A Supplementary Material: Proof Appendix

Lemma 2. *Let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be the ones in the definition of section 3.1. Then, for each $i \geq 0$,*

(i) $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$;

(ii) there is a (smallest) $k \geq 0$ such that $\mathcal{B}_{k+1} = \mathcal{B}_k$;

(iii) $\mathcal{B}_i$ is reflexive, symmetric and transitive.

Proof. (i) Base: it is clear for $i = 0$ by the above definition.

Step: suppose it holds for $i = n$, i.e., $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.
$(s, s') \in \mathcal{B}_{n+2}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s_1') \in R'$ such that $(s_1, s_1') \in \mathcal{B}_{n+1}$, and (c) for every $(s', s_1') \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s_1') \in \mathcal{B}_{n+1}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s_1') \in R'$ such that $(s_1, s_1') \in \mathcal{B}_n$ by inductive assumption, and (c) for every $(s', s_1') \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s_1') \in \mathcal{B}_n$ by inductive assumption
$\Rightarrow (s, s') \in \mathcal{B}_{n+1}$.

(ii) and (iii) are evident from (i) and the definition of $\mathcal{B}_i$. $\square$

Lemma 1 The relation $\leftrightarrow_V$ is an equivalence relation.

Proof. It is clear from Lemma 2 (ii) such that there is a $k \geq 0$ where $\mathcal{B}_k = \mathcal{B}_{k+1}$ which is $\leftrightarrow_V$, and it is reflexive, symmetric and transitive by (iii). $\square$

Proposition 1 Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures (Ind-structures) such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:

(i) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(ii) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Proof. In order to distinguish the relations $\mathcal{B}_0, \mathcal{B}_1, \ldots$ for different set $V \subseteq \mathcal{A}$, by $\mathcal{B}_i^V$ we mean the relation $\mathcal{B}_1, \mathcal{B}_2, \ldots$ for $V \subseteq \mathcal{A}$. Denote as $\mathcal{B}_0, \mathcal{B}_1, \ldots$ when the underlying set V is clear from the context. Moreover, for the ease of notation, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$ (i.e., without subindex).

The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$, and

(c) for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$,

where $\mathcal{K}_i' = (\mathcal{M}_i, s_i')$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

$(\Rightarrow)$ (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s_1') \in R_1$, there is a $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s_1') - V = L_2(s_2') - V$. Therefore, $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}$. (c) This is similar with (b).

$(\Leftarrow)$ (a) $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) Condition (ii) implies that for every $(s_1, s_1') \in R_1$, there is $(s_2, s_2') \in R_2$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$; (c) Condition (iii) implies that for every $(s_2, s_2') \in R_2$, there is $(s_1, s_1') \in R_1$ such that $(\mathcal{K}_1', \mathcal{K}_2') \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(i) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing (s_1, s_3) from the (a), (b) and (c) of the previous steps of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(ii) Let $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$ and $(s_{i,k}, s_{i,k+1}) \in R_i$ mean that $s_{i,k+1}$ is the $(k + 2)$-th node in the path $(s_i, s_{i,1}, s_{i,2}, \ldots, s_{i,k+1}, \ldots)$ $(i = 1, 2)$. We will show that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_2}$ for all $n \geq 0$ inductively.

Base: $L_1(s_1) - V_1 = L_2(s_2) - V_1$
$\Rightarrow \forall q \in \mathcal{A} - V_1$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$
$\Rightarrow \forall q \in \mathcal{A} - V_2$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$ due to $V_1 \subseteq V_2$
$\Rightarrow L_1(s_1) - V_2 = L_2(s_2) - V_2$, i.e., $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0^{V_2}$.

Step: Supposing that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ for all $0 \leq i \leq k$ $(k > 0)$, we will show $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{k+1}^{V_2}$.

(a) It is apparent that $L_1(s_1) - V_2 = L_2(s_2) - V_2$ by base.

(b) $\forall (s_1, s_{1,1}) \in R_1$, we will show that there is a $(s_2, s_{2,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_{k-1}^{V_2}$ by inductive assumption, we need only to prove the following points:
(a) $\forall (s_{1,k}, s_{1,k+1}) \in R_1$ there is a $(s_{2,k}, s_{2,k+1}) \in R_2$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. It is easy to see that $L_1(s_{1,k+1}) - V_1 = L_1(s_{2,k+1}) - V_1$, then there is $L_1(s_{1,k+1}) - V_2 = L_1(s_{2,k+1}) - V_2$. Therefore, $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$.
(b) $\forall (s_{2,k}, s_{2,k+1}) \in R_1$ there is a $(s_{1,k}, s_{1,k+1}) \in R_1$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. This can be proved as (a).

