# Forgeting in $\mu$-calculus*

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**Abstract.** Propositional $\mu$-calculus is an expressive logic, and also an important specification language in verification. One of the common phenomenons in both the verification and the system design is some information content of such specification might become irrelevant for the system due to various reasons e.g., it might be discarded or become obsolete by time, or just become infeasible due to practical difficulties. Then, the problem arises on how to distill the information without altering the relevant system behavior or violating the original specification over a given signature. Moreover, three crucial notions are vital: the *strongest necessary condition* (SNC), the *weakest sufficient condition* (WSC) and the knowledge update. To address these scenarios and to target the relevant notions SNC (WSC) and knowledge update in a principled way. In this paper, we explore the knowledge update and SNC (WSC) of $\mu$-calculus from the point of *forgetting*. We study its theoretical properties and also show that our notion of forgetting satisfies existing essential postulates of knowledge forgetting. Furthermore, we show that the reasoning problems of the forgetting are EXPTIME-complete.

**Keywords:** Weakest precondition · Forgetting · Knowledge update.

## 1 Introduction

Propositional $\mu$-calculus is an expressive logic, on binary trees it is as expressive as the monadic second order logic of two successors (S2S) [14,26]. Subsequent research showed that the $\mu$-calculus is an important logic when specification and verification is concerned. One of the common phenomenons in both the verification and the system design is some information content of such specification might become irrelevant for the system due to various reasons e.g., it might be discarded or become obsolete by time, or just become infeasible due to practical difficulties. However, In this case it is expensive in time and space to re-extract the specification that meets the requirements. The problem arises on how to remove it without altering the relevant system behaviour or violating the existing system specifications over a given signature. Let's consider the following example.

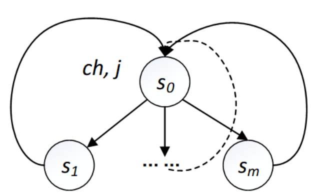
* Supported by organization x.

*Example 1 (Playing PingPong).* Assume *John* plays PingPong with $n+1$ people (i.e. $n+2$ people in total) and there are $n+2$ chairs. At the beginning, John plays PingPong with one of the $n+1$ people, and the other $n$ people are sitting on their chairs at this time. In our scenario, we assume *John* can not win anyone, which means that *John* has always been beaten by other people every turn. In this case, after $n$ games it will be *John*'s turn. This process can be modeled by Fig. 1 (a).

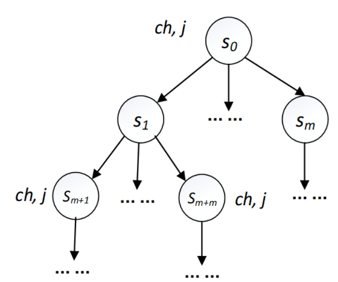
In Fig. 1 (a), this scenario is represented by the Kripke structure $\mathcal{M} = (S, s_0, R, L)$ with the corresponding atomic variables $V = \{j, ch\}$, in which $j$ means *John*'s turn and $ch$ means the *John*'s chair is free. Besides, $m = 2^n$, this means there are $m$ orders to *John*'s turn.

In this scenario, we can see from the unwinding of $\mathcal{M}$ in Fig. 1 (b) that for each (some) path $\pi = (s_0, s_1, ...)$ starting from $s_0$ we have $\forall i \in \mathbb{N}$ there is $(\mathcal{M}, s_{2*i}) \models ch \wedge j$ and $(\mathcal{M}, s_{2*i+1}) \models \neg ch \wedge \neg j$, i.e. for each *even state* in each (some) path that starting from $s_0$ the $ch \wedge j$ holds and $\neg ch \wedge \neg j$ holds for *old states* in this path. This property can be represented by a $\mu$-calculus formula $\varphi = \nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$ ($\varphi = \nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$), this is not expressible with other temporal logics.

Now assume a situation in which due to some problems (i.e. the venue changed or the chair broke down), John does no longer have a chair. This means, all the playing processes concerning "$ch$" no more necessary and should be dropped from both the specifications (e.g. $\varphi$) and the Kripke structure for simplification.




(a) Kripke structure $\mathcal{M}$




(b) The unwinding of $\mathcal{M}$

Fig. 1: The PingPong's Model

Similar scenarios like the one presented in Example 1 may arise in many different domains such as business-process modelling, software development, concurrent systems and more [2]. Yet dropping some restrictions in a large and complex system or specification, without affecting the working system components or violating dependent specifications over a given signature, is a non-trivial task. Moreover, in such a scenario, two logical notions introduced by E. Dijkstra in [11] are highly informative: the strongest post-condition (SP) and the weakest precondition (WP) of a given specification, which are corresponding with the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC), proposed by Lin [24], of the specification, respectively, and have been central to a wide variety of tasks and studies, e.g. in generating counterexamples [10] and refinement of system [29]. These correspond to the *most general consequence* and the *most specific abduction* of such specification, respectively.

Besides, *belief updates* and *belief revision*, as two well-studied issues in artificial intelligence, are concerned with the update and revision aspects of an agent's belief with respect to new beliefs [19]. Intuitively, if $\varphi$ represents the agent's belief about the world and the agent performs an action that is supposed to make $\psi$ true in the resulting world, then the agent's belief about the resulting world can be described by $\varphi \diamond \psi$, where $\diamond$ is the update operator of choice. We can see that the theory of belief updates does not tell us how to do updates with respect to such gain in knowledge due to a sensing action. In this sense, as an analogous notion of belief update, the *knowledge update* was proposed by Chitta in [3] to solve the belief updates caused by sensing actions, in which the effect of a sensing action is expressed by introducing modal operator (K)nows. Nevertheless, there are no approaches to solve the knowledge update in logic languages which contain *temporal operators*.

To address these scenarios and to target the relevant notions SNC (WSC) and knowledge update in a pricipled way. Inspired by [24,16], in this paper we explore the knowledge update and SNC (WSC) of $\mu$-calculus from the point of forgetting. In particular, we will give the definition of forgetting in $\mu$-calculus by using the bisimulation [5,2,30] and show whether this notion satisfies the general principles or postulates proposed by Zhang [31]. We then study the relationship between SNC (WSC) and forgetting, and we also demonstrate how forgetting can be used in knowledge update in $\mu$-calculus.

*Forgetting*, which is a dual concept of *uniform interpolant* [27,21] and was first formally defined in propositional and FOL by Lin and Reiter [25,12], can be traced back to the work of Boole on propositional variable elimination and the seminal work of Ackermann [1]. In classical propositional logic (CPL) the result of forgetting atom $p$ from formula $\varphi$ is $\varphi[p/\top] \vee \varphi[p/\bot]$, that is the disjunction of formulas obtained from $\varphi$ by replacing $p$ with $\top$ and $\bot$ respectively.

However, existing forgetting definitions in PL and answer set programming are not directly applicable in modal logics. And we can also not directly use the method of forgetting in CTL [17] since it will not work when the models of the formula are infinite. Hopefully, it has been proved that the modal $\mu$-calculus has *Uniform interpolation* [7]. Informally, for every $\mu$ sentence $\varphi$ and every finite set $V$ of atoms, there exists an $\mu$ sentence $\exists V \varphi$ which does not contain atoms from $V$ but is logically closest to $\varphi$ in some sense. This means that the result of forgetting some atoms from a $\mu$-calculus sentence always exists. In this sense, showing the semantic of forgetting in $\mu$-calculus through *general principles or postulates* is important to make it clearer to understand.

Informally, the four postulates proposed by Zhang [31] show that the result $\psi$ of forgetting some set $V$ of atoms from a formula $\varphi$ is not only weaker than the $\varphi$, i.e. $\varphi \models \psi$, irrelevant to $V$, i.e. exists some formula that do not contain atoms in $V$ and equivalent with $\psi$, and also has the same "logic content" with $\varphi$, i.e. for each formula $\phi$ that irrelevant to $V$, $\phi$ can be implied by $\varphi$ iff $\phi$ can be implied by $\psi$. In this paper we explore the forgetting of $\mu$-calculus under infinite models (to distinguish that of CTL [17]) from both the postulates and the algebraic properties of the forgetting operator. The complexities of the reasoning problems of the forgetting operator are also explored from the point of automaton and it is shown that these are in EXPTIME-complete. It is worth mentioning that we restrict the models are finite in the knowledge update part in order to express the models of a formula. And we show that our definition of $\diamond_\mu$ by forgetting satisfies Katsuno and Mendelzon's update postulates [20].

The rest of the paper is organised as follows.

## 2 Modal $\mu$-calculus

We start with some technical and notational preliminaries. Given a formula $\varphi$, the *language* of $\varphi$, denoted $L(\varphi)$, is the set of all propositional constants appearing in the formula.

