

On the Weakest Sufficient Conditions in Propositional μ -calculus

No Author Given

No Institute Given

Abstract. The μ -calculus is one of the most important logics describing specifications of transition systems. It has been extensively explored for formal verification in model checking due to its exceptional balance between expressiveness and algorithmic properties. On the one hand, some information content in a specification might become irrelevant or unnecessary due to various reasons from the perspective of knowledge representation. On the other hand, a weakest precondition of a specification is badly necessary in verification, where a (weakest) precondition is sufficient for a transition system to enjoy a desire property. This paper is to address these scenarios for μ -calculus in a principle way in terms of knowledge *forgetting*. In particular, it proposes a notion of forgetting by a generalized bisimilar equivalence (over a signature) and explores its important properties as a knowledge distilling operator, besides some reasoning complexity results. It then shows that how the weakest sufficient condition and the strongest necessary condition can be established via forgetting. It also discusses knowledge update for μ -calculus in terms of forgetting.

Keywords: Weakest precondition · Forgetting · Knowledge update.

1 Introduction

Propositional μ -calculus is an expressive logic, on binary trees it is as expressive as the monadic second order logic of two successors (S2S) [14,24]. Subsequent research showed that the μ -calculus is an important logic when specification and verification is concerned. One of the common phenomena in both the verification and the system design is some information content of such specification might become irrelevant for the system due to various reasons e.g., it might be discarded or become obsolete by time, or just become infeasible due to practical difficulties. However, In this case it is expensive in time and space to re-extract the specification that meets the requirements. The problem arises on how to remove it without altering the relevant system behaviour or violating the existing system specifications over a given signature. Let's consider the following scenario.

Example 1 (Playing PingPong). Assume *John* plays PingPong with $n + 1$ people (i.e. $n + 2$ people in total) and there are $n + 2$ chairs. At the beginning, *John* plays PingPong with one of the $n + 1$ people, and the other n people are sitting on their chairs at this time. In our scenario, we assume *John* can not win anyone, which means that *John* has always been beaten by other people every turn. In this case, after n games it will be *John's* turn. This process can be modeled by Fig. 1 (a).

In Fig. 1 (a), this scenario is represented by the Kripke structure $\mathcal{M} = (S, s_0, R, L)$ with the corresponding atomic variables $V = \{j, ch\}$, in which j means *John's turn* and ch means the *John's chair is free*. Besides, $m = 2^n$, this means there are m orders to *John's turn*.

In this scenario, we can see from the unwinding of \mathcal{M} in Fig. 1 (b) that for each (some) path $\pi = (s_0, s_1, \dots)$ starting from s_0 we have $\forall i \in \mathbb{N}$ there is $(\mathcal{M}, s_{2*i}) \models ch \wedge j$ and $(\mathcal{M}, s_{2*i+1}) \models \neg ch \wedge \neg j$, i.e. for each *even state* in each (some) path that starting from s_0 the $ch \wedge j$ holds and $\neg ch \wedge \neg j$ holds for *old states* in this path. This property can be represented by a μ -calculus formula $\varphi = \nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$ ($\varphi = \nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$), this is not expressible with other temporal logics.

Now assume a situation in which due to some problems (i.e. the venue changed or the chair broke down), John does no longer have a chair. This means, all the playing processes concerning “ ch ” no more necessary and should be dropped from both the specifications (e.g. φ) and the Kripke structure for simplification.

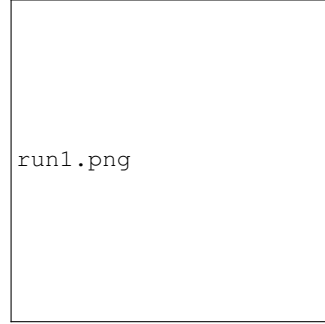
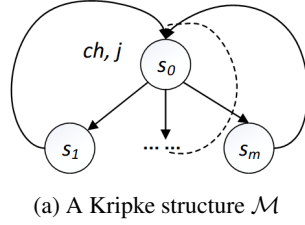


Fig. 1: The PingPong's Model

Similar scenarios like the one presented in Example 1 may arise in many different domains such as business-process modelling, software development, concurrent systems and more [2]. Yet dropping some restrictions in a large and complex system or specification, without affecting the working system components or violating dependent specifications over a given signature, is a non-trivial task. Moreover, in such a scenario, two logical notions introduced by E. Dijkstra in [11] are highly informative: the strongest post-condition (SP) and the weakest precondition (WP) of a given specification, which are corresponding with the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC), proposed by Lin [22], of the specification, respectively, and have been central to a wide variety of tasks and studies, e.g. in generating counterexamples [10] and refinement of system [27]. These correspond to the *most general consequence* and the *most specific abduction* of such specification, respectively.

Besides, *belief updates* and *belief revision*, as two well-studied issues in artificial intelligence, are concerned with the update and revision aspects of an agent's belief with respect to new beliefs [18]. Intuitively, if φ represents the agent's belief about the world and the agent performs an action that is supposed to make ψ true in the resulting world, then the agent's belief about the resulting world can be described by $\varphi \diamond \psi$, where \diamond is the update operator of choice. We can see that the theory of belief updates does not tell us how to do updates with respect to such gain in knowledge due to a sensing action. In this sense, as an analogous notion of belief update, the *knowledge update* was proposed by Chitta in [3] to solve the belief updates caused by sensing actions, in which the effect of a sensing action is expressed by introducing modal operator (K)nows. Nevertheless, there are no approaches to solve the knowledge update in logic languages which contain *temporal operators*.

To address these scenarios and to target the relevant notions SNC (WSC) and knowledge update in a principle way. Inspired by [22,16], in this paper we explore the knowledge update and SNC (WSC) of μ -calculus from the point of forgetting. In particular, we will give the definition of forgetting in μ -calculus by using the bisimulation [5,2,28] and show whether this notion satisfies the general principles or postulates proposed by Zhang [28]. We then study the relationship between SNC (WSC) and forgetting, and we also demonstrate how forgetting can be used in knowledge update in μ -calculus.

Forgetting, which is a dual concept of *uniform interpolation* [25,19] and was first formally defined in propositional and FOL by Lin and Reiter [23,12], can be traced back to the work of Boole on propositional variable elimination and the seminal work of Ackermann [1]. Moreover, it has been extended to various logic systems. See [?,12,?] for a recent and comprehensive survey. Particularly, in classical propositional logic (C-PL) the result of forgetting atom p from formula φ is $\varphi[p/\top] \vee \varphi[p/\perp]$, that is the disjunction of formulas obtained from φ by replacing p with \top and \perp respectively.

However, existing forgetting definitions in PL and answer set programming are not directly applicable in modal logics. And we can also not directly use the method of forgetting in CTL [16] since it will not work when the models of the formula are infinite. Hopefully, it has been proved that the modal μ -calculus has *Uniform interpolation* [7]. Informally, for every μ sentence φ and every finite set V of atoms, there exists a μ sentence $\exists V \varphi$ which does not contain atoms from V but is logically closest to φ in some sense. This means that the result of forgetting some atoms from a μ -calculus sentence always exists. In this sense, showing the semantic of forgetting in μ -calculus through *general principles or postulates* is important to make it clearer to understand.

Informally, the four postulates proposed by Zhang [28] show that the result ψ of forgetting some set V of atoms from a formula φ is not only weaker than the φ , i.e. $\varphi \models \psi$, irrelevant to V , i.e. exists some formula that do not contain atoms in V and equivalent with ψ , and also has the same “logic content” with φ , i.e. for each formula ϕ that irrelevant to V , ϕ can be implied by φ iff ϕ can be implied by ψ . In this paper we explore the forgetting of μ -calculus under infinite models (to distinguish that of CTL [16]) from both the postulates and the algebraic properties of the forgetting operator. The complexities of the reasoning problems of the forgetting operator are also explored from the point of automaton and it is shown that these are in EXPTIME-complete. It is worth mentioning that we restrict the models are finite in the knowledge update part

in order to express the models of a formula. And we show that our definition of \diamond_μ by forgetting satisfies Katsuno and Mendelzon's update postulates [18].

2 Modal μ -calculus

We start with some technical and notational preliminaries. Given a formula φ , the *language* of φ , denoted $L(\varphi)$, is the set of all propositional constants appearing in the formula.

2.1 Syntax

Modal μ -calculus is an extension of modal logic, we consider the propositional μ -calculus as introduced by Kozen [20]. Let $\mathcal{A} = \{p, q, \dots\}$ be a set of propositional letters (atoms) and $\mathcal{V} = \{X, Y, \dots\}$ a set of variables. Formulas of the μ -calculus over these sets can be defined by the following grammar:

$$\varphi := p \mid \neg p \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \diamond \varphi \mid \square \varphi \mid \mu X. \varphi \mid \nu X. \varphi$$

where $p \in \mathcal{A}$ and $X \in \mathcal{V}$. \top and \perp are also μ -calculus formulae, which express 'true' and 'false' respectively. Note that we allow negations only before propositional letters. All the results presented here extend to the general case when negation before variables is also allowed, restricting as usual to positive occurrences of bound variables, that is to say variables appear after an even number of negations. Variables, propositional letters and their negations will be called *literals*. For convenience, in the following, $\varphi, \varphi_1, \dots, \psi, \psi_1, \dots$ will denote formulas. And by $\text{Var}(\varphi)$ we mean the set of atoms appearing in formula φ .

We call a formula *well named* iff every variable is bound at most once in the formula and free variables are distinct from bound variables. For a variable X bound in a well named formula φ there exists a unique subterm of φ of the form $\delta X. \varphi(X)$ with $\delta \in \{\nu, \mu\}$, from now on called the *binding definition* of X in φ . We call X a μ -variable when $\delta = \mu$, otherwise we call X a ν -variable.

Variable X in $\delta X. \varphi(X)$ is *guarded* iff every occurrence of X in $\varphi(X)$ is in the scope of some modality operator \diamond or \square . For convenience, we mix the two symbols \diamond and EX (\square and AX). A formula is guarded iff every bound variable in the formula is guarded.

Proposition 1. *Every formula is equivalent to some guarded formula.*

This proposition allows us to restrict ourselves to **guarded, well-named** formulas. From now on, we shall only consider formulas of this kind.

2.2 Semantic

Formulas are interpreted in transition systems of the form $\mathcal{M} = (S, r, R, L)$, we call it a Kripke structure, where:

- S is a nonempty set of states,

- $r \in S$,
- R is a binary relation on S , i.e. $R \subseteq S \times S$, called a transition relation, and
- $L : S \rightarrow 2^{\mathcal{A}}$ is a labeling function.

A Kripke structure \mathcal{M} is finite if S is finite and $q \notin L(s)$ (for each $s \in S$) for almost all $q \in \mathcal{A}$.

Given a Kripke structure \mathcal{M} and a valuation $v : \mathcal{V} \rightarrow 2^S$, the set of states in which a formula φ is true, denoted $\|\varphi\|_v^{\mathcal{M}}$, is defined inductively as follows (we will omit superscript \mathcal{M} when it causes no ambiguity):

$$\begin{aligned}
\|p\|_v &= \{s \mid p \in L(s)\} \quad \|\top\|_v = S \quad \|\perp\|_v = \emptyset \\
\|\neg p\|_v &= S - \|p\|_v \\
\|X\|_v &= v(X) \\
\|\varphi_1 \vee \varphi_2\|_v &= \|\varphi_1\|_v \cup \|\varphi_2\|_v \\
\|\varphi_1 \wedge \varphi_2\|_v &= \|\varphi_1\|_v \cap \|\varphi_2\|_v \\
\|\Diamond \varphi\|_v &= \{s \mid \exists s'. (s, s') \in R \wedge s' \in \|\varphi\|_v\} \\
\|\Box \varphi\|_v &= \{s \mid \forall s'. (s, s') \in R \Rightarrow s' \in \|\varphi\|_v\} \\
\|\mu X. \varphi\|_v &= \bigcap \{S' \subseteq S \mid \|\varphi\|_{v[X:=S']} \subseteq S'\} \\
\|\nu X. \varphi\|_v &= \bigcup \{S' \subseteq S \mid S' \subseteq \|\varphi\|_{v[X:=S']}\}
\end{aligned}$$

where $v[X := S']$ is same to the valuation function v except that S' is assigned to X .

