Artificial Intelligence 218 (2015) 1–22

Contents lists available at ScienceDirect

Artificial Intelligence

www.elsevier.com/locate/artint

Belief revision in Horn theories ☆

James P. Delgrande [a], Pavlos Peppas [b,c,*]

[a] School of Computing Science, Simon Fraser University, Burnaby, B.C., V5A 1S6, Canada
[b] Centre for Quantum Computation and Intelligent Systems, FEIT, University of Technology, Sydney, NSW 2007, Australia
[c] Dept of Business Administration, University of Patras, Patras 265 00, Greece

A R T I C L E I N F O

Article history:
Received 1 May 2013
Received in revised form 24 July 2014
Accepted 27 August 2014
Available online 10 September 2014

Keywords:
Knowledge representation
Belief revision
Horn logic

A B S T R A C T

This paper investigates belief revision where the underlying logic is that governing Horn clauses. We show that classical (AGM) belief revision doesn't immediately generalise to the Horn case. In particular, a standard construction based on a total preorder over possible worlds may violate the accepted (AGM) postulates. Conversely, in the obvious extension to the AGM approach, Horn revision functions are not captured by total preorders over possible worlds. We address these difficulties by introducing two modifications to the AGM approach. First, the semantic construction is restricted to "well behaved" orderings, what we call *Horn compliant* orderings. Second, the revision postulates are augmented by an additional postulate. Both restrictions are redundant in the AGM approach, but not in the Horn case. In a representation result we show that the class of revision functions captured by Horn compliant total preorders over possible worlds is precisely that given by the (extended) set of Horn revision postulates. Further, we show that Horn revision is compatible with work in iterated revision and work concerning relevance in revision. We also consider specific revision operators. Arguably this work is interesting for several reasons. It extends AGM revision to inferentially-weaker Horn theories; hence it sheds light on the theoretical underpinnings of belief change, as well as generalising the AGM paradigm. Thus, this work is relevant to revision in areas that employ Horn clauses, such as deductive databases and logic programming, as well as areas in which inference is weaker than classical logic, such as in description logic.

1. Introduction

The area of belief change studies how an agent may modify its beliefs given new information about its environment. The best-known approach to belief change is the AGM paradigm [1,16], named after the original developers. This work focussed on belief *revision*, in which new information is incorporated into an agent's belief corpus, as well as belief *contraction*, in which an agent may reduce its set of beliefs. The AGM approach addresses belief change at an abstract level, in which an agent's beliefs are characterised by *belief sets* or deductively closed sets of sentences, and where the underlying logic includes classical propositional logic. In this approach to revision, a set of rationality postulates is given which arguably any revision function should satisfy. As well, a semantic construction of revision functions has been given, in terms of a total

☆ This article is an extension and elaboration of the work published in [9].
* Corresponding author.
E-mail addresses: jim@cs.sfu.ca (J.P. Delgrande), pavlos@upatras.gr (P. Peppas).

http://dx.doi.org/10.1016/j.artint.2014.08.006

preorder over possible worlds, called a *faithful ranking* [18]. These syntactic and semantic approaches have been shown to capture the same set of revision functions.

In this paper we address AGM-style belief revision in the language of *Horn clauses*, where a Horn clause can be expressed as a rule of the form $a_1 \wedge a_2 \wedge \cdots \wedge a_n \rightarrow a$ for $n \geq 0$, and where a, a_i $(1 \leq i \leq n)$ are atoms, and a is an atom or the constant falsum $\perp$. (Thus, expressed in conjunctive normal form, a Horn clause is a clause with at most one positive literal.) In our approach, an agent's beliefs are represented by a Horn clause belief set, and the input is a Horn formula, consisting of a conjunction of Horn clauses. It proves to be the case that AGM-style belief revision doesn't transfer directly to Horn knowledge bases. On the one hand, in the Horn case the AGM postulate set is unsound with respect to faithful rankings over possible worlds. On the other hand, given a Horn revision function that satisfies the AGM postulates, there may be no corresponding faithful ranking that captures the revision function or, alternately, there may be several faithful rankings that capture the function.

Nonetheless, we show that the AGM approach can be extended to the Horn case naturally and satisfactorily. On the semantic side, we impose a "well-behaved" condition on faithful rankings, expressing that a ranking must be coherent with respect to Horn revision. We call such rankings *Horn compliant*. On the syntactic, postulational, side, we add a postulate to the standard suite of AGM postulates. Interestingly, in the AGM approach this additional postulate is redundant, in that it follows as a theorem from the other AGM postulates. In the Horn case, in which the language is less expressive than in the classical case, this postulate is independent of the others. Given these adjustments to the AGM approach, we then prove a representation result, showing that the class of Horn revision functions conforming to the extended postulate set is the same as those capturable by Horn compliant faithful rankings. Moreover, we prove Horn revision, as modelled herein, is consistent with Darwiche and Pearl postulates for iterated revision [8] and with Parikh's postulate for relevance [22]. A final contribution of our work is the development of two specific Horn revision operators, called *basic Horn revision* and *canonical Horn revision*, with *polynomial* time complexity ($O(n)$ and $O(n^2 \log n)$ respectively).

This topic is interesting for several reasons. It sheds light on the theory of belief change, in that it weakens the assumption that the underlying logic contains propositional logic. In doing so, it shows that the AGM approach is more generally applicable than perhaps originally believed. That is, our results provide a *broadening* of the AGM approach to include Horn reasoning, and not just a modification of the AGM approach to accommodate Horn reasoning. Horn clauses are a very useful restriction of classical logic, and have found widespread application in artificial intelligence and database theory. As well, results here may also be relevant to belief change in description logics, a topic that has also received recent attention.

The next section gives basic notation and definitions used in the paper. The third section introduces belief change and related work that has been carried out in belief change in Horn clause reasoning. This is followed by a discussion of issues that arise in Horn clause belief revision (Section 4). Section 5 develops the approach, and in particular presents the representation result for Horn formula revision. Section 6 discusses iterated Horn revision, while Section 7 discusses relevance and Horn revision, and Section 8 introduces specific Horn revision operators and examines their computational complexity. The paper concludes with a discussion of future work and a brief conclusion.

2. Formal preliminaries

We introduce here the terminology that we will use in the rest of the paper. $\mathcal{P} = \{a, b, c, \ldots\}$ is a finite set of propositional variables. $\mathcal{L}_H$ denotes the Horn formula language over $\mathcal{P} \cup \{\perp\}$. That is, $\mathcal{L}_H$ is the least set given by:

1. $a_1 \wedge a_2 \wedge \cdots \wedge a_n \rightarrow a$ is a *Horn clause*, where $n \geq 0$ and $a, a_i \in \mathcal{P} \cup \{\perp\}$ for $1 \leq i \leq n$.
   If $n = 0$ then $\rightarrow a$ is also written a, and is a *fact*.
2. If ϕ is a Horn clause then (ϕ) is a *Horn formula*.
3. If ϕ and ψ are Horn formulas then so is $(\phi \wedge \psi)$.

In our approach, we deal exclusively with Horn formulas, and so *formula* will refer to a Horn formula; the only exception is when we discuss background work, in which case the context is clear. Formulas are denoted by lower case Greek letters; sets of formulas are denoted by upper case Greek letters. Parentheses are required in 2 above to distinguish, for example $(p \wedge q \rightarrow r)$ from $(p) \wedge (q \rightarrow r)$. Nonetheless we freely drop parentheses when the meaning is clear.

An *interpretation* w is a subset of $\mathcal{P}$, where $a \in w$ means that a is assigned *true* in w and $a \notin w$ means that a is assigned *false* in w. Occasionally we will explicitly list negated atoms in an interpretation; for example for $\mathcal{P} = \{p, q\}$ the interpretation $\{p\}$ will sometimes be written $\{p, \neg q\}$ or more briefly $p\bar{q}$. The symbol $\perp$ is always assigned *false*. $\mathcal{M}$ is the set of *interpretations* or *(possible) worlds* (we use these terms interchangeably). Sentences of $\mathcal{L}_H$ are *true* or *false* in an interpretation according to the standard rules in propositional logic. Truth of ϕ in w is denoted $w \models \phi$. As well, for $W \subseteq \mathcal{M}$, $W \models \phi$ iff for every $w \in W$, $w \models \phi$. For formula ϕ, $[\phi]$ is the set of models of ϕ. For set of worlds W, $t_H(W)$ denotes the set of formulas satisfied by all worlds in W, i.e.

$$t_H(W) = \{\phi \in \mathcal{L}_H \mid m \models \phi \text{ for every } m \in W\}.$$

Note that this means that $t_H(\emptyset) = \mathcal{L}_H$.

$\mathit{\Gamma} \vdash \phi$ iff ϕ is derivable from the set of formulas $\mathit{\Gamma}$. Again, members of $\mathit{\Gamma}$ and ϕ are Horn, and $\vdash$ is defined in terms of Horn formulas; see [12] for details. $\psi \vdash \phi$ is an abbreviation for $\{\psi\} \vdash \phi$, and $\psi \equiv \phi$ is logical equivalence, i.e. $\psi \vdash \phi$

and $\phi \vdash \psi$. This extends in the obvious fashion to sets of formulas. For a set of formulas Γ, the closure of Γ under Horn derivability is denoted $\mathcal{C}n_H(\Gamma)$. A (Horn) theory H is a set of formulas such that $H = \mathcal{C}n_H(H)$, also referred to as a *belief set*. $\mathcal{H}$ is the set of Horn theories. For theory H and formula ϕ,

$$H + \phi = \mathcal{C}n_H\big(H \cup \{\phi\}\big)$$

is the *expansion* of H. $H_\perp = \mathcal{L}_H$ is the inconsistent belief set.

Models of Horn formulas are distinguished by the fact that they are closed under intersection of positive atoms in an interpretation. That is:

$$\text{If } w_1, w_2 \in [\phi] \quad \text{then } w_1 \cap w_2 \in [\phi].$$

Note that the converse is also true; i.e., if a set of models W is closed under intersection of positive atoms in an interpretation, then there is a Horn formula ϕ such that $[\phi] = W$.

A (partial) preorder $\preceq$ is a reflexive, transitive binary relation. A total preorder is a partial preorder such that $w_1 \preceq w_2$ or $w_2 \preceq w_1$ for every w_1, w_2. The strict part of the preorder $\preceq$ is denoted by $\prec$, that is, $w_1 \prec w_2$ just if $w_1 \preceq w_2$ and $w_2 \not\preceq w_1$. As usual, $w_1 \approx w_2$ abbreviates $w_1 \preceq w_2$ and $w_2 \preceq w_1$. Finally, for a set of interpretations $W \subseteq \mathcal{M}$, we define the set $\min(W, \preceq)$ by

$$\min(W, \preceq) = \{w_1 \in W \mid \text{for all } w_2 \in W,\ \text{if } w_2 \preceq w_1 \text{ then } w_1 \preceq w_2\}.$$

3. Background

3.1. Belief revision

In the AGM approach to belief change [1,16], beliefs of an agent are modelled by a deductively closed set of formulas, or *belief set*. Thus a belief set is a set of formulas K such that $K = \mathcal{C}n(K)$, where $\mathcal{C}n(K)$ denotes the closure of K under classical logical consequence. It is assumed that the underlying logic contains classical propositional logic. Belief revision is modelled as a function from a belief set K and a formula ϕ to a belief set K' such that ϕ is believed in K', i.e. $\phi \in K'$. Since ϕ may be inconsistent with K, and since it is desirable to maintain consistency whenever possible (i.e. whenever ϕ is consistent) then some formulas may need to be dropped from K before ϕ can be consistently added. Formally, a revision operator $*$ maps a belief set K and formula ϕ to a revised belief set $K * \phi$. The AGM postulates for revision specify conditions that arguably should hold for any rational revision operator. These postulates can be expressed as follows, where $\equiv_{PC}$ and $+_{PC}$ stand for logical equivalence and expansion, respectively, in classical propositional logic.

(K*1) $K * \phi = \mathcal{C}n(K * \phi)$
(K*2) $\phi \in K * \phi$
(K*3) $K * \phi \subseteq K +_{PC} \phi$
(K*4) If $\neg\phi \notin K$ then $K +_{PC} \phi \subseteq K * \phi$
(K*5) $K * \phi$ is inconsistent only if ϕ is inconsistent
(K*6) If $\phi \equiv_{PC} \psi$ then $K * \phi = K * \psi$
(K*7) $K * (\phi \wedge \psi) \subseteq K * \phi +_{PC} \psi$
(K*8) If $\neg\psi \notin K * \phi$ then $K * \phi +_{PC} \psi \subseteq K * (\phi \wedge \psi)$

Thus, the result of revising K by ϕ yields a belief set in which ϕ is believed ((K*1), (K*2)); whenever the result is consistent, the revised belief set consists of the expansion of K by ϕ ((K*3), (K*4)); the only time that K is inconsistent is when ϕ is inconsistent ((K*5)); and revision is independent of the syntactic form of the formula for revision ((K*6)). The last two postulates deal with the relation between revising by a conjunction and expansion: whenever consistent, revision by a conjunction corresponds to revision by one conjunct and expansion by the other. Motivation for these postulates can be found in [16,25]. We shall call any function $*$ that satisfies (K*1)–(K*8) an *AGM revision function*.

Katsuno and Mendelzon [18] have shown that a necessary and sufficient condition for constructing an AGM revision operator is that there is a function that associates a total preorder on the set of possible worlds with any belief set K, as follows:[1]

Definition 1. (See [18].) A *faithful assignment* is a function that maps each belief set K to a total preorder $\preceq_K$ on $\mathcal{M}$ such that for any possible worlds w_1, w_2:

1. If $w_1, w_2 \in [K]$ then $w_1 \approx_K w_2$.
2. If $w_1 \in [K]$ and $w_2 \notin [K]$, then $w_1 \prec_K w_2$.

[1] In fact, Katsuno and Mendelzon deal with formulas instead of belief sets. Since we deal with finite languages only, the difference is immaterial. We use belief sets in order to adhere more closely to the original AGM approach.

The resulting preorder is referred to as a *faithful ranking* associated with K. Intuitively, $w_1 \preceq_K w_2$ if w_1 is at least as plausible as w_2. Katsuno and Mendelzon then provide the following representation result, where $t(W)$ is the set of formulas of classical logic true in the set of possible worlds W:

Theorem 1. *(See [18].) A revision operator * satisfies postulates (K^*1)–(K^*8) iff there exists a faithful assignment that maps each belief set K to a total preorder $\preceq_K$ such that*

$$K * \phi = t\big(\min\big([\phi], \preceq_K\big)\big).$$

Thus the revision of K by ϕ is characterised by those models of ϕ that are most plausible according to the agent. Given that we are working with a finite language, this construction is in fact equivalent to the earlier *system of spheres* approach due to Grove [17]. It is easier to present our results in terms of faithful assignments, and so we do so here.

Another form of belief change in the AGM approach is belief *contraction*, in which an agent's beliefs decrease. Thus in the contraction of ϕ from K, written $K - \phi$, one has $\phi \notin K - \phi \subseteq K$, while $\neg\phi$ is not necessarily believed. There are two primary means of constructing contraction functions. Using *remainder sets*, a contraction $K - \phi$ is defined in terms of maximal subsets of K that fail to imply ϕ. Via *epistemic entrenchment*, an ordering is defined on sentences of K, and a contraction $K - \phi$ is (roughly) defined in terms of the most entrenched set of sentences that does not imply ϕ. Of interest, and pertinent to the approach at hand, these various constructions are all in a certain sense interdefinable, as are revision and contraction functions. Hence, given a contraction function $-$, one may define a revision function by the so-called Levi identity:

$$K * \phi = (K - \neg\phi) +_{PC} \phi. \tag{1}$$

See [16,25] for details.

