# Causal models have no complete axiomatic characterization

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#### Abstract

Markov networks and Bayesian networks are effective graphic representations of the dependencies embedded in probabilistic models. It is well known that independencies captured by Markov networks (called graph-isomorphs) have a finite axiomatic characterization. This paper, however, shows that independencies captured by Bayesian networks (called causal models) have no axiomatization by using even countably many Horn or disjunctive clauses. This is because a sub-independency model of a causal model may be not causal, while graph-isomorphs are closed under sub-models.

#### Keywords:

causal model; axiomatization; sub-model; conditional independence; graph-isomorph

## 1 Introduction

The notion of conditional independence (CI) plays a fundamental role in probabilistic reasoning. In traditional theories of probability, to decide if a CI statement holds, we need to check whether two conditional probabilities are equal, which require summations over exponentially large number of variable combinations. This numerical approach is clearly impractical. An alternative qualitative approach is very popular in artificial intelligence, where new CI statements can be derived logically without reference to numerical quantities. Given an initial set of independence relations, a fixed (finite) set of axioms can be used to infer new independencies by logical manipulations.

A natural question arises: can CI relations be completely characterized by a finite set of axioms (or called inference rules)? Pearl and Paz [5] introduced the concept of semi-graphoid as an independency model that satisfies four specific axioms, and showed that each CI relation is a semi-graphoid. Later, Studený [6] gave a negative answer to this question. But, more positively, he also showed that (i) CI relations have a characterization by a countable set of axioms [6]; and (ii) every probabilistically sound axiom with at most two antecedents is a consequence of the semi-graphoid axioms [7].

Although CI relations in general have no complete axiomatic characterization, Geiger and Pearl [2] developed complete axiomatizations for saturated independence and marginal independence – two special families of CI relations.

Graphs are the most common metaphors for communicating and reasoning about dependencies. It is not surprising that graphical models is a very popular way of specifying independence constraints. There are in general two kinds of graphical models: Markov networks and Bayesian networks. A Markov network is an undirected graph, while a Bayesian network is a directed acyclic graph (DAG). Geiger and Pearl [2] developed an axiomatic basis for the relationships between CI and graphic models in statistic analysis. They showed in particular that (i) every axiom for conditional independence is also an axiom for graph separation; and (ii) every graph represents a consistent set of independence and dependence constraints. Moreover, an early work of Pearl and Paz [5] gave an axiomatic characterization for CI relations captured by undirected graphs (called graph-isomorphs). It was also conjectured [3] that CI relations captured by DAGs (called causal models) may have no finite axiomatic characterization.

In this paper, we confirm this conjecture and show that causal models have no complete characterization by any (finite or countable) set of (Horn or disjunctive) axioms. We achieve this by

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showing that a sub-model of a causal model can be not causal. This is contrasted by CI relations and graph-isomorphs. Both are closed under sub-models.

It came to us very late that the same observation has been made in [8, Remark 3.5], where Studený gave just basic argument. This paper will provide a complete proof for this observation.

The remainder part of this paper proceeds as follows. Section 2 provides preliminary definitions for independency models, CI relations, graph-isomorphs, and causal models. Section 3 gives syntactic and semantic descriptions of independency logic, and then formalizes the notion of axiomatization. Then in Section 4 we discuss heredity property of independency models. Further discussions are given in the last section.

#### 2 Preliminaries

In this section we introduce the basic notions used in this paper. Our reference is [3, 4]. In what follows, if not otherwise stated we assume U is a finite set, and write  $\wp(U)$  for the powerset of U. The notion of conditional independency (CI) plays a fundamental role in probabilistic reasoning.

**Definition 2.1** (conditional independency, CI). Let U be a finite set of variables with discrete values. Let  $P(\cdot)$  be a joint probability function over the variables in U. For three disjoint subsets X, Y, Z of U, X and Y are said to be *conditional independent given* Z if for all values x, y and z such that P(y, z) > 0 we have P(x|y, z) = P(x|z).

