposition 5 *Let φ, φ_i, ψ_i $(i = 1, 2)$ be formulas and $V \subseteq \mathcal{A}$. We have*

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ *is satisfiable iff φ is;*

(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1, V) \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi_2, V)$*;*

(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1, V) \models \mathrm{F}_{\mathrm{CTL}}(\varphi_2, V)$*;*

(iv) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$*;*

(v) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \wedge \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$*;*

Another interesting result is that the forgetting of $PT\varphi$ ($P \in \{\mathrm{E}, \mathrm{A}\}$, $T \in \{\mathrm{F}, \mathrm{X}\}$) on $V \subseteq \mathcal{A}$ can be computed by $PT\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. This gives us a convenient method to compute forgetting since we can push the forgetting operator to a subformula without affecting the semantics.

Proposition 6 *Let $V \subseteq \mathcal{A}$ and ϕ a formula.*

(i) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(ii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(iii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AF}\phi, V) \equiv \mathrm{AF}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(iv) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EF}\phi, V) \equiv \mathrm{EF}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

3.4 Complexity Results

In the following, we outline the computational complexity of the various tasks regarding the forgetting in CTL and its popular fragment $\mathrm{CTL}_{\mathrm{AF}}$. It turns out that the model-checking on forgetting without any restriction is NP-complete.

Proposition 7 (Model Checking on Forgetting) *Let $(\mathcal{M}, s_0)$ be an initial* K*-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of* $\mathrm{F}_{CTL}(\varphi, V)$ *is NP-complete.*

The fragment of CTL, in which each formula contains only AF temporal connective correspond to specification descriptions for properties that is desired to hold in all branches eventually. Such properties are of special interest in concurrent systems e.g., mutual exclusion and waiting events [Baier and Katoen, 2008]. In the following, we report various complexity results concerning forgetting and the logical entailment in this fragment.

Theorem 5 *Let φ and ψ be two CTL_{AF} formulas and V a set of atoms. Then, results:*

(i) deciding $\mathrm{F}_{CTL}(\varphi, V) \models^? \psi$ *is co-NP-complete,*

(ii) deciding $\psi \models^? \mathrm{F}_{CTL}(\varphi, V)$ *is* Π_2^P*-complete,*

(iii) deciding $\mathrm{F}_{CTL}(\varphi, V) \models^? \mathrm{F}_{CTL}(\psi, V)$ *is* Π_2^P*-complete.*

The following results follow from Theorem 5 and extends them to semantic equivalence.

Corollary 6 *Let φ and ψ be two CTL_{AF} formulas and V a set of atoms. Then*

(i) deciding $\psi \equiv^? \mathrm{F}_{CTL}(\varphi, V)$ *is* Π_2^P*-complete,*

(ii) deciding $\mathrm{F}_{CTL}(\varphi, V) \equiv^? \varphi$ *is co-NP-complete,*

(iii) deciding $\mathrm{F}_{CTL}(\varphi, V) \equiv^? \mathrm{F}_{CTL}(\psi, V)$ *is* Π_2^P*-complete.*

4 Strongest Necessary and Weakest Sufficient Conditions

In this section, we will give the definition of SNC (WSC) and show that the SNC (WSC) of a specification (a CTL formula) under a given initial K-structure and set V of atoms can be obtained from forgetting in CTL. The SNC (WSC) of a proposition will be given at first:

Definition 4 (sufficient and necessary condition) *Let ϕ be a formula or an initial* K*-structure, ψ be a formula, $V \subseteq Var(\phi)$, $q \in Var(\phi) \setminus V$ and $Var(\psi) \subseteq V$.*

- *ψ is a* necessary condition *(NC in short) of q on V under ϕ if $\phi \models q \to \psi$.*
- *ψ is a* sufficient condition *(SC in short) of q on V under ϕ if $\phi \models \psi \to q$.*
- *ψ is a* strongest necessary condition *(SNC in short) of q on V under ϕ if it is a NC of q on V under ϕ and $\phi \models \psi \to \psi'$ for any NC ψ' of q on V under ϕ.*
- *ψ is a* weakest sufficient condition *(SNC in short) of q on V under ϕ if it is a SC of q on V under ϕ and $\phi \models \psi' \to \psi$ for any SC ψ' of q on V under ϕ.*

Note that if both ψ and ψ' are SNC (WSC) of q on V under ϕ then $Mod(\psi) = Mod(\psi')$, *i.e.* ψ and ψ' have the same models. In the sense of equivalence the SNC (WSC) is unique (up to equivalence).

Proposition 8 ***(dual)*** *Let V, q, φ and ψ are the ones in Definition 4. The ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.*

This show that the SNC and WSC are in fact dual conditions. Under the dual property, we can consider the SNC party only in sometimes, while the WSC part can be talked similarly.

In order to generalise Definition 4 to arbitrary formulas, one can replace q (in the definition) by any formula α, and redefine V as a subset of $Var(\alpha) \cup Var(\phi)$.

It is seems that the SNC and WSC of any formula can be reduced to that of a proposition.

Proposition 9 *Let Γ and α be two formulas, $V \subseteq Var(\alpha) \cup Var(\phi)$ and q is a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$.*

We propose the theorem of computing the SNC (WSC) of an atom due to the SNC (WSC) of a formula can be changed to the SNC (WSC) of an atom by Proposition 9.

Theorem 7 *Let φ be a formula, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) \setminus V$.*

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) \setminus V)$ *is a SNC of q on V under φ.*

(ii) $\neg\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) \setminus V)$ *is a WSC of q on V under φ.*

As we have said before that any initial K-structure can be characterized by a CTL formula, we can obtain the SNC (WSC) of an initial K-structure for satisfy some needed property (formula) by forgetting.

Algorithm 1: Model-based: Computing forgetting

Input: A CTL formula φ and a set V of atoms
Output: $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$

$T = \emptyset$ // the set of models of φ ;
$T' = \emptyset$ // the set of possible initial K-structures ;
$n = |\mathcal{A}|$;
for $i = 1, ..., 2^n$ **do**
  for $s_j \in \{s_1, \dots, s_i\}$ **do**
    Let s_j be an initial state, construct $\mathcal{M} = (S, R, L, s_j)$ by the definition of model structure with $S = \{s_1, \dots, s_i\}$;
    for $\mathcal{K} \in \mathcal{T}'$ **do**
      if $(\mathcal{M}, s_j) \leftrightarrow_{\overline{Var(\varphi)}} \mathcal{K}$ **then**
        Let $T' \leftarrow T' \cup \{(\mathcal{M}, s_j)\}$;
      end
    end
  end
  for $(\mathcal{M}, s_0) \in T'$ **do**
    if $(\mathcal{M}, s_0) \models \varphi$ **then**
      $T \leftarrow T \cup \{(\mathcal{M}, s_0)\}$;
    end
  end
end
return $\bigvee_{(\mathcal{M}', s_0') \in T} \mathcal{F}_{\overline{V}}(\mathcal{M}', s_0')$.