3.2. Related work

Earlier work on belief change involving Horn formulas dealt with the Horn fragment of a propositional theory, rather than Horn clause belief change as a distinct phenomenon. For example, the complexity of specific approaches to revising knowledge bases is addressed by Eiter and Gottlob [14], including the case where the knowledge base and formula for revision are Horn formulas. Liberatore [20] considers the problem of compact representation for revision in the Horn case. Given a knowledge base K and formula ϕ, both Horn, the main problem considered is whether a revised knowledge base can be expressed by a propositional formula whose size is polynomial with respect to the sizes of K and ϕ. More recently, belief revision in other fragments of propositional logic, including Krom and affine formulas, has been addressed in [6].

Langlois et al. [19] approach the study of revising Horn formulas by characterizing the existence of a complement of a Horn consequence; such a complement corresponds to the result of a contraction operator. This work may be seen as a specific instance of a general framework developed by Flouris et al. [15].

The main difference between our work and the above approaches to revision, is that in our approach revision functions *always* produce Horn theories (they are postulated to do so). This of course adds an extra burden to the revision process since it now needs to comply with both the *principle of minimal change* (see [16,25]), and the requirement to produce Horn theories (which in this context can be seen as an instance of the *principle of categorical matching*). Our results show that, with some adjustments to the original AGM framework, this double objective can indeed be achieved.

With respect to AGM-style belief change in Horn theories, most work has focussed on Horn contraction. Delgrande and Pearl [12] addresses maxichoice belief contraction in Horn clause theories, where contraction is defined in terms of remainder sets. Booth et al. [2] generalise this to so-called infra-remainder sets, while Delgrande and Wassermann [11] link Horn contraction to AGM contraction via weak remainder sets. In a series of papers, Zhuang and Pagnucco [29–31] and Booth et al. [3] further explore Horn contraction by considering other constructions including epistemic entrenchment, partial meet, and kernel contraction. Zhuang et al. [32] present a technique for obtaining a Horn revision in terms of contraction. The difficulty in any approach to defining Horn revision in terms of contraction is that one must deal with the negation of a Horn formula which, in general, is not Horn. Zhuang et al. circumvent this difficulty by contracting by a sequence of Horn strengthenings [28] of the negation of the formula for revision.

4. Horn revision: preliminary considerations

4.1. Expressing revision in the context of Horn theories

The postulates and semantic construction of Section 3.1 are easily adapted to Horn theories. For the postulates, we have the following, expressed in terms of Horn theories.

An *AGM (Horn) revision* function $*$ is a function from $\mathcal{H} \times \mathcal{L}_H$ to $\mathcal{H}$ satisfying the following postulates.

(H*1) $H * \phi = Cn_H(H * \phi)$.
(H*2) $\phi \in H * \phi$.
(H*3) $H * \phi \subseteq H + \phi$.
(H*4) If $\perp \notin H + \phi$ then $H + \phi \subseteq H * \phi$.
(H*5) If ϕ is consistent then $\perp \notin H * \phi$.
(H*6) If $\psi \equiv \phi$ then $H * \psi = H * \phi$.
(H*7) $H * (\psi \wedge \phi) \subseteq (H * \psi) + \phi$.
(H*8) If $\perp \notin (H * \psi) + \phi$ then $(H * \psi) + \phi \subseteq H * (\psi \wedge \phi)$.

As well, faithful assignments can be defined for the Horn case, basically by changing notation:

Definition 2. A *faithful assignment* is a function that maps each Horn theory H to a total preorder $\preceq_H$ on $\mathcal{M}$ such that for any possible worlds w_1, w_2:

1. If $w_1, w_2 \in [H]$ then $w_1 \approx_H w_2$.
2. If $w_1 \in [H]$ and $w_2 \notin [H]$, then $w_1 \prec_H w_2$.

The resulting preorder is referred to as the *faithful ranking* associated with H. Finally, one can define a function $*$ in terms of a faithful ranking by:

$$H * \phi = t_H\big(\min\big([\phi], \preceq_H\big)\big). \tag{2}$$

The use of $*$ in Eq. (2) is suggestive; ideally one would next establish a correspondence between functions that satisfy the postulates and those that can be specified via Eq. (2). However, there are significant difficulties in immediately establishing such a representation result. We review these problems next, and then present our solution in the following section.

4.2. Problems with naïve AGM Horn revision

While Horn revision is naturally expressible in terms of the (Horn) AGM postulates on the one hand, and faithful assignments on the other, it is perhaps not surprising that results differ from revision with respect to classical logic. Below we review some issues that arise, ranging from the inconvenient to the highly problematic.

4.2.1. Interdefinability results do not hold in Horn belief change

As mentioned in Section 3.1, in the AGM approach revision may be defined in terms of contraction via the Levi Identity (1). However, previous work [10] suggests that Horn contraction is unsuitable for specifying a revision operator. As well, if one considers the Levi Identity, revision by a Horn formula ϕ is defined in terms of the contraction by $\neg\phi$. Since ϕ is a conjunction of Horn clauses, $\neg\phi$ in general will not be Horn, and so the Levi identity would seem to be inapplicable for Horn theories.[2]

These points are not definitive (there is, after all, no formal result stating an impossibility of interdefinability of Horn contraction and revision), but they do suggest the overall difficulty in obtaining such a result. Consequently, we focus on a direct definition of Horn revision, in terms of ranking functions, in the next section. Having developed such an approach, we then suggest that the relation between Horn contraction and revision is a suitable and interesting topic for future research.

4.2.2. Distinct faithful rankings may yield the same revision function

Consider the Horn language defined by $\mathcal{P} = \{p, q\}$, and the following three total preorders:

$$pq \prec \overline{pq} \prec p\bar{q} \prec \bar{p}q \tag{3}$$

$$pq \prec \overline{pq} \prec \bar{p}q \prec p\bar{q} \tag{4}$$

$$pq \prec \overline{pq} \prec \bar{p}q \approx p\bar{q} \tag{5}$$

It can be verified that if one defines revision via Eq. (2), the three total preorders yield the same Horn revision function. In particular, there is no way in which the relative ranking of worlds $p\bar{q}$ and $\bar{p}q$ can be distinguished. This is because any Horn formula ϕ consistent with $p\bar{q}$ and $\bar{p}q$ is also consistent with $\overline{pq}$ (i.e. if $p\bar{q}, \bar{p}q \in [\phi]$ then $\overline{pq} \in [\phi]$). Hence for all Horn formulas ϕ, the minimal ϕ-worlds are identical under all three preorders.

[2] Zhuang et al. [32] circumvent this difficulty by employing Horn strengthenings of a non-Horn formula. However their approach to contraction is with respect to faithful orderings, and not the more common approaches of remainder sets or epistemic entrenchment.

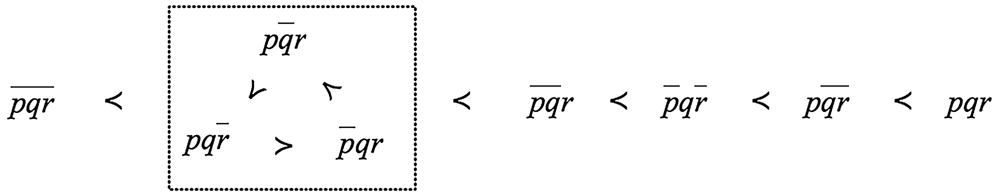

Fig. 1. Pseudo-preorder that induces function satisfying (H*1)–(H*8).

4.2.3. Postulates may not be satisfied in a faithful ranking

Consider the Horn language with atoms $\mathcal{P} = \{p, q, r\}$ and the ranking:

$$pqr \prec \bar{p}q\bar{r} \approx p\overline{qr} \prec \overline{pq}r \prec \overline{pqr} \prec \text{all other worlds} \tag{6}$$

The agent's belief set H is given by $Cn_H(p \wedge q \wedge r)$. Let μ be $p \wedge q \to \bot$ and ϕ be $\neg p \wedge \neg q$. Defining $*$ as in Eq. (2), it can be verified that:

$$H * \mu = Cn_H\big((p \wedge q \to \bot) \wedge \neg r\big),$$
$$(H * \mu) + \phi = Cn_H(\neg p \wedge \neg q \wedge \neg r),$$
$$H * (\mu \wedge \phi) = Cn_H(\neg p \wedge \neg q \wedge r).$$

Thus $(H * \mu) + \phi$ and $H * (\mu \wedge \phi)$ are not equivalent and violate both (H*7) and (H*8).

Informally, the culprit is the set of worlds $\{\bar{p}q\bar{r}, p\overline{qr}\}$. This set (as with the previous problem) is not expressible by a Horn formula, since it is not closed under intersection of (positive) atoms. It can be observed that the "missing" interpretation is given by $\overline{pqr}$, where in our ordering (6) we have $\bar{p}q\bar{r} \approx p\overline{qr} \prec \overline{pqr}$. The problem arises then because one may revise by a Horn formula (viz. $\mu = p \wedge q \to \bot$) that yields the set of minimal models $\{\bar{p}q\bar{r}, p\overline{qr}\}$, but in producing the corresponding Horn theory $t_H(\{\bar{p}q\bar{r}, p\overline{qr}\}) = Cn_H((p \wedge q \to \bot) \wedge \neg r)$, a new *non-minimal* model $\overline{pqr}$ creeps in.[3]

*4.2.4. There is a Horn AGM revision function satisfying (H*1)–(H*8) that cannot be modelled by a preorder on worlds*

Consider the following pseudo-preorder on worlds:

That is, the most preferred world is $\overline{pqr}$, followed by $pq\bar{r}, \bar{p}qr, p\bar{q}r$ which form a *cycle* (i.e. $pq\bar{r} \prec \bar{p}qr \prec p\bar{q}r \prec pq\bar{r}$), followed by the sequence of worlds $\overline{pq}r \prec \bar{p}q\bar{r} \prec p\overline{qr} \prec pqr$.

Clearly, $\prec$ is not transitive and therefore is not a preorder. Nevertheless, we can still use Definition 2 to induce a function $*$ from $\prec$. Perhaps surprisingly, $*$ satisfies all eight postulates (H*1)–(H*8).

Proposition 1. *The function $*$ induced via Definition 2 from the binary relation $\prec$ of Fig. 1 satisfies (H*1)–(H*8).*

Proof. Postulates (H*1), (H*2), (H*3), (H*4), and (H*6) follow trivially from Definition 2. For (H*5), it suffices to show that any nonempty set of worlds S which is closed under intersection (and therefore is definable by a Horn formula), has a minimal element with respect to the pseudo-preorder $\prec$. If S is a subset of $\mathcal{M} - \{\bar{p}qr\}$ this is indeed the case since, as it can be easily observed, the restriction of $\prec$ to $\mathcal{M} - \{\bar{p}qr\}$ is a total preorder (the problematic cycle $pq\bar{r} \prec \bar{p}qr \prec p\bar{q}r \prec pq\bar{r}$ is no longer present). In fact, the restriction of $\prec$ to $\mathcal{M} - \{\bar{p}qr\}$ is not only a total preorder; it is a *linear preorder*, and therefore if $S \subseteq \mathcal{M} - \{\bar{p}qr\}$, S has a *unique* minimal element. The same of course is true in the case where $S \subseteq \mathcal{M} - \{p\bar{q}r\}$ or $S \subseteq \mathcal{M} - \{pq\bar{r}\}$. This leaves us with the case where $\{\bar{p}qr, p\bar{q}r, pq\bar{r}\} \subseteq S$. Recall that we are only interested in sets S that are closed under intersection, since only for such sets is there a Horn formula ϕ such that $[\phi] = S$. Notice however that if S is closed under intersection, and all three worlds $\bar{p}qr$, $p\bar{q}r$, and $pq\bar{r}$ belong to S, then $\overline{pqr}$ also belongs to S and therefore, by the definition of $\prec$, $\overline{pqr}$ is the (unique) minimal element of S. Hence we have shown that any nonempty, closed under intersection, set of worlds S has a (unique) minimal element with respect to $\prec$. From this, (H*5) trivially follows.

For (H*7) and (H*8), consider any two Horn formulas ϕ, ψ. If ϕ is inconsistent with $Cn_H(\overline{pqr}) * \psi$ then (H*7) and (H*8) are trivially true. Assume therefore that ϕ is consistent with $Cn_H(\overline{pqr}) * \psi$, or equivalently with $t_H(\min([\psi], \prec))$. Moreover, notice that $\min([\psi], \prec)$ is a singleton.[4] Hence from our assumption that ϕ is consistent with $t_H(\min([\psi], \prec))$ it follows that the unique minimal ψ-world also satisfies ϕ. Therefore $\min([\psi], \prec) = \min([\psi \wedge \phi], \prec)$ and consequently $(Cn_H(\overline{pqr}) * \psi) + \phi = Cn_H(\overline{pqr}) * \psi = Cn_H(\overline{pqr}) * (\psi \wedge \phi)$. Thus (H*7) and (H*8) are true. □

[3] As a subtlety, this doesn't imply that, when we come to define our approach, an equivalently-ranked set of worlds must be definable by a Horn formula. For example (5) contains a set of worlds that isn't definable by a Horn formula. However, in our approach this ranking will prove to be an acceptable ordering on worlds.

[4] As argued above, every nonempty, closed under intersection, set of worlds S—like $[\psi]$—has a *unique* minimal world.

Although (H*1)–(H*8) are satisfied by a function induced from a non-preorder, the connection between the postulates and total preorders can still be rescued if we can find *another* binary relation $\prec'$, different from $\prec$, that also induces $*$ via Definition 2 and which *is* a total preorder. It turns out though that this is not the case.

Proposition 2. *Let $*$ be the function induced via Definition 2 from the binary relation $\prec$ of Fig. 1. Every binary relation $\preceq'$ that also induces $*$ via Definition 2 contains the cycle $pq\bar{r} \prec' \bar{p}qr \prec' p\bar{q}r \prec' pq\bar{r}$.*

Proof. Let $\preceq'$ be a binary relation that induced $*$ via Definition 2. From Fig. 1 it follows that $Cn(\overline{pqr}) * p = Cn(pq\bar{r})$ and consequently, $pq\bar{r} \prec' p\bar{q}r$. Similarly, $Cn(\overline{pqr}) * r = Cn(p\bar{q}r)$ and therefore $p\bar{q}r \prec' \bar{p}qr$. Finally, from $Cn(\overline{pqr}) * q = Cn(\bar{p}qr)$ we derive $\bar{p}qr \prec' pq\bar{r}$. Hence we obtain the cycle $pq\bar{r} \prec' \bar{p}qr \prec' p\bar{q}r \prec' pq\bar{r}$. $\square$

5. Horn revision: the approach

As is clear from the previous discussion, there are substantial differences between classical AGM revision and Horn revision. These differences come about from the weakened expressibility of Horn clause theories.

Consider again the issues discussed in the previous section. The first issue isn't a problem with Horn revision *per se*. Rather, it suggests that, when we look at Horn theories, belief change operators are not interdefinable, or at best are not readily interdefinable. The second issue also isn't a problem as such. Instead, it indicates that a ranking may be underconstrained by a revision function.

The third issue, that a ranking may violate the postulates (H*7) and (H*8), is indeed a problem. As discussed, the difficulty essentially is that some orderings are unsuitable with respect to Horn revision. The solution then is to add a constraint to faithful orderings such that these "unsuitable orderings" are ruled out. This is covered by the notion of *Horn compliance*, defined below.