We use the notation  $I(X, Z, Y)_P$  to denote the conditional independency of X and Y given Z. The set of all these CI statements form a ternary relation on  $\wp(U)$ , called a CI relation. In general, we have

**Definition 2.2** (independency model [3]). An independency model M defined on U is a ternary relation on  $\wp(U)$  which satisfies the following condition:

$$(A, C, B) \in M \implies A, B, C$$
 are pairwise disjoint. (1)

A tuple (A, C, B) in M (out of M, resp.) is called an independence statement (a dependence statement, resp.). We write  $I(A, C, B)_M$  to indicate the fact that (A, C, B) is in M.

Two other classes of independency models arise from graphs, where the notion of separation plays a key role.

**Definition 2.3** (graph separation [3]). If A, B and C are three disjoint subsets of nodes in an undirected graph G, then C is said to separate A from B, denoted  $\langle A|C|B\rangle_G$ , if along every path between a node in A and a node in B there is a node in C.

The independency model consisting of all graph separation instances in G is a graph-isomorph.

**Definition 2.4** (graph-isomorph [3]). An independency model M is said to be a graph-isomorph if there exists an undirected graph G = (U, E) such that for every three disjoint subsets A, B, C of U, we have

$$I(A, C, B)_M \Leftrightarrow \langle A|C|B\rangle_G.$$
 (2)

For directed acyclic graphs, a similar separation property was defined.

**Definition 2.5** (*d*-separation [3]). If A, B and C are three disjoint subsets of nodes in a DAG D, then C is said to *d*-separate A from B, denoted  $\langle A|C|B\rangle_D$ , if along every path between a node in A and a node in B there is a node w satisfying one of the following two conditions:

- w has converging arrows and none of w or its descendants are in C; or
- w does not have converging arrows and w is in C.

The independency model consisting of all d-separation instances in a DAG D is a causal model.

**Definition 2.6** (causal model [3]). An independency model M is said to be *causal* if there is a DAG D such that for every three disjoint subsets A, B, C of U, we have

$$I(A, C, B)_M \Leftrightarrow \langle A|C|B\rangle_D.$$
 (3)

It was proved by Geiger and Pearl that, for every graph-isomorph (causal model) M on U, there is a probability distribution P on U such that M is precisely the CI relation induced by P [1, 2].

# 3 Independency logic

To formalize the notion of axiomatization, we introduce the independency logic  $\mathcal{IL}$ . Although  $\mathcal{IL}$  is a fragment of first-order logic, we are mainly concerned with its propositional counterpart.

The language of  $\mathcal{IL}$  has as its alphabet of symbols:

- variables  $X_1, X_2, \cdots$ ;
- the constant  $\emptyset$ ;
- the ternary predicate *I*;
- three function letters:  $-, \cup, \cap$ ;
- the punctuation symbols (,) and ,;
- the connectives  $\neg, \lor, \land$

Terms in the independency logic are defined as follows.

**Definition 3.1** (term). A term in  $\mathcal{IL}$  is defined as follows.

- (i) Constant and Variables are terms.
- (ii) If  $T_1, T_2$  are terms in  $\mathcal{IL}$ , then  $-T_1, T_1 \cup T_2, T_1 \cap T_2$  are terms in  $\mathcal{IL}$ .
- (iii) The set of all terms is generated as in (i) and (ii).

Using the unique predicate I, we can form atomic formulas.

**Definition 3.2** (atom, literal, clause). An atom in  $\mathcal{IL}$  is defined by: if  $T_i$  (i = 1, 2, 3) are terms in  $\mathcal{IL}$ , then  $I(T_1, T_2, T_3)$  is an atom. A literal is defined to be an atom (called positive literal) or its negation (called negative literal). A clause is the disjunction of a finite set of literals.

Formulas in  $\mathcal{IL}$  are defined in the standard way.

**Definition 3.3** (formula). A formula in  $\mathcal{IL}$  is an expression involving atoms and connectives  $\neg, \wedge, \vee$ , which can be formed using the rules:

- (i) Any atom is a formula.
- (ii) If  $\mathcal{A}$  and  $\mathcal{B}$  are formulas, then so are  $(\neg \mathcal{A})$ ,  $(\mathcal{A} \wedge \mathcal{B})$ , and  $(\mathcal{A} \vee \mathcal{B})$ .

In the rest of this paper, we shall sometimes omit parentheses, as long as no ambiguity is introduced.