(c) $\forall (s_2, s_{2,1}) \in R_1$, we will show that there is a $(s_1, s_{1,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. This can be proved as (ii).

$\square$

Theorem1 Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.

Proof. This theorem can be proved by inducting on the formula ϕ and supposing *Var*$(\phi) \cap V = \varnothing$. Let $\mathcal{K}_1 = (\mathcal{M}, s)$ and $\mathcal{K}_2 = (\mathcal{M}', s')$.

Case $\phi = p$ where $p \in \mathcal{A} - V$:
$(\mathcal{M}, s) \models \phi$ iff $p \in L(s)$ (by the definition of satisfiability)
$\Leftrightarrow p \in L'(s')$ $(s \leftrightarrow_V s')$
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \neg\psi$:
$(\mathcal{M}, s) \models \phi$ iff $(\mathcal{M}, s) \not\models \psi$
$\Leftrightarrow (\mathcal{M}', s') \not\models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \psi_1 \vee \psi_2$:
$(\mathcal{M}, s) \models \phi$
$\Leftrightarrow (\mathcal{M}, s) \models \psi_1$ or $(\mathcal{M}, s) \models \psi_2$
$\Leftrightarrow (\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{EX}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s, s_1, ...)$ such that $\mathcal{M}, s_1 \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s', s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_1 \leftrightarrow_V s'_1$ $(\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s'_1) \models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{EG}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that for each $i \geq 0$ there is $(\mathcal{M}, s_i) \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s' = s'_0, s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_i \leftrightarrow_V s'_i$ for each $i \geq 0$ $(\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s'_i) \models \psi$ for each $i \geq 0$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{E}[\psi_1 \mathrm{U} \psi_2]$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that there is $i \geq 0$ such that $(\mathcal{M}, s_i) \models \psi_2$, and for all $0 \leq j < i$, $(\mathcal{M}, s_j) \models \psi_1$
$\Leftrightarrow$ There is a path $\pi' = (s = s'_0, s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow (\mathcal{M}', s'_i) \models \psi_2$, and for all $0 \leq j < i$ $(\mathcal{M}', s'_j) \models \psi_1$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$ $\square$

Proposition 3 Let φ be a CTL formula, then $\varphi \equiv_{\langle V', I\rangle} T_\varphi$.

Proof. (sketch) This can be proved from T_i to T_{i+1} $(0 \leq i < n)$ by using one transformation rule on T_i. We will prove this proposition from the following several aspects:

(1) $\varphi \equiv_{\langle \{p\}, \varnothing\rangle} T_0$.

$(\Rightarrow)$ $\forall(\mathcal{M}_1, s_1) \in$ *Mod*(φ), *i.e.* $(\mathcal{M}_1, s_1) \models \varphi$. We can construct an Ind-model structure $\mathcal{M}_2$ is identical to $\mathcal{M}_1$ except $L_2(s_2) = L_1(s_1) \cup \{p\}$. It is apparent that $(\mathcal{M}_2, s_2) \models T_0$ and $(\mathcal{M}_1, s_1) \leftrightarrow_{\langle \{p\}, \varnothing\rangle} (\mathcal{M}_2, s_2)$.

$(\Leftarrow)$ $\forall(\mathcal{M}_1, s_1) \in$ *Mod*(T_0), it is apparent that $(\mathcal{M}_1, s_1) \models \varphi$ by the sematic of **start**.

By $\psi \rightarrow_t R_i$ we mean using transformation rules t on formula ψ (the formulae ψ as the premises of rule t) and obtaining the set R_i of transformation results. Let X be a set of formulas we will show $T_i \equiv_{\langle V', I\rangle} T_{i+1}$ by using the transformation rule t. Where $T_i = X \cup \{\psi\}$, $T_{i+1} = X \cup R_i$, V' is the set of atoms introduced by t and I is the set of indexes introduced by t. (We will prove this result in $t \in$ {Trans(1), Trans(4), Trans(6)}, other cases can be proved similarly.)

