On the Weakest Sufficient Conditions in Propositional μ-calculus

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Abstract. The μ-calculus is one of the most important logics describing specifications of transition systems. It has been extensively explored for formal verification in model checking due to its exceptional balance between expressiveness and algorithmic properties. On the one hand, some information content in a specification might become irrelevant or unnecessary due to various reasons from the perspective of knowledge representation. On the other hand, a weakest precondition of a specification is badly necessary in verification, where a (weakest) precondition is sufficient for a transition system to enjoy a desire property. This paper is to address these scenarios for μ-calculus in a principle way in terms of knowledge *forgetting*. In particular, it proposes a notion of forgetting by a generalized bisimilar equivalence (over a signature) and explores its important properties as a knowledge distilling operator, besides some reasoning complexity results. It then shows that how the weakest sufficient condition and the strongest necessary condition can be established via forgetting. It also discusses knowledge update for μ-calculus in terms of forgetting.

Keywords: Weakest precondition · Forgetting · Knowledge update.

1 Introduction

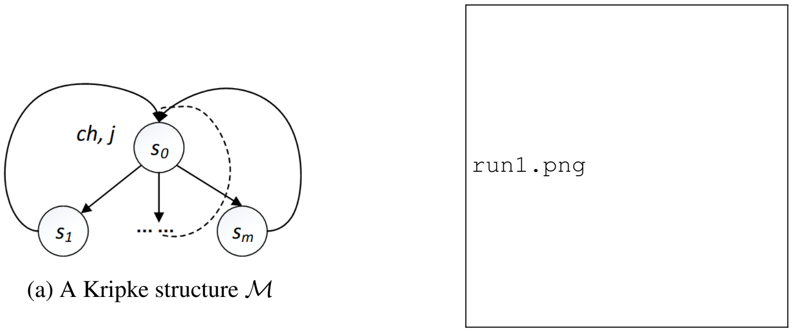
Propositional μ-calculus is an expressive logic, on binary trees it is as expressive as the monadic second order logic of two successors (S2S) [16,29]. Subsequent research showed that the μ-calculus is an important logic when specification and verification is concerned. One of the common phenomenons in both the verification and the system design is some information content of such specification might become irrelevant for the system due to various reasons e.g., it might be discarded or become obsolete by time, or just become infeasible due to practical difficulties. However, In this case it is expensive in time and space to re-extract the specification that meets the requirements. The problem arises on how to remove it without altering the relevant system behaviour or violating the existing system specifications over a given signature. Let's consider the following scenario.

Example 1 (Playing PingPong). Assume *John* plays PingPong with $n + 1$ people (i.e. $n + 2$ people in total) and there are $n + 2$ chairs. At the beginning, John plays PingPong with one of the $n + 1$ people, and the other n people are sitting on their chairs at this time. In our scenario, we assume *John* can not win anyone, which means that *John* has always been beaten by other people every turn. In this case, after n games it will be *John*'s turn. This process can be modeled by Fig. 1 (a).

In Fig. 1 (a), this scenario is represented by the Kripke structure $\mathcal{M} = (S, s_0, R, L)$ with the corresponding atomic variables $V = \{j, ch\}$, in which j means *John*'s turn and ch means the *John*'s chair is free. Besides, $m = 2^n$, this means there are m orders to *John*'s turn.

In this scenario, we can see from the unwinding of $\mathcal{M}$ in Fig. 1 (b) that for each (some) path $\pi = (s_0, s_1, ...)$ starting from s_0 we have $\forall i \in \mathbb{N}$ there is $(\mathcal{M}, s_{2*i}) \models ch \wedge j$ and $(\mathcal{M}, s_{2*i+1}) \models \neg ch \wedge \neg j$, i.e. for each *even state* in each (some) path that starting from s_0 the $ch \wedge j$ holds and $\neg ch \wedge \neg j$ holds for *old states* in this path. This property can be represented by a μ-calculus formula $\varphi = \nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$ ($\varphi = \nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$), this is not expressible with other temporal logics.

Now assume a situation in which due to some problems (i.e. the venue changed or the chair broke down), John does no longer have a chair. This means, all the playing processes concerning "ch" no more necessary and should be dropped from both the specifications (e.g. φ) and the Kripke structure for simplification.



(a) A Kripke structure $\mathcal{M}$

(b) The unwinding of $\mathcal{M}$

Fig. 1: The PingPong's Model

Similar scenarios like the one presented in Example 1 may arise in many different domains such as business-process modelling, software development, concurrent systems and more [2]. Yet dropping some restrictions in a large and complex system or specification, without affecting the working system components or violating dependent specifications over a given signature, is a non-trivial task. Moreover, in such a scenario, two logical notions introduced by E. Dijkstra in [12] are highly informative: the strongest post-condition (SP) and the weakest precondition (WP) of a given specification, which are corresponding with the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC), proposed by Lin [25], of the specification, respectively, and have been central to a wide variety of tasks and studies, e.g. in generating counterexamples [10] and refinement of system [36]. These correspond to the *most general consequence* and the *most specific abduction* of such specification, respectively.

Besides, *belief updates* and *belief revision*, as two well-studied issues in artificial intelligence, are concerned with the update and revision aspects of an agent's belief with respect to new beliefs [21]. Intuitively, if φ represents the agent's belief about the world and the agent performs an action that is supposed to make ψ true in the resulting world, then the agent's belief about the resulting world can be described by $\varphi \diamond \psi$, where $\diamond$ is the update operator of choice. We can see that the theory of belief updates does not tell us how to do updates with respect to such gain in knowledge due to a sensing action. In this sense, as an analogous notion of belief update, the *knowledge update* was proposed by Chitta in [3] to solve the belief updates caused by sensing actions, in which the effect of a sensing action is expressed by introducing modal operator (K)nows. Nevertheless, there are no approaches to solve the knowledge update in logic languages which contain *temporal operators*.

To address these scenarios and to target the relevant notions SNC (WSC) and knowledge update in a principle way. Inspired by [25,18], in this paper we explore the knowledge update and SNC (WSC) of μ-calculus from the point of forgetting. In particular, we will give the definition of forgetting in μ-calculus by using the bisimulation [5,2,38] and show whether this notion satisfies the general principles or postulates proposed by Zhang [38]. We then study the relationship between SNC (WSC) and forgetting, and we also demonstrate how forgetting can be used in knowledge update in μ-calculus.

Forgetting, which is a dual concept of *uniform interpolation* [30,22] and was first formally defined in propositional and FOL by Lin and Reiter [27,13], can be traced back to the work of Boole on propositional variable elimination and the seminal work of Ackermann [1]. Moreover, it has been extended to various logic systems. See [13] for a recent and comprehensive survey.

However, existing forgetting definitions in PL and answer set programming are not directly applicable in modal logics. And we can also not directly use the method of forgetting in CTL [18] since it will not work when the models of the formula are infinite. Hopefully, it has been proved that the modal μ-calculus has *Uniform interpolation* [7]. Informally, for every μ sentence φ and every finite set V of atoms, there exists an μ sentence $\exists V \varphi$ which does not contain atoms from V but is logically closest to φ in some sense. This means that the result of forgetting some atoms from a μ-calculus sentence always exists. In this sense, showing the semantic of forgetting in μ-calculus through *general principles or postulates* is important to make it clearer to understand.

Informally, the four postulates proposed by Zhang [38] show that the result ψ of forgetting some set V of atoms from a formula φ is not only weaker than the φ, i.e. $\varphi \models \psi$, irrelevant to V, i.e. exists some formula that do not contain atoms in V and equivalent with ψ, and also has the same "logic content" with φ, i.e. for each formula ϕ that irrelevant to V, ϕ can be implied by φ iff ϕ can be implied by ψ. In this paper we explore the forgetting of μ-calculus under infinite models (to distinguish that of CTL [18]) from both the postulates and the algebraic properties of the forgetting operator. The complexities of the reasoning problems of the forgetting operator are also explored from the point of automaton and it is shown that these are in EXPTIME-complete. It is worth mentioning that we restrict the models are finite in the knowledge update part in order to express the models of a formula. And we show that our definition of $\diamond_\mu$ by forgetting satisfies Katsuno and Mendelzon's update postulates [21].

The rest of the paper is organised as follows. Next section reports about the related work. Section 3 introduces the notation and technical preliminaries. As key contributions, Section 4, introduces the notion of forgetting in μ-calculus. Moreover, it studies the semantic properties of forgetting. In addition, a complexity analysis, concerning model checking and reasoning problems of forgetting, is carried out. Section 5 explores the relation between forgetting and SNC (WSC). Section 6 gives how to Represent knowledge update via forgetting. Conclusion closes the paper.

Due to space restrictions, for most of the technical results, the actual proof is moved to the supplementary material in Appendix, and instead an intuitive justification is put in place.

2 Related work

In this section, we shall discuss briefly about published matter that technically relates to our work.

2.1 The Weakest precondition condition

The Weakest precondition, as an important concept in formal verification, was proposed in [11] at first by Dijkstra to solve the problem of computing or approximating invariant appearing in *verification of computer programs and systems*, particularly in “Hoare triple” [19]. Afterward, it has been wildly studied by researchers, especially in generating counterexamples [10] and refinement of system [36].

In the filed of knowledge expression and reasoning (KR), there is a similar concept, called *weakest sufficient condition*(WSC), had been introduced in [26] by Lin to generate successor state axioms from causal theories. Moreover, the SNC and WSC for a proposition q on a restricted subset of the propositional variables under a propositional theory T are computed based on the notion of forgetting. Besides, the SNC and WSC are generalized to first order logic (FOL) and a direct method that is based on *Second-Order Quantifier Elimination* (SOQE) technique has been proposed to automatically generate SNC and WSC in [**?**]. Besides, the paper [18] also use a forgetting-based method to compute the SNC and WSC in CTL.

2.2 The notion of forgetting

Forgetting, which was first formally defined in propositional and first order logics by Lin and Reiter [27], can be traced back to the work of Boole on propositional variable elimination and the seminal work of Ackermann [1] who recognised that the problem amounts to the elimination of existential second-order quantifiers. Usually, the definition of forgetting can be defined from the perspective of Strong/Semantic Forgetting and Weak Forgetting respectively [**?**].

In First-Order Logic (FOL), forgetting has often been studied as an instance of the SOQE problem. It shows in [27] that the result of (Strongly) forgetting an n-ary predicate P from a first-order formula φ is $\exists R\varphi[P/R]$, in which R is an n-ary predicate variable and $\varphi[X/Y]$ is a result of replacing every occurrence of X in φ by Y. The task of forgetting in first-order logic, as a computational problem, is to find a first-order formula that is equivalent to $\exists R\varphi[P/R]$. It is evident that this is a SOQE problem. However, it shows in [?] that the solution of SOQE problem is not always expressible in FOL, which means that the results of forgetting in FOL is not always expressible in FOL. Nonetheless, the solution of weak forgetting is always expressible in FOL, though there are cases the forgetting solution can only be represented by an infinite set of FOL formulas [?]. Similarly, the forgetting in description logics are also explored to create restricted views of ontologies by eliminating concept and role symbols from description logic-based ontologies [34,28,39].

In Propositional Logic (PL), forgetting has often been studied under the name of variable elimination. In particular, the solution of forgetting a propositional variable p from a PL formula φ is $\varphi[p/\bot] \vee \varphi[p/\top]$ [27], where $\varphi[p/\bot]$ and $\varphi[p/\top]$ denote the formulas obtained from φ by replacing atom p with $\bot$ and $\top$ respectively. In [38], the authors define the knowledge forgetting of S5 modal logic from the Strong Forgetting point of view to explore the relationship between knowledge forgetting and knowledge update. Besides, they have proposed four general postulates for knowledge forgetting and show that these four postulates precisely characterize the notion of knowledge forgetting described above in S5 modal logic. What's more, forgetting in logic programs under answer-set semantics, which is a non-monotonic logic program, are also considered, see [37,14,35,33,32,?,?] for more details.