As the fourth problem demonstrates, the Horn AGM postulates may fail to rule out undesirable relations on sets of worlds, which is to say, the postulate set is too weak to eliminate certain undesirable non-preorders. This then requires adding an additional postulate, which we call (Acyc), to the postulate set (H*1)–(H*8) to (semantically) further constrain the set of allowable orderings.

5.1. Initial considerations

In accordance with the previous discussion, on the one hand we add a condition to restrict rankings on worlds; on the other hand we add a postulate to the set of Horn AGM postulates.

On the semantic side, we restrict rankings to those that yield coherent results with respect to Horn revision. That is, we want to allow only those orderings where revision by a Horn formula will yield a set of worlds corresponding to a Horn formula. Call a set of worlds W *Horn elementary* iff it is definable via a Horn formula, i.e. if there is a Horn formula ϕ such that $W = [\phi]$. So W is Horn elementary iff $W = Cl_\cap(W)$. A preorder $\preceq_H$ is *Horn compliant* iff for every formula $\phi \in \mathcal{L}_H$, $\min([\phi], \preceq_H)$ is Horn elementary.

For example, the preorder in (5) is Horn compliant. Note that, while the set $\{\bar{p}q, p\bar{q}\}$ is not Horn elementary, there is no Horn formula ϕ over $\mathcal{P} = \{p, q\}$ such that $\min([\phi], \preceq_H) = \{\bar{p}q, p\bar{q}\}$. On the other hand, the ordering in (6) is not Horn compliant since $\min([p \wedge q \rightarrow \bot], \preceq_H) = \{\bar{p}q\bar{r}, p\overline{qr}\}$, and $\{\bar{p}q\bar{r}, p\overline{qr}\}$ is not Horn elementary.

With respect to postulates, we want to rule out pseudo-preorders such as is shown in Fig. 1. This problem does not arise in standard AGM revision due to the expressivity of classical propositional logic. A (very) informal argument is as follows: Consider where we are given a function that satisfies the AGM postulates, and we wish to construct a corresponding faithful ordering. In a finite language (which is what is assumed in the Katsuno and Mendelzon approach) it is straightforward to determine the relative position of two possible worlds, w_1 and w_2: Simply determine the result of revising by a formula with models given by $\{w_1, w_2\}$. If the result corresponds to (the literals in) w_1 then one has $w_1 \prec w_2$; if the result corresponds to w_2 then $w_2 \prec w_1$; otherwise $w_1 \approx w_2$. This breaks down in the Horn case because the formula corresponding to $\{w_1, w_2\}$ will in general not be Horn.

However, it proves to be the case that we can still rule out pseudo-preorders such as given in Fig. 1. That is, our language is expressive enough to enforce the condition that, if there is a $\preceq$ cycle among possible worlds, then none of the $\preceq$ relations can be strict. Or, in other words, every $\preceq$ cycle is in fact an $\approx$ cycle. This then proves sufficient (after some more work) to construct a total preorder that captures a specific revision function. To this end, we introduce the following schema:

(Acyc) If for $0 \le i < n$ we have $(H * \mu_{i+1}) + \mu_i \nvdash \bot$, and $(H * \mu_0) + \mu_n \nvdash \bot$, then $(H * \mu_n) + \mu_0 \nvdash \bot$.

Informally, (Acyc) rules out cycles (of any length n) as found for example in Fig. 1. To see this, consider the instance of (Acyc) for $n = 2$:

$$\begin{array}{ll} \text{If} & (H * \mu_1) + \mu_0 \nvdash \bot \quad \text{and} \\ & (H * \mu_2) + \mu_1 \nvdash \bot \quad \text{and} \\ & (H * \mu_0) + \mu_2 \nvdash \bot \end{array}$$

then

$$(H * \mu_2) + \mu_0 \nvdash \bot.$$

If revision is defined via Definition 2 and Eq. (2), then $(H * \mu_1) + \mu_0 \nvdash \bot$ iff $\min([\mu_1], \preceq_H) \cap [\mu_0] \neq \emptyset$. This last relation implies that, for $w_1 \in \min([\mu_1], \preceq_H)$ and $w_0 \in \min([\mu_0], \preceq_H)$ it must be that $w_0 \preceq_H w_1$. Consequently, the postulate can then be read as requiring that if $w_0 \preceq_H w_1 \preceq_H w_2 \preceq_H w_0$ then $w_0 \preceq_H w_2$, and with some further deliberation, $w_0 \approx_H w_1 \approx_H w_2$ (thus ruling out the pseudo-preorder of Fig. 1).

We note that in the presence of the AGM postulates, (Acyc) is redundant for any underlying language/logic that contains classical propositional logic:

Proposition 3. *If the underlying language/logic contains classical propositional logic, then (Acyc) is derivable from the AGM postulates (K*1)–(K*8).*

Proof. Assume that the underlying language $\mathcal{L}$ and the entailment relationship $\vdash$ contain classical propositional logic. Let K be a theory of $\mathcal{L}$. We prove (Acyc) by induction on n.

If $n = 1$, (Acyc) is trivially true.

Assume that (Acyc) holds for all $n \leq k$.

Consider now any sequence of sentences $\mu_0, \mu_1, \ldots, \mu_k, \mu_{k+1}$, such that:

$$\begin{array}{l} (K * \mu_1) + \mu_0 \nvdash \bot \\ (K * \mu_2) + \mu_1 \nvdash \bot \\ \qquad \vdots \\ (K * \mu_{k+1}) + \mu_k \nvdash \bot \\ (K * \mu_0) + \mu_{k+1} \nvdash \bot \end{array}$$

From $(K * \mu_2) + \mu_1 \nvdash \bot$ we derive, via the AGM postulates, that $K * (\mu_1 \vee \mu_2) \subseteq K * \mu_1$ (refer to properties (3.15) and (3.16) in [16]). Moreover, from $(K * \mu_1) + \mu_0 \nvdash \bot$ it follows that $\neg\mu_0 \notin K * \mu_1$. Consequently, $K * (\mu_1 \vee \mu_2) + \mu_0 \nvdash \bot$. Finally, from $(K * \mu_3) + \mu_2 \nvdash \bot$ we conclude that $(K * \mu_3) + (\mu_1 \vee \mu_2) \nvdash \bot$. Hence,

$$\begin{array}{l} \big(K * (\mu_1 \vee \mu_2)\big) + \mu_0 \nvdash \bot \\ (K * \mu_3) + (\mu_1 \vee \mu_2) \nvdash \bot \\ (K * \mu_4) + \mu_3 \nvdash \bot \\ \qquad \vdots \\ (K * \mu_{k+1}) + \mu_k \nvdash \bot \\ (K * \mu_0) + \mu_{k+1} \nvdash \bot \end{array}$$

From the induction hypothesis we then derive $(K * \mu_{k+1}) + \mu_0 \nvdash \bot$ as desired. □

While, as Proposition 3 shows, (Acyc) follows from the AGM postulates in a classical setting, when we restrict the underlying language to Horn formulas, (Acyc) becomes independent from the corresponding postulates (H*1)–(H*8). Fig. 1 suffices to show this: Defining revision according to Definition 2 yields a revision function that satisfies (H*1)–(H*8) (see Proposition 1) and yet violates (Acyc) (simply consider the revisions by the Horn formulas $\mu_0 = r$, $\mu_1 = q$, and $\mu_2 = p$).

Corollary 1. *If the underlying language/logic is Horn, then (Acyc) is independent of the (modified) AGM postulates (H*1)–(H*8).*

Finally, we note that while (Acyc) does indeed rule out undesirable orderings such as that given in Fig. 1, nonetheless there are pseudo-orderings that satisfy (Acyc) along with the postulates (H*1)–(H*8). For example, consider Fig. 2.[5]

While this pseudo-preorder has the same structure as that in Fig. 1, notably, each world in the cycle contains two negated literals rather than one negated literals, as in Fig. 1. It can be readily verified that the revision function induced by the pseudo-preorder in Fig. 2 satisfies postulates (H*1)–(H*8) and (Acyc); we omit the details.

This is not problematic for the approach, for the following reasons. In Fig. 1, the binary relation $\prec$ induces a revision function $*$ such that *every* binary relation that induces $*$ also contains the undesirable $\prec$-cycle. In Fig. 2, this is not the case. The pseudo-preorder shown in Fig. 2 induces a revision function $*'$ satisfying (H*1)–(H*8) and (Acyc), but in this case there is a (non-pseudo) total preorder that also induces $*'$ (simply erase any one of the three inequalities in the cycle). In fact,

[5] We thank Adrian Haret and Stefan Woltran for pointing out this example to us.

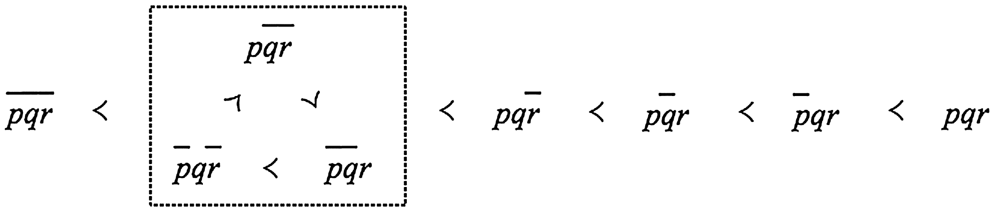

Fig. 2. Pseudo-preorder that induces function satisfying (H*1)–(H*8) and (Acyc).

the "completeness" part of the representation result given in Theorem 3 shows how to construct such a satisfying preorder in the general case.

5.2. A representation result

The notion of Horn compliance on the one hand, and the postulate (Acyc) on the other, prove to be sufficient to extend the AGM approach to capture revision in Horn theories. We obtain the following results:

Theorem 2. *Let H be a Horn belief set and $\preceq$ a Horn compliant faithful ranking associated with H. Define an operator $*: \mathcal{H} \times \mathcal{L}_H \mapsto \mathcal{H}$ by*

$$H * \phi = t_H\big(\min([\phi], \preceq)\big).$$

*Then $*$ satisfies postulates (H*1)–(H*8) and (Acyc).*

Proof. The proof is much the same as for classical AGM revision. Postulates (H*1), (H*2), and (H*6) follow immediately from the definition of $*$. For (H*3) and (H*4), if ϕ is inconsistent with H, then both postulates are trivially satisfied. If on the other hand ϕ is consistent with H, then from $\preceq$ being faithful, we derive that $\min([\phi], \preceq) = [H] \cap [\phi]$, and therefore (H*3) and (H*4) are satisfied.

For (H*5), assume that ϕ is consistent. Then clearly, since $\preceq$ is a finite total preorder, $\min([\phi], \preceq) \neq \emptyset$. Consequently, $\perp \notin t_H(\min([\phi], \preceq))$ and therefore $\perp \notin H * \phi$.

For (H*7) and (H*8), if ϕ is inconsistent with $H * \psi$, then both postulates are trivially satisfied. Assume therefore that ϕ is consistent with $H * \psi$. This entails, since $\preceq$ is Horn compliant, that $[\phi] \cap \min([\psi], \preceq) \neq \emptyset$. Hence $\min([\phi \wedge \psi], \preceq) = [\phi] \cap \min([\psi], \preceq) \neq \emptyset$ and consequently, $H * (\phi \wedge \psi) = (H * \psi) + \phi$. Thus (H*7) and (H*8) are satisfied.

For (Acyc), let $\mu_0, \mu_1, \ldots, \mu_n$, be a sequence of Horn formulas such that,

$$\begin{array}{c}
(H * \mu_1) + \mu_0 \nvdash \perp \\
(H * \mu_2) + \mu_1 \nvdash \perp \\
\vdots \\
(H * \mu_n) + \mu_{n-1} \nvdash \perp \\
\\
(H * \mu_0) + \mu_n \nvdash \perp
\end{array}$$

From $(H * \mu_1) + \mu_0 \nvdash \perp$ it follows that $[\mu_0] \cap [t_H(\min([\mu_1], \preceq))] \neq \emptyset$. Moreover, since $\preceq$ is Horn compliant, it follows that $[t_H(\min([\mu_1], \preceq))] = \min([\mu_1], \preceq)$. Hence we derive that $[\mu_0] \cap \min([\mu_1], \preceq) \neq \emptyset$, and therefore there is a μ_0-world, call it w_0', such that $w_0' \preceq w_1$, for all $w_1 \in [\mu_1]$. Similarly, from $(H * \mu_2) + \mu_1 \nvdash \perp$ we conclude that there is a $w_1' \in [\mu_1]$ such that $w_1' \preceq w_2$, for all $w_2 \in [\mu_2], \ldots,$ and from $(H * \mu_n) + \mu_{n-1} \nvdash \perp$ we conclude that there is a $w_{n-1}' \in [\mu_{n-1}]$ such that $w_{n-1}' \preceq w_n$, for all $w_n \in [\mu_n]$. Then, from transitivity we then derive that $w_0' \preceq w_n$, for all $w_n \in [\mu_n]$. Finally, from $(H * \mu_0) + \mu_n \nvdash \perp$ it follows that there is a minimal μ_0-world, call it w_0'', that satisfies μ_n. Moreover, from $w_0'' \preceq w_0' \preceq w_n$ (for all $w_n \in [\mu_n]$), it follows that w_0'' is also a minimal μ_n-world; i.e. $w_0'' \in \min([\mu_n], \preceq)$. Since $\min([\mu_n], \preceq)$ contains a μ_0-world, it follows that μ_0 is consistent with $H * \mu_n$ as desired. □

Theorem 3. *Let $*: \mathcal{H} \times \mathcal{L}_H \mapsto \mathcal{H}$ be a function satisfying postulates (H*1)–(H*8) and (Acyc). Then for fixed theory H, there is a faithful ranking $\preceq$ on $\mathcal{M}$ such that $\preceq$ is Horn compliant and $H * \phi = t_H(\min([\phi], \preceq))$.*

Proof. Let H be a Horn belief set. We shall progressively construct the preorder $\preceq$ alluded to in the statement of Theorem 3. First we define, using H and $*$, a binary relation $\preceq'$ in $\mathcal{M}$ for which we show that $[H * \mu] = \min([\mu], \preceq')$ for all Horn formulas μ. In general, $\preceq'$ is neither transitive nor total (although it is reflexive). The transitive closure of $\preceq'$, denoted $\preceq^*$, is of course a preorder, but in general it is not total. We therefore construct an extension of $\preceq^*$, denoted $\preceq$, which is total and moreover preserves the minimal μ-worlds (as defined with respect to $\preceq'$), for all Horn formulas μ. The total preorder $\preceq$ is shown to be the desired preorder.

In progressing from $\preceq'$ to $\preceq$ we shall prove a number of supplementary results that will help us establish the main line of the argument.

First some extra notation and terminology. For any two worlds w_1, w_2, we shall denote by $\phi(w_1, w_2)$ a Horn formula such that $[\phi(w_1, w_2)] = \{w_1, w_2, w_1 \cap w_2\}$.[6] We define the binary relation $\preceq'$ in $\mathcal{M}$ as follows (using infix notation):

$$w_1 \preceq' w_2 \quad \text{iff } w_1 \in \big[H * \phi(w_1, w_2)\big].$$

As usual, $\prec'$ denotes the strict part of $\preceq'$; i.e. $w_1 \prec' w_2$ iff $w_1 \preceq' w_2$ and $w_2 \not\preceq' w_1$. Moreover, for any set of worlds V, $\min(V, \preceq')$ is defined as in Section 2; i.e. $\min(V, \preceq') = \{w \in V \mid \text{for all } w' \in V, \text{ if } w' \preceq' w \text{ then } w \preceq' w'\}$.