Since implication statement are convenient for expressing inference rules, we define  $(A \to B)$  as an abbreviation of  $((\neg A) \lor B)$ . As a consequence, each clause

$$\bigvee_{i=1}^{k} \neg I(\mathsf{T}_{1i}, \mathsf{T}_{2i}, \mathsf{T}_{3i}) \lor \bigvee_{j=1}^{l} I(\mathsf{T}_{1,k+j}, \mathsf{T}_{2,k+j}, \mathsf{T}_{3,k+j}) \tag{4}$$

can be equivalently represented as an implication (or rule)

$$\bigwedge_{i=1}^{k} I(\mathsf{T}_{1i}, \mathsf{T}_{2i}, \mathsf{T}_{3i}) \to \bigvee_{j=1}^{l} I(\mathsf{T}_{1,k+j}, \mathsf{T}_{2,k+j}, \mathsf{T}_{3,k+j}). \tag{5}$$

Clauses are of particular importance in axiomatization of independency models.

**Definition 3.4** (Horn and disjunctive clauses). For a clause C of form Eq. 5, C is called a Horn clause if  $l \leq 1$ , and called disjunctive otherwise.

Above we introduced the syntactic part of  $\mathcal{IL}$ . Next we turn to semantic notions.

**Definition 3.5** (valuation). Let M be an independency model defined on U. A valuation in M is a function  $v: \{X_1, X_2, \dots\} \to 2^U$ .

Valuations can be extended in a natural way to terms in  $\mathcal{IL}$ .

**Definition 3.6** (valid valuation). Let  $\mathcal{A}$  be a formula, and let M be an independency model defined on U. A valuation v in M is valid for  $\mathcal{A}$  if for each atom  $I(T_1, T_2, T_3)$  appeared in  $\mathcal{A}$ ,  $v(T_1), v(T_2)$ , and  $v(T_3)$  are pairwise disjoint, where v(T) is the valuation of T in M.

The notion of satisfaction is defined in the standard way. Note that if v is valid for  $\mathcal{A}$  in M, then it is also valid for any sub-formula  $\mathcal{B}$  of  $\mathcal{A}$  in M. The following definition is therefore well-defined.

**Definition 3.7** (satisfaction). Let  $\mathcal{A}$  be a formula, and let M be an independency model defined on U. A valuation v in M is said to satisfy  $\mathcal{A}$  if v is valid for  $\mathcal{A}$  and it can be shown inductively to do so under the following conditions.

- v satisfies atom  $I(\mathsf{T}_1,\mathsf{T}_2,\mathsf{T}_3)$  if  $(v(\mathsf{T}_1),v(\mathsf{T}_2),v(\mathsf{T}_3))\in M$ .
- v satisfies  $\neg \mathcal{B}$  if v does not satisfies  $\mathcal{B}$ .
- v satisfies  $\mathcal{B} \vee \mathcal{C}$  if either v satisfies  $\mathcal{B}$  or v satisfies  $\mathcal{C}$ .
- v satisfies  $\mathcal{B} \wedge \mathcal{C}$  if v satisfies both  $\mathcal{B}$  and  $\mathcal{C}$ .

We say M satisfies  $\mathcal{A}$ , in notation  $M \models \mathcal{A}$ , if all valid valuations of  $\mathcal{A}$  in M satisfy  $\mathcal{A}$ .

The following proposition is a consequence of the definition of  $\rightarrow$ .

**Proposition 3.1.** Let A, B be two formulas, and let M be an independency model defined on U. Then  $M \models A \rightarrow B$  iff for any valid valuation v of  $A \rightarrow B$  in M, v satisfies A implies v satisfies B.

For a clause we have the following characterization.

**Corollary 3.1.** Let C be a clause of form Eq. 5, and let M be an independency model defined on U. Then  $M \models C$  iff the following condition holds:

• for any valid valuation v of C in M, if  $(v(T_{1i}), v(T_{2i}), v(T_{3i})) \in M$  for <u>all</u>  $1 \leq i \leq k$ , then  $(v(T_{1j}), v(T_{2j}), v(T_{3j})) \in M$  for <u>some</u>  $k + 1 \leq j \leq k + l$ .