In the following, we denote $s \in \|\varphi\|_v$ by $(\mathcal{M}, s, v) \models \varphi$ and we may leave out the valuation v , if φ is a sentence (i.e. no variables in φ are free). $(\mathcal{M}, v) \models \varphi$ is used to denote $(\mathcal{M}, r, v) \models \varphi$, and in this case we say that (\mathcal{M}, v) is a model of φ . By $Mod(\varphi)$ we mean the set of models of φ . In particular, if φ is a sentence, then $Mod(\varphi) = \{\mathcal{M} \mid (\mathcal{M}, r) \models \varphi\}$. Similarly, let Σ be a set of sentences, we define $Mod(\Sigma)$ as the set of Kripke structure \mathcal{M} such that $\mathcal{M} \models \varphi$ for each $\varphi \in \Sigma$. $\varphi \models \psi$ denotes logical consequence: if $(\mathcal{M}, v) \models \varphi$ then $(\mathcal{M}, v) \models \psi$ for every model \mathcal{M} and for every valuation v . Especially, given two sentences (or set of sentences) Σ and Π , $\Sigma \models \Pi$ if $Mod(\Sigma) \subseteq Mod(\Pi)$ and $\Sigma \equiv \Pi$ whenever $Mod(\Sigma) = Mod(\Pi)$. A formula ϕ is *irrelevant* to the atoms in a set V (or simply *V-irrelevant*), written $IR(\phi, V)$ of atoms, if there is a formula ψ with $Var(\psi) \cap V = \emptyset$ such that $\phi \equiv \psi$. The *V-irrelevance* of a set of formulas can be defined similarly, i.e. a set Σ of formulas is irrelevant to the atoms in V , written $IR(\Sigma, V)$, if $IR(\varphi, V)$ for each $\varphi \in \Sigma$.

By the Tarski-Knaster theorem, the least and greatest fixpoints of monotonic functions f over subsets of a set U can be obtained by transfinite induction, i.e., the least fixpoint $\mu(f) = \bigcup \mu_\alpha(f)$, where

- $\mu_0(f) = \emptyset$,
- $\mu_{\alpha+1} = f(\mu_\alpha(f))$,
- $\mu_\lambda = \bigcup_{\alpha < \lambda} \mu_\alpha(f)$, for λ a limit ordinal.

Similarly, the greatest fixpoint $\nu(f) = \bigcap \nu_\alpha(f)$, where

- $\nu_0(f) = U$,

- $\nu_{\alpha+1} = f(\nu_\alpha(f))$,
- $\nu_\lambda = \bigcap_{\alpha < \lambda} \nu_\alpha(f)$, for λ a limit ordinal.

Apparently, in $\mu X.\varphi(X)$, $\varphi(X)$ is a monotonic function about X since X appearing in $\varphi(X)$ positively. Therefore, $\|\mu X.\varphi(X)\|_v$ is the least fixpoint of the monotone operator $\|\varphi\|_v : 2^S \rightarrow 2^S$ (it is also written as $Y \mapsto \|\varphi\|_{v[X:=Y]}$ sometimes). Let's see an example to show how to compute the greatest fixpoints of the μ -formula φ talked in Example 1.

Example 2. (1) For $\varphi = \nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$, let $\mathcal{M} = (S, r, R, L)$ be a Kripke structure, then we have:

- $\nu_0(\|\varphi\|) = S$,
- $\nu_1(\|\varphi\|) = \|j\| \cap \|ch\| \cap \|\Box(\neg j \wedge \neg ch)\| \cap \|\Box\Box\nu_0(\|\varphi\|)\| = \{s \mid s \in L(j) \cap L(ch) \text{ and } \forall t.(s, t) \in R, t \in S - (L(j) \cup L(ch))\}$,
- ...

(2) For $\varphi = \nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$, let $\mathcal{M} = (S, r, R, L)$ be a Kripke structure, then we have:

- $\nu_0(\|\varphi\|) = S$,
- $\nu_1(\|\varphi\|) = \|j\| \cap \|ch\| \cap \|\Box(\neg j \wedge \neg ch)\| \cap \|\Box\Box\nu_0(\|\varphi\|)\| = \{s \mid s \in L(j) \cap L(ch) \text{ and } \exists t.(s, t) \in R, t \in S - (L(j) \cup L(ch))\}$,
- $\nu_2(\|\varphi\|) = \|j\| \cap \|ch\| \cap \|\Box(\neg j \wedge \neg ch)\| \cap \|\Box\Box\nu_1(\|\varphi\|)\|$
- ...

For the Kripke structure \mathcal{M} in (a) of Example 1, we have $\|\varphi\| = \{s_0\}$.

An alternative syntax for the μ -calculus is obtained by substituting the \Diamond operator with a set of *cover operators*, one for each natural number n . For $n \geq 1$ these operators are defined as follows: if $\varphi_1, \dots, \varphi_n$ are formulas, then

$$Cover(\varphi_1, \dots, \varphi_n)$$

is a formula. The constant operator $Cover(\emptyset)$ is also allowed. The cover operators are interpreted in a Kripke structure $\mathcal{M} = (S, r, R, L)$ as follows: $Cover(\emptyset)$ is true in \mathcal{M} if and only if the root of \mathcal{M} does not have any successor, while $Cover(\varphi_1, \dots, \varphi_n)$ is true in \mathcal{M} if and only if the successors of the root are covered by $\varphi_1, \dots, \varphi_n$. More formally, $(\mathcal{M}, s, v) \models Cover(\varphi_1, \dots, \varphi_n)$ if and only if:

- for every $i = 1, \dots, n$ there exists t with $(s, t) \in R$ and $(\mathcal{M}, t, v) \models \varphi_i$;
- for every t with $(s, t) \in R$ there exists $i \in \{1, \dots, n\}$ with $(\mathcal{M}, t, v) \models \varphi_i$.

We call this syntax the *covers-syntax* to distinguish it from the original \Diamond -syntax.

Since $Cover(\varphi_1, \dots, \varphi_n)$ is equivalent to

$$\Diamond\varphi_1 \wedge \dots \wedge \Diamond\varphi_n \wedge \Box(\varphi \vee \dots \vee \varphi_n),$$

cover operators are definable in the \Diamond syntax. Conversely,

$$\Diamond\varphi \Leftrightarrow Cover(\varphi, \top).$$

Hence, the μ -calculus obtained from the covers-syntax is equivalent to the familiar μ -calculus constructed using the \Diamond -syntax. In this paper we use a mixture of the two syntax because, as we shall see, cover operators behave nicely with respect to the definition of disjunctive formula. Specially, the μ -calculus formulas have a normal form, called disjunctive formula [17] as follows.

Definition 1 (disjunctive formula). *The set of disjunctive formulas, \mathcal{F}_d is the smallest set defined by the following clauses:*

- *disjunctions and non-contradictory conjunction of literals are disjunctive formulas;*
- *special conjunctions: if $\varphi_1, \dots, \varphi_n \in \mathcal{F}_d$ and δ is a non contradictory conjunction of literals, then $\delta \wedge \text{Cover}(\varphi_1, \dots, \varphi_n) \in \mathcal{F}_d$;*
- *fixpoint operators: if $\varphi \in \mathcal{F}_d$, φ does not contain $X \wedge \psi$ as a subformula for any formula ψ , and X is positive in φ , then $\mu X.\varphi$, $\nu X.\varphi$ are in \mathcal{F}_d .*

Example 3. It is easy to check that both $\nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond \Diamond X$ and $\nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box \Box X$ are not disjunctive formulas. While $j \wedge ch \wedge \Diamond(\neg j \wedge \neg ch)$, $\mu X.(j \wedge ch) \wedge \Diamond X$ and $\nu X.(j \wedge ch) \wedge \Diamond \Diamond X$ are disjunctive formulas because we have:

$$j \wedge ch \wedge \Diamond(\neg j \wedge \neg ch) \equiv j \wedge ch \wedge \text{Cover}(\neg j \wedge \neg ch, \top),$$

$$\mu X.(j \wedge ch) \wedge \Diamond X \equiv \mu X.(j \wedge ch) \wedge \text{Cover}(X, \top)$$

and

$$\nu X.(j \wedge ch) \wedge \Diamond \Diamond X \equiv \nu X.(j \wedge ch) \wedge \text{Cover}(\text{Cover}(X, \top), \top).$$

Theorem 1 ([17]). *Any μ -calculus formula is equivalent to a disjunctive one.*

Theorem 1 means that for each μ -calculus formula φ there is a disjunctive formula $\psi \in \mathcal{F}_d$ such that $\varphi \equiv \psi$. Deciding whether a disjunctive formula is satisfiable can be done in polynomial time, hence changing the μ -calculus formula into its disjunctive form will increase the efficiency of deciding its satisfiability since the satisfiability of μ -calculus is in EXPTIME-complete.

Another important concept is *uniform interpolation*, which has been widely explored in various of logic languages. Some logics enjoy uniform interpolation, but some are not. Formally, the uniform interpolation under μ -calculus is defined as follows.

Definition 2 (Uniform interpolation [9]). *Given a μ -sentence φ and a set of atoms $L' \subseteq \text{Var}(\varphi)$, the uniform interpolant of φ with respect to L' is a μ -sentence ψ such that:*

- $\varphi \models \psi$;
- *whenever $\varphi \models \varphi_1$ and $\text{Var}(\varphi) \cap \text{Var}(\varphi_1) \subseteq L'$ then $\psi \models \varphi_1$;*
- $\text{Var}(\psi) \subseteq L'$.

Notice that uniform interpolation is stronger than Craig interpolation, which states that for any two formulas φ, ψ with $\varphi \models \psi$ there exists a formula θ (called the Craig interpolant of φ, ψ) in the common language with $\varphi \models \theta$ and $\theta \models \psi$. Clearly, if the logic enjoys uniform interpolation then the Craig interpolant of φ, ψ does not depend on

ψ but only on the common language: it is simply the uniform interpolant of φ relative to $L' = \text{Var}(\varphi) \cap \text{Var}(\psi)$.

It has been proved that the μ -calculus has uniform interpolation [8,7]. Moreover, when the given formula is a disjunctive formula then the uniform interpolant of it can be obtained by replacing both the pointed atoms and their negation with \top at the same time [9]. Formally:

Theorem 2. *The uniform interpolant $\tilde{\exists}p\varphi$ ($p \in \mathcal{A}$) of a disjunctive formula φ is equivalent to the μ -formula $\varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\top]$ is defined from φ by simultaneously substituting the literals p and $\neg p$ with \top .*

3 Forgetting in μ -calculus

In this section we present the definition of forgetting in μ -calculus and investigate its semantic properties. First, we give the definition of V -bisimulation between Kripke structures. The notion of V -bisimulation captures the idea that the two systems are behaviourally the same except for the atoms in V . In this way we give the definition of forgetting by using the V -bisimulation.

Second, the related properties, e.g. Modularity, Commutativity and Homogeneity, of the forgetting operator will be explored. And last, we show that the model checking problem of forgetting V from a disjunctive formula is $\text{NP} \cap \text{co-NP}$ and the reasoning problems are EXPTIME -complete.

3.1 Definition of Forgetting

In this subsection we present the definition of forgetting in μ -calculus by the bisimulation technique. For convenience, in the following we let $\mathcal{M}_i = (S_i, r_i, R_i, L_i)$ with $i \in \mathbb{N}$ be Kripke structures.

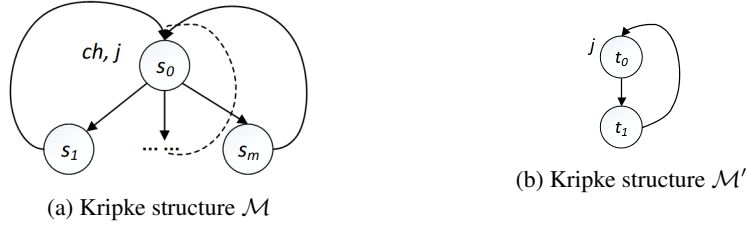
Definition 3 (V-bisimulation). *Let $V \subseteq \mathcal{A}$, \mathcal{M}_1 and \mathcal{M}_2 be two Kripke structures. $\mathcal{B} \subseteq S_1 \times S_2$ is a V -bisimulation between \mathcal{M}_1 and \mathcal{M}_2 if:*

- $r_1 \mathcal{B} r_2$,
- for each $s \in S_1$ and $t \in S_2$, if $s \mathcal{B} t$ then $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A} - V$,
- $(s, s') \in R_1$ and $s \mathcal{B} t$ imply that there is a t' such that $s' \mathcal{B} t'$ and $(t, t') \in R_2$, and
- vice versa: if $s \mathcal{B} t$ and $(t, t') \in R_2$ then there is an s' with $(s, s') \in R_1$ and $t' \mathcal{B} s'$.