Lemma 1. *Let $w_1, w_2 \in \mathcal{M}$ be any two worlds such that $w_1 \preceq' w_2$. Then for all Horn formulas μ, if $w_1 \in [\mu]$ and $w_2 \in [H * \mu]$, then $w_1 \in [H * \mu]$.*

Proof. Let μ be any Horn formula μ such that $w_1 \in [\mu]$, $w_2 \in [H * \mu]$. Then from (H*7) and (H*8) we derive that $H * (\mu \wedge \phi(w_1, w_2)) = (H * \mu) + \phi(w_1, w_2)$. Moreover, from $w_2 \in [H * \mu]$ and (H*2) it follows that $w_2 \in [\mu]$. Consequently, since μ is Horn, $w_1 \cap w_2 \in [\mu]$, and therefore $[\phi(w_1, w_2)] \subseteq [\mu]$. Postulate (H*6) then entails that $H * (\mu \wedge \phi(w_1, w_2)) = H * \phi(w_1, w_2)$. Hence, $H * \phi(w_1, w_2) = (H * \mu) + \phi(w_1, w_2)$. This, together with $w_1 \preceq' w_2$, entails $w_1 \in [H * \mu]$. $\square$

Lemma 2. *For all Horn formulas μ,* $\min([\mu], \preceq') = [H * \mu]$.

Proof.

LHS ⊆ RHS

Let $\mu \in \mathcal{L}_H$ be any Horn formula and assume towards a contradiction that there is a $w_1 \in \min([\mu], \preceq')$ such that $w_1 \notin [H * \mu]$. From $w_1 \in \min([\mu], \preceq')$ it follows that μ is consistent, and therefore, by (H*5), $[H * \mu] \neq \emptyset$. Let w_2 be any world in $[H * \mu]$. By Lemma 1, we then derive that $w_1 \not\preceq' w_2$. This again entails that $w_2 \not\preceq' w_1$ (for otherwise w_1 wouldn't be minimal in $[\mu]$). Let us denote by w_3 the intersection of w_1 and w_2; i.e. $w_3 = w_1 \cap w_2$. From $w_1 \not\preceq' w_2$ and $w_2 \not\preceq' w_1$ we derive, given (H*2) and (H*5), that $[H * \phi(w_1, w_2)] = \{w_3\}$ and moreover, $w_3 \subset w_1$ and $w_3 \subset w_2$.

Since $[H * \phi(w_1, w_2)] = \{w_3\}$, it follows that $\phi(w_1, w_3)$ is consistent with $H * \phi(w_1, w_2)$, and therefore, by (H*7) and (H*8), $[H * (\phi(w_1, w_2) \wedge \phi(w_1, w_3))] = [H * (\phi(w_1, w_2) + \phi(w_1, w_3))] = \{w_3\}$. Moreover, it is not hard to see that $\phi(w_1, w_3) \vdash \phi(w_1, w_2)$, and therefore, by (H*6), $H * (\phi(w_1, w_2) \wedge \phi(w_1, w_3)) = H * \phi(w_1, w_3)$. Thus $[H * \phi(w_1, w_3)] = \{w_3\}$ and consequently, $w_3 \prec' w_1$. Moreover, from $w_1, w_2 \in [\mu]$, and since μ is Horn, it follows that $w_3 \in [\mu]$. This however contradicts our assumption that w_1 is minimal in $[\mu]$ with respect to $\preceq'$.

RHS ⊆ LHS

Let $\mu \in \mathcal{L}_H$ be any Horn formula and let w_1 be any world in $[H * \mu]$. We show that $w_1 \preceq' w_2$ for all $w_2 \in [\mu]$. Let w_2 be any world in $[\mu]$. Clearly, since $w_1 \in [H * \mu]$, $\phi(w_1, w_2)$ is consistent with $(H * \mu)$, and consequently, by (H*7) and (H*8), $H * (\mu \wedge \phi(w_1, w_2)) = (H * \mu) + \phi(w_1, w_2)$. On the other hand, since μ is Horn, the world $w_3 = w_1 \cap w_2$ also belongs to $[\mu]$, and therefore $\phi(w_1, w_2) \vdash \mu$. Hence by (H*6), $H * \phi(w_1, w_2) = H * (\mu \wedge \phi(w_1, w_2)) = (H * \mu) + \phi(w_1, w_2)$. Consequently, from $w_1 \in [H * \mu]$ we derive that $w_1 \in [H * \phi(w_1, w_2)]$, and therefore, $w_1 \preceq' w_2$. Since w_2 was chosen arbitrarily, it follows that $w_1 \in \min([\mu], \preceq')$. $\square$

Lemma 3. *If $w_1 \preceq' w_2 \preceq' \ldots \preceq' w_n \preceq' w_1$ then $w_1 \preceq' w_n$.*

Proof. If $n = 1$, the lemma is trivially true.

Let $w_1, w_2, \ldots, w_n$ be any sequence of worlds, with $n > 1$, such that $w_1 \preceq' w_2 \preceq' \ldots \preceq' w_n \preceq' w_1$.

Then $w_1 \in [H * \phi(w_1, w_2)]$, $w_2 \in [H * \phi(w_2, w_3)], \ldots, w_{n-1} \in [H * \phi(w_{n-1}, w_n)]$, and $w_n \in [H * \phi(w_1, w_n)]$. Hence,

$$\begin{aligned}
H * \phi(w_2, w_3) \ &+\ \phi(w_1, w_2) \nvdash \bot \\
&\ \vdots \\
H * \phi(w_{n-1}, w_n) \ &+\ \phi(w_{n-2}, w_{n-1}) \nvdash \bot \\
H * \phi(w_1, w_n) \ &+\ \phi(w_{n-1}, w_n) \nvdash \bot \\
\\
H * \phi(w_1, w_2) \ &+\ \phi(w_1, w_n) \nvdash \bot
\end{aligned}$$

Then by (Acyc) we derive that $(H * \phi(w_1, w_n)) + \phi(w_1, w_2) \nvdash \bot$. Consequently, by (H*7) and (H*8), $H * (\phi(w_1, w_n) \wedge \phi(w_1, w_2)) = (H * \phi(w_1, w_n)) + \phi(w_1, w_2)$. Moreover, from $(H * \phi(w_1, w_2)) + \phi(w_1, w_n) \nvdash \bot$ and (H*7)–(H*8) we derive

[6] Note that the set $\{w_1, w_2, w_1 \cap w_2\}$ is closed under intersection and therefore there is always a Horn formula ϕ such that $[\phi(w_1, w_2)] = \{w_1, w_2, w_1 \cap w_2\}$. Moreover observe that in the limiting case where $w_1 \subseteq w_2$ (or $w_2 \subseteq w_1$), the set $\{w_1, w_2, w_1 \cap w_2\}$ reduces to $\{w_1, w_2\}$.

$H * (\phi(w_1, w_n) \wedge \phi(w_1, w_2)) = (H * \phi(w_1, w_2)) + \phi(w_1, w_n)$. Hence, from $w_1 \preceq' w_2$, it follows that $w_1 \in [H * (\phi(w_1, w_n) \wedge \phi(w_1, w_2))]$. Consequently, from $H * (\phi(w_1, w_n) \wedge \phi(w_1, w_2)) = (H * \phi(w_1, w_n)) + \phi(w_1, w_2)$ we conclude that $w_1 \in [H * \phi(w_1, w_n)]$, and therefore $w_1 \preceq' w_n$. □

Let us now define $\preceq^*$ to be the transitive closure of $\preceq'$; i.e. $w \preceq^* w'$ iff there exist worlds $u_1, \dots, u_n$, such that $w \preceq' u_1 \preceq' \cdots \preceq' u_n \preceq' w'$. By construction, $\preceq^*$ is reflexive and transitive; i.e. $\preceq^*$ is a partial preorder.

For a set of worlds $S \subseteq \mathcal{M}$, we shall say that w is *maximal* in S with respect to $\preceq^*$ iff $w \in S$, and for all $w' \in S$, if $w \preceq^* w'$ then $w' \preceq^* w$. We shall denote the set of all maximal worlds in S (with respect to $\preceq^*$) by $\max(S, \preceq^*)$. The following auxiliary results establish some interesting properties of $\preceq^*$:

Lemma 4. *For any $w, w' \in \mathcal{M}$, if $w \prec' w'$ then $w \prec^* w'$.*

Proof. Let $w, w' \in \mathcal{M}$ be any two worlds such that $w \prec' w'$. Assume towards a contradiction that $w' \preceq^* w$. Then there exist $u_1, \dots, u_n \in \mathcal{M}$ such that $w' \preceq' u_1 \preceq' \cdots \preceq' u_n \preceq' w$. Consequently, $w' \preceq' u_1 \preceq' \cdots \preceq' u_n \preceq' w \prec' w'$. This however contradicts Lemma 3. □

Lemma 5. *For every nonempty set of worlds S, $\max(S, \preceq^*) \neq \emptyset$.*

Proof. Let S be any nonempty set of worlds. Assume towards a contradiction that $\max(S, \preceq^*) = \emptyset$. Then for every world $w \in S$ there is a $w' \in S$ such that $w \prec^* w'$. Hence, we can start with any world $w_0 \in S$, and build a sequence of worlds $w_0, w_1, w_2, \ldots \in S$, such that, $w_0 \prec^* w_1 \prec^* w_2 \prec^* \ldots$. Given that there are only finitely many worlds, we will eventually reach a w_j that also appears earlier in the sequence; i.e. for some $i < j$, $w_i = w_j$. Hence we get a *cycle* $w_i \prec^* w_{i+1} \prec^* \cdots \prec^* w_{i+k} \prec^* w_i$. From the definition of $\preceq^*$ and Lemma 3 we then derive that $w_i \preceq' w_{i+k}$. Consequently, $w_i \preceq^* w_{i+k}$. This of course contradicts $w_{i+k} \prec^* w_i$. □

As mentioned in the beginning of the proof, the next step is to define an extension of $\preceq^*$ which is total, and preserves the μ-minimal worlds, for all Horn formulas μ. To this end, consider the sequence of sets of worlds $S_0, S_1, \ldots,$ defined recursively as follows:

$$\begin{aligned} S_0 &= \max(\mathcal{M}, \preceq^*) \\ S_1 &= \max(\mathcal{M} - S_0, \preceq^*) \\ &\vdots \\ S_{i+1} &= \max\left(\mathcal{M} - \bigcup_{j=0}^{i} S_j, \preceq^*\right) \\ &\vdots \end{aligned}$$

Since there are finitely many worlds in $\mathcal{M}$, from Lemma 5 we derive that the sequence S_0, S_1, ... will eventually reach the empty set, and will stay empty from then onwards. We shall denote by m the index of the last nonempty set in the sequence; i.e. $S_m \neq \emptyset$ and $S_{m+i} = \emptyset$ for all $i \geq 1$. It is not hard to see that the sets $S_0, \ldots, S_m$ form a partition of $\mathcal{M}$. Based on this partition $\{S_i\}_{i \in [0,m]}$, we finally define the binary relation $\preceq \subseteq \mathcal{M} \times \mathcal{M}$ as follows:

$$w \preceq w' \quad \text{iff there are sets } S_i \text{ and } S_j \text{ such that } w \in S_i,\ w' \in S_j, \text{ and } i \geq j$$

Clearly, since $\{S_i\}_{i \in [0,m]}$ is a partition of $\mathcal{M}$, $\preceq$ is a total preorder. Also, it is not hard to verify that $\preceq$ is an extension of $\preceq^*$. More importantly however, as shown by the following lemma, $\preceq$ preserves the μ-minimal worlds, for all Horn formulas μ:

Lemma 6. *For all Horn formulas μ, $\min([\mu], \preceq) = \min([\mu], \preceq')$.*

Proof. Let μ be any Horn formula. If $\mu \vdash \bot$, then the lemma is trivially true. Assume therefore that μ is consistent. Let S_k be the last set in the sequence $S_0, \ldots, S_m$ that intersects $[\mu]$. It is not hard to verify that $\min([\mu], \preceq) = S_k \cap [\mu]$. Hence we need to show that $S_k \cap [\mu] = \min([\mu], \preceq')$.

LHS $\subseteq$ *RHS*

Assume that $w \in S_k \cap [\mu]$ and suppose towards a contradiction that $w \notin \min([\mu], \preceq')$. Then for some $w_0 \in [\mu]$, $w_0 \prec' w$. Lemma 4 then entails that $w_0 \prec^* w$. Then, w_0 is not maximal (with respect to $\preceq^*$) in any set containing w, including of course S_k. Moreover, since S_k is the last set in the sequence $S_0, \ldots, S_m$ that intersects $[\mu]$, it follows that $w_0 \in S_j$ for some $j < k$. Hence w_0 is maximal (with respect to $\preceq^*$) in $\mathcal{M} - \bigcup_{i=0}^{j-1} S_i$, and therefore, since $w_0 \prec' w$, we conclude that $w \notin \mathcal{M} - \bigcup_{i=0}^{j-1} S_i$. This however contradicts $w \in S_k$ and $k > j$.

RHS $\subseteq$ *LHS*

Assume that $w_0 \in \min([\mu], \preceq')$ and suppose towards a contradiction that $w_0 \notin S_k \cap [\mu]$. Then, clearly, $w_0 \in [\mu]$ and $w_0 \notin S_k$. Moreover, since S_k is the last set in the sequence $S_0, \ldots, S_m$ intersecting $[\mu]$, it follows that $w_0 \in S_j$ for some $j < k$. Let w_1 be any world in $S_k \cap [\mu]$. Since $j < k$ we derive that $w_1 \in \mathcal{M} - \bigcup_{i=0}^{j-1} S_i$, and since w_0 is maximal in $\mathcal{M} - \bigcup_{i=0}^{j-1} S_i$, we have that $w_0 \not\prec^* w_1$ and consequently, by Lemma 4, $w_0 \not\prec' w_1$. Next we show that $w_0 \in [H * \phi(w_0, w_1)]$.

Assume on the contrary that this is not the case. Then $w_0 \not\preceq' w_1$, and since w_0 is minimal in $[\mu]$, we derive that $w_1 \not\preceq' w_0$, which in turn entails $w_1 \notin [H * \phi(w_0, w_1)]$. That is, neither w_0 nor w_1 belong to $[H * \phi(w_0, w_1)]$. Let w_2 be the world such that $w_2 = w_0 \cap w_1$. From $w_0, w_1 \notin [H * \phi(w_0, w_1)]$, and given (H*2), (H*5), we derive that $[H * \phi(w_0, w_1)] = \{w_2\}$. Moreover it clearly follows that $w_2 \subset w_0$ and $w_2 \subset w_1$.

From $[H * \phi(w_0, w_1)] = \{w_2\}$ and postulates (H*7) and (H*8), we derive that $H * (\phi(w_0, w_1) \wedge \phi(w_0, w_2)) = (H * \phi(w_0, w_1)) + \phi(w_0, w_2) = \{w_2\}$. Next observe that, since $w_2 = w_0 \cap w_1$, it follows that $\phi(w_0, w_2) \vdash \phi(w_0, w_1)$, and therefore, by (H*6), $H * (\phi(w_0, w_1) \wedge \phi(w_1, w_2)) = H * \phi(w_0, w_2)$. Consequently, $[H * \phi(w_0, w_2)] = \{w_2\}$, and therefore $w_2 \prec' w_0$. Recall however that μ is Horn, and therefore from $w_0, w_1 \in [\mu]$, it follows that $w_2 \in [\mu]$, and consequently, $w_2 \prec' w_0$ contradicts the minimality of w_0 in $[\mu]$. Thus we have shown that $w_0 \in [H * \phi(w_0, w_1)]$, and consequently, $w_0 \preceq' w_1$.

