On the Weakest Sufficient Conditions in Propositional μ-calculus

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Abstract. The μ-calculus is one of the most important logics describing specifications of transition systems. It has been extensively explored for formal verification in model checking due to its exceptional balance between expressiveness and algorithmic properties. On the one hand, some information content in a specification might become irrelevant or unnecessary due to various reasons from the perspective of knowledge representation. On the other hand, a weakest precondition of a specification is badly necessary in verification, where a (weakest) precondition is sufficient for a transition system to enjoy a desire property. This paper is to address these scenarios for μ-calculus in a principle way in terms of knowledge *forgetting*. In particular, it proposes a notion of forgetting by a generalized bisimilar equivalence (over a signature) and explores its important properties as a knowledge distilling operator, besides some reasoning complexity results. It then shows that how the weakest sufficient condition and the strongest necessary condition can be established via forgetting. It also discusses knowledge update for μ-calculus in terms of forgetting.

Keywords: Weakest precondition · Forgetting · Knowledge update.

1 Introduction

Propositional μ-calculus is an expressive logic, on binary trees it is as expressive as the monadic second-order logic of two successors (S2S) [18,36]. Subsequent research has shown that the μ-calculus is an important logic when formal specification and verification are concerned. Let's consider the *model checking* problem, i.e., deciding whether $\mathcal{M} \models \varphi$ for a given model $\mathcal{M}$ and a specification φ (a μ-formula). Currently, most solvers will output "Yes" if $\mathcal{M}$ satisfies φ, otherwise, they will output "No" and a counterexample. In this case, a problem is raised on how to compute a *precondition* ψ over a signature such that $\mathcal{M} \models \psi \to \varphi$. This condition is limited to the weakest one, i.e., the weakest precondition (WP), when addressing "how to verify a 'Hoare triple' $\{\phi\}P\{\psi\}$ where P is a program". For this reason, E. Dijkstra explored computing the WP for a specification ψ and a given program P [13]. In this method, it requires the program P will be terminating. However, the given model $\mathcal{M}$ in model checking problem may be non-terminating, and we cannot use Dijkstra's method to compute WP.

Moreover, whether in verification or system design, a common phenomenon is that some informative content in a specification may be system-independent for a variety of reasons e.g., it may be discarded or become obsolete over time, or it might become unworkable because of practical difficulties. However, in this case, it is usually expensive and tedious to redesign the specifications meeting the given requirements. An alterative approach is to remove the irrelevant information without changing the behaviour of the associated system or violating the existing system specification for the given signature.

However, it is a non-trivial task [1] to remove some constraints (atoms) from large and complex systems or specifications without affecting working system components or violating the relevant specifications for a given signature. Moreover, as we have seen that the strongest post-condition (SP) and the weakest precondition (WP) of a given specification are central to a wide variety of tasks and studies, e.g. in generating counterexamples [11] and in the refinement of system [45] in verification. It should be noted that the SP and WP correspond to the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC), proposed by Lin [32], of the specification, respectively.

To address these scenarios and to target the relevant notions of the SNC (WSC) in a principled way inspired by [32,20], in this paper, we explore the SNC (WSC) of μ-calculus from the aspect of forgetting. In particular, we give the definition of forgetting in μ-calculus by using a bisimulation [6,2,47] under infinite models (to distinguish it from computation tree logic (CTL) [20]). Then, we show whether this notion satisfies the general principles or postulates proposed by Zhang and Zhou [47] and the algebraic properties (including modularity, commutativity and homogeneity). Moreover, we study the relationship between the SNC (WSC) and forgetting.

Forgetting in propositional logic (PL) and knowledge forgetting in S5 modal logic have been defined and used in the field of *belief update/revision* and *knowledge update*, respectively[33,30,37,3,48]. Intuitively, if φ represents the agent's belief about the world and the agent performs an action that is supposed to make ψ true in the resulting world, then the agent's belief about the resulting world can be described by $\varphi \diamond \psi$, where $\diamond$ is the selected update operator. We can see that the theory of belief updates does not tell us how to perform updates with respect to this gain in knowledge due to a sensing action. In this sense, analogous to the notion of belief update, the *knowledge update* was proposed by Baral and Zhang in [3] to solve the belief updates caused by sensing actions, in which the effect of a sensing action is expressed by introducing the modal operator (K)nows. And then knowledge update was defined by forgetting in S5 modal logic [48].

After exploring the definition and properties of forgetting in μ-calculus, we demonstrate how forgetting can be used in knowledge update in μ-calculus in this paper. In addition, we also show that our definition of knowledge update operator $\diamond_\mu$ by forgetting satisfies the update postulates of Katsuno and Mendelzon [27].

The rest of the paper is organized as follows. After discussing the related work in the next section, the basic notation and technical preliminaries are introduced in Section 3. The formal definition of forgetting in μ-calculus, its various properties and the computational complexities are presented in Section 4. Section 5 shows the forgetting

[1] Although the model checking for unrestricted CTL can be decided in P (more precisely, even $O(|S|.|\varphi|)$ [Schnoebelen, P., 2002]), in regard to the forgetting version of the problem, even for the lighter fragment CTL_{AF}, it is NP-complete (i.e., $(\mathcal{M}, s_0) \models \text{F}_{\text{CTL}}(\varphi, V)$ is NP-complete [Prop. 7 of Feng et al. 2020]). Hence, one can expect the general problem to be *at least as hard as* NP.

can be used to comupte both WSC (SNC) and knowledge update. Finally, concluding remarks are given in Section 6.

To avoid hindering the flow of content, detailed proofs of the technical results are provided in the Appendix.

2 Related work

In this section, we briefly discuss published matter that is technically related to our work.

2.1 The Weakest Precondition

The *weakest precondition*, as an important concept in formal verification, was first proposed by Dijkstra to solve the problem of computing or approximating invariants appearing in the *verification of computer programs and systems* [13], particularly in the "Hoare triple" [25]. Afterwards, it was wildly studied in various fields, especially in generating counterexamples [11] and refining system [45].

In the field of AI, there is a similar concept called the *weakest sufficient condition* (WSC), which was introduced by Lin to generate successor state axioms from causal theories [32]. Moreover, the SNC and WSC for proposition q on a restricted subset of the propositional variables under propositional theory T are computed based on the notion of forgetting. Afterwards the SNC and WSC were generalized to first-order logic (FOL) and a direct method based on the *second-order quantifier elimination* (SOQE) technique was proposed to automatically generate the SNC and WSC [14]. In addition, a forgetting-based method is used to compute the SNC and WSC in CTL [20].

2.2 Forgetting

Forgetting was first formally defined in PL and FOL by Lin and Reiter [34,15]. As a technique for distilling knowledge, it has been explored in various of logic languages and widely used in AI. Except for the WSC (SNC), belief update/revision, and knowledge update talked about in the Introduction, forgetting has been used for conflict solving [46,31] and knowledge compilation [4]. Informally, forgetting is used to abstract from a knowledge base $\mathcal{T}$ only the part that is relevant to a subset of alphabet $\mathcal{P}$ while not affecting the results of $\mathcal{T}$ on $\mathcal{P}$.

The concept of forgetting can be traced back to the work of Boole on *propositional variable elimination* and the seminal work of Ackermann [1], who recognised that the problem amounts to *the elimination of existential second-order quantifiers*. Moreover, it has been extended to various logic systems, including modal logics [47,19] and nonmonotonic logics [41,24].

In PL, forgetting has often been studied under the name 'variable elimination'. Formally, the solution of forgetting a propositional variable p from a PL formula φ is $\varphi[p/\bot] \vee \varphi[p/\top]$ [34], where $\varphi[p/\bot]$ and $\varphi[p/\top]$ denote the formulas obtained from φ by replacing atom p with $\bot$ and $\top$, respectively.

In FOL, the definition of forgetting was defined from the perspective of *strong (or semantic) forgetting* and *weak forgetting* [49]. Although weak forgetting and strong forgetting are not exactly the same, they coincide when the result of strong forgetting exists. We consider semantic forgetting (abbreviated to forgetting) and give its definition in PL and FOL here. Forgetting is considered an instance of the SOQE problem in FOL. In that case, the result of forgetting an n-ary predicate P from a first-order formula φ is $\exists R\varphi[P/R]$ [34], in which R is an n-ary predicate variable and $\varphi[X/Y]$ is a result of replacing every occurrence of X in φ by Y. The task of forgetting in FOL, as a computational problem, is to find a first-order formula that is equivalent to $\exists R\varphi[P/R]$. It is evident that this is an SOQE problem. However, the solution to the SOQE problem is not always expressible in FOL [22],which means that the results of forgetting in FOL are not always expressible in FOL, i.e., forgetting in FOL is *not closed*. Nonetheless, the solution of weak forgetting is always expressible in FOL, although there are cases in which the forgetting solution can be represented only by an infinite set of FOL formulas [49]. See [15] for a recent and comprehensive survey.

In non-classical logics, the knowledge forgetting of S5 modal logic was defined [47]. In this paper, the authors shown that the knwoledge update can be represented through knowledge forgetting. In addition, they proposed four general postulates for knowledge forgetting and showed that these four postulates precisely characterize the notion of knowledge forgetting in S5 modal logic. Moreover, they show that *uniform interpolation* [38] is a dual concept of forgetting in S5 and PL. These notions were recently extended to multi-agent modal logics [19] and CTL [21]. Furthermore, in the scenario of non-monotonic reasoning, forgetting in logic programs under answer set semantics has been extensively investigated from the perspective of various forgetting postulates [46,16,44,42,40,12,24], see [15,23] for a comprehensive survey. Similarly, the forgetting in description logics (DL) are also explored with the motivation of constructing restricted ontologies by eliminating concept and role symbols from DL-based ontologies [43,35,28,50,51].

It should be mentioned that the existing forgetting definitions in PL and answer set programming are not directly applicable in modal logics. We cannot directly use the method of forgetting in CTL [20] since it will not work when the models of the formula are infinite. Hopefully, it has been proved that the modal μ-calculus has *uniform interpolation* [8]. Informally, for every μ-sentence φ and every finite set V of atoms, there exists a μ-sentence $\widetilde{\exists} V\varphi$ that does not contain atoms from V but is logically closest to φ in some sense. As we will show that our forgetting definition is equivalent to the semantic definition of uniform interpolation in [8,10]. In the sense, showing the semantics of forgetting in μ-calculus through *general principles or postulates* is important to make it easier to understand.

3 Preliminaries

In this section, we introduce the technical and notational preliminaries, i.e., the syntax and semantics of μ-calculus, closely related to this paper. Moreover, throughout this paper, we denote by $\overline{V}$ the complement of $V \subseteq B$ on a given set B, i.e., $\overline{V} = B - V$.

3.1 The syntax of μ-calculus

Modal μ-calculus is an extension of modal logic, and we consider the propositional μ-calculus introduced by Kozen [29]. Let $\mathcal{A} = \{p, q, \dots\}$ be a set of propositional letters (atoms) and $\mathcal{V} = \{X, Y, \dots\}$ be a set of variables. Then, the formulas of the μ-calculus, called μ-formulas (or formulas), over these sets can be inductively defined in Backus-Naur form:

$$\varphi := p \mid \neg p \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathrm{EX}\varphi \mid \mathrm{AX}\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

where $p \in \mathcal{A}$ and $X \in \mathcal{V}$. $\top$ and $\bot$ are also μ-calculus formulas, which express 'true' and 'false', respectively. The variable X is said to be bound in $\mu X.\varphi$, $\nu X.\varphi$.