Two Kripke structures \mathcal{M}_1 and \mathcal{M}_2 are V -bisimilar, denoted $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$, if there exists a V -bisimulation \mathcal{B} between them. The V -bisimulation is similar with the \mathcal{L} -bisimulation in [7], which is a relation satisfying the above clauses just for the symbols in language \mathcal{L} . From this definition we know that the \leftrightarrow_V is an equivalence relation between Kripke structures. Formally:

Proposition 2. *Let $V, V_1 \subseteq \mathcal{A}$, \mathcal{M}_1 , \mathcal{M}_2 and \mathcal{M}_3 be three Kripke structures, then we have:*

- (i) *the \leftrightarrow_V is an equivalence relation between Kripke structures;*

Fig. 2: Two $\{ch\}$ -bisimilar Kripke structures

(ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$, then $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.

Example 4. In Fig. 2 we can check that $\mathcal{M} \leftrightarrow_{\{ch\}} \mathcal{M}'$ because there is a $\{ch\}$ -bisimulation $\mathcal{B} = \{(s_0, t_0), (s_1, t_1), (s_2, t_1), \dots, (s_m, t_1)\}$ between \mathcal{M} and \mathcal{M}' .

As it has been said in [7] that any \mathcal{L} -sentence φ (that is: a μ -sentence that only uses symbols from the language \mathcal{L}) is invariant for \mathcal{L} -bisimulation, i.e. if there is a \mathcal{L} -bisimulation between \mathcal{M} and \mathcal{M}' then φ holds in \mathcal{M} iff it holds in \mathcal{M}' . In this case, it should then be obvious that if $\text{IR}(\varphi, V)$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ then φ holds in \mathcal{M} iff it holds in \mathcal{M}' . **Therefore, in the following we only consider μ -sentences.**

We define the forgetting in μ -calculus in the follows, although forgetting is a dual concept in S5 and Propositional logic [15], it is important to give the formal definition of forgetting in μ -calculus since as we will see that it is useful to knowledge update of μ -calculus. Moreover, as shown in [16], forgetting in CTL can also be used to compute the SNC and the WSC and we shall report this in μ -calculus.

Definition 4 (Forgetting). Let $V \subseteq \mathcal{A}$ and ϕ be a μ -sentence. A formula ψ with $\text{Var}(\psi) \cap V = \emptyset$ is a result of forgetting V from ϕ , if

$$\text{Mod}(\psi) = \{\mathcal{M} \mid \exists \mathcal{M}' \in \text{Mod}(\phi) \ \& \ \mathcal{M}' \leftrightarrow_V \mathcal{M}\}.$$

For convenience, we denote the result of forgetting V from ϕ as $F_\mu(\phi, V)$.

3.2 Semantic Properties of Forgetting in μ -calculus

In this part we show the semantic properties of forgetting in μ -calculus. In particular, we show that our forgetting is closed in μ -calculus, satisfies the general postulates, i.e. the representation theorem, and the algebraic properties, including Modularity, Commutativity and Homogeneity.

Theorem 3. Let $q \in \mathcal{A}$ and ϕ be a μ -sentence. There is a μ -sentence ψ such that $\text{IR}(\psi, \{q\})$ and $\psi \equiv F_\mu(\phi, \{q\})$.

Proof. This can be obtained from the Theorem 3.1 in [7].

This means that the uniform interpolant $\widetilde{\exists}q\phi$ ($q \in \mathcal{A}$) of ϕ with respect to $\text{Var}(\phi) - \{q\}$ is the result of forgetting $\{q\}$ from ϕ . In this case, we also say that the forgetting of μ -calculus is *closed*, that is the result of forgetting some set of atoms from a μ -sentence is also a μ -sentence. Note that if both ψ and ψ' are results of forgetting V from ϕ , then $\text{Mod}(\psi) = \text{Mod}(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the forgetting result is unique (up to equivalence).

At this point, it is important to emphasize that, the notion of forgetting we have defined for μ -calculus extends the classical forgetting defined for propositional logic (PL) [23]. Assuming φ is a PL formula and $p \in \mathcal{A}$, then $\text{Forget}(\varphi, p)$ is a result of forgetting p from φ ; that is, $\text{Forget}(\varphi, p) \equiv \varphi[p/\perp] \vee \varphi[p/\top]$. That way, given a set $V \subseteq \mathcal{A}$, one can recursively define $\text{Forget}(\varphi, V \cup \{p\}) = \text{Forget}(\text{Forget}(\varphi, p), V)$, where $\text{Forget}(\varphi, \emptyset) = \varphi$. Using this insight, the following result shows that the classical notion of forgetting (for PL [23]) is a special case of forgetting in μ -calculus.

Theorem 4. *Let φ be a PL formula and $V \subseteq \mathcal{A}$, then*

$$F_\mu(\varphi, V) \equiv \text{Forget}(\varphi, V).$$

Proof. On the one hand, for each $\mathcal{M} \in \text{Mod}(F_\mu(\varphi, V))$ there exists a $\mathcal{M}' \in \text{Mod}(\varphi)$ such that $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Thus, $r\mathcal{B}r'$. Hence, \mathcal{M} is a model of $\text{Forget}(\varphi, V)$ due to $\text{IR}(\text{Forget}(\varphi, V), V)$.

On the other hand, for each $\mathcal{M} \in \text{Mod}(\text{Forget}(\varphi, V))$ with $\mathcal{M} = (S, r, R, L)$ there exists a $\mathcal{M}' \in \text{Mod}(\varphi)$ such that $r\mathcal{B}r'$. Construct a Kripke structure \mathcal{M}_1 such that $\mathcal{M}_1 = (S_1, r_1, R_1, L_1)$ with $S_1 = (S - \{r\}) \cup \{r_1\}$, R_1 is the same as R except that r is replaced by r_1 , and L_1 is the same as L except $L_1(r_1) = L'(r')$, where L' is the label function of \mathcal{M}' . It is clear that \mathcal{M}_1 is a model of φ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}$. Hence, \mathcal{M} is a model of $F_\mu(\varphi, V)$ due to $\text{IR}(F_\mu(\varphi, V), V)$.

A general description is important for understanding the concept of forgetting. To doing so, authors give four postulates (also called *forgetting postulates*), which can be considered as desirable properties of such a notion, concerning knowledge forgetting in **S5** modal logic [28]. In the following, we first list these postulates, and then show that our notion of forgetting in μ -calculus satisfies them.

Forgetting postulates [28] are:

- (**W**) Weakening: $\varphi \models \varphi'$;
- (**PP**) Positive Persistence: for any formula η , if $\text{IR}(\eta, V)$ and $\varphi \models \eta$ then $\varphi' \models \eta$;
- (**NP**) Negative Persistence : for any formula η , if $\text{IR}(\eta, V)$ and $\varphi \not\models \eta$ then $\varphi' \not\models \eta$;
- (**IR**) Irrelevance: $\text{IR}(\varphi', V)$

where $V \subseteq \mathcal{A}$, φ is a μ -sentence and φ' is the result of forgetting V from φ . Intuitively, the postulate (**W**) says, forgetting weakens the original formula, i.e. φ' is a logical consequence of φ ; the postulates (**PP**) and (**NP**) say that forgetting results have no effect on formulas that are irrelevant to forgotten atoms; the postulate (**IR**) states that forgetting result is irrelevant to forgotten atoms. It is noteworthy that they are not all orthogonal e.g., (**NP**) is a consequence of (**W**) and (**PP**). Nonetheless, we prefer to list them all, in order to outline the basic intuition behind them.

The following says that the forgetting postulates above indeed precisely characterize the underling forgetting semantics of μ -calculus.

Theorem 5 (Representation Theorem). *Let φ , φ' and ϕ be μ -sentences and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

- (i) $\varphi' \equiv F_\mu(\varphi, V)$,
- (ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi \text{ and } \text{IR}(\phi, V)\}$,
- (iii) *Postulates (W), (PP), (NP) and (IR) hold if φ , φ' and V are as in (i) and (ii).*

Proof. (i) \Leftrightarrow (ii). To prove this, we will show that:

$$\text{Mod}(F_\mu(\varphi, V)) = \text{Mod}(\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\}).$$

Firstly, suppose that \mathcal{M}' is a model of $F_\mu(\varphi, V)$. Then there exists a Kripke structure \mathcal{M} such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Therefore, we have $\mathcal{M}' \models \phi$ for all ϕ that $\varphi \models \phi$ and $\text{IR}(\phi, V)$. Thus, \mathcal{M}' is a model of $\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\}$.

It is evident that $\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\} \models F_\mu(\varphi, V)$ since $\text{IR}(F_\mu(\varphi, V), V)$ and $\varphi \models F_\mu(\varphi, V)$ by Theorem 3.

See the appendix for the proofs of the other parts.

Theorem 5 means that for a given μ -sentence φ and a set of atoms V , a μ -sentence φ' represents a result of forgetting V from φ if φ' satisfies the four forgetting postulates, and vice versa. That is, the representation theorem gives an “if and only if” characterization on forgetting in μ -calculus, which is in accordance with that in **S5**.

Excepting for the representation theorem, postulate IR is also of crucial importance for computing SNC and WSC. Consider the $\psi = \varphi \wedge (q \leftrightarrow \alpha)$. If $\text{IR}(\varphi \wedge \alpha, \{q\})$, then the result of forgetting q from ψ is φ . Formally, it can be described in the following lemma, and as we will see later in Section 4, it is the base of reducing the SNC (WSC) of any μ -sentence to that of a proposition.

Lemma 1. *Let φ and α be two μ -sentences and $q \in \overline{\text{Var}(\varphi) \cup \text{Var}(\alpha)}$. Then $F_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.*

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model \mathcal{M} of $F_\mu(\varphi', q)$ there is a Kripke structure \mathcal{M}' s.t. $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$ and $\mathcal{M}' \models \varphi'$. It's evident that $\mathcal{M}' \models \varphi$, and then $\mathcal{M} \models \varphi$ since $\text{IR}(\varphi, \{q\})$ and $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$.

Let $\mathcal{M} \in \text{Mod}(\varphi)$ with $\mathcal{M} = (S, s, R, L)$. We construct \mathcal{M}' with $\mathcal{M}' = (S, s, R, L')$ as follows:

$$\begin{aligned} L' : S &\rightarrow 2^{\mathcal{A}} \text{ and } \forall s^* \in S, L'(s^*) = L(s^*) - \{q\} \text{ if } (\mathcal{M}, s^*) \not\models \alpha, \\ &\text{else } L'(s^*) = L(s^*) \cup \{q\}, \\ L'(s) &= L(s) \cup \{q\} \text{ if } (\mathcal{M}, s) \models \alpha, \text{ and } L'(s) = L(s) \text{ otherwise.} \end{aligned}$$

It is clear that $\mathcal{M}' \models \varphi$, $\mathcal{M}' \models q \leftrightarrow \alpha$ and $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$. Therefore $\mathcal{M}' \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $\mathcal{M} \models F_\mu(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$ and $\text{IR}(F_\mu(\varphi \wedge (q \leftrightarrow \alpha), q), \{q\})$.

We will list other interesting properties of the forgetting operator in the follows. Most importantly, the following property guarantees that we can modularly apply forgetting one by one to the atoms to be forgotten, instead of forgetting the set of atoms as a whole, which is spoken in the definition of forgetting.

Proposition 3. (Modularity) *Given a μ -sentence φ , V a set of atoms and p an atom such that $p \notin V$. Then,*

$$F_\mu(\varphi, \{p\} \cup V) \equiv F_\mu(F_\mu(\varphi, p), V).$$

Proof. Let \mathcal{M}_1 with $\mathcal{M}_1 = (S_1, s_1, R_1, L_1)$ be a model of $F_\mu(\varphi, \{p\} \cup V)$. By the definition of forgetting, there exists a model \mathcal{M} with $\mathcal{M} = (S, s, R, L)$ of φ , such that $\mathcal{M}_1 \leftrightarrow_{\{p\} \cup V} \mathcal{M}$. We construct a Kripke structure \mathcal{M}_2 with $\mathcal{M}_2 = (S_2, s_2, R_2, L_2)$ as follows:

- (1) for s_2 : let s_2 be the state such that:
 - $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
 - for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
 - for all other atoms q' , $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.
- (2) for another:
 - (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\} \cup V} w_1$, let $w_2 \in S_2$ and
 - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
 - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
 - for all other atoms q' , $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.
 - (ii) if $(w'_1, w_1) \in R_1$, w_2 is constructed based on w_1 and $w'_2 \in S_2$ is constructed based on w'_1 , then $(w'_2, w_2) \in R_2$.
- (3) delete duplicated states in S_2 and pairs in R_2 .

