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Syntax Splitting = Relevance + Independence: New Postulates for Nonmonotonic Reasoning From Conditional Belief Bases

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Abstract

Syntax splitting, first introduced by Parikh in 1999, is a natural and desirable property of KR systems. Syntax splitting combines two aspects: it requires that the outcome of a certain epistemic operation should only depend on relevant parts of the underlying knowledge base, where relevance is given a syntactic interpretation (relevance). It also requires that strengthening antecedents by irrelevant information should have no influence on the obtained conclusions (independence). In the context of belief revision the study of syntax splitting already proved useful and led to numerous new insights. In this paper we analyse syntax splitting in a different setting, namely nonmonotonic reasoning based on conditional knowledge bases. More precisely, we analyse inductive inference operators which, like system P, system Z, or the more recent c-inference, generate an inference relation from a conditional knowledge base. We axiomatize the two aforementioned aspects of syntax splitting, relevance and independence, as properties of such inductive inference operators. Our main results show that system P and system Z, whilst satisfying relevance, fail to satisfy independence. C-inference, in contrast, turns out to satisfy both relevance and independence and thus fully complies with syntax splitting.

1 Introduction

Postulates have become standard tools in knowledge representation. They limit the range of approaches considered to be interesting and help us to evaluate and to classify them. This also applies to nonmonotonic reasoning, where the axiomatics of system P (Adams 1966; Kraus, Lehmann, and Magidor 1990) has been established as kind of a standard while system Z (Pearl 1990) is often perceived as a particularly convenient and intuitive method of nonmonotonic reasoning. As examples of such postulates which are crucial to assess the quality of a nonmonotonic inference relation $\mid\!\sim$, we recall the well-known postulates of *Reflexivity, Cautious Monotony (CM)*, and *Cut* which form a core of many axiomatic systems and together are also known as *Cumulativity* (Makinson 1989) (A, B, C propositional formulas) :

(Reflexivity) $A \mid\!\sim A$

(CM) If $A \mid\!\sim B$ and $A \mid\!\sim C$ then also $A \wedge B \mid\!\sim C$

(Cut) If $A \mid\!\sim B$ and $A \wedge B \mid\!\sim C$ then also $A \mid\!\sim C$

We would like to emphasize that these postulates focus entirely on the inference relation. How this relation is obtained is not addressed at all. This is somewhat surprising, since both system P and system Z, as well as many other inference systems like, e.g., rational closure (Lehmann and Magidor 1992), make use of a (conditional) belief base from which the inference relation is derived. Except for some few more general works on nonmonotonic reasoning like (Weydert 2003), this belief base does not show up explicitly in most of the postulates considered so far.

In this paper, we focus not only on (nonmonotonic) inference relations, but also on how they are generated. Consequently, we consider inference relations $\mid\!\sim_\Delta$ which are parameterized by the conditional belief base Δ they are induced from, i.e., Δ serves as a starting point for inferences that extend (like system P) or complete (like system Z and rational closure) the beliefs in Δ. We call operators that assign inference relations to conditional belief bases *inductive inference operators*. We will propose several postulates for inductive inference operators which shed new light on the relationships between nonmonotonic inference systems. The new postulates allow us to point out differences which cannot be expressed in terms of properties of the induced inference relations alone. An existing example of the kind of postulate we have in mind is *Direct Inference (DI)* (Lukasiewicz 2005). This postulate links conditional beliefs $(B|A) \in \Delta$ to nonmonotonic inferences via $\mid\!\sim_\Delta$:

(DI) If Δ is a conditional belief base and $\mid\!\sim_\Delta$ is an inference relation that is induced on Δ, then $(B|A) \in \Delta$ implies $A \mid\!\sim_\Delta B$.

In a very fundamental way, (DI) justifies to call $\mid\!\sim_\Delta$ an *inference relation induced from* Δ. For this reason we will take (DI) as part of the definition of inductive inference operators. (DI) is satisfied by system Z and rational closure, and also by inferences via c-representations (Kern-Isberner 2001; Kern-Isberner 2004). For inductive reasoning, (DI) plays a role that is similar to the *success conditions* in belief change (Gärdenfors and Rott 1994).

Beyond (DI), it is hard to specify exactly what influence the conditional beliefs in Δ should have on the inferences derived from them. However, there are cases in which we are able to state quite clearly that we do not want to have interferences: if the belief base splits into two subbases over disjoint languages, then first, we expect the inferences between propositions from one sublanguage to be determined only by the respective subbase, and second, we do not want these inferences to be affected by additional information coming from the other sublanguage. The first property sets a scope of *relevance* for inferences, while the second property keeps inferences regarding one sublanguage *independent* from propositions over the other sublanguage, if no justification for a dependence can be found in the belief base. Briefly stated, if the belief base splits then we also expect the inferences to be split (over sublanguages). For example, if one subbase talks about birds (b) usually being able to fly (f), i.e., $\Delta_1 = \{(f|b)\}$, and the other expresses beliefs about dark objects (d) usually being hardly resp. not visible in the night ($\neg v$), i.e., $\Delta_2 = \{(\neg v|d)\}$, then we would expect that the inferences $b \mid\!\sim f$ and $d \mid\!\sim \neg v$ can be drawn (this is what (DI) ensures). Regarding information coming from the respective other subbase, we might also expect that they would not affect these inferences, i.e., we might accept that dark birds also are able to fly and not visible in the night. Following this line of thought, $db \mid\!\sim f$ and $db \mid\!\sim \neg v$ seem to be reasonable inferences, given (only) $\Delta_1 \cup \Delta_2$. However, such cases are not covered by standard axioms of nonmonotonic inference but would require to take the (syntactic) structure of the belief base explicitly into account. Of course, there are cases where we deem interferences to be possible and even expected, but in those cases we would expect to see conditionals in the belief base which somehow link atoms from one sublanguage to atoms from the other.

We formalize the properties *Relevance* and *Independence* which together axiomatize the property of *Syntax Splitting* for (nonmonotonic) inductive inference relations, and present semantic counterparts for two popular semantic frameworks on which such inference relations can be based: total preorders (Makinson 1989), and ordinal conditional functions (OCF, also called ranking functions) (Spohn 1988). For total preorders, we show a representation theorem, while for ranking functions, *Relevance* and *Independence* can be strengthened into one combined axiom. We also point out connections to work in belief revision theory, showing how syntax splitting for inductive inference relations can be considered as a special case of syntax splitting properties in belief revision, which have also been addressed under the name *Relevance*. Our *Independence* property extends notions of irrelevance that have been addressed in various places before (e.g., (Benferhat, Dubois, and Prade 2002; Delgrande and Pelletier 1998)).

The main contributions of this paper are as follows:

- We axiomatize syntax splitting for inductive inference relations from conditional belief bases by the formal properties of *Relevance* and *Independence*.
- We present a representation theorem for inductive inference relations based on total preorders, and we present a stronger, more concise definition of syntax splitting for ranking functions which is downward compatible with the qualitative notions.
- In particular, we introduce a qualitative notion of independence for total preorders, defining when two disjoint subsets of variables are independent from one another, in accordance to notions of independence for ranking functions, and probabilities.
- As a byproduct of this last point, we introduce a qualitative conditioning operator for total preorders, and characterize qualitative independence in terms of qualitative conditionalization in a way that is analogous to the correspondence of both concepts in probability theory.
- We introduce the notion of *selection strategies for c-representations* and show that they provide inductive inference operators that fully comply with syntax splitting; furthermore, we show that also skeptical inference over all c-representations obeys syntax splitting.

The paper is organized as follows. Sect. 2 introduces basic formal notions. Sect. 3 provides the relevant background on nonmonotonic inference relations based on conditional belief bases. Inductive inference operators are defined in Sect. 4, together with several specializations of this notion. Sect. 5 introduces our axiomatization of syntax splitting for inductive inference operators, as composed of relevance and independence. It studies several fundamental properties of the axiomatization and analyzes system P and system Z in the light of the postulates. It turns out that these systems satisfy relevance, but not independence. Sect. 6 extends the analysis to c-representations and c-inference and establishes another main result, namely full satisfaction of syntax splitting for c-inference. Relevant related work is discussed in Sect. 7. Sect. 8 concludes and points to future work.

2 Formal Basics

Let $\mathcal{L}$ be a finitely generated propositional language over an alphabet Σ with atoms $a, b, c, \ldots$, and with formulas $A, B, C, \ldots$. For conciseness of notation, we will omit the logical *and*-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of *possible worlds* over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $Mod\,(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $Mod\,(A) \subseteq Mod\,(B)$, as usual. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to use ω both as an interpretation and a proposition, which will ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise.