Given a family of independency models  $\mathbb{M}$  and a (finite or countable) set of formulas  $\mathbb{F}$  in  $\mathcal{IL}$ , we now formalize the notion that  $\mathbb{B}$  can be axiomatically characterized by  $\mathbb{F}$ .

**Definition 3.8** (axiomatization). A family of independency models  $\mathbb{M}$  can be completely characterized by a set of formulas  $\mathbb{F}$  in  $\mathcal{IL}$  if the following condition holds for any independency model M:

$$M \in \mathbb{M} \Leftrightarrow (\forall \mathcal{B} \in \mathbb{F})M \models \mathcal{B}. \tag{6}$$

We say  $\mathbb{M}$  has a finite (countable, resp.) axiomatization if it can be completely characterized by a finite (countable, resp.) set of *formulas* in  $\mathcal{IL}$ .

Since each formula in  $\mathcal{IL}$  is semantically equivalent to the conjunction of a set of finite clauses, we need only consider clauses.

**Proposition 3.2.** A family of independency models  $\mathbb{M}$  has a finite (countable, resp.) axiomatization iff it can be completely characterized by a finite (countable, resp.) set of clauses in  $\mathcal{IL}$ .

Analogous to propositional calculus, we have the completeness theorem for  $\mathcal{IL}$ .

**Theorem 3.1.** Suppose  $\mathbb{M}$  is axiomatically characterized by  $\mathbb{F}$ . Let  $\Sigma$  be a set of formulas,  $\mathcal{A}$  be a formula. Then the following two conditions are equivalent.

- (1)  $\Sigma \models_{\mathbb{M}} A$ : for any model M in  $\mathbb{M}$ , if M satisfies all formulas in  $\Sigma$ , it also satisfies A;
- (2)  $\Sigma \vdash_{\mathbb{F}} A$ : A is deducible from  $\Sigma$  by using axioms in  $\mathbb{F}$ .

In particular, we have

**Corollary 3.2.** Let  $\mathbb{M}$  and  $\mathbb{F}$  be as in the above theorem. For a set  $\Gamma$  of independence statements  $\{I(T_{i1}, T_{i2}, T_{i3}) : 1 \leq i \leq k\}$  and an independence statement  $\gamma = I(T_{k+1,1}, T_{k+1,2}, T_{k+1,3})$ , we have  $\Gamma \models_{\mathbb{M}} \gamma$  iff  $\gamma$  is deducible from  $\Gamma$  by using axioms in  $\mathbb{F}$ .

### 4 Sub-models

In this section, we consider sub-independency models.

**Definition 4.1** (sub-model). Let M be an independency model defined on U, and let V be a subset of U. We call  $M|_{V} = \{(A, C, B) \in M : A, B, C \subseteq V\}$  the sub-independency model (or simply sub-model) of M on V.

The following result asserts that if an independency model satisfies a formula, so does its sub-model.

**Proposition 4.1.** Let A be a formula, and let M be an independency model defined on U. For any subset V of U, if M satisfies A, then so does  $M|_{V}$ .

*Proof.* This is because any valuation v in  $M|_V$  is also a valuation in M.

An interesting question arises naturally. Given an independency model M on U, suppose M is a CI relation (or graph-isomorph, or causal model), and  $V \subseteq U$ . Is sub-model  $M|_V$  also a CI relation (or graph-isomorph, or causal model)? This is important for a family of independency models  $\mathbb{M}$  to be axiomatizable. Actually, if  $\mathbb{M}$  is not closed under sub-models, then it cannot be axiomatically characterized by any set of formulas.

Given a joint probability  $P(\cdot)$ , write M for the CI relation on U induced by  $P(\cdot)$ , i.e. for any pairwise disjoint subsets A, B, C of U the tuple (A, C, B) is an instance of M if and only if A and B are conditionally independent given C (see Def. 2.1 and Def. 2.2). We claim that, for a nonempty subset V of U,  $M|_V$ , the restriction of M on V, is a CI relation on V. This is because  $M|_V$  is induced by the joint probability  $P|_V(\cdot)$ , which is obtained from  $P(\cdot)$  by computing the marginal probability of P on V.

A similar conclusion holds for graph-isomorphs.