It is obvious that negations, i.e., '$\neg$', are allowed only before propositional letters. All the results presented here extend to the general case where negation before variables is also allowed, restricted as usual to positive occurrences of bound variables; that is variables appear after an even number of negations. Variables, propositional letters and their negations are called *literals*. For convenience, in the following, φ, φ_1, $\dots$, ψ, ψ_1, $\dots$ are used to denote μ-formulas. By *Var*(φ) we mean the set of atoms appearing in formula φ.

A formula is *well named* iff every variable is bound at most once in the formula, and free variables are distinct from bound variables. For a variable X bound in a well named formula φ there exists a unique subformula of φ of the form $\delta X.\psi(X)$ with $\delta \in \{\nu, \mu\}$.

Variable X in $\delta X.\varphi(X)$ is *guarded* iff every occurrence of X in $\varphi(X)$ is within the scope of some modality operators EX or AX. A formula is guarded iff every bound variable in the formula is guarded. Furthermore, a *μ-sentence* is a formula containing no free variables.

In the following, we restrict ourselves to **guarded**, **well-named** μ-sentences.

3.2 The semantics of μ-calculus

Generally, μ-formulas are interpreted in transition systems of the form $\mathcal{M} = (S, r, R, L)$, which we call a Kripke structure, where:

- S is a nonempty set of states,
- $r \in S$,
- R is a binary relation on S, i.e. $R \subseteq S \times S$, called a transition relation, and
- $L : S \to 2^{\mathcal{A}}$ is a labeling function.

Sometimes, r is called the '*root*' of $\mathcal{M}$ [9]. A Kripke structure $\mathcal{M}$ is finite if S is finite and for each state $s \in S$, there is $q \notin L(s)$ for almost all $q \in \mathcal{A}$.

Given a Kripke structure $\mathcal{M}$ and a valuation $v : \mathcal{V} \to 2^S$, the set of states in which a formula φ is true, denoted as $\|\varphi\|_v^{\mathcal{M}}$, is defined inductively as follows (the superscript

$\mathcal{M}$ is omitted when doing so causes no ambiguity):

$$\begin{aligned}
&\|p\|_v = \{s \mid p \in L(s)\}\ ;\ \|\top\|_v = S\ ;\ \|\bot\|_v = \emptyset;\\
&\|\neg p\|_v = S - \|p\|_v\ ;\\
&\|X\|_v = v(X);\\
&\|\varphi_1 \vee \varphi_2\|_v = \|\varphi_1\|_v \cup \|\varphi_2\|_v\ ;\\
&\|\varphi_1 \wedge \varphi_2\|_v = \|\varphi_1\|_v \cap \|\varphi_2\|_v\ ;\\
&\|\mathrm{EX}\varphi\|_v = \{s|\exists s'.(s,s') \in R \wedge s' \in \|\varphi\|_v\};\\
&\|\mathrm{AX}\varphi\|_v = \{s|\forall s'.(s,s') \in R \Rightarrow s' \in \|\varphi\|_v\};\\
&\|\mu X.\varphi\|_v = \bigcap\{S' \subseteq S|\ \|\varphi\|_{v[X:=S']} \subseteq S'\};\\
&\|\nu X.\varphi\|_v = \bigcup\{S' \subseteq S|S' \subseteq \|\varphi\|_{v[X:=S']}\}.
\end{aligned}$$

where $v[X := S']$ is the same as the valuation function v except that S' is assigned to X, i.e., for each $Y \in \mathcal{V}$:

$$v[X := S'](Y) = \begin{cases} S', & \text{if } Y = X;\\ v(Y), & \text{otherwise.}\end{cases}$$

In the following, we denote $s \in \|\varphi\|_v$ by $(\mathcal{M}, s, v) \models \varphi$ and we may leave out the valuation v if φ is a μ-sentence. $(\mathcal{M}, v) \models \varphi$ is used to denote $(\mathcal{M}, r, v) \models \varphi$. $(\mathcal{M}, v)$ is a *model* of φ whenever $(\mathcal{M}, v) \models \varphi$. In this case, *Mod*$(\varphi)$ denotes the set of models of φ. Particularly, if φ is a μ-sentence, then we use $\mathcal{M} \models \varphi$ to replace $(\mathcal{M}, v) \models \varphi$ and *Mod*$(\varphi) = \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$. Similarly, let Σ be a set of μ-sentences; we define *Mod*(Σ) as the set of Kripke structures $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ for each $\varphi \in \Sigma$. Moreover, ψ is a *logical consequence* of φ, denoted by $\varphi \models \psi$, if $(\mathcal{M}, v) \models \varphi$ then $(\mathcal{M}, v) \models \psi$ for every Kripke structure $\mathcal{M}$ and valuation v. Particularly, given two sentences (or set of sentences) Σ and Π, $\Sigma \models \Pi$ if *Mod*$(\Sigma) \subseteq$ *Mod*(Π). And $\Sigma \equiv \Pi$ whenever *Mod*$(\Sigma) =$ *Mod*(Π); in this case we also call Σ and Π *semantically equivalent*.

A μ-sentence ϕ is *irrelevant to* the atoms in a set V (or simply V*-irrelevant*), written IR(ϕ, V), if there is a μ-sentence ψ with *Var*$(\psi) \cap V = \emptyset$ such that $\phi \equiv \psi$. The V-*irrelevance* of a set of μ-sentences can be defined similarly, i.e., a set Σ of μ-sentences is irrelevant to the atoms in V, written IR(Σ, V), if IR(φ, V) for each $\varphi \in \Sigma$.

3.3 Disjunctive μ-formula

The disjunctive formula of μ-formula originate from the work in [26]. In this paper, we use the definition of disjunctive μ-formula in [10].

An alternative syntax for the μ-calculus, called *covers-syntax*, is obtained by substituting the EX operator with a set of *cover operators*, one for each natural n. In this way, $Cover(\emptyset)$ is a μ-formula and for $n \geq 1$, if $\varphi_1, \ldots, \varphi_n$ are formulas, then

$$Cover(\varphi_1, \ldots, \varphi_n)$$

is a formula. For a given Kripke structure $\mathcal{M} = (S, r, R, L)$, $Cover(\emptyset)$ is true in $\mathcal{M}$ if and only if the root of $\mathcal{M}$ does not have any successor, while $Cover(\varphi_1, \ldots, \varphi_n)$ is true in $\mathcal{M}$ if and only if the successors of the root are covered by $\varphi_1, \ldots, \varphi_n$. More formally, $(\mathcal{M}, s, v) \models Cover(\varphi_1, \ldots, \varphi_n)$ with $s \in S$ if and only if:

- for every $i = 1, ..., n$, there exists t with $(s, t) \in R$ and $(\mathcal{M}, t, v) \models \varphi_i$;
- for every t with $(s, t) \in R$ there exists $i \in \{1, ..., n\}$ with $(\mathcal{M}, t, v) \models \varphi_i$.

It has shown that the μ-calculus obtained from the covers-syntax is equivalent to the familiar μ-calculus talked in subsection 3.1[10].

Definition 1 (disjunctive μ-formula [10]). *The set of disjunctive μ-formulas, $\mathcal{F}_d$ is the smallest set containing $\top$, $\bot$, and non-contradictory conjunciton of literals which is closed under:*

(1) disjunctions: if $\alpha, \beta \in \mathcal{F}_d$, then $\alpha \vee \beta \in \mathcal{F}_d$;
(2) special conjunctions: if $\varphi_1, \ldots, \varphi_n \in \mathcal{F}_d$ and δ is a non-contradictory conjunction of literals, then $\delta \wedge Cover(\varphi_1, \ldots, \varphi_n) \in \mathcal{F}_d$;
(3) fixpoint operators: if $\varphi \in \mathcal{F}_d$, φ does not contain $X \wedge \psi$ as a subformula for any formula ψ, and X is positive in φ, then $\mu X.\varphi$ and $\nu X.\varphi$ are in $\mathcal{F}_d$.

The disjunctive μ-formulas are representative of the whole μ-calculus, i.e., any μ-calculus formula is equivalent to a disjuntive one.

4 Forgetting in μ-calculus

As has been shown in [20], the WSC of a given CTL formula (property) under a set of atoms and a non-terminating system (called an initial K-structure[2]) can be computed by using the forgetting technique. However, that work does not discuss how to obtain the WSC when the given property is a μ-formula. In this section, we extend the forgetting in CTL to μ-calculus from two aspects: (1) the language discussed is extended from CTL to μ-calculus; and (2) the Kripke structures are more general, i.e., the Kripke structures can contain infinite states, multiple initial states, and so on.

In particular, we present the definition of forgetting in μ-calculus and investigate its semantic properties in this section. First, we give the definition of V-bisimulation between Kripke structures, in which $V \subseteq \mathcal{A}$ is a set of atoms. The notion of V-bisimulation captures the idea that the two systems are behaviourally the same except for the atoms in V. In this way, we define forgetting by using V-bisimulation.

Second, representation theorem and the related properties (i.e., modularity, commutativity, and homogeneity) of the forgetting operator are explored. Finally, it shows that the model checking problem of forgetting V from a disjunctive formula is in NP $\cap$ co-NP, and the reasoning problems are EXPTIME-complete.

[2] An *initial structure* is a Kripke structure $\mathcal{M} = (S, sr, R, L)$ with S being a finite set of states, sr being an initial state (i.e., for each state $s' \in S$, the sr can arrive at s'), R being a total relation and $L : S \to 2^{\mathcal{A}}$ being a label function, where $\mathcal{A}$ is restricted to a finite set. See [21] for more details.

4.1 Definition of Forgetting

Recalling the meaning of forgetting in the explored logic languages, "forgetting" some atoms from a given formula should not violate the existing specification over the remaining signature. That is, the models of the formula will be extended to some other Kripke structures such that those Kripke structures and the existing models simulate each other on the remaining signature. This reminds us to think about the notion of *bisimulation*.

A bisimulation is a binary relation between state transition systems (they are expressed by Kripke structures in this paper) and associating systems that behave in the same way, in the sense that two systems mimic each other. More clearly, if two Kripke structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar, then they satisfy the same formula, i.e., for each formula φ, $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$. The result is that neither of the systems can be distinguished from the other by an observer

To clarify the meaning of "forgetting" presented earlier, we define the bisimilar relation on a given signature between Kripke structures. That is, we extend the bisimulation to one under a given set of atoms, i.e., V-bisimulation with V being a set of atoms. For convenience, let $\mathcal{M}_i = (S_i, r_i, R_i, L_i)$, in which i is in the natural set $\mathbf{N}$, be Kripke structures.