### 2.1 Syntax

Modal $\mu$-calculus is an extension of modal logic, we consider the propositional $\mu$-calculus as introduced by Kozen [22]. Let $\mathcal{A} = \{p, q, \dots\}$ be a set of propositions letters and $\mathcal{V} = \{X, Y, \dots\}$ a set of variables. Formulas of the $\mu$-calculus over these sets can be defined by the following grammar:

$$\varphi := p \mid \neg p \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

Where $p \in \mathcal{A}$ and $X \in \mathcal{V}$. $\top$ and $\bot$ are also $\mu$-calculus formulae, which express 'true' and 'false' respectively. Note that we allow negations only before propositional letters. All the results presented here extend to the general case when negation before variables is also allowed, restricting as usual to positive occurrences of bound variables, that is to say variables appear after an even number of negations. Variables, propositional letters and their negations will be called *literals*. For convenience, in the following, $\varphi$, $\varphi_1, \dots,$ $\psi$, $\psi_1, \dots$ will denote formulas. And by *Var*$(\varphi)$ we mean the set of atoms appearing in formula $\varphi$.

We call a formula *well named* iff every variable is bound at most once in the formula and free variables are distinct from bound variables. For a variable $X$ bound in a well named formula $\varphi$ there exists a unique subterm of $\varphi$ of the form $\delta X.\varphi(X)$ with $\delta \in \{\nu, \mu\}$, from now on called the *binding definition* of $X$ in $\varphi$. We call $X$ a $\mu$-variable when $\delta = \mu$, otherwise we call $X$ a $\nu$-variable.

Variable X in $\delta X.\varphi(X)$ is *guarded* iff every occurrence of $X$ in $\varphi(X)$ is in the scope of some modality operator $\Diamond$ or $\Box$. For convenience, we mix the two symbols $\Diamond$ and EX ($\Box$ and AX). A formula is guarded iff every bound variable in the formula is guarded.

**Proposition 1.** *Every formula is equivalent to some guarded formula.*

This proposition allows us to restrict ourselves to **guarded**, **well-named** formulas. From now on, we shall only consider formulas of this kind.

### 2.2 Semantic

Formulas are interpreted in transition systems of the form $\mathcal{M} = (S, r, R, L)$, we call it a Kripke structure, where:

- $S$ is a nonempty set of states,
- $r \in S$,
- $R$ is a binary relation on $S$, i.e. $R \subseteq S \times S$, called transition relation, and
- $L : S \to 2^{\mathcal{A}}$ is a labeling function.

A Kripke structure $\mathcal{M}$ is finite if $S$ is finite and $L(q) = \emptyset$ for almost all $q \in \mathcal{A}$.

Given a Kripke structure $\mathcal{M}$ and a valuation $V : \mathcal{V} \to 2^S$, the set of states in which a formula $\varphi$ is true, denoted $\| \varphi \|_V^{\mathcal{M}}$, is defined inductively as follows (we will omit superscript $\mathcal{M}$ when it causes no ambiguity):

$$
\begin{aligned}
&\| p \|_V = \{s \mid p \in L(s)\} \quad \| \top \|_V = S \quad \| \bot \|_V = \emptyset \\
&\| \neg p \|_V = S - \| p \|_V \\
&\| X \|_V = V(X) \\
&\| \varphi_1 \vee \varphi_2 \|_V = \| \varphi_1 \|_V \cup \| \varphi_2 \|_V \\
&\| \varphi_1 \wedge \varphi_2 \|_V = \| \varphi_1 \|_V \cap \| \varphi_2 \|_V \\
&\| \Diamond\varphi \|_V = \{s | \exists s'.(s, s') \in R \wedge s' \in \| \varphi \|_V\} \\
&\| \Box\varphi \|_V = \{s | \forall s'.(s, s') \in R \Rightarrow s' \in \| \varphi \|_V\} \\
&\| \mu X.\varphi \|_V = \bigcap \{S' \subseteq S | \| \varphi \|_{V[X:=S']} \subseteq S'\} \\
&\| \nu X.\varphi \|_V = \bigcup \{S' \subseteq S | S' \subseteq \| \varphi \|_{V[X:=S']}\}
\end{aligned}
$$

here $V[X := S']$ is equal to the valuation function $V$ except that $S'$ is assigned to $X$. Note that $\| \mu X.\varphi \|_V$ is the least fixpoint of the $\varphi(X)$.

In the following, we denote $s \in \| \varphi \|_V$ by $(\mathcal{M}, s, V) \models \varphi$ and we may leave out the valuation $V$, if $\varphi$ is a sentence (i.e. no variables in $\varphi$ are free). $(\mathcal{M}, V) \models \varphi$ is used to denote $(\mathcal{M}, r, V) \models \varphi$, and in this case we say that $(\mathcal{M}, V)$ is a model of $\varphi$. By $Mod(\varphi)$ we mean the set of models of $\varphi$. In particular, if $\varphi$ is a sentence, then $Mod(\varphi) = \{\mathcal{M} \mid (\mathcal{M}, r) \models \varphi\}$. Similarly, let $\Sigma$ be a set of sentences, we define $Mod(\Sigma)$ as the set of Kripke structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ for each $\varphi \in \Sigma$. $\varphi \models \psi$ denotes logical consequence: if $(\mathcal{M}, V) \models \varphi$ then $(\mathcal{M}, V) \models \psi$ for every model $\mathcal{M}$ and for every valuation $V$. Especially, given two sentences (or set of sentences) $\Sigma$ and $\Pi$, $\Sigma \models \Pi$ if $Mod(\Sigma) \subseteq Mod(\Pi)$ and $\Sigma \equiv \Pi$ whenever $Mod(\Sigma) = Mod(\Pi)$.

By the Tarski-Knaster theorem, least and greatest fixpoints of monotonic functions $f$ over subsets of a set $U$ can be defined by transfinite induction, i.e., the least fixpoint $\mu(f) = \bigcup \mu_\alpha(f)$, where

- $\mu_0(f) = \emptyset$,
- $\mu_{\alpha+1} = f(\mu_\alpha(f))$,
- $\mu_\lambda = \bigcup_{\alpha<\lambda} \mu_\alpha(f)$, for $\lambda$ a limit ordinal.

Similarly, the greatest fixpoint $\nu(f) = \bigcap \nu_\alpha(f)$, where

- $\nu_0(f) = U$,
- $\nu_{\alpha+1} = f(\nu_\alpha(f))$,
- $\nu_\lambda = \bigcap_{\alpha<\lambda} \nu_\alpha(f)$, for $\lambda$ a limit ordinal.

Apparently, in $\mu X.\varphi(X)$, $\varphi(X)$ is a monotonic function about $X$ since $X$ appearing in $\varphi(X)$ positively. Therefore, $\| \mu X.\varphi(X) \|_V$ is the least fixpoint of the monotone operator $\| \varphi \|_V$: $2^S \to 2^S$ (it also be written as $Y \mapsto \| \varphi \|_{V[X:=Y]}$ sometimes). Let's see an example to show how to compute the greatest fixpoints of the $\mu$-formula $\varphi$ talked in Example 1.

*Example 2.* (1) For $\varphi = \nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$, let $\mathcal{M} = (S, r, R, L)$ be a Kripke structure, then we have:

- $\nu_0(\| \varphi \|) = S$,
- $\nu_1(\| \varphi \|) = \| j \| \cap \| ch \| \cap \| \Box(\neg j \wedge \neg ch) \| \cap \| \Box\Box\nu_0(\| \varphi \|) \| = \{s | s \in L(j) \cap L(ch)$ and $\forall t.(s,t) \in R, t \in S - (L(j) \cup L(ch))\}$,
- ...

(2) For $\varphi = \nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$, let $\mathcal{M} = (S, r, R, L)$ be a Kripke structure, then we have:

- $\nu_0(\| \varphi \|) = S$,
- $\nu_1(\| \varphi \|) = \| j \| \cap \| ch \| \cap \| \Box(\neg j \wedge \neg ch) \| \cap \| \Box\Box\nu_0(\| \varphi \|) \| = \{s | s \in L(j) \cap L(ch)$ and $\exists t.(s,t) \in R, t \in S - (L(j) \cup L(ch))\}$,
- $\nu_2(\| \varphi \|) = \| j \| \cap \| ch \| \cap \| \Box(\neg j \wedge \neg ch) \| \cap \| \Box\Box\nu_1(\| \varphi \|) \|$
- ...

For the Kripke structure $\mathcal{M}$ in (a) of Example 1, we have $\| \varphi \| = \{s_0\}$.