3 Modal μ-calculus

We start with some technical and notational preliminaries. Given a formula φ, the *language* of φ, denoted $L(\varphi)$, is the set of all propositional constants appearing in the formula.

3.1 Syntax

Modal μ-calculus is an extension of modal logic, we consider the propositional μ-calculus as introduced by Kozen [23]. Let $\mathcal{A} = \{p, q, \dots\}$ be a set of propositional letters (atoms) and $\mathcal{V} = \{X, Y, \dots\}$ a set of variables. Formulas of the μ-calculus over these sets can be defined by the following grammar:

$$\varphi := p \mid \neg p \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

where $p \in \mathcal{A}$ and $X \in \mathcal{V}$. $\top$ and $\bot$ are also μ-calculus formulae, which express 'true' and 'false' respectively. Note that we allow negations only before propositional letters. All the results presented here extend to the general case when negation before variables is also allowed, restricting as usual to positive occurrences of bound variables, that is to say variables appear after an even number of negations. Variables, propositional letters and their negations will be called *literals*. For convenience, in the following, φ, $\varphi_1, \dots$, ψ, $\psi_1, \dots$ will denote formulas. And by *Var*(φ) we mean the set of atoms appearing in formula φ.

We call a formula *well named* iff every variable is bound at most once in the formula and free variables are distinct from bound variables. For a variable X bound in a well named formula φ there exists a unique subterm of φ of the form $\delta X.\varphi(X)$ with $\delta \in \{\nu, \mu\}$, from now on called the *binding definition* of X in φ. We call X a μ-variable when $\delta = \mu$, otherwise we call X a ν-variable.

Variable X in $\delta X.\varphi(X)$ is *guarded* iff every occurrence of X in $\varphi(X)$ is in the scope of some modality operator $\Diamond$ or $\Box$. For convenience, we mix the two symbols $\Diamond$ and EX ($\Box$ and AX). A formula is guarded iff every bound variable in the formula is guarded.

Proposition 1. *Every formula is equivalent to some guarded formula.*

This proposition allows us to restrict ourselves to **guarded**, **well-named** formulas. From now on, we shall only consider formulas of this kind.

3.2 Semantic

Formulas are interpreted in transition systems of the form $\mathcal{M} = (S, r, R, L)$, we call it a Kripke structure, where:

- S is a nonempty set of states,
- $r \in S$,
- R is a binary relation on S, i.e. $R \subseteq S \times S$, called a transition relation, and
- $L : S \to 2^{\mathcal{A}}$ is a labeling function.

A Kripke structure $\mathcal{M}$ is finite if S is finite and $q \notin L(s)$ (for each $s \in S$) for almost all $q \in \mathcal{A}$.

Given a Kripke structure $\mathcal{M}$ and a valuation $v : \mathcal{V} \to 2^S$, the set of states in which a formula φ is true, denoted $\|\varphi\|_v^{\mathcal{M}}$, is defined inductively as follows (we will omit superscript $\mathcal{M}$ when it causes no ambiguity):

$$\begin{aligned}
&\|p\|_v = \{s \mid p \in L(s)\} \quad \|\top\|_v = S \quad \|\bot\|_v = \emptyset\\
&\|\neg p\|_v = S - \|p\|_v\\
&\|X\|_v = v(X)\\
&\|\varphi_1 \vee \varphi_2\|_v = \|\varphi_1\|_v \cup \|\varphi_2\|_v\\
&\|\varphi_1 \wedge \varphi_2\|_v = \|\varphi_1\|_v \cap \|\varphi_2\|_v\\
&\|\Diamond\varphi\|_v = \{s | \exists s'.(s, s') \in R \wedge s' \in \|\varphi\|_v\}\\
&\|\Box\varphi\|_v = \{s | \forall s'.(s, s') \in R \Rightarrow s' \in \|\varphi\|_v\}\\
&\|\mu X.\varphi\|_v = \bigcap\{S' \subseteq S | \, \|\varphi\|_{v[X:=S']} \subseteq S'\}\\
&\|\nu X.\varphi\|_v = \bigcup\{S' \subseteq S | S' \subseteq \|\varphi\|_{v[X:=S']}\}
\end{aligned}$$

where $v[X := S']$ is same to the valuation function v except that S' is assigned to X.

In the following, we denote $s \in \|\varphi\|_v$ by $(\mathcal{M}, s, v) \models \varphi$ and we may leave out the valuation v, if φ is a sentence (i.e. no variables in φ are free). $(\mathcal{M}, v) \models \varphi$ is used to denote $(\mathcal{M}, r, v) \models \varphi$, and in this case we say that $(\mathcal{M}, v)$ is a model of φ. By *Mod*(φ) we mean the set of models of φ. In particular, if φ is a sentence, then

$\mathit{Mod}(\varphi) = \{\mathcal{M} \mid (\mathcal{M}, r) \models \varphi\}$. Similarly, let Σ be a set of sentences, we define $\mathit{Mod}(\Sigma)$ as the set of Kripke structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ for each $\varphi \in \Sigma$. $\varphi \models \psi$ denotes logical consequence: if $(\mathcal{M}, v) \models \varphi$ then $(\mathcal{M}, v) \models \psi$ for every model $\mathcal{M}$ and for every valuation v. Especially, given two sentences (or set of sentences) Σ and Π, $\Sigma \models \Pi$ if $\mathit{Mod}(\Sigma) \subseteq \mathit{Mod}(\Pi)$ and $\Sigma \equiv \Pi$ whenever $\mathit{Mod}(\Sigma) = \mathit{Mod}(\Pi)$. A formula ϕ is *irrelevant to* the atoms in a set V (or simply V*-irrelevant*), written $\mathrm{IR}(\phi, V)$ of atoms, if there is a formula ψ with $\mathit{Var}(\psi) \cap V = \emptyset$ such that $\phi \equiv \psi$. The V*-irrelevance* of a set of formulas can be defined similarly, i.e. a set Σ of formulas is irrelevant to the atoms in V, written $\mathrm{IR}(\Sigma, V)$, if $\mathrm{IR}(\varphi, V)$ for each $\varphi \in \Sigma$.

By the Tarski-Knaster theorem, the least and greatest fixpoints of monotonic functions f over subsets of a set U can be obtained by transfinite induction, i.e., the least fixpoint $\mu(f) = \bigcup \mu_\alpha(f)$, where

- $\mu_0(f) = \emptyset$,
- $\mu_{\alpha+1} = f(\mu_\alpha(f))$,
- $\mu_\lambda = \bigcup_{\alpha<\lambda} \mu_\alpha(f)$, for λ a limit ordinal.

Similarly, the greatest fixpoint $\nu(f) = \bigcap \nu_\alpha(f)$, where

- $\nu_0(f) = U$,
- $\nu_{\alpha+1} = f(\nu_\alpha(f))$,
- $\nu_\lambda = \bigcap_{\alpha<\lambda} \nu_\alpha(f)$, for λ a limit ordinal.

Apparently, in $\mu X.\varphi(X)$, $\varphi(X)$ is a monotonic function about X since X appearing in $\varphi(X)$ positively. Therefore, $\|\mu X.\varphi(X)\|_v$ is the least fixpoint of the monotone operator $\|\varphi\|_v : 2^S \to 2^S$ (it is also written as $Y \mapsto \|\varphi\|_{v[X:=Y]}$ sometimes). Let's see an example to show how to compute the greatest fixpoints of the μ-formula φ talked in Example 1.

Example 2. (1) For $\varphi = \nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$, let $\mathcal{M} = (S, r, R, L)$ be a Kripke structure, then we have:

- $\nu_0(\|\varphi\|) = S$,
- $\nu_1(\|\varphi\|) = \|j\| \cap \|ch\| \cap \|\Box(\neg j \wedge \neg ch)\| \cap \|\Box\Box\nu_0(\|\varphi\|)\| = \{s | s \in L(j) \cap L(ch)$ and $\forall t.(s,t) \in R, t \in S - (L(j) \cup L(ch))\}$,
- ...

(2) For $\varphi = \nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$, let $\mathcal{M} = (S, r, R, L)$ be a Kripke structure, then we have:

- $\nu_0(\|\varphi\|) = S$,
- $\nu_1(\|\varphi\|) = \|j\| \cap \|ch\| \cap \|\Box(\neg j \wedge \neg ch)\| \cap \|\Box\Box\nu_0(\|\varphi\|)\| = \{s | s \in L(j) \cap L(ch)$ and $\exists t.(s,t) \in R, t \in S - (L(j) \cup L(ch))\}$,
- $\nu_2(\|\varphi\|) = \|j\| \cap \|ch\| \cap \|\Box(\neg j \wedge \neg ch)\| \cap \|\Box\Box\nu_1(\|\varphi\|)\|$
- ...

For the Kripke structure $\mathcal{M}$ in (a) of Example 1, we have $\|\varphi\| = \{s_0\}$.

An alternative syntax for the μ-calculus is obtained by substituting the $\Diamond$ operator with a set of *cover operators*, one for each natural number n. For $n \geq 1$ these operators are defined as follows: if $\varphi_1, \ldots, \varphi_n$ are formulas, then

$$Cover(\varphi_1, \ldots, \varphi_n)$$

is a formula. The constant operator $Cover(\emptyset)$ is also allowed. The cover operators are interpreted in a Kripke structure $\mathcal{M} = (S, r, R, L)$ as follows: $Cover(\emptyset)$ is true in $\mathcal{M}$ if and only if the root of $\mathcal{M}$ does not have any successor, while $Cover(\varphi_1, \ldots, \varphi_n)$ is true in $\mathcal{M}$ if and only if the successors of the root are covered by $\varphi_1, \ldots, \varphi_n$. More formally, $(\mathcal{M}, s, v) \models Cover(\varphi_1, \ldots, \varphi_n)$ if and only if:

- for every $i = 1, ..., n$ there exists t with $(s, t) \in R$ and $(\mathcal{M}, t, v) \models \varphi_i$;
- for every t with $(s, t) \in R$ there exists $i \in \{1, ..., n\}$ with $(\mathcal{M}, t, v) \models \varphi_i$.

We call this syntax the covers-syntax to distinguish it from the original $\Diamond$-syntax. Since $Cover(\varphi_1, \ldots, \varphi_n)$ is equivalent to

$$\Diamond\varphi_1 \wedge \cdots \wedge \Diamond\varphi_n \wedge \Box(\varphi \vee \cdots \vee \varphi_n),$$

cover operators are definable in the $\Diamond$ syntax. Conversely,

$$\Diamond\varphi \Leftrightarrow Cover(\varphi, \top).$$

Hence, the μ-calculus obtained from the covers-syntax is equivalent to the familiar μ-calculus constructed using the $\Diamond$-syntax. In this paper we use a mixture of the two syntax because, as we shall see, cover operators behave nicely with respect to the definition of disjunctive formula. Specially, the μ-calculus formulas have a normal form, called disjunctive formula [20] as follows.