For subsets Θ of Σ, let $\mathcal{L}(\Theta)$ denote the propositional language defined by Θ, with associated set of interpretations $\Omega(\Theta)$. Note that while each sentence of $\mathcal{L}(\Theta)$ can also be considered as a sentence of $\mathcal{L}$, the interpretations $\omega^{\Theta} \in \Omega(\Theta)$ are not elements of $\Omega(\Sigma)$ if $\Theta \neq \Sigma$. But each interpretation $\omega \in \Omega$ can be written uniquely in the form $\omega = \omega^{\Theta}\omega^{\overline{\Theta}}$ with concatenated $\omega^{\Theta} \in \Omega(\Theta)$ and $\omega^{\overline{\Theta}} \in \Omega(\overline{\Theta})$, where $\overline{\Theta} = \Sigma \backslash \Theta$ is the complement of Θ in Σ. Note that the syntactical reading of interpretations as conjunctions makes perfect sense here: According to this reading, ω is a conjunction of ω^{Θ} and $\omega^{\overline{\Theta}}$ (with omitted $\wedge$ symbol). ω^{Θ} is called the *reduct* of ω to Θ (Delgrande 2017). If $\Omega' \subseteq \Omega$ is a subset of models, then $\Omega'|_\Theta = \{\omega^\Theta | \omega \in \Omega'\} \subseteq \Omega(\Theta)$ restricts Ω' to a subset of $\Omega(\Theta)$. In the following, we will often consider the case that Σ_1, Σ_2 are disjoint subsignatures of Σ, then we write ω^i instead of ω^{Σ_i} for the reducts to ease notation.

By making use of a conditional operator $|$, we introduce the language $(\mathcal{L}|\mathcal{L})$ of *conditionals* over $\mathcal{L}$:

$$(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}.$$

Conditionals $(B|A)$ are meant to express plausible, defeasible rules "If A then plausibly (usually, possibly, probably, typically etc.) B". A popular semantic framework that is often used for interpreting conditionals is provided by ordinal conditional functions. *Ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$, were introduced (in a more general form) first by (Spohn 1988). They express degrees of plausibility of propositional formulas A by specifying degrees of disbeliefs of their negations $\overline{A}$. More formally, we have $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$, so that $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. A proposition A is believed if $\kappa(\overline{A}) > 0$ (which implies particularly $\kappa(A) = 0$). The *uniform OCF* κ_u is defined by $\kappa_u(\omega) = 0$ for all $\omega \in \Omega$.

Degrees of plausibility can also be assigned to conditionals by setting $\kappa(B|A) = \kappa(AB) - \kappa(A)$. A conditional $(B|A)$ is *accepted* in the epistemic state represented by κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff AB is more plausible than $A\overline{B}$.

The *marginal of κ on $\Theta \subseteq \Sigma$*, denoted by $\kappa|_\Theta$, is defined by $\kappa|_\Theta(\omega^\Theta) = \kappa(\omega^\Theta)$ for any $\omega^\Theta \in \Omega(\Theta)$. Let Σ_1, Σ_2 be disjoint subsignatures of Σ, let κ be an OCF. Σ_1, Σ_2 are κ-*independent* iff for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, $\kappa(\omega^1\omega^2) = \kappa(\omega^1) + \kappa(\omega^2)$ holds, which is the same as $\kappa|_{\Theta_1}(\omega^1) + \kappa|_{\Theta_2}(\omega^2)$. Lemma 1 is straightforward:

Lemma 1. *Let Σ_1, Σ_2 be disjoint subsignatures of Σ, let κ be an OCF. Then Σ_1, Σ_2 are κ-independent iff for all $A \in \mathcal{L}(\Sigma_1), B \in \mathcal{L}(\Sigma_2)$, $\kappa(AB) = \kappa(A) + \kappa(B)$ holds.*

So, Σ_1, Σ_2 are κ-independent iff all propositions A, B of the respective different sublanguages are κ-independent in the sense of (Kern-Isberner and Huvermann 2017).

3 Inference Relations

Nonmonotonic inference relations are relations $\mathrel{|\!\sim}$ between (sets of) sentences of a given logical language $\mathcal{L}$. System P (Adams 1966) is a set of axioms that can be used to evaluate the logical quality of such an inference relation. Besides *Reflexivity, Cautious Monotony*, and *Cut* mentioned in Sec. 1, system P encompasses the following three more axioms:

(Left Logical Equivalence) If $A \equiv B$ and $A \mathrel{|\!\sim} C$, then $B \mathrel{|\!\sim} C$.

(Right Weakening) If $A \mathrel{|\!\sim} B$ and $B \models C$, then $A \mathrel{|\!\sim} C$.

(Or) If $A \mathrel{|\!\sim} C$ and $B \mathrel{|\!\sim} C$, then $A \vee B \mathrel{|\!\sim} C$.

On the semantic side, preferential entailment (Makinson 1989) is a well-known form of nonmonotonic inference that makes use of relations over possible worlds and defines nonmonotonic inferences from $A \in \mathcal{L}$ via its most preferred models. For instance, system P can be characterized by preferential models (Makinson 1989). Particularly well-behaved inference relations arise if total preorders (TPO) $\preceq$ over possible worlds are used. As usual, $\omega \prec \omega'$ iff $\omega \preceq \omega'$ and not $\omega' \preceq \omega$, and $\omega \approx_\Psi \omega'$ iff both $\omega \preceq \omega'$ and $\omega' \preceq \omega$. The *uniform total preorder* $\preceq_u$ is defined via $\omega \preceq_u \omega'$ for all $\omega, \omega' \in \Omega$. Such TPOs can be lifted to total preorders on the set of propositions via $A \preceq B$ iff there is a (minimal) $\omega \in Mod(A)$ such that $\omega \preceq \omega'$ for all $\omega' \in Mod(B)$. If $\Omega' \subseteq \Omega$, then $\min_\preceq(\Omega') = \{\omega' \in \Omega' \,|\, \omega' \preceq \omega'' \text{ for all } \omega'' \in \Omega'\}$ denotes the set of $\preceq$-minimal models in Ω'. If $\Omega' = \Omega$, then we simply write $\min(\preceq)$ instead of $\min_\preceq(\Omega)$. If $A \in \mathcal{L}$, then $\min_\preceq(A) = \min_\preceq(Mod(A))$. The agent believes exactly the propositions that are valid in all most plausible models.

Applying preferential entailment to a preferential model which is given by a TPO $\preceq$ yields the nonmonotonic inference relation $\mathrel{|\!\sim}_\preceq$ which is given by

$$A \mathrel{|\!\sim}_\preceq B \text{ iff } AB \prec A\overline{B}. \tag{1}$$

Conversely, from this we obtain

$$A \prec B \text{ iff } A \vee B \mathrel{|\!\sim}_\preceq \overline{B}, \tag{2}$$

because $A \prec B$ is equivalent to $\min\{AB, A\overline{B}\} \prec \min\{AB, \overline{A}B\}$, and this can only hold if $A\overline{B} \prec \min\{AB, \overline{A}B\} \equiv B$.

Note that all these definitions depend crucially on the given language, i.e., if the logical language changes, the format of (inferred) beliefs will change, too. Similar to OCFs, we can also *marginalize* total preorders and even inference relations, i.e., restricting them to sublanguages, in a natural way: If $\Theta \subseteq \Sigma$ then any TPO $\preceq$ on $\Omega(\Sigma)$ induces uniquely a *marginalized TPO* $\preceq_{|\Theta}$ on $\Omega(\Theta)$ by setting

$$\omega_1^\Theta \preceq_{|\Theta} \omega_2^\Theta \text{ iff } \omega_1^\Theta \preceq \omega_2^\Theta. \tag{3}$$

Note that on the right hand side of the *iff* condition above $\omega_1^\Theta, \omega_2^\Theta$ are considered as propositions in the superlanguage $\mathcal{L}(\Omega)$, hence $\omega_1^\Theta \preceq_\Psi \omega_2^\Theta$ is well defined (Kern-Isberner and Brewka 2017).

In (Beierle and Kern-Isberner 2012), many different semantics for conditionals logics, including TPOs, OCFs, probablity distributions, possibility measures, conditional objects, and variants thereof, are formalized as *institutions* (Goguen and Burstall 1992). The marginalization for OCFs and TPOs presented above are special cases of the general forgetful functor $Mod(\sigma)$ from Σ-models to Σ'-models given in (Beierle and Kern-Isberner 2012) where $\Sigma' \subseteq \Sigma$ and σ is the inclusion from Σ' to Σ.

Similarly, any inference relation $\mathrel{|\!\sim}$ on $\mathcal{L}(\Sigma)$ induces a *marginalized inference relation* $\mathrel{|\!\sim}_{|\Theta}$ on $\mathcal{L}(\Theta)$ by setting

$$A \mathrel{|\!\sim}_{|\Theta} B \text{ iff } A \mathrel{|\!\sim} B \tag{4}$$

for any $A, B \in \mathcal{L}(\Theta)$. Marginalization helps us to focus on relevant parts of the language.