An alternative syntax for the $\mu$-calculus is obtained by substituting the $\Diamond$ operator with a set of *cover operators*, one for each natural n. For $n \geq 1$ these operators are defined as follows: if $\varphi_1, \ldots, \varphi_n$ are formulas, then

$$Cover(\varphi_1, \ldots, \varphi_n)$$

is a formula. The constant operator $Cover(\emptyset)$ is also allowed. The cover operators are interpreted in a Kripke structure $\mathcal{M}$ as follows: $Cover(\emptyset)$ is true in $\mathcal{M}$ if and only if the root of $\mathcal{M}$ does not have any successor, while $Cover(\varphi_1, \ldots, \varphi_n)$ is true in $\mathcal{M}$ if and only if the successors of the root are covered by $\varphi_1, \ldots, \varphi_n$. More formally, $(\mathcal{M}, s, V) \models Cover(\varphi_1, \ldots, \varphi_n)$ if and only if:

- for every $i = 1, ..., n$ there exists $t$ with $(s, t) \in \mathcal{M}$ and $(\mathcal{M}, t, V) \models \varphi_i$;
- for every $t$ with $(s, t) \in \mathcal{M}$ there exists $i \in \{1, ..., n\}$ with $(\mathcal{M}, t, V) \models \varphi_i$.

We call this syntax the covers-syntax to distinguish it from the original $\Diamond$-syntax.
Since $Cover(\varphi_1, \ldots, \varphi_n)$ is equivalent to

$$\Diamond\varphi_1 \wedge \cdots \wedge \Diamond\varphi_n \wedge \Box(\varphi \vee \cdots \vee \varphi_n),$$

cover operators are definable in the $\Diamond$ syntax. Conversely,

$$\Diamond\varphi \Leftrightarrow Cover(\varphi, \top).$$

Hence, the $\mu$-calculus obtained from the covers-syntax is equivalent to the familiar $\mu$-calculus constructed using the $\Diamond$-syntax. In this paper we use a mixture of the two syntax because, as we shall see, cover operators behave nicely with respect to the definition of disjunctive formula. Specially, the $\mu$-calculus formula have a normal form, called disjunctive formula [18] as follows.

**Definition 1 (disjunctive formula).** *The set of disjunctive formulas, $\mathcal{F}_d$ is the smallest set defined by the following clauses:*

- *disjunctions and non-contradictory conjunction of literals are disjunctive formulas;*
- *special conjunctions: if $\varphi_1, \ldots, \varphi_n \in \mathcal{F}_d$ and $\delta$ is a non contradictory conjunction of literals, then $\delta \wedge Cover(\varphi_1, \ldots, \varphi_n) \in \mathcal{F}_d$;*
- *fixpoint operators: if $\varphi \in \mathcal{F}_d$, $\varphi$ does not contain $X \wedge \psi$ as a subformula for any formula $\psi$, and $X$ is positive in $\varphi$, then $\mu X.\varphi$, $\nu X.\varphi$ are in $\mathcal{F}_d$.*

*Example 3.* It is easy to check that both $\nu X.(j \wedge ch) \wedge \Diamond(\neg j \wedge \neg ch) \wedge \Diamond\Diamond X$ and $\nu X.(j \wedge ch) \wedge \Box(\neg j \wedge \neg ch) \wedge \Box\Box X$ are not disjunctive formulas. While $j \wedge ch \wedge \Diamond(\neg j \wedge \neg ch)$, $\mu X.(j \wedge ch) \wedge \Diamond X$ and $\nu X.(j \wedge ch) \wedge \Diamond\Diamond X$ are disjunctive formulas because we have:

$$j \wedge ch \wedge \Diamond(\neg j \wedge \neg ch) \equiv j \wedge ch \wedge Cover(\neg j \wedge \neg ch, \top),$$

$$\mu X.(j \wedge ch) \wedge \Diamond X \equiv \mu X.(j \wedge ch) \wedge Cover(X, \top)$$

and

$$\nu X.(j \wedge ch) \wedge \Diamond\Diamond X \equiv \nu X.(j \wedge ch) \wedge Cover(Cover(X, \top), \top).$$

**Theorem 1 ([18]).** *Any $\mu$-calculus formula is equivalent to a disjunctive one.*

Theorem 1 means that for each $\mu$-calculus formula $\varphi$ there is a disjunctive formula $\psi \in \mathcal{F}_d$ such that $\varphi \equiv \psi$. Deciding whether a disjunctive formula is satisfiable can be done in polynomial time, hence changing the $\mu$-calculus formula into its disjunctive form will increase the efficiency of deciding its satisfiability since the satisfiability of $\mu$-calculus is in EXPTIME-complete.

Another important conept is *uniform interpolant*, which has been widely explored in various of logic languages. Some logics enjoy uniform interpolation, but some are not. Formally, the uniform interpolant under $\mu$-calculus is defined as follows.

**Definition 2 (Uniform interplant [9]).** *Given a $\mu$-sentence $\varphi$ and a language $L' \subseteq L(\varphi)$, the uniform interpolant of $\varphi$ with respect to $L'$ is a $\mu$-sentence $\psi$ such that:*

- *$\varphi \models \psi$;*
- *whenever $\varphi \models \varphi_1$ and $L(\varphi) \cap L(\varphi_1) \subseteq L'$ then $\psi \models \varphi_1$;*
- *$L(\psi) \subseteq L'$.*

Notice that uniform interpolation is stronger than Craig interpolation, which states that for any two formulas $\varphi$, $\psi$ with $\varphi \models \psi$ there exists a formula $\theta$ (called the Craig interpolant of $\varphi$, $\psi$) in the common language with $\varphi \models \theta$ and $\theta \models \psi$. Clearly, if the logic enjoys uniform interpolation then the Craig interpolant of $\varphi, \psi$ does not depend on $\psi$ but only on the common language: it is simply the uniform interpolant of $\varphi$ relative to $L' = L(\varphi) \cap L(\psi)$.

It has been proved that the $\mu$-calculus has uniform interpolation [8,7]. Moreover, when the given formula is a disjunctive formula then the uniform interpolant of it can be obtained by replacing both the pointed atoms and their negation with $\top$ at the same time [9]. Formally:

**Theorem 2.** *The uniform interpolant $\widetilde{\exists} p\varphi$ ($p \in \mathcal{A}$) of a disjunctive formula $\varphi$ is equivalent to the $\mu$-formula $\varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\top]$ is defined from $\varphi$ by simultaneously substituting the literals $p$ and $\neg p$ with $\top$.*

## 3 Forgetting in $\mu$-calculus

In this section we present the definition of forgetting in $\mu$-calculus and investigate its semantic properties. First, we give the definition of $V$-bisimulation between Kripke structures. The notion of $V$-bisimulation capture the idea that the two systems are behaviourally same except the atoms in $V$. In this way we give the definition of forgetting by using the $V$-bisimulation.

Second, the related properties, e.g. Modularity, Commutativity and Homogeneity, of the forgetting operator will be explored. And last, we show that the Model checking problem of forgetting $V$ from a disjunctive formula is in NP $\cap$ co-NP and the reasoning problems are in EXPTIME-complete.

### 3.1 Definition of Forgetting

In this subsection we present the definition of forgetting in $\mu$-calculus by the bisimulation technique. For convenience, in the following we let $\mathcal{M}_i = (S_i, r_i, R_i, L_i)$ with $i \in \mathbf{N}$ be Kripke structure.

**Definition 3 (V-bisimulation).** *Let $V \subseteq \mathcal{A}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Kripke structures. $\mathcal{B} \subseteq S_1 \times S_2$ is a $V$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ if:*

- *$r_1 \mathcal{B} r_2$,*
- *for each $s \in S_1$ and $t \in S_2$, if $s\mathcal{B}t$ then $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A} - V$,*
- *$(s, s') \in R_1$ and $s\mathcal{B}t$ implies that there is a $t'$ such that $s'\mathcal{B}t'$ and $(t, t') \in R_2$, and*
- *vice versa: if $s\mathcal{B}t$ and $(t, t') \in R_2$ then there is an $s'$ with $(s, s') \in R_1$ and $t'\mathcal{B}s'$.*

Two Kripke structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are $V$-*bisimilar*, denoted $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$, if there exists a $V$-bisimulation $\mathcal{B}$ between them. The $V$-bisimulation is similar with the $\mathcal{L}$-bisimulation in [7], which is a relation satisfying the above clauses just for the symbols in language $\mathcal{L}$. From this definition we know that the $\leftrightarrow_V$ is an equivalence relation between Kripke structures. Formally:

**Proposition 2.** *Let $V, V_1 \subseteq \mathcal{A}$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three Kripke structures, then we have:*

*(i) the $\leftrightarrow_V$ is an equivalence relation between Kripke structures;*
*(ii) if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$, then $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$.*

*Proof.* (i) We prove it form the reflexivity, symmetry and transitivity.