(2) For t=Trans(1):
$(\Rightarrow)$ $\forall(\mathcal{M}_1, s_1) \in$ *Mod*(T_i) *i.e.* $(\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \mathrm{EX}\varphi)$
$\Rightarrow (\mathcal{M}_1, s_1) \models X$ and for every π starting from s_1 and every state $s_1^j \in \pi$, $(\mathcal{M}, s_1^j) \models \neg q$ or there exists a path π' starting from s_1^j such that there exists a state s_1^{j+1} such that $(s_1^j, s_1^{j+1}) \in R_1$ and $(\mathcal{M}, s_1^{j+1}) \models \varphi$
We can construct an Ind-model structure $\mathcal{M}_2$ is identical to $\mathcal{M}_1$ except $[ind]_2 = \bigcup_{s \in S} R_s \cup R_y$, where $R_{s_1^j} = \{(s_1^j, s_1^{j+1}), (s_1^{j+1}, s_1^{j+2}), \dots\}$ and $R_y = \{(s_x, s_y) | \forall s_x \in S$ if $\forall(s'_1, s'_2) \in \bigcup_{s \in S} R_s, s'_1 \neq s_x$ then find a unique $s_y \in S$ such that $(s_x, s_y) \in R\}$. It is apparent that $(\mathcal{M}_1, s_1) \leftrightarrow_{\langle \varnothing, \{ind\}\rangle} (\mathcal{M}_2, s_2)$ (let $s_2 = s_1$).
$\Rightarrow$ for every path starting from s_1 and every state s_1^j in this path, $(\mathcal{M}_2, s_1^j) \models \neg q$ or $(\mathcal{M}_2, s_1^j) \models \mathrm{EX}\varphi_{\langle ind\rangle}$ (by the semantic of EX)
$\Rightarrow (\mathcal{M}_2, s_1) \models \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}\varphi)$
$\Rightarrow (\mathcal{M}_2, s_1) \models X \wedge \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}\varphi)$

$(\Leftarrow)$ $\forall(\mathcal{M}_1, s_1) \in$ *Mod*(T_{i+1}) *i.e.* $(\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}\varphi)$
$\Rightarrow (\mathcal{M}_1, s_1) \models X$ and $(\mathcal{M}_1, s_1) \models \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}\varphi)$
$\Rightarrow$ for every path starting from s_1 and every state s_1^j in this path, $(\mathcal{M}_1, s_1^j) \models \neg q$ or there exits a state s' such that $(s_1^j, s') \in [ind]_1$ and $(\mathcal{M}_1, s') \models \varphi$ (by the semantic of $\mathrm{E}_{\langle ind\rangle}\mathrm{X}$)
$\Rightarrow$ for every path starting from s_1 and every state s_1^j in this path, $(\mathcal{M}_1, s_1^j) \models \neg q$ or $(\mathcal{M}_1, s_1^j) \models \mathrm{EX}\varphi$ (by the semantic of EX)
$\Rightarrow (\mathcal{M}_1, s_1) \models \mathrm{AG}(q \supset \mathrm{EX}\varphi)$
$\Rightarrow (\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \mathrm{EX}\varphi)$
It is apparent that $(\mathcal{M}_1, s_1) \leftrightarrow_{\langle \varnothing, \{ind\}\rangle} (\mathcal{M}_1, s_1)$.

(3) For t=Trans(4):
$(\Rightarrow)$ $\forall(\mathcal{M}_1, s_1) \in$ *Mod*(T_i), *i.e.* $(\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \varphi_1 \vee \varphi_2)$
$\Rightarrow (\mathcal{M}_1, s_1) \models X$ and $\forall s'_1 \in S, (\mathcal{M}_1, s'_1) \models q \supset \varphi_1 \vee \varphi_2$
$\Rightarrow (\mathcal{M}_1, s'_1) \models \neg q$ or $(\mathcal{M}_1, s'_1) \models \varphi_1 \vee \varphi_2$
The we can construct an Ind-model structure $\mathcal{M}_2$ as follows. $\mathcal{M}_2$ is the same with $\mathcal{M}_1$ when $(\mathcal{M}_1, s'_1) \models \neg q$. When $(\mathcal{M}_1, s'_1) \models q$, $\mathcal{M}_2$ is identical to $\mathcal{M}_1$ except if $(\mathcal{M}_1, s'_1) \models \varphi_1$ then $L_2(s'_1) = L_1(s'_1)$ else $L_2(s'_1) = L_1(s'_1) \cup \{p\}$. It is apparent that $(\mathcal{M}_2, s'_1) \models (q \supset \varphi_1 \vee p) \wedge (p \supset \varphi_2)$, then $(\mathcal{M}_2, s_1) \models T_{i+1}$ and $(\mathcal{M}_1, s_1) \leftrightarrow_{\langle \{p\}, \varnothing\rangle} (\mathcal{M}_2, s_2)$.