Conditionals from $(\mathcal{L} \mid \mathcal{L})$ can be considered to encode nonmonotonic inferences, briefly by stating that a conditional $(B|A)$ is *accepted* if $A \mathrel{|\!\sim} B$ holds. Roughly, the basic prerequisite for both statements is that AB is more plausible than $A\overline{B}$ resp. preferred to $A\overline{B}$. This nicely matches the intuitive understanding of conditionals. So, we say that a total preorder $\preceq$ *accepts* a conditional $(B|A)$, denoted as $\preceq \models (B|A)$, if $AB \prec A\overline{B}$. Note that also OCFs κ induce total preorders on Ω via $\omega_1 \preceq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$, so everything we state on total preorders will apply to OCFs, but OCFs allow for more expressive statements because of their usage of natural numbers and the corresponding arithmetics. Moreover, the nonmonotonic inference relation $\mid\!\sim_\kappa$ induced by $\preceq_\kappa$ is given by (Spohn 1988)

$$A \mid\!\sim_\kappa B \text{ iff } \kappa(AB) < \kappa(A\overline{B}) \ \ (i.e., \text{ iff } \kappa \models (B|A)). \quad (5)$$

Conditional belief bases Δ (over $\mathcal{L}$) consist of finitely many conditionals from $(\mathcal{L} \mid \mathcal{L})$. Consistency of such a conditional belief base Δ can be defined in terms of OCFs (Pearl 1990): Δ is consistent iff there is an OCF κ such that $\kappa \models \Delta$.

A well-known inference relation that is based on OCFs and conditional belief bases is system Z (Pearl 1990) which makes use of a so-called *tolerance partition* of the conditional belief base Δ. First, a conditional $(B|A)$ is *tolerated* by Δ iff there is a world $\omega \in \Omega$ such that $\omega \models AB$ and ω does not falsify any conditional in Δ. Then, the first partitioning set $\mathcal{D}_0 \subseteq \Delta$ contains all conditionals from Δ that are tolerated by Δ. Recursively, continuing with $i = 1$, $\mathcal{D}_i$ contains all conditionals which are tolerated by $\Delta \setminus \cup_{j<i} \mathcal{D}_j$, until all conditionals from Δ are contained in some $\mathcal{D}_i$, or no tolerated conditional can be found. Δ is consistent iff such a so-called Z-partition $\Delta = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_m$ exists (Pearl 1990). Note that for the unique Z-partition, all $\mathcal{D}_i$'s have to be maximal, so that for a conditional $(B|A) \in \Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$, $Z_\Delta((B|A)) = j$ iff $(B|A) \in \mathcal{D}_j$ is well-defined. The OCF κ^z_Δ is defined as (Pearl 1990)

$$\kappa^z_\Delta(\omega) = \max_{1 \leqslant i \leqslant n} \{Z_\Delta((B_i|A_i)) \mid \omega \models A_i\overline{B_i}\} + 1, \quad (6)$$

where $\max \emptyset = -1$. The *system-Z inference relation* $\mid\!\sim^z_\Delta$ is then defined by $A \mid\!\sim^z_\Delta B$ iff $\kappa^z_\Delta(AB) < \kappa^z_\Delta(A\overline{B})$.

4 Inductive Reasoning from Belief Bases

We focus on consistent conditional belief bases $\Delta \subset (\mathcal{L} \mid \mathcal{L})$ over a propositional language $\mathcal{L}$ (Goldszmidt and Pearl 1996) in this paper and consider (nonmonotonic) inference relations which are *induced* by such bases. Our approach and results can be generalized to cover also the inconsistent case by slight, suitable extensions of the definitions, but for the sake of clarity, we will presuppose that all conditional belief bases are consistent; moreover, we assume that all conditionals $(B|A)$ are non-contradictory, i.e., $AB \not\equiv \bot$.

We start with the most general case of such inductive inference operators. Besides the (DI) axiom from the introduction, we will also presuppose that empty belief bases should not license any non-trivial inferences:

(Trivial Vacuity) If $\Delta = \emptyset$, then $A \mid\!\sim B$ only if $A \models B$.

Definition 1 (inductive inference operator $\mathbf{C}$)**.** *An* inductive inference operator (from conditional belief bases) (on $\mathcal{L}$) *is a mapping* $\mathbf{C}$ *that assigns to each conditional belief base* $\Delta \subseteq (\mathcal{L} \mid \mathcal{L})$ *an inference relation* $\mid\!\sim_\Delta$ *on* $\mathcal{L}$ *such that (DI) and (Trivial Vacuity) are satisfied:*

$$\mathbf{C} : \Delta \mapsto \ \mid\!\sim_\Delta .$$

Example 1. *System P can be defined in terms of inductive inference from conditional belief bases as follows: Given a conditional belief base* Δ*, let* $\mid\!\sim^P_\Delta$ *be the minimal inference relation that satisfies (DI) and is closed under all axioms of system P. Then the inductive inference operator* $\mathbf{C}^P$ *can be defined via*

$$\mathbf{C}^P : \Delta \mapsto \ \mid\!\sim^P_\Delta . \quad (7)$$

It is clear that $\mathbf{C}^P$ *satisfies both (DI) and (Trivial Vacuity).*

Mentioning Δ *explicitly helps checking P-inferences for consistent belief bases:* $A \mid\!\sim^P_\Delta B$ *iff* $\Delta \cup \{(\overline{B}|A)\}$ *is inconsistent (Adams 1966). A further characterization of P-inferences can be obtained via OCFs:* $A \mid\!\sim^P_\Delta B$ *iff* $\kappa \models \Delta$ *implies* $\kappa \models (B|A)$ *(Goldszmidt and Pearl 1996).*

We take up the example from the introduction and extend it a bit to use it as a running example throughout the paper.

Example 2. *Let* $\Sigma = \{p, b, f, d, v\}$ *with the atoms having the following meaning, respectively: penguins (p), birds (b), being able to fly (f), dark objects (d), being visible in the night (v), and let* $\Delta = \{(f|b), (b|p), (\overline{f}|p), \overline{v}|d)\}$*. Then we have, e.g.,* $pb \mid\!\sim^P_\Delta \overline{f}$ *(with (DI) and (CM)), and* $d \mid\!\sim^P_\Delta \overline{v}$ *(with (DI)). However, no inferences regarding dark birds (bd) can be drawn, i.e., system P is undecided about their ability to fly, or whether they are plausibly visible in the night or not.*

More well-behaved inference relations are usually obtained by associating one specific model (i.e., a TPO, or an OCF) to a conditional belief base, and then base inferences on that model:

Definition 2 (inductive inference operator $\mathbf{C}^{tpo}$ for TPOs)**.** *A* model-based inductive inference operator (from conditional belief bases) for total preorders (on Ω) *is a mapping* $\mathbf{C}^{tpo}$ *that assigns to each conditional belief base* Δ *a total preorder* $\preceq_\Delta$ *on* Ω *such that* $\preceq_\Delta \models \Delta$ *and* $\preceq_\emptyset = \preceq_u$*, i.e., such that (DI) and (Trivial Vacuity) are ensured:*

$$\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta .$$

The appertaining inference relation $\mid\!\sim_{\preceq_\Delta}$ *is obtained via (1), i.e.,* $A \mid\!\sim_{\preceq_\Delta} B$ *iff* $AB \prec_\Delta A\overline{B}$*.*

The most expressive TPOs that we consider in this paper are given by OCFs, so OCFs will also serve as models of conditional belief bases:

Definition 3 (inductive inference operator $\mathbf{C}^{ocf}$ for OCFs)**.** *A* model-based inductive inference operator (from conditional belief bases) for OCFs (on $\mathcal{L}$) *is a mapping* $\mathbf{C}^{ocf}$ *that assigns to each conditional belief base* $\Delta \subseteq (\mathcal{L} \mid \mathcal{L})$ *an OCF* κ_Δ *on* $\mathcal{L}$ *such that* $\kappa_\Delta \models \Delta$ *and* $\kappa_\emptyset = \kappa_u$*, i.e., such that (DI) and (Trivial Vacuity) are ensured:*

$$\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta .$$

The appertaining inference relation $\mid\!\sim_{\kappa_\Delta}$ *is obtained via (5), i.e.,* $A \mid\!\sim_{\kappa_\Delta} B$ *iff* $\kappa(AB) < \kappa(A\overline{B})$*.*

Example 3. *System Z (6) yields an inductive inference operator based on* κ^z_Δ *as follows:*

$$\mathbf{C}^z : \Delta \mapsto \kappa^z_\Delta . \quad (8)$$

Also $\mathbf{C}^z$ *satisfies both (DI) and (Trivial Vacuity).*

Note that each inductive inference operator $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ for OCFs induces an inductive inference operator

$$\mathbf{C}^{tpo} : \Delta \mapsto \preceq_{\kappa_\Delta} \tag{9}$$

for TPOs, i.e., by mapping Δ to the total preorder $\preceq_{\kappa_\Delta}$ induced by κ_Δ. In an analogous way, each inductive inference operator $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ for TPOs induces an inductive inference operator $\mathbf{C}$ on $\mathcal{L}$ by setting

$$\mathbf{C} : \Delta \mapsto \mid\!\sim_{\preceq_\Delta}. \tag{10}$$

For the inference relation that we obtain in this way from Δ via system Z, we write simply $\mid\!\sim^z_\Delta$.