(1) $\leftrightarrow_V$ is reflexive. It is easy to check that $\mathcal{M} \leftrightarrow_V \mathcal{M}$ for any Kripke structure.

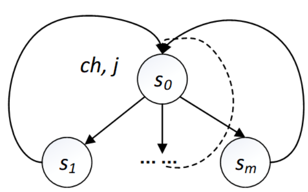
(2) $\leftrightarrow_V$ is symmetric. We will show that for each $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ then $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the $V$-bisimulation $\mathcal{B}$, we construct a relation $\mathcal{B}_1$ as follows: $\mathcal{B}_1 = \{(s,t)|(t,s) \in \mathcal{B}\}$. We will show that $\mathcal{B}_1$ is a $V$-bisimulation between $\mathcal{M}_2$ and $\mathcal{M}_1$ from the following several points:

- $r_2 \mathcal{B}_1 r_1$ since $r_1 \mathcal{B} r_2$,
- for each $s \in S_1$ and $t \in S_2$, if $t\mathcal{B}_1 s$ then we have $s\mathcal{B}t$ and hence $p \in L_1(s)$ iff $p \in L_2(t)$ for each $p \in \mathcal{A} - V$, and
- the third and forth points in the definition of $V$-bisimulation can be checked easily for $\mathcal{B}_1$.

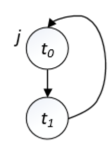
(3) $\leftrightarrow_V$ is transitive. We will show that for each $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, if $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ then $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the $V$-bisimulation $\mathcal{B}_1$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_3$ by the $V$-bisimulation $\mathcal{B}_2$, we construct a relation $\mathcal{B}$ as follows: $\mathcal{B} = \{(s,z)|(s,t) \in \mathcal{B}_1 \ and \ (t,z) \in \mathcal{B}_2\}$. We can also prove silimarly with (2) that $\mathcal{B}$ is a $V$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$. Therefore, $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_3$.

(ii) In order to prove $\mathcal{M}_1 \leftrightarrow_{V \cup V_1} \mathcal{M}_3$, we only need to find a binary relation $\mathcal{B}$ such that $\mathcal{B}$ is a $(V \cup V_1)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$. Supposing $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ by the $V$-bisimulation $\mathcal{B}_1$ and $\mathcal{M}_2 \leftrightarrow_{V_1} \mathcal{M}_3$ by the $V_1$-bisimulation $\mathcal{B}_2$. Let $\mathcal{B} = \{(s_1,s_3)|(s_1,s_2) \in \mathcal{B}_1 \ and \ (s_2,s_3) \in \mathcal{B}_2\}$. We can easily check that $\mathcal{B}$ is a $(V \cup V_1)$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_3$.

*Example 4.* In Fig. 2 we can check that $\mathcal{M} \leftrightarrow_{\{ch\}} \mathcal{M}'$ because there is a $\{ch\}$-bisimulation $\mathcal{B} = \{(s_0,t_0),(s_1,t_1),(s_2,t_1),\ldots,(s_m,t_1)\}$ between $\mathcal{M}$ and $\mathcal{M}'$.




(a) Kripke structure $\mathcal{M}$




(b) Kripke structure $\mathcal{M}'$

Fig. 2: Two $\{ch\}$-bisimilar Kripke structures

As it has been said in [7] that any $\mathcal{L}$-sentence $\varphi$ (that is: a $\mu$-sentence that only uses symbols from the language $\mathcal{L}$) is invariant for $\mathcal{L}$-bisimulation, i.e. if there is a $\mathcal{L}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ then $\varphi$ holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$. It should then be obvious that if IR$(\varphi, V)$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ then $\varphi$ holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$. **Therefore, in the following we only consider the $\mu$-sentence.**

We define the forgetting in $\mu$-calculus in the follows, although forgetting is a dual concept in S5 and Propositional logic [15], it is important to give the formal definition of forgetting in $\mu$-calculus since as we will see that it is useful to knowledge update of $\mu$-calculus. Moreover, as shown in [17], forgetting in CTL can also be used to compute the SNC and the WSC.

**Definition 4 (Forgetting).** *Let $V \subseteq \mathcal{A}$ and $\phi$ be a $\mu$-sentence. A formula $\psi$ with $Var(\psi) \cap V = \emptyset$ is a* result of forgetting $V$ from $\phi$*, if*

$$Mod(\psi) = \{\mathcal{M} \mid \exists \mathcal{M}' \in Mod(\phi) \ \&\ \mathcal{M}' \leftrightarrow_V \mathcal{M}\}.$$

For convenience, we denote the result of forgetting $V$ from $\phi$ as $\mathrm{F}_\mu(\phi, V)$. A formula $\phi$ is *irrelevant to* the atoms in a set $V$ (or simply $V$*-irrelevant*), written $\mathrm{IR}(\phi, V)$, if there is a formula $\psi$ with $Var(\psi) \cap V = \emptyset$ such that $\phi \equiv \psi$. The $V$*-irrelevant* of a set of formulas can be defined similarly, i.e. a set $\Sigma$ of formulas is irrelevant to the atoms in $V$, written $\mathrm{IR}(\Sigma, V)$, if $\mathrm{IR}(\varphi, V)$ for each $\varphi \in \Sigma$.

### 3.2 Semantic Properties of Forgetting in $\mu$-calculus

In this part we show the semantic properties of forgetting in $\mu$-calculus. In particular, we show that our forgetting is closed in $\mu$-calculus, satisfies the general postulates, i.e. the representation theorem, and the algebraic properties, including Modularity, Commutativity and Homogeneity.

**Theorem 3.** *Let $q \in \mathcal{A}$ and $\phi$ be a $\mu$-sentence. There is a $\mu$-sentence $\psi$ such that $IR(\psi, \{q\})$ and $\psi \equiv \mathrm{F}_\mu(\phi, \{q\})$.*

*Proof.* This can be obtained from the Theorem 3.1 in [7].

This means that the uniform interpolant $\widetilde{\exists} q\phi$ ($q \in \mathcal{A}$) of $\phi$ with respect to $Var(\phi) - \{q\}$ is the result of forgetting $\{q\}$ from $\phi$. In this case, we also say that the forgetting of $\mu$-calculus is *closed*, that is the result of forgetting some set of atoms from a $\mu$-sentence is also a $\mu$-sentence. Note that if both $\psi$ and $\psi'$ are results of forgetting $V$ from $\phi$, then $Mod(\psi) = Mod(\psi')$, i.e., $\psi$ and $\psi'$ have the same models. In this sense, the forgetting result is unique (up to equivalence).

At this point, it is important to emphasize that, the notion of forgetting we have defined for $\mu$-calculus respects the classical forgetting defined for propositional logic (PL) [25]. Assuming $\varphi$ is a PL formula and $p \in \mathcal{A}$, then $Forget(\varphi, p)$ is a result of forgetting $p$ from $\varphi$; that is, $Forget(\varphi, p) \equiv \varphi[p/\bot] \vee \varphi[p/\top]$. That way, given a set $V \subseteq \mathcal{A}$, one can recursively define $Forget(\varphi, V \cup \{p\}) = Forget(Forget(\varphi, p), V)$, where $Forget(\varphi, \emptyset) = \varphi$. Using this insight, the following result shows that the classical notion of forgetting (for PL [25]) is a special case of forgetting in $\mu$-calculus.

**Theorem 4.** *Let $\varphi$ be a PL formula and $V \subseteq \mathcal{A}$, then*

$$\mathrm{F}_\mu(\varphi, V) \equiv Forget(\varphi, V).$$

*Proof.* On one hand, for each $\mathcal{M} \in Mod(\mathrm{F}_\mu(\varphi, V))$ there exists a $\mathcal{M}' \in Mod(\varphi)$ such that $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Thus, $r\mathcal{B}r'$. Hence, $\mathcal{M}$ is a model of $Forget(\varphi, V)$ due to $\mathrm{IR}(Forget(\varphi, V), V)$.

On the other hand, for each $\mathcal{M} \in \mathit{Mod}(\mathit{Forget}(\varphi, V))$ with $\mathcal{M} = (S, r, R, L)$ there exists a $\mathcal{M}' \in \mathit{Mod}(\varphi)$ such that $r\mathcal{B}r'$. Construct a Kripke structure $\mathcal{M}_1$ such that $\mathcal{M}_1 = (S_1, r_1, R_1, L_1)$ with $S_1 = (S - \{r\}) \cup \{r_1\}$, $R_1$ is the same as $R$ except replace $r$ with $r_1$, and $L_1$ is the same as $L$ except $L_1(r_1) = L'(r')$, where $L'$ is the label function of $M'$. It is clear that $\mathcal{M}_1$ is a model of $\varphi$ and $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}$. Hence, $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\varphi, V)$ due to $\mathrm{IR}(\mathrm{F}_\mu(\varphi, V), V)$.