Theorem 8 *Let $\mathcal{K} = (\mathcal{M}, s)$ be an initial* K*-structure with $\mathcal{M} = (S, R, L, s_0)$ on the finite set $\mathcal{A}$ of atoms, $V \subseteq \mathcal{A}$ and $q \in V'$ ($V' = \mathcal{A} \setminus V$). Then:*

(i) the SNC of q on V under $\mathcal{K}$ is $\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q on V under $\mathcal{K}$ is $\neg\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge \neg q, V')$.

Example 3 For the Example 1, the WSC of φ on V under $\mathcal{K} = (\mathcal{M}, s_0)$ is $\neg\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge (q \equiv \varphi) \wedge \neg q, \mathcal{A} \setminus V)$.

5 Algorithm to Compute Forgetting

To compute the forgetting in CTL, we propose a model-based method in this part. Literally speaking, the model-based method means that we can obtain the result of forgetting in CTL by obtain all the possible finite models of this result. By the definition of forgetting in CTL, the set of models of the result of forgetting is also a finite set of initial K-structures.

Then we have the following model-based Algorithm 1 to compute the forgetting under CTL. By Lemma 3 and Theorem 2 we can prove the correctness of this algorithm.

Example 4 Let $\varphi = \mathrm{AGAF}(p \wedge r)$, $\mathcal{A} = \{p, r\}$ and $V = \{r\}$. For convenience, we use the label of a state to express the state and then remove the label function in a model structure. Let $\mathcal{M}_1 = (\{\{p, r\}, \{(\{p, r\}, \{p, r\})\}, \{p, r\})$ and $\mathcal{M}_2 = (\{\emptyset, \{p, r\}\}, \{(\emptyset, \{p, r\}), (\{p, r\}, \{p, r\})\}, \emptyset)$. The set of models of$\varphi$ is $Mod(\varphi) = \{(\mathcal{M}_1, \{p\}), (\mathcal{M}_2, \emptyset), \dots\}$.

Let $\mathcal{M}_1' = (\{\{p\}, \{(\{p\}, \{p\})\}, \{p\})$ and $\mathcal{M}_2' = (\{\emptyset, \{p\}\}, \{(\emptyset, \{p\}), (\{p\}, \{p\})\}, \emptyset)$ Then we can obtain all the possible initial K-structure that is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$, *i.e.* $Mod(\mathrm{F}_{\mathrm{CTL}}(\varphi, V)) = \{\mathcal{K}_1 = (\mathcal{M}_1', \{p\}), \mathcal{K}_2 = (\mathcal{M}_2', \emptyset), \dots\}$.

Let $V' = \{p\}$, then $\mathcal{F}_{V'}(\mathcal{K}_1) = p \wedge \text{AG}(p \supset \text{EX}p \wedge \text{AX}p)$, and $\mathcal{F}_{V'}(\mathcal{K}_2) = \neg p \wedge \text{AG}(p \supset \text{EX}\neg p \wedge \text{AX}\neg p) \wedge \text{AG}(\neg p \supset \text{EX}p \wedge \text{AX}p)$. Similarly, we can obtain the characteristic formula of other models and then the $\text{F}_{\text{CTL}}(\varphi, V)$.

Proposition 10 *Let φ be a CTL formula and $V \subseteq \mathcal{A}$. The time and space complexity of Algorithm 1 are $O(2^{m*2^m})$.*

6 Concluding Remark

Based on the proposed V-bisimulation between K-structures, forgetting in CTL and characteristic formula on V on an initial K-structure $\mathcal{K}$, a method compute the WSC (SNC) of a property φ (a CTL formula) on $\mathcal{K}$ and V has been introduced by computing forgetting in CTL. Besides, we have shown that the CTL system is close under our definition of forgetting, and this definition satisfies those four postulates of forgetting. As we have said the complexity of Algorithm 1 is $O(2^{m*2^m})$ (very inefficient), a future work is to find an efficient algorithm to compute forgetting in CTL and then WSC (SNC).

7 Proof

Lemma 5 *Let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be the ones in the definition of section 3.1. Then, for each $i \geq 0$,*

(i) $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$;

(ii) there is a (smallest) $k \geq 0$ such that $\mathcal{B}_{k+1} = \mathcal{B}_k$;

(iii) $\mathcal{B}_i$ is reflexive, symmetric and transitive.

Proof: (i) Base: it is clear for $i = 0$ by the above definition.

Step: suppose it holds for $i = n$, *i.e.* $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.
$(s, s') \in \mathcal{B}_{n+2}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s_1') \in R'$ such that $(s_1, s_1') \in \mathcal{B}_{n+1}$, and (c) for every $(s', s_1') \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s_1') \in \mathcal{B}_{n+1}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s_1') \in R'$ such that $(s_1, s_1') \in \mathcal{B}_n$ by inductive assumption, and (c) for every $(s', s_1') \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s_1') \in \mathcal{B}_n$ by inductive assumption
$\Rightarrow (s, s') \in \mathcal{B}_{n+1}$.

(ii) and (iii) are evident by the above definition. ∎

Lemma 1 The relation $\leftrightarrow_V$ is an equivalence relation.
Proof: It is clear from Lemma 5 (ii) such that there is a $k \geq 0$ where $\mathcal{B}_k = \mathcal{B}_{k+1}$ which is $\leftrightarrow_V$, and it is reflexive, symmetric and transitive by (iii). ∎

Proposition 1 Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s_i's be two states and π_i's be two pathes, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:

(i) $s_1' \leftrightarrow_{V_i} s_2'$ $(i = 1, 2)$ implies $s_1' \leftrightarrow_{V_1 \cup V_2} s_2'$;

(ii) $\pi_1' \leftrightarrow_{V_i} \pi_2'$ $(i = 1, 2)$ implies $\pi_1' \leftrightarrow_{V_1 \cup V_2} \pi_2'$;

(iii) for each path π_{s_1} of $\mathcal{M}_1$ there is a path π_{s_2} of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Proof: In order to distinguish the relations $\mathcal{B}_0, \mathcal{B}_1, \dots$ for different set $V \subseteq \mathcal{A}$, by $\mathcal{B}_i^V$ we mean the relation $\mathcal{B}_1, \mathcal{B}_2, \dots$ for $V \subseteq \mathcal{A}$. Denote as $\mathcal{B}_0, \mathcal{B}_1, \dots$ when the underlying set V is clear from the context. Moreover, for the ease of notation, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$ (i.e., without subindex).