**Lemma 4.1.** Let G = (U, E) be an undirected graph on U, and let V be a nonempty proper subset of U. Define an undirected graph G' = (V, E') as follows: for any two nodes  $\alpha, \beta \in V$ ,  $(\alpha, \beta) \in E'$  iff there is a path p from  $\alpha$  to  $\beta$  in G such that all other nodes in p are contained in U - V. Then

$$\langle \alpha | C | \beta \rangle_G \Leftrightarrow \langle \alpha | C | \beta \rangle_{G'} \tag{7}$$

for any  $\alpha, \beta \in V$ , and any  $C \subset V$ .

*Proof.* Suppose  $\langle \alpha | C | \beta \rangle_{G'}$ . We show C separates  $\alpha$  from  $\beta$  in G. For each path

$$p = \alpha \gamma_1 \gamma_2 \cdots \gamma_m \beta \quad (m \ge 0)$$

in G, we show  $m \geq 1$  and some  $\gamma_i$  is contained in C. Since  $\langle \alpha | C | \beta \rangle_{G'}$ ,  $(\alpha, \beta)$  is not an edge in G'. By definition, we know (i)  $(\alpha, \beta)$  is not an edge in G, hence  $m \geq 1$ ; and (ii) some node  $\gamma_i$  must be contained in V. Suppose  $\gamma_{i_1}, \gamma_{i_2}, \cdots, \gamma_{i_k}$   $(1 \leq i_1 < i_2 < \cdots < i_k \leq m)$  are all those nodes in V. Since nodes between  $\gamma_{i_u}$  and  $\gamma_{i_{u+1}}$  (and those between  $\alpha$  and  $\gamma_{i_1}$ , and between  $\gamma_{i_k}$  and  $\beta$ ) must be

<sup>&</sup>lt;sup>1</sup>In the sense of logic deduction.

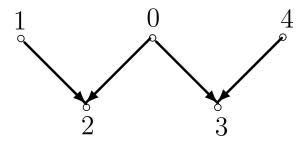


Figure 1: A DAG D on  $U = \{0, 1, 2, 3, 4\}$ .

contained in U-V. By definition of G', we know  $(\alpha, \gamma_{i_1}), (\gamma_{i_u}, \gamma_{i_{u+1}}), (\gamma_{i_k}, \beta)$  are all edges in G'. Therefore

$$p' = \alpha \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k} \beta$$

is a path in G'. By  $\langle \alpha | C | \beta \rangle_{G'}$ , we know some  $\gamma_{i_u}$  must be in C. This shows that C separates  $\alpha$  from  $\beta$  for every path p in G.

On the other hand, suppose  $\langle \alpha | C | \beta \rangle_G$ . We show C separates  $\alpha$  from  $\beta$  in G'. For each path

$$p = \alpha \gamma_1 \gamma_2 \cdots \gamma_m \beta \quad (m \ge 0)$$

in G', we show some  $\gamma_i$  is contained in C.

Write  $\gamma_0$  and  $\gamma_{m+1}$  for  $\alpha$  and  $\beta$ . Note that if  $(\gamma_i, \gamma_{i+1})$  is not an edge in G, then by definition there is a path  $p_i$  in G from  $\gamma_i$  to  $\gamma_{i+1}$  such that all other nodes in  $p_i$  are contained in G. Concatenating paths  $p_0, p_1, \dots, p_m$  we obtain a new 'path' in G from  $\alpha$  to  $\beta$  which satisfies the following condition:

Each node is either in p or in U - V.

In this 'path' identical nodes may occur several times. With proper modifications, we obtain a shortened path p' in G which also satisfies condition (4). By our assumption that  $\langle \alpha | C | \beta \rangle_G$ , we know some node in p' must be contained in C. But by  $C \subseteq V$ , this shows some  $\gamma_i$  must be in C. Hence C separates  $\alpha$  from  $\beta$  for every path p in G'.