Definition 1 (disjunctive formula). *The set of disjunctive formulas, $\mathcal{F}_d$ is the smallest set defined by the following clauses:*

- *disjunctions and non-contradictory conjunction of literals are disjunctive formulas;*
- *special conjunctions: if $\varphi_1, \ldots, \varphi_n \in \mathcal{F}_d$ and δ is a non contradictory conjunction of literals, then $\delta \wedge Cover(\varphi_1, \ldots, \varphi_n) \in \mathcal{F}_d$;*
- *fixpoint operators: if $\varphi \in \mathcal{F}_d$, φ does not contain $X \wedge \psi$ as a subformula for any formula ψ, and X is positive in φ, then $\mu X.\varphi$, $\nu X.\varphi$ are in $\mathcal{F}_d$.*

Example 3. It is easy to check that both $\nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$ and $\nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$ are not disjunctive formulas. While $j \wedge ch \wedge \Diamond(\neg j \wedge \neg ch)$, $\mu X.(j \wedge ch) \wedge \Diamond X$ and $\nu X.(j \wedge ch) \wedge \Diamond\Diamond X$ are disjunctive formulas because we have:

$$j \wedge ch \wedge \Diamond(\neg j \wedge \neg ch) \equiv j \wedge ch \wedge Cover(\neg j \wedge \neg ch, \top),$$

$$\mu X.(j \wedge ch) \wedge \Diamond X \equiv \mu X.(j \wedge ch) \wedge Cover(X, \top)$$

and

$$\nu X.(j \wedge ch) \wedge \Diamond\Diamond X \equiv \nu X.(j \wedge ch) \wedge Cover(Cover(X, \top), \top).$$

Theorem 1 ([20]). *Any μ-calculus formula is equivalent to a disjunctive one.*

Theorem 1 means that for each μ-calculus formula φ there is a disjunctive formula $\psi \in \mathcal{F}_d$ such that $\varphi \equiv \psi$. Deciding whether a disjunctive formula is satisfiable can be done in polynomial time, hence changing the μ-calculus formula into its disjunctive form will increase the efficiency of deciding its satisfiability since the satisfiability of μ-calculus is in EXPTIME-complete.

Another important concept is *uniform interpolation*, which has been widely explored in various of logic languages. Some logics enjoy uniform interpolation, but some are not. Formally, the uniform interpolation under μ-calculus is defined as follows.

Definition 2 (Uniform interpolation [9]). *Given a μ-sentence φ and a set of atoms $L' \subseteq Var(\varphi)$, the uniform interpolant of φ with respect to L' is a μ-sentence ψ such that:*

- $\varphi \models \psi$*;*
- *whenever $\varphi \models \varphi_1$ and $Var(\varphi) \cap Var(\varphi_1) \subseteq L'$ then $\psi \models \varphi_1$;*
- $Var(\psi) \subseteq L'$*.*

Notice that uniform interpolation is stronger than Craig interpolation, which states that for any two formulas φ, ψ with $\varphi \models \psi$ there exists a formula θ (called the Craig interpolant of φ, ψ) in the common language with $\varphi \models \theta$ and $\theta \models \psi$. Clearly, if the logic enjoys uniform interpolation then the Craig interpolant of φ, ψ does not depend on ψ but only on the common language: it is simply the uniform interpolant of φ relative to $L' = Var(\varphi) \cap Var(\psi)$.

It has been proved that the μ-calculus has uniform interpolation [8,7]. Moreover, when the given formula is a disjunctive formula then the uniform interpolant of it can be obtained by replacing both the pointed atoms and their negation with $\top$ at the same time [9]. Formally:

Theorem 2. *The uniform interpolant $\widetilde{\exists} p \varphi$ ($p \in \mathcal{A}$) of a disjunctive formula φ is equivalent to the μ-formula $\varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\top]$ is defined from φ by simultaneously substituting the literals p and $\neg p$ with $\top$.*

4 Forgetting in μ-calculus

In this section we present the definition of forgetting in μ-calculus and investigate its semantic properties. First, we give the definition of V-bisimulation between Kripke structures. The notion of V-bisimulation captures the idea that the two systems are behaviourally the same except for the atoms in V. In this way we give the definition of forgetting by using the V-bisimulation.

Second, the related properties, e.g. Modularity, Commutativity and Homogeneity, of the forgetting operator will be explored. And last, we show that the model checking problem of forgetting V from a disjunctive formula is NP $\cap$ co-NP and the reasoning problems are EXPTIME-complete.

4.1 Definition of Forgetting

In this subsection we present the definition of forgetting in μ-calculus by the bisimulation technique. A bisimulation is a binary relation between state transition systems (they are expressed by Kripke structures in this paper), associating systems that behave in the same way, in the sense that two systems mimic each other. The result is that each of the systems cannot be distinguished from the other by an observer. Intuitively, if two Kripke structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar, then they satisfy the same formula, i.e. for each formula φ, $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$.

Recall the scenario introduced in the Introduction, we should "forget" something that is obsolete from the given specification without violating the existing specification over the remaining signature. That is, the models of such specification will be extend to some other Kripke structures such that those Kripke structures simulates the existing models on the remaining signature and vice versa. To doing so, we should extend the bisimulation into the one under a given set of atoms, i.e. V-bisimulation with V is a set of atoms. For convenience, in the following we let $\mathcal{M}_i = (S_i, r_i, R_i, L_i)$ with $i \in \mathbf{N}$ be Kripke structures.

Definition 3 (V-bisimulation). *Let $V \subseteq \mathcal{A}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Kripke structures. $\mathcal{B} \subseteq S_1 \times S_2$ is a V-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ if:*

- $r_1 \mathcal{B} r_2$,
- *for each $s \in S_1$ and $t \in S_2$, if $s\mathcal{B}t$ then $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A}-V$,*
- *$(s, s') \in R_1$ and $s\mathcal{B}t$ imply that there is a t' such that $s'\mathcal{B}t'$ and $(t, t') \in R_2$, and*
- *vice versa: if $s\mathcal{B}t$ and $(t, t') \in R_2$ then there is an s' with $(s, s') \in R_1$ and $t'\mathcal{B}s'$.*

On one hand, the V-bisimulation is similar with the $\mathcal{L}$-bisimulation in [7], which is a relation satisfying the above clauses just for the symbols in language $\mathcal{L}$.

On the other hand, it is easy to see that our definition is similar with the one introduced in [18]. That is, they are the same whenever the state s in the K-structure $(\mathcal{M}, s)$ in that definition is limited to its initial state, and each state in $\mathcal{M}_i$ is reachable from the r_i. Moreover, the V-bisimulation defined in [18], the classical bisimulation-equivalence of Definition 7.1 in [2], state equivalence (i.e., E_n) in [5] and state-based bisimulation notion of Definition 7.7 in [2] are closely related [1]. In this sense, we can see that our V-bisimulation is also similar to those definitions to some extent.

We say that two Kripke structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are *V-bisimilar*, denoted $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$, if there exists a V-bisimulation $\mathcal{B}$ between them. In this case, we also say that $\mathcal{M}_1$ and $\mathcal{M}_2$ on V. Moreover, from this Definition 3 we can see that the $\leftrightarrow_V$ between Kripke structures has some interesting properties besides the equivalence relation. Formally:

Proposition 2. *Let $V, V_1 \subseteq \mathcal{A}$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three Kripke structures, then we have:*

(i) the $\leftrightarrow_V$ is an equivalence relation between Kripke structures;

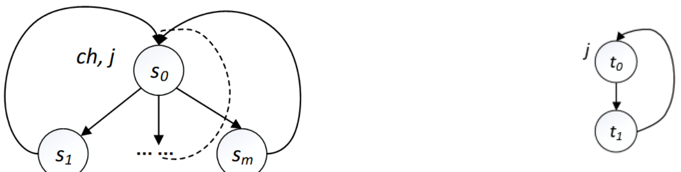
[1] the V-bisimulation defined in [18] is similar to the state equivalence (i.e., E_n) in [5], yet it is different in the sense that V-bisimulation is defined on K-structures, while state-equivalence is defined on states. Moreover, V-bisimulation is also different from the state-based bisimulation notion of Definition 7.7 in [2], which is defined for states of a given K-structure.

(ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ *and* $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$*, then* $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.

Intuitively, the (i) in Proposition 2 means that $\leftrightarrow_V$ is reflexive, symmetric and transitive. (ii) refers to that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar on two different sets V and V_1 respectively, then they are bisimilar on the union of those two sets. As we will show in the next context that this is important to demonstrate the *modularity*, one of the important properties of forgetting in μ-calculus.

To get some intuition of the V-bisimulation, let's look at the following example.

Example 4. In Fig. 2 we can check that $\mathcal{M} \leftrightarrow_{\{ch\}} \mathcal{M}'$ because there is a $\{ch\}$-bisimulation $\mathcal{B} = \{(s_0, t_0), (s_1, t_1), (s_2, t_1), \ldots, (s_m, t_1)\}$ between $\mathcal{M}$ and $\mathcal{M}'$.



(a) Kripke structure $\mathcal{M}$

(b) Kripke structure $\mathcal{M}'$

Fig. 2: Two $\{ch\}$-bisimilar Kripke structures

As it has been said in [7] that any $\mathcal{L}$-sentence φ (that is: a μ-sentence that only uses symbols from the language $\mathcal{L}$) is invariant for $\mathcal{L}$-bisimulation, i.e. if there is a $\mathcal{L}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ then φ holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$. In this case, it should then be obvious that if IR(φ, V) and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ then φ holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$. **Therefore, in the following we only consider μ-sentences.**

We define the forgetting in μ-calculus in the follows, although forgetting is a dual concept in S5 and Propositional logic [17], it is important to give the formal definition of forgetting in μ-calculus since as we will see that it is useful to knowledge update of μ-calculus. Moreover, as shown in [18], forgetting in CTL can also be used to compute the SNC and the WSC and we shall report this in μ-calculus.

Definition 4 (Forgetting). *Let* $V \subseteq \mathcal{A}$ *and* ϕ *be a* μ*-sentence. A formula* ψ *with* $Var(\psi) \cap V = \emptyset$ *is a* result of forgetting V from ϕ*, if*

$$Mod(\psi) = \{\mathcal{M} \mid \exists \mathcal{M}' \in Mod(\phi) \ \& \ \mathcal{M}' \leftrightarrow_V \mathcal{M}\}.$$

For convenience, we denote the result of forgetting V from ϕ as $\mathrm{F}_\mu(\phi, V)$. In Definition 4, instead of giving a syntactic definition like uniform interpolation above, we define the forgetting from the point of the semantics. Moreover, it is not difficult to see that Definition 4 implies if both ψ and ψ' are results of forgetting V from ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the result of forgetting V from ϕ is unique (up to semantic equivalence). In the following subsection we will show that such μ-sentence (for each given μ-sentence and set of atoms) always exists.

4.2 Semantic Properties of Forgetting in μ-calculus

In this part we show the semantic properties of forgetting in μ-calculus. In particular, we show that our forgetting is closed in μ-calculus, satisfies the general postulates, i.e. the representation theorem, and the algebraic properties, including Modularity, Commutativity and Homogeneity.

Theorem 3. *Let $q \in \mathcal{A}$ and ϕ be a μ-sentence. There is a μ-sentence ψ such that $IR(\psi, \{q\})$ and $\psi \equiv \mathrm{F}_\mu(\phi, \{q\})$.*

Proof. (sketch) This can be obtained from the Theorem 3.1 in [7].

This means that the uniform interpolant $\widetilde{\exists} q\phi$ $(q \in \mathcal{A})$ of ϕ with respect to $Var(\phi) - \{q\}$ is the result of forgetting $\{q\}$ from ϕ. In this case, we also say that the forgetting of μ-calculus is *closed*, that is the result of forgetting some set of atoms from a μ-sentence is also a μ-sentence. Note that if both ψ and ψ' are results of forgetting V from ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models.