(i) Base: it is clear for $n = 0$.
Step: For $n > 0$, supposing if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1 \cup V_2}$ for all $0 \leq i \leq n$. We will show that if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1 \cup V_2}$.
(a) It is evident that $L_1(s_1) \setminus (V_1 \cup V_2) = L_2(s_2) \setminus (V_1 \cup V_2)$.
(b) We will show that for each $(s_1, s_1^1) \in R_1$ there is a $(s_2, s_2^1) \in R_2$ such that $(s_1^1, s_2^1) \in \mathcal{B}_n^{V_1 \cup V_2}$. There is $(\mathcal{K}_1^1, \mathcal{K}_2^1) \in \mathcal{B}_{n-1}^{V_1 \cup V_2}$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_1 \cup V_2}$ by inductive assumption. Then we only need to prove for each $(s_1^1, s_1^2) \in R_1$ there is a $(s_2^1, s_2^2) \in R_2$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$ and for each $(s_2^1, s_2^2) \in R_2$ there is a $(s_1^1, s_1^2) \in R_1$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$. Therefore, we only need to prove that for each $(s_1^n, s_1^{n+1}) \in R_1$ there is a $(s_2^n, s_2^{n+1}) \in R_2$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$ and for each $(s_2^n, s_2^{n+1}) \in R_2$ there is a $(s_1^n, s_1^{n+1}) \in R_1$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$. It is apparent that $L_1(s_1^{n+1}) \setminus (V_1 \cup V_2) = L_1(s_2^{n+1}) \setminus (V_1 \cup V_2)$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$. Where $\mathcal{K}_i^j = (\mathcal{M}_i, s_i^j)$ with $i \in \{1, 2\}$ and $0 < j \leq n+1$.
(c) It is similar with (b).

(ii) It is clear from (i).

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) \setminus V = L_2(s_2) \setminus V$,

(b) for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$, and

(c) for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$,

where $\mathcal{K}_i' = (\mathcal{M}_i, s_i')$ with $i \in \{1, 2\}$.

We prove it form the following two aspects:

($\Rightarrow$) (a) It is apparent that $L_1(s_1) \setminus V = L_2(s_2) \setminus V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s_1') \in R_1$, there is a $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s_1') \setminus V = L_2(s_2') \setminus V$. Therefore, $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$. (c) This is similar with (b).

($\Leftarrow$) (a) $L_1(s_1) \setminus V = L_2(s_2) \setminus V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) Condition (ii) implies that for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$; (c) Condition (iii) implies that for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' \subseteq S_1 \times S_3$ and $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation between s_1 and s_3 from the (a), (b) and (c) of the previous step (iii) of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3)\mathcal{B}'$ by the definition of $\mathcal{B}'$.

(v) We will show that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^W$ for all $n \geq 0$ inductively.

Base: $L_1(s_1) \setminus V = L_2(s_2) \setminus V$
$\Rightarrow \forall q \in A \setminus V$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$
$\Rightarrow \forall q \in A \setminus W$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$ due to $V \subseteq W$
$\Rightarrow L_1(s_1) \setminus W = L_2(s_2) \setminus W$, *i.e.* $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0^W$.

Step: Supposing that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^W$ for all $0 \leq i \leq k$ $(k > 0)$, we will show $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{k+1}^W$.

(a) It is apparent that $L_1(s_1) \setminus W = L_2(s_2) \setminus W$ by base.

(b) $\forall (s_1, s_{1,1}) \in R_1$, we will show that there is a $(s_2, s_{2,1}) \in R_2$ $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^W$. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_{k-1}^W$ by inductive assumption, we need only to prove the following points:
(a) $\forall (s_{1,k}, s_{1,k+1}) \in R_1$ there is a $(s_{2,k}, s_{2,k+1}) \in R_2$ $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^W$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^V$. It is easy to see that $L_1(s_{1,k+1}) \setminus V = L_1(s_{2,k+1}) \setminus V$, then there is $L_1(s_{1,k+1}) \setminus W = L_1(s_{2,k+1}) \setminus W$. Therefore, $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^W$.
(b) $\forall (s_{2,k}, s_{2,k+1}) \in R_1$ there is a $(s_{1,k}, s_{1,k+1}) \in R_1$ $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^W$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^V$. This can be proved as (a).

(c) $\forall (s_2, s_{2,1}) \in R_1$, we will show that there is a $(s_1, s_{1,1}) \in R_2$ $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^W$. This can be proved as (ii).

Where $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$ and $(s_{i,k}, s_{i,k+1}) \in R_i$ means that $s_{i,k+1}$ is the $(k+2)$-th node in the path $(s_i, s_{i,1}, s_{i,2}, \ldots, s_{i,k+1}, \ldots)$ $(i = 1, 2)$. ∎

Theorem1 Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.

Proof: This theorem can be proved by inducting on the formula φ and supposing $Var(\varphi) \cap V = \emptyset$.

Here we only prove the only-if direction. The other direction can be similarly proved.

Case $\varphi = p$ where $p \in \mathcal{A} \setminus V$:
$(\mathcal{M}, s) \models \varphi$ iff $p \in L(s)$ (by the definition of satisfiability)
$\Leftrightarrow p \in L'(s')$ $(s \leftrightarrow_V s')$
$\Leftrightarrow (\mathcal{M}', s') \models \varphi$

Case $\varphi = \neg\psi$:
$(\mathcal{M}, s) \models \varphi$ iff $(\mathcal{M}, s) \not\models \psi$
$\Leftrightarrow (\mathcal{M}', s') \not\models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \varphi$

Case $\varphi = \psi_1 \vee \psi_2$:
$(\mathcal{M}, s) \models \varphi$
$\Leftrightarrow (\mathcal{M}, s) \models \psi_1$ or $(\mathcal{M}, s) \models \psi_2$
$\Leftrightarrow (\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \varphi$

Case $\varphi = \text{EX}\psi$:
$\mathcal{M}, s \models \varphi$
$\Leftrightarrow$ There is a path $\pi = (s, s_1, ...)$ such that $\mathcal{M}, s_1 \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s', s_1', ...)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_1 \leftrightarrow_V s_1'$ $(\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s_1') \models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \varphi$