In [30], authors give four postulates concerning knowledge forgetting in **S5** modal logic (also called *forgetting postulates*) which can be considered as desirable properties of such a notion. In the following, we first list these postulates, and then show that our notion of forgetting in $\mu$-calculus satisfies them.

**Forgetting postulates** [30] are:

(**W**) Weakening: $\varphi \models \varphi'$;
(**PP**) Positive Persistence: for any formula $\eta$, if $\mathrm{IR}(\eta, V)$ and $\varphi \models \eta$ then $\varphi' \models \eta$;
(**NP**) Negative Persistence : for any formula $\eta$, if $\mathrm{IR}(\eta, V)$ and $\varphi \not\models \eta$ then $\varphi' \not\models \eta$;
(**IR**) Irrelevance: $\mathrm{IR}(\varphi', V)$

where $V \subseteq \mathcal{A}$, $\varphi$ is a $\mu$-sentence and $\varphi'$ is the result of forgetting $V$ from $\varphi$. Intuitively, the postulate (**W**) says, forgetting weakens the original formula; the postulates (**PP**) and (**NP**) say that forgetting results have no effect on formulas that are irrelevant to forgotten atoms; the postulate (**IR**) states that forgetting result is irrelevant to forgotten atoms. It is noteworthy that they are not all orthogonal e.g., (**NP**) is a consequence of (**W**) and (**PP**). Nonetheless, we prefer to list them all, in order to outline the basic intuition behind them.

The following says that the forgetting postulates above indeed precisely characterize the underling forgetting semantics of $\mu$-calculus.

**Theorem 5 (Representation Theorem).** *Let $\varphi$, $\varphi'$ and $\phi$ be $\mu$-sentences and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:*

*(i)* $\varphi' \equiv \mathrm{F}_\mu(\varphi, V)$,
*(ii)* $\varphi' \equiv \{\phi \mid \varphi \models \phi$ *and* $IR(\phi, V)\}$,
*(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold if $\varphi$, $\varphi'$ and $V$ are as in (i) and (ii).*

*Proof.* $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$\mathit{Mod}(\mathrm{F}_\mu(\varphi, V)) = \mathit{Mod}(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}).$$

Firstly, suppose that $\mathcal{M}'$ is a model of $\mathrm{F}_\mu(\varphi, V)$. Then there exists a Kripke structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Therefore, we have $\mathcal{M}' \models \phi$ for all $\phi$ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $\mathcal{M}'$ is a model of $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

It is evident that $\{\phi \mid \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \mathrm{F}_\mu(\varphi, V)$ since $\mathrm{IR}(\mathrm{F}_\mu(\varphi, V), V)$ and $\varphi \models \mathrm{F}_\mu(\varphi, V)$ by Theorem 3.

The other parts can be similarly proved as Theorem 4 in [17].

Theorem 5 means that for a given $\mu$-sentence $\varphi$ and a set of atoms $V$, a $\mu$-sentence $\varphi'$ represents a result of forgetting $V$ from $\varphi$ if $\varphi'$ satisfies the four forgetting postulates, and vice versa. That is, the representation theorem gives an "if and only if" characterization on forgetting in $\mu$-calculus, which is in accordance with that in **S5**.

Excepting for the representation theorem, postulate IR is also of crucial importance for computing SNC and WSC. Consider the $\psi = \varphi \wedge (q \leftrightarrow \alpha)$. If $\mathrm{IR}(\varphi \wedge \alpha, \{q\})$, then the result of forgetting $q$ from $\psi$ is $\varphi$. Formally, it can be described in the following lemma, and as we will later see in Section 4, it is the base of reducing the SNC (WSC) of any $\mu$-sentence to that of a proposition.

**Lemma 1.** *Let $\varphi$ and $\alpha$ be two $\mu$-sentences and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then* $\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

*Proof.* Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $\mathcal{M}$ of $\mathrm{F}_\mu(\varphi', q)$ there is a Kripke structure $\mathcal{M}'$ s.t. $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$ and $\mathcal{M}' \models \varphi'$. It's evident that $\mathcal{M}' \models \varphi$, and then $\mathcal{M} \models \varphi$ since $\mathrm{IR}(\varphi, \{q\})$ and $\mathcal{M} \leftrightarrow_{\{q\}} \mathcal{M}'$.

Let $\mathcal{M} \in Mod(\varphi)$ with $\mathcal{M} = (S, s, R, L)$. We construct $\mathcal{M}'$ with $\mathcal{M}' = (S, s, R, L')$ as follows:

$$
\begin{aligned}
&L' : S \to 2^{\mathcal{A}} \; and \; \forall s^* \in S, L'(s^*) = L(s^*) - \{q\} \; if \; (\mathcal{M}, s^*) \not\models \alpha,\\
&else \; L'(s^*) = L(s^*) \cup \{q\},\\
&L'(s) = L(s) \cup \{q\} \; if \; (\mathcal{M}, s) \models \alpha, \; and \; L'(s) = L(s) \; otherwise.
\end{aligned}
$$

It is clear that $\mathcal{M}' \models \varphi$, $\mathcal{M}' \models q \leftrightarrow \alpha$ and $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$. Therefore $\mathcal{M}' \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $\mathcal{M} \models \mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $\mathcal{M}' \leftrightarrow_{\{q\}} \mathcal{M}$ and $\mathrm{IR}(\mathrm{F}_\mu(\varphi \wedge (q \leftrightarrow \alpha), q), \{q\})$.

We will list other interesting properties of the forgetting operator in the follows. Most importantly, the following property guarantees that we can modularly apply forgetting one by one to the atoms to be forgotten, instead of forgetting the set of atoms as a whole, which is spoken in the definition of forgetting.

**Proposition 3.** ***(Modularity)*** *Given a $\mu$-sentence $\varphi$, $V$ a set of atoms and $p$ an atom such that $p \notin V$. Then,*

$$\mathrm{F}_\mu(\varphi, \{p\} \cup V) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V).$$

*Proof.* Let $\mathcal{M}_1$ with $\mathcal{M}_1 = (S_1, s_1, R_1, L_1)$ be a model of $\mathrm{F}_\mu(\varphi, \{p\} \cup V)$. By the definition of forgetting, there exists a model $\mathcal{M}$ with $\mathcal{M} = (S, s, R, L)$ of $\varphi$, such that $\mathcal{M}_1 \leftrightarrow_{\{p\} \cup V} \mathcal{M}$. We construct a Kripke structure $\mathcal{M}_2$ with $\mathcal{M}_2 = (S_2, s_2, R_2, L_2)$ as follows:

(1) for $s_2$: let $s_2$ be the state such that:
  - $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
  - for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
  - for all other atoms $q'$, $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:
  - (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\} \cup V} w_1$, let $w_2 \in S_2$ and
    - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
    - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,

– for all other atoms $q'$, $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

(ii) if $(w_1', w_1) \in R_1$, $w_2$ is constructed based on $w_1$ and $w_2' \in S_2$ is constructed based on $w_1'$, then $(w_2', w_2) \in R_2$.

(3) delete duplicated states in $S_2$ and pairs in $R_2$.

Then we have $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$. Thus, $(\mathcal{M}_2, s_2) \models \mathrm{F}_\mu(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V)$.

On the other hand, suppose that $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, p), V)$, then there exists a Kripke structure $\mathcal{M}_2$ such that $\mathcal{M}_2 \models \mathrm{F}_\mu(\varphi, p)$ and $\mathcal{M}_2 \leftrightarrow_V \mathcal{M}_1$, and there exists $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M} \leftrightarrow_{\{p\}} \mathcal{M}_2$. Therefore, $\mathcal{M} \leftrightarrow_{\{p\} \cup V} \mathcal{M}_1$ by (ii) of Proposition 2, and consequently, $\mathcal{M}_1 \models \mathrm{F}_\mu(\varphi, \{p\} \cup V)$.

The next property follows from the above proposition.

**Corollary 1 (Commutativity).** *Let $\varphi$ be a $\mu$-sentence and $V_i \subseteq \mathcal{A}$ $(i = 1, 2)$. Then:*

$$\mathrm{F}_\mu(\varphi, V_1 \cup V_2) \equiv \mathrm{F}_\mu(\mathrm{F}_\mu(\varphi, V_1), V_2).$$

The following properties show that the forgetting respects the basic semantic notions of logic. They hold in classical propositional logic, modal logic **S5** [30] and CTL [17]. Below we show that they are also satisfied in our notion forgetting in $\mu$-calculus.