At this point, it is important to emphasize that, the notion of forgetting we have defined for μ-calculus extends the classical forgetting defined for propositional logic (PL) [27]. Assuming φ is a PL formula and $p \in \mathcal{A}$, then $Forget(\varphi, p)$ is a result of forgetting p from φ; that is, $Forget(\varphi, p) \equiv \varphi[p/\bot] \vee \varphi[p/\top]$. That way, given a set $V \subseteq \mathcal{A}$, one can recursively define $Forget(\varphi, V \cup \{p\}) = Forget(Forget(\varphi, p), V)$, where $Forget(\varphi, \emptyset) = \varphi$. Using this insight, the following result shows that the classical notion of forgetting (for PL [27]) is a special case of forgetting in μ-calculus.

Theorem 4. *Let φ be a PL formula and $V \subseteq \mathcal{A}$, then*

$$\mathrm{F}_\mu(\varphi, V) \equiv Forget(\varphi, V).$$

Let's see the following example.

Example 5. Let $\alpha = (a \wedge b) \vee (a \wedge \neg b \wedge c)$ and $V = \{b\}$. We show how to compute forgetting V from α by *forget* and F_μ respectively.

(1) The step of computing forgetting by *forget* is as follows:

$$\begin{aligned} forget(\alpha, V) &\equiv \alpha[b/\bot] \vee \alpha[b/\top] \\ &\equiv (a \wedge c) \vee a \\ &\equiv a \end{aligned}$$

(2) We can use Theorem 2 and Theorem 3 to compute forgetting V from α since α is a disjunctive formula:

$$\begin{aligned} \mathrm{F}_\mu(\alpha, V) &\equiv \alpha[b/\top, \neg b/\top] \\ &\equiv (a \wedge \top) \vee (a \wedge \neg\top \wedge c) \\ &\equiv a \end{aligned}$$

It is obvious that *forget*$(\alpha, V) \equiv \mathrm{F}_\mu(\alpha, V)$.

A general description is important for understanding the concept of forgetting. To doing so, authors give four postulates (also called *forgetting postulates*), which can be considered as desirable properties of such a notion, concerning knowledge forgetting in **S5** modal logic [38]. In the following, we first list these postulates, and then show that our notion of forgetting in μ-calculus satisfies them.

Forgetting postulates [38] are:

(**W**) Weakening: $\varphi \models \varphi'$;
(**PP**) Positive Persistence: for any formula η, if $\mathrm{IR}(\eta, V)$ and $\varphi \models \eta$ then $\varphi' \models \eta$;
(**NP**) Negative Persistence : for any formula η, if $\mathrm{IR}(\eta, V)$ and $\varphi \not\models \eta$ then $\varphi' \not\models \eta$;
(**IR**) Irrelevance: $\mathrm{IR}(\varphi', V)$

where $V \subseteq \mathcal{A}$, φ is a μ-sentence and φ' is the result of forgetting V from φ. Intuitively, the postulate (**W**) says, forgetting weakens the original formula, i.e. φ' is a logical consequence of φ; the postulates (**PP**) and (**NP**) say that forgetting results have no effect on formulas that are irrelevant to forgotten atoms; the postulate (**IR**) states that forgetting result is irrelevant to forgotten atoms. It is noteworthy that they are not all orthogonal e.g., (**NP**) is a consequence of (**W**) and (**PP**). Nonetheless, we prefer to list them all, in order to outline the basic intuition behind them.

The following says that the forgetting postulates above indeed precisely characterize the underling forgetting semantics of μ-calculus.

Theorem 5 (Representation Theorem). *Let φ, φ' and ϕ be μ-sentences and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

(i) $\varphi' \equiv \mathrm{F}_\mu(\varphi, V)$,
(ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi$ *and* $IR(\phi, V)\}$,
*(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold if φ, φ' and V are as in (i) and (ii).*

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\mathrm{F}_\mu(\varphi, V)) = Mod(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}).$$

Firstly, suppose that $\mathcal{M}'$ is a model of $\mathrm{F}_\mu(\varphi, V)$. Then there exists a Kripke structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Therefore, we have $\mathcal{M}' \models \phi$ for all ϕ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $\mathcal{M}'$ is a model of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

It is evident that $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \mathrm{F}_\mu(\varphi, V)$ since $\mathrm{IR}(\mathrm{F}_\mu(\varphi, V), V)$ and $\varphi \models \mathrm{F}_\mu(\varphi, V)$ by Theorem 3.

See the appendix for the proofs of the other parts.

Theorem 5 means that for a given μ-sentence φ and a set of atoms V, a μ-sentence φ' represents a result of forgetting V from φ if φ' satisfies the four forgetting postulates, and vice versa. That is, the representation theorem gives an “if and only if” characterization on forgetting in μ-calculus, which is in accordance with that in **S5**.

Excepting for the representation theorem, postulate IR is also of crucial importance for computing SNC and WSC. Consider the $\psi = \varphi \wedge (q \leftrightarrow \alpha)$. If $\mathrm{IR}(\varphi \wedge \alpha, \{q\})$, then the result of forgetting q from ψ is φ. Formally, it can be described in the following lemma, and as we will see later in Section 4, it is the base of reducing the SNC (WSC) of any μ-sentence to that of a proposition.

Lemma 1. *Let φ and α be two μ-sentences and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then* $\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

We will list other interesting properties of the forgetting operator in the follows. Most importantly, the following property guarantees that we can modularly apply forgetting one by one to the atoms to be forgotten, instead of forgetting the set of atoms as a whole, which is spoken in the definition of forgetting.

Proposition 3. ***(Modularity)*** *Given a μ-sentence φ, V a set of atoms and p an atom such that $p \notin V$. Then,*

$$\mathrm{F}_\mu(\varphi, \{p\} \cup V) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V).$$

The next property follows from the above proposition.

Corollary 1 (Commutativity). *Let φ be a μ-sentence and $V_i \subseteq \mathcal{A}$ $(i = 1, 2)$. Then:*

$$\mathrm{F}_\mu(\varphi, V_1 \cup V_2) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, V_1), V_2).$$

The following properties show that the forgetting respects the basic semantic notions of logic. They hold in classical propositional logic, modal logic **S5** [38] and CTL [18]. Below we show that they are also satisfied in our notion forgetting in μ-calculus.

Proposition 4. *Let φ, φ_i, ψ_i ($i = 1, 2$) be formulas in CTL and $V \subseteq \mathcal{A}$. We have*

(i) $\mathrm{F}_\mu(\varphi, V)$ *is satisfiable iff* φ *is;*
(ii) If $\varphi_1 \equiv \varphi_2$, *then* $\mathrm{F}_\mu(\varphi_1, V) \equiv \mathrm{F}_\mu(\varphi_2, V)$*;*
(iii) If $\varphi_1 \models \varphi_2$, *then* $\mathrm{F}_\mu(\varphi_1, V) \models \mathrm{F}_\mu(\varphi_2, V)$*;*
(iv) $\mathrm{F}_\mu(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$*;*
(v) $\mathrm{F}_\mu(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_\mu(\psi_1, V) \wedge \mathrm{F}_\mu(\psi_2, V)$*;*

Intuitively, in Proposition 4, (i) means that forgetting some set of atoms from a sentence do not affect the satisfiability of this sentence. In (ii) we can see that if two sentence are equivalent then the results of forgetting the same set of atoms from both of them are also equivalent. The intuitive meaning of (iii) is obvious. (iv) refers to the result of forgetting V from a disjunctive formula $\varphi_1 \vee \varphi_2$ is equivalent with the disjunction of the results of forgetting V from φ_1 and φ_2 respectively. While (v) points out that it is not the case for a conjunctive formula.

Another important property of F_μ operator is about the formulas with that all the sub-formula are in the domain of AX ($\Box$) or EX ($\Diamond$). Formally:

Proposition 5 (Homogeneity). *Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence, then we have:*

(i) $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_\mu(\phi, V)$.
(ii) $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_\mu(\phi, V)$.

The homogeneity of AX (or EX) on forgetting says we can move the operator F_μ afterward to the AX (or EX) to compute the forgetting of formula in the form AXφ (or EXφ).

Although the homogeneity in Proposition 5 gives a convenience for computing forgetting. However, it is noteworthy that it is easy to compute a forgetting results for a set of atoms from a μ-sentence when it is a disjunctive formula. By Theorem 3 we know that the set of models of the result of forgetting a proposition p from a μ-sentence φ is equal to the set of models of the uniform interpolant of φ with respect to $\mathit{Var}(\varphi) - \{p\}$, i.e. $\mathit{Mod}(\mathrm{F}_\mu(\varphi, \{p\})) = \mathit{Mod}(\widetilde{\exists} p \varphi)$. This means that when the given formula is a disjunctive formula, we can compute the result of forgetting p from just by replacing the p and $\neg p$ with $\top$ at the same time. Then we have the following proposition.

Proposition 6. *Let φ be a μ-sentence and $p \in \mathcal{A}$. If φ be a disjunctive formula [9], then $\mathrm{F}_\mu(\varphi, \{p\})$ can be computed in linear time.*

Proof. By Theorem 3.6. in [9], we have $\mathrm{F}_\mu(\varphi, \{p\}) \equiv \varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\bot]$ is obtained from φ by simultaneously substituting the literals p and $\neg p$ with $\top$.

In this sense, we can transform a formula into its disjunctive form offline, and then compute its result of forgetting some atoms from it, this will be efficient in some situations. Let's recall the three disjunctive formulas, which obtained from μ-sentences, in Example 3. And in the following example, we shall show how to compute forgetting ch from those formulas.

Example 6. Let's denote those formulas as follows: $\varphi_1 = j \wedge ch \wedge Cover(\neg j \wedge \neg ch, \top)$, $\varphi_2 = \mu X.(j \wedge ch) \wedge Cover(X, \top)$ and $\varphi_3 = \nu X.(j \wedge ch) \wedge Cover(Cover(X, \top), \top)$. Let $V = \{ch\}$, we can easily compute their results of forgetting V from those formulas.

(1) $\mathrm{F}_\mu(\varphi_1, V) \equiv j \wedge Cover(\bot, \top)$;
(2) $\mathrm{F}_\mu(\varphi_2, V) \equiv \mu X.j \wedge Cover(X, \top)$;
(3) $\mathrm{F}_\mu(\varphi_2, V) \equiv \nu X.j \wedge Cover(Cover(X, \top), \top)$.

4.3 Complexity Results

Computational complexity theory focuses on classifying computational problems according to their resource usage, and relating these classes to each other. A problem is regarded as inherently difficult if its solution requires significant resources. Hence, classify the forgetting operator from its complexity is important to explore an efficient algorithm to compute it. In this section, we will explore the complexity of the forgetting operator from both its model checking and reasoning problems. Recall that the result of forgetting some atoms from a disjunctive formula can be obtained in linear time, nevertheless, as we will show that the model checking problem is intractable even the given formula is a disjunctive formula.

Before talk about the complexity results of the forgetting operator, let's recall the μ-automaton, which is an important technique in μ-calculus to compute the uniform interpolation of a given formula and demonstrate the complexity of model checking of μ-calculus. In this part we will show that the μ-automaton can also make a big convenience to show our complexity results.