Definition 2 (V-bisimulation). *Let $V \subseteq \mathcal{A}$ and $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Kripke structures. $\mathcal{B} \subseteq S_1 \times S_2$ is a V-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ if:*

- $r_1 \mathcal{B} r_2$,
- *for each $s \in S_1$ and $t \in S_2$, if $s\mathcal{B}t$ then $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A}-V$,*
- *$(s, s') \in R_1$ and $s\mathcal{B}t$ imply that there is a t' such that $s'\mathcal{B}t'$ and $(t, t') \in R_2$, and*
- *vice versa: if $s\mathcal{B}t$ and $(t, t') \in R_2$, then there is an s' with $(s, s') \in R_1$ and $t'\mathcal{B}s'$.*

On the one hand, the V-bisimulation is the same with the $\mathcal{L}$-bisimulation[3] in [8], but on the complement. In this case, as was stated in [8] that any $\mathcal{L}$-sentence φ (that is, a μ-sentence that uses only symbols from the language $\mathcal{L}$) is invariant for $\mathcal{L}$-bisimulation, i.e., if there is an $\mathcal{L}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, then φ holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$. Therefore, if $\mathrm{IR}(\varphi, V)$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$, then φ holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$. In this paper, we call this property *V-invariant*.

On the other hand, it is easy to see that our definition is similar to that introduced in [20]. That is, the definitions are the same whenever $(\mathcal{M}_i, r_i)$ is limited to an initial K-structure. Moreover, the V-bisimulation defined in [20], the classical bisimulation-equivalence of Definition 7.1 in [2], the state equivalence (i.e., E_n) in [6], and the state-based bisimulation notion of Definition 7.7 in [2] are closely related.[4] In this sense, one can see that our V-bisimulation is also closely related to those definitions to some extent.

[3] It is a relation satisfying the clauses in Definition 2 just for the symbols in language $\mathcal{L}$

[4] The V-bisimulation defined in [20] is similar to the state equivalence (i.e., E_n) in [6], yet it is different in the sense that the one in [20] is defined on K-structures, while state equivalence is defined on states. Moreover, V-bisimulation is different from the state-based bisimulation notion of Definition 7.7 in [2], which is defined for states of a given K-structure.

Two Kripke structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are V-*bisimilar*, denoted as $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$, if there exists a V-bisimulation $\mathcal{B}$ between them. In this case, $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar on V. To obtain some intuition of the V-bisimulation, let us consider the following example.

Example 1. In Fig.1, we can check that $\mathcal{M} \leftrightarrow_{\{ch\}} \mathcal{M}'$ because there is a $\{ch\}$-bisimulation $\mathcal{B} = \{(s_0, t_0), (s_1, t_1), (s_2, t_1)\}$ between $\mathcal{M}$ and $\mathcal{M}'$.

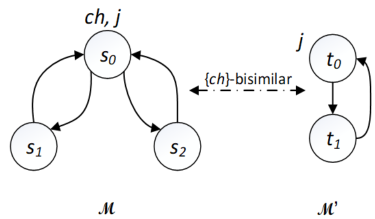

Fig.1. Two $\{ch\}$-bisimilar Kripke structures.

Moreover, we can see that the relation $\leftrightarrow_V$ has some interesting properties in addition to the equivalence relation. Formally:

Proposition 1. *Let $V, V_1 \subseteq \mathcal{A}$ and $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three Kripke structures, then we have:*

(i) $\leftrightarrow_V$ *is an equivalence relation between Kripke structures;*
(ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ *and* $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$*, then* $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.

Intuitively, property (i) in Proposition 1 means that $\leftrightarrow_V$ is reflexive, symmetric, and transitive. (ii) indicates that if a Kripke structure is V and V_1-bisimilar to the other two Kripke structures respectively, then those two Kripke structures are $V \cup V_1$-bisimilar. As we will show in the following context, it is important to demonstrate the *modularity*, one of the important properties of forgetting in μ-calculus.

We now define forgetting in μ-calculus.

Definition 3 (Forgetting). *Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence. A μ-sentence ψ with* $Var(\psi) \cap V = \emptyset$ *is a* result of forgetting V from ϕ *if*

$$Mod(\psi) = \{\mathcal{M} \mid \exists \mathcal{M}' \in Mod(\phi) \ \&\ \mathcal{M}' \leftrightarrow_V \mathcal{M}\}.$$

We denote the result of forgetting V from ϕ as $\mathrm{F}_\mu(\phi, V)$. It is not difficult to see that Definition 3 implies that if both ψ and ψ' are results of forgetting V from ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the result of forgetting V from ϕ is unique up to semantic equivalence.

It is worthy of note that D'Agostino at al. studied the notion of *uniform interpolation* in μ-calculus, and inicated that μ-calculus has the uniform interpolation property [8,10]. Informally, this means that for every μ-sentence φ and every finite set $V \subseteq Var(\varphi)$, there exists a μ-sentenc $\widetilde{\exists} V \varphi$ which does not contain atoms from V but is logically closest to φ in some sense.

We should mention that our forgetting definition $\mathrm{F}_\mu(\phi, V)$ is equivalent to the semantic definition of the formula of uniform interpolation in [10].

4.2 Semantic Properties of Forgetting in μ-calculus

In this part, we show the semantic properties of forgetting in μ-calculus. In particular, we show that our forgetting is closed in μ-calculus. Moreover, we demonstrate that the notion of forgetting satisfies the general postulates, i.e., the *representation theorem*, and the algebraic properties, including modularity, commutativity, and homogeneity.

It has been proved that the μ-calculus has uniform interpolation [9,8]. Now, we show that the forgetting in μ-calculus is *closed* [5]. Formally:

Theorem 1. *Let $q \in \mathcal{A}$ and ϕ be a μ-sentence. There is a μ-sentence ψ such that IR$(\psi, \{q\})$ and $\psi \equiv \mathrm{F}_\mu(\phi, \{q\})$.*

Theorem 1 means that the result of forgetting some set of atoms from a μ-sentence is also a μ-sentence; that is, the forgetting in μ-calculus is *closed*.

A general description is important for understanding the concept of forgetting. To achieve this, four postulates (also called *forgetting postulates*), which indeed precisely characterize the underlying knowledge forgetting semantics **S5** modal logic, is proposed [47]. In the following, we first list these postulates and then show that it also provides an "if and only if" characterization on our notion of forgetting in μ-calculus.

Forgetting postulates [47] are:

(**W**) Weakening: $\varphi \models \varphi'$;
(**PP**) Positive Persistence: for any formula η, if IR(η, V) and $\varphi \models \eta$ then $\varphi' \models \eta$;
(**NP**) Negative Persistence : for any formula η, if IR(η, V) and $\varphi \not\models \eta$ then $\varphi' \not\models \eta$;
(**IR**) Irrelevance: IR(φ', V)

where $V \subseteq \mathcal{A}$, φ is a μ-sentence and φ' is the result of forgetting V from φ. We prefer to list those properties all to outline the basic intuition of forgetting, although they are not all independent, e.g., (**NP**) is a consequence of (**W**) and (**PP**). Intuitively, the postulate (**W**) states that forgetting weakens the original formula, i.e., φ' is a logical consequence of φ; the postulates (**PP**) and (**NP**) state that the forgetting results have no effect on formulas that are independent of the atoms to be forgotten; and the postulate (**IR**) states that the forgetting result is irrelevant to forgotten atoms.

The following theorem states that the forgetting postulates above indeed precisely characterize the underling forgetting semantics of μ-calculus.

Theorem 2 (Representation Theorem). *Let φ, φ' and ϕ be μ-sentences and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

(i) $\varphi' \equiv \mathrm{F}_\mu(\varphi, V)$,
(ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi$ *and* IR$(\phi, V)\}$,
*(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold if φ, φ' and V are as in (i) and (ii).*

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\mathrm{F}_\mu(\varphi, V)) = Mod(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}).$$

[5] Intuitively, given a logic language $\mathcal{L}$, we say some operator $\mathcal{O}$ in $\mathcal{L}$ is closed whenever the result of using the $\mathcal{O}$ on the elements of $\mathcal{L}$ is also in $\mathcal{L}$.

($\Rightarrow$) For each model $\mathcal{M}'$ of $\mathrm{F}_\mu(\varphi, V)$
$\Rightarrow$ there exists a Kripke structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ (Def. 3)
$\Rightarrow \mathcal{M}' \models \phi$ for all ϕ with $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$
$\Rightarrow \mathcal{M}' \models \{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$

($\Leftarrow$) It is evident that $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \mathrm{F}_\mu(\varphi, V)$ since $\mathrm{IR}(\mathrm{F}_\mu(\varphi, V), V)$ and $\varphi \models \mathrm{F}_\mu(\varphi, V)$ by Theorem 1.

$(ii) \Rightarrow (iii)$. For convenience, let $A = \{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Firstly, is easy to see that $\mathrm{IR}(A, V)$ since for any $\phi' \in A$ there is $\mathrm{IR}(\phi', V)$. Therefore, we have $\mathrm{IR}(\varphi', V)$. Second, $\varphi \models \phi'$ for any $\phi' \in A$, hence $\varphi \models \varphi'$. Third, $\forall \phi$ with $\mathrm{IR}(\phi, V)$, if $\varphi \models \phi$ then $\phi \in A$ by the definition of A. And hence, $\varphi' \models \phi$. Last but not least, $\forall \phi$ with $\mathrm{IR}(\phi, V)$, if $\varphi \not\models \phi$ then $\phi \notin A$ by the definition of A. And hence, $\varphi' \not\models \phi$ by Definition 3 and V-invarianty.

$(iii) \Rightarrow (ii)$. (1) $\varphi' \models \{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$ ((**PP**))
(2) $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \varphi'$ ((**W**) and (**IR**))
$\Rightarrow \varphi' \equiv \{\phi \mid \varphi \models \phi$ and $\mathrm{IR}(\phi, V)\}$ ((1) and (2)).

Theorem 2 means that for a given μ-sentence φ and a set of atoms V, a μ-sentence φ' represents a result of forgetting V from φ if φ' satisfies the forgetting postulates, and vice versa. That is, the representation theorem gives an "if and only if" characterization on forgetting in μ-calculus, which is in accordance with that in **S5** and that in CTL.

As we mentioned in Related work, the notion of forgetting has been defined and used in a variety of contexts under PL. It is important to know the relationship between forgetting in PL and forgetting in μ-calculus.

To show this, let us recall the following notations for a given atom p, a set $V \subseteq \mathcal{A}$ of atoms and a PL formula φ: *Forget*$(\varphi, \{p\}) \equiv \varphi[p/\bot] \vee \varphi[p/\top]$ is a result of forgetting p from φ, and *Forget*$(\varphi, V \cup \{p\})$ is recursively defined as *Forget*$($*Forget*$(\varphi, \{p\}), V)$, with *Forget*$(\varphi, \emptyset) = \varphi$. Using this insight, the following result shows that the notion of forgetting (for PL [34]) is a special case of forgetting in μ-calculus.

Theorem 3. *Let φ be a PL formula and $V \subseteq \mathcal{A}$, then*

$$\mathrm{F}_\mu(\varphi, V) \equiv \textit{Forget}(\varphi, V).$$

Theorem 3 reveals that the forgetting in μ-calculus is an extension of the forgetting in PL. This gives us a sense that some of the properties of forgetting in PL also exist in forgetting of μ-calculus. The following properties hold in PL, modal logic **S5** [47] and CTL [20]. Below we show that they are also satisfied in our notion of forgetting in μ-calculus.