Case $\varphi = \text{EG}\psi$:
$\mathcal{M}, s \models \varphi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that for each $i \geq 0$ there is $(\mathcal{M}, s_i) \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s' = s_0', s_1', ...)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_i \leftrightarrow_V s_i'$ for each $i \geq 0$ $(\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s_i') \models \psi$ for each $i \geq 0$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \varphi$

Case $\varphi = \text{E}[\psi_1 \text{U} \psi_2]$:
$\mathcal{M}, s \models \varphi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that there is $i \geq 0$ such that $(\mathcal{M}, s_i) \models \psi_2$, and for all $0 \leq j < i$, $(\mathcal{M}, s_j) \models \psi_1$
$\Leftrightarrow$ There is a path $\pi' = (s = s_0', s_1', ...)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow (\mathcal{M}', s_i') \models \psi_2$, and for all $0 \leq j < i$ $(\mathcal{M}', s_j') \models \psi_1$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \varphi$ ∎

Proposition 2 Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two K-structures. Then

$$(s_1, s_2) \in \mathcal{B}_n \text{ iff } \text{Tr}_j(s_1) \leftrightarrow_V \text{Tr}_j(s_2) \text{ for every } 0 \leq j \leq n.$$

Proof: We will prove this from two aspects:

$(\Rightarrow)$ If $s\mathcal{B}_n s'$, then $Tr_j(s) \leftrightarrow_V Tr_j(s')$ for all $0 \leq j \leq n$. $s\mathcal{B}_n s'$ implies both roots of $Tr_n(s)$ and $Tr_n(s')$ have the same atoms except those atoms in V. Besides, for any s_1 with $s \to s_1$, there is a s_1' with $s' \to s_1'$ s.t. $s_1\mathcal{B}_{n-1}s_1'$ and vice versa. Then we have $Tr_1(s) \leftrightarrow_V Tr_1(s')$. Therefore, $Tr_n(s) \leftrightarrow_V Tr_n(s')$ by use such method recursively, and then $Tr_j(s) \leftrightarrow_V Tr_j(s')$ for all $0 \leq j \leq n$.

$(\Leftarrow)$ If $Tr_j(s) \leftrightarrow_V Tr_j(s')$ for all $j \leq n$, then $s\mathcal{B}_n s'$. $Tr_0(s) \leftrightarrow_V Tr_0(s')$ implies $L(s) \setminus V = L'(s') \setminus V$ and then $s\mathcal{B}_0 s'$. $Tr_1(s) \leftrightarrow_V Tr_1(s')$ implies $L(s) \setminus V = L'(s') \setminus V$ and for every successors s_1 of the root of one, it is possible to find a successor of the root of the other s_1' such that $s_1\mathcal{B}_0 s_1'$. Therefore $s\mathcal{B}_1 s'$, and then we will have $s\mathcal{B}_n s'$ by use such method recursively. ∎

Proposition 3 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least k such that $\text{Tr}_k(s)$ and $\text{Tr}_k(s')$ are not V-bisimilar.

Proof: If $s \not\leftrightarrow_V s'$, then there exists a least constant k

such that $(s_i, s_j) \notin \mathcal{B}_k$, and then there is a least constant m ($m \leq k$) such that $\mathrm{Tr}_m(s_i)$ and $\mathrm{Tr}_m(s_j)$ are not V-corresponding by Proposition 2. Let $c = m$, the lemma is proved. ■

Lemma2 Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $\mathrm{Tr}_n(s) \leftrightarrow_{\overline{V}} \mathrm{Tr}_n(s')$, then $\mathcal{F}_V(\mathrm{Tr}_n(s)) \equiv \mathcal{F}_V(\mathrm{Tr}_n(s'))$.

Proof: This result can be proved by inducting on n.

Base. It is apparent that for any $s_n \in S$ and $s'_n \in S'$, if $\mathrm{Tr}_0(s_n) \leftrightarrow_{\overline{V}} \mathrm{Tr}_0(s'_n)$ then $\mathcal{F}_V(\mathrm{Tr}_0(s_n)) \equiv \mathcal{F}_V(\mathrm{Tr}_0(s'_n))$ due to $L(s_n) \setminus \overline{V} = L'(s'_n) \setminus \overline{V}$ by known.

Step. Supposing that for $k = m$ $(0 < m \leq n)$ there is if $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ then $\mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k)) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))$, then we will show if $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$ then $\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1}))$. Apparent that:
$\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) = \left(\bigwedge_{(s_{k-1},s_k)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_{k-1},s_k)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k))\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s_{k-1}))$
$\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1})) = \left(\bigwedge_{(s'_{k-1},s'_k)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))\right) \wedge \mathrm{AX}\left(\bigvee_{(s'_{k-1},s'_k)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s'_{k-1}))$ by the definition of characterizing formula of the computation tree. Then we have for any $(s_{k-1}, s_k) \in R$ there is $(s'_{k-1}, s'_k) \in R'$ such that $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ by $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$. Besides, for any $(s'_{k-1}, s'_k) \in R'$ there is $(s_{k-1}, s_k) \in R$ such that $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ by $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$. Therefore, we have $\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1}))$ by induction hypothesis. ■

Lemma 3 Let φ be a formula. We have

$$\varphi \equiv \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \tag{4}$$

Proof: Let $(\mathcal{M}', s'_0)$ be a model of φ. Then $(\mathcal{M}', s'_0) \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$ due to $(\mathcal{M}', s'_0) \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}', s'_0)$. On the other hand, suppose that $(\mathcal{M}', s'_0)$ is a model of $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. Then there is a $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s'_0) \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. And then $(\mathcal{M}, s_0) \leftrightarrow_{\emptyset} (\mathcal{M}', s'_0)$ by Theorem 2. Therefore, $(\mathcal{M}, s_0)$ is also a model of φ by Theorem 1. ■
bisimilar

Theorem 2 Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures. Then,

(a) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s'_0)$.

(b) $s_0 \leftrightarrow_{\overline{V}} s'_0$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s'_0)$.

In order to prove Theorem 2, we prove the following two lemmas at first.

Lemma 6 *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s))$.

(ii) If $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s'))$ then $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$.