Definition 5 (**μ-automaton [7]**)**.** *A μ-automaton A, also called modal automaton [4], is a tuple $(Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ such that:*

(i) Q *is a finite set of states;*
(ii) Σ_p *is a finite subset of* $\mathcal{A}$*;*
(iii) Σ_r *is a finite subset of the set of actions, in this paper it is an empty set;*
(iv) $q_0 \in Q$ *is the initial state;*
(v) $\delta : Q \times \mathcal{P}(\Sigma_p) \to \mathcal{PP}(\Sigma_r \times Q)$*;*
(vi) $\Omega : Q \to \mathcal{N}$*.*

Although this automaton differs slightly from those given in [20,4], but the automata in their various guises are equivalent. Moreover, modal automata are essentially alternating automata [4]. And it has been shown that constructing a μ-automaton from a μ-calculus formula can be done in exponential time, while it is in polynomial time when the μ-calculus formula is a disjunctive formula.

In addition, it had been shown in [7] that for each μ-automaton A_φ obtained from a μ-formula φ and a proposition p it is easy to construct another μ-automaton B such that, for any Kripke structure $\mathcal{N}$, B accepts $\mathcal{N}$ iff there is a model $\mathcal{N}'$ of φ such that $\mathcal{N} \leftrightarrow_{\{p\}} \mathcal{N}'$, i.e. B corresponds to a μ-sentence which is equivalent to $\mathrm{F}_\mu(\varphi, V)$ by the definition of forgetting in μ-calculus. Then we have the following complexity result.

Proposition 7 (Model Checking). *Given a finite Kripke structure* $\mathcal{M}$*, a disjunctive formula* φ *and* $V \subseteq \mathcal{A}$*, deciding* $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ *is* NP $\cap$ *co*-NP.

More importantly, from the perspective of knowledge bases evolving, we are also interested in the following reasoning problems about forgetting, which are explored in CPL [31].

(i) [Var-weak] if the restriction of φ on the signature of ψ is at most as strong as ψ, i.e. $\psi \models \mathrm{F}_\mu(\varphi, V)$,
(ii) [Var-strong] if the restriction of φ on the signature of ψ is at least as strong as ψ, i.e. $\mathrm{F}_\mu(\varphi, V) \models \psi$,
(iii) [Var-entailment] if the restriction of one knowledge base on its original signature is at most as strong as that of the other, i.e. $\mathrm{F}_\mu(\varphi, V) \models \mathrm{F}_\mu(\psi, V)$

where φ, ψ are μ-sentences, and V a set of atoms. Besides, in (i) and (ii) there is $Var(\varphi) - V = Var(\psi)$ and in (iii) there is $V \subseteq Var(\varphi) \cap Var(\psi)$. Then we have the following results.

Theorem 6 (Entailment). *Let* φ *and* ψ *be two* μ*-sentences and* V *be a set of atoms. Then, the following problems are* EXPTIME*-complete.*

(i) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \psi$*,*
(ii) deciding $\psi \models^? \mathrm{F}_\mu(\varphi, V)$*,*
(iii) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \mathrm{F}_\mu(\psi, V)$*.*

Similarly with the reasoning problems talked above, the following equivalent problem are also important, in which "var-independence" and "var-equivalence" under CPL are proposed [24].

(i) [Var-independence] If a formula φ is independent of a set V of atoms, i.e. $\mathrm{F}_\mu(\varphi, V) \equiv \varphi$,

(ii) [Var-match] if the restriction of φ on the signature of ψ perfectly matches ψ, i.e. $\mathrm{F}_\mu(\varphi, V) \equiv \psi$.
(iii) [Var-equivalence] if the restriction of the two formulas on a common signature are equivalent, i.e. $\mathrm{F}_\mu(\varphi, V) \equiv \mathrm{F}_\mu(\psi, V)$.

The following results are implications of Theorem 6.

Corollary 2. *Let φ and ψ be two μ-sentences and V be a set of atoms. Then, the following problems are* EXPTIME*-complete.*

(i) deciding $\psi \equiv^? \mathrm{F}_\mu(\varphi, V)$,
(ii) deciding $\mathrm{F}_\mu(\varphi, V) \equiv^? \varphi$,
(iii) deciding $\mathrm{F}_\mu(\varphi, V) \equiv^? \mathrm{F}_\mu(\psi, V)$.

5 Necessary and Sufficient Conditions

In this section, we present two key notions of our work: namely, the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC) of a given μ-calculus specification, which corresponds to the *most general consequence* and the *most specific abduction* of a specification, respectively. As aforementioned in the introduction, these notions respectively are accordance with the *strongest precondition* (SP) and the *weakest post-condition* (WP) (introduced by E. Dijkstra in [12]), which have been central to a wide variety of tasks and studies, e.g. generating counterexamples and refinement of system in verification. Our contribution, in particular, will be on computing SNC and WSC via forgetting under a given μ-sentence and a set V of atoms. Let us give the formal definition.

Definition 6 (sufficient and necessary condition). *Let ϕ, ψ be two μ-sentences, $V \subseteq Var(\phi)$, $q \in Var(\phi) - V$ and $Var(\psi) \subseteq V$.*

- *ψ is a* necessary condition *(NC in short) of q on V under ϕ if $\phi \models q \to \psi$.*
- *ψ is a* sufficient condition *(SC in short) of q on V under ϕ if $\phi \models \psi \to q$.*
- *ψ is a* strongest necessary condition *(SNC in short) of q on V under ϕ if it is a NC of q on V under ϕ, and $\phi \models \psi \to \psi'$ for any NC ψ' of q on V under ϕ.*
- *ψ is a* weakest sufficient condition *(WSC in short) of q on V under ϕ if it is a SC of q on V under ϕ, and $\phi \models \psi' \to \psi$ for any SC ψ' of q on V under ϕ.*

Intuitively, the SNC (WSC) is the strongest (weakest) one among the NCs (SCs) of q on V under ϕ. Note that if both ψ and ψ' are SNC (WSC) of q on V under ϕ, then $Mod(\psi) = Mod(\psi')$, i.e., ψ and ψ' have the same models. In this sense, the SNC (WSC) of q on V under ϕ is unique (up to semantic equivalence). The following result shows that the SNC and WSC are in fact dual notions.

Proposition 8 (Dual). *Let V, q, φ and ψ are defined as in Definition 6. Then, ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.*

In order to generalise Definition 6 to arbitrary formulas, one can replace q (in the definition) by any formula α, and redefine V as a subset of $Var(\alpha) \cup Var(\phi)$.

It turns out that the previous notions of SNC and WSC for an atomic variable can be lifted to any formula, or, conversely, the SNC and WSC of any formula can be reduced to that of an atomic variable, as the following result shows.

Proposition 9. *Let Γ and α be two μ-sentences, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q be a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.*

The following result establishes the bridge between forgetting and the notion of SNC (WSC) which are central to our contribution.

Theorem 7. *Let φ be a μ-sentence, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.*

(i) $\mathrm{F}_\mu(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ *is a SNC of q on V under φ.*
(ii) $\neg\mathrm{F}_\mu(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ *is a WSC of q on V under φ.*

Following Theorem 7, assume that $\beta = \mathrm{F}_\mu(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$. Then, $\varphi \wedge q \models \beta$ by (**W**). Moreover, $\varphi \wedge q \models \beta$, and then β is a NC of q on V under φ.

In addition, for any μ-sentence ψ with IR$(\psi, (Var(\varphi) \cup \{q\}) - V)$ and $\varphi \wedge q \models \psi$, we have $\beta \models \psi$ by (**PP**). Therefore, β is the SNC of q on V under φ. This shows the intuition of how the SNC can be obtained from the forgetting.

Although we can use forgetting to compute the SNC and WSC, it had been found that sometimes, it is a lot easier to compute the strongest necessary condition than the weakest sufficient condition, and sometimes, it is the other way around [26]. For instance, the author had found that for many theories, strongest necessary conditions are easier to compute than weakest sufficient conditions. When one condition is much easier to compute, the following proposition will be very helpful in computing the other one.

Proposition 10. *Let Γ be a μ-sentence, q a proposition, and V a set of propositions.*

(i) If φ is a necessary condition of q on V under Γ, and ψ the weakest sufficient condition of q on V under $\Gamma \cup \{\varphi\}$, then $\varphi \wedge \psi$ is the weakest sufficient condition of q on V under Γ.

(ii) If ψ is a sufficient condition of q on V under Γ, and φ a strongest necessary condition of q on V under $\Gamma \cup \{\neg\psi\}$, then $\varphi \vee \psi$ is a strongest necessary condition of q on V under Γ.

6 Representing knowledge update via forgetting

In this section, we present the final key notion of our work: knowledge update. In particular, we will propose a method of defining knowledge update via forgetting which will satisfy all the following Katsuno and Mendelzon's postulates (U1)-(U8) proposed in [21]:

(U1) $\Gamma \diamond \phi \models \phi$.
(U2) If $\Gamma \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma$.
(U3) If both Γ and ϕ are satisfiable, then $\Gamma \diamond \phi$ is also satisfiable.
(U4) If $\Gamma_1 \equiv \Gamma_2$ and $\phi_1 \equiv \phi_2$, then $\Gamma_1 \diamond \phi_1 \equiv \Gamma_2 \diamond \phi_2$.
(U5) $(\Gamma \diamond \phi) \wedge \psi \models \Gamma \diamond (\phi \wedge \psi)$.
(U6) If $\Gamma \diamond \phi \models \psi$ and $\Gamma \diamond \psi \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma \diamond \psi$.

(U7) If Γ has a unique model, then $(\Gamma \diamond \phi) \wedge (\Gamma \diamond \psi) \models \Gamma \diamond (\phi \vee \psi)$.
(U8) $(\Gamma_1 \vee \Gamma_2) \diamond \phi \equiv (\Gamma_1 \diamond \phi) \vee (\Gamma_2 \diamond \phi)$.

Here, $\varphi \diamond \psi$ expresses the result of updating φ with ψ and $\diamond$ is the knowledge update operator.

For this purpose, in this part we suppose the models of a μ-sentence are initial structures, in which an initial structure is a Kripke structure $\mathcal{M} = (S, sr, R, L)$ with S is a finite set of states, sr is an initial state (i.e. for each state $s' \in S$ the sr can arrive at s') and R is a total relation. Besides, we also restrict the definition of forgetting on initial structures, i.e. the models mentioned in Definition 4 are initial structures. In this case, we have:

Theorem 8. *Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence. Then there is a μ-sentence ψ such that:*

$$\mathcal{M} \models \psi \textit{ iff there is a model } \mathcal{M}' \textit{ of } \phi \textit{ such that } \mathcal{M} \leftrightarrow_V \mathcal{M}'$$

where both $\mathcal{M}$ and $\mathcal{M}'$ are initial structures.

Intuitively, given a logic language $\mathcal{L}$, we say some operator $\mathcal{O}$ in $\mathcal{L}$ is closed whenever the result of using the $\mathcal{O}$ on the elements of $\mathcal{L}$ is also in $\mathcal{L}$. Theorem 8 shows that the forgetting in μ-calculus is also closed when restrict the models of μ-sentence to initial structures. Formally:

Corollary 3. *The forgetting of μ-calculus is closed under the initial structure semantic, i.e. we only consider the initial structures as the models of μ-sentence.*

According to [18], we can see that any initial structure $\mathcal{M}$ on $\mathcal{A}$ can be captured by a CTL formula, i.e. the characterizing formula $\mathcal{F}_{\mathcal{A}}(\mathcal{M})$ [18], and hence a μ-sentence [15]. In this case, we define the knowledge update operator $\diamond_\mu$ in μ-calculus as follows.

Definition 7. *Let Γ and ϕ be μ-sentences. The knowledge update operator $\diamond_\mu$ is defined as follows:*

$$\mathit{Mod}(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} \bigcup_{V_{min}} \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi),$$

where $\mathcal{F}_{\mathcal{A}}(\mathcal{M})$ is the characterizing formula of $\mathcal{M}$ on $\mathcal{A}$, and $V_{min} \subseteq \mathcal{A}$ is a minimal subset of atoms that makes $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi$ consistent.