Proposition 2. *Let φ, φ_i, ψ_i ($i = 1, 2$) be μ-sentences and $V \subseteq \mathcal{A}$. We have*

(i) $\mathrm{F}_\mu(\varphi, V)$ *is satisfiable iff φ is;*
(ii) *If $\varphi_1 \equiv \varphi_2$, then* $\mathrm{F}_\mu(\varphi_1, V) \equiv \mathrm{F}_\mu(\varphi_2, V)$*;*
(iii) *If $\varphi_1 \models \varphi_2$, then* $\mathrm{F}_\mu(\varphi_1, V) \models \mathrm{F}_\mu(\varphi_2, V)$*;*
(iv) $\mathrm{F}_\mu(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$*;*
(v) $\mathrm{F}_\mu(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_\mu(\psi_1, V) \wedge \mathrm{F}_\mu(\psi_2, V)$*;*

Intuitively, in Proposition 2, (i) means that forgetting some set of atoms from a sentence does not affect the satisfiability of this sentence. In (ii), we can see that if two sentences are equivalent, then the results of forgetting the same set of atoms from both of them are also equivalent. The intuitive meaning of (iii) is obvious. (iv) indicates that the result of forgetting V from a disjunctive formula $\varphi_1 \vee \varphi_2$ is equivalent to the disjunction of the results of forgetting V from φ_1 and φ_2. (v) points out that (iv) is not true for the case of conjunctive formula.

Postulate (**IR**) is also of crucial importance for computing the SNC and WSC, in addition to the representation theorem. Consider the μ-sentence $\psi = \varphi \wedge (q \leftrightarrow \alpha)$. If $\varphi \wedge \alpha$ is $\{q\}$-irrelevant, then the result of forgetting q from ψ is φ. Formally, this can be described as in the following lemma, and as we will see later in Section 4, it is the basis of reducing the SNC (WSC) of any μ-sentence to that of a proposition.

Lemma 1. *Let φ and α be two μ-sentences and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then,* $\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

We will list other interesting properties of the forgetting operator in the following. Most importantly, the following property guarantees that we can modularly apply forgetting one by one to the atoms to be forgotten, instead of forgetting the set of atoms as a whole, which is stated in the definition of forgetting.

Proposition 3 (Modularity). *Given a μ-sentence φ, a set of atoms V and an atom p such that $p \notin V$, then,*

$$\mathrm{F}_\mu(\varphi, \{p\} \cup V) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V).$$

The next property follows from the above proposition.

Corollary 1 (Commutativity). *Let φ be a μ-sentence and $V_i \subseteq \mathcal{A}$ $(i = 1, 2)$. Then,*

$$\mathrm{F}_\mu(\varphi, V_1 \cup V_2) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, V_1), V_2).$$

Another important property of F_μ is about the formulas with that all the sub-formulas appear in the scope of the same temporal operators AX or EX. Formally:

Proposition 4 (Homogeneity). *Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence; then, we have*

(i) $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_\mu(\phi, V)$.
(ii) $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_\mu(\phi, V)$.

The homogeneity of AX (or EX) on forgetting indicates that we can move the operator F_μ afterward to the AX (or EX) to forget a set V from a formula in the form $\mathrm{AX}\varphi$ (or $\mathrm{EX}\varphi$). Especially, when the formula ϕ in Proposition 4 is a PL formula, then forgetting of formulas with the form $Q\mathrm{X}\phi$ $(Q \in \{\mathrm{E}, \mathrm{A}\})$ can be achieved through the corresponding forgetting in PL.

4.3 Complexity Results

Computational complexity theory focuses on classifying computational problems according to their resource usage, and relating these classes to each other. A problem is regarded as inherently difficult if its solution requires significant resources. Hence, classifying the forgetting operator from its complexity is important to explore an efficient algorithm to compute it. In this section, we explore the complexity of the forgetting operator from both its model checking and reasoning problems.

Recall that the uniform interpolant $\widetilde{\exists} p\varphi$ ($p \in \mathcal{A}$) of a disjunctive μ-formula φ is equivalent to the μ-formula $\varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\top]$ is defined from φ by simultaneously substituting the literals p and $\neg p$ with $\top$ [10]. Moreover, as we have talked above, our forgetting definition $\mathrm{F}_\mu(\varphi, V)$ is equivalent to the semantic definition of uniform interpolant $\widetilde{\exists} V\varphi$[10]. Therefore, the following result is trivial.

Proposition 5. *Let φ be a μ-sentence and $p \in \mathcal{A}$. If φ is a disjunctive μ-formula, then $\mathrm{F}_\mu(\varphi, \{p\})$ can be computed in linear time.*

In this sense, we can transform a formula into its disjunctive form offline and then compute the result of forgetting some atoms from it, which will be efficient in some situations. In the following example, we show how to compute forgetting "ch" from those disjunctive μ-formulas.

Example 2. Let us consider the following formulas: $\varphi_1 = j \wedge ch \wedge Cover(\neg j \wedge \neg ch, \top)$, $\varphi_2 = \mu X.(j \wedge ch) \wedge Cover(X, \top)$ and $\varphi_3 = \nu X.(j \wedge ch) \wedge Cover(Cover(X, \top), \top)$. Let $V = \{ch\}$, we can easily compute the results of forgetting V from these formulas.

(1) $\mathrm{F}_\mu(\varphi_1, V) \equiv j \wedge Cover(\neg j, \top) \equiv j \wedge \text{EX}(\neg j)$;

(2) $\mathrm{F}_\mu(\varphi_2, V) \equiv \mu X.j \wedge Cover(X, \top) \equiv \mu X.j \wedge \text{EX}X$;

(3) $\mathrm{F}_\mu(\varphi_2, V) \equiv \nu X.j \wedge Cover(Cover(X, \top), \top) \equiv \nu X.j \wedge \text{EX}(\text{EX}X)$.

Nevertheless, we will show that the model checking problem of forgetting is intractable even if the given formula is a disjunctive formula.

Proposition 6 (Model Checking). *Given a finite Kripke structure $\mathcal{M}$, a μ-sentence φ and $V \subseteq \mathcal{A}$. We have:*

(i) Deciding $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is EXPTIME*;*
(ii) If φ is a disjunctive μ-formula, then deciding $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is in NP$\cap$*co*-NP.

More importantly, from the perspective of knowledge base evolution, the following reasoning problems about forgetting, which are explored in PL [39], are also of interest.

(i) [Var-weak] if the restriction of φ on the signature of ψ is at most as strong as ψ, i.e., $\psi \models \mathrm{F}_\mu(\varphi, V)$,
(ii) [Var-strong] if the restriction of φ on the signature of ψ is at least as strong as ψ, i.e., $\mathrm{F}_\mu(\varphi, V) \models \psi$,
(iii) [Var-entailment] if the restriction of one knowledge base on its original signature is at most as strong as that of the other, i.e., $\mathrm{F}_\mu(\varphi, V) \models \mathrm{F}_\mu(\psi, V)$

where φ, ψ are μ-sentences, and V is a set of atoms. In addition, in (i) and (ii), there is $Var(\varphi) - V = Var(\psi)$, and in (iii), there is $V \subseteq Var(\varphi) \cap Var(\psi)$. Then, we have the following results.

Theorem 4 (Entailment). *Let φ and ψ be two μ-sentences and V be a set of atoms. Then, the following problems are* EXPTIME*-complete.*

(i) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \psi$,
(ii) deciding $\psi \models^? \mathrm{F}_\mu(\varphi, V)$,
(iii) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \mathrm{F}_\mu(\psi, V)$.

Similar to the reasoning problems discussed above, the following equivalent problems are also important, in which "var-independence" and "var-equivalence" under PL are proposed in [30]:

(i) [Var-independence] if a formula φ is independent of a set V of atoms, i.e., $\mathrm{F}_\mu(\varphi, V) \equiv \varphi$,
(ii) [Var-match] if the restriction of φ on the signature of ψ perfectly matches ψ, i.e., $\mathrm{F}_\mu(\varphi, V) \equiv \psi$,
(iii) [Var-equivalence] if the restriction of the two formulas on a common signature are equivalent, i.e., $\mathrm{F}_\mu(\varphi, V) \equiv \mathrm{F}_\mu(\psi, V)$.

The following results are implications of Theorem 4.

Corollary 2. *Let φ and ψ be two μ-sentences and V be a set of atoms. Then, the following problems are* EXPTIME*-complete.*

(i) deciding $\psi \equiv^? \mathrm{F}_\mu(\varphi, V)$,
(ii) deciding $\mathrm{F}_\mu(\varphi, V) \equiv^? \varphi$,
(iii) deciding $\mathrm{F}_\mu(\varphi, V) \equiv^? \mathrm{F}_\mu(\psi, V)$.

5 Applications of Forgetting

As mentioned above, forgetting in PL has been used in a variety of contexts, we show how forgetting in μ-calculus can be used in computing WSC (SNC) and knowledge update in this section.

5.1 Necessary and Sufficient Conditions

In this section, we present two key notions of our work: the SNC and the WSC of a given μ-calculus specification, which correspond to the *most general consequence* and the *most specific abduction* of a specification, respectively. As mentioned in the introduction, these notions are in accordance with the SP and the WP (introduced by E. Dijkstra in [13]), which have been central to a wide variety of tasks and studies, e.g. generating counterexamples and refining system in verification. Our contribution, in particular, will be to compute the SNC and WSC by forgetting under a given μ-sentence and a set V of atoms. Let us give the formal definition.

Definition 4 (sufficient and necessary condition). *Let ϕ, ψ be two μ-sentences, $V \subseteq Var(\phi)$, $q \in Var(\phi) - V$ and $Var(\psi) \subseteq V$.*

– *ψ is a* necessary condition *(NC) of q on V under ϕ if $\phi \models q \to \psi$.*
– *ψ is a* sufficient condition *(SC) of q on V under ϕ if $\phi \models \psi \to q$.*
– *ψ is a* strongest necessary condition *(SNC in short) of q on V under ϕ if it is an NC of q on V under ϕ, and $\phi \models \psi \to \psi'$ for any NC ψ' of q on V under ϕ.*
– *ψ is a* weakest sufficient condition *(WSC in short) of q on V under ϕ if it is an SC of q on V under ϕ, and $\phi \models \psi' \to \psi$ for any SC ψ' of q on V under ϕ.*

Intuitively, the SNC (WSC) is the strongest (weakest) among the NCs (SCs) of q on V under ϕ, i.e., for each ψ' with $\phi \models q \to \psi'$ ($\phi \models \psi' \to q$), $\phi \models \text{SNC} \to \psi'$ ($\phi \models \psi' \to \text{WSC}$). Note that if both ψ and ψ' are SNCs (WSCs) of q on V under ϕ, then $\psi \equiv \psi'$. In this sense, the SNC (WSC) of q on V under ϕ is unique (up to semantic equivalence). Furthermore, the following result shows that the SNC and WSC are dual notions.