**Proposition 4.2.** Let M be a graph-isomorph on U. For a nonempty subset V of U,  $M|_{V}$  is also a graph-isomorph.

Proof. Suppose M is represented by an undirected graph G = (U, E). We show  $M|_V$  can be represented by the undirected graph G' = (V, E') constructed in Lemma 4.1, i.e. for for any pairwise disjoint subsets A, B, C of V, we have  $I(A, C, B)_{M|_V}$  iff  $\langle A|C|B\rangle_{G'}$ . By definition of graph separation, for a graph  $G^*$  we know  $\langle A|C|B\rangle_{G^*}$  iff  $(\forall \alpha \in A)(\forall \beta \in B)\langle \alpha|C|\beta\rangle_{G^*}$ . By Lemma 4.1, for any  $\alpha, \beta \in V$  and any  $C \subset V$  we have  $\langle \alpha|C|\beta\rangle_G \Leftrightarrow \langle \alpha|C|\beta\rangle_{G'}$ . Therefore  $\langle A|C|B\rangle_{G'}$  iff  $\langle A|C|B\rangle_G$  for any pairwise disjoint subsets A, B, C of V. Since M is representable by G, it is clear that  $M|_V$  is also representable by G'.

But the following example shows that this is not true for causal models.

**Example 4.1.** Let M be the causal model representable by the DAG D given in Fig. 1, and let  $V = \{1, 2, 3, 4\}$ . The sub-independency model  $M|_V$  is not representable by any DAG.

To prove this conclusion, we use the notation  $D(\alpha, \beta)$  to express the fact that in  $M|_V$  there is no  $C \subset V$  such that  $I(\alpha, C, \beta)$  is true. It is clear that the following independency statements holds in  $M|_V$ :

- D(1,2), D(3,4), D(2,3);
- $I(1,\emptyset,3)$ , I(1,4,3),  $I(2,\emptyset,4)$ , I(2,1,4).

In a DAG  $D=(U,\overrightarrow{E})$ , for any two nodes  $\alpha,\beta\in U$ , it is well known that  $(\alpha,\beta)\in\overrightarrow{E}$  or  $(\beta,\alpha)\in\overrightarrow{E}$  iff no  $C\subseteq U$  can d-separates  $\alpha$  from  $\beta$ .

Suppose  $M|_V$  is representable by some DAG D' defined on V. By D(1,2), D(3,4), and D(2,3) we know in D' node 1 is connected to node 2, node 2 is connected to node 3, and node 3 is connected to node 4. This shows that p=1234 is a path from node 1 to node 4. But by  $I(1,\varnothing,3)_{M|_V}$  and p'=123 is a path from node 1 to node 3, we know in D' we should have  $1\to 2\leftarrow 3$ . Similarly, for nodes 2 and 4, we should also have  $2\to 3\leftarrow 4$  in D'. This is impossible since  $2\to 3$  and  $2\leftarrow 3$  cannot appear together in the same DAG.

This proves that  $M|_V$  has no DAG representation, hence is not causal.

As a corollary of this example and Prop. 4.1 we have

**Theorem 4.1.** Causal models have no complete axiomatic characterization.

*Proof.* Let  $\Gamma = \{A_1, A_2, \dots\}$  be the set of clauses that are satisfied by all causal models. In particular, the causal model M given in Example 4.1 satisfies each  $A_i$ . By Prop. 4.1 we know  $M|_V$  also satisfies each  $A_i$ . Since  $M|_V$  is not causal, the infinite set  $\Gamma$  (let alone finite subsets of  $\Gamma$ ) cannot provide a complete characterization for causal models.

# 5 Discussion

We have shown that it is impossible to give a complete axiomatic characterization for causal models. This is different from the results obtained in [5] and [6]. In [5], Pearl and Paz proved that graph-isomorphs have a complete characterization by five axioms (4 Horn, 1 disjunctive). Since a sub-model of a graph-isomorph also satisfies these axioms, it is clear that sub-models of graph-isomorphs are graph-isomorphs. We gave a method for constructing such a graph representation.

Studený [6] showed that there is no finite axiomatization for CI relations by using Horn clauses. More positively, he also showed that there exist an infinite set of Horn clauses that completely characterize CI relations. But it is still unknown whether CI relations have finite axiomatization by using arbitrary clauses (Horn or disjunctive).

The class of sub-models of causal models seems useful when (unknown) hidden variables are involved. As for axiomatization, a result by Geiger (see [3, Exercises 3.7]) suggests that it may have no finite characterization by Horn axioms.

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