Proof: (i) It is apparent from the definition of $\mathcal{F}_V(\mathrm{Tr}_n(s))$. Base. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s))$.
Step. For $k \geq 0$, supposing the result talked in (i) is correct in $k-1$, we will show that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_{k+1}(s))$, *i.e.* :

$$(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \mathrm{EX}T(s')\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s')\in R} T(s')\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s)).$$

Where $T(s') = \mathcal{F}_V(\mathrm{Tr}_k(s'))$. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s))$ by Base. It is apparent that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \mathcal{F}_V(\mathrm{Tr}_k(s'))$ by inductive assumption. Then we have $(\mathcal{M}, s) \models \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_k(s')$, and then $(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_k(s'))\right)$. Similarly, we have that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \bigvee_{(s,s'')\in R} \mathcal{F}_V(\mathrm{Tr}_k(s''))$. Therefore, $(\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s,s'')\in R} \mathcal{F}_V(\mathrm{Tr}_k(s''))\right)$.

(ii) **Base**. If $n = 0$, then $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s'))$ implies $L(s) \setminus \overline{V} = L'(s') \setminus \overline{V}$. Hence, $\mathrm{Tr}_0(s) \leftrightarrow_{\overline{V}} \mathrm{Tr}_0(s')$.
Step. Supposing $n > 0$ and the result talked in (ii) is correct in $n-1$.
(a) It is easy to see that $L(s) \setminus \overline{V} = L'(s') \setminus \overline{V}$.
(b) We will show that for each $(s, s_1) \in R$, there is a $(s', s'_1) \in R'$ such that $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s',s'_1)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))\right)$. Therefore, for each $(s, s_1) \in R$ there is a $(s', s'_1) \in R'$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))$. Hence, $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$ by inductive hypothesis.
(c) We will show that for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $\mathrm{Tr}_{n-1}(s'_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \bigwedge_{(s',s'_1)\in R'} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))$. Therefore, for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1)$. Hence, $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$ by inductive hypothesis. ■

A consequence of the previous lemma is:

Lemma 7 *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ a model structure, $k = ch(\mathcal{M}, V)$ and $s \in S$.*

- *$(\mathcal{M}, s) \models \mathcal{F}_V(Tr_k(s))$, and*
- *for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(Tr_k(s))$.*

Proof: Let $\phi = \mathcal{F}_V(\mathrm{Tr}_k(s))$, where k is the V-characteristic number of $\mathcal{M}$. $(\mathcal{M}, s) \models \phi$ by the definition of $\mathcal{F}$, and then $\forall s' \in S$, if $s \leftrightarrow_{\overline{V}} s'$ there is $(\mathcal{M}, s') \models \phi$ by Theorem 1 due to $\mathrm{IR}(\phi, \mathcal{A} \setminus V)$. Supposing $(\mathcal{M}, s') \models \phi$, if $s \not\leftrightarrow_{\overline{V}} s'$, then $\mathrm{Tr}_k(s) \not\leftrightarrow_{\overline{V}} \mathrm{Tr}_k(s')$, and then $(\mathcal{M}, s') \not\models \phi$ by Lemma 6, a contradiction. ■

Now we are in the position of proving Theorem 2.

Proof: (a) Let $\mathcal{F}_V(\mathcal{M}, s_0)$ be the characterizing formula of $(\mathcal{M}, s_0)$ on V. It is apparent that $\mathrm{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. We will show that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ at first.

It is apparent that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s_0))$ by Lemma 6. We must show that $(\mathcal{M}, s_0) \models \bigwedge_{s\in S} G(\mathcal{M}, s)$. Let $\mathcal{X} = \mathcal{F}_V(\mathrm{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$, we will show $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$. Where $G(\mathcal{M}, s) = \mathrm{AG}\mathcal{X}$. There are two cases we should consider:

- If $(\mathcal{M}, s_0) \not\models \mathcal{F}_V(\mathrm{Tr}_c(s))$, it is apparent that $(\mathcal{M}, s_0) \models \mathcal{X}$;
- If $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s))$:
  $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s))$
  $\Rightarrow s_0 \leftrightarrow_{\overline{V}} s$ by the definition of characteristic number and Lemma 7.
  For each $(s, s_1) \in R$ there is:
  $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_c(s_1))$ $(s_1 \leftrightarrow_{\overline{V}} s_1)$
  $\Rightarrow (\mathcal{M}, s) \models \bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$
  $\Rightarrow (\mathcal{M}, s_0) \models \bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (by $\mathrm{IR}(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1)), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$).
  For each (s, s_1) there is:
  $\mathcal{M}, s_1 \models \bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))$
  $\Rightarrow (\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$
  $\Rightarrow (\mathcal{M}, s_0) \models \mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ (by $\mathrm{IR}(\mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$)
  $\Rightarrow (\mathcal{M}, s_0) \models \mathcal{X}$.

For any other states s' which can reach from s_0 can be proved similarly, *i.e.* , $(\mathcal{M}, s') \models \mathcal{X}$. Therefore, $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$, and then $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$.

We will prove this theorem from the following two aspects:

$(\Leftarrow)$ If $s_0 \leftrightarrow_{\overline{V}} s'_0$, then $(\mathcal{M}', s'_0) \models \mathcal{F}_V(M, s_0)$. Since $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and $\mathrm{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$, hence $(\mathcal{M}', s'_0) \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

$(\Rightarrow)$ If $(\mathcal{M}', s'_0) \models \mathcal{F}_V(M, s_0)$, then $s_0 \leftrightarrow_{\overline{V}} s'_0$. We will prove this by showing that $\forall n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s'_0)$.

Base. It is apparent that $Tr_0(s_0) \equiv Tr_0(s'_0)$.

Step. Supposing $\mathrm{Tr}_k(s_0) \leftrightarrow_{\overline{V}} \mathrm{Tr}_k(s'_0)$ $(k > 0)$, we will prove $\mathrm{Tr}_{k+1}(s_0) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{k+1}(s'_0)$. We should only show that $\mathrm{Tr}_1(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_1(s'_k)$. Where $(s_0, s_1), (s_1, s_2), \ldots, (s_{k-1}, s_k) \in R$ and $(s'_0, s'_1), (s'_1, s'_2), \ldots, (s'_{k-1}, s'_k) \in R'$, *i.e.* s_{i+1} (s'_{i+1}) is an immediate successor of s_i (s'_i) for all $0 \leq i \leq k-1$.

(i) It is apparent that $L(s_k) \setminus \overline{V} = L'(s'_k) \setminus \overline{V}$ by inductive assumption.