Proposition 7 (Dual). *Let V, q, φ and ψ be defined as in Definition 4. Then, ψ is an SNC (a WSC) of q on V under φ iff $\neg\psi$ is a WSC (an SNC) of $\neg q$ on V under φ.*

Replacing q with any μ-sentence α and redefining V as a subset of $Var(\alpha) \cup Var(\phi)$ in Definition 4, we can generalize this definition to arbitrary formulas.

It turns out that we can lift previous concepts of the SNC and WSC for an atomic variable to any formula or, on the contrary, reduce the SNC and WSC of any formula to that of an atomic variable, as shown in the following results.

Proposition 8. *Let Γ and α be two μ-sentences, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q be a new proposition not in Γ and α. Then, a formula φ with $Var(\varphi) \subseteq V$ is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.*

The following result shows that the forgetting and notion of the SNC (WSC) are closely related.

Theorem 5. *Let φ be a μ-sentence, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.*

(i) $\mathrm{F}_\mu(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ *is a SNC of q on V under φ.*
(ii) $\neg\mathrm{F}_\mu(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ *is a WSC of q on V under φ.*

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 7. And let $\mathcal{F} = \mathrm{F}_\mu(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \to \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on V under φ.

The "SNC" part: We shall show that for all ψ' with ψ' is the NC of q on V under φ, i.e. $\varphi \models q \to \psi'$ and $Var(\psi') \subseteq V$, there is $\varphi \models \mathcal{F} \to \psi'$. Suppose that there exists a NC ψ of q on V under φ with ψ is not logic equivalence with $\mathcal{F}$ under φ, such that $\varphi \models \psi \to \mathcal{F}$.

$\Rightarrow \varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ ((**PP**) and $\mathrm{IR}(\psi, (Var(\varphi) \cup \{q\})) - V$)
$\Rightarrow \varphi \wedge \mathcal{F} \models \psi$ ($\varphi \wedge q \models \psi$ (by suppose))
$\Rightarrow \varphi \wedge \psi \models \mathcal{F}$ (by suppose)
$\Rightarrow \varphi \models \psi \leftrightarrow \mathcal{F}$ (Contradict with the suppose)

Therefore, $\mathcal{F}$ is the SNC of q on V under φ.

Recall that any initial K-structure can be expressed as a CTL formula [20]. In this sense, when the given system $\mathcal{M}$ is an initial K-structure, it is easy to compute the WSC of φ on some set of atoms under $\mathcal{M}$ in the model checking problem $\mathcal{M} \models \varphi$ by using forgetting.

Another important point is that although we can use forgetting to compute the SNC and WSC, it has been found that sometimes, it is much easier to compute the SNC than the WSC, and sometimes, it is the other way around [32]. For instance, the author found that for many formulas, the SNCs are easier to compute than WSCs [32]. Therefore, when one condition is much easier to compute, the following proposition will be very helpful in computing the other condition.

Proposition 9. *Let Γ be a μ-sentence, q be a proposition, and V be a set of propositions.*

(i) If φ is an NC of q on V under Γ, and ψ is the WSC of q on V under $\Gamma \cup \{\varphi\}$, then $\varphi \wedge \psi$ is the WSC of q on V under Γ.
(ii) If ψ is an SC of q on V under Γ and φ is the SNC of q on V under $\Gamma \cup \{\neg\psi\}$, then $\varphi \vee \psi$ is the SNC of q on V under Γ.

Even so, deciding whether a μ-sentence ψ is an SNC (a WSC) of the given q on a set V of atoms under μ-sentence φ is intractable since the problem of deciding whether $\varphi \models q \rightarrow \psi$ is EXPTIME-complete [5]. Thus, deciding whether ψ is an NC of q on V under φ is EXPTIME-complete. Formally, we have the following result.

Theorem 6. *Let ψ and φ be two μ-sentences, $V \subseteq Var(\varphi)$ be a set of atoms, $q \in Var(\varphi) - V$ and $Var(\psi) \subseteq V$. Then, the following problems are* EXPTIME*-complete.*

(i) Deciding whether ψ is an SNC of q on V under φ,
(ii) Deciding whether ψ is a WSC of q on V under φ.

5.2 Representing a Knowledge Update Via Forgetting

As talked in Introduction, knowledge update, originates from belief revision and update, has been defined by knowledge forgetting in S5 modal logic [48]. This section presents the final key notion of our work: the knowledge update. In particular, it proposes a method of defining knowledge update by forgetting in μ-calculus that will satisfy all the following Katsuno and Mendelzon's postulates (U1)-(U8) proposed in [27]:

(U1) $\Gamma \diamond \phi \models \phi$.
(U2) If $\Gamma \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma$.
(U3) If both Γ and ϕ are satisfiable, then $\Gamma \diamond \phi$ is also satisfiable.
(U4) If $\Gamma_1 \equiv \Gamma_2$ and $\phi_1 \equiv \phi_2$, then $\Gamma_1 \diamond \phi_1 \equiv \Gamma_2 \diamond \phi_2$.
(U5) $(\Gamma \diamond \phi) \wedge \psi \models \Gamma \diamond (\phi \wedge \psi)$.
(U6) If $\Gamma \diamond \phi \models \psi$ and $\Gamma \diamond \psi \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma \diamond \psi$.
(U7) If Γ has a unique model, then $(\Gamma \diamond \phi) \wedge (\Gamma \diamond \psi) \models \Gamma \diamond (\phi \vee \psi)$.
(U8) $(\Gamma_1 \vee \Gamma_2) \diamond \phi \equiv (\Gamma_1 \diamond \phi) \vee (\Gamma_2 \diamond \phi)$.

where $\varphi \diamond \psi$ expresses the result of updating φ with ψ and $\diamond$ is the knowledge update operator.

For this purpose, in this part, we assume that the models of a μ-sentence are initial structures. In addition, we also limit the definition of forgetting on initial structures, i.e., the models mentioned in Definition 3 are initial structures. In this case, we have:

Theorem 7. *Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence. Then there is a μ-sentence ψ such that:*

$$\mathcal{M} \models \psi \textit{ iff there exists } \mathcal{M}' \in \textit{Mod}(\phi) \textit{ s.t. } \mathcal{M} \leftrightarrow_V \mathcal{M}'$$

where both $\mathcal{M}$ and $\mathcal{M}'$ are initial structures.

Theorem 7 shows that the forgetting in μ-calculus is also closed when restricting the models of the μ-sentence to initial structures. Formally:

Corollary 3. *The forgetting of μ-calculus is closed under the initial structure semantic, i.e., we consider only the initial structures as the models of the μ-sentence.*

According to [20], we can see that any initial structure $\mathcal{M}$ on $\mathcal{A}$ can be captured by a CTL formula, i.e., the characterizing formula $\mathcal{F}_{\mathcal{A}}(\mathcal{M})$ [20], and hence a μ-sentence [17]. For the set $\mathcal{A}$, V_{min}, and $\varphi = \mathrm{F}_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min})$, by *Mod*$(\varphi)$, we mean the set of models of φ, where $V_{min} \subseteq \mathcal{A}$ is a minimal subset of atoms that makes φ consistent. Besides,

$$\bigcup_{V_{min} \subseteq \mathcal{A}} \textit{Mod}(\varphi)$$

denotes the union of *Mod*(φ) with $V_{min} \subseteq \mathcal{A}$. Then, we define the knowledge update operator $\diamond_{\mu}$ in μ-calculus as follows.

Definition 5. *Let Γ and ϕ be μ-sentences. The knowledge update operator $\diamond_{\mu}$ is defined as follows:*

$$\textit{Mod}(\Gamma \diamond_{\mu} \phi) = \bigcup_{\mathcal{M} \in \textit{Mod}(\Gamma)} \bigcup_{V_{min} \subseteq \mathcal{A}} \textit{Mod}(\mathrm{F}_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi),$$

where $\mathcal{F}_{\mathcal{A}}(\mathcal{M})$ is the characterizing formula of $\mathcal{M}$ on $\mathcal{A}$ and $V_{min} \subseteq \mathcal{A}$ is a minimal subset of atoms that makes $\mathrm{F}_{\mu}(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi$ consistent.

Intuitively, $\Gamma \diamond_{\mu} \phi$ is the result of updating Γ with ϕ by minimally fixing the models of Γ to that of ϕ. In other words, the knowledge update defined in Definition 5 is achieved by minimally changing every model of Γ to make it consistent with ϕ. In this sense, this method can be viewed as a model-based update approach.

Recall the definition of knowledge update in PL: Let I, J_1 and J_2 be three interpretations; then J_1 is closer to I than J_2, written $J_1 \leq_{I,pam} J_2$, iff $\mathrm{Diff}(I, J_1) \subseteq \mathrm{Diff}(I, J_2)$, where $\mathrm{Diff}(X, Y) = \{p \in \mathcal{A} \mid X(p) \neq Y(p)\}$. The set of models of knowledge updating ψ on Γ is exactly the union of the minimal models of ψ under the partial order $\leq_{I,pam}$, where I is a model of Γ, i.e.,

$$\textit{Mod}(\Gamma \diamond_{pam} \psi) = \bigcup_{I \in \textit{Mod}(\Gamma)} Min(\textit{Mod}(\psi), \leq_{I,pam}).$$

Here, $Min(Mod(\psi), \leq_{I,pam})$ is the set of models of ψ that are minimal with respect to $\leq_{I,pam}$.

Similarly, we define a partial ordering over the set of initial structures that link to the knowledge operator $\diamond_\mu$.

Definition 6. *let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be three initial structures, then $\mathcal{M}_1$ is closer to $\mathcal{M}$ than $\mathcal{M}_2$, written $\mathcal{M}_1 \leq_{\mathcal{M}} \mathcal{M}_2$, iff for any $V_2 \subseteq \mathcal{A}$ such that $\mathcal{M}_2 \leftrightarrow_{V_2} \mathcal{M}$, there exists a $V_1 \subseteq V_2$ such that $\mathcal{M}_1 \leftrightarrow_{V_1} \mathcal{M}$. We denote $\mathcal{M}_1 <_{\mathcal{M}} \mathcal{M}_2$ iff $\mathcal{M}_1 \leq_{\mathcal{M}} \mathcal{M}_2$ and $\mathcal{M}_2 \not\leq_{\mathcal{M}} \mathcal{M}_1$.*

Let M be a set of initial structures and $\mathcal{M}$ be an initial structure. We also use $Min(M, \leq_{\mathcal{M}})$ to denote the set of all minimal initial structures with respect to $\leq_{\mathcal{M}}$. Then, the following theorem is important for relating $\diamond_\mu$ and $\leq_{\mathcal{M}}$.

Theorem 8. *Let Γ and ϕ be μ-sentences. Then, we have:*

$$Mod(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in Mod(\Gamma)} Min(Mod(\phi), \leq_{\mathcal{M}}).$$

Theorem 8 indicates that our definition of knowledge update in μ-calculus by forgetting is in accordance with the one defined by the $\leq_{\mathcal{M}}$, which is similar to the partial order $\leq_{I,pam}$ in PL.

More importantly, the following theorem shows that our definition of $\diamond_\mu$ by forgetting satisfies the Katsuno and Mendelzon update postulates.