Then we have $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Thus, $(\mathcal{M}_2, s_2) \models F_\mu(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models F_\mu(F_\mu(\varphi, p), V)$.

On the other hand, suppose that \mathcal{M}_1 is a model of $F_\mu(F_\mu(\varphi, p), V)$, then there exists a Kripke structure \mathcal{M}_2 such that $\mathcal{M}_2 \models F_\mu(\varphi, p)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$, and there exists \mathcal{M} such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$. Therefore, $\mathcal{M} \leftrightarrow_{\{p\} \cup V} \mathcal{M}_1$ by (ii) of Proposition 2, and consequently, $\mathcal{M}_1 \models F_\mu(\varphi, \{p\} \cup V)$.

The next property follows from the above proposition.

Corollary 1 (Commutativity). *Let φ be a μ -sentence and $V_i \subseteq \mathcal{A}$ ($i = 1, 2$). Then:*

$$F_\mu(\varphi, V_1 \cup V_2) \equiv F_\mu(F_\mu(\varphi, V_1), V_2).$$

The following properties show that the forgetting respects the basic semantic notions of logic. They hold in classical propositional logic, modal logic **S5** [28] and CTL [16]. Below we show that they are also satisfied in our notion forgetting in μ -calculus.

Proposition 4. *Let $\varphi, \varphi_i, \psi_i$ ($i = 1, 2$) be formulas in CTL and $V \subseteq \mathcal{A}$. We have*

- (i) $F_\mu(\varphi, V)$ is satisfiable iff φ is;
- (ii) If $\varphi_1 \equiv \varphi_2$, then $F_\mu(\varphi_1, V) \equiv F_\mu(\varphi_2, V)$;
- (iii) If $\varphi_1 \models \varphi_2$, then $F_\mu(\varphi_1, V) \models F_\mu(\varphi_2, V)$;
- (iv) $F_\mu(\psi_1 \vee \psi_2, V) \equiv F_\mu(\psi_1, V) \vee F_\mu(\psi_2, V)$;
- (v) $F_\mu(\psi_1 \wedge \psi_2, V) \equiv F_\mu(\psi_1, V) \wedge F_\mu(\psi_2, V)$;

Proof. (i) (\Rightarrow) Supposing \mathcal{M} is a model of $F_\mu(\varphi, V)$, then there is a model \mathcal{M}' of φ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by the definition of F_μ .

(\Leftarrow) Supposing \mathcal{M} is a model of φ , then there is a Kripke structure \mathcal{M}' s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$, and then $\mathcal{M}' \models F_\mu(\varphi, V)$ by the definition of F_μ .

The (ii) and (iii) can be proved similarly.

(iv) (\Rightarrow) For all $\mathcal{M} \in \text{Mod}(F_\mu(\psi_1 \vee \psi_2, V))$, there exists $\mathcal{M}' \in \text{Mod}(\psi_1 \vee \psi_2)$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ and $\mathcal{M}' \models \psi_1$ or $\mathcal{M}' \models \psi_2$

\Rightarrow there exists $\mathcal{M}_1 \in \text{Mod}(F_\mu(\psi_1, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_1$ or there exists $\mathcal{M}_2 \in \text{Mod}(F_\mu(\psi_2, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_2$

$\Rightarrow \mathcal{M} \models F_\mu(\psi_1, V) \vee F_\mu(\psi_2, V)$.

(\Leftarrow) for all $\mathcal{M} \in \text{Mod}(F_\mu(\psi_1, V) \vee F_\mu(\psi_2, V))$

$\Rightarrow \mathcal{M} \models F_\mu(\psi_1, V)$ or $\mathcal{M} \models F_\mu(\psi_2, V)$

\Rightarrow there is a Kripke structure \mathcal{M}_1 s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}_1$ and $\mathcal{M}_1 \models \psi_1$ or $\mathcal{M}_1 \models \psi_2$

$\Rightarrow \mathcal{M}_1 \models \psi_1 \vee \psi_2$

\Rightarrow there is an initial K-structure \mathcal{M}_2 s.t. $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \models F_\mu(\psi_1 \vee \psi_2, V)$

$\Rightarrow \mathcal{M} \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M} \models F_\mu(\psi_1 \vee \psi_2, V)$.

The (v) can be proved as (iv).

Intuitively, in Proposition 4, (i) means that forgetting some set of atoms from a sentence do not affect the satisfiability of this sentence. In (ii) we can see that if two sentence are equivalent then the results of forgetting the same set of atoms from both of them are also equivalent. The intuitive meaning of (iii) is obvious. (iv) refers to the result of forgetting V from a disjunctive formula $\varphi_1 \vee \varphi_2$ is equivalent with the disjunction of the results of forgetting V from φ_1 and φ_2 respectively. While (v) points out that it is not the case for a conjunctive formula.

Proposition 5 (Homogeneity). *Let $V \subseteq \mathcal{A}$ and ϕ be a μ -sentence, then we have:*

(i) $F_\mu(\text{AX}\phi, V) \equiv \text{AX}F_\mu(\phi, V)$.

(ii) $F_\mu(\text{EX}\phi, V) \equiv \text{EX}F_\mu(\phi, V)$.

The homogeneity of AX (or EX) on forgetting says we can move the operator F_μ afterward to the AX (or EX) to compute the forgetting of formula in the form $\text{AX}\varphi$ (or $\text{EX}\varphi$).

It is noteworthy that it is easy to compute a forgetting results for a set of atoms from a μ -sentence when it is a disjunctive formula.

Proposition 6. *Let φ be a μ -sentence and $p \in \mathcal{A}$. If φ be a disjunctive formula [9], then $F_\mu(\varphi, \{p\})$ can be computed in linear time.*

Proof. By Theorem 3.6. in [9], we have $F_\mu(\varphi, \{p\}) \equiv \varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\perp]$ is obtained from φ by simultaneously substituting the literals p and $\neg p$ with \top .

3.3 Complexity Results

Before talk about the complexity results of the forgetting operator, let's recall the μ -automaton, which is important in μ -calculus to show the uniform interpolation and model checking of μ -calculus. In this part we will show that the μ -automaton can also make a big convenience to show our complexity results.

Definition 5 (μ -automaton [7]). A μ -automaton A , also called modal automaton [4], is a tuple $(Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ such that:

- (i) Q is a finite set of states;
- (ii) Σ_p is a finite subset of \mathcal{A} ;
- (iii) Σ_r is a finite subset of the set of actions, in this paper it is an empty set;
- (iv) $q_0 \in Q$ is the initial state;
- (v) $\delta : Q \times \mathcal{P}(\Sigma_p) \rightarrow \mathcal{PP}(\Sigma_r \times Q)$;
- (vi) $\Omega : Q \rightarrow \mathcal{N}$.

Although this automaton differs slightly from those given in [17,4], but the automata in their various guises are equivalent. Moreover, modal automata are essentially alternating automata [4]. It has been shown that constructing a μ -automaton from a μ -calculus formula can be done in exponential time, while it is in polynomial time when the μ -calculus formula is a disjunctive formula. Then we have the following complexity result.

Proposition 7 (Model Checking). Given a finite Kripke structure \mathcal{M} , a disjunctive formula φ and $V \subseteq \mathcal{A}$, deciding $\mathcal{M} \models^? F_\mu(\varphi, V)$ is $\text{NP} \cap \text{co-NP}$.

Proof. Let A_φ be a μ -automaton such that, for any Kripke structure \mathcal{N} , A_φ accepts \mathcal{N} iff $\mathcal{N} \models \varphi$, where $A_\varphi = (Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ with $\text{Var}(\varphi) = \Sigma_p \cup \Sigma_r$. Without loss of generality, we assume $V \subseteq \text{Var}(\varphi)$ and $V = \{p\}$. Therefore we can construct a μ -automaton $B = (Q, \Sigma_p - V, \Sigma_r, q_0, \delta', \Omega)$ with:

$$\delta'(q, L) := \delta(q, L) \cup \delta(q, L \cup \{p\}).$$

It has been proved in [7] that, for each Kripke structure \mathcal{N} , B accepts \mathcal{N} iff there is a model \mathcal{N}' of φ such that $\mathcal{N} \leftrightarrow_{\{p\}} \mathcal{N}'$, i.e. B corresponds to a μ -sentence which is equivalent to $F_\mu(\varphi, V)$ by the definition of forgetting in μ -calculus.

In this case, the problem $\mathcal{M} \models^? F_\mu(\varphi, V)$ is reduced to decide whether B accepts \mathcal{M} , which is $\text{NP} \cap \text{co-NP}$ [4].

More importantly, from the perspective of knowledge bases evolving, we are also interested in the following reasoning problems about forgetting, which are explored in CPL [26].

- (i) [Var-weak] if the restriction of φ on the signature of ψ is at most as strong as ψ , i.e. $\psi \models F_\mu(\varphi, V)$,
- (ii) [Var-strong] if the restriction of φ on the signature of ψ is at least as strong as ψ , i.e. $F_\mu(\varphi, V) \models \psi$,
- (iii) [Var-entailment] if the restriction of one knowledge base on its original signature is at most as strong as that of the other, i.e. $F_\mu(\varphi, V) \models F_\mu(\psi, V)$

where φ, ψ are μ -sentences, and V a set of atoms. Besides, in (i) and (ii) there is $\text{Var}(\varphi) - V = \text{Var}(\psi)$ and in (iii) there is $V \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$. Then we have the following results.

Theorem 6 (Entailment). *Let φ and ψ be two μ -sentences and V be a set of atoms. Then, the following problems are EXPTIME-complete.*

- (i) *deciding $F_\mu(\varphi, V) \models^? \psi$,*
- (ii) *deciding $\psi \models^? F_\mu(\varphi, V)$,*
- (iii) *deciding $F_\mu(\varphi, V) \models^? F_\mu(\psi, V)$.*

Proof. We prove the (i), there other two results can be proved similarly.

Let A_φ and A_ψ be the μ -automaton of φ and ψ respectively, we can construct the μ -automaton B of $F_\mu(\varphi, V)$ from A_φ by the proof of Proposition 7. By Proposition 7.3.2 in [6], we can obtain the complement C of A_ψ in linear time, and then the intersection $A_{C \cap B}$ between C and B in linear time. In this case, the $F_\mu(\varphi, V) \models^? \psi$ is reduced to decide whether the language accepted by $A_{C \cap B}$ is empty, which is EXPTIME-complete [6].

Similarly with the reasoning problems talked above, the following equivalent problem are also important, in which “var-independence” and “var-equivalence” under CPL are proposed [21].

- (i) [Var-independence] If a formula φ is independent of a set V of atoms, i.e. $F_\mu(\varphi, V) \equiv \varphi$,
- (ii) [Var-match] if the restriction of φ on the signature of ψ perfectly matches ψ , i.e. $F_\mu(\varphi, V) \equiv \psi$.
- (iii) [Var-equivalence] if the restriction of the two formulas on a common signature are equivalent, i.e. $F_\mu(\varphi, V) \equiv F_\mu(\psi, V)$.

The following results are implications of Theorem 6.

Corollary 2. *Let φ and ψ be two μ -sentences and V be a set of atoms. Then, the following problems are EXPTIME-complete.*

- (i) *deciding $\psi \equiv^? F_\mu(\varphi, V)$,*
- (ii) *deciding $F_\mu(\varphi, V) \equiv^? \varphi$,*
- (iii) *deciding $F_\mu(\varphi, V) \equiv^? F_\mu(\psi, V)$.*

4 Necessary and Sufficient Conditions

In this section, we present two key notions of our work: namely, the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC) of a given μ -calculus specification, which corresponds to the *most general consequence* and the *most specific abduction* of a specification, respectively. As aforementioned in the introduction, these notions respectively are accordance with the *strongest precondition* (SP) and the *weakest post-condition* (WP) (introduced by E. Dijkstra in [11]), which have been central to a wide variety of tasks and studies, e.g. generating counterexamples and refinement of system in verification. Our contribution, in particular, will be on computing SNC and WSC via forgetting under a given μ -sentence and a set V of atoms. Let us give the formal definition.

Definition 6 (sufficient and necessary condition). Let ϕ, ψ be two μ -sentences, $V \subseteq \text{Var}(\phi)$, $q \in \text{Var}(\phi) - V$ and $\text{Var}(\psi) \subseteq V$.