Before talking about the other points, note the following fact that:

$(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$
$\Rightarrow \forall s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\mathrm{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ for any $s \in S$. **(fact)**
(I) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s_0)) \rightarrow \left(\bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ **(fact)**
(II) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s_0)))$ (known)
(III) $(\mathcal{M}', s'_0) \models \left(\bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ ((I),(II))

(ii) We will show that for each $(s_k, s_{k+1}) \in R$ there is a $(s'_k, s'_{k+1}) \in R'$ such that $L(s_{k+1}) \setminus \overline{V} = L'(s'_{k+1}) \setminus \overline{V}$.
(1) $(\mathcal{M}', s'_0) \models \bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (III)
(2) $\forall(s_0, s_1) \in R$, $\exists(s'_0, s'_1) \in R'$ $(\mathcal{M}', s'_1) \models \mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (2)
(3) $\mathrm{Tr}_c(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s'_1)$ ((2), Lemma 6)
(4) $L(s_1) \setminus \overline{V} = L'(s'_1) \setminus \overline{V}$ ((3), $c \geq 0$)
(5) $(\mathcal{M}', s'_1) \models \mathcal{F}_V(\mathrm{Tr}_c(s_1)) \rightarrow \left(\bigwedge_{(s_1,s_2)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_2))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ **(fact)**
(6) $(\mathcal{M}', s'_1) \models \left(\bigwedge_{(s_1,s_2)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_2))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ ((2), (5))
(7) ......
(8) $(\mathcal{M}', s'_k) \models \left(\bigwedge_{(s_k,s_{k+1})\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right)$ (similar with (6))
(9) $\forall(s_k, s_{k+1}) \in R$, $\exists(s'_k, s'_{k+1}) \in R'$ $(\mathcal{M}', s'_{k+1}) \models \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))$ (8)
(10) $\mathrm{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s'_{k+1})$ ((9), Lemma 6)
(11) $L(s_{k+1}) \setminus \overline{V} = L'(s'_{k+1}) \setminus \overline{V}$ ((10), $c \geq 0$)

(iii) We will show that for each $(s'_k, s'_{k+1}) \in R'$ there is a $(s_k, s_{k+1}) \in R$ such that $L(s_{k+1}) \setminus \overline{V} = L'(s'_{k+1}) \setminus \overline{V}$.
(1) $(\mathcal{M}', s'_k) \models \mathrm{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right)$ (by (8) talked above)
(2) $\forall(s'_k, s'_{k+1}) \in R'$, $\exists(s_k, s_{k+1}) \in R$ $(\mathcal{M}', s'_{k+1}) \models \mathcal{F}_V(\mathrm{Tr}_c(s'_{k+1}))$ (1)
(3) $\mathrm{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s'_{k+1})$ ((2), Lemma 6)
(4) $L(s_{k+1}) \setminus \overline{V} = L'(s'_{k+1}) \setminus \overline{V}$ ((3), $c \geq 0$)

(b) This is following Lemma 2 and the definition of the characterizing formula of initial K-structure $\mathcal{K}$ on V.

■

Theorem 3 (Representation theorem). Let φ and ψ be two formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

(i) $\psi \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$,
(ii) $\psi \equiv \{\phi | \varphi \models \phi \& \mathrm{IR}(\phi, V)\}$,
(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.

Proof: $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$\mathit{Mod}(\mathrm{F}_{\mathrm{CTL}}(\varphi, V)) = \mathit{Mod}(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\})$$
$$= \mathit{Mod}(\bigvee_{\mathcal{M}, s_0 \in \mathit{Mod}(\varphi)} \mathcal{F}_{\mathcal{A}\setminus V}(\mathcal{M}, s_0)).$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. Then there exists an an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of φ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all ϕ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s_0')$ is a models of $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}\setminus V}(\mathcal{M}, s_0)$ due to $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}\setminus V}(\mathcal{M}, s_0)$ is irrelevant to V.

Finally, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}\setminus V}(\mathcal{M}, s_0)$. Then there exists $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}\setminus V}(\mathcal{M}, s_0)$. Hence, $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ by Theorem 2. Thus $(\mathcal{M}', s_0')$ is also a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. Suppose that all postulates hold. By Positive Persistence, we have $\psi \models \{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Now we show that $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \psi$. Otherwise, there exists formula ϕ' such that $\psi \models \phi'$ but $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\} \not\models \phi'$. There are three cases:

- ϕ' is relevant to V. Thus, ψ is also relevant to V, a contradiction to Irrelevance.
- ϕ' is irrelevant to V and $\varphi \models \phi'$. This contradicts to our assumption.
- ϕ' is irrelevant to V and $\varphi \not\models \phi'$. By Negative Persistence, $\psi \not\models \phi'$, a contradiction.

Thus, ψ is equivalent to $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi, V)\}$. ■

Lemma 4 Let φ and α be two CTL formulae and $q \in \overline{Var(\varphi \cup \{\alpha\})}$. Then $\mathrm{F}_{\mathrm{CTL}}(\varphi \cup \{q \leftrightarrow \alpha\}, q) \equiv \varphi$.

Proof: Let $\varphi' = \varphi \cup \{q \leftrightarrow \alpha\}$. For any model $(\mathcal{M}, s)$ of $\mathrm{F}_{\mathrm{CTL}}(\Gamma', q)$ there is an initial K-structure $(\mathcal{M}', s')$ s.t. $(\mathcal{M}, s) \leftrightarrow_{\{q\}} (\mathcal{M}', s')$ and $(\mathcal{M}', s') \models \varphi'$. It's apparent that $(\mathcal{M}', s') \models \varphi$, and then $(\mathcal{M}, s) \models \varphi$ since $\mathrm{IR}(\varphi, \{q\})$ and $(\mathcal{M}, s) \leftrightarrow_{\{q\}} (\mathcal{M}', s')$ by Theorem 1.

Let $(\mathcal{M}, s) \in Mod(\varphi)$ with $\mathcal{M} = (S, R, L, s)$. We construct $(\mathcal{M}', s)$ with $\mathcal{M}' = (S, R, L', s)$ as follows:

$$L' : S \to \mathcal{A} \; and \; \forall s^* \in S, L'(s^*) = L(s^*) \; if \; (\mathcal{M}, s^*) \not\models \alpha,$$

$$else \; L'(s^*) = L(s^*) \cup \{q\},$$

$$L'(s) = L(s) \cup \{q\} \; if \; (\mathcal{M}, s) \models \alpha, \; and \; L'(s) = L(s)$$

$$otherwise.$$

It is clear that $(\mathcal{M}', s) \models \varphi$, $(\mathcal{M}', s) \models q \leftrightarrow \alpha$ and $(\mathcal{M}', s) \leftrightarrow_{\{q\}} (\mathcal{M}, s)$. Therefore $(\mathcal{M}', s) \models \varphi \cup \{q \leftrightarrow \alpha\}$, and then $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi \cup \{q \leftrightarrow \alpha\}, q)$ by $(\mathcal{M}', s) \leftrightarrow_{\{q\}} (\mathcal{M}, s)$. ■