We continue our running example with system Z.

Example 4. *Let Σ, Δ be as in Example 2. First, we set up the tolerance partitioning. Since (only) the penguin conditionals are not tolerated by the other conditionals in Δ but tolerate one another, we obtain $\mathcal{D}_0 = \{(f|b), (\overline{v}|d)\}$ and $\mathcal{D}_1 = \{(b|p), (\overline{f}|p)\}$. Therefore, for κ^z_Δ, we compute $\kappa^z_\Delta(\omega) = 2$ iff $\omega \models p\overline{b}$ or $\omega \models pf$, and for the rest of the possible worlds, we have $\kappa^z_\Delta(\omega) = 1$ iff $\omega \models b\overline{f}$ or $\omega \models vd$, i.e., $\kappa^z_\Delta(\omega) = 1$ iff $\omega \models b\overline{f} \vee vd\overline{p}$. For all other possible worlds, we obtain $\kappa^z_\Delta(\omega) = 0$. System Z yields all system-P inferences, and more. In particular, we are now able to conclude that dark birds fly and are not visible in the night: $bd \mid\!\sim^z_\Delta f$ and $bd \mid\!\sim^z_\Delta \overline{v}$ because $\kappa^z_\Delta(bdf) = 0 = \kappa^z_\Delta(bd\overline{v})$ and $\kappa^z_\Delta(bd\overline{f}) = 1 = \kappa^z_\Delta(bdv)$, hence both $\kappa^z_\Delta(bdf) < \kappa^z_\Delta(bd\overline{f})$ and $\kappa^z_\Delta(bd\overline{v}) < \kappa^z_\Delta(bdv)$.*

5 Syntax Splitting

In the following, we focus on the case that the conditional belief base Δ splits into subbases Δ_1, Δ_2 over disjoint sublanguages, i.e., $\Delta = \Delta_1 \cup \Delta_2, \Delta_i \subset (\mathcal{L}_i|\mathcal{L}_i), \mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i = 1, 2$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\Sigma_1 \cup \Sigma_2 = \Sigma$, writing

$$\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$$

in this case. In this scenario, for any world $\omega = \omega^1\omega^2$ with $\omega^i \in \Omega(\Sigma_i), i \in \{1, 2\}$, and for any $A \in \mathcal{L}_i$, we have

$$\omega^1\omega^2 \models A \text{ iff } \omega^i \models A, \tag{11}$$

$i \in \{1, 2\}$. Note that we do not assume the sublanguages $\mathcal{L}_i$ to be chosen minimally, i.e., there may be atoms in $\mathcal{L}_i$ that are not mentioned in Δ_i, and that any of Δ_i may be empty. When only one of Δ_1, Δ_2 is empty, this means that the conditionals in Δ make use only of atoms of a sublanguage of $\mathcal{L}$. So, this base case covers the most general case of syntax splitting which can be recursively applied to finer syntax splittings (in the sense of Parikh (Parikh 1999)).

Example 5. *In our Example 2, $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ with $\Sigma_1 = \{p, b, f\}, \Sigma_2 = \{d, v\}$ and $\Delta_1 = \{(f|b), (b|p), (\overline{f}|p)\}$, $\Delta_2 = \{(\overline{v}|d)\}$. Since $p\overline{f} \in \mathcal{L}_1 = \mathcal{L}(\Sigma_1)$, we have $\omega = pb\overline{f}\overline{v}d \models p\overline{f}$ iff $\omega^1 = pb\overline{f} \models p\overline{f}$.*

We start with considering general inductive inferences. If an atom of the language does not occur in a belief base at all, then we would expect the inferences based on that belief base not to depend on that atom. This can be generalized to propositions, or even conditionals using only atoms that do not occur in the belief base, leading in quite a natural way to the following relevance property:

(Rel) An inductive inference operator $\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta$ on $\mathcal{L}$ satisfies **(Rel)** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_i$ $(i \in \{1, 2\})$,

$$A \mid\!\sim_\Delta B \quad \text{iff} \quad A \mid\!\sim_{\Delta_i} B. \tag{12}$$

This relevance property means that only conditionals from the respective sublanguage are relevant for inductive inferences, and that conditionals of the respective other subbase are irrelevant for these inferences. In the paper (Weydert 2003), a corresponding axiom can be found under the name *Strong Irrelevance*.

Example 6. *Taking up the setting from Example 5, (Rel) implies $pb \mid\!\sim_\Delta \overline{f}$ iff $pb \mid\!\sim_{\Delta_1} \overline{f}$. However, (Rel) does not say anything about dark penguins, pd.*

Moreover, we do not want any sentence from the respective other sublanguage to interfere with inductive inferences regarding one sublanguage, i.e., such inferences should be independent from any proposition over the other sublanguage:

(Ind) An inductive inference operator $\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta$ on $\mathcal{L}$ satisfies **(Ind)** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_i, C \in \mathcal{L}_j$ $(i, j \in \{1, 2\}, j \neq i)$,

$$A \mid\!\sim_\Delta B \text{ iff } AC \mid\!\sim_\Delta B. \tag{13}$$

These two basic properties make up jointly the property of syntax splitting:

(SynSplit) An inductive inference operator $\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta$ on $\mathcal{L}$ satisfies **(SynSplit)** if it satisfies (Rel) and (Ind).

We illustrate how (SynSplit) can help to reduce and focus inference problems.

Example 7. *Continuing Example 5, if $\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta$ satisfies (SynSplit) then we obtain $pbd \mid\!\sim_\Delta \overline{f}$ iff $pb \mid\!\sim_\Delta \overline{f}$ (by (Ind)), iff $pb \mid\!\sim_{\Delta_1} \overline{f}$ (by (Rel)).*

Actually, the (Rel)-property is equivalent to stating that the inductive inference is compatible with marginalisation:

Proposition 1. *An inductive inference operator $\mathbf{C}$ satisfies (Rel) iff for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, $\mathbf{C}(\Delta_i) = \mathbf{C}(\Delta)|_{\Sigma_i}$.*

The first inductive inference operator that we evaluate with respect to syntax splitting, is the system P operator $\mathbf{C}^P$ defined by (7). Since system P takes into account all ranking functions accepting Δ, it is easy to find counterexamples to (Ind) (e.g., see Example 2). Nevertheless, the next proposition shows that system P satisfies (Rel):

Proposition 2. $\mathbf{C}^P$ *satisfies (Rel).*

Lemma 2 will prove useful for the proof of Proposition 2:

Lemma 2. *$\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ is consistent iff both of Δ_1, Δ_2 are consistent.*

Proof. The "only-if" direction is trivial: If there is an OCF accepting Δ, then it also accepts both Δ_1, Δ_2. For the other direction, from OCFs κ_1, κ_2 accepting Δ_1 resp. Δ_2, we can construct $\kappa(\omega^1\omega^2) = \kappa_1(\omega^1) + \kappa_2(\omega^2)$ (i.e., by considering Σ_1, Σ_2 to be independent). It is straightforward to show that κ is an OCF accepting Δ, hence Δ is consistent. □

Proof of Proposition 2. Let $A, B \in \mathcal{L}_i$. $A \mid\!\sim^P_\Delta B$ is equivalent to $\Delta \cup \{(\overline{B}|A)\}$ being inconsistent (see Ex. 1), which is equivalent to $\Delta_i \cup \{(\overline{B}|A)\}$ being inconsistent according to Lemma 2 and the fact that $(\overline{B}|A)$ cannot be inconsistent with Δ_j. But this is equivalent to $A \mid\!\sim^P_{\Delta_i} B$. □

Given that both total preorders and ranking functions induce inference relations, we will now lift the properties (Rel), (Ind), and (SynSplit), respectively, to TPOs and OCFs so that the induced inference relations (via (1) and (5)) satisfy the respective property.

(Rel^{tpo}) An inductive inference operator for TPOs $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ on $\mathcal{L}$ satisfies **(Rel^{tpo})** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_i$ $(i = 1, 2)$,

$$A \preceq_\Delta B \quad \text{iff} \quad A \preceq_{\Delta_i} B. \tag{14}$$

(Ind^{tpo}) An inductive inference operator for TPOs $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ on $\mathcal{L}$ satisfies **(Ind^{tpo})** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_i, C \in \mathcal{L}_j$ $(i, j \in \{1, 2\}, j \neq i)$,

$$A \preceq_\Delta B \quad \text{iff} \quad AC \preceq_\Delta BC. \tag{15}$$

We obtain the syntax-splitting property again as a combination of relevance and independence:

(SynSplit^{tpo}) An inductive inference operator $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ for total preorders on $\mathcal{L}$ satisfies **(SynSplit^{tpo})** if it satisfies (Rel^{tpo}) and (Ind^{tpo}).

The next example illustrates how relations holding between two worlds over a sublanguage with respect to a subbase can be lifted to relations between worlds over the full language with respect to the full base.