We are only one step away from reaching a contradiction and thus proving the lemma. This final step is to show that w_1 is maximal in $\mathcal{M} - \bigcup_{i=0}^{j-1} S_i$. Indeed, consider any world $w_3 \in \mathcal{M} - \bigcup_{i=0}^{j-1} S_i$, such that $w_1 \preceq^* w_3$. Then, $w_0 \preceq' w_1 \preceq^* w_3$ implies $w_0 \preceq^* w_3$. Since w_0 is maximal in $\mathcal{M} - \bigcup_{i=0}^{j-1} S_i$, we then derive that $w_3 \preceq^* w_0$. Hence from $w_0 \preceq' w_1$ we conclude that $w_3 \preceq^* w_1$. Since w_3 was chosen arbitrarily, it follows that w_1 is maximal in $\mathcal{M} - \bigcup_{i=0}^{j-1} S_i$.

This however makes w_1 an element of S_j, and since $j < k$, we derive a contradiction with $w_1 \in S_k$. $\square$

We can now put all the pieces together to complete the proof of the theorem. From Lemma 6 and Lemma 2 we have that the total preorder $\preceq$ is such that $[H * \mu] = \min([\mu], \preceq)$, for all Horn formulas μ. This makes $\preceq$ Horn compliant. Finally, notice that $\preceq$ is also faithful with respect to H. Indeed, if H is inconsistent, this is trivially the case. Assume therefore that H is consistent, and let p be any propositional variable. By (H*3) and (H*4) we have that $H * (p \to p) = H$. Hence, $[H] = \min([p \to p], \preceq) = \min(\mathcal{M}, \preceq)$. Hence $\preceq$ is faithful with respect to H. $\square$

6. Iteration and Horn belief revision

The results of the previous section prove that the classical AGM postulates can be recast in a Horn framework. In this section we show that this is also the case for the Darwiche and Pearl postulates for iterated revision [8].

First a note on Horn compliance. It was shown in [31] that Horn compliance is equivalent to the following condition:

(H$\preceq$) If $w \approx w'$ then $(w \cap w') \preceq w$.

Condition (H$\preceq$) says that whenever two worlds w and w' are equidistant from the start of a preorder $\preceq$, then the world $w \cap w'$ resulting from their intersection cannot appear later in the preorder. Zhuang and Pagnucco show that any preorder that satisfies (H$\preceq$) is Horn compliant, and conversely, every Horn compliant preorder satisfies (H$\preceq$). Observe that (H$\preceq$) makes no reference to an input Horn formula μ. It is therefore a useful characterization of Horn compliance that will be used extensively in the rest of the article.

The postulates proposed by Darwiche and Pearl for iterated revision, call them the *DP postulates*, are shown below.[7]

(DP1) If $\phi \vdash \psi$ then $(K * \psi) * \phi = K * \phi$.
(DP2) If $\phi \vdash \neg\psi$ then $(K * \psi) * \phi = K * \phi$.
(DP3) If $\psi \in K * \phi$ then $\psi \in (K * \psi) * \phi$.
(DP4) If $\neg\psi \notin K * \phi$ then $\neg\psi \notin (K * \psi) * \phi$.

We recall that the DP postulates have been characterized by corresponding restrictions on faithful rankings. In particular, let H be a belief set and $\preceq$ a faithful ranking with respect to H. Moreover, let us denote by $\preceq_\mu$ the total preorder assigned to the belief set $H * \mu$ resulting from the revision of H by μ. In [8] it was shown that the conditions (IR1)–(IR4) below characterize (DP1)–(DP4) respectively:

(IR1) If $w, w' \in [\mu]$ then $w \prec_\mu w'$ iff $w \prec w'$.
(IR2) If $w, w' \in [\neg\mu]$ then $w \prec_\mu w'$ iff $w \prec w'$.

[7] We note that the symbol K in (DP1)–(DP4) denotes a *belief state* rather than a *belief set* (see [8] for details). Although this is an important distinction, it will not affect the discussion herein since we will be working with the semantic characterization of the DP postulates—see conditions (IR1)–(IR4) below—rather than with the DP postulates themselves.

(IR3) If $w \in [\mu]$ and $w' \in [\neg\mu]$ then $w \prec w'$ entails $w \prec_\mu w'$.
(IR4) If $w \in [\mu]$ and $w' \in [\neg\mu]$ then $w \preceq w'$ entails $w \preceq_\mu w'$.

Thus to show that (DP1)–(DP4) are consistent with (H*1)–(H*8) and (Acyc), it suffices to prove the following result:

Theorem 4. *Let H be a Horn belief set, and $\preceq$ a Horn compliant, faithful ranking with respect to H. Moreover, let $*$ be the Horn revision function induced from $\preceq$ via Definition 2. For every Horn formula μ, there exists a Horn compliant, total preorder $\preceq_\mu$, that is faithful with respect to $H * \mu$, and such that (IR1)–(IR4) are satisfied.*

Proof. Let μ be any Horn formula. Consider first the case where μ is inconsistent. Define $\preceq_\mu$ to be equal to $\preceq$. Clearly, in this case $\preceq_\mu$ satisfies (IR1)–(IR4). Moreover, since $\preceq$ is Horn compliant, so is $\preceq_\mu$. Finally for faithfulness, since μ is inconsistent, by (H*2), $[H * \mu] = \emptyset$ and therefore $\preceq_\mu$ is trivially faithful with respect to $H * \mu$. Hence the theorem is true when μ is inconsistent.

Assume now that μ is consistent. We define $\preceq_\mu$ as follows:

$$w \preceq_\mu w' \quad \text{iff } w \in \min\big([\mu], \preceq\big) \quad \text{or} \quad w \preceq w' \quad \text{and} \quad w' \notin \min\big([\mu], \preceq\big).$$

According to the above definition, to construct $\preceq_\mu$, one starts with $\preceq$ and simply places the minimal μ-worlds (with respect to $\preceq$) at the beginning of the ranking; everything else remains the same. We note that this construction is not new. It has been proposed by Boutilier [4,5], in his treatment of iterated revision, and it is known to satisfy (IR1)–(IR4) [8]. Moreover, $\preceq_\mu$ is clearly faithful with respect to $H * \mu$. Finally for Horn compliance, let w, w' be any two worlds such that $w \approx_\mu w'$. Since (H$\preceq$) entails Horn compliance, it suffices to show that $(w \cap w') \preceq_\mu w$.

We distinguish between two cases. First assume that $w \in \min([\mu], \preceq)$. From $w \approx_\mu w'$ we derive that $w' \in \min([\mu], \preceq)$. Since $\preceq$ is Horn compliant, this entails that $(w \cap w') \in \min([\mu], \preceq)$, and therefore, by the construction of $\preceq_\mu$, $(w \cap w') \preceq_\mu w$ as desired.

Next assume that $w \notin \min([\mu], \preceq)$. Then, by construction, w' also does not belong to $\min([\mu], \preceq)$. Therefore from $w \approx_\mu w'$ we derive that $w \approx w'$. Since $\preceq$ is Horn compliant, by (H$\preceq$) it follows that $(w \cap w') \preceq w$. Consequently, since $w \notin \min([\mu], \preceq)$, $(w \cap w') \preceq_\mu w$. $\square$

7. Relevance and Horn belief revision

Apart from iteration, the notion of *relevance* was also not fully addressed by the AGM postulates. It was as tackled later with the addition of a new axiom due to Parikh [22]. In this section we shall examine the compatibility of Horn revision with Parikh's axiom for relevance.

In [22], Parikh argues that the following intuitive rule should be adhered to during the revision of a theory H by a sentence ϕ: the beliefs in H that are not relevant to ϕ should not be affected. To formally encode this rule, Parikh proposed a new axiom, which in [23] was shown to be equivalent to the following condition:[8]

(R1) If $H = Cn(\chi, \psi)$, $P_\chi \cap P_\psi = \emptyset$, and $\phi \in \mathcal{L}_\chi$, then $(H * \phi) \cap \overline{\mathcal{L}_\chi} = H \cap \overline{\mathcal{L}_\chi}$.

Some explanations are due. For a sentence γ, by P_γ we denote the minimum set of propositional variables in which γ can be expressed, and by $\mathcal{L}_\gamma$ we denote the propositional language built over $P_\gamma \cup \{\bot\}$.[9] Moreover, by $\overline{\mathcal{L}_\gamma}$ we denote the propositional language built over $(\mathcal{P} - P_\gamma) \cup \{\bot\}$.

Condition (R1) essentially encodes the intuitive rule discussed earlier for the special case of a "composite" belief set H: if H can be split into two disjoint parts χ, ψ, and the new information ϕ belongs entirely to the minimum language of the first part χ, then the second part is not affected by the revision of H by ϕ.

Condition (R1) was characterised semantically in [23] in terms of two constraints on faithful assignments named (Q1) and (Q2). We shall use this characterisation to prove the consistency between (R1) and (H1)–(H8), (Acyc). First however we need some additional notation related to the *difference* between a theory and a possible world.

Let H be a theory and let $Q = \{Q_1, \ldots, Q_n\}$ be a partition of the set P of propositional variables; i.e. $\bigcup Q = P$, $Q_i \neq \emptyset$, and $Q_i \cap Q_j = \emptyset$, for all $1 \le i \neq j \le n$. We say that $Q = \{Q_1, \ldots, Q_n\}$ is an *H-splitting* iff there exist sentences $\phi_1 \in \mathcal{L}^{Q_1}, \ldots, \phi_n \in \mathcal{L}^{Q_n}$, such that $H = Cn(\phi_1, \ldots, \phi_n)$.[10] Parikh has shown in [22] that for every theory H there is a unique *finest* H-splitting, i.e. one which refines every other H-splitting.[11]

[8] The format of axiom (R1) used herein is slightly different from that used in [23], but the two are clearly equivalent. It should be noted that there are in fact two readings of Parikh's axiom for relevance, a weak version and a strong version (see [23] for a discussion). Herein we focus on the weak version of Parikh's axiom.

[9] It was shown in [22] that for a sentence γ, P_γ is *unique*.

[10] For a set of propositional variables Q, by $\mathcal{L}^Q$ we denote the propositional language built from Q and the special symbol $\bot$.

[11] A partition Q' refines another partition Q, iff for every $Q'_i \in Q'$ there is $Q_j \in Q$, such that $Q'_i \subseteq Q_j$.

Consider now a world w and a theory H whose finest H-splitting is F. The difference, $\mathit{Diff}(H, w)$, between H and w is defined as follows [23]:

$$\mathit{Diff}(H, w) = \bigcup \{ F_i \in F \mid \text{for some } \phi \in \mathcal{L}^{F_i},\ H \models \phi \text{ and } w \models \neg\phi \}$$

Observe that whenever H is not a complete theory, there can be two (or more) distinct worlds with the same difference from H. Consider for example the Horn theory $H = Cn(\{p \leftrightarrow q, r \leftrightarrow u\})$, whose finest splitting is, clearly, the set $F = \{\{p, q\}, \{r, u\}\}$. Then the worlds $w_1 = p\bar{q}ru$ and $w_2 = \bar{p}qru$ have the same difference from H; namely, $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w_2) = \{p, q\}$.

Notice that w_1, w_2 have also another thing in common: they agree in all variables *outside* their difference from H. On the other hand, the worlds $w_3 = pq\bar{r}u$ and $w_4 = \overline{pqr}u$, which also have the same difference from H, namely $\{r, u\}$, disagree on the variables outside $\{r, u\}$. Yet w_3 and w_4 agree on the variables *inside* their (common) difference from H. We call the first pair of worlds w_1, w_2 *external H-duals*, and the second pair w_3, w_4, *internal H-duals*.

More precisely, given a theory H, two worlds w_1, w_2 are called *external H-duals* iff $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w_2)$ and for all $x \in \mathcal{P} - \mathit{Diff}(H, w_1)$, $w_1 \models x$ iff $w_2 \models x$. Moreover, two worlds w_1, w_2 are called *internal H-duals* iff $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w_2)$ and for all $x \in \mathit{Diff}(H, w_1)$, $w_1 \models x$ iff $w_2 \models x$. See [23] for a discussion on internal and external duals.

The final notion that we need to recall from [23] is that of a *cover*. Given a theory H and any three worlds w_1, w_2, w_3, we say that w_3 is a *w_2-cover* of w_1 iff $\mathit{Diff}(H, w_1) \subset \mathit{Diff}(H, w_2)$, w_1 and w_3 are external H-duals, and moreover, for all $x \in \mathit{Diff}(H, w_1)$, $w_3 \models x$ iff $w_2 \models x$.

To give an example of a cover, consider again the theory $H = Cn(\{p \leftrightarrow q, r \leftrightarrow u\})$, and let w_1, w_2, w_3 be the worlds $w_1 = p\bar{q}ru$, $w_2 = \bar{p}qr\bar{u}$, and $w_3 = \bar{p}qru$. Clearly w_1 and w_3 are external H-duals. Moreover, $\mathit{Diff}(H, w_1) = \{p, q\}$, $\mathit{Diff}(H, w_2) = \{p, q, r, u\}$, and therefore $\mathit{Diff}(H, w_1) \subset \mathit{Diff}(H, w_2)$. Finally notice that w_3 agrees with w_2 on the values of p, q. Hence w_3 is a *w_2-cover* of w_1. Refer to [23] for a discussion on the notion of a cover.

It was shown in [23] that for a theory H and a total preorder $\preceq$ that is faithful with respect to H, the revision function $*$ induced from $\preceq$ satisfies (R1) iff $\preceq$ satisfies the following two conditions:[12]

(Q1) If $\mathit{Diff}(H, w_1) \subset \mathit{Diff}(H, w_2)$ then there is a world w_3 that is a w_2-cover of w_1, such that $w_3 \prec w_2$.
(Q2) If w_1 and w_2 are internal H-duals, then $w_1 \approx w_2$.

With the aid of this characterisation of (R1), Theorem 5 below shows that Parikh's axiom for relevance is consistent with Horn revision:

Theorem 5. *Let H be a consistent Horn theory. There exists a revision function $*$ that at H satisfies all the postulates (H^*1)–(H^*8), (Acyc), and (R1).*

Proof. To prove the theorem it suffices to construct a total preorder $\preceq$, satisfying ($H\preceq$), (Q1)–(Q2), and such that $\min(\mathcal{M}, \preceq) = [H]$. We construct $\preceq$ with the aid of an auxiliary preorder order $\ll$ defined below.

Let F be the (unique) finest H-splitting, and let $F_1 \ldots F_k$ be an enumeration of the elements of F. For a world w, let us denote by $\Delta(H, w)$ the elements of F that are contained in $\mathit{Diff}(H, w)$; i.e.

$$\Delta(H, w) = \{ F_i \in F \mid F_i \subseteq \mathit{Diff}(H, w) \}.$$

Let us call the set $\Delta(H, w)$ the *disparity* of w from H. Finally, let $p_1, \ldots, p_m$ be an enumeration of the propositional variables in $\mathcal{P}$.

We define $\ll$ to be the following binary relation in $\mathcal{M}$:

$w_1 \ll w_2$ iff:

(a) $|\Delta(H, w_1)| < |\Delta(H, w_2)|$, or
(b) $|\Delta(H, w_1)| = |\Delta(H, w_2)|$, and for some $1 \le i \le k$, $\{F_1, \ldots, F_{i-1}\} \subseteq \Delta(H, w_1) \cap \Delta(H, w_2)$, $F_i \in \Delta(H, w_1)$, and $F_i \notin \Delta(H, w_2)$, or
(c) $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w_2)$, and for some $1 \le i \le n$, $\{p_1, \ldots, p_{i-1}\} \subseteq w_1 \cap w_2 \cap \mathit{Diff}(H, w_1)$, $p_i \in \mathit{Diff}(H, w_1)$, $p_i \in w_1$, and $p_i \notin w_2$.