- ψ is a necessary condition (NC in short) of q on V under ϕ if $\phi \models q \rightarrow \psi$.
- ψ is a sufficient condition (SC in short) of q on V under ϕ if $\phi \models \psi \rightarrow q$.
- ψ is a strongest necessary condition (SNC in short) of q on V under ϕ if it is a NC of q on V under ϕ , and $\phi \models \psi \rightarrow \psi'$ for any NC ψ' of q on V under ϕ .
- ψ is a weakest sufficient condition (WSC in short) of q on V under ϕ if it is a SC of q on V under ϕ , and $\phi \models \psi' \rightarrow \psi$ for any SC ψ' of q on V under ϕ .

Note that if both ψ and ψ' are SNC (WSC) of q on V under ϕ , then $\text{Mod}(\psi) = \text{Mod}(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the SNC (WSC) of q on V under ϕ is unique (up to semantic equivalence). The following result shows that the SNC and WSC are in fact dual notions.

Proposition 8 (Dual). Let V, q, φ and ψ are defined as in Definition 6. Then, ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ .

In order to generalise Definition 6 to arbitrary formulas, one can replace q (in the definition) by any formula α , and redefine V as a subset of $\text{Var}(\alpha) \cup \text{Var}(\Gamma)$.

It turns out that the previous notions of SNC and WSC for an atomic variable can be lifted to any formula, or, conversely, the SNC and WSC of any formula can be reduced to that of an atomic variable, as the following result shows.

Proposition 9. Let Γ and α be two μ -sentences, $V \subseteq \text{Var}(\alpha) \cup \text{Var}(\Gamma)$ and q be a new proposition not in Γ and α . Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

The following result establishes the bridge between forgetting and the notion of SNC (WSC) which are central to our contribution.

Theorem 7. Let φ be a μ -sentence, $V \subseteq \text{Var}(\varphi)$ and $q \in \text{Var}(\varphi) - V$.

- (i) $F_\mu(\varphi \wedge q, (\text{Var}(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ .
- (ii) $\neg F_\mu(\varphi \wedge \neg q, (\text{Var}(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ .

Following Theorem 7, assume that $\beta = F_\mu(\varphi \wedge q, (\text{Var}(\varphi) \cup \{q\}) - V)$. Then, $\varphi \wedge q \models \beta$ by (W). Moreover, $\varphi \wedge q \models \beta$, and then β is a NC of q on V under φ .

In addition, for any μ -sentence ψ with $\text{IR}(\psi, (\text{Var}(\varphi) \cup \{q\}) - V)$ and $\varphi \wedge q \models \psi$, we have $\beta \models \psi$ by (PP). Therefore, β is the SNC of q on V under φ . This shows the intuition of how the SNC can be obtained from the forgetting.

5 Representing knowledge update via forgetting

In this section, we present the final key notion of our work: knowledge update. In particular, we will propose a method of defining knowledge update via forgetting which will satisfy all the following Katsuno and Mendelzon's postulates (U1)-(U8) proposed in [18]:

- (U1) $\Gamma \diamond \phi \models \phi$.
- (U2) If $\Gamma \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma$.
- (U3) If both Γ and ϕ are satisfiable, then $\Gamma \diamond \phi$ is also satisfiable.
- (U4) If $\Gamma_1 \equiv \Gamma_2$ and $\phi_1 \equiv \phi_2$, then $\Gamma_1 \diamond \phi_1 \equiv \Gamma_2 \diamond \phi_2$.
- (U5) $(\Gamma \diamond \phi) \wedge \psi \models \Gamma \diamond (\phi \wedge \psi)$.
- (U6) If $\Gamma \diamond \phi \models \psi$ and $\Gamma \diamond \psi \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma \diamond \psi$.
- (U7) If Γ has a unique model, then $(\Gamma \diamond \phi) \wedge (\Gamma \diamond \psi) \models \Gamma \diamond (\phi \vee \psi)$.
- (U8) $(\Gamma_1 \vee \Gamma_2) \diamond \phi \equiv (\Gamma_1 \diamond \phi) \vee (\Gamma_2 \diamond \phi)$.

Here, $\varphi \diamond \psi$ expresses the result of updating φ with ψ and \diamond is the knowledge update operator.

For this purpose, in this part we suppose the models of a μ -sentence are initial structures, in which an initial structure is a Kripke structure $\mathcal{M} = (S, sr, R, L)$ with S is a finite set of states, sr is an initial state (i.e. for each state $s' \in S$ the sr can arrive at s') and R is a total relation. Besides, we also restrict the definition of forgetting on initial structures, i.e. the models mentioned in Definition 4 are initial structures. In this case, we have:

Theorem 8. *Let $V \subseteq \mathcal{A}$ and ϕ be a μ -sentence. Then there is a μ -sentence ψ such that:*

$$\mathcal{M} \models \psi \text{ iff there is a model } \mathcal{M}' \text{ of } \phi \text{ such that } \mathcal{M} \leftrightarrow_V \mathcal{M}'$$

where both \mathcal{M} and \mathcal{M}' are initial structures.

Proof. Let $\psi = F_\mu(\phi, V)$. We have that for each $\mathcal{M} \models \psi$ there is a $\mathcal{M}' \models \phi$ with $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by Theorem 3 and for each $\mathcal{M}' \in Mod(\phi)$ there is $\phi \models \psi$. In this case, we can easy prove that for each initial structure \mathcal{M} , if $\mathcal{M} \models \psi$ then we can obtain an initial structure \mathcal{M}' such that $\mathcal{M}' \models \phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Besides, for each $\mathcal{M}' \in Mod(\phi)$ there is $\mathcal{M}' \models \psi$ by $\phi \models \psi$.

Intuitively, given a logic language \mathcal{L} , we say some operator \mathcal{O} in \mathcal{L} is closed whenever the result of using the \mathcal{O} on the elements of \mathcal{L} is also in \mathcal{L} . Theorem 8 shows that the forgetting in μ -calculus is also closed when restrict the models of μ -sentence to initial structures. Formally:

Corollary 3. *The forgetting of μ -calculus is closed under the initial structure semantic, i.e. we only consider the initial structures as the models of μ -sentence.*

According to [16], we can see that any initial structure \mathcal{M} on \mathcal{A} can be captured by a CTL formula, i.e. the characterizing formula $\mathcal{F}_\mathcal{A}(\mathcal{M})$ [16], and hence a μ -sentence [13]. In this case, we define the knowledge update operator \diamond_μ in μ -calculus as follows.

Definition 7. *Let Γ and ϕ be μ -sentences. The knowledge update operator \diamond_μ is defined as follows:*

$$Mod(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in Mod(\Gamma)} \bigcup_{V_{min}} Mod(F_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_{min}) \wedge \phi),$$

where $\mathcal{F}_\mathcal{A}(\mathcal{M})$ is the characterizing formula of \mathcal{M} on \mathcal{A} , and $V_{min} \subseteq \mathcal{A}$ is a minimal subset of atoms that makes $F_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_{min}) \wedge \phi$ consistent.

Intuitively, $\Gamma \diamond_\mu \phi$ means the result of updating Γ with ϕ by minimally fixing the models of Γ into that of ϕ .

Recall the definition of knowledge update in CPL, let I , J_1 and J_2 be three interpretations, then J_1 is closer to I than J_2 , written $J_1 \leq_{I,pam} J_2$, iff $\text{Diff}(I, J_1) \subseteq \text{Diff}(I, J_2)$, where $\text{Diff}(X, Y) = \{p \in \mathcal{A} \mid X(p) \neq Y(p)\}$. The set of models of knowledge updating ψ on Γ is exactly the union of minimal models of ψ under the partial order $\leq_{I,pam}$ where I is a model of Γ , i.e.

$$\text{Mod}(\Gamma \diamond_{pam} \psi) = \bigcup_{I \in \text{Mod}(\Gamma)} \text{Min}(\text{Mod}(\psi), \leq_{I,pam}).$$

Here, $\text{Min}(\text{Mod}(\psi), \leq_{I,pam})$ is the set of models of ψ that are minimal with respect to $\leq_{I,pam}$.

Similarly, we can define a partial ordering over the set of initial structures that links to knowledge operator \diamond_μ .

Definition 8. let \mathcal{M} , \mathcal{M}_1 and \mathcal{M}_2 be three initial structures, then \mathcal{M}_1 is closer to \mathcal{M} than \mathcal{M}_2 , written $\mathcal{M}_1 \leq_{\mathcal{M}} \mathcal{M}_2$, iff for any $V_2 \subseteq \mathcal{A}$ such that $\mathcal{M}_2 \leftrightarrow_{V_2} \mathcal{M}$, there exists a $V_1 \subseteq V_2$ such that $\mathcal{M}_1 \leftrightarrow_{V_1} \mathcal{M}$. We denote $\mathcal{M}_1 <_{\mathcal{M}} \mathcal{M}_2$ iff $\mathcal{M}_1 \leq_{\mathcal{M}} \mathcal{M}_2$ and $\mathcal{M}_2 \not\leq_{\mathcal{M}} \mathcal{M}_1$.

Let M be a set of initial structures and \mathcal{M} an initial structure, we also use $\text{Min}(M, \leq_{\mathcal{M}})$ to denote the set of all minimal initial structures with respect to $\leq_{\mathcal{M}}$. Then we have the following theorem.

Theorem 9. Let Γ and ϕ be μ -sentences. Then we have:

$$\text{Mod}(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in \text{Mod}(\Gamma)} \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}}).$$

Proof. For each initial structure $\mathcal{M}' \in \text{Mod}(\Gamma \diamond_\mu \phi)$, we will show that there exists some $\mathcal{M} \in \text{Mod}(\Gamma)$ such that $\mathcal{M}' \in \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$. According to Definition 7, we know that there exists some $\mathcal{M} \in \text{Mod}(\Gamma)$ such that $\mathcal{M}' \in \text{Mod}(F_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$. Further, there is a particular $V' \subseteq \mathcal{A}$ (i.e. $V' = V_{min}$) such that $\mathcal{M}' \leftrightarrow_{V'} \mathcal{M}$ and $\mathcal{M}' \in \text{Mod}(\phi)$. Since such V' is a minimal subset of \mathcal{A} satisfying these properties, it concludes that for any other models \mathcal{M}'' of ϕ with $\mathcal{M}'' \leftrightarrow_{V_{min}} \mathcal{M}$, we have $\mathcal{M}' \leq_{\mathcal{M}} \mathcal{M}''$ by the definitions of forgetting and characterizing formula. Therefore, $\mathcal{M}' \in \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$.

For each initial structure $\mathcal{M}' \in \bigcup_{\mathcal{M} \in \text{Mod}(\Gamma)} \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$, there exists some $\mathcal{M} \in \text{Mod}(\Gamma)$ such that $\mathcal{M}' \in \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$. Let V_{min} be a minimal subset of atoms such that $\mathcal{M}' \leftrightarrow_{V_{min}} \mathcal{M}$. Then according to the definition of $\leq_{\mathcal{M}}$, we know that there does not exist another $\mathcal{M}'' \in \text{Mod}(\phi)$ such that $\mathcal{M}'' \leftrightarrow_{V'} \mathcal{M}$ and $V' \subset V_{min}$. This follows that $\mathcal{M}' \in \text{Mod}(F_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$ and hence $\mathcal{M}' \in \text{Mod}(\Gamma \diamond_\mu \phi)$.

Theorem 9 means that our definition of knowledge update in μ -calculus by forgetting is accordance with the one defined by the $\leq_{\mathcal{M}}$, which is similar with the partial order $\leq_{I,pam}$ in CPL.

More important, the following theorem shows that our definition of \diamond_μ by forgetting satisfies Katsuno and Mendelzon's update postulates.

Theorem 10. *Knowledge update operator \diamond_μ satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).*

Proof. For (U1), we know that $Mod(\Gamma \diamond_\mu \phi) \subseteq Mod(\phi)$ by Theorem 9, hence $\Gamma \diamond_\mu \phi \models \phi$.

For (U2), we will prove $\Gamma \diamond_\mu \phi \models \Gamma$ at first. For any model \mathcal{M} of $\Gamma \diamond_\mu \phi$ there is a $\mathcal{M}_1 \in Mod(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Then we have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Similarly, for any model \mathcal{M} of Γ , there is a $\mathcal{M}_1 \in Mod(\Gamma \diamond_\mu \phi)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. We have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Hence $\Gamma \models \Gamma \diamond_\mu \phi$.