Example 8. *We continue our running example from Example 5. The following table shows a TPO $\preceq_{\Delta_1}$ on $\Omega_1 = \Omega(\Sigma_1)$ for Δ_1 (where the most plausible worlds are in the lowermost layer):*

$pbf, p\overline{b}f, p\overline{b}\,\overline{f}$
$\overline{p}b\overline{f}, pb\overline{f}$
$\overline{p}bf, \overline{p}\overline{b}f, \overline{p}\overline{b}\,\overline{f}$

Note that $\preceq_{\Delta_1}$ corresponds to the ranking $\kappa^z_{\Delta_1}$ on Ω_1 (see also Example 4), so (DI) is satisfied. Now, for example, from $\overline{p}bf \prec_{\Delta_1} \overline{p}b\overline{f}$, we may obtain $\overline{p}bf \prec_\Delta \overline{p}b\overline{f}$ by (Rel), and furthermore $\overline{p}bf\dot{d}\dot{v} \prec_\Delta \overline{p}b\overline{f}\dot{d}\dot{v}$ via (Ind) for any literals $\dot{d} \in \{d, \overline{d}\}, \dot{v} \in \{v, \overline{v}\}$ (please note that $\dot{d}, \dot{v}$ can be arbitrary but must be fixed, i.e., the same literals must occur on both sides of the inequality).

Before turning to ranking functions, we analyze the relevance and independence properties for total preorders a bit closer. Again, immediately from the definition of marginalization of total preorders (3), (Rel^{tpo}) claims that $\mathbf{C}^{tpo}$ is compatible with marginalisation:

Proposition 3. *An inductive inference operator $\mathbf{C}^{tpo}$ satisfies (Rel^{tpo}) iff for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, $\mathbf{C}^{tpo}(\Delta_i) = \mathbf{C}^{tpo}(\Delta)|_{\Sigma_i}$, i.e., iff $\preceq_{\Delta_i} = \preceq_\Delta|_{\Sigma_i}$.*

The (Ind^{tpo})-property needs to be exploited a bit more. We first present a purely semantic independence property for total preorders, and then show how (Ind^{tpo}) can be based on it.

Definition 4. *Let $\preceq$ be a total preorder on $\Omega(\Sigma)$, and let Σ_1, Σ_2 be two (disjoint) subsignatures of Σ. Let $\Omega_i = \Omega(\Sigma_i)$ be the respective sets of worlds, containing the reducts ω^i, $i \in \{1, 2\}$. Then Σ_1, Σ_2 are* independent with respect to *$\preceq$, or simply $\preceq$-*independent*, if for any $i \in \{1, 2\}$, for any $\omega^i_1, \omega^i_2 \in \Omega_i$, $\omega^j \in \Omega_j$, $j \in \{1, 2\}, j \neq i$, $\omega^i_1 \preceq \omega^i_2$ iff $\omega^i_1\omega^j \preceq \omega^i_2\omega^j$.*

So, $\preceq$-independence means that there should not be any interference between (marginalized) $\preceq$-relationships on world reducts of one sublanguage and information (expressed as world reducts) from the respective other sublanguage. This is kind of a ceteris paribus-condition since the same ω^j has to be taken into account on both sides; otherwise, $\preceq$-relationships from Ω_j might have an influence.

It is now straightforward to relate (Ind^{tpo}) with independence with respect to total preorders:

Proposition 4. *An inductive inference operator for TPOs $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ on $\mathcal{L}$ satisfies (**Ind**^{tpo}) iff for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, Σ_1, Σ_2 are $\preceq_\Delta$-independent.*

This shows that inductive inference operator for TPOs respect independence of sublanguages unless dependencies are justified by the belief base.

So, for OCFs, we can now directly express relevance and independence by the concepts of marginalization and independence which are well-known in the OCF-framework (and inherited by probabilities):

(Rel^{ocf}) An inductive inference operator for OCFs $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ on $\mathcal{L}$ satisfies **(Rel^{ocf})** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$,

$$\mathbf{C}^{ocf}(\Delta_i) = \mathbf{C}^{ocf}(\Delta)|_{\Sigma_i}, \tag{16}$$

i.e., if $\kappa_{\Delta_i} = \kappa_\Delta|_{\Sigma_i}$.

(Ind^{ocf}) An inductive inference operator for OCFs $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ on $\mathcal{L}$ satisfies **(Ind^{ocf})** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$, Σ_1, Σ_2 are κ-independent, i.e. for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, $\kappa_\Delta(\omega^1\omega^2) = \kappa_\Delta(\omega^1) + \kappa_\Delta(\omega^2)$.

(SynSplit^{ocf}) An inductive inference operator $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ for OCFs on $\mathcal{L}$ satisfies **(SynSplit^{ocf})** if it satisfies (Rel^{ocf}) and (Ind^{ocf}).

In the case of OCFs, we obtain a particularly concise characterization of syntax splitting:

Proposition 5. *An inductive inference operator $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ for OCFs on $\mathcal{L}$ satisfies **(SynSplit^{ocf})** iff for any $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, the following equation holds:*

$$\kappa_\Delta = \kappa_{\Delta_1} + \kappa_{\Delta_2}. \tag{17}$$

Proof. From (17), (Rel^{ocf}) and (Ind^{ocf}) can be easily verified. The other way round, if $\mathbf{C}^{ocf}$ satisfies (SynSplit^{ocf}), i.e., (Rel^{ocf}) and (Ind^{ocf}), then we have

$$\begin{array}{rcll} \kappa_\Delta(\omega^1\omega^2) & = & \kappa_\Delta(\omega^1) + \kappa_\Delta(\omega^2) & ((\text{Ind}^{ocf})) \\ & = & \kappa_\Delta|_{\Sigma_1}(\omega^1) + \kappa_\Delta|_{\Sigma_2}(\omega^2) & \\ & = & \kappa_{\Delta_1}(\omega^1) + \kappa_{\Delta_2}(\omega^2) & ((\text{Rel}^{ocf})), \end{array}$$

so (17) holds. □

Since system-Z inferences are not changed by atoms not at all mentioned in the belief base ((Goldszmidt and Pearl 1996), see also Example 4), we might hope that it is also a candidate for the (SynSplit^{ocf}) axiom. However, system Z does not satisfy (Ind^{ocf}), as the following example shows:

Example 9. *We continue Example 4. For the world $\omega = \omega^1\omega^2$ with $\omega^1 = \overline{p}b\overline{f}$ and $\omega^2 = dv$ (i.e., both conditionals from $\mathcal{D}_0$ are falsified, but none of the conditionals from $\mathcal{D}_1$), we obtain $\kappa^z_\Delta(\omega) = 1 \neq 2 = \kappa^z_\Delta(\omega^1) + \kappa^z_\Delta(\omega^2)$, hence (Ind^{ocf}) does not hold.*

Note that this failure was implicitly noticed in an even simpler, unconditional example in (Klassen, McIlraith, and Levesque 2018) in the setting of belief revision, but it can also be modelled in terms of nonmonotonic inferences here: In that paper, the authors considered the belief base $\Delta = \{(a|\top), (b|\top)\}$ with atoms a, b, and observed that when revising by $\neg a$, then also belief in b is lost. Simulating this in our framework, this would mean that $\neg a \not\mathrel{|\!\sim}_{\kappa^z_\Delta} b$ although $\top \mathrel{|\!\sim}_{\kappa^z_\Delta} b$ holds, and indeed, for the same reasons as above, we have $\kappa^z_\Delta(\overline{a}\overline{b}) = 1 = \kappa^z_\Delta(\overline{a}b)$.

Nevertheless, as for system P, system Z satisfies the relevance axiom:

Proposition 6. $\mathbf{C}^z$ *satisfies (Rel^{ocf}).*

Proof. Let $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$. The crucial point in this proof is the observation that a conditional $(B|A) \in \Delta_i, i \in \{1,2\}$, is tolerated by a subset Δ' of Δ iff it is tolerated by the intersection $\Delta' \cap \Delta_i$ because of (11). This implies that the partitioning sets of the tolerance partition of Δ are (disjoint) unions of the partitioning sets of the respective partitioning sets of the two subbases. Therefore, for $(B|A) \in \Delta_i$, we have $Z_\Delta((B|A)) = Z_{\Delta_i}((B|A)), i \in \{1,2\}$. Now it is straightforward to check that $\kappa^z_{\Delta_i} = \kappa^z_\Delta|_{\Sigma_i}$, hence (Rel^{ocf}) holds for system Z. □

All postulates are downward compatible:

Proposition 7. *(Rel^{ocf}) resp. (Ind^{ocf}) resp. (SynSplit^{ocf}) implies (Rel^{tpo}) resp. (Ind^{tpo}) resp. (SynSplit^{tpo}) for the induced inference operator according to (9). (Rel^{tpo}) resp. (Ind^{tpo}) resp. (SynSplit^{tpo}) implies (Rel) resp. (Ind) resp. (SynSplit) for the induced inference operator according to (10).*

In particular, the independence notions for total preorders and inference relations are inherited from OCF independence. Furthermore, we can define a qualitative conditionalization of total preorders which does essentially the same for total preorders what conditionalization does for probability distributions with respect to comparisons.