**Proposition 4.** *Let $\varphi$, $\varphi_i$, $\psi_i$ ($i = 1, 2$) be formulas in CTL and $V \subseteq \mathcal{A}$. We have*

*(i)* $\mathrm{F}_\mu(\varphi, V)$ *is satisfiable iff* $\varphi$ *is;*
*(ii) If* $\varphi_1 \equiv \varphi_2$*, then* $\mathrm{F}_\mu(\varphi_1, V) \equiv \mathrm{F}_\mu(\varphi_2, V)$*;*
*(iii) If* $\varphi_1 \models \varphi_2$*, then* $\mathrm{F}_\mu(\varphi_1, V) \models \mathrm{F}_\mu(\varphi_2, V)$*;*
*(iv)* $\mathrm{F}_\mu(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$*;*
*(v)* $\mathrm{F}_\mu(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_\mu(\psi_1, V) \wedge \mathrm{F}_\mu(\psi_2, V)$*;*

*Proof.* (i) ($\Rightarrow$) Supposing $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\varphi, V)$, then there is a model $\mathcal{M}'$ of $\varphi$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by the definition of $\mathrm{F}_\mu$.

($\Leftarrow$) Supposing $\mathcal{M}$ is a model of $\varphi$, then there is a Kripke structure $\mathcal{M}'$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$, and then $\mathcal{M}' \models \mathrm{F}_\mu(\varphi, V)$ by the definition of $\mathrm{F}_\mu$.

The (ii) and (iii) can be proved similarly.

(iv) ($\Rightarrow$) For all$\mathcal{M} \in$ *Mod*$(\mathrm{F}_\mu(\psi_1 \vee \psi_2, V))$, there exists $\mathcal{M}' \in$ *Mod*$(\psi_1 \vee \psi_2)$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ and $\mathcal{M}' \models \psi_1$ or $\mathcal{M}' \models \psi_2$
$\Rightarrow$ there exists $\mathcal{M}_1 \in$ *Mod*$(\mathrm{F}_\mu(\psi_1, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_1$ or there exists $\mathcal{M}_2 \in$ *Mod*$(\mathrm{F}_\mu(\psi_2, V))$ s.t. $\mathcal{M}' \leftrightarrow_V \mathcal{M}_2$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V)$.

($\Leftarrow$) for all $\mathcal{M} \in$ *Mod*$(\mathrm{F}_\mu(\psi_1, V) \vee \mathrm{F}_\mu(\psi_2, V))$
$\Rightarrow \mathcal{M} \models \mathrm{F}_\mu(\psi_1, V)$ or $\mathcal{M} \models \mathrm{F}_\mu(\psi_2, V)$
$\Rightarrow$ there is a Kripke structure $\mathcal{M}_1$ s.t. $\mathcal{M} \leftrightarrow_V \mathcal{M}_1$ and $\mathcal{M}_1 \models \psi_1$ or $\mathcal{M}_1 \models \psi_2$
$\Rightarrow \mathcal{M}_1 \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $\mathcal{M}_2$ s.t. $\mathcal{M}_1 \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M}_2 \models \mathrm{F}_\mu(\psi_1 \vee \psi_2, V)$
$\Rightarrow \mathcal{M} \leftrightarrow_V \mathcal{M}_2$ and $\mathcal{M} \models \mathrm{F}_\mu(\psi_1 \vee \psi_2, V)$.

The (v) can be proved as (iv).

Intuitively, in Proposition 4, (i) means that forgetting some set of atoms from a sentence do not affect the satisfiability of this sentence. In (ii) we can see that if two sentence are equivalent then the results of forgetting the same set of atoms from both of them are also equivalent. The intuitive meaning of (iii) is obvious. (iv) refers to the result of forgetting $V$ from a disjunctive formula $\varphi_1 \vee \varphi_2$ is equivalent with the disjunction of the results of forgetting $V$ from $\varphi_1$ and $\varphi_2$ respectively. While (v) points out that it is not the case for a conjunctive formula.

**Proposition 5 (Homogeneity).** *Let $V \subseteq \mathcal{A}$ and $\phi$ be a $\mu$-sentence, then we have:*

*(i)* $\mathrm{F}_\mu(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_\mu(\phi, V)$.
*(ii)* $\mathrm{F}_\mu(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_\mu(\phi, V)$.

The homogeneity of AX (or EX) on forgetting says we can move the operator $\mathrm{F}_\mu$ afterward to the AX (or EX) to compute the forgetting of formula of the form $\mathrm{AX}\varphi$ (or $\mathrm{EX}\varphi$).

It is noteworthy that it is easy to compute forgetting a set of atoms from a $\mu$-sentence when it is a disjunctive formula.

**Proposition 6.** *Let $\varphi$ be a $\mu$-sentence and $p \in \mathcal{A}$. If $\varphi$ be a disjunctive formula [9], then $\mathrm{F}_\mu(\varphi, \{p\})$ can be computed in linear time.*

*Proof.* By Theorem 3.6. in [9], we have $\mathrm{F}_\mu(\varphi, \{p\}) \equiv \varphi[p/\top, \neg p/\top]$, where $\varphi[p/\top, \neg p/\bot]$ is obtained from $\varphi$ by simultaneously substituting the literals $p$ and $\neg p$ with $\top$.

### 3.3 Complexity Results

Before talk about the complexity results of the forgetting operator, let's recall the $\mu$-automaton, which is important in $\mu$-calculus to show the uniform interpolation and model checking of $\mu$-calculus. In this part we will show that the $\mu$-automaton can also make a big convenience to show our complexity results.

**Definition 5 ($\mu$-automaton [7]).** *A $\mu$-automaton $A$, also called modal automaton [4], is a tuple $(Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ such that:*

*(i)* *$Q$ is a finite set of states;*
*(ii)* *$\Sigma_p$ is a finite subset of $\mathcal{A}$;*
*(iii)* *$\Sigma_r$ is a finite subset of the set of actions, in this paper it is an empty set;*
*(iv)* *$q_0 \in Q$ is the initial state;*
*(v)* *$\delta : Q \times \mathcal{P}(\Sigma_p) \to \mathcal{P}\mathcal{P}(\Sigma_r \times Q)$;*
*(vi)* *$\Omega : Q \to \mathcal{N}$.*

Although this automata differ slightly from those given in [18,4], but the automata in their various guises are equivalent. Moreover, modal automata are essentially alternating automata [4]. It is worth mentioning that it has been shown that construct a $\mu$-automaton from a $\mu$-calculus formula can be done in exponential time, while it is in polynomial time when the $\mu$-calculus formula is a disjunctive formula. Then we have the following complexity result.

**Proposition 7 (Model Checking).** *Given a finite Kripke structure $\mathcal{M}$, a disjunctive formula $\varphi$ and $V \subseteq \mathcal{A}$, deciding $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is* NP $\cap$ *co*-NP.

*Proof.* Let $A_\varphi$ be a $\mu$-automaton such that for any Kripke structure $\mathcal{N}$ there is $A_\varphi$ accept $\mathcal{N}$ iff $\mathcal{N} \models \varphi$. Where $A_\varphi = (Q, \Sigma_p, \Sigma_r, q_0, \delta, \Omega)$ with $Var(\varphi) = \Sigma_p \cup \Sigma_r$. Without loss of generality, we assume $V \subseteq Var(\varphi)$ and $V = \{p\}$. Therefore we can constructure a $\mu$-automaton $B = (Q, \Sigma_p - V, \Sigma_r, q_0, \delta', \Omega)$ with:

$$\delta'(q, L) := \delta(q, L) \cup \delta(q, L \cup \{p\}).$$

It has been proved in [7] that for each Kripke structure $\mathcal{N}$, $B$ accept $\mathcal{N}$ iff there is a model $\mathcal{N}'$ of $\varphi$ such that $\mathcal{N} \leftrightarrow_{\{p\}} \mathcal{N}'$, i.e. $B$ corresponds to a $\mu$-sentence which is equivalent to $\mathrm{F}_\mu(\varphi, V)$ by the definition of forgetting in $\mu$-calculus.

In this case, the problem $\mathcal{M} \models^? \mathrm{F}_\mu(\varphi, V)$ is reduced to decide whether $B$ accept $\mathcal{M}$, which is NP $\cap$ co-NP [4].

More importantly, from the perspective of knowledge bases evolving, we are also interested in the following reasoning problems about forgetting, which are explored in CPL [28].

(i) [Var-weak] if the restriction of $\varphi$ on the signature of $\psi$ is at most as strong as $\psi$, i.e. $\psi \models \mathrm{F}_\mu(\varphi, V)$,
(ii) [Var-strong] if the restriction of $\varphi$ on the signature of $\psi$ is at least as strong as $\psi$, i.e. $\mathrm{F}_\mu(\varphi, V) \models \psi$,
(iii) [Var-entailment] if the restriction of one knowledge base on its original signature is at most as strong as that of the other, i.e. $\mathrm{F}_\mu(\varphi, V) \models \mathrm{F}_\mu(\psi, V)$.