Intuitively the above definition works in three steps as follows: first worlds are ordered according to the cardinality of their disparity from H (condition (a)). Then, worlds with the same cardinality of disparity are ordered lexicographically wrt the enumeration $F_1 \ldots F_k$ (condition (b)). Finally, worlds with exactly the same disparity are ordered lexicographically wrt the enumeration $p_1 \ldots p_n$ (condition (c)).

[12] To be precise, the representation result in [23] is expressed in terms of systems of spheres, but it can be easily translated in terms of faithful total preorders as presented herein.

From the definition of $\ll$ it follows immediately that $\ll$ is irreflexive. Next we show that $\ll$ is also transitive.

Let w_1, w_2, w_3 be any three worlds such that $w_1 \ll w_2 \ll w_3$. By the definition of $\ll$ it follows that $|\Delta(H, w_1)| \leq |\Delta(H, w_2)| \leq |\Delta(H, w_3)|$. Hence, if $|\Delta(H, w_1)| < |\Delta(H, w_2)|$ or $|\Delta(H, w_2)| < |\Delta(H, w_3)|$ then $|\Delta(H, w_1)| < |\Delta(H, w_3)|$ and consequently $w_1 \ll w_3$ as desired. Hence assume that $|\Delta(H, w_1)| = |\Delta(H, w_2)| = |\Delta(H, w_3)|$. We need to examine four cases depending on which of the conditions (b) or (c) are satisfied by the two pair of worlds w_1, w_2, and w_2, w_3.

For the first case assume that both pairs of worlds satisfy (b); i.e. $\Delta(H, w_1) \neq \Delta(H, w_2)$, $\Delta(H, w_2) \neq \Delta(H, w_3)$, the first $F_i \in F$ in which $\Delta(H, w_1)$ and $\Delta(H, w_2)$ differ belongs to $\Delta(H, w_1)$, and the first $F_j \in F$ in which $\Delta(H, w_2)$ and $\Delta(H, w_3)$ differ belongs to $\Delta(H, w_2)$. Since $F_i \notin \Delta(H, w_2)$ and $F_j \in \Delta(H, w_2)$, clearly $i \neq j$. If $j < i$, then F_j is also the first member of F in which $\Delta(H, w_1)$ and $\Delta(H, w_3)$ differ and moreover $F_j \in w_1$; hence $w_1 < w_3$ as desired. If on the other hand $i < j$, then from $F_i \notin \Delta(H, w_2)$ it follows that $F_i \notin \Delta(H, w_3)$, and therefore F_i is the first member of F in which $\Delta(H, w_1)$ and $\Delta(H, w_3)$ differ. Given that $F_i \in \Delta(H, w_1)$ we once again derive that $w_1 \ll w_3$ as desired.

For the second case, assume that w_1, w_2 satisfy (b), and w_2, w_3 satisfy (c). Let F_i be the first member of F in which $\Delta(H, w_1)$ and $\Delta(H, w_2)$ differ. Then since $\Delta(H, w_2) = \Delta(H, w_3)$, it follows that the first member of F in which $\Delta(H, w_1)$ and $\Delta(H, w_3)$ differ is also F_i, which moreover it belongs to $\Delta(H, w_1)$. Hence $w_1 \ll w_3$ as desired.

For the third case, assume that w_1, w_2 satisfy (c), and w_2, w_3 satisfy (b). Let F_i be the first member of F in which $\Delta(H, w_2)$ and $\Delta(H, w_3)$ differ. Then since $\Delta(H, w_1) = \Delta(H, w_2)$, it follows that the first member of F in which $\Delta(H, w_1)$ and $\Delta(H, w_3)$ differ is also F_i, which moreover it belongs to $\Delta(H, w_1)$. Hence, once again, $w_1 \ll w_3$ as desired.

Finally for the forth case assume that both pairs of worlds satisfy (c). Then $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w_2) = \mathit{Diff}(H, w_3)$. Let p_i be the first variable in $\mathit{Diff}(H, w_1)$ that has a different value in w_1 and w_2, and let p_j be the first value in $\mathit{Diff}(H, w_2)$ that has a different value in w_2 and w_3. From $w_1 \ll w_2 \ll w_3$ we have that $p_i \in w_1$, $p_i \notin w_2$, $p_j \in w_2$, and $p_j \notin w_3$. Clearly then $i \neq j$. If $i < j$, then it is not hard to see that p_i is the first variable in $\mathit{Diff}(H, w_1)$ that has a different value in w_1 and w_3. Hence from $p_i \in w_1$ it follows that $w_1 \ll w_3$. If on the other hand $j < i$ then given that, up to p_i, w_1 agrees with w_2 (within $\mathit{Diff}(H, w_1)$), it follows that p_j is the first variable in $\mathit{Diff}(H, w_1)$ that has a different value in w_1 and w_3, and moreover $p_j \in w_1$. Once again we then derive that $w_1 \ll w_3$.

This concludes the proof of the transitivity of $\ll$. Transitivity together with irreflexivity entail that $\ll$ is acyclic.

Next we define recursively the sequence $S_0, S_1, \ldots$ of sets of worlds as follows:

$$\begin{aligned}
S_0 &= \{w \in \mathcal{M} : \text{ for all } w' \in \mathcal{M},\ w' \not\ll w\}, \\
S_1 &= \{w \in (\mathcal{M} - S_0) : \text{ for all } w' \in (\mathcal{M} - S_0),\ w' \not\ll w\}, \\
&\vdots \\
S_{i+1} &= \left\{ w \in \left(\mathcal{M} - \bigcup_{j=0}^{i} S_j \right) : \text{ for all } w' \in \left(\mathcal{M} - \bigcup_{j=0}^{i} S_j \right),\ w' \not\ll w \right\}, \\
&\vdots
\end{aligned}$$

A few words about the above sequence before we proceed with the definition of $\preceq$.

Clearly by construction, the sets in the above sequence are pairwise disjoint. Moreover, since $\ll$ is acyclic, it follows that for any nonempty set of worlds Q, the set $\{w \in Q: \text{ for all } w' \in Q,\ w' \not\ll w\}$ is also nonempty. This again entails that there is an $m \in \aleph$ such that for all $0 \leq i \leq m$, $S_i \neq \emptyset$ and $\bigcup_{j=0}^{m} S_j = \mathcal{M}$. In other words, $\{S_0, \ldots, S_m\}$ is a partition of $\mathcal{M}$; we shall call the elements of this partition *cells*.

Next we show that $S_0 = [H]$. Since H is assumed to be consistent, then $[H] \neq \emptyset$. Let w_1 by any world in $[H]$. Clearly $\mathit{Diff}(H, w_1) = \emptyset$ and therefore by the definition of $\ll$ there is no world w_2 such that $w_2 \ll w_1$; hence $[H] \subseteq S_0$. For the converse let w_2 be any world in $\mathcal{M} - [H]$. Then $\mathit{Diff}(H, w_2) \neq \emptyset$ and therefore $\emptyset = |\Delta(H, w_1)| < |\Delta(H, w_2)|$ for any $w_1 \in [H]$. This again entails $w_1 \ll w_2$ and consequently $w_2 \notin S_0$. Hence $S_0 \subseteq [H]$ as desired.

We can now define the preorder $\preceq$ alluded in the beginning of the proof as follows:

$$w_1 \preceq w_2 \quad \text{iff for some } 0 \leq i, j \leq m, w_1 \in S_i, w_2 \in S_j, \text{ and } i \leq j.$$

Since $\{S_0, \ldots, S_m\}$ is a partition of $\mathcal{M}$, then by construction $\preceq$ is a total preorder in $\mathcal{M}$. Moreover, since $S_0 = [H]$, it follows that $\preceq$ is faithful wrt H. Hence to conclude the proof of the theorem we need to show that $\preceq$ satisfies (Q1), (Q2), and (H$\preceq$).

For (Q1), consider any two worlds w_1, w_2 such that $\mathit{Diff}(H, w_1) \subset \mathit{Diff}(H, w_2)$. Then clearly, $|\Delta(H, w_1)| < |\Delta(H, w_2)|$. Define w_3 to be the world that agrees with w_2 in all variables in $\mathit{Diff}(H, w_1)$, and it agrees with w_1 outside $\mathit{Diff}(H, w_1)$; i.e. $\mathit{Diff}(H, w_1) \cap w_3 = \mathit{Diff}(H, w_1) \cap w_2$ and $(\mathcal{P} - \mathit{Diff}(H, w_1)) \cap w_3 = (\mathcal{P} - \mathit{Diff}(H, w_1)) \cap w_1$. Clearly w_3 is a w_2-cover of w_1. Moreover by construction, $\mathit{Diff}(H, w_3) = \mathit{Diff}(H, w_1)$, and therefore $|\Delta(H, w_3)| < |\Delta(H, w_2)|$. Consequently $w_3 \ll w_2$. Let S_i and S_j be the cells that the two worlds w_3 and w_2 respectively belong. Since $w_3 \ll w_2$ it follows that $i < j$. Hence $w_3 \prec w_2$ as desired.

For (Q2), let w_1, w_2 be two internal H-duals. It suffices to show that both worlds belong to the same cell. Assume towards contradiction that $w_1 \in S_i$, $w_2 \in S_j$ and $i \neq j$. Moreover, without loss of generality, assume that $i < j$. Then

$w_1, w_2 \in (\mathcal{P} - \bigcup_{k=0}^{i-1} S_k)$. Moreover, since $w_2 \notin S_i$ it follows that there is a $w_3 \in (\mathcal{P} - \bigcup_{k=0}^{i-1} S_k)$ such that $w_3 \ll w_2$. Notice however that since w_2 and w_1 are internal H-duals, from the definition of $\ll$ it follows that $w_3 \ll w_1$. This however contradicts our assumption that $w_1 \in S_i$. Hence w_1 and w_2 belong to the same cell and therefore (Q2) is satisfied.

Finally for (H$\preceq$), consider any two worlds w_1, w_2 such that $w_1 \approx w_2$. Then clearly, by the construction of $\preceq$, $w_1 \not\ll w_2$ and $w_2 \not\ll w_1$. From the definition of $\ll$ this entails that w_1 and w_2 are internal H-duals, or equivalently, $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w_2)$ and $\mathit{Diff}(H, w_1) \cap \mathit{Diff}(w_1, w_2) = \emptyset$.

Define $w = w_1 \cap w_2$. Clearly, $\mathit{Diff}(H, w_1) \subseteq \mathit{Diff}(H, w)$, and $\mathit{Diff}(H, w_1) \cap \mathit{Diff}(w_1, w) = \emptyset$. Next we show that it is also the case that $\mathit{Diff}(H, w) \subseteq \mathit{Diff}(H, w_1)$.

Since w_1 and w_2 agree with H outside $\mathit{Diff}(H, w_1)$, it follows that there exist worlds $w_1', w_2' \in [H]$ such that, w_1' agrees with w_1 outside $\mathit{Diff}(H, w_1)$, and w_2' agrees with w_2 outside $\mathit{Diff}(H, w_1)$; i.e. $w_1' \cap (\mathcal{P} - \mathit{Diff}(H, w_1)) = w_1 \cap (\mathcal{P} - \mathit{Diff}(H, w_1))$, and $w_2' \cap (\mathcal{P} - \mathit{Diff}(H, w_1)) = w_2 \cap (\mathcal{P} - \mathit{Diff}(H, w_1))$. Consequently, $(w_1' \cap w_2') \cap (\mathcal{P} - \mathit{Diff}(H, w_1)) = (w_1 \cap w_2) \cap (\mathcal{P} - \mathit{Diff}(H, w_1)) = w \cap (\mathcal{P} - \mathit{Diff}(H, w_1))$. Moreover observe that, since H is Horn, its models are closed under intersection, and therefore $w_1' \cap w_2' \in [H]$. Hence we have shown that there is an H-world, namely $w_1' \cap w_2'$, that agrees with w on all variables outside $\mathit{Diff}(H, w_1)$. Clearly then $\mathit{Diff}(H, w) \subseteq \mathit{Diff}(H, w_1)$.

Combining the above we have that $\mathit{Diff}(H, w_1) = \mathit{Diff}(H, w)$, and moreover $\mathit{Diff}(H, w_1) \cap \mathit{Diff}(w_1, w) = \emptyset$. By the construction of $\preceq$ we then derive that $w_1 \approx w$. $\square$

We conclude this section with a note on relevance and iterated revision. It was shown in [24] that the DP postulates contradict Parikh's axiom for relevance. It may therefore be surprising at first that Horn revision is consistent with *both*. On reflection thought the reader will realise that the results herein guarantee only that Horn revision is consistent with *each* of the two, not with their conjunction.

8. Specific Horn revision operators

One of the main reasons for studying belief revision in a Horn setting is because of the attractive computational properties of Horn logic: Horn satisfiability is decidable in *linear time* [13]. In this section, we take advantage of this feature of Horn logic, and develop two specific Horn revision operators, called *basic* and *canonical* revision, which also have polynomial time complexity; in particular, $O(n)$ and $O(n^2 \log n)$ respectively.

First however, let us examine two of the most popular revision operators in the classical AGM setting, namely the ones introduced by Dalal [7] and Satoh [27]. It turns out that neither of these two operators can be applied directly in a Horn setting; nevertheless, they provide a guiding intuition for developing our own Horn revision functions.

Recall that Dalal and Satoh both define belief revision in terms of specific preorders on possible worlds, which we denote $\preceq_D$ and $\preceq_S$ respectively. In particular, for any two worlds w and w' the *difference* between w and w', denoted $\mathit{Diff}(w, w')$, is defined as the set of propositional variables that have different truth values in the two worlds; in symbols,

$$\mathit{Diff}\big(w, w'\big) = \big(w - w'\big) \cup \big(w' - w\big).$$

Given a theory H, Dalal defines the faithful ranking associated with H as follows:

$$w \preceq_D w_1 \quad \text{iff there is a } w' \in [H] \text{ such that,} \quad \big|\mathit{Diff}\big(w', w\big)\big| \le \big|\mathit{Diff}\big(w_1', w_1\big)\big|, \quad \text{for all } w_1' \in [H].$$

In the above definition, $|S|$ denotes the *cardinality* of a set S. Satoh's definition is very similar to Dalal's, except that instead of cardinality, Satoh uses set inclusion:

$$\begin{aligned} w \preceq_S w_1 \quad & \text{iff there is a } w' \in [H] \text{ such that for all } w_1' \in [H], \\ & \text{if } \mathit{Diff}\big(w_1', w_1\big) \subseteq \mathit{Diff}\big(w', w\big) \text{ then } \mathit{Diff}\big(w', w\big) \subseteq \mathit{Diff}\big(w_1', w_1\big). \end{aligned}$$

Proposition 4. *There exist Horn theories H for which neither Dalal's preorder $\preceq_D$ nor Satoh's preorder $\preceq_S$ are Horn compliant.*

Proof. Assume that the underlying propositional language is that built from the propositional variables p, q. Define H to be the Horn theory $H = \mathcal{C}n_H(\{p, q\})$. Then both Dalal and Satoh, assign the following preorder $\preceq$ to H: $pq \prec p\bar{q} \approx \bar{p}q \prec \overline{pq}$. Notice however that the worlds $p\bar{q}$ and $\bar{p}q$ violate condition (H$\preceq$), and therefore $\preceq$ is not Horn compliant. $\square$

In the classical AGM framework, Dalal's and Satoh's operators can be viewed as "off-the-shelf" domain independent revision functions that can be used when no information is available about the relative plausibility of possible worlds. In particular, in the absence of any other information, both Dalal and Satoh assume that the plausibility of a possible world w, is defined in terms of the truth values of the *atoms* in w, and their relation with the corresponding values at the initial belief set. Although, as shown above, Dalal's and Satoh's operators cannot be applied directly in a Horn setting, their intuition of an *atom-based* plausibility is used below to develop our own operators for Horn revision.