Definition 5. *Let $\preceq$ be a total preorder on a set of worlds $\Omega = \Omega(\Sigma)$, and let $A \in \mathcal{L} = \mathcal{L}(\Sigma)$. The* conditionalization *of $\preceq$ by A, in terms $\preceq|^A$, is defined by*

$$\omega_1 \preceq|^A \omega_2 \text{ iff } \omega_1 \wedge A \leqslant \omega_2 \wedge A. \tag{18}$$

Note that by (18) indeed, the $\preceq$-relationships between models of A are preserved, while all models of $\neg A$ are shifted to the uppermost layer. Qualitative conditionalization can be lifted as usual to the level of propositions by stating $B \preceq|^A C$ iff $AB \preceq AC$ for any $B, C \in \mathcal{L}$. With the conditionalization operator, (Ind^{tpo}) can now be expressed as follows: $\mathbf{C}^{tpo}$ satisfies (Ind^{tpo}) iff for any $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, and for any $A, B \in \mathcal{L}_i, C \in \mathcal{L}_j,\ i, j \in \{1,2\}, j \neq i$,

$$A \preceq_\Delta B \quad \text{iff} \quad A \preceq_\Delta|^C B \tag{19}$$

i.e., iff for all $C \in \mathcal{L}_j,\ \preceq_\Delta|^C$ coincides with $\preceq_\Delta$ on $\mathcal{L}(\Sigma_i)$, $i, j \in \{1,2\}, j \neq i$. Note that we do not need $\preceq_\Delta$ to be induced by a conditional belief base here, and regarding Definition 4 and Proposition 4, (19) expresses the $\preceq$-independence of two sets of variables Σ_1, Σ_2 in terms of qualitative conditionalization.

We continue with investigating relationships between our syntax-splitting axioms, and we obtain a representation theorem for inductive inference operators based on TPOs:

Theorem 1. *Let $\mathbf{C} : \Delta \mapsto \mathrel{|\!\sim}_\Delta$ be an inductive inference operator on $\mathcal{L}$ that is implemented via total preorders, i.e., there is an inductive inference operator for TPOs $\mathbf{C}^{tpo} : \Delta \mapsto \preceq_\Delta$ such that*

$$\mathbf{C}(\Delta) = \mathrel{|\!\sim}_\Delta = \mathrel{|\!\sim}_{\preceq_\Delta} = \mathrel{|\!\sim}_{\mathbf{C}^{tpo}(\Delta)}. \tag{20}$$

Then $\mathbf{C}$ satisfies (SynSplit) iff $\mathbf{C}^{tpo}$ satisfies (SynSplit^{tpo}).

Proof. Let $\mathbf{C}$ be as given by (20). We show that (Ind) and (Rel) of $\mathbf{C}$ are equivalent to (Ind^{tpo}) and (Rel^{tpo}) of $\mathbf{C}^{tpo}$, respectively. Let $\Delta = \Delta_1 \bigcup\limits_{\Sigma_1,\Sigma_2} \Delta_2$, and let $A, B \in \mathcal{L}_i$ $(i = 1,2)$. We use (Ind^{tpo}) and (Rel^{tpo}) in their contrapositive versions:

$$\begin{array}{lll} A \prec_\Delta B & \text{iff} & AC \prec_\Delta BC, \\ & & \text{where } C \in \mathcal{L}_j,\ j \in \{1,2\}, j \neq i, \end{array} \tag{21}$$

$$A \prec_\Delta B \quad \text{iff} \quad A \prec_{\Delta_i} B. \tag{22}$$

Using the connection between $\preceq_\Delta$ and $\mathrel{|\!\sim}_\Delta$ provided by (1), (2), and (20), it is straightforward to translate (21) and (22) into (Ind) and (Rel), respectively, and vice versa. □

None of the inference operators considered so far was able to fully comply with the syntax splitting axioms. In the next section, we present inference operators based on so-called c-revisions (Kern-Isberner 2001; Kern-Isberner 2004) that satisfy both the independence and relevance axioms that are needed for full syntax splitting.

6 c-Representations and c-Inference

Among the OCF models of Δ, c-representations are special ranking models obtained by assigning individual integer impacts to the conditionals in Δ and generating the world ranks as the sum of impacts of falsified conditionals.

Definition 6 (c-representation (Kern-Isberner 2001; Kern-Isberner 2004))**.** *A* c-representation *of a conditional knowledge base* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ *is an OCF* κ *constructed from non-negative integer impacts* $\eta_j \in \mathbb{N}_0$ *assigned to each* $(B_j|A_j)$ *such that* κ *accepts* Δ *and is given by:*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B}_j}} \eta_j \tag{23}$$

C-representations can conveniently be specified using a constraint satisfaction problem (for detailed explanations, see (Kern-Isberner 2001; Kern-Isberner 2004; Beierle et al. 2018)):

Definition 7 ($CR(\Delta)$)**.** *Let* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. *The* constraint satisfaction problem for c-representations of Δ, *denoted by* $CR(\Delta)$, *is given by the conjunction of the constraints, for all* $j \in \{1, \ldots, n\}$*:*

$$\eta_j \geqslant 0 \tag{24}$$

$$\eta_j > \min_{\omega \models A_j B_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k - \min_{\omega \models A_j \overline{B}_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k \tag{25}$$

Constraint (24) expresses that falsification of conditionals should make worlds less plausible, and (25) ensures that κ as specified by (23) accepts Δ.

A solution of $CR(\Delta)$ is a vector $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ of natural numbers. $Sol(CR(\Delta))$ denotes the set of all solutions of $CR(\Delta)$. For $\vec{\eta} \in Sol(CR(\Delta))$ and κ as in Equation (23), κ is the *OCF induced by* $\vec{\eta}$ and is denoted by $\kappa_{\vec{\eta}}$. $CR(\Delta)$ is sound and complete (Kern-Isberner 2001; Beierle et al. 2018): For every $\vec{\eta} \in Sol(CR(\Delta))$, $\kappa_{\vec{\eta}}$ is a c-representation with $\kappa_{\vec{\eta}} \models \Delta$, and for every c-representation κ with $\kappa \models \Delta$, there is $\vec{\eta} \in Sol(CR(\Delta))$ such that $\kappa = \kappa_{\vec{\eta}}$.

We will now define model-based inductive inference operators assigning a c-representation κ to each Δ. Since every c-representation κ with $\kappa \models \Delta$ yields an inference relation expanding the beliefs in Δ, we employ a selection function for modelling the different possible choices of which c-representation should be selected.

Definition 8 (selection strategy σ)**.** *A* selection strategy (for c-representations) *is a function* σ

$$\sigma : \Delta \mapsto \vec{\eta}$$

assigning to each conditional belief base Δ *an impact vector* $\vec{\eta} \in Sol(CR(\Delta))$.

Example 10. *We take up Example 5 and compute the c-representations of* Δ_1 *and* Δ_2*, and their respective constraint satisfaction problems. Using (23), we obtain the schema for c-representations* $\kappa_{\vec{\eta}^1}$ *for* Δ_1 *and c-representations* $\kappa_{\vec{\eta}^2}$ *for* Δ_2*, respectively, as follows:*

ω	$\kappa_{\vec{\eta}^1}(\omega)$	ω	$\kappa_{\vec{\eta}^1}(\omega)$	ω	$\kappa_{\vec{\eta}^2}(\omega)$
pbf	η_3^1	$\overline{p}bf$	0	vd	η_1^2
$pb\overline{f}$	η_1^1	$\overline{p}b\overline{f}$	η_1^1	$v\overline{d}$	0
$p\overline{b}f$	$\eta_2^1 + \eta_3^1$	$\overline{p}\overline{b}f$	0	$\overline{v}d$	0
$p\overline{b}\,\overline{f}$	η_2^1	$\overline{p}\overline{b}\,\overline{f}$	0	$\overline{v}\overline{d}$	0

(26)

The vectors $\vec{\eta}^1 = (\eta_1^1, \eta_2^1, \eta_3^1)$ *and* $\vec{\eta}^2 = (\eta_1^2)$ *are constrained by the inequalities* $\eta_1^1 > 0; \eta_2^1 > \eta_1^1; \eta_3^1 > \eta_1^1; \eta_1^2 > 0$*; so* $\vec{\eta}^1 = (1, 2, 2)$ *and* $\vec{\eta}^2 = (1)$ *would be solutions with minimal* η_i^j*. A selection strategy* σ *then might reasonably choose* $\sigma(\Delta_1) = (1, 2, 2)$ *and* $\sigma(\Delta_2) = (1)$.