Proposition 4 Let φ be a formula, V a set of atoms and p an atom such that $p \notin V$. Then:

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V).$$

Proof: Let $(\mathcal{M}_1, s_1)$ with $\mathcal{M}_1 = (S_1, R_1, L_1, s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V)$. By the definition, there exists a model $(\mathcal{M}, s)$ with $\mathcal{M} = (S, R, L, s)$ of φ, such that $(\mathcal{M}_1, s_1) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}, s)$ via a binary relation $\mathcal{B}$. We construct an initial K-structure $(\mathcal{M}_2, s_2)$ with $\mathcal{M}_2 = (S_2, R_2, L_2, s_2)$ as follows:

(1) for s_2: let s_2 be the state such that:

- $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
- for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
- for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:

(i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w\mathcal{B}w_1$, let $w_2 \in S_2$ and
- $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
- for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
- for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

(ii) if $w_1'\mathcal{R}_1 w_1$, w_2 is constructed based on w_1 and $w_2' \in S_2$ is constructed based on w_1', then $w_2'\mathcal{R}_2 w_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$. Thus, $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V)$.

On the other hand, suppose that $(\mathcal{M}_1, s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi, p), V)$, then there exists an initial Kripke structure $(\mathcal{M}_2, s_2)$ such that $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, p)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$, and there exists $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \varphi$ and $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$. Therefore, $(\mathcal{M}, s) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}_1, s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V)$. ■

Proposition 5 Let φ, φ_i, ψ_i $(i = 1, 2)$ be formulas and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is satisfiable iff φ is;

(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1, V) \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi_2, V)$;

(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1, V) \models \mathrm{F}_{\mathrm{CTL}}(\varphi_2, V)$;

(iv) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$;

(v) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \wedge \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$;

Proof: (i) $(\Rightarrow)$ Supposing $(\mathcal{M}, s)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$, then there is a model $(\mathcal{M}', s')$ of φ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

$(\Leftarrow)$ Supposing $(\mathcal{M}, s)$ is a model of φ, then there is an initial Kripke structure $(\mathcal{M}', s')$ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$, and then $(\mathcal{M}', s') \models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

The (ii) and (iii) can be proved similarly.

(iv) $(\Rightarrow)$ $\forall(\mathcal{M}, s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2, V))$, $\exists(\mathcal{M}', s') \in Mod(\psi_1 \vee \psi_2)$ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$ and $(\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$
$\Rightarrow \exists(\mathcal{M}_1, s_1) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1, V))$ s.t. $(\mathcal{M}', s') \leftrightarrow_V (\mathcal{M}_1, s_1)$ or $\exists(\mathcal{M}_2, s_2) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_2, V))$ s.t. $(\mathcal{M}', s') \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$ by Theorem 1.

$(\Leftarrow)$ $\forall(\mathcal{M}, s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1, V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2, V))$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1, V)$ or $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_2, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}, s) \leftrightarrow_V$

$(\mathcal{M}_1, s_1)$ and $(\mathcal{M}_1, s_1) \models \psi_1$ or $(\mathcal{M}_1, s_1) \models \psi_2$
$\Rightarrow (\mathcal{M}_1, s_1) \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$ and $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2, V)$
$\Rightarrow (\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ and $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2, V)$.

The (v) can be proved as (iV). ∎

Proposition 6 Let $V \subseteq \mathcal{A}$ and ϕ a formula.

(i) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(ii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(iii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AF}\phi, V) \equiv \mathrm{AF}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(iv) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EF}\phi, V) \equiv \mathrm{EF}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

Proof: Let $\mathcal{M} = (S, R, L, s_0)$ with initial state s_0 and $\mathcal{M}' = (S', R', L', s'_0)$ with initial state s'_0, then we call $\mathcal{M}', s'_0$ be a sub-structure of $\mathcal{M}, s_0$ if:

- $S' = \{s' | s'$ is reachable from $s'_0\}$ and $S' \subseteq S$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to \mathcal{A}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- there is a stete $s \in S$ reachable from s_0 such that $(\mathcal{M}, s) \leftrightarrow_\emptyset (\mathcal{M}', s'_0)$.

(i) In order to prove $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$, we only need to prove $Mod(\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)) = Mod(\mathrm{AX}\mathrm{F}_{\mathrm{CTL}}(\phi, V))$:

($\Rightarrow$) $\forall(\mathcal{M}', s') \in Mod(\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \mathrm{AX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ for any sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ there is $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ that s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow \mathcal{M}_3, s_3 \models \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$, especially, let $\mathcal{M}_3, s_3 = \mathcal{M}', s'$, we have $\mathcal{M}', s' \models \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$.

($\Leftarrow$) $\forall$ $(\mathcal{M}_3, s_3) \in Mod(\mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V)))$, then for any sub-structure $(\mathcal{M}_2, s_2)$ whit s_2 is a directed successor s_3 of $(\mathcal{M}_3, s_3)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ that s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{AX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$, we only need to prove Mod $(\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi,\ V)) = Mod(\mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V))$:

($\Rightarrow$) $\forall \mathcal{M}', s' \in Mod(\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \mathrm{EX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ there is a sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ that s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow (\mathcal{M}_3, s_3) \models \mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$, especially, let $(\mathcal{M}_3, s_3) = (\mathcal{M}', s')$, we have $(\mathcal{M}', s') \models \mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$.

($\Leftarrow$) $\forall$ $(\mathcal{M}_3, s_3) \in Mod(\mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V)))$, then there exists a sub-structure $(\mathcal{M}_2, s_2)$ of $(\mathcal{M}_3, s_3)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ that s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{EX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V)$.

(iii) and (iV) can be proved as (i) and (ii) respectively. ∎

Proposition7 Let $(\mathcal{M}, s_0)$ be an initial K-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is NP-complete.

Proof: The problem can be determined by the following two things: (1) guessing an initial K-structure $(\mathcal{M}', s'_0)$ satisfying φ; and (2) checking if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$. Both two steps can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard [Zhang and Zhou, 2008]. ∎

Theorem 5 Let φ and ψ be two $\mathrm{CTL}_{\mathrm{AF}}$ (a fragment of CTL, in which each formula contains only AF temporal connective) formulas and V a set of atoms. Then we have the results:

(i) deciding if $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \psi$ is co-NP-complete,

(ii) deciding if $\psi \models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is Π_2^P-complete,

(iii) deciding if $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ is Π_2^P-complete.