Intuitively, $\Gamma \diamond_\mu \phi$ means the result of updating Γ with ϕ by minimally fixing the models of Γ into that of ϕ.

Recall the definition of knowledge update in CPL, let I, J_1 and J_2 be three interpretations, then J_1 is closer to I than J_2, written $J_1 \leq_{I,pam} J_2$, iff $\mathrm{Diff}(I, J_1) \subseteq \mathrm{Diff}(I, J_2)$, where $\mathrm{Diff}(X, Y) = \{p \in \mathcal{A} \mid X(p) \neq Y(p)\}$. The set of models of knowledge updating ψ on Γ is exactly the union of minimal models of ψ under the partial order $\leq_{I,pam}$ where I is a model of Γ, i.e.

$$\mathit{Mod}(\Gamma \diamond_{pam} \psi) = \bigcup_{I \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\psi), \leq_{I,pam}).$$

Here, $Min(\mathit{Mod}(\psi), \leq_{I,pam})$ is the set of models of ψ that are minimal with respect to $\leq_{I,pam}$.

Similarly, we can define a partial ordering over the set of initial structures that links to knowledge operator $\diamond_\mu$.

Definition 8. *let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be three initial structures, then $\mathcal{M}_1$ is closer to $\mathcal{M}$ than $\mathcal{M}_2$, written $\mathcal{M}_1 \leq_{\mathcal{M}} \mathcal{M}_2$, iff for any $V_2 \subseteq \mathcal{A}$ such that $\mathcal{M}_2 \leftrightarrow_{V_2} \mathcal{M}$, there exists a $V_1 \subseteq V_2$ such that $\mathcal{M}_1 \leftrightarrow_{V_1} \mathcal{M}$. We denote $\mathcal{M}_1 <_{\mathcal{M}} \mathcal{M}_2$ iff $\mathcal{M}_1 \leq_{\mathcal{M}} \mathcal{M}_2$ and $\mathcal{M}_2 \not\leq_{\mathcal{M}} \mathcal{M}_1$.*

Let M be a set of initial structures and $\mathcal{M}$ an initial structure, we also use $Min(M, \leq_{\mathcal{M}})$ to denote the set of all minimal initial structures with respect to $\leq_{\mathcal{M}}$. Then we have the following theorem.

Theorem 9. *Let Γ and ϕ be μ-sentences. Then we have:*

$$\mathit{Mod}(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}}).$$

Theorem 9 means that our definition of knowledge update in μ-calculus by forgetting is accordance with the one defined by the $\leq_{\mathcal{M}}$, which is similar with the partial order $\leq_{I,pam}$ in CPL.

More important, the following theorem shows that our definition of $\diamond_\mu$ by forgetting satisfies Katsuno and Mendelzon's update postulates.

Theorem 10. *Knowledge update operator $\diamond_\mu$ satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).*

7 Concluding Remarks

Summary In this paper, we targeted the strongest necessary and weakest sufficient conditions SNC (WSC) and knowledge update in μ-calculus using a forgetting-based approach. In doing so, we introduced and employed the notion of V-bisimulation which is similar with the $\mathcal{L}$-bisimulation, in which any μ-sentence is invariant for $\mathcal{L}$-bisimulation. Furthermore, we have studied formal properties about forgetting, among them, homogeneity, modularity and commutativity. In particular, we have shown that our notion of forgetting satisfies the well-known postulates of forgetting, which means it faithfully extends the notion of forgetting from classical propositional logic, modal logic **S5**, CTL to μ-calculus. We also showed that for any μ-sentence, if it is a disjunctive formula, then forgetting a set of atoms from it can be down in linear time. On the complexity theory side, we have investigated that the Model checking problem of forgetting V from a disjunctive formula is NP $\cap$ co-NP and the entailment problems of forgetting are EXPTIME-complete from the point of μ-automaton. And finally, we showed that the knowledge update in terms of forgetting in μ-calculus under initial structures satisfies the Katsuno and Mendelzon's postulates (U1)-(U8).

Future work As we can see that the model checking problem of forgetting is NP $\cap$ co-NP and the reasoning problems of forgetting are EXPTIME-complete, which means there is no algorithm can compute the result of forgetting a set of atoms from a μ-sentence in polynomial time. It is worthwhile to explore sub-classes of μ-calculus such that we can compute the forgetting in polynomial time. Moreover, when a finite transition system $\mathcal{M}$ does not satisfy a specification φ, one can evaluate the weakest sufficient condition ψ over a signature V under which $\mathcal{M} \models \varphi$, viz. expressing $\mathcal{M}$ with its characterizing formula, to explore how the condition ψ can guide the design of a new transition system $\mathcal{M}'$ satisfying φ.

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A Supplementary Material: Proof Appendix

Proposition 2 Let $V, V_1 \subseteq \mathcal{A}$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three Kripke structures, then we have:

(i) the $\leftrightarrow_V$ is an equivalence relation between Kripke structures;
(ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$, then $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.

Proof. (i) We prove it form the reflexivity, symmetry and transitivity.

(1) $\leftrightarrow_V$ is reflexive. It is easy to check that $\mathcal{M} \leftrightarrow_V \mathcal{M}$ for any Kripke structure.

(2) $\leftrightarrow_V$ is symmetric. We will show that for each $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ then $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V-bisimulation $\mathcal{B}$, we construct a relation $\mathcal{B}_1$ as follows: $\mathcal{B}_1 = \{(s,t)|(t,s) \in \mathcal{B}\}$. We will show that $\mathcal{B}_1$ is a V-bisimulation between $\mathcal{M}_2$ and $\mathcal{M}_1$ from the following several points:

- $r_2\mathcal{B}_1 r_1$ since $r_1\mathcal{B}r_2$,
- for each $s \in S_1$ and $t \in S_2$, if $t\mathcal{B}_1 s$ then we have $s\mathcal{B}t$ and hence $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A} - V$, and
- the third and forth points in the definition of V-bisimulation can be checked easily for $\mathcal{B}_1$.

(3) $\leftrightarrow_V$ is transitive. We will show that for each $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ then $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V-bisimulation $\mathcal{B}_1$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ by the V-bisimulation $\mathcal{B}_2$, we construct a relation $\mathcal{B}$ as follows: $\mathcal{B} = \{(s,z)|(s,t) \in \mathcal{B}_1$ and $(t,z) \in \mathcal{B}_2\}$. We can also prove similarly with (2) that $\mathcal{B}$ is a V-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$. Therefore, $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$.

(ii) In order to prove $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$, we only need to find a binary relation $\mathcal{B}$ such that $\mathcal{B}$ is a $(V \cup V_1)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the V-bisimulation $\mathcal{B}_1$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$ by the V_1-bisimulation $\mathcal{B}_2$. Let $\mathcal{B} = \{(s_1,s_3)|(s_1,s_2) \in \mathcal{B}_1$ and $(s_2,s_3) \in \mathcal{B}_2\}$. We can easily check that $\mathcal{B}$ is a $(V \cup V_1)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$.

Theorem4 Let φ be a PL formula and $V \subseteq \mathcal{A}$, then

$$\mathrm{F}_\mu(\varphi, V) \equiv \mathit{Forget}(\varphi, V).$$

Proof. On the one hand, for each $\mathcal{M} \in \mathit{Mod}(\mathrm{F}_\mu(\varphi, V))$ there exists a $\mathcal{M}' \in \mathit{Mod}(\varphi)$ such that $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Thus, $r\mathcal{B}r'$. Hence, $\mathcal{M}$ is a model of $\mathit{Forget}(\varphi, V)$ due to $\mathrm{IR}(\mathit{Forget}(\varphi, V), V)$.

On the other hand, for each $\mathcal{M} \in \mathit{Mod}(\mathit{Forget}(\varphi, V))$ with $\mathcal{M} = (S, r, R, L)$ there exists a $\mathcal{M}' \in \mathit{Mod}(\varphi)$ such that $r\mathcal{B}r'$. Construct a Kripke structure $\mathcal{M}_1$ such that $\mathcal{M}_1 = (S_1, r_1, R_1, L_1)$ with $S_1 = (S - \{r\}) \cup \{r_1\}$, R_1 is the same as R except that r is replaced by r_1, and L_1 is the same as L except $L_1(r_1) = L'(r')$, where L' is the label function of M'. It is clear that $\mathcal{M}_1$ is a model of φ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}$. Hence, $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\varphi, V)$ due to $\mathrm{IR}(\mathrm{F}_\mu(\varphi, V), V)$.

Theorem 5 **(Representation Theorem)** Let φ, φ' and ϕ be μ-sentences and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

(i) $\varphi' \equiv \mathrm{F}_\mu(\varphi, V)$,
(ii) $\varphi' \equiv \{\phi \mid \varphi \models \phi$ and $\mathrm{IR}(\phi, V)\}$,
(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold if φ, φ' and V are as in (i) and (ii).

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$\mathit{Mod}(\mathrm{F}_\mu(\varphi, V)) = \mathit{Mod}(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}).$$

Firstly, suppose that $\mathcal{M}'$ is a model of $\mathrm{F}_\mu(\varphi, V)$. Then there exists a Kripke structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Therefore, we have $\mathcal{M}' \models \phi$ for all ϕ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $\mathcal{M}'$ is a model of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

It is evident that $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \mathrm{F}_\mu(\varphi, V)$ since $\mathrm{IR}(\mathrm{F}_\mu(\varphi, V), V)$ and $\varphi \models \mathrm{F}_\mu(\varphi, V)$ by Theorem 3.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. Suppose that all postulates hold. By Positive Persistence, we have $\varphi' \models \{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. The $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \varphi'$ can be obtained from (**W**) and (**IR**). Thus, φ' is equivalent to $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

Lemma 1 Let φ and α be two μ-sentences and $q \in \overline{\mathit{Var}(\varphi) \cup \mathit{Var}(\alpha)}$. Then $\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $\mathcal{M}$ of $\mathrm{F}_\mu(\varphi', q)$ there is a Kripke structure $\mathcal{M}'$ s.t. $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$ and $\mathcal{M}' \models \varphi'$. It's evident that $\mathcal{M}' \models \varphi$, and then $\mathcal{M} \models \varphi$ since $\mathrm{IR}(\varphi, \{q\})$ and $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$.

Let $\mathcal{M} \in \mathit{Mod}(\varphi)$ with $\mathcal{M} = (S, s, R, L)$. We construct $\mathcal{M}'$ with $\mathcal{M}' = (S, s, R, L')$ as follows:

$$L' : S \to 2^{\mathcal{A}}\ and\ \forall s^* \in S, L'(s^*) = L(s^*) - \{q\}\ if\ (\mathcal{M}, s^*) \not\models \alpha,$$
$$else\ L'(s^*) = L(s^*) \cup \{q\},$$
$$L'(s) = L(s) \cup \{q\}\ if\ (\mathcal{M}, s) \models \alpha,\ and\ L'(s) = L(s)\ otherwise.$$

It is clear that $\mathcal{M}' \models \varphi$, $\mathcal{M}' \models q \leftrightarrow \alpha$ and $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$. Therefore $\mathcal{M}' \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $\mathcal{M} \models \mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$ and $\mathrm{IR}(\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q), \{q\})$.