Definition 9 (inductive inference operator $\mathbf{C}_\sigma^{c\text{-}rep}$)**.** *An* inductive inference operator for c-representations with selection strategy σ *is a function*

$$\mathbf{C}_\sigma^{c\text{-}rep} : \Delta \mapsto \kappa_{\sigma(\Delta)}$$

where σ *is a selection strategy for c -representations; as before,* $\mathrel{|\!\sim}_{\kappa_{\sigma(\Delta)}}$ *is obtained via Equation (5).*

Note that $\mathbf{C}_\sigma^{c\text{-}rep}$ is an inductive inference operator because each $\mathrel{|\!\sim}_{\kappa_{\sigma(\Delta)}}$ satisfies both (DI) and (Trivial Vacuity).

In principle, for every Δ, a selection strategy may choose some impact vector independently from the choices for all other belief bases. The following property characterizes selection strategies that preserve the impacts chosen for subbases if Δ splits into these subbases. In accordance with our notation in Example 10, for an impact vector $\vec{\eta}$, we will simply write $\vec{\eta}^1$ and $\vec{\eta}^2$ for the corresponding projections $\vec{\eta}|_{\Delta_1}$ and $\vec{\eta}|_{\Delta_2}$, and $(\vec{\eta}^1, \vec{\eta}^2)$ for their composition.

(IP$^{c\text{-}rep}$) A selection strategy σ is *impact preserving* if $\sigma(\Delta) = (\sigma(\Delta_1), \sigma(\Delta_2))$ for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$.

Example 11. *If the selection strategy* σ *in Example 10 is impact preserving then* $\sigma(\Delta) = (1, 2, 2, 1)$.

The following proposition provides the basis for impact preserving selection strategies. It shows the fundamental property of c-representations stating that the composition of *any* impact vectors for subbases Δ_1, Δ_2 which Δ splits into yields an impact vector for Δ, and vice versa.

Proposition 8. *For any* $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ *we have* $Sol(CR(\Delta)) = \{(\vec{\eta}^1, \vec{\eta}^2) \mid \vec{\eta}^i \in Sol(CR(\Delta_i)), i = 1, 2\}$*, i.e.:*

$$Sol(CR(\Delta)) = Sol(CR(\Delta_1)) \times Sol(CR(\Delta_2))$$

Proof. (sketch) Due to lack of space, we outline the proof. Let $\Delta_1 = \{(B_1|A_1), \ldots, (B_{n_1}|A_{n_2})\}$, $\Delta_2 = \{(B_{n_1+1}|A_{n_1+1}), \ldots, (B_{n_1+n_2}|A_{n_1+n_2})\}$; $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$, thus $n = n_1 + n_2$. Let us denote the constraint variables in $CR(\Delta_1)$ with $\eta_1^1, \ldots, \eta_{n_1}^1$ and in $CR(\Delta_2)$ with $\eta_{n_1+1}^2, \ldots, \eta_n^2$. Hence, the following statements

(S1) $\vec{\eta}^1 \in Sol(CR(\Delta_1)), \vec{\eta}^2 \in Sol(CR(\Delta_2))$

(S2) $(\vec{\eta}^1, \vec{\eta}^2) \in Sol(\Gamma), \quad \Gamma = CR(\Delta_1) \cup CR(\Delta_2)$

are obviously equivalent. Let us use $\eta^1_1, \ldots, \eta^1_{n_1}, \ldots, \eta^2_{n_1+1}, \ldots, \eta^2_n$ also as the constraint variables for expressing $CR(\Delta)$. By exploiting in particular that any atom from Σ_1 (resp. Σ_2) can not influence the verification of falsification of a conditional from Δ_2 (resp. Δ_1), we can stepwise transform the right-hand side of the constraint (25) for η^i_j in Γ to the corresponding constraint for η^i_j in $CR(\Delta)$ without changing the set of solutions for the constraint. This yields that (S2) and thus also (S1) is equivalent to $(\vec{\eta}^1, \vec{\eta}^2) \in Sol(CR(\Delta))$, completing the proof. □

For every impact preserving selection strategy, c-representations satisfy **(SynSplit^{ocf})**. Moreover, **(IP$^{c\text{-}rep}$)** precisely characterizes the inductive operators based on single c-representations that satisfy syntax splitting.

Proposition 9. $\mathbf{C}^{c\text{-}rep}_\sigma$ *satisfies **(SynSplit^{ocf})** iff there is a selection strategy σ' such that σ' satisfies **(IP$^{c\text{-}rep}$)** and* $\mathbf{C}^{c\text{-}rep}_\sigma = \mathbf{C}^{c\text{-}rep}_{\sigma'}$.

Proof. First, let σ satisfy (**IP**$^{c\text{-}rep}$); we will show that $\mathbf{C}^{c\text{-}rep}_\sigma$ satisfies **(SynSplit^{ocf})**. Let $\Delta, \Delta_1, \Delta_2$ be as in the proof of Proposition 8, $\sigma(\Delta) = \vec{\eta}, \sigma(\Delta_i) = \vec{\eta}^i$, and hence $\mathbf{C}^{c\text{-}rep}_\sigma(\Delta) = \kappa_{\vec{\eta}}, \mathbf{C}^{c\text{-}rep}_\sigma(\Delta_i) = \kappa_{\vec{\eta}^i}$, for $i \in \{1, 2\}$, and $\vec{\eta} = (\vec{\eta}^1, \vec{\eta}^2)$. Furthermore, let $\eta^1_1, \ldots, \eta^2_n$ be as in the proof of Proposition 8. According to Proposition 5, it suffices to show, for every ω,

$$\kappa_{\vec{\eta}}(\omega) = \kappa_{\vec{\eta}^1}(\omega) + \kappa_{\vec{\eta}^2}(\omega) \tag{27}$$

which we obtain by the following derivation:

$$\begin{aligned}\kappa_{\vec{\eta}}(\omega) &= \sum_{\substack{\omega \models A_j \overline{B}_j \\ (B_j|A_j) \in \Delta}} \eta^i_j = \sum_{\substack{\omega \models A_j \overline{B}_j \\ (B_j|A_j) \in \Delta_1}} \eta^1_j + \sum_{\substack{\omega \models A_j \overline{B}_j \\ (B_j|A_j) \in \Delta_2}} \eta^2_j \\ &= \kappa_{\vec{\eta}^1}(\omega) + \kappa_{\vec{\eta}^2}(\omega)\end{aligned}$$

Hence, $\mathbf{C}^{c\text{-}rep}_\sigma$ satisfies **(SynSplit^{ocf})**. As an immediate consequence, also every $\mathbf{C}^{c\text{-}rep}_{\sigma'}$ with $\mathbf{C}^{c\text{-}rep}_{\sigma'} = \mathbf{C}^{c\text{-}rep}_\sigma$ satisfies **(SynSplit^{ocf})**. The proof of the other direction is obtained by using similar arguments and Proposition 8. □

The next proposition provides useful splitting properties of c-representations regarding formulas from a sublanguage.

Proposition 10. *For any* $\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2$, *for all* $\vec{\eta} \in Sol(CR(\Delta))$, $F_i \in \mathcal{L}(\Sigma_i)$, $i \in \{1, 2\}$, *we have* $\kappa_{\vec{\eta}^1}(F_2) = \kappa_{\vec{\eta}^2}(F_1) = 0$ *and* $\kappa_{\vec{\eta}}(F_i) = \kappa_{\vec{\eta}^i}(F_i)$.

Proof. (sketch) Similar as in Proposition 8, the crucial point is again to exploit the fact that any atom from one sublanguage can not influence the verification/falsification of a conditional over the other sublanguage. □

Inference based on single c-representations does not only satisfy syntax splitting as laid out in Prop. 9, but also exhibits other desirable inference properties (Kern-Isberner 2001; Kern-Isberner 2004). *C-inference* was introduced in (Beierle, Eichhorn, and Kern-Isberner 2016; Beierle et al. 2018) as the skeptical inference relation obtained by taking all c-representations of a belief base Δ into account.

Definition 10 (c-inference, $\mathord{\mid\!\sim}^{c\text{-}sk}_{\mathbf{\Delta}}$)**.** *Let Δ be a belief base and let A, B be formulas. B is* a (skeptical) c-inference from A in the context of Δ, *denoted by* $A \mathrel{\mid\!\sim}^{c\text{-}sk}_\Delta B$, *iff* $A \mathrel{\mid\!\sim}_\kappa B$ *holds for all c-representations κ of Δ.*

Because $\mathord{\mid\!\sim}^{c\text{-}sk}_\Delta$ satisfies (DI) and (Trivial Vacuity) the following proposition holds.

Proposition 11 (inductive inference operator $\mathbf{C}^{c\text{-}sk}$)**.** *The function*

$$\mathbf{C}^{c\text{-}sk} : \Delta \mapsto \mathord{\mid\!\sim}^{c\text{-}sk}_\Delta$$

assigning to each Δ the c-inference relation $\mathord{\mid\!\sim}^{c\text{-}sk}_\Delta$ is an inductive inference operator from conditional belief bases.

The next proposition shows that skeptical c-inference satisfies syntax splitting. Note that since in general, the inference relation $\mathord{\mid\!\sim}^{c\text{-}sk}_\Delta$ can neither be represented by a TPO nor by an OCF, the corresponding syntax splitting characterisations are not applicable to it.