Where $\varphi$, $\psi$ are $\mu$-sentences, and $V$ a set of atoms. Besides, in (i) and (ii) there is $Var(\varphi) - V = Var(\psi)$ and in (iii) there is $V \subseteq Var(\varphi) \cap Var(\psi)$. Then we have the following results.

**Theorem 6 (Entailment).** *Let $\varphi$ and $\psi$ be two $\mu$-sentences and $V$ be a set of atoms. Then, the following problems are* EXPTIME-*complete.*

*(i) deciding* $\mathrm{F}_\mu(\varphi, V) \models^? \psi$,
*(ii) deciding* $\psi \models^? \mathrm{F}_\mu(\varphi, V)$,
*(iii) deciding* $\mathrm{F}_\mu(\varphi, V) \models^? \mathrm{F}_\mu(\psi, V)$.

*Proof.* We prove the (i), there other two results can be proved similarly.

Let $A_\varphi$ and $A_\psi$ be the $\mu$-automaton of $\varphi$ and $\psi$ respectively, we can construct the $\mu$-automaton $B$ of $\mathrm{F}_\mu(\varphi, V)$ from $A_\varphi$ by the proof of Proposition 7. By Proposition 7.3.2 in [6] , we can obtain the complement $C$ of $A_\psi$ in linear time, and then the intersection $A_{C \cap B}$ between $C$ and $B$ in linear time. In this case, the $\mathrm{F}_\mu(\varphi, V) \models^? \psi$ is reduced to decide whether the language accpeted by $A_{C \cap B}$ is empty, which is EXPTIME-complete [6].

Similarly with the reasoning problems talked above, the following equivalent problem are also important, in which "var-independence" and "var-equivalence" under CPL are proposed [23].

(i) [Var-independence] If a formula $\varphi$ is independent of a set $V$ of atoms, i.e. $\mathrm{F}_\mu(\varphi, V) \equiv \varphi$,
(ii) [Var-match] if the restriction of $\varphi$ on the signature of $\psi$ perfectly matches $\psi$, i.e. $\mathrm{F}_\mu(\varphi, V) \equiv \psi$.
(iii) [Var-equivalence] if the restriction of the two formulas on a common signature are equivalent, i.e. $\mathrm{F}_\mu(\varphi, V) \equiv \mathrm{F}_\mu(\psi, V)$.

The following results are implications of Theorem 6.

**Corollary 2.** *Let $\varphi$ and $\psi$ be two $\mu$-sentences and $V$ be a set of atoms. Then, the following problems are* EXPTIME*-complete.*

*(i) deciding* $\psi \equiv^? \mathrm{F}_\mu(\varphi, V)$,
*(ii) deciding* $\mathrm{F}_\mu(\varphi, V) \equiv^? \varphi$,
*(iii) deciding* $\mathrm{F}_\mu(\varphi, V) \equiv^? \mathrm{F}_\mu(\psi, V)$.

## 4 Necessary and Sufficient Conditions

In this section, we present two key notions of our work: namely, the *strongest necessary condition* (SNC) and the *weakest sufficient condition* (WSC) of a given $\mu$-calculus specification, which correspond to the *most general consequence* and the *most specific abduction* of a specification, respectively. As aforementioned in the introduction, these notions respectively are accordance with the *strongest precondition* (SP) and the *weakest post-condition* (WP) (introduced by E. Dijkstra in [11]), which have been central to a wide variety of tasks and studies, e.g. generating counterexamples and refinement of system in verification. Our contribution, in particular, will be on computing SNC and WSC via forgetting under a given $\mu$-sentence and a set $V$ of atoms. Let us give the formal definition.

**Definition 6 (sufficient and necessary condition).** *Let $\phi$, $\psi$ be two $\mu$-sentences, $V \subseteq Var(\phi)$, $q \in Var(\phi) - V$ and $Var(\psi) \subseteq V$.*

- *$\psi$ is a* necessary condition *(NC in short) of $q$ on $V$ under $\phi$ if $\phi \models q \rightarrow \psi$.*
- *$\psi$ is a* sufficient condition *(SC in short) of $q$ on $V$ under $\phi$ if $\phi \models \psi \rightarrow q$.*
- *$\psi$ is a* strongest necessary condition *(SNC in short) of $q$ on $V$ under $\phi$ if it is a NC of $q$ on $V$ under $\phi$, and $\phi \models \psi \rightarrow \psi'$ for any NC $\psi'$ of $q$ on $V$ under $\phi$.*
- *$\psi$ is a* weakest sufficient condition *(WSC in short) of $q$ on $V$ under $\phi$ if it is a SC of $q$ on $V$ under $\phi$, and $\phi \models \psi' \rightarrow \psi$ for any SC $\psi'$ of $q$ on $V$ under $\phi$.*

Note that if both $\psi$ and $\psi'$ are SNC (WSC) of $q$ on $V$ under $\phi$, then $Mod(\psi) = Mod(\psi')$, i.e., $\psi$ and $\psi'$ have the same models. In this sense, the SNC (WSC) of $q$ on $V$ under $\phi$ is unique (up to semantic equivalence). The following result shows that the SNC and WSC are in fact dual notions.

**Proposition 8 (Dual).** *Let $V, q, \varphi$ and $\psi$ are defined as in Definition 6. Then, $\psi$ is a SNC (WSC) of $q$ on $V$ under $\varphi$ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on $V$ under $\varphi$.*

In order to generalise Definition 6 to arbitrary formulas, one can replace $q$ (in the definition) by any formula $\alpha$, and redefine $V$ as a subset of $\mathit{Var}(\alpha) \cup \mathit{Var}(\phi)$.

It turns out that the previous notions of SNC and WSC for an atomic variable can be lifted to any formula, or, conversely, the SNC and WSC of any formula can be reduced to that of an atomic variable, as the following result shows.

**Proposition 9.** *Let $\Gamma$ and $\alpha$ be two $\mu$-sentences, $V \subseteq \mathit{Var}(\alpha) \cup \mathit{Var}(\Gamma)$ and $q$ be a new proposition not in $\Gamma$ and $\alpha$. Then, a formula $\varphi$ of $V$ is the SNC (WSC) of $\alpha$ on $V$ under $\Gamma$ iff it is the SNC (WSC) of $q$ on $V$ under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.*

The following result establishes the bridge between forgetting and the notion of SNC (WSC) which are central to our contribution.

**Theorem 7.** *Let $\varphi$ be a $\mu$-sentence, $V \subseteq \mathit{Var}(\varphi)$ and $q \in \mathit{Var}(\varphi) - V$.*

*(i)* $\mathrm{F}_\mu(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ *is a SNC of $q$ on $V$ under $\varphi$.*
*(ii)* $\neg\mathrm{F}_\mu(\varphi \wedge \neg q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ *is a WSC of $q$ on $V$ under $\varphi$.*

Following Theorem 7, assume that $\beta = \mathrm{F}_\mu(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$. Then, $\varphi \wedge q \models \beta$ by (**W**). Moreover, $\varphi \wedge q \models \beta$, and then $\beta$ is a NC of $q$ on $V$ under $\varphi$.

In addition, for any $\mu$-sentence $\psi$ with $\mathrm{IR}(\psi, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ and $\varphi \wedge q \models \psi$, we have $\beta \models \psi$ by (**PP**). Therefore, $\beta$ is the SNC of $q$ on $V$ under $\varphi$. This shows the intuition of how the SNC can be obtained from the forgetting.

## 5 Representing knowledge update via forgetting

In this section, we present the final key notion of our work: knowledge update. In particular, we will propose a method of defining knowledge update via forgetting which will satisfy all the following Katsuno and Mendelzon's postulates (U1)-(U8) proposed in [20]:

(U1) $\Gamma \diamond \phi \models \phi$.
(U2) If $\Gamma \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma$.
(U3) If both $\Gamma$ and $\phi$ are satisfiable, then $\Gamma \diamond \phi$ is also satisfiable.
(U4) If $\Gamma_1 \equiv \Gamma_2$ and $\phi_1 \equiv \phi_2$, then $\Gamma_1 \diamond \phi_1 \equiv \Gamma_2 \diamond \phi_2$.
(U5) $(\Gamma \diamond \phi) \wedge \psi \models \Gamma \diamond (\phi \wedge \psi)$.
(U6) If $\Gamma \diamond \phi \models \psi$ and $\Gamma \diamond \psi \models \phi$, then $\Gamma \diamond \phi \equiv \Gamma \diamond \psi$.
(U7) If $\Gamma$ has a unique model, then $(\Gamma \diamond \phi) \wedge (\Gamma \diamond \psi) \models \Gamma \diamond (\phi \vee \psi)$.
(U8) $(\Gamma_1 \vee \Gamma_2) \diamond \phi \equiv (\Gamma_1 \diamond \phi) \vee (\Gamma_2 \diamond \phi)$.