Theorem 9. *The knowledge update operator $\diamond_\mu$ satisfies the Katsuno and Mendelzon's update postulates (U1)-(U8).*

6 Concluding Remarks

Summary In this paper, we targeted the strongest necessary and weakest sufficient conditions (SNC and WSC) and knowledge update in μ-calculus using a forgetting-based approach. In doing so, we introduced and employed the notion of V-bisimulation, which is similar to the $\mathcal{L}$-bisimulation, in which any μ-sentence is invariant for $\mathcal{L}$-bisimulation. Furthermore, we studied formal properties about forgetting, including homogeneity, modularity, and commutativity. In particular, we showed that our notion of forgetting satisfies the well-known forgetting postulates. Thus, it faithfully extends the notion of forgetting in classical propositional logic, modal logic **S5** and CTL. We also showed that for any μ-sentence, if it is a disjunctive formula, then forgetting a set of atoms from it can be done in linear time. From the perspective of computational complexity, we investigated whether the model checking problem for forgetting results from a disjunctive formula is in NP∩co-NP and whether the entailment problems relating to forgetting are EXPTIME-complete. Moreover, deciding whether a given μ-sentence is a SNC (WSC) of a specification is EXPTIME-complete. We finally showed that the knowledge update in terms of forgetting for μ-calculus under initial structures satisfies the Katsuno and Mendelzon's postulates (U1)-(U8).

Future work In the future, an algorithm to compute forgetting will be explored and implemented. As shown in this work, the forgetting of a disjunctive formula can be done in linear time. In this sense, a method of transforming a formula into its disjunctive form will be useful for the forgetting algorithm. In addition, it is worthwhile to explore sub-classes of μ-calculus for which the forgetting can be computed easily. Moreover, when a finite transition system $\mathcal{M}$ does not satisfy a specification φ, it is important to calculate the weakest sufficient condition ψ over a signature V under which $\mathcal{M}$ satisfies φ. It is also crucial to explore how the weakest sufficient condition ψ can be used to revise or update the transition system $\mathcal{M}$ for the specification φ.

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A Supplementary Material: Proof Appendix

Proposition 1 Let $V, V_1 \subseteq \mathcal{A}$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three Kripke structures, then we have:

(i) the $\leftrightarrow_V$ is an equivalence relation between Kripke structures;
(ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$, then $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.

Proof. (i) We prove it form the reflexivity, symmetry and transitivity.

(1) $\leftrightarrow_V$ is reflexive. It is easy to check that $\mathcal{M} \leftrightarrow_V \mathcal{M}$ for any Kripke structure.

(2) $\leftrightarrow_V$ is symmetric. We will show that for each $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ then $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V-bisimulation $\mathcal{B}$, we construct a relation $\mathcal{B}_1$ as follows: $\mathcal{B}_1 = \{(s,t)|(t,s) \in \mathcal{B}\}$. We will show that $\mathcal{B}_1$ is a V-bisimulation between $\mathcal{M}_2$ and $\mathcal{M}_1$ from the following several points:

- $r_2 \mathcal{B}_1 r_1$ since $r_1 \mathcal{B} r_2$,
- for each $s \in S_1$ and $t \in S_2$, if $t\mathcal{B}_1 s$ then we have $s\mathcal{B}t$ and hence $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A} - V$, and
- the third and forth points in the definition of V-bisimulation can be checked easily for $\mathcal{B}_1$.

(3) $\leftrightarrow_V$ is transitive. We will show that for each $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ then $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V-bisimulation $\mathcal{B}_1$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ by the V-bisimulation $\mathcal{B}_2$, we construct a relation $\mathcal{B}$ as follows: $\mathcal{B} = \{(s,z)|(s,t) \in \mathcal{B}_1 \text{ and } (t,z) \in \mathcal{B}_2\}$. We can also prove similarly with (2) that $\mathcal{B}$ is a V-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$. Therefore, $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$.

(ii) In order to prove $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$, we only need to find a binary relation $\mathcal{B}$ such that $\mathcal{B}$ is a $(V \cup V_1)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V-bisimulation $\mathcal{B}_1$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$ by the V_1-bisimulation $\mathcal{B}_2$. Let $\mathcal{B} = \{(s_1,s_3)|(s_1,s_2) \in \mathcal{B}_1 \text{ and } (s_2,s_3) \in \mathcal{B}_2\}$. We can easily check that $\mathcal{B}$ is a $(V \cup V_1)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$.

Theorem 1 Let $q \in \mathcal{A}$ and ϕ be a μ-sentence. There is a μ-sentence ψ such that $\mathrm{IR}(\psi, \{q\})$ and $\psi \equiv \mathrm{F}_\mu(\phi, \{q\})$.

Proof. It has been proved that for each μ-sentence ϕ and atom p, there is a μ-sentence ϕ' with $\mathrm{IR}(\phi', \{p\})$ such that (Theorem 3.1 in [8]):

$$\mathcal{M} \models \phi' \text{ iff } \exists \mathcal{M}' \in \phi \text{ such that } \mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}'.$$

This is accordance with the definition of forgetting.

Theorem 3 Let φ be a PL formula and $V \subseteq \mathcal{A}$, then

$$\mathrm{F}_\mu(\varphi, V) \equiv \mathit{Forget}(\varphi, V).$$

Proof. Let $\mathcal{M} = (S, r, R, L)$ and $\mathcal{M}' = (S', r', R', L')$ be two Kripke structure. It is easy to see that a Kripke structure $\mathcal{M}$ is a model of a PL formula ψ if $L(r)$ satisfy ψ, and also can be denoted as $\mathcal{M} \models \psi$.

($\Rightarrow$) For each $\mathcal{M} \in \mathit{Mod}(\mathrm{F}_\mu(\varphi, V))$
$\Rightarrow \exists \mathcal{M}' \in \mathit{Mod}(\varphi)$ such that $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by the V-bisimulation $\mathcal{B}$ (Def. 3)
$\Rightarrow r\mathcal{B}r'$
$\Rightarrow \mathcal{M} \models \mathit{Forget}(\varphi, V)$ (IR($\mathit{Forget}(\varphi, V), V$) and V-invariant)
($\Leftarrow$) For each $\mathcal{M} \in \mathit{Mod}(\mathit{Forget}(\varphi, V))$
$\Rightarrow \exists \mathcal{M}' \in \mathit{Mod}(\varphi)$ such that for each $p \in \mathcal{A} - V$, $p \in L(r)$ iff $p \in L'(r')$ (by the definition of *Forget*)
$\Rightarrow$ Constructing a Kripke structure $\mathcal{M}_1 = (S_1, r_1, R_1, L_1)$ such that:

* $S_1 = (S - \{r\}) \cup \{r_1\}$,
* R_1 is the same as R except that r is replaced by r_1, and
* L_1 is the same as L except $L_1(r_1) = L'(r')$.

$\Rightarrow \mathcal{M}_1 \models \varphi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\varphi, V)$ (IR($\mathrm{F}_\mu(\varphi, V), V$) and V-invariant)

Proposition 2 Let φ, φ_i, ψ_i ($i = 1, 2$) be formulas in μ-calculus and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_\mu(\varphi, V)$ is satisfiable iff φ is;
(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_\mu(\varphi_1, V) \equiv \mathrm{F}_\mu(\varphi_2, V)$;
(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_\mu(\varphi_1, V) \models \mathrm{F}_\mu(\varphi_2, V)$;
(iv) $\mathrm{F}_\mu(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$;
(v) $\mathrm{F}_\mu(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_\mu(\psi_1, V) \wedge \mathrm{F}_\mu(\psi_2, V)$;

Proof. (i) ($\Rightarrow$) Supposing $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\varphi, V)$, then there is a model $\mathcal{M}'$ of φ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by the definition of F_μ.

($\Leftarrow$) Supposing $\mathcal{M}$ is a model of φ, then there is a Kripke structure $\mathcal{M}'$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$, and then $\mathcal{M}' \models \mathrm{F}_\mu(\varphi, V)$ by the definition of F_μ.

The (ii) and (iii) can be proved similarly.

(iv) ($\Rightarrow$) For all$\mathcal{M} \in \mathit{Mod}(\mathrm{F}_\mu(\psi_1 \vee \psi_2, V))$, there exists $\mathcal{M}' \in \mathit{Mod}(\psi_1 \vee \psi_2)$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ and $\mathcal{M}' \models \psi_1$ or $\mathcal{M}' \models \psi_2$
$\Rightarrow$ there exists $\mathcal{M}_1 \in \mathit{Mod}(\mathrm{F}_\mu(\psi_1, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_1$ or there exists $\mathcal{M}_2 \in \mathit{Mod}(\mathrm{F}_\mu(\psi_2, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$.
($\Leftarrow$) for all $\mathcal{M} \in \mathit{Mod}(\mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V))$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\psi_1, V)$ or $\mathcal{M} \models \mathrm{F}_\mu(\psi_2, V)$
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}_1$ and $\mathcal{M}_1 \models \psi_1$ or $\mathcal{M}_1 \models \psi_2$
$\Rightarrow \mathcal{M}_1 \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $\mathcal{M}_2$ s.t. $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \models \mathrm{F}_\mu(\psi_1 \vee \psi_2, V)$
$\Rightarrow \mathcal{M} \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M} \models \mathrm{F}_\mu(\psi_1 \vee \psi_2, V)$.
The (v) can be proved as (iv).

Lemma 1 Let φ and α be two μ-sentences and $q \in \overline{\mathit{Var}(\varphi) \cup \mathit{Var}(\alpha)}$. Then $\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $\mathcal{M}$ of $\mathrm{F}_\mu(\varphi', q)$ there is a Kripke structure $\mathcal{M}'$ s.t. $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$ and $\mathcal{M}' \models \varphi'$. It's evident that $\mathcal{M}' \models \varphi$, and then $\mathcal{M} \models \varphi$ since IR($\varphi, \{q\}$) and $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$.

Let $\mathcal{M} \in Mod(\varphi)$ with $\mathcal{M} = (S, s, R, L)$. We construct $\mathcal{M}'$ with $\mathcal{M}' = (S, s, R, L')$ as follows:

$$L' : S \to 2^{\mathcal{A}} \; and \; \forall s^* \in S, L'(s^*) = L(s^*) - \{q\} \; if \; (\mathcal{M}, s^*) \not\models \alpha,$$
$$else \; L'(s^*) = L(s^*) \cup \{q\},$$
$$L'(s) = L(s) \cup \{q\} \; if \; (\mathcal{M}, s) \models \alpha, \; and \; L'(s) = L(s) \; otherwise.$$

It is clear that $\mathcal{M}' \models \varphi$, $\mathcal{M}' \models q \leftrightarrow \alpha$ and $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$. Therefore $\mathcal{M}' \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $\mathcal{M} \models \mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$ and IR$(\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q), \{q\})$.