Proposition3 **(Modularity)** Given a μ-sentence φ, V a set of atoms and p an atom such that $p \notin V$. Then,

$$\mathrm{F}_\mu(\varphi, \{p\} \cup V) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V).$$

Proof. Let $\mathcal{M}_1$ with $\mathcal{M}_1 = (S_1, s_1, R_1, L_1)$ be a model of $\mathrm{F}_\mu(\varphi, \{p\} \cup V)$. By the definition of forgetting, there exists a model $\mathcal{M}$ with $\mathcal{M} = (S, s, R, L)$ of φ, such that $\mathcal{M}_1 \leftrightarrow_{\{p\} \cup V} \mathcal{M}$. We construct a Kripke structure $\mathcal{M}_2$ with $\mathcal{M}_2 = (S_2, s_2, R_2, L_2)$ as follows:

(1) for s_2: let s_2 be the state such that:
- $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
- for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
- for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:
  (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\} \cup V} w_1$, let $w_2 \in S_2$ and
    - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
    - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
    - for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.
  (ii) if $(w_1', w_1) \in R_1$, w_2 is constructed based on w_1 and $w_2' \in S_2$ is constructed based on w_1', then $(w_2', w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Thus, $(\mathcal{M}_2, s_2) \models \mathrm{F}_\mu(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V)$.

On the other hand, suppose that $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V)$, then there exists a Kripke structure $\mathcal{M}_2$ such that $\mathcal{M}_2 \models \mathrm{F}_\mu(\varphi, p)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$, and there exists $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$. Therefore, $\mathcal{M} \leftrightarrow_{\{p\} \cup V} \mathcal{M}_1$ by (ii) of Proposition 2, and consequently, $\mathcal{M}_1 \models \mathrm{F}_\mu(\varphi, \{p\} \cup V)$.

Proposition 4 Let φ, φ_i, ψ_i $(i = 1, 2)$ be formulas in CTL and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_\mu(\varphi, V)$ is satisfiable iff φ is;
(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_\mu(\varphi_1, V) \equiv \mathrm{F}_\mu(\varphi_2, V)$;
(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_\mu(\varphi_1, V) \models \mathrm{F}_\mu(\varphi_2, V)$;
(iv) $\mathrm{F}_\mu(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$;
(v) $\mathrm{F}_\mu(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_\mu(\psi_1, V) \wedge \mathrm{F}_\mu(\psi_2, V)$;

Proof. (i) ($\Rightarrow$) Supposing $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\varphi, V)$, then there is a model $\mathcal{M}'$ of φ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by the definition of F_μ.

($\Leftarrow$) Supposing $\mathcal{M}$ is a model of φ, then there is a Kripke structure $\mathcal{M}'$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$, and then $\mathcal{M}' \models \mathrm{F}_\mu(\varphi, V)$ by the definition of F_μ.

The (ii) and (iii) can be proved similarly.

(iv) ($\Rightarrow$) For all$\mathcal{M} \in$ *Mod*$(\mathrm{F}_\mu(\psi_1 \vee \psi_2, V))$, there exists $\mathcal{M}' \in$ *Mod*$(\psi_1 \vee \psi_2)$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ and $\mathcal{M}' \models \psi_1$ or $\mathcal{M}' \models \psi_2$
$\Rightarrow$ there exists $\mathcal{M}_1 \in$ *Mod*$(\mathrm{F}_\mu(\psi_1, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_1$ or there exists $\mathcal{M}_2 \in$ *Mod*$(\mathrm{F}_\mu(\psi_2, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$.

($\Leftarrow$) for all $\mathcal{M} \in$ *Mod*$(\mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V))$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\psi_1, V)$ or $\mathcal{M} \models \mathrm{F}_\mu(\psi_2, V)$
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}_1$ and $\mathcal{M}_1 \models \psi_1$ or $\mathcal{M}_1 \models \psi_2$
$\Rightarrow \mathcal{M}_1 \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $\mathcal{M}_2$ s.t. $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \models \mathrm{F}_\mu(\psi_1 \vee \psi_2, V)$
$\Rightarrow \mathcal{M} \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M} \models \mathrm{F}_\mu(\psi_1 \vee \psi_2, V)$.

The (v) can be proved as (iv).

Proposition 5 **(Homogeneity)** Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence, then we have:

(i) $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_\mu(\phi, V)$.
(ii) $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_\mu(\phi, V)$.

Proof. Let $\mathcal{M} = (S, R, s, L)$, $M_i = (S_i, R_i, s_i, L_i)$ with $i \in \mathrm{N}$ and $\mathcal{M}' = (S', R', s', L')$, then we call $\mathcal{M}' = (S', R', s', L')$ be a sub-structure of $\mathcal{M}$ if:

- $S' \subseteq S$ and $S' = \{s' | s'$ is reachable from $s'\} \cup A$ with $A = \{s'' \mid s''$ can not be reached from s' and there is not such a sequence of states $(s, \dots, s'', s')\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to 2^{\mathcal{A}}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- s' is s or a state reachable from s.

(i) In order to prove $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}(\mathrm{F}_\mu(\phi, V))$, we only need to prove $\mathit{Mod}(\mathrm{F}_\mu(\mathrm{AX}\phi, V)) = \mathit{Mod}(\mathrm{AX}\mathrm{F}_\mu(\phi, V))$:

($\Rightarrow$) $\forall \mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathrm{AX}\phi, V))$ there exists a Kripke structure $\mathcal{M}$ s.t. $\mathcal{M} \models \mathrm{AX}\phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$
$\Rightarrow$ for any sub-structure $\mathcal{M}_1$ of $\mathcal{M}$ there is $\mathcal{M}_1 \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_2$ s.t. $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}_3$ by $\mathcal{M}_2$ s.t. $\mathcal{M}_2$ is a sub-structure of $\mathcal{M}_3$ with s_2 is a direct successor of s_3 and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}$
$\Rightarrow \mathcal{M}_3 \models \mathrm{AX}(\mathrm{F}_\mu(\phi, V))$ and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}'$
$\Rightarrow \mathcal{M}' \models \mathrm{AX}(\mathrm{F}_\mu(\phi, V))$.

($\Leftarrow$) $\forall\ \mathcal{M}_3 \in \mathit{Mod}(\mathrm{AX}(\mathrm{F}_\mu(\phi, V)))$, then for any sub-structure $\mathcal{M}_2$ with s_2 is a directed successor of s_3 there is $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$
$\Rightarrow$ for any $\mathcal{M}_2$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M}_1 \models \phi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}$ by $\mathcal{M}_1$ s.t. $\mathcal{M}_1$ is a sub-structure of $\mathcal{M}$ with s_1 is a direct successor of s and $\mathcal{M} \leftrightarrow_V \mathcal{M}_3$
$\Rightarrow \mathcal{M} \models \mathrm{AX}\phi$ and then $\mathcal{M}_3 \models \mathrm{F}_\mu(\mathrm{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_\mu(\phi, V)$, we only need to prove Mod $(\mathrm{F}_\mu(\mathrm{EX}\phi, V)) = \mathit{Mod}(\mathrm{EX}\mathrm{F}_\mu(\phi, V))$:

($\Rightarrow$) $\forall \mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathrm{EX}\phi, V))$ there exists a Kripke structure $\mathcal{M}$ s.t. $\mathcal{M} \models \mathrm{EX}\phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$
$\Rightarrow$ there is a sub-structure $\mathcal{M}_1$ of $\mathcal{M}$ s.t. $\mathcal{M}_1 \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_2$ s.t. $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}_3$ by $\mathcal{M}_2$ s.t. $\mathcal{M}_2$ is a sub-structure of $\mathcal{M}_3$ that s_2 is a direct successor of s_3 and $\mathcal{M}_3 \leftrightarrow_V \mathcal{M}$
$\Rightarrow \mathcal{M}_3 \models \mathrm{EX}(\mathrm{F}_\mu(\phi, V))$
$\Rightarrow \mathcal{M}' \models \mathrm{EX}(\mathrm{F}_\mu(\phi, V))$.

($\Leftarrow$) $\forall\ \mathcal{M}_3 \in \mathit{Mod}(\mathrm{EX}(\mathrm{F}_\mu(\phi, V)))$, then there exists a sub-structure $\mathcal{M}_2$ of $\mathcal{M}_3$ s.t. $\mathcal{M}_2 \models \mathrm{F}_\mu(\phi, V)$
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M}_1 \models \phi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow$ it is easy to construct a Kripke structure $\mathcal{M}$ by $\mathcal{M}_1$ s.t. $\mathcal{M}_1$ is a sub-structure of $\mathcal{M}$ that s_1 is a direct successor of s and $\mathcal{M} \leftrightarrow_V \mathcal{M}_3$
$\Rightarrow \mathcal{M} \models \mathrm{EX}\phi$ and then $\mathcal{M}_3 \models \mathrm{F}_\mu(\mathrm{EX}\phi, V)$.

Proposition7 **(Model Checking)** Given a finite Kripke structure $\mathcal{M}$, a disjunctive formula φ and $V \subseteq \mathcal{A}$, deciding $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is NP $\cap$ co-NP.

Proof. Let A_φ be a μ-automaton such that, for any Kripke structure $\mathcal{N}$, A_φ accepts $\mathcal{N}$ iff $\mathcal{N} \models \varphi$, where $A_\varphi = (Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ with $\mathit{Var}(\varphi) = \Sigma_p \cup \Sigma_r$. Without loss of generality, we assume $V \subseteq \mathit{Var}(\varphi)$ and $V = \{p\}$. Therefore we can construct a μ-automaton $B = (Q, \Sigma_p - V, \Sigma_r, q_0, \delta', \Omega)$ with:

$$\delta'(q, L) := \delta(q, L) \cup \delta(q, L \cup \{p\}).$$

It has been proved in [7] that, for each Kripke structure $\mathcal{N}$, B accepts $\mathcal{N}$ iff there is a model $\mathcal{N}'$ of φ such that $\mathcal{N} \leftrightarrow_{\{p\}} \mathcal{N}'$, i.e. B corresponds to a μ-sentence which is equivalent to $\mathrm{F}_\mu(\varphi, V)$ by the definition of forgetting in μ-calculus.

In this case, the problem $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is reduced to decide whether B accepts $\mathcal{M}$, which is NP $\cap$ co-NP [4].

Theorem6 **(Entailment)** Let φ and ψ be two μ-sentences and V be a set of atoms. Then, the following problems are EXPTIME-complete.

(i) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \psi$,
(ii) deciding $\psi \models^? \mathrm{F}_\mu(\varphi, V)$,
(iii) deciding $\mathrm{F}_\mu(\varphi, V) \models^? \mathrm{F}_\mu(\psi, V)$.

Proof. We prove the (i), there other two results can be proved similarly.

Let A_φ and A_ψ be the μ-automaton of φ and ψ respectively, we can construct the μ-automaton B of $\mathrm{F}_\mu(\varphi, V)$ from A_φ by the proof of Proposition 7. By Proposition 7.3.2 in [6] , we can obtain the complement C of A_ψ in linear time, and then the intersection $A_{C\cap B}$ between C and B in linear time. In this case, the $\mathrm{F}_\mu(\varphi, V) \models^? \psi$ is reduced to decide whether the language accepted by $A_{C\cap B}$ is empty, which is EXPTIME-complete [6].

Proposition 8 **(dual)** Let V, q, φ and ψ are defined as in Definition 6. Then, ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \rightarrow \psi$. Thus $\varphi \models \neg\psi \rightarrow \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \rightarrow \neg q$. Then $\varphi \models q \rightarrow \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \rightarrow \neg\psi'$ by assumption. So $\varphi \models \psi' \rightarrow \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case.