$(\Leftarrow)$ $\forall(\mathcal{M}_1, s_1) \in$ *Mod*(T_{i+1}), *i.e.* $(\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \varphi_1 \vee p) \wedge \mathrm{AG}(p \supset \varphi_2)$. It is apparent that $(\mathcal{M}_1, s_1) \models T_i$.

(4) For t=Trans(6):
We prove for $\mathrm{E}_{\langle ind\rangle}\mathrm{X}$, while for the AX can be proved similarly.

($\Rightarrow$) $\forall(\mathcal{M}_1, s_1) \in Mod(T_i)$, *i.e.* $(\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}\varphi)$
$\Rightarrow (\mathcal{M}_1, s_1) \models X$ and $\forall s_1' \in S, (\mathcal{M}_1, s_1') \models q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}\varphi$
$\Rightarrow (\mathcal{M}_1, s_1') \models \neg q$ or there exists a state s' such that $(s_1', s') \in [ind]$ and $(\mathcal{M}_1, s') \models \varphi$
We can construct an Ind-model structure $\mathcal{M}_2$ as follows. $\mathcal{M}_2$ is the same with $\mathcal{M}_1$ when $(\mathcal{M}_1, s_1') \models \neg q$. When $(\mathcal{M}_1, s_1') \models q$, $\mathcal{M}_2$ is identical to $\mathcal{M}_1$ except for s' there is $L_2(s') = L_1(s') \cup \{p\}$. It is apparent that $(\mathcal{M}_2, s_1) \models \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}p) \wedge \mathrm{AG}(p \supset \varphi)$, $(\mathcal{M}_2, s_2) \models T_{i+1}$ and $(\mathcal{M}_1, s_1) \leftrightarrow_{\langle\{p\},\emptyset\rangle} (\mathcal{M}_2, s_2)$ $(s_2 = s_1)$.

($\Leftarrow$) $\forall(\mathcal{M}_1, s_1) \in Mod(T_{i+1})$, *i.e.* $(\mathcal{M}_1, s_1) \models X \wedge \mathrm{AG}(q \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}p) \wedge \mathrm{AG}(p \supset \varphi)$. It is apparent that $(\mathcal{M}_1, s_1) \models T_i$.

□

Proposition 4 Let φ be a CTL formula, then $T_\varphi \equiv_{\langle V\cup V',\emptyset\rangle} Res$.

Proof. (sketch) This can be proved from T_i to T_{i+1} $(0 \le i < n)$ by using one resolution rule on T_i.

By $\psi \to_r R_i$ we mean using resolution rules r on set ψ (the formulae in ψ as the premises of rule r) and obtaining the set R_i of resolution results. we will show $T_i \equiv_{\langle V,I\rangle} T_{i+1}$ by using the resolution rule r. Where $T_i = X \cup \psi$, $T_{i+1} = X \cup R_i$, X be a set of $\mathrm{SNF}^g_{\mathrm{CTL}}$ clauses, p be the proposition corresponding with literal l used to do resolution in r.

(1) If $\psi \to_r R_i$ by an application of $r \in \{$**(SRES1)**$, \ldots,$ **(SRES8)**, **RW1**, **RW2**$\}$, then $T_i \equiv_{\langle\{p\},\emptyset\rangle} T_{i+1}$.

On one hand, it is apparent that $\psi \models R_i$ and then $T_i \models T_{i+1}$. On the other hand, $T_i \subseteq T_{i+1}$ and then $T_{i+1} \models T_i$.

(2) If $\psi \to_r R_i$ by an application of $r =$**(ERES1)**, then $T_i \equiv_{\langle\{l,w^{\mathrm{A}}_{\neg l}\},\emptyset\rangle} T_{i+1}$.

It has been proved that $\psi \models R_i$ in (Bolotov 2000), then there is $T_{i+1} = T_i \cup \Lambda^{\mathrm{A}}_{\neg l}$ and then $\forall(\mathcal{M}_1, s_1) \in Mod(T_i = X \cup \psi)$ there is a $(\mathcal{M}_2, s_2) \in Mod(T_{i+1} = T_i \cup \Lambda^{\mathrm{A}}_{\neg l})$ s.t. $(\mathcal{M}_1, s_1) \leftrightarrow_{\langle\{p,w^{\mathrm{A}}_{\neg l}\},\emptyset\rangle} (\mathcal{M}_2, s_2)$ and vice versa by Proposition 3.

For rule **(ERES2)** we have the same result.