Proposition 3 **(Modularity)** Given a μ-sentence φ, a set of atoms V and an atom p such that $p \notin V$, then,

$$\mathrm{F}_\mu(\varphi, \{p\} \cup V) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V).$$

Proof. Let $\mathcal{M}_1$ with $\mathcal{M}_1 = (S_1, s_1, R_1, L_1)$ be a model of $\mathrm{F}_\mu(\varphi, \{p\} \cup V)$. By the definition of forgetting, there exists a model $\mathcal{M}$ with $\mathcal{M} = (S, s, R, L)$ of φ, such that $\mathcal{M}_1 \leftrightarrow_{\{p\} \cup V} \mathcal{M}$. We construct a Kripke structure $\mathcal{M}_2$ with $\mathcal{M}_2 = (S_2, s_2, R_2, L_2)$ as follows:

(1) for s_2: let s_2 be the state such that:
- $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
- for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
- for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another: supposing $\mathcal{M}_1 \leftrightarrow_{\{p\} \cup V} \mathcal{M}$ by the $\{p\} \cup V$-bisimulation $\mathcal{B}$.
   (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $(w, w_1) \in \mathcal{B}$, let $w_2 \in S_2$ and
   - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
   - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
   - for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

   (ii) if $(w_1', w_1) \in R_1$, w_2 is constructed based on w_1 and $w_2' \in S_2$ is constructed based on w_1', then $(w_2', w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Thus, $(\mathcal{M}_2, s_2) \models \mathrm{F}_\mu(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V)$.

On the other hand, suppose that $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V)$, then there exists a Kripke structure $\mathcal{M}_2$ such that $\mathcal{M}_2 \models \mathrm{F}_\mu(\varphi, p)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$, and there exists $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$. Therefore, $\mathcal{M} \leftrightarrow_{\{p\} \cup V} \mathcal{M}_1$ by (ii) of Proposition 1, and consequently, $\mathcal{M}_1 \models \mathrm{F}_\mu(\varphi, \{p\} \cup V)$.

Proposition 4 **(Homogeneity)** Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence; then, we have

(i) $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_\mu(\phi, V)$.
(ii) $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_\mu(\phi, V)$.

Proof. Let $\mathcal{M} = (S, R, s, L)$, $M_i = (S_i, s_i, R_i, L_i)$ with $i \in \mathrm{N}$ and $\mathcal{M}' = (S', s', R', L')$, then we call $\mathcal{M}' = (S', s', R', L')$ be a sub-structure of $\mathcal{M}$ if:

- $S' \subseteq S$ and $S' = \{s' | s'$ is reachable from $s'\} \cup A$ with $A = \{s'' \mid s''$ can not be reached from s' and there is not such a sequence of states $(s, \ldots, s'', s')\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to 2^{\mathcal{A}}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- s' is s or a state reachable from s.

(i) To prove $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}(\mathrm{F}_\mu(\phi, V))$, we only need to prove $Mod(\mathrm{F}_\mu(\mathrm{AX}\phi, V)) = Mod(\mathrm{AXF}_\mu(\phi, V))$:

($\Rightarrow$) $\forall \mathcal{M}' \in Mod(\mathrm{F}_\mu(\mathrm{AX}\phi, V))$ there exists a Kripke structure $\mathcal{M}$ s.t. $\mathcal{M} \models \mathrm{AX}\phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$
$\Rightarrow$ for any sub-structure $\mathcal{M}_1$ of $\mathcal{M}$ there is $\mathcal{M}_1 \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_2$ s.t. $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}_3$ by $\mathcal{M}_2$ s.t. $\mathcal{M}_2$ is a sub-structure of $\mathcal{M}_3$ with s_2 is a direct successor of s_3 and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}$
$\Rightarrow \mathcal{M}_3 \models \mathrm{AX}(\mathrm{F}_\mu(\phi, V))$ and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}'$
$\Rightarrow \mathcal{M}' \models \mathrm{AX}(\mathrm{F}_\mu(\phi, V))$.

($\Leftarrow$) $\forall\ \mathcal{M}_3 \in Mod(\mathrm{AX}(\mathrm{F}_\mu(\phi, V)))$, then for any sub-structure $\mathcal{M}_2$ with s_2 is a directed successor of s_3 there is $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$
$\Rightarrow$ for any $\mathcal{M}_2$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M}_1 \models \phi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}$ by $\mathcal{M}_1$ s.t. $\mathcal{M}_1$ is a sub-structure of $\mathcal{M}$ with s_1 is a direct successor of s and $\mathcal{M} \leftrightarrow_V \mathcal{M}_3$
$\Rightarrow \mathcal{M} \models \mathrm{AX}\phi$ and then $\mathcal{M}_3 \models \mathrm{F}_\mu(\mathrm{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EXF}_\mu(\phi, V)$, we only need to prove $Mod(\mathrm{F}_\mu(\mathrm{EX}\phi, V)) = Mod(\mathrm{EXF}_\mu(\phi, V))$:

($\Rightarrow$) $\forall \mathcal{M}' \in Mod(\mathrm{F}_\mu(\mathrm{EX}\phi, V))$ there exists a Kripke structure $\mathcal{M}$ s.t. $\mathcal{M} \models \mathrm{EX}\phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$
$\Rightarrow$ there is a sub-structure $\mathcal{M}_1$ of $\mathcal{M}$ s.t. $\mathcal{M}_1 \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_2$ s.t. $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}_3$ by $\mathcal{M}_2$ s.t. $\mathcal{M}_2$ is a sub-structure of $\mathcal{M}_3$ that s_2 is a direct successor of s_3 and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}$
$\Rightarrow \mathcal{M}_3 \models \mathrm{EX}(\mathrm{F}_\mu(\phi, V))$
$\Rightarrow \mathcal{M}' \models \mathrm{EX}(\mathrm{F}_\mu(\phi, V))$.

($\Leftarrow$) $\forall\ \mathcal{M}_3 \in Mod(\mathrm{EX}(\mathrm{F}_\mu(\phi, V)))$, then there exists a sub-structure $\mathcal{M}_2$ of $\mathcal{M}_3$ s.t. $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M}_1 \models \phi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}$ by $\mathcal{M}_1$ s.t. $\mathcal{M}_1$ is a sub-structure of $\mathcal{M}$ that s_1 is a direct successor of s and $\mathcal{M} \leftrightarrow_V \mathcal{M}_3$
$\Rightarrow \mathcal{M} \models \mathrm{EX}\phi$ and then $\mathcal{M}_3 \models \mathrm{F}_\mu(\mathrm{EX}\phi, V)$.

Proposition 6 **(Model Checking)** Given a finite Kripke structure $\mathcal{M}$, a μ-sentence φ and $V \subseteq \mathcal{A}$. We have:

(i) Deciding $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is EXPTIME;
(ii) If φ is a disjunctive μ-formula, then deciding $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is in NP$\cap$co-NP.

Proof. For a given μ-formula, constructing a μ-automaton (also called modal automaton [5]) A_φ can be done in polynomial time if it is a disjunctive μ-formula, else in EXPTIME [6]. We prove (ii), then (i) can be proved similarly.

Let A_φ be a μ-automaton such that, for any Kripke structure $\mathcal{N}$, A_φ accepts $\mathcal{N}$ iff $\mathcal{N} \models \varphi$, where $A_\varphi = (Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ with $Var(\varphi) = \Sigma_p \cup \Sigma_r$. Without loss of generality, we assume $V \subseteq Var(\varphi)$ and $V = \{p\}$. Therefore we can construct a μ-automaton $\mathcal{B} = (Q, \Sigma_p - V, \Sigma_r, q_0, \delta', \Omega)$ such that

$$\delta'(q, L) = \delta(q, L) \cup \delta(q, L \cup \{p\}).$$

It has been proved in [8] that, for each Kripke structure $\mathcal{N}$, $\mathcal{B}$ accepts $\mathcal{N}$ iff there is a model $\mathcal{N}'$ of φ such that $\mathcal{N} \leftrightarrow_{\{p\}} \mathcal{N}'$, i.e. $\mathcal{B}$ corresponds to a μ-sentence which is equivalent to $\mathrm{F}_\mu(\varphi, V)$ by the definition of forgetting in μ-calculus.

In this case, the problem $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is reduced to decide whether $\mathcal{B}$ accepts $\mathcal{M}$. The automaton $\mathcal{B}$ accepts a Kripke structure $\mathcal{M} = (S, r, R, L)$ from the root r iff Eve has a winning strategy in the parity game $\mathcal{G}(\mathcal{M}, \mathcal{A})$ from the position (r, q^0), which is in NP∩co-NP [5].

Theorem 4 **(Entailment)** Let φ and ψ be two μ-sentences and V be a set of atoms. Then, the following problems are EXPTIME-complete.

(i) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \psi$,
(ii) deciding $\psi \models^? \mathrm{F}_\mu(\varphi, V)$,
(iii) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \mathrm{F}_\mu(\psi, V)$.

Proof. We prove the (i), there other two results can be proved similarly.

Let A_φ and A_ψ be the μ-automaton of φ and ψ respectively, we can construct the μ-automaton B of $\mathrm{F}_\mu(\varphi, V)$ from A_φ by the proof of Proposition 6. By Proposition 7.3.2 in [7] , we can obtain the complement C of A_ψ in linear time, and then the intersection $A_{C\cap B}$ between C and B in linear time. In this case, the $\mathrm{F}_\mu(\varphi, V) \models^? \psi$ is reduced to decide whether the language accepted by $A_{C\cap B}$ is empty, which is EXPTIME-complete (Theorem 7.5.1 of [7]).

Proposition 7 **(dual)** Let V, q, φ and ψ be defined as in Definition 4. Then, ψ is an SNC (a WSC) of q on V under φ iff $\neg\psi$ is a WSC (an SNC) of $\neg q$ on V under φ.

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \to \psi$. Thus $\varphi \models \neg\psi \to \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \to \neg q$. Then $\varphi \models q \to \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \to \neg\psi'$ by assumption. So $\varphi \models \psi' \to \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case.

Proposition 8 Let Γ and α be two μ-sentences, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q be a new proposition not in Γ and α. Then, a formula φ with $Var(\varphi) \subseteq V$ is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

[6] Personal communication: Giovanna D'Agostino, Renyan Feng, 2020.

Proof. We prove this for SNC. The case for WSC is similar. Let $\mathit{SNC}(\varphi, \beta, V, \Gamma)$ denote that φ is the SNC of β on V under Γ, and $\mathit{NC}(\varphi, \beta, V, \Gamma)$ denote that φ is the NC of β on V under Γ, in which β is a formula.

($\Rightarrow$) We will show that if $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds, then $\mathit{SNC}(\varphi, q, V, \Gamma')$ will be true. According to $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \rightarrow \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F}_{\mathrm{CTL}}(\Gamma', q) \models \alpha \rightarrow \varphi'$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \rightarrow \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi'$ by Lemma 1, this means $\mathit{NC}(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \rightarrow \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \rightarrow \varphi'$. Hence, $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds, then $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ will be true. According to $\mathit{SNC}(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F}_{\mathrm{CTL}}(\Gamma', \{q\}) \models \alpha \rightarrow \varphi$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \rightarrow \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi$ by Lemma 1, this means $\mathit{NC}(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \rightarrow \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $\mathit{NC}(\varphi', q, V, \Gamma')$. According to $\mathit{SNC}(\varphi, q, V, \Gamma')$, $\mathit{IR}(\varphi \rightarrow \varphi', \{q\})$ and (**PP**), we have $\mathrm{F}_{\mathrm{CTL}}(\Gamma', \{q\}) \models \varphi \rightarrow \varphi'$, and $\Gamma \models \varphi \rightarrow \varphi'$ by Lemma 1. Hence, $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds.