Proof: (1) It is proved that deciding whether ψ is satisfiable is NP-Complete [Meier *et al.*, 2015]. The hardness is easy to see by setting $\mathrm{F}_{\mathrm{CTL}}(\varphi, Var(\varphi)) \equiv \top$, *i.e.* deciding whether ψ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $IR(\psi, V)$. Clearly, in $\mathrm{CTL}_{\mathrm{AF}}$, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $IR(\psi, V)$ is also in co-NP. Without loss of generality, we assume that ψ is satisfiable. We consider the complement of the problem: deciding whether ψ is not irrelevant to V. It is easy to see that ψ is not irrelevant to V iff there exist a model $(\mathcal{M}, s_0)$ of ψ and an initial K-structure $(\mathcal{M}', s'_0)$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$ and $(\mathcal{M}', s'_0) \not\models \psi$. So checking whether ψ is not irrelevant to V can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s'_0)$, (2) check if $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s'_0) \not\models \psi$, and (3) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$. Obviously (1) can be done in polynomial time and also (2) and (3) can be done in polynomial time.

(2) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ and check whether $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}, s_0) \not\models \mathrm{F}_{\mathrm{CTL}}(\ \varphi, V)$. From Proposition 7, we know that this is in Σ_2^P. So the original problem is in Π_2^P. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case temporal forgetting,

the hardness is directly followed from the proof of Proposition 24 in [Lang *et al.*, 2003].

(3) Membership. If $\mathrm{F_{CTL}}(\varphi, V) \not\models \mathrm{F_{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F_{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F_{CTL}}(\psi, V)$, *i.e.* , there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M}, s)$, $(\mathcal{M}_1, s_1)$ with $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ and checking $(\mathcal{M}_1, s_1) \models \varphi$ are feasible while checking $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ can be done in polynomial time. Thus the problem is in Π_2^P.

Hardness. It follows from (2) due to the fact that $\mathrm{F_{CTL}}(\varphi, V) \models \mathrm{F_{CTL}}(\psi, V)$ iff $\varphi \models \mathrm{F_{CTL}}(\psi, V)$ thanks to $IR(\mathrm{F_{CTL}}(\psi, V), V)$.

■

Proposition 8 Let V, q, φ and ψ are the ones in Definition 4. The ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.

Proof: (i) Suppose ψ is the SNC of q. Then $\varphi \models q \to \psi$. Thus $\varphi \models \neg\psi \to \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \to \neg q$. Then $\varphi \models q \to \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \to \neg\psi'$ by assumption. So $\varphi \models \psi' \to \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case.
■

Proposition 9 Let Γ and α be two formulas, $V \subseteq \mathit{Var}(\alpha) \cup \mathit{Var}(\phi)$ and q is a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$.

Proof: We prove this for SNC. The case for WSC is similar. Let $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ denote that φ is the SNC of α on V under Γ, and $\mathit{NC}(\varphi, \alpha, V, \Gamma)$ denote that φ is the NC of α on V under Γ.

($\Rightarrow$) if $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds, then $\mathit{SNC}(\varphi, q, V, \Gamma')$ will be true. According to $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \to \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F_{CTL}}(\Gamma', q) \models \alpha \to \varphi'$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \to \varphi', \{q\})$ and **(pp)**, *i.e.* $\Gamma \models \alpha \to \varphi'$ by Lemma 4, this means $\mathit{NC}(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \to \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \to \varphi'$. Hence, $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) if $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds, then $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ will be true. According to $\mathit{SNC}(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F_{CTL}}(\Gamma', \{q\}) \models \alpha \to \varphi$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \to \varphi, \{q\})$ and **(pp)**, *i.e.* $\Gamma \models \alpha \to \varphi$ by Lemma 4, this means $\mathit{NC}(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \to \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $\mathit{NC}(\varphi', q, V, \Gamma')$. According to $\mathit{SNC}(\varphi, q, V, \Gamma')$, $\mathit{IR}(\varphi \to \varphi', \{q\})$ and **(pp)**, we have $\mathrm{F_{CTL}}(\Gamma', \{q\}) \models \varphi \to \varphi'$, and $\Gamma \models \varphi \to \varphi'$ by Lemma 4. Hence, $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds. ■

Theorem 7 Let φ be a formula, $V \subseteq \mathit{Var}(\varphi)$ and $q \in \mathit{Var}(\varphi) \setminus V$.

(i) $\mathrm{F_{CTL}}(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) \setminus V)$ is a SNC of q on V under φ.

(ii) $\neg\mathrm{F_{CTL}}(\varphi \wedge \neg q, (\mathit{Var}(\varphi) \cup \{q\}) \setminus V)$ is a WSC of q on V under φ.

Proof: We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F_{CTL}}(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) \setminus V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \to \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \to \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \to \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $\mathit{IR}(\psi, (\mathit{Var}(\varphi) \cup \{q\}) \setminus V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ. ■

Theorem 8 Let $\mathcal{K} = (\mathcal{M}, s)$ be an initial K-structure with $\mathcal{M} = (S, R, L, s_0)$ on the finite set $\mathcal{A}$ of atoms, $V \subseteq \mathcal{A}$ and $q \in V'$ ($V' = \mathcal{A} \setminus V$). Then:

(i) the SNC of q on V under $\mathcal{K}$ is $\mathrm{F_{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q on V under $\mathcal{K}$ is $\neg\mathrm{F_{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge \neg q, V')$.

Proof: (i) As we know that any initial K-structure $\mathcal{K}$ can be described as a characterizing formula $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$, then the SNC of q on V under $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$ is $\mathrm{F_{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, \mathcal{A} \setminus V)$.

(ii) This is proved by the dual property. ■

Proposition10 Let φ be a CTL formula and $V \subseteq \mathcal{A}$. The time and space complexity of Algorithm 1 are $O(2^{m*2^m})$.

Proof: The time and space spent by Algorithm 1 is mainly the **for** cycles between lines 4 and 16. Under a given number i of states, there are i^i number of relations, i^{2^m} number of label functions and i number of possible initial states. In this case, we need the memory for the initial K-model in each time is $(i + i^i + i^{2^m} + 1)$.

For each $1 \leq i \leq 2^m$, there is at most $i{*}i^i{*}i^{2^m}{*}i = i^2{*}i^{(i+2^m)}$ possible initial K-models. Suppose that we can obtain an initial K-models in unit time (at each step), then we require $(2^m)^2 * (2^m)^{(2^m+2^m)} = (2^m)^{2+2*2^m}$ steps in the worst case. Therefore, the time and space complexity are $O(2^{m*2^m})$. ■

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