It is easy to show \diamond_μ satisfies (U3) and (U4). We now prove (U5). For any model \mathcal{M} of $(\Gamma \diamond_\mu \phi) \wedge \psi$ there is a $\mathcal{M}_1 \in Mod(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Besides, we can see that $\mathcal{M} \models \phi \wedge \psi$. Therefore, we have $\mathcal{M} \models \Gamma \diamond_\mu (\phi \wedge \psi)$.

For (U6), we will prove $\Gamma \diamond_\mu \phi \models \Gamma \diamond_\mu \psi$, and the other direction can be proved similarly. For any model \mathcal{M} of $\Gamma \diamond_\mu \phi$, \mathcal{M} is also a model of ψ . There is a $\mathcal{M}_1 \in Mod(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore \mathcal{M} is a model of $F_\mu(\mathcal{F}_A(\mathcal{M}_1), V_{min}) \wedge \psi$. This shows that $F_\mu(\mathcal{F}_A(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Moreover, V_{min} is also the minimal set such that $F_\mu(\mathcal{F}_A(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Otherwise, suppose that $V \subset V_{min}$ such that $F_\mu(\mathcal{F}_A(\mathcal{M}_1), V) \wedge \psi$ is consistent as well. Then, $F_\mu(\mathcal{F}_A(\mathcal{M}_1), V) \wedge \phi$ should also be consistent by $\Gamma \diamond_\mu \psi \models \phi$, which contradicts to the fact that V_{min} is the minimal set of atoms such that $F_\mu(\mathcal{F}_A(\mathcal{M}_1), V_{min}) \wedge \phi$ is consistent. Hence, \mathcal{M} is also a model of $\Gamma \diamond_\mu \psi \models \psi$.

Now we prove (U7). Suppose that Γ has the unique model \mathcal{M} . For each $\mathcal{M}_1 \in Mod((\Gamma \diamond_\mu \phi) \wedge (\Gamma \diamond_\mu \psi))$ there exists V_1 and V_2 which are minimal such that $\mathcal{M} \leftrightarrow_{V_1} \mathcal{M}_1$ and $\mathcal{M} \leftrightarrow_{V_2} \mathcal{M}_1$, i.e. \mathcal{M}_1 is a model of both $F_\mu(\mathcal{F}_A(\mathcal{M}), V_1) \wedge \phi$ and $F_\mu(\mathcal{F}_A(\mathcal{M}), V_2) \wedge \psi$. Therefore $\mathcal{M}_1 \leftrightarrow_{V_1 \cap V_2} \mathcal{M}$. Thus, \mathcal{M}_1 is a model of $F_\mu(\mathcal{F}_A(\mathcal{M}), V_1 \cap V_2)$. Then we have $V_1 = V_2$, otherwise V_1 (or V_2) is not the minimal set. \mathcal{M}_1 is a model of $F_\mu(\mathcal{F}_A(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ as well. Moreover, V_1 is the minimal set such that $F_\mu(\mathcal{F}_A(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ is satisfiable. Otherwise, suppose that $V_3 \subset V_1$ such that $F_\mu(\mathcal{F}_A(\mathcal{M}), V_3) \wedge (\phi \vee \psi)$ is satisfiable. Then $F_\mu(\mathcal{F}_A(\mathcal{M}), V_3) \wedge \phi$ or $F_\mu(\mathcal{F}_A(\mathcal{M}), V_3) \wedge \psi$ is satisfiable. Without loss of generality, suppose that $F_\mu(\mathcal{F}_A(\mathcal{M}), V_3) \wedge \phi$ is satisfiable, V_1 is not the minimal set, a contradiction. Therefore \mathcal{M}_1 is also a model of $\Gamma \diamond_\mu (\phi \vee \psi)$.

For (U8), we will prove $(\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi \models (\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi)$ at first. For each $\mathcal{M} \in Mod((\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi)$, there is a $\mathcal{M}_1 \in Mod(\Gamma_1)$ (or $\mathcal{M}_1 \in Mod(\Gamma_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore, we have $\mathcal{M} \models (\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi)$. Similarly, for each $\mathcal{M} \in Mod((\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi))$ there is a $\mathcal{M}_1 \in Mod(\Gamma_1)$ (or $\mathcal{M}_1 \in Mod(\Gamma_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Hence, $\mathcal{M} \models (\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi$.

6 Concluding Remarks

Summary In this paper, we targeted the strongest necessary and weakest sufficient conditions SNC (WSC) and knowledge update in μ -calculus using a forgetting-based approach. In doing so, we introduced and employed the notion of V -bisimulation which is similar with the \mathcal{L} -bisimulation, in which any μ -sentence is invariant for \mathcal{L} -bisimulation. Furthermore, we have studied formal properties about forgetting, among them, homogeneity, modularity and commutativity. In particular, we have shown that our notion

of forgetting satisfies the well-known postulates of forgetting, which means it faithfully extends the notion of forgetting from classical propositional logic, modal logic **S5**, CTL to μ -calculus. We also showed that for any μ -sentence, if it is a disjunctive formula, then forgetting a set of atoms from it can be done in linear time. On the complexity theory side, we have investigated that the Model checking problem of forgetting V from a disjunctive formula is $\text{NP} \cap \text{co-NP}$ and the entailment problems of forgetting are EXPTIME-complete from the point of μ automaton. And finally, we showed that the knowledge update in terms of forgetting in μ -calculus under initial structures satisfies the Katsuno and Mendelzon's postulates (U1)-(U8).

Future work As we can see that the reasoning problems of forgetting are EXPTIME-complete, which means there is no algorithm can compute the result of forgetting a set of atoms from a μ -sentence in polynomial time. It is worthwhile to explore sub-classes of μ -calculus such that we can compute the forgetting in polynomial time. Moreover, when a finite transition system \mathcal{M} does not satisfy a specification φ , one can evaluate the weakest sufficient condition ψ over a signature V under which $\mathcal{M} \models \varphi$, viz. expressing \mathcal{M} with its characterizing formula, to explore how the condition ψ can guide the design of a new transition system \mathcal{M}' satisfying φ .

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A Supplementary Material: Proof Appendix

Proposition 2 Let $V, V_1 \subseteq \mathcal{A}$, $\mathcal{M}_1, \mathcal{M}_2$ and \mathcal{M}_3 be three Kripke structures, then we have:

- (i) the \leftrightarrow_V is an equivalence relation between Kripke structures;
- (ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$, then $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.

Proof. (i) We prove it from the reflexivity, symmetry and transitivity.

(1) \leftrightarrow_V is reflexive. It is easy to check that $\mathcal{M} \leftrightarrow_V \mathcal{M}$ for any Kripke structure.

(2) \leftrightarrow_V is symmetric. We will show that for each \mathcal{M}_1 and \mathcal{M}_2 , if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ then $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V -bisimulation \mathcal{B} , we construct a relation \mathcal{B}_1 as follows: $\mathcal{B}_1 = \{(s, t) | (t, s) \in \mathcal{B}\}$. We will show that \mathcal{B}_1 is a V -bisimulation between \mathcal{M}_2 and \mathcal{M}_1 from the following several points:

- $r_2 \mathcal{B}_1 r_1$ since $r_1 \mathcal{B} r_2$,
- for each $s \in S_1$ and $t \in S_2$, if $t \mathcal{B}_1 s$ then we have $s \mathcal{B} t$ and hence $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A} - V$, and
- the third and forth points in the definition of V -bisimulation can be checked easily for \mathcal{B}_1 .

(3) \leftrightarrow_V is transitive. We will show that for each $\mathcal{M}_1, \mathcal{M}_2$ and \mathcal{M}_3 , if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ then $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V -bisimulation \mathcal{B}_1 and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ by the V -bisimulation \mathcal{B}_2 , we construct a relation \mathcal{B} as follows: $\mathcal{B} = \{(s, z) | (s, t) \in \mathcal{B}_1 \text{ and } (t, z) \in \mathcal{B}_2\}$. We can also prove similarly with (2) that \mathcal{B} is a V -bisimulation between \mathcal{M}_1 and \mathcal{M}_3 . Therefore, $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$.

(ii) In order to prove $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$, we only need to find a binary relation \mathcal{B} such that \mathcal{B} is a $(V \cup V_1)$ -bisimulation between \mathcal{M}_1 and \mathcal{M}_3 . Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V -bisimulation \mathcal{B}_1 and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$ by the V_1 -bisimulation \mathcal{B}_2 . Let $\mathcal{B} = \{(s_1, s_3) | (s_1, s_2) \in \mathcal{B}_1 \text{ and } (s_2, s_3) \in \mathcal{B}_2\}$. We can easily check that \mathcal{B} is a $(V \cup V_1)$ -bisimulation between \mathcal{M}_1 and \mathcal{M}_3 .

Theorem 4 Let φ be a PL formula and $V \subseteq \mathcal{A}$, then

$$F_\mu(\varphi, V) \equiv \text{Forget}(\varphi, V).$$

Proof. On the one hand, for each $\mathcal{M} \in \text{Mod}(F_\mu(\varphi, V))$ there exists a $\mathcal{M}' \in \text{Mod}(\varphi)$ such that $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Thus, $r \mathcal{B} r'$. Hence, \mathcal{M} is a model of $\text{Forget}(\varphi, V)$ due to $\text{IR}(\text{Forget}(\varphi, V), V)$.

On the other hand, for each $\mathcal{M} \in \text{Mod}(\text{Forget}(\varphi, V))$ with $\mathcal{M} = (S, r, R, L)$ there exists a $\mathcal{M}' \in \text{Mod}(\varphi)$ such that $r \mathcal{B} r'$. Construct a Kripke structure \mathcal{M}_1 such that $\mathcal{M}_1 = (S_1, r_1, R_1, L_1)$ with $S_1 = (S - \{r\}) \cup \{r_1\}$, R_1 is the same as R except that r is replaced by r_1 , and L_1 is the same as L except $L_1(r_1) = L'(r')$, where L' is the label function of \mathcal{M}' . It is clear that \mathcal{M}_1 is a model of φ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}$. Hence, \mathcal{M} is a model of $F_\mu(\varphi, V)$ due to $\text{IR}(F_\mu(\varphi, V), V)$.

Theorem 5 (Representation Theorem) Let φ, φ' and ϕ be μ -sentences and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

- (i) $\varphi' \equiv F_\mu(\varphi, V)$,
- (ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi \text{ and } \text{IR}(\phi, V)\}$,
- (iii) Postulates **(W)**, **(PP)**, **(NP)** and **(IR)** hold if φ, φ' and V are as in (i) and (ii).

Proof. (i) \Leftrightarrow (ii). To prove this, we will show that:

$$\text{Mod}(F_\mu(\varphi, V)) = \text{Mod}(\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\}).$$

Firstly, suppose that \mathcal{M}' is a model of $F_\mu(\varphi, V)$. Then there exists a Kripke structure \mathcal{M} such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Therefore, we have $\mathcal{M}' \models \phi$ for all ϕ that $\varphi \models \phi$ and $\text{IR}(\phi, V)$. Thus, \mathcal{M}' is a model of $\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\}$.

It is evident that $\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\} \models F_\mu(\varphi, V)$ since $\text{IR}(F_\mu(\varphi, V), V)$ and $\varphi \models F_\mu(\varphi, V)$ by Theorem 3.

(ii) \Rightarrow (iii). It is not difficult to prove it.

(iii) \Rightarrow (ii). Suppose that all postulates hold. By Positive Persistence, we have $\varphi' \models \{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\}$. The $\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\} \models \varphi'$ can be obtained from **(W)** and **(IR)**. Thus, φ' is equivalent to $\{\phi \mid \varphi \models \phi, \text{IR}(\phi, V)\}$.

Lemma 1 Let φ and α be two μ -sentences and $q \in \overline{\text{Var}(\varphi) \cup \text{Var}(\alpha)}$. Then $F_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model \mathcal{M} of $F_\mu(\varphi', q)$ there is a Kripke structure \mathcal{M}' s.t. $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$ and $\mathcal{M}' \models \varphi'$. It's evident that $\mathcal{M}' \models \varphi$, and then $\mathcal{M} \models \varphi$ since $\text{IR}(\varphi, \{q\})$ and $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$.