8.1. Basic Horn revision

Like Dalal and Satoh, we define basic Horn revision in terms of preorders on possible worlds (one per Horn belief set). In particular, let H be a belief set. We define the *basic Horn ranking* associated with H, denoted $\preceq$, as follows:

$$w \preceq w' \quad \text{iff either } w \in [H] \text{ or } w, w' \notin [H] \text{ and } |w| \leq |w'|.$$

This ordering reflects an intuition deriving from the logic programming community, that an atom is false unless it is (depending on the underlying approach) "required" to be true. Here we give preference to worlds with fewer true atomic propositions. We will show that $\preceq$ is a total preorder, faithful with respect to H, and also Horn compliant. First however, let us familiarize ourselves with the definition of $\preceq$ by considering a specific example.

Suppose that the underlying language $\mathcal{L}_H$ is built from the propositional variables p, q, r and that the initial belief set is $H = Cn(\bar{p} \wedge (q \vee \bar{r}))$. The worlds in $[H]$, namely, $\bar{p}qr$, $\bar{p}q\bar{r}$, and $\overline{pqr}$, will be placed in the beginning of the preorder $\preceq$; the rest will be ranked according to the number of atoms satisfied by each world:

$$\begin{array}{ccccccc}
\bar{p}qr & & & & & & \\
 & & \overline{pq}r & & p\bar{q}r & & \\
\bar{p}q\bar{r} & \preceq & & \preceq & & \preceq & pqr \\
 & & p\overline{qr} & & pq\bar{r} & & \\
\overline{pqr} & & & & & &
\end{array}$$

Theorem 6. *Let H be a Horn belief set, and let $\preceq$ be the basic Horn ranking associated with H. Then $\preceq$ is a total preorder, faithful with respect to H, and Horn compliant.*

Proof. By definition, $\preceq$ is clearly reflexive. For transitivity, let w_0, w_1, and w_2 be any three worlds such that $w_0 \preceq w_1$ and $w_1 \preceq w_2$. If $w_0 \in [H]$, then clearly $w_0 \preceq w_2$. Assume therefore that $w_0 \notin [H]$. Then $w_0 \preceq w_1$ entails $w_1 \notin [H]$, and $|w_0| \leq |w_1|$. Similarly, from $w_1 \notin [H]$ and $w_1 \preceq w_2$, we derive that $w_2 \notin [H]$, and $|w_1| \leq |w_2|$. Hence $|w_0| \leq |w_2|$ and $w_0, w_2 \notin [H]$. Consequently $w_0 \preceq w_2$ as desired.

For totality, let w_0 and w_1 be any two worlds. If either of them is in $[H]$, then clearly the two worlds are comparable with respect to $\preceq$. Assume therefore that neither of them are members of $[H]$. If $|w_0| \leq |w_1|$ then $w_0 \preceq w_1$; otherwise $w_1 \preceq w_0$. In either case the two worlds are comparable with respect to $\preceq$ and hence $\preceq$ is total.

Faithfulness with respect to H follows immediately from the definition of $\preceq$. Hence to complete the proof we need to show that $\preceq$ is Horn compliant. We do so by proving that (H$\preceq$) is satisfied. Consider therefore any two worlds w, w' such that $w \approx w'$. We will show that $(w \cap w') \preceq w$. If $w \cap w' \in [H]$ this is clearly true. Assume therefore that $w \cap w' \notin [H]$. Then, since H is Horn, not both w and w' can be members of $[H]$. Since one of w, w' is not in $[H]$, from $w \approx w'$ we derive that neither of the two worlds belong to $[H]$. Moreover, clearly, $|w \cap w'| \leq |w|$. Therefore, $(w \cap w') \preceq w$ as desired. $\square$

We define the *basic Horn revision function* $\circ$ to be the function induced from the basic Horn ranking(s) $\preceq$ via Definition 2. Clearly, in view of Theorem 6, $\circ$ satisfies (H*1)–(H*8), (Acyc). Moreover, as we will show later in this section, $\circ$ has nice computational properties. It therefore seems that $\circ$ is the ideal "off-the-shelf" Horn revision operator: it is simple to understand, easy to compute (see below), and has all the right theoretical properties. There is however one feature of $\circ$ that may be undesirable in some cases: whenever the new information μ is inconsistent with the initial belief set H, the new belief set $H \circ \mu$ is *complete*. This behaviour is a consequence of the following result:

Proposition 5. *Let μ be a consistent Horn formula, and let w_μ be the world $w_\mu = \{p_i \in \mathcal{P} \mid \mu \vdash p_i\}$. Then $w_\mu \in [\mu]$, and moreover, every world in $[\mu]$ is a superset of w_μ.*

Proof. We show that $w_\mu \in [\mu]$ by showing that w_μ satisfies every clause of μ. Let c_j be an arbitrary clause of μ, where a clause is represented as a *set* of literals. If $w_\mu \cap c_j \neq \emptyset$ or if c_j is a singleton, then clearly $w_\mu \models c_j$.[13] Assume therefore that c_j is not a singleton and moreover $w_\mu \cap c_j = \emptyset$. We distinguish between two cases depending on whether c_j is a positive or negative Horn clause.[14] Assume first that c_j is a positive Horn clause. Then for some atoms p_j, $p_{k_1}, \ldots, p_{k_m} \in \mathcal{P}$, $c_j = \{p_j, \neg p_{k_1}, \ldots, \neg p_{k_m}\}$. Notice that if all of the atoms $p_{k_1}, \ldots, p_{k_m}$ belong to w_μ, then $\mu \vdash p_j$, and consequently, $p_j \in w_\mu$. This however contradicts our assumption $w_\mu \cap c_j = \emptyset$. Hence, at least one of the atoms $p_{k_1}, \ldots, p_{k_m}$, say the atom p_{k_r}, does not belong to w_μ. Consequently, $w_\mu \models \neg p_{k_r}$, and therefore $w_\mu \models c_j$.

Next assume that c_j is a negative Horn clause; i.e., for some atoms $p_{k_1}, \ldots, p_{k_m} \in \mathcal{P}$, $c_j = \{\neg p_{k_1}, \ldots, \neg p_{k_m}\}$. Since μ is consistent, not all of the atoms $p_{k_1}, \ldots, p_{k_m}$ belong to w_μ. Therefore one of these atoms, call it p_{k_r}, is such that $w_\mu \models \neg p_{k_r}$. Consequently, once again, $w_\mu \models c_j$. Since c_j was chosen as an arbitrary clause of μ, we derive that $w_\mu \models \mu$ as desired.

[13] If c_j is a singleton, then for some atom $p_j \in \mathcal{P}$, either $c_j = \{p_j\}$ or $c_j = \{\neg p_j\}$. In the first case, $p_j \in w_\mu$, and trivially $w_\mu \models c_j$. In the second case, since μ is consistent, $p_j \notin w_\mu$, and therefore, once again, $w_\mu \models c_j$.

[14] A Horn clause is *positive* if it contains a positive (unnegated) atom. If all literals in a Horn clause are negated atoms then the clause is called *negative*.

Finally, for the second part of the result, let w' be any world in $[\mu]$. We will show that $w_\mu \subseteq w'$. Let p_i be any atom in w_μ. By construction, $\mu \vdash p_i$. Hence, since $w' \models \mu$ it follows that $w' \models p_i$, or equivalently, $p_i \in w'$. Consequently, $w_\mu \subseteq w'$ as desired. □

An immediate consequence of the above result is that, for any consistent Horn formula μ, there is *only one* smallest μ-world (in terms of cardinality); namely the world w_μ as defined in Proposition 5. Consequently, given the definition of the basic ranking $\preceq$ associated with a Horn belief set H, if $H + \mu \vdash \bot$, then $\min([\mu], \preceq)$ is a singleton. Clearly too, this will be the same as the *minimal* μ-model. Therefore,

Corollary 2. *Let H be a Horn belief set and μ be a Horn formula. If $H + \mu \vdash \bot$, then*

1. *$H \circ \mu$ is complete.*[15]
2. *$H \circ \mu$ is equivalent to the minimal μ-model.*

As mentioned earlier, these features of basic Horn revision could be undesirable in many situations. In fact, this is the main reason for introducing an alternative Horn revision operator in the next section. Nevertheless, in situations where completeness can be tolerated, basic Horn revision can be computed in *polynomial*, in fact *linear time*. Let ϕ, μ be Horn formulas and let n be the size of both these formulas. Moreover, assume that $\mathcal{P} = \{p_1, \ldots, p_m\}$ are the atoms that appear in ϕ, μ. Below is an algorithm for computing $Cn(\phi) \circ \mu$ in time $O(n)$.

Algorithm A1

Input: Horn formulas ϕ, μ
Output: A Horn formula ψ such that $Cn(\psi) = Cn(\phi) \circ \mu$.

if μ is unsatisfiable
  then return μ
if $\phi \wedge \mu$ is satisfiable
  then return $\phi \wedge \mu$
$w_\mu := \{p_i \in \mathcal{P} \mid \mu \vdash p_i\}$
return w_μ

Algorithm $A1$ first deals with the limiting cases. In particular, it checks if the epistemic input μ is inconsistent, or consistent with the initial belief set $Cn(\phi)$, in which case it returns respectively, μ and $\phi \wedge \mu$. If on the other hand μ is consistent and $\phi \wedge \mu \vdash \bot$, then $A1$ computes the minimal μ-model, denoted w_μ, and returns w_μ as the result. In view of Proposition 5, this is indeed correct.

In terms of computational complexity, in [13] it has been shown that Horn satisfiability, as well as the computation of w_μ (if μ is satisfiable), can be accomplished in linear time. Hence the overall time complexity of $A1$ is also linear.

8.2. Canonical Horn revision

As discussed above, the "maximalist" behaviour of basic Horn revision (as stated in Corollary 2), may not be a desirable feature in some cases. Therefore in this section we introduce a second Horn revision operator, denoted $\star$, which we call *canonical Horn revision*. Contrary to basic Horn revision, which is defined semantically, canonical Horn revision is defined syntactically. In particular, starting with a set of Horn clauses H_0 representing the initial belief set, we shall define a sequence of progressively weaker sets $H_1, H_2, \ldots$, called *fallbacks*, through which $H \star \mu$ is defined for any Horn formula μ.

To be precise, canonical Horn revision is not a single revision function, but rather a whole family of functions parameterized by a total preorder on atoms. In particular, let $\leqq$ be a total preorder on the set $\mathcal{P}$ of atoms. We define the subsets $P_1, P_2, \ldots$ of $\mathcal{P}$ recursively as follows:

$$\begin{aligned}
P_1 &= \min(\mathcal{P}, \leqq) \\
P_2 &= \min(\mathcal{P} - P_1, \leqq) \\
&\vdots \\
P_{j+1} &= \min\left(\mathcal{P} - \left(\bigcup_{i=1}^{j} P_i\right), \leqq\right) \\
&\vdots
\end{aligned}$$

[15] It is worth noting that this maximalist behaviour of basic Horn revision is induced despite the fact that the basic Horn ranking is *not* linear.

Clearly, since $\mathcal{P}$ is finite, at some point the above sequence will reach the empty set and will remain equal to the empty set from then onwards. Let us denote by z the index of the last nonempty set in the above sequence. By construction, P_1, $P_2, \ldots, P_z$ is a partition of $\mathcal{P}$. Based on this partition, we define the fallback sets $H_1, \ldots, H_z$ (relative to $\leqq$) as follows:

$$H_i = \left\{ c \cup \left(\bigcup_{j=1}^{i} \overline{P_j} \right) \middle| \, c \in H_0 \right\}$$

Some explanations are due. Firstly, for a set of atoms P_i, by $\overline{P_i}$ we denote the set of negated atoms of P_i; i.e. $\overline{P_i} = \{\neg p \mid p \in P_i\}$. Secondly, all H_i in the above sequence are sets of clauses, which in turn are represented as sets of literals. Hence each H_i is constructed from H_0 by adding the negated atoms of the partitions $P_0, \ldots, P_i$, to all clauses of H_0. Notice that only negated atoms are added to the clauses of H_i, and consequently, since H_0 is Horn, so is H_i. It should be clear from the construction that $H_i \vdash H_j$ for all $0 \leq i \leq j \leq z$. Finally notice that it is possible during the above construction that a clause degenerate to a tautology; such clauses can be safely removed from the fallback sets.

An example will help us clarify the above observations. Assume that the language $\mathcal{L}_H$ is built from five atoms, i.e. $\mathcal{P} = \{p, q, r, u, v\}$. Let the initial belief base H_0 be $H_0 = \{\{p, \bar{q}\}, \{\bar{q}, \bar{r}\}\}$ and let the preorder $\leqq$ on atoms be $u, v \leqq p \leqq q, r$. Then the sets P_i, H_i are defined as follows:

$$P_1 = \{u, v\}$$

$$P_2 = \{p\}$$

$$P_3 = \{q, r\}$$

$$H_0 = \big\{\{p, \bar{q}\}, \{\bar{q}, \bar{r}\}\big\}$$

$$H_1 = \big\{\{p, \bar{q}, \bar{u}, \bar{v}\}, \{\bar{q}, \bar{r}, \bar{u}, \bar{v}\}\big\}$$

$$H_2 = \big\{\{p, \bar{q}, \bar{u}, \bar{v}, \bar{p}\}, \{\bar{q}, \bar{r}, \bar{u}, \bar{v}, \bar{p}\}\big\}$$

$$H_3 = \big\{\{\bar{q}, \bar{r}, \bar{u}, \bar{v}, \bar{p}\}\big\}$$

The notion of a *fallback* used for the sets H_i is not new in the belief revision literature (see [21]). Loosely speaking, the intuition is that, besides the initial belief base H_0, there exists a sequence of progressively weaker belief bases $H_1, \ldots, H_z$, to which the agent can *fall back* to if the new information μ contradicts H_0. Herein the fallback sets are induced by a preorder $\leqq$ on atoms which can be understood as representing their *a priori* comparative plausibility: the earlier an atom p appears in the preorder, the more plausible *its negation* is.[16]

The way fallback sets are used to define canonical Horn revision is quite simple. In particular, for any Horn sentence μ, let us denote by H_μ the first set in the sequence $H_0, H_1, \ldots, H_z$ that is consistent with μ; if all sets in the sequence are inconsistent with μ, we define H_μ to be the empty set. The canonical Horn revision of $Cn(H_0)$ by μ (parameterized by $\leqq$) is defined as follows:[17]

$$Cn(H_0) \star \mu = Cn(H_\mu \cup \mu)$$

The first thing we note about canonical Horn revision is that it doesn't have the maximalist attitude of basic Horn revision. Continuing with our previous example, suppose that we revise the initial belief base $H_0 = \{\{p, \bar{q}\}, \{\bar{q}, \bar{r}\}\}$ with $\mu = \{\{\bar{p}\}, \{q\}\}$. Then, $H_\mu = H_1$, and therefore the revised belief base is $\{\{p, \bar{q}, \bar{u}, \bar{v}\}, \{\bar{q}, \bar{r}, \bar{u}, \bar{v}\}, \{\bar{p}\}, \{q\}\}$, which clearly is not complete. The next result shows that canonical Horn revision also has all the essential theoretical properties.