Theorem 5 Let φ be a μ-sentence, $V \subseteq \mathit{Var}(\varphi)$ and $q \in \mathit{Var}(\varphi) - V$.

(i) $\mathrm{F}_\mu(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ.
(ii) $\neg\mathrm{F}_\mu(\varphi \wedge \neg q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ.

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 7. And let $\mathcal{F} = \mathrm{F}_\mu(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by (**W**). Hence, $\varphi \models q \rightarrow \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on V under φ.

The "SNC" part: We shall show that for all ψ' with ψ' is the NC of q on V under φ, i.e. $\varphi \models q \rightarrow \psi'$ and $\mathit{Var}(\psi') \subseteq V$, there is $\varphi \models \mathcal{F} \rightarrow \psi'$. Suppose that there exists a NC ψ of q on V under φ with ψ is not logic equivalence with $\mathcal{F}$ under φ, such that $\varphi \models \psi \rightarrow \mathcal{F}$.

$\Rightarrow \varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ ((**PP**) and $\mathrm{IR}(\psi, (\mathit{Var}(\varphi) \cup \{q\})) - V)$
$\Rightarrow \varphi \wedge \mathcal{F} \models \psi$ ($\varphi \wedge q \models \psi$ (by suppose))
$\Rightarrow \varphi \wedge \psi \models \mathcal{F}$ (by suppose)
$\Rightarrow \varphi \models \psi \leftrightarrow \mathcal{F}$ (Contradict with the suppose)

Therefore, $\mathcal{F}$ is the SNC of q on V under φ.

Proposition 9 Let Γ be a μ-sentence, q be a proposition, and V be a set of propositions.

(i) If φ is a necessary condition of q on V under Γ, and ψ the weakest sufficient condition of q on V under $\Gamma \cup \{\varphi\}$, then $\varphi \wedge \psi$ is the weakest sufficient condition of q on V under Γ.

(ii) If ψ is a sufficient condition of q on V under Γ, and φ a strongest necessary condition of q on V under $\Gamma \cup \{\neg\psi\}$, then $\varphi \vee \psi$ is a strongest necessary condition of q on V under Γ.

Proof. (i) First of all, $\varphi \wedge \psi$ is an SC due to $\Gamma \cup \{\varphi\} \models \psi \rightarrow q$, i.e. $\Gamma \models \varphi \wedge \psi \rightarrow q$. We will show it is the weakest one: suppose φ' is an SC, i.e. $\Gamma \models \varphi' \rightarrow q$. We need to show that $\Gamma \models \varphi' \to \varphi \wedge \psi$. Thanks to $\Gamma \models q \to \varphi$. We have $\Gamma \models \varphi' \to \varphi$. However, φ' is also an SC of q under $\Gamma \cup \{\varphi\}$, hence $\Gamma \cup \{\varphi\} \models \varphi' \to \psi$ because ψ is the WSC of q under $\Gamma \cup \{\varphi\}$. Therefore $\Gamma \models (\varphi \wedge \psi)$.

(ii) Suppose φ' is an NC of q under Γ, i.e. $\Gamma \models q \to \varphi'$. We have $\Gamma \models \psi \to \varphi'$ since $\Gamma \models \psi \to q$. And there is $\Gamma \cup \{\neg\psi\} \models \varphi \to \varphi'$ due to $\Gamma \cup \{\neg\psi\} \models q \to \varphi$ and $\Gamma \cup \{\neg\psi\} \models q \to \varphi'$. Therefore we have $\Gamma \models \varphi \to \varphi'$ by $\Gamma \models \neg\psi \wedge \varphi \to \varphi'$ and $\Gamma \models \psi \to \varphi'$.

Theorem 6 Let ψ and φ be two μ-sentence, $V \subseteq \mathit{Var}(\varphi)$ be a set of atoms, $q \in \mathit{Var}(\varphi) - V$ and $\mathit{Var}(\psi) \subseteq V$. Then, the following problems are EXPTIME-complete.

(i) Deciding whether ψ is a SNC of q on V under φ,
(ii) Deciding whether ψ is a wsc of q on V under φ.

Proof. We prove (i) and (ii) can be proved similarly.

For the "NC" part, deciding $\varphi \models q \to \psi$ can be down in EXPTIME-complete. In this sense, in order to prove the "SNC" part, we only need to decide $\psi \equiv^? \mathrm{F}_\mu(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$. According to Corollary 2, it is in EXPTIME-complete.

Theorem 7 Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence. Then there is a μ-sentence ψ such that:

$$\mathcal{M} \models \psi \text{ iff there is a model } \mathcal{M}' \text{ of } \phi \text{ such that } \mathcal{M} \leftrightarrow_V \mathcal{M}'$$

where both $\mathcal{M}$ and $\mathcal{M}'$ are initial structures.

Proof. Let $\psi = \mathrm{F}_\mu(\phi, V)$. We have that for each $\mathcal{M} \models \psi$ there is a $\mathcal{M}' \models \phi$ with $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by Theorem 1 and for each $\mathcal{M}' \in \mathit{Mod}(\phi)$ there is $\phi \models \psi$. In this case, we can easy prove that for each initial structure $\mathcal{M}$, if $\mathcal{M} \models \psi$ then we can obtain an initial structure $\mathcal{M}'$ such that $\mathcal{M}' \models \phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Besides, for each $\mathcal{M}' \in \mathit{Mod}(\phi)$ there is $\mathcal{M}' \models \psi$ by $\phi \models \psi$.

Theorem 8 Let Γ and ϕ be μ-sentences. Then, we have:

$$\mathit{Mod}(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}}).$$

Proof. For each initial structure $\mathcal{M}' \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$, we will show that there exists some $\mathcal{M} \in \mathit{Mod}(\Gamma)$ such that $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$. According to Definition 5, we know that there exists some $\mathcal{M} \in Mod(\Gamma)$ such that $\mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$. Further, there is a particular $V' \subseteq \mathcal{A}$ (i.e. $V' = V_{min}$) such that $\mathcal{M}' \leftrightarrow_{V'} \mathcal{M}$ and $\mathcal{M}' \in \mathit{Mod}(\phi)$. Since such V' is a minimal subset of $\mathcal{A}$ satisfying these properties, it concludes that for any other models $\mathcal{M}''$ of ϕ with $\mathcal{M}'' \leftrightarrow_{V_{min}} \mathcal{M}$, we have $\mathcal{M}' \leq_{\mathcal{M}} \mathcal{M}''$ by the definitions of forgetting and characterizing formula. Therefore, $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$.

For each initial structure $\mathcal{M}' \in \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$, there exists some $\mathcal{M} \in \mathit{Mod}(\Gamma)$ such that $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$. Let V_{min} be a minimal subset of atoms such that $\mathcal{M}' \leftrightarrow_{V_{min}} \mathcal{M}$. Then according to the definition of $\leq_{\mathcal{M}}$, we know that there does not exist another $\mathcal{M}'' \in \mathit{Mod}(\phi)$ such that $\mathcal{M}'' \leftrightarrow_{V'} \mathcal{M}$ and $V' \subset V_{min}$. This follows that $\mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$ and hence $\mathcal{M}' \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$.

Theorem 9 Knowledge update operator $\diamond_\mu$ satisfies the Katsuno and Mendelzon's update postulates (U1)-(U8).

Proof. For (U1), we know that $Mod(\Gamma \diamond_\mu \phi) \subseteq Mod(\phi)$ by Theorem 8, hence $\Gamma \diamond_\mu \phi \models \phi$.

For (U2), we will prove $\Gamma \diamond_\mu \phi \models \Gamma$ at first. For any model $\mathcal{M}$ of $\Gamma \diamond_\mu \phi$ there is a $\mathcal{M}_1 \in Mod(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Then we have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Similarly, for any model $\mathcal{M}$ of Γ, there is a $\mathcal{M}_1 \in Mod(\Gamma \diamond_\mu \phi)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. We have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Hence $\Gamma \models \Gamma \diamond_\mu \phi$.

It is easy to show $\diamond_\mu$ satisfies (U3) and (U4). We now prove (U5). For any model $\mathcal{M}$ of $(\Gamma \diamond_\mu \phi) \wedge \psi$ there is a $\mathcal{M}_1 \in Mod(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Besides, we can see that $\mathcal{M} \models \phi \wedge \psi$. Therefore, we have $\mathcal{M} \models \Gamma \diamond_\mu (\phi \wedge \psi)$.

For (U6), we will prove $\Gamma \diamond_\mu \phi \models \Gamma \diamond_\mu \psi$, and the other direction can be proved similarly. For any model $\mathcal{M}$ of $\Gamma \diamond_\mu \phi$, $\mathcal{M}$ is also a model of ψ. There is a $\mathcal{M}_1 \in Mod(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \psi$. This shows that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Moreover, V_{min} is also the minimal set such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Otherwise, suppose that $V \subset V_{min}$ such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V) \wedge \psi$ is consistent as well. Then, $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V)$ $\wedge \phi$ should also be consistent by $\Gamma \diamond_\mu \psi \models \phi$, which contradicts to the fact that V_{min} is the minimal set of atoms such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \phi$ is consistent. Hence, $\mathcal{M}$ is also a model of $\Gamma \diamond_\mu \psi \models \psi$.

Now we prove (U7). Suppose that Γ has the unique model $\mathcal{M}$. For each $\mathcal{M}_1 \in Mod((\Gamma \diamond_\mu \phi) \wedge (\Gamma \diamond_\mu \psi))$ there exists V_1 and V_2 which are minimal such that $\mathcal{M} \leftrightarrow_{V_1} \mathcal{M}_1$ and $\mathcal{M} \leftrightarrow_{V_2} \mathcal{M}_1$, i.e. $\mathcal{M}_1$ is a model of both $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge \phi$ and $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_2) \wedge \psi$. Therefore $\mathcal{M}_1 \leftrightarrow_{V_1 \cap V_2} \mathcal{M}$. Thus, $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1 \cap V_2)$. Then we have $V_1 = V_2$, otherwise V_1 (or V_2) is not the minimal set. $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ as well. Moreover, V_1 is the minimal set such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ is satisfiable. Otherwise, suppose that $V_3 \subset V_1$ such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge (\phi \vee \psi)$ is satisfiable. Then $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \phi$ or $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \psi$ is satisfiable. Without loss of generality, suppose that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \phi$ is satisfiable, V_1 is not the minimal set, a contradiction. Therefore $\mathcal{M}_1$ is also a model of $\Gamma \diamond_\mu (\phi \vee \psi)$.

For (U8), we will prove $(\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi \models (\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi)$ at first. For each $\mathcal{M} \in Mod((\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi)$, there is a $\mathcal{M}_1 \in Mod(\Gamma_1)$ (or $\mathcal{M}_1 \in Mod(\Gamma_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore, we have $\mathcal{M} \models (\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi)$. Similarly, for each $\mathcal{M} \in Mod((\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi))$ there is a $\mathcal{M}_1 \in Mod(\Gamma_1)$ (or $\mathcal{M}_1 \in Mod(\Gamma_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Hence, $\mathcal{M} \models (\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi$.