Let $\mathcal{M} \in \text{Mod}(\varphi)$ with $\mathcal{M} = (S, s, R, L)$. We construct \mathcal{M}' with $\mathcal{M}' = (S, s, R, L')$ as follows:

$$\begin{aligned} L' : S &\rightarrow 2^A \text{ and } \forall s^* \in S, L'(s^*) = L(s^*) - \{q\} \text{ if } (\mathcal{M}, s^*) \not\models \alpha, \\ &\text{else } L'(s^*) = L(s^*) \cup \{q\}, \\ L'(s) &= L(s) \cup \{q\} \text{ if } (\mathcal{M}, s) \models \alpha, \text{ and } L'(s) = L(s) \text{ otherwise.} \end{aligned}$$

It is clear that $\mathcal{M}' \models \varphi$, $\mathcal{M}' \models q \leftrightarrow \alpha$ and $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$. Therefore $\mathcal{M}' \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $\mathcal{M} \models F_\mu(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$ and $\text{IR}(F_\mu(\varphi \wedge (q \leftrightarrow \alpha), q), \{q\})$.

Proposition 3 (Modularity) Given a μ -sentence φ , V a set of atoms and p an atom such that $p \notin V$. Then,

$$F_\mu(\varphi, \{p\} \cup V) \equiv F_\mu(F_\mu(\varphi, p), V).$$

Proof. Let \mathcal{M}_1 with $\mathcal{M}_1 = (S_1, s_1, R_1, L_1)$ be a model of $F_\mu(\varphi, \{p\} \cup V)$. By the definition of forgetting, there exists a model \mathcal{M} with $\mathcal{M} = (S, s, R, L)$ of φ , such that $\mathcal{M}_1 \leftrightarrow_{\{p\} \cup V} \mathcal{M}$. We construct a Kripke structure \mathcal{M}_2 with $\mathcal{M}_2 = (S_2, s_2, R_2, L_2)$ as follows:

- (1) for s_2 : let s_2 be the state such that:
 - $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
 - for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
 - for all other atoms q' , $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:

- (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\} \cup V} w_1$, let $w_2 \in S_2$ and
 - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
 - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
 - for all other atoms q' , $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.
- (ii) if $(w'_1, w_1) \in R_1$, w_2 is constructed based on w_1 and $w'_2 \in S_2$ is constructed based on w'_1 , then $(w'_2, w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2 .

Then we have $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Thus, $(\mathcal{M}_2, s_2) \models F_\mu(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models F_\mu(F_\mu(\varphi, p), V)$.

On the other hand, suppose that \mathcal{M}_1 is a model of $F_\mu(F_\mu(\varphi, p), V)$, then there exists a Kripke structure \mathcal{M}_2 such that $\mathcal{M}_2 \models F_\mu(\varphi, p)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$, and there exists \mathcal{M} such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$. Therefore, $\mathcal{M} \leftrightarrow_{\{p\} \cup V} \mathcal{M}_1$ by (ii) of Proposition 2, and consequently, $\mathcal{M}_1 \models F_\mu(\varphi, \{p\} \cup V)$.

Proposition 4 Let $\varphi, \varphi_i, \psi_i$ ($i = 1, 2$) be formulas in CTL and $V \subseteq \mathcal{A}$. We have

- (i) $F_\mu(\varphi, V)$ is satisfiable iff φ is;
- (ii) If $\varphi_1 \equiv \varphi_2$, then $F_\mu(\varphi_1, V) \equiv F_\mu(\varphi_2, V)$;
- (iii) If $\varphi_1 \models \varphi_2$, then $F_\mu(\varphi_1, V) \models F_\mu(\varphi_2, V)$;
- (iv) $F_\mu(\psi_1 \vee \psi_2, V) \equiv F_\mu(\psi_1, V) \vee F_\mu(\psi_2, V)$;
- (v) $F_\mu(\psi_1 \wedge \psi_2, V) \equiv F_\mu(\psi_1, V) \wedge F_\mu(\psi_2, V)$;

Proof. (i) (\Rightarrow) Supposing \mathcal{M} is a model of $F_\mu(\varphi, V)$, then there is a model \mathcal{M}' of φ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by the definition of F_μ .

(\Leftarrow) Supposing \mathcal{M} is a model of φ , then there is a Kripke structure \mathcal{M}' s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$, and then $\mathcal{M}' \models F_\mu(\varphi, V)$ by the definition of F_μ .

The (ii) and (iii) can be proved similarly.

(iv) (\Rightarrow) For all $\mathcal{M} \in \text{Mod}(F_\mu(\psi_1 \vee \psi_2, V))$, there exists $\mathcal{M}' \in \text{Mod}(\psi_1 \vee \psi_2)$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ and $\mathcal{M}' \models \psi_1$ or $\mathcal{M}' \models \psi_2$
 \Rightarrow there exists $\mathcal{M}_1 \in \text{Mod}(F_\mu(\psi_1, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_1$ or there exists $\mathcal{M}_2 \in \text{Mod}(F_\mu(\psi_2, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_2$
 $\Rightarrow \mathcal{M} \models F_\mu(\psi_1, V) \vee F_\mu(\psi_2, V)$.

(\Leftarrow) for all $\mathcal{M} \in \text{Mod}(F_\mu(\psi_1, V) \vee F_\mu(\psi_2, V))$
 $\Rightarrow \mathcal{M} \models F_\mu(\psi_1, V)$ or $\mathcal{M} \models F_\mu(\psi_2, V)$
 \Rightarrow there is a Kripke structure \mathcal{M}_1 s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}_1$ and $\mathcal{M}_1 \models \psi_1$ or $\mathcal{M}_1 \models \psi_2$
 $\Rightarrow \mathcal{M}_1 \models \psi_1 \vee \psi_2$
 \Rightarrow there is an initial K-structure \mathcal{M}_2 s.t. $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \models F_\mu(\psi_1 \vee \psi_2, V)$
 $\Rightarrow \mathcal{M} \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M} \models F_\mu(\psi_1 \vee \psi_2, V)$.

The (v) can be proved as (iv).

Proposition 5 (Homogeneity) Let $V \subseteq \mathcal{A}$ and ϕ be a μ -sentence, then we have:

- (i) $F_\mu(\text{AX}\phi, V) \equiv \text{AX}F_\mu(\phi, V)$.
- (ii) $F_\mu(\text{EX}\phi, V) \equiv \text{EX}F_\mu(\phi, V)$.

Proof. Let $\mathcal{M} = (S, R, s, L)$, $\mathcal{M}_i = (S_i, R_i, s_i, L_i)$ with $i \in \mathbb{N}$ and $\mathcal{M}' = (S', R', s', L')$, then we call $\mathcal{M}' = (S', R', s', L')$ be a sub-structure of \mathcal{M} if:

- $S' \subseteq S$ and $S' = \{s' | s' \text{ is reachable from } s'\} \cup A$ with $A = \{s'' | s'' \text{ can not be reached from } s' \text{ and there is not such a sequence of states } (s, \dots, s'', s')\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S' \text{ and } (s_1, s_2) \in R\}$,
- $L' : S' \rightarrow 2^{\mathcal{A}}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- s' is s or a state reachable from s .

(i) In order to prove $F_\mu(\text{AX}\phi, V) \equiv \text{AX}(F_\mu(\phi, V))$, we only need to prove $\text{Mod}(F_\mu(\text{AX}\phi, V)) = \text{Mod}(\text{AX}F_\mu(\phi, V))$:

$(\Rightarrow) \forall \mathcal{M}' \in \text{Mod}(F_\mu(\text{AX}\phi, V))$ there exists a Kripke structure \mathcal{M} s.t. $\mathcal{M} \models \text{AX}\phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$

\Rightarrow for any sub-structure \mathcal{M}_1 of \mathcal{M} there is $\mathcal{M}_1 \models \phi$, where s_1 is a directed successor of s

\Rightarrow there is a Kripke structure \mathcal{M}_2 s.t. $\mathcal{M}_2 \models F_\mu(\phi, V)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$

\Rightarrow it is easy to construct a Kripke structure \mathcal{M}_3 by \mathcal{M}_2 s.t. \mathcal{M}_2 is a sub-structure of \mathcal{M}_3 with s_2 is a direct successor of s_3 and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}$

$\Rightarrow \mathcal{M}_3 \models \text{AX}(F_\mu(\phi, V))$ and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}'$

$\Rightarrow \mathcal{M}' \models \text{AX}(F_\mu(\phi, V))$.

$(\Leftarrow) \forall \mathcal{M}_3 \in \text{Mod}(\text{AX}(F_\mu(\phi, V)))$, then for any sub-structure \mathcal{M}_2 with s_2 is a directed successor of s_3 there is $\mathcal{M}_2 \models F_\mu(\phi, V)$

\Rightarrow for any \mathcal{M}_2 there is a Kripke structure \mathcal{M}_1 s.t. $\mathcal{M}_1 \models \phi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$

\Rightarrow it is easy to construct a Kripke structure \mathcal{M} by \mathcal{M}_1 s.t. \mathcal{M}_1 is a sub-structure of \mathcal{M} with s_1 is a direct successor of s and $\mathcal{M} \leftrightarrow_V \mathcal{M}_3$

$\Rightarrow \mathcal{M} \models \text{AX}\phi$ and then $\mathcal{M}_3 \models F_\mu(\text{AX}\phi, V)$.

(ii) In order to prove $F_\mu(\text{EX}\phi, V) \equiv \text{EX}F_\mu(\phi, V)$, we only need to prove $\text{Mod}(F_\mu(\text{EX}\phi, V)) = \text{Mod}(\text{EX}F_\mu(\phi, V))$:

$(\Rightarrow) \forall \mathcal{M}' \in \text{Mod}(F_\mu(\text{EX}\phi, V))$ there exists a Kripke structure \mathcal{M} s.t. $\mathcal{M} \models \text{EX}\phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$

\Rightarrow there is a sub-structure \mathcal{M}_1 of \mathcal{M} s.t. $\mathcal{M}_1 \models \phi$, where s_1 is a directed successor of s

\Rightarrow there is a Kripke structure \mathcal{M}_2 s.t. $\mathcal{M}_2 \models F_\mu(\phi, V)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$

\Rightarrow it is easy to construct a Kripke structure \mathcal{M}_3 by \mathcal{M}_2 s.t. \mathcal{M}_2 is a sub-structure of \mathcal{M}_3 that s_2 is a direct successor of s_3 and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}$

$\Rightarrow \mathcal{M}_3 \models \text{EX}(F_\mu(\phi, V))$

$\Rightarrow \mathcal{M}' \models \text{EX}(F_\mu(\phi, V))$.

$(\Leftarrow) \forall \mathcal{M}_3 \in \text{Mod}(\text{EX}(F_\mu(\phi, V)))$, then there exists a sub-structure \mathcal{M}_2 of \mathcal{M}_3 s.t. $\mathcal{M}_2 \models F_\mu(\phi, V)$

\Rightarrow there is a Kripke structure \mathcal{M}_1 s.t. $\mathcal{M}_1 \models \phi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$

\Rightarrow it is easy to construct a Kripke structure \mathcal{M} by \mathcal{M}_1 s.t. \mathcal{M}_1 is a sub-structure of \mathcal{M} that s_1 is a direct successor of s and $\mathcal{M} \leftrightarrow_V \mathcal{M}_3$

$\Rightarrow \mathcal{M} \models \text{EX}\phi$ and then $\mathcal{M}_3 \models F_\mu(\text{EX}\phi, V)$.

Proposition7 (Model Checking) Given a finite Kripke structure \mathcal{M} , a disjunctive formula φ and $V \subseteq \mathcal{A}$, deciding $\mathcal{M} \models^? F_\mu(\varphi, V)$ is $\text{NP} \cap \text{co-NP}$.

Proof. Let A_φ be a μ -automaton such that, for any Kripke structure \mathcal{N} , A_φ accepts \mathcal{N} iff $\mathcal{N} \models \varphi$, where $A_\varphi = (Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ with $\text{Var}(\varphi) = \Sigma_p \cup \Sigma_r$. Without loss

of generality, we assume $V \subseteq \text{Var}(\varphi)$ and $V = \{p\}$. Therefore we can construct a μ -automaton $B = (Q, \Sigma_p - V, \Sigma_r, q_0, \delta', \Omega)$ with:

$$\delta'(q, L) := \delta(q, L) \cup \delta(q, L \cup \{p\}).$$

It has been proved in [7] that, for each Kripke structure \mathcal{N} , B accepts \mathcal{N} iff there is a model \mathcal{N}' of φ such that $\mathcal{N} \leftrightarrow_{\{p\}} \mathcal{N}'$, i.e. B corresponds to a μ -sentence which is equivalent to $F_\mu(\varphi, V)$ by the definition of forgetting in μ -calculus.

In this case, the problem $\mathcal{M} \models^? F_\mu(\varphi, V)$ is reduced to decide whether B accepts \mathcal{M} , which is $\text{NP} \cap \text{co-NP}$ [4].