Theorem 7. *Let $\leqq$ be a total preorder on the set of propositional variables $\mathcal{P}$. The canonical Horn revision function $\star$, parameterized by $\leqq$, satisfies (H*1)–(H*8), and (Acyc).*

Proof. Let H_0 be a set of Horn clauses representing the initial belief set $H = Cn(H_0)$, and let $H_1, \ldots, H_z$, be the fallback sets induced from $\leqq$ and H_0. We shall construct a total preorder on worlds $\preceq$, which we will show to be faithful with respect to H, Horn compliant, and such that $\min([\mu], \preceq) = [(H_\mu \cup \mu)]$. In view of Theorem 3, the existence of such a preorder $\preceq$ suffices to prove Theorem 7. We define the preorder $\preceq$ in $\mathcal{M}$ as follows:

$$w \preceq w' \quad \text{iff for every fallback set } H_i,\ w' \in [H_i] \text{ entails } w \in [H_i].$$

[16] Observe that a single preorder on atoms suffices to construct the fallback sets for *all* initial belief bases.

[17] There is some ambiguity in the use of μ in this equation. In particular, on the left side of the equation, μ is treated as a (Horn) *formula*, whereas in the right hand side, it is treated as a *set* of (Horn) clauses. We have allowed this dual representation of μ for the sake of readability; any ambiguity is resolved from the context.

Reflexivity and transitivity of $\preceq$ follow immediately from the definition. Hence $\preceq$ is indeed a preorder. For totality, consider any two worlds w, w', such that $w' \not\preceq w$. We show that $w \preceq w'$. Consider any fallback set H_j such that $w' \in H_j$. From $w' \not\preceq w$ we derive that there is a fallback set H_i such that $w \in [H_i]$ and $w' \notin [H_i]$. Moreover, by construction, the fallback sets become progressively weaker, and therefore, for any k, m, $[H_k] \subseteq [H_m]$ iff $k \leq m$. Hence, from $w' \notin [H_i]$ and $w' \in [H_j]$ we derive that $i \leq j$. This again, together with $w \in [H_i]$, entails that $w \in [H_j]$. Thus $w \preceq w'$, and $\preceq$ is total.

For faithfulness, if H is inconsistent, then $\preceq$ is trivially faithful with respect to H. Assume therefore that H is consistent and let w be any world in $[H]$. Then, by construction, $w \in [H_i]$, for all $0 \leq i \leq n$. Hence $w \preceq w'$ for all $w' \in \mathcal{M}$, and therefore $[H] \subseteq \min(\mathcal{M}, \preceq)$. For the converse, let w be any world in $\min(\mathcal{M}, \preceq)$, and let w' be a world in $[H]$. Since w is minimal, it follows that $w \preceq w'$. Hence, by the definition of $\preceq$ and since $w' \in [H]$, we derive that $w \in [H]$.

For Horn compliance, we will once again use its equivalent condition (H$\preceq$). Consider any two worlds w, w' such that $w \approx w'$, and let w'' be the world resulting from their intersection; i.e. $w'' = w \cap w'$. Consider now any fallback set H_i such that $w \in [H_i]$. Then from $w \approx w'$, it follows that $w' \in [H_i]$, and consequently, since by construction H_i is Horn, we derive that $w'' \in [H_i]$. Thus $w'' \preceq w$ as desired.

Finally we show that for every Horn formula μ, $\min([\mu], \preceq) = [(H_\mu \cup \mu)]$. If μ is inconsistent, then this is clearly true. Assume therefore that μ is consistent. If $H_\mu = \emptyset$ then no fallback set is consistent with μ, and consequently, by the definition of $\preceq$, all μ-worlds are equally ranked with respect to $\preceq$, which again entails that $\min([\mu], \preceq) = [\mu] = [H_\mu \cap \mu]$. Assume therefore that $H_\mu \neq \emptyset$.

Let w be any world in $\min([\mu], \preceq)$. Clearly, $w \in [\mu]$. Moreover, by the definition of $\preceq$, and the minimality of w in $[\mu]$, it follows that w is a model of every fallback set H_i such that $[H_i]$ contains at least one μ-world w'. In other words, w is a model of every fallback set consistent with μ, and consequently, w is also a model of H_μ. Thus $w \in [(H_\mu \cup \mu)]$. This proves $\min([\mu], \preceq) \subseteq [(H_\mu \cup \mu)]$. For the converse, assume that w is a world in $[(H_\mu \cup \mu)]$. Then w is a μ-world that is also a model of H_μ. Consider now any world $w' \in [\mu]$ and assume that w' is a model of some fallback set H_i; i.e. $w' \in [H_i]$. Since H_μ is the *first* fallback set consistent with μ, it follows that H_i doesn't appear earlier than H_μ in the sequence of fallback sets. Moreover, since fallback sets become progressively (logically) weaker, and given that $w \in [H_\mu]$, it follows that $w \in [H_i]$. Consequently, $w \preceq w'$. Since w' was chosen as an arbitrary μ-world we derive that $w \in \min([\mu], \preceq)$. Hence $[(H_\mu \cup \mu)] \subseteq \min([\mu], \preceq)$. $\square$

We conclude this section with a discussion on the computational complexity of canonical Horn revision. Assume that the set $\mathcal{P}$ of propositional variables has m elements, and let k be the number of Horn clauses of the initial belief base H_0. Then, by construction, every fallback set H_i has no more than k clauses. Given that each Horn clause can have up to $m+1$ literals,[18] it follows that each H_i can be constructed in time $O(k * m)$. Hence a crude algorithm for computing the canonical Horn revision of H_0 by a Horn formula μ would be to construct sequentially the fallback sets $H_1, H_2, \ldots,$ until we reach a set H_μ that is consistent with μ (or define $H_\mu = \emptyset$ if all fallback sets are inconsistent with μ); then return $H_\mu \cup \mu$ as the result of revision. Let us denote by n the length of the entire input.[19] Since the construction of each fallback set, and testing its consistency with μ can be performed in time $O(n^2)$, and moreover, there are at most m fallback sets, the time complexity of the whole computation is no greater than $O(n^3)$. We can improve on that if instead of searching sequentially for H_μ, we use *binary search*; doing so reduces the complexity to $O(n^2 \log n)$.

9. Discussion

In this paper, we have investigated belief revision in Horn clause theories. We showed that AGM belief revision doesn't immediately generalise to the Horn case and that, in the naïve extension of AGM revision to Horn clause theories, several problems arise. We address these issues by first restricting the semantic construction involving faithful rankings to "well behaved" orderings, those that we call *Horn compliant*. As well, we augment the revision postulates by an additional postulate (Acyc).

Notably, these results represent an *extension* rather than a *modification* of the AGM approach. That is, we have redefined (AGM-style) revision in the context of a logic, Horn logic, that is *weaker* than propositional logic. Postulates consist of (H*1)–(H*8) and (Acyc), while the construction is in terms of Horn compliant faithful rankings. This however subsumes classical AGM revision: classical propositional logic obviously is stronger than Horn logic. In classical propositional logic the notion corresponding to Horn compliance is trivial, since (over a finite alphabet) for any formula ϕ of propositional logic, $\min([\phi], \preceq)$ is definable via a formula of propositional logic. On the postulational side, as we showed, (Acyc) is derivable from the other postulates in the context of classical propositional logic. What this means, in other words, is that if one takes the approach developed herein, but replaces Horn logic by classical propositional logic, one ends up with the standard AGM approach.

We also showed that Horn revision is compatible with work in iterated revision and work concerning relevance in revision. We also considered specific revision operators: while the analogues of Dalah- and Satoh-style revision are incompatible with Horn revision, we proposed two specific revision operators, both with good complexity characteristics.

[18] The only case where Horn clauses have $m+1$ literals are tautologies. All other Horn clauses have up to m literals.

[19] In this case, the input includes the initial belief base H_0, the new information μ, as well as the preorder on atoms $\leqq$.

These results suggest several (in our opinion) very interesting directions for future work. First, we argued that AGM revision can be extended by weakening the underlying logic to that of Horn logic. This raises the issue of whether the overall framework can be generalised to subsume other weakened inference relations, while maintaining the overall AGM character as reflected in the standard AGM postulates.

The broader area of belief change in Horn theories is in the process of being mapped out. Other research has characterized Horn contraction, while the present paper has addressed revision. However, with the exception of [32], which considers the definition of Horn revision in terms of contraction, there has been no work that we are aware of in linking the areas of Horn contraction and revision. Moreover, the constructions in Horn contraction have focussed on the standard contraction constructions of remainder sets and epistemic entrenchment, while the present work has used the standard revision construction of a faithful ranking. Hence there is also a disconnect in the underlying formal characterizations. Consequently, research on linking Horn contraction and revision would help shed further light on the foundations of belief change.

Last, there is burgeoning interest in addressing belief change in description logics (see [26] for instance) or in analogous areas such as ontology evolution. Given that a Horn clause may also find interpretation as a subsumption, by mapping a rule $p \rightarrow q$ to a subsumption of the form $P \sqsubseteq Q$, the present approach may also shed light on approaches to revision in description logics.

Acknowledgements

The first author was partially supported by a Canadian NSERC Discovery Grant, No. 611011. We thank the anonymous reviewers for their helpful comments.

References

[1] C.E. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change: partial meet contraction and revision functions, J. Symb. Log. 50 (2) (1985) 510–530.
[2] R. Booth, T. Meyer, I.J. Varzinczak, Next steps in propositional Horn contraction, in: Proceedings of the International Joint Conference on Artificial Intelligence, Pasadena, CA, 2009, pp. 702–707.
[3] R. Booth, T. Meyer, I. Varzinczak, R. Wassermann, On the link between partial meet, kernel, and infra contraction and its application to Horn logic, J. Artif. Intell. Res. 42 (2011) 31–53.
[4] C. Boutilier, Revision sequences and nested conditionals, in: Proceedings of the International Joint Conference on Artificial Intelligence, 1993, pp. 519–531.
[5] C. Boutilier, Iterated revision and minimal change of conditional beliefs, J. Log. Comput. 25 (1996) 262–305.
[6] Nadia Creignou, Odile Papini, Reinhard Pichler, Stefan Woltran, Belief revision within fragments of propositional logic, in: Gerhard Brewka, Thomas Eiter, Sheila A. McIlraith (Eds.), Proceedings of the Thirteenth International Conference on the Principles of Knowledge Representation and Reasoning, AAAI Press, 2012.
[7] M. Dalal, Investigations into theory of knowledge base revision, in: Proceedings of the AAAI National Conference on Artificial Intelligence, St. Paul, Minnesota, 1988, pp. 449–479.
[8] A. Darwiche, J. Pearl, On the logic of iterated belief revision, Artif. Intell. 89 (1997) 1–29.
[9] James Delgrande, Pavlos Peppas, Revising Horn theories, in: Proceedings of the International Joint Conference on Artificial Intelligence, Barcelona, Spain, 2011, pp. 839–844.
[10] James Delgrande, Renata Wassermann, Horn clause contraction functions: belief set and belief base approaches, in: Fangzhen Lin, Uli Sattler (Eds.), Proceedings of the Twelfth International Conference on the Principles of Knowledge Representation and Reasoning, Toronto, AAAI Press, 2010, pp. 143–152.
[11] James P. Delgrande, Renata Wassermann, Horn clause contraction functions, J. Artif. Intell. Res. 48 (2013) 475–511.
[12] J.P. Delgrande, Horn clause belief change: contraction functions, in: Gerhard Brewka, Jérôme Lang (Eds.), Proceedings of the Eleventh International Conference on the Principles of Knowledge Representation and Reasoning, Sydney, Australia, AAAI Press, 2008, pp. 156–165.
[13] W.F. Dowling, J.H. Gallier, Linear-time algorithms for testing the satisfiability of propositional Horn formulae, Log. Program. 1 (3) (1984) 267–284.
[14] T. Eiter, G. Gottlob, On the complexity of propositional knowledge base revision, updates, and counterfactuals, Artif. Intell. 57 (2–3) (1992) 227–270.
[15] Giorgos Flouris, Dimitris Plexousakis, Grigoris Antoniou, Generalizing the AGM postulates: preliminary results and applications, in: Proceedings of the 10th International Workshop on Non-Monotonic Reasoning, NMR-04, Whistler BC, Canada, June 2004, 2004, pp. 171–179.
[16] P. Gärdenfors, Knowledge in Flux: Modelling the Dynamics of Epistemic States, MIT Press, Cambridge, MA, 1988.
[17] A. Grove, Two modellings for theory change, J. Philos. Log. 17 (1988) 157–170.
[18] H. Katsuno, A. Mendelzon, Propositional knowledge base revision and minimal change, Artif. Intell. 52 (3) (1991) 263–294.
[19] M. Langlois, R.H. Sloan, B. Szörényi, G. Turán, Horn complements: towards Horn-to-Horn belief revision, in: Proceedings of the AAAI National Conference on Artificial Intelligence, Chicago, IL, July 2008, 2008.
[20] Paolo Liberatore, Compilability and compact representations of revision of Horn knowledge bases, ACM Trans. Comput. Log. 1 (1) (2000) 131–161.
[21] Sten Lindström, Wlodzimierz Rabinowicz, Epistemic entrenchment with incomparabilities and relational belief revision, in: The Logic of Theory Change, in: Lecture Notes in Computer Science, vol. 465, 1991, pp. 93–126.
[22] Rohit Parikh, Beliefs, belief revision, and splitting languages, in: L.S. Moss, J. Ginzburg, M. Rijke (Eds.), Logic, Language and Computation, vol. 2, CSLI Publications, 1999, pp. 266–278.
[23] P. Peppas, S. Chopra, N. Foo, Distance semantics for relevance-sensitive belief revision, in: KR2004: Principles of Knowledge Representation and Reasoning, Morgan Kaufmann, San Francisco, 2004.
[24] P. Peppas, A. Fotinopoulos, S. Seremetaki, Conflicts between relevance-sensitive and iterated belief revision, in: Proceedings of the 18th European Conference on Artificial Intelligence, ECAI-08, 2008.
[25] P. Peppas, Belief revision, in: F. van Harmelen, V. Lifschitz, B. Porter (Eds.), Handbook of Knowledge Representation, Elsevier Science, San Diego, USA, 2008, pp. 317–359.
[26] Guilin Qi, Fangkai Yang, A survey of revision approaches in description logics, in: Proceedings of the 2nd International Conference on Web Reasoning and Rule Systems, Springer-Verlag, Berlin, Heidelberg, 2008, pp. 74–88.
[27] K. Satoh, Nonmonotonic reasoning by minimal belief revision, in: Proceedings of the International Conference on Fifth Generation Computer Systems, Tokyo, 1988, pp. 455–462.

[28] B. Selman, H. Kautz, Knowledge compilation and theory approximation, J. ACM 43 (2) (1996) 193–224.
[29] Z. Zhuang, Maurice Pagnucco, Horn contraction via epistemic entrenchment, in: Tomi Janhunen, Ilkka Niemelä (Eds.), Logics in Artificial Intelligence—12th European Conference, JELIA 2010, in: Lecture Notes in Artificial Intelligence, vol. 6341, Springer Verlag, 2010, pp. 339–351.
[30] Zhiqiang Zhuang, Maurice Pagnucco, Transitively relational partial meet Horn contraction, in: Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence, Barcelona, Spain, 2011, pp. 1132–1138.
[31] Zhiqiang Zhuang, Maurice Pagnucco, Model based Horn contraction, in: Proceedings of the Thirteenth International Conference on the Principles of Knowledge Representation and Reasoning, Rome, Italy, 2012.
[32] Zhiqiang Zhuang, Maurice Pagnucco, Yan Zhang, Definability of Horn revision from Horn contraction, in: Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence, Beijing, China, 2013.