Proposition 9 Let Γ and α be two μ-sentences, $V \subseteq \mathit{Var}(\alpha) \cup \mathit{Var}(\Gamma)$ and q be a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

Proof. We prove this for SNC. The case for WSC is similar. Let $\mathit{SNC}(\varphi, \beta, V, \Gamma)$ denote that φ is the SNC of β on V under Γ, and $\mathit{NC}(\varphi, \beta, V, \Gamma)$ denote that φ is the NC of β on V under Γ, in which β is a formula.

($\Rightarrow$) We will show that if $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds, then $\mathit{SNC}(\varphi, q, V, \Gamma')$ will be true. According to $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \rightarrow \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F}_{\mathrm{CTL}}(\Gamma', q) \models \alpha \rightarrow \varphi'$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \rightarrow \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi'$ by Lemma 1, this means $\mathit{NC}(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \rightarrow \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \rightarrow \varphi'$. Hence, $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $SNC(\varphi, q, V, \Gamma')$ holds, then $SNC(\varphi, \alpha, V, \Gamma)$ will be true. According to $SNC(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F}_{\mathrm{CTL}}(\Gamma', \{q\}) \models \alpha \rightarrow \varphi$ due to $\alpha \equiv q$, $IR(\alpha \rightarrow \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi$ by Lemma 1, this means $NC(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \rightarrow \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $NC(\varphi', q, V, \Gamma')$. According to $SNC(\varphi, q, V, \Gamma')$, $IR(\varphi \rightarrow \varphi', \{q\})$ and (**PP**), we have $\mathrm{F}_{\mathrm{CTL}}(\Gamma', \{q\}) \models \varphi \rightarrow \varphi'$, and $\Gamma \models \varphi \rightarrow \varphi'$ by Lemma 1. Hence, $SNC(\varphi, \alpha, V, \Gamma)$ holds.

Theorem 7 Let φ be a μ-sentence, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.

(i) $\mathrm{F}_{\mu}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ.
(ii) $\neg\mathrm{F}_{\mu}(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ.

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F}_{\mu}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \rightarrow \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \rightarrow \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \rightarrow \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $IR(\psi, (Var(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ.

Proposition10 Let Γ be a μ-sentence, q a proposition, and V a set of propositions.

(i) If φ is a necessary condition of q on V under Γ, and ψ the weakest sufficient condition of q on V under $\Gamma \cup \{\varphi\}$, then $\varphi \wedge \psi$ is the weakest sufficient condition of q on V under Γ.

(ii) If ψ is a sufficient condition of q on V under Γ, and φ a strongest necessary condition of q on V under $\Gamma \cup \{\neg\psi\}$, then $\varphi \vee \psi$ is a strongest necessary condition of q on V under Γ.

Proof. (i) First of all, $\varphi \wedge \psi$ is an SC due to $\Gamma \cup \{\varphi\} \models \psi \supset q$, i.e. $\Gamma \models \varphi \wedge \supset q$. We will show it is the weakest one: suppose φ' is an SC, i.e. $\Gamma \models \varphi' \supset q$. We need to show that $\Gamma \models \varphi' \supset \varphi \wedge \psi$. Thanks to $\Gamma \models q \supset \varphi$. We have $\Gamma \models \varphi' \supset \varphi$. However, φ' is also an SC of q under $\Gamma \cup \{\varphi\}$, hence $\Gamma \cup \{\varphi\} \models \varphi' \supset \psi$ because ψ is the WSC of q under $\Gamma \cup \{\varphi\}$. Therefore $\Gamma \models (\varphi \wedge \psi)$.

(ii) Suppose φ' is an NC of q under Γ, i.e. $\Gamma \models q \supset \varphi'$. We have $\Gamma \models \psi \supset \varphi'$ since $\Gamma \models \psi \supset q$. And there is $\Gamma \cup \{\neg\psi\} \models \varphi \supset \varphi'$ due to $\Gamma \cup \{\neg\psi\} \models q \supset \varphi$ and $\Gamma \cup \{\neg\psi\} \models q \supset \varphi'$. Therefore we have $\Gamma \models \varphi \supset \varphi'$ by $\Gamma \models \neg\psi \wedge \varphi \supset \varphi'$ and $\Gamma \models \psi \supset \varphi'$.

Theorem8 Let $V \subseteq \mathcal{A}$ and ϕ be a μ-sentence. Then there is a μ-sentence ψ such that:

$$\mathcal{M} \models \psi \text{ iff there is a model } \mathcal{M}' \text{ of } \phi \text{ such that } \mathcal{M} \leftrightarrow_V \mathcal{M}'$$

where both $\mathcal{M}$ and $\mathcal{M}'$ are initial structures.

Proof. Let $\psi = \mathrm{F}_\mu(\phi, V)$. We have that for each $\mathcal{M} \models \psi$ there is a $\mathcal{M}' \models \phi$ with $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by Theorem 3 and for each $\mathcal{M}' \in \mathit{Mod}(\phi)$ there is $\phi \models \psi$. In this case, we can easy prove that for each initial structure $\mathcal{M}$, if $\mathcal{M} \models \psi$ then we can obtain an initial structure $\mathcal{M}'$ such that $\mathcal{M}' \models \phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Besides, for each $\mathcal{M}' \in \mathit{Mod}(\phi)$ there is $\mathcal{M}' \models \psi$ by $\phi \models \psi$.

Theorem9 Let Γ and ϕ be μ-sentences. Then we have:

$$\mathit{Mod}(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}}).$$

Proof. For each initial structure $\mathcal{M}' \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$, we will show that there exists some $\mathcal{M} \in \mathit{Mod}(\Gamma)$ such that $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$. According to Definition 7, we know that there exists some $\mathcal{M} \in Mod(\Gamma)$ such that $\mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$. Further, there is a particular $V' \subseteq \mathcal{A}$ (i.e. $V' = V_{min}$) such that $\mathcal{M}' \leftrightarrow_{V'} \mathcal{M}$ and $\mathcal{M}' \in \mathit{Mod}(\phi)$. Since such V' is a minimal subset of $\mathcal{A}$ satisfying these properties, it concludes that for any other models $\mathcal{M}''$ of ϕ with $\mathcal{M}'' \leftrightarrow_{V_{min}} \mathcal{M}$, we have $\mathcal{M}' \leq_{\mathcal{M}} \mathcal{M}''$ by the definitions of forgetting and characterizing formula. Therefore, $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$.

For each initial structure $\mathcal{M}' \in \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$, there exists some $\mathcal{M} \in \mathit{Mod}(\Gamma)$ such that $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$. Let V_{min} be a minimal subset of atoms such that $\mathcal{M}' \leftrightarrow_{V_{min}} \mathcal{M}$. Then according to the definition of $\leq_{\mathcal{M}}$, we know that there does not exist another $\mathcal{M}'' \in \mathit{Mod}(\phi)$ such that $\mathcal{M}'' \leftrightarrow_{V'} \mathcal{M}$ and $V' \subset V_{min}$. This follows that $\mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}), V_{min}) \wedge \phi)$ and hence $\mathcal{M}' \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$.

Theorem 10 Knowledge update operator $\diamond_\mu$ satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).

Proof. For (U1), we know that $\mathit{Mod}(\Gamma \diamond_\mu \phi) \subseteq \mathit{Mod}(\phi)$ by Theorem 9, hence $\Gamma \diamond_\mu \phi \models \phi$.

For (U2), we will prove $\Gamma \diamond_\mu \phi \models \Gamma$ at first. For any model $\mathcal{M}$ of $\Gamma \diamond_\mu \phi$ there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Then we have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Similarly, for any model $\mathcal{M}$ of Γ, there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. We have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Hence $\Gamma \models \Gamma \diamond_\mu \phi$.

It is easy to show $\diamond_\mu$ satisfies (U3) and (U4). We now prove (U5). For any model $\mathcal{M}$ of $(\Gamma \diamond_\mu \phi) \wedge \psi$ there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Besides, we can see that $\mathcal{M} \models \phi \wedge \psi$. Therefore, we have $\mathcal{M} \models \Gamma \diamond_\mu (\phi \wedge \psi)$.

For (U6), we will prove $\Gamma \diamond_\mu \phi \models \Gamma \diamond_\mu \psi$, and the other direction can be proved similarly. For any model $\mathcal{M}$ of $\Gamma \diamond_\mu \phi$, $\mathcal{M}$ is also a model of ψ. There is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma)$ and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \psi$. This shows that $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Moreover, V_{min} is also the minimal set such that $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Otherwise, suppose that $V \subset V_{min}$ such that $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V) \wedge \psi$ is consistent as well. Then, $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V)$ $\wedge \phi$ should also be consistent by $\Gamma \diamond_\mu \psi \models \phi$, which contradicts to the fact that V_{min} is the minimal set of atoms such that $\mathrm{F}_\mu(\mathcal{F}_{\mathcal{A}}(\mathcal{M}_1), V_{min}) \wedge \phi$ is consistent. Hence, $\mathcal{M}$ is also a model of $\Gamma \diamond_\mu \psi \models \psi$.

Now we prove (U7). Suppose that Γ has the unique model $\mathcal{M}$. For each $\mathcal{M}_1 \in \mathit{Mod}((\Gamma \diamond_\mu \phi) \wedge (\Gamma \diamond_\mu \psi))$ there exists V_1 and V_2 which are minimal such that $\mathcal{M} \leftrightarrow_{V_1}$

$\mathcal{M}_1$ and $\mathcal{M} \leftrightarrow_{V_2} \mathcal{M}_1$, i.e. $\mathcal{M}_1$ is a model of both $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge \phi$ and $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_2) \wedge \psi$. Therefore $\mathcal{M}_1 \leftrightarrow_{V_1 \cap V_2} \mathcal{M}$. Thus, $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1 \cap V_2)$. Then we have $V_1 = V_2$, otherwise V_1 (or V_2) is not the minimal set. $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ as well. Moreover, V_1 is the minimal set such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ is satisfiable. Otherwise, suppose that $V_3 \subset V_1$ such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge (\phi \vee \psi)$ is satisfiable. Then $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \phi$ or $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \psi$ is satisfiable. Without loss of generality, suppose that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \phi$ is satisfiable, V_1 is not the minimal set, a contradiction. Therefore $\mathcal{M}_1$ is also a model of $\mathit{\Gamma} \diamond_\mu (\phi \vee \psi)$.

For (U8), we will prove $(\mathit{\Gamma}_1 \vee \mathit{\Gamma}_2) \diamond_\mu \phi \models (\mathit{\Gamma}_1 \diamond_\mu \phi) \vee (\mathit{\Gamma}_2 \diamond_\mu \phi)$ at first. For each $\mathcal{M} \in \mathit{Mod}((\mathit{\Gamma}_1 \vee \mathit{\Gamma}_2) \diamond_\mu \phi)$, there is a $\mathcal{M}_1 \in \mathit{Mod}(\mathit{\Gamma}_1)$ (or $\mathcal{M}_1 \in \mathit{Mod}(\mathit{\Gamma}_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore, we have $\mathcal{M} \models (\mathit{\Gamma}_1 \diamond_\mu \phi) \vee (\mathit{\Gamma}_2 \diamond_\mu \phi)$. Similarly, for each $\mathcal{M} \in \mathit{Mod}((\mathit{\Gamma}_1 \diamond_\mu \phi) \vee (\mathit{\Gamma}_2 \diamond_\mu \phi))$ there is a $\mathcal{M}_1 \in \mathit{Mod}(\mathit{\Gamma}_1)$ (or $\mathcal{M}_1 \in \mathit{Mod}(\mathit{\Gamma}_2)$) and V_{min} such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Hence, $\mathcal{M} \models (\mathit{\Gamma}_1 \vee \mathit{\Gamma}_2) \diamond_\mu \phi$.