□

Proposition 6 Let $V'' = V \cup V'$, $\Gamma = $ *Instantiate*(Res, V') and $\Gamma_1 = $ *Removing_atoms* (*Connect*$(\Gamma), \Gamma)$, then $\Gamma_1 \equiv_{\langle V'',\emptyset\rangle} Res$ and $\Gamma_1 \equiv_{\langle V'',I\rangle} \varphi$.

Proof. Take note the fact that for each clause $C = T \supset H$ in *Connect*(Γ), if $\Gamma \cap Var(C) \neq \emptyset$ then there must be an atom $p \in \Gamma \cap Var(H)$. It is apparent that *Connect*$(\Gamma) \models \Gamma_1$, we will show $\forall(\mathcal{M}, s_0) \in Mod(\Gamma_1)$ there is a $(\mathcal{M}', s_0)$ such that $(\mathcal{M}', s_0) \models$ *Connect*(Γ) and $(\mathcal{M}, s_0) \leftrightarrow_{\langle\Gamma,\emptyset\rangle} (\mathcal{M}', s_0)$. Let $C = T \supset H$ in *Connect*(Γ) with $\Gamma \cap Var(C) \neq \emptyset$, $\forall(\mathcal{M}, s_0) \in Mod(\Gamma_1)$ we construct $(\mathcal{M}', s_0)$ as $(\mathcal{M}, s_0)$ except for each $s \in S$, if $(\mathcal{M}, s) \not\models T$ then $L'(s) = L(s)$, else:

(i) if $(\mathcal{M}, s) \models H$, then $L'(s) = L(s)$;

(ii) else if $(\mathcal{M}, s) \models T$ with $p \in Var(H) \cap \Gamma$, then if p appearing in H negatively, then if C is a global (or an initial) clause then let $L'(s) = L(s) \setminus \{p\}$ else let $L'(s^*) = L(s^*) \setminus \{p\}$ for (each (if C is an A-step or A-sometime clause)) $s^* \in \pi_s$, else if C is a global (or an initial) clause then let $L'(s) = L(s) \cup \{p\}$ else let $L'(s^*) = L(s^*) \cup \{p\}$ for (each (if C is a A-step or A-sometime clause)) $s^* \in \pi_s$. Where s^* is a next or future state of s (it depends on the type of the clause: if the clause is a X-step ($X \in \{\mathrm{A}, \mathrm{E}\}$) clause then s^* is the next state, else if the clause is a X-sometime clause then s^* is a future state).

It is apparent that $(\mathcal{M}, s_0) \leftrightarrow_{\langle\Gamma,\emptyset\rangle} (\mathcal{M}', s_0)$, we will show that $(\mathcal{M}', s_0) \models$ *Connect*(Γ) from the following two points:

(1) For (i), it is apparent $(\mathcal{M}', s_0) \models C$;
(2) For (ii) talked-above, we show it from the form of $\mathrm{SNF}^g_{\mathrm{CTL}}$ clauses. Supposing C_1 and C_2 are instantiate formula of Γ:
   (a) If C is a global clause, i.e. $C = \top \supset p \vee C_1$ with C_1 is a disjunction of literals (we suppose p appearing in C positively). If there is a $C' = \top \supset \neg p \vee C_2 \in$ *Connect*(Γ), then there is $\top \supset C_1 \vee C_2 \in$ *Connect*(Γ) by the resolution ($(\mathcal{M}, s) \models C_2$ due to we have suppose $(\mathcal{M}, s) \not\models C$). It is apparent that $(\mathcal{M}', s_0) \models C \wedge C'$.
   (b) If $C = T \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}(p \vee C_1)$. If there is a $C' = T' \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}(\neg p \vee C_2) \in$ *Connect*(Γ), then there is $T \wedge T' \supset \mathrm{E}_{\langle ind\rangle}\mathrm{X}(C_1 \vee C_2) \in$ *Connect*(Γ) by the resolution ($(\mathcal{M}, s) \models \mathrm{E}_{\langle ind\rangle}\mathrm{X}C_2$ due to we have suppose $(\mathcal{M}, s) \not\models C$). It is apparent that $(\mathcal{M}', s_0) \models C \wedge C'$.
   (c) Other cases can be proved similarly.

Therefore, we have $\Gamma_1 \equiv_{\langle V'',\emptyset\rangle} Res$ by Proposition 2 and Proposition 5.

And then $\Gamma_1 \equiv_{\langle V'',I\rangle} \varphi$ follows. □