Proposition 12. $\mathbf{C}^{c\text{-}sk}$ *satisfies **(SynSplit)**.*

Proof. Let $\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2$. W.l.o.g. assume $A, B \in \mathcal{L}_1$ and $C \in \mathcal{L}_2$, and let $\dot{B} \in \{B, \overline{B}\}$. For proving **(Rel)**, we have to show $A \mathrel{\mid\!\sim}^{c\text{-}sk}_\Delta B$ iff $A \mathrel{\mid\!\sim}^{c\text{-}sk}_{\Delta_1} B$. Thus, due to Proposition 8, it suffices to show

$$\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B}) \quad \text{iff} \quad \kappa_{\vec{\eta}^1}(AB) < \kappa_{\vec{\eta}^1}(A\overline{B}) \tag{28}$$

for all $\vec{\eta} = (\vec{\eta}^1, \vec{\eta}^2) \in Sol(CR(\Delta))$. Due to Proposition 10, (28) holds because $\kappa_{\vec{\eta}}(A\dot{B}) = \kappa_{\vec{\eta}^1}(A\dot{B})$. For proving **(Ind)**, we have to show $A \mathrel{\mid\!\sim}^{c\text{-}sk}_\Delta B$ iff $AC \mathrel{\mid\!\sim}^{c\text{-}sk}_\Delta B$. Due to Proposition 8, it suffices to show

$$\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B}) \text{ iff } \kappa_{\vec{\eta}}(ACB) < \kappa_{\vec{\eta}}(AC\overline{B}) \tag{29}$$

for all $\vec{\eta} \in Sol(CR(\Delta))$. Due to Lemma 1, (29) holds because Σ_1 and Σ_2 are $\kappa_{\vec{\eta}}$-independent and thus $\kappa_{\vec{\eta}}(A\dot{B}) + \kappa_{\vec{\eta}}(C) = \kappa_{\vec{\eta}}(AC\dot{B})$. □

In this way, c-representations help us defining an inference relation that fully complies with **(SynSplit)**. Note that $\mathbf{C}^{c\text{-}sk}$ does not make use of selection strategies any more.

7 Related Work

Research works that are most closely related to this approach are papers dealing with syntax splitting in belief revision, in particular, (Kern-Isberner and Brewka 2017). The connection between inductive reasoning from conditional belief bases and advanced, iterated belief revision becomes quite obvious if one considers revision tasks that aim at revising an epistemic state Ψ by a set of conditionals Δ, i.e., $\Psi * \Delta$. While this is far beyond classical AGM belief revision theory (Alchourrón, Gärdenfors, and Makinson 1985), some approaches have been presented to solve this problem nevertheless. For example, *c-revisions* (Kern-Isberner 2004) provide a suitable methodology for this. If we can solve the above-mentioned revision task $\Psi * \Delta$ for (basically) any Ψ and any Δ, then we can apply such a revision method to any Δ and a uniform epistemic state Ψ_u not making any difference between possible worlds. From this, we can obtain an inductive, model-based inference relation based on the inferences that $(\Psi_u * \Delta)$ provides. Note that c-representations (Def. 6) arise from c-revisions in this way.

The paper (Kern-Isberner and Brewka 2017) presents syntax splitting axioms for the revision of epistemic states Ψ represented by total preorders $\preceq_\Psi$ by sets of propositions, generalizing and extending the seminal work of Parikh (Parikh 1999) and the paper (Peppas et al. 2015). The central axioms of (Kern-Isberner and Brewka 2017), *Marginalized Revision (MR)* and *Strong Iterated P, P^{it}*, are extended to sets of conditionals by the central axioms in this paper, *Relevance* and *Independence*, for inductive inferences. Note that also the axiomatization of syntax splitting presented in (Kern-Isberner and Brewka 2017) can be extended in a straightforward way to also deal with revisions by sets of conditionals. The non-equivalence of (MR) and P^{it} that has been noticed in that paper becomes quite clear regarding the results of this paper: *Relevance* and *Independence* are two independent postulates which get connected in the OCF framework by (17), but are not related for general inductive inference operations, nor for total preorders. However, as shown by (Peppas et al. 2015) and (Kern-Isberner and Brewka 2017), basic versions of these two postulates referring only to belief sets are found to be equivalent in the AGM framework. Note also that our concept of $\preceq$-independence is basically equivalent to $\preceq$-splitting from (Kern-Isberner and Brewka 2017), more precisely, a total preorder $\preceq$ splits over (Σ_1, Σ_2) iff Σ_1, Σ_2 are $\preceq$-independent; for further details, please see (Kern-Isberner and Brewka 2017).

The two readings of Parikh's syntax splitting axiom (P) (Parikh 1999) that are mentioned in (Peppas et al. 2015), the *weak* and the *strong* reading, appear to be "two sides of the same coin" in our approach to inductive reasoning from conditional belief bases where we consider a splitting of the belief base Δ into two peer subbases Δ_1, Δ_2 over disjoint subsignatures. Weak (P) means that only the relevant part of the language should be revised, leaving the other parts unchanged (for technical details, see (Peppas et al. 2015)). This is covered in our approach by allowing one of the subbases Δ_i to be empty, and by ensuring that an empty belief base should not yield any inferences, i.e., our axiom (Trivial Vacuity). Note that (Trivial Vacuity) in this paper corresponds roughly to (Trivial Vacuity) in (Kern-Isberner and Brewka 2017) when $\Psi = \Psi_u$. Strong (P), i.e., how the relevant part of the epistemic state is changed, is covered in our framework by the (Relevance) axioms.

Our (Independence) axioms generalize axioms of *Irrelevance* that have been mentioned in various publications (e.g., (Goldszmidt and Pearl 1996; Delgrande and Pelletier 1998; Benferhat, Dubois, and Prade 2002)):

(Irrelevance) If c is an atom that does not occur in Δ, and $\mathrel{|\!\sim}$ is based on Δ, then $A \mathrel{|\!\sim} B$ implies $A \wedge c \mathrel{|\!\sim} B$.

(Irrelevance) is covered in our approach by allowing the "irrelevant" subbase to be empty. But note that our (Independence) axioms are significantly stronger than (Irrelevance) because, e.g., system Z satisfies (Irrelevance) (Goldszmidt and Pearl 1996), but not (Ind^{ocf}) (see section 5). On the other hand, the (Relevance) axiom for inductive inference operators here corresponds to *Strong Irrelevance* for inference relations in (Weydert 2003). Also Lehmann (Lehmann 1995) emphasizes the importance of *independence* for nonmonotonic inferences but he did not give a clear definition of his independence concept. Obviously, the concept he had in mind is broader than (Independence) as defined here because most of his examples require more general considerations than syntax splitting.

In general, the notions of *(ir)relevance* and *independence* have not been used coherently throughout the literature, nor is there a general agreement what suitable formal implementations would be, as this discussion shows. Although dealing only with syntactic aspects, our clear formal definitions might help sharpening and discriminating these concepts.

The paper (Kern-Isberner 2008) pursues a more general idea than this paper by associating inference operators with conditional belief bases which are parametrized by epistemic states (serving as background beliefs), but it focusses on more classical axioms like cumulativity. Also Weydert (Weydert 2003) considers inference relations using conditional belief bases more broadly, where syntax splitting only plays a minor role; interestingly, he also uses selections strategies in the form of choice functions on OCF-like semantic structures. Much more spefically, we apply selection strategies to c-representations which themselves implement selection principles, most fundamentally the principle of conditional preservation (Kern-Isberner 2004).

There are also weak relations to works on the topic of forgetting: we apply ideas from *variable independence* (Lang, Liberatore, and Marquis 2003) to conditional belief bases, and Delgrande's general approach to forgetting (Delgrande 2017) corresponds to marginalisation (on a propositional level) as used in this paper.

8 Conclusion

Syntax splitting has been analyzed successfully in the context of belief revision. To the best of our knowledge, this is the first paper that conducts a similar investigation for nonmonotonic inference systems based on conditional belief bases. The analysis turns out to be at least as fruitful as for belief revision. Our formal definition of syntax splitting for inductive inference operators not only gives a precise characterization of this notion and clearly reveals its two inherent aspects, namely relevance and independence. It also leads to new insights regarding the best-known systems for reasoning from conditional belief bases, namely system P, system Z and c-inference. In particular, whilst all three systems satisfy relevance, the second important aspect of syntax splitting, independence, is only satisfied by c-inference. We believe this result not only increases our understanding of these inference systems, it also provides additional support for c-inference, since independence seems to be a desirable property to have.

As to future work, there are various open research topics that suggest themselves. First of all, it would be interesting to find possible modifications of system P and system Z leading to satisfaction of the independence property. Secondly, there might be further reasonable postulates for inductive inference operators which haven't been identified so far. And finally, a study of syntax splitting for nonmonotonic inference systems which are not inductive in the sense of Def. 1 since they are *not* based on conditional belief bases, like answer set programming or abstract argumentation, might also reveal interesting new insights.

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