Theorem6 (Entailment)

Let φ and ψ be two μ -sentences and V be a set of atoms. Then, the following problems are EXPTIME-complete.

- (i) deciding $F_\mu(\varphi, V) \models^? \psi$,
- (ii) deciding $\psi \models^? F_\mu(\varphi, V)$,
- (iii) deciding $F_\mu(\varphi, V) \models^? F_\mu(\psi, V)$.

Proof. We prove the (i), there other two results can be proved similarly.

Let A_φ and A_ψ be the μ -automaton of φ and ψ respectively, we can construct the μ -automaton B of $F_\mu(\varphi, V)$ from A_φ by the proof of Proposition 7. By Proposition 7.3.2 in [6], we can obtain the complement C of A_ψ in linear time, and then the intersection $A_{C \cap B}$ between C and B in linear time. In this case, the $F_\mu(\varphi, V) \models^? \psi$ is reduced to decide whether the language accepted by $A_{C \cap B}$ is empty, which is EXPTIME-complete [6].

Proposition 8 (dual) Let V, q, φ and ψ are defined as in Definition 6. Then, ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ .

Proof. (i) Suppose ψ is the SNC of q . Then $\varphi \models q \rightarrow \psi$. Thus $\varphi \models \neg\psi \rightarrow \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \rightarrow \neg q$. Then $\varphi \models q \rightarrow \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ . Thus $\varphi \models \psi \rightarrow \neg\psi'$ by assumption. So $\varphi \models \psi' \rightarrow \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case.

Proposition 9 Let Γ and α be two μ -sentences, $V \subseteq \text{Var}(\alpha) \cup \text{Var}(\Gamma)$ and q be a new proposition not in Γ and α . Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

Proof. We prove this for SNC. The case for WSC is similar. Let $\text{SNC}(\varphi, \beta, V, \Gamma)$ denote that φ is the SNC of β on V under Γ , and $\text{NC}(\varphi, \beta, V, \Gamma)$ denote that φ is the NC of β on V under Γ , in which β is a formula.

(\Rightarrow) We will show that if $\text{SNC}(\varphi, \alpha, V, \Gamma)$ holds, then $\text{SNC}(\varphi, q, V, \Gamma')$ will be true. According to $\text{SNC}(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \rightarrow \varphi$, which means φ is a NC of q on V under Γ' . Suppose φ' is any NC of q on V under Γ' , then $F_{\text{CTL}}(\Gamma', q) \models \alpha \rightarrow \varphi'$ due to $\alpha \equiv q$, $\text{IR}(\alpha \rightarrow \varphi', \{q\})$ and **(PP)**, i.e., $\Gamma \models \alpha \rightarrow \varphi'$ by Lemma

1, this means $NC(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \rightarrow \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \rightarrow \varphi'$. Hence, $SNC(\varphi, q, V, \Gamma')$ holds.

(\Leftarrow) We will show that if $SNC(\varphi, q, V, \Gamma')$ holds, then $SNC(\varphi, \alpha, V, \Gamma)$ will be true. According to $SNC(\varphi, q, V, \Gamma')$, it's not difficult to know that $F_{CTL}(\Gamma', \{q\}) \models \alpha \rightarrow \varphi$ due to $\alpha \equiv q$, $IR(\alpha \rightarrow \varphi, \{q\})$ and **(PP)**, i.e., $\Gamma \models \alpha \rightarrow \varphi$ by Lemma 1, this means $NC(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ . Then $\Gamma' \models q \rightarrow \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $NC(\varphi', q, V, \Gamma')$. According to $SNC(\varphi, q, V, \Gamma')$, $IR(\varphi \rightarrow \varphi', \{q\})$ and **(PP)**, we have $F_{CTL}(\Gamma', \{q\}) \models \varphi \rightarrow \varphi'$, and $\Gamma \models \varphi \rightarrow \varphi'$ by Lemma 1. Hence, $SNC(\varphi, \alpha, V, \Gamma)$ holds.

Theorem 7 Let φ be a μ -sentence, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.

- (i) $F_\mu(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ .
- (ii) $\neg F_\mu(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ .

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = F_\mu(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The “NC” part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \rightarrow \mathcal{F}$, this means \mathcal{F} is a NC of q on P under φ .

The “SNC” part: for all ψ', ψ' is the NC of q on V under φ , s.t. $\varphi \models \mathcal{F} \rightarrow \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with \mathcal{F} under φ , s.t. $\varphi \models \psi \rightarrow \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $IR(\psi, (Var(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with \mathcal{F} under φ . This is contradict with the suppose. Then \mathcal{F} is the SNC of q on P under φ .

Theorem8 Let $V \subseteq \mathcal{A}$ and ϕ be a μ -sentence. Then there is a μ -sentence ψ such that:

$$\mathcal{M} \models \psi \text{ iff there is a model } \mathcal{M}' \text{ of } \phi \text{ such that } \mathcal{M} \leftrightarrow_V \mathcal{M}'$$

where both \mathcal{M} and \mathcal{M}' are initial structures.

Proof. Let $\psi = F_\mu(\phi, V)$. We have that for each $\mathcal{M} \models \psi$ there is a $\mathcal{M}' \models \phi$ with $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by Theorem 3 and for each $\mathcal{M}' \in Mod(\phi)$ there is $\phi \models \psi$. In this case, we can easy prove that for each initial structure \mathcal{M} , if $\mathcal{M} \models \psi$ then we can obtain an initial structure \mathcal{M}' such that $\mathcal{M}' \models \phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Besides, for each $\mathcal{M}' \in Mod(\phi)$ there is $\mathcal{M}' \models \psi$ by $\phi \models \psi$.

Theorem9 Let Γ and ϕ be μ -sentences. Then we have:

$$Mod(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in Mod(\Gamma)} Min(Mod(\phi), \leq_{\mathcal{M}}).$$

Proof. For each initial structure $\mathcal{M}' \in Mod(\Gamma \diamond_\mu \phi)$, we will show that there exists some $\mathcal{M} \in Mod(\Gamma)$ such that $\mathcal{M}' \in Min(Mod(\phi), \leq_{\mathcal{M}})$. According to Definition 7, we know that there exists some $\mathcal{M} \in Mod(\Gamma)$ such that $\mathcal{M}' \in Mod(F_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$. Further, there is a particular $V' \subseteq \mathcal{A}$ (i.e. $V' = V_{min}$) such that $\mathcal{M}' \leftrightarrow_{V'}$

\mathcal{M} and $\mathcal{M}' \in \text{Mod}(\phi)$. Since such V' is a minimal subset of \mathcal{A} satisfying these properties, it concludes that for any other models \mathcal{M}'' of ϕ with $\mathcal{M}'' \leftrightarrow_{V_{min}} \mathcal{M}$, we have $\mathcal{M}' \leq_{\mathcal{M}} \mathcal{M}''$ by the definitions of forgetting and characterizing formula. Therefore, $\mathcal{M}' \in \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$.

For each initial structure $\mathcal{M}' \in \bigcup_{\mathcal{M} \in \text{Mod}(\Gamma)} \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$, there exists some $\mathcal{M} \in \text{Mod}(\Gamma)$ such that $\mathcal{M}' \in \text{Min}(\text{Mod}(\phi), \leq_{\mathcal{M}})$. Let V_{min} be a minimal subset of atoms such that $\mathcal{M}' \leftrightarrow_{V_{min}} \mathcal{M}$. Then according to the definition of $\leq_{\mathcal{M}}$, we know that there does not exist another $\mathcal{M}'' \in \text{Mod}(\phi)$ such that $\mathcal{M}'' \leftrightarrow_{V'} \mathcal{M}$ and $V' \subset V_{min}$. This follows that $\mathcal{M}' \in \text{Mod}(F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$ and hence $\mathcal{M}' \in \text{Mod}(\Gamma \diamond_{\mu} \phi)$.

Theorem 10 Knowledge update operator \diamond_{μ} satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).

Proof. For (U1), we know that $\text{Mod}(\Gamma \diamond_{\mu} \phi) \subseteq \text{Mod}(\phi)$ by Theorem 9, hence $\Gamma \diamond_{\mu} \phi \models \phi$.

For (U2), we will prove $\Gamma \diamond_{\mu} \phi \models \Gamma$ at first. For any model \mathcal{M} of $\Gamma \diamond_{\mu} \phi$ there is a $\mathcal{M}_1 \in \text{Mod}(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Then we have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Similarly, for any model \mathcal{M} of Γ , there is a $\mathcal{M}_1 \in \text{Mod}(\Gamma \diamond_{\mu} \phi)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. We have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Hence $\Gamma \models \Gamma \diamond_{\mu} \phi$.

It is easy to show \diamond_{μ} satisfies (U3) and (U4). We now prove (U5). For any model \mathcal{M} of $(\Gamma \diamond_{\mu} \phi) \wedge \psi$ there is a $\mathcal{M}_1 \in \text{Mod}(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Besides, we can see that $\mathcal{M} \models \phi \wedge \psi$. Therefore, we have $\mathcal{M} \models \Gamma \diamond_{\mu} (\phi \wedge \psi)$.

For (U6), we will prove $\Gamma \diamond_{\mu} \phi \models \Gamma \diamond_{\mu} \psi$, and the other direction can be proved similarly. For any model \mathcal{M} of $\Gamma \diamond_{\mu} \phi$, \mathcal{M} is also a model of ψ . There is a $\mathcal{M}_1 \in \text{Mod}(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore \mathcal{M} is a model of $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \psi$. This shows that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Moreover, V_{min} is also the minimal set such that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Otherwise, suppose that $V \subset V_{min}$ such that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V) \wedge \psi$ is consistent as well. Then, $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V) \wedge \phi$ should also be consistent by $\Gamma \diamond_{\mu} \psi \models \phi$, which contradicts to the fact that V_{min} is the minimal set of atoms such that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \phi$ is consistent. Hence, \mathcal{M} is also a model of $\Gamma \diamond_{\mu} \psi \models \psi$.

Now we prove (U7). Suppose that Γ has the unique model \mathcal{M} . For each $\mathcal{M}_1 \in \text{Mod}((\Gamma \diamond_{\mu} \phi) \wedge (\Gamma \diamond_{\mu} \psi))$ there exists V_1 and V_2 which are minimal such that $\mathcal{M} \leftrightarrow_{V_1} \mathcal{M}_1$ and $\mathcal{M} \leftrightarrow_{V_2} \mathcal{M}_1$, i.e. \mathcal{M}_1 is a model of both $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_1) \wedge \phi$ and $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_2) \wedge \psi$. Therefore $\mathcal{M}_1 \leftrightarrow_{V_1 \cap V_2} \mathcal{M}$. Thus, \mathcal{M}_1 is a model of $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_1 \cap V_2)$. Then we have $V_1 = V_2$, otherwise V_1 (or V_2) is not the minimal set. \mathcal{M}_1 is a model of $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ as well. Moreover, V_1 is the minimal set such that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ is satisfiable. Otherwise, suppose that $V_3 \subset V_1$ such that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_3) \wedge (\phi \vee \psi)$ is satisfiable. Then $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_3) \wedge \phi$ or $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_3) \wedge \psi$ is satisfiable. Without loss of generality, suppose that $F_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_3) \wedge \phi$ is satisfiable, V_1 is not the minimal set, a contradiction. Therefore \mathcal{M}_1 is also a model of $\Gamma \diamond_{\mu} (\phi \vee \psi)$.

For (U8), we will prove $(\Gamma_1 \vee \Gamma_2) \diamond_{\mu} \phi \models (\Gamma_1 \diamond_{\mu} \phi) \vee (\Gamma_2 \diamond_{\mu} \phi)$ at first. For each $\mathcal{M} \in \text{Mod}((\Gamma_1 \vee \Gamma_2) \diamond_{\mu} \phi)$, there is a $\mathcal{M}_1 \in \text{Mod}(\Gamma_1)$ (or $\mathcal{M}_1 \in \text{Mod}(\Gamma_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore, we have $\mathcal{M} \models (\Gamma_1 \diamond_{\mu} \phi) \vee (\Gamma_2 \diamond_{\mu} \phi)$. Similarly, for each $\mathcal{M} \in \text{Mod}((\Gamma_1 \diamond_{\mu} \phi) \vee (\Gamma_2 \diamond_{\mu} \phi))$ there is a $\mathcal{M}_1 \in \text{Mod}(\Gamma_1)$ (or $\mathcal{M}_1 \in \text{Mod}(\Gamma_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Hence, $\mathcal{M} \models (\Gamma_1 \vee \Gamma_2) \diamond_{\mu} \phi$.