Where $\varphi \diamond \psi$ expresses the result of updating $\varphi$ with $\psi$ and $\diamond$ is the knowledge update operator.

For this purpose, in this part we suppose the models of a $\mu$-sentence are initial structures, in which an initial structure is a Kripke structure $\mathcal{M} = (S, sr, R, L)$ with $S$ is a finite set of states, $sr$ is an initial state (i.e. for each state $s' \in S$ the $sr$ can arrive at $s'$) and $R$ is a total relation. Besides, we also restrict the definition of forgetting on initial structures, i.e. the models mentioned in Definition 4 are initial structures. In this case, we have:

**Theorem 8.** *Let $V \subseteq \mathcal{A}$ and $\phi$ be a $\mu$-sentence. Then there is a $\mu$-sentence $\psi$ such that:*

$$\mathcal{M} \models \psi \text{ iff there is a model } \mathcal{M}' \text{ with } \mathcal{M} \leftrightarrow_V \mathcal{M}' \text{ and } \mathcal{M}' \models \phi.$$

*Where both $\mathcal{M}$ and $\mathcal{M}'$ are initial structures.*

*Proof.* Let $\psi = \mathrm{F}_\mu(\phi, V)$. We have that for each $\mathcal{M} \models \psi$ there is a $\mathcal{M}' \models \phi$ with $\mathcal{M} \leftrightarrow_V \mathcal{M}'$ by Theorem 3 and for each $\mathcal{M}' \in Mod(\phi)$ there is $\phi \models \psi$. In this case, we can easy prove that for each initial structure $\mathcal{M}$, if $\mathcal{M} \models \psi$ then we can obtain an initial structure $\mathcal{M}'$ such that $\mathcal{M}' \models \phi$ and $\mathcal{M} \leftrightarrow_V \mathcal{M}'$. Besides, for each $\mathcal{M}' \in Mod(\phi)$ there is $\mathcal{M}' \models \psi$ by $\phi \models \psi$.

Intuitively, given a logic language $\mathcal{L}$, we say some operator $\mathcal{O}$ in $\mathcal{L}$ is closed whenever the result of using the $\mathcal{O}$ on the elements of $\mathcal{L}$ is also in $\mathcal{L}$. Theorem 8 shows that the forgetting in $\mu$-calculus is also closed when restrict the models of $\mu$-sentence to initial structures. Formally:

**Corollary 3.** *The forgetting of $\mu$-calculus is closed under the initial structure semantic, i.e. we only consider the initial structures as the models of $\mu$-sentence.*

According to [17], we can see that any initial structure $\mathcal{M}$ on $\mathcal{A}$ can be captured by a CTL formula, i.e. the characterizing formula $\mathcal{F}_\mathcal{A}(\mathcal{M})$ [17], and hence a $\mu$-sentence [13]. In this case, we define the knowledge update operator $\diamond_\mu$ in $\mu$-calculus as follows.

**Definition 7.** *Let $\Gamma$ and $\phi$ be $\mu$-sentences, the knowledge update operator $\diamond_\mu$ is defined as follows:*

$$Mod(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in Mod(\Gamma)} \bigcup_{V_{min}} Mod(\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_{min}) \wedge \phi),$$

*where $\mathcal{F}_\mathcal{A}(\mathcal{M})$ is the characterizing formula of $\mathcal{M}$ on $\mathcal{A}$, and $V_{min} \subseteq \mathcal{A}$ is a minimal subset of atoms that makes $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_{min}) \wedge \phi$ consistent.*

Intuitively, $\Gamma \diamond_\mu \phi$ means the result of updating $\Gamma$ with $\phi$ by minimally fixing the models of $\Gamma$ into that of $\phi$.

Recall the definition of knowledge update in CPL, let $I$, $J_1$ and $J_2$ be three interpretations, then $J_1$ is closer to $I$ than $J_2$, written $J_1 \leq_{I,pam} J_2$, iff $\mathrm{Diff}(I, J_1) \subseteq \mathrm{Diff}(I, J_2)$, where $\mathrm{Diff}(X, Y) = \{p \in \mathcal{A} \mid X(p) \neq Y(p)\}$. The set of models of knowledge update $\psi$ to $\Gamma$ is equal with the set of union of sets of models which are minimal models of $\psi$ about the partial order $\leq_{I,pam}$ with $I$ is a model of $\Gamma$, i.e. $Mod(\Gamma \diamond_{pam} \psi) = \bigcup_{I \in Mod(\Gamma)} Min(Mod(\psi), \leq_{I,pam})$. Where $Min(Mod(\psi), \leq_{I,pam})$ expresses the set of models $J$ of $\psi$ such that $J$ is minimal with respect to $\leq_{I,pam}$.

Similarly, we can define a partial ordering over the set of initial structures that links to knowledge operator $\diamond_\mu$.

**Definition 8.** *let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be three initial structures, then $\mathcal{M}_1$ is closer to $\mathcal{M}$ than $\mathcal{M}_2$, written $\mathcal{M}_1 \leq_\mathcal{M} \mathcal{M}_2$, iff for any $V_2 \subseteq \mathcal{A}$ such that $\mathcal{M}_2 \leftrightarrow_{V_2} \mathcal{M}$, there exists a $V_1 \subseteq V_2$ such that $\mathcal{M}_1 \leftrightarrow_{V_1} \mathcal{M}$. We denote $\mathcal{M}_1 <_\mathcal{M} \mathcal{M}_2$ iff $\mathcal{M}_1 \leq_\mathcal{M} \mathcal{M}_2$ and $\mathcal{M}_2 \not\leq_\mathcal{M} \mathcal{M}_1$.*

Let $M$ be a set of initial structures and $\mathcal{M}$ an initial structure, we also use $Min(M, \leq_{\mathcal{M}})$ to denote the set of all minimal initial structures with respect to $\leq_{\mathcal{M}}$. Then we have the following theorem.

**Theorem 9.** *Let $\Gamma$ and $\phi$ be $\mu$-sentences. Then we have:*

$$\mathit{Mod}(\Gamma \diamond_\mu \phi) = \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}}).$$

*Proof.* For each initial structure $\mathcal{M}' \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$, we will show that there exists some $\mathcal{M} \in \mathit{Mod}(\Gamma)$ such that $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$. According to Definition 7, we know that there exists some $\mathcal{M} \in Mod(\Gamma)$ such that $\mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_{min}) \wedge \phi)$. Further, there is a particular $V' \subseteq \mathcal{A}$ (i.e. $V' = V_{min}$) such that $\mathcal{M}' \leftrightarrow_{V'} \mathcal{M}$ and $\mathcal{M}' \in \mathit{Mod}(\phi)$. Since such $V'$ is a minimal subset of $\mathcal{A}$ satisfying these properties, it concludes that for any other models $\mathcal{M}''$ of $\phi$ with $\mathcal{M}'' \leftrightarrow_{V_{min}} \mathcal{M}$, we have $\mathcal{M}' \leq_{\mathcal{M}} \mathcal{M}''$ by the definitions of forgetting and characterizing formula. Therefore, $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$.

For each initial structure $\mathcal{M}' \in \bigcup_{\mathcal{M} \in \mathit{Mod}(\Gamma)} Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$, there exists some $\mathcal{M} \in \mathit{Mod}(\Gamma)$ such that $\mathcal{M}' \in Min(\mathit{Mod}(\phi), \leq_{\mathcal{M}})$. Let $V_{min}$ be a minimal subset of atoms such that $\mathcal{M}' \leftrightarrow_{V_{min}} \mathcal{M}$. Then according to the definition of $\leq_{\mathcal{M}}$, we know that there does not exist another $\mathcal{M}'' \in \mathit{Mod}(\phi)$ such that $\mathcal{M}'' \leftrightarrow_{V'} \mathcal{M}$ and $V' \subset V_{min}$. This follows that $\mathcal{M}' \in \mathit{Mod}(\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_{min}) \wedge \phi)$ and hence $\mathcal{M}' \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$.

Theorem 9 means that our definition of knowledge update in $\mu$-calculus by forgetting is accordance with that by the $\leq_{\mathcal{M}}$, which is similar with the partial order $\leq_{I,pam}$ in CPL.

More important, the following theorem shows that our definition of $\diamond_\mu$ by forgetting satisfies Katsuno and Mendelzon's update postulates.

**Theorem 10.** *Knowledge update operator $\diamond_\mu$ satisfies Katsuno and Mendelzon's update postulates (U1)-(U8).*

*Proof.* For (U1), we know that $\mathit{Mod}(\Gamma \diamond_\mu \phi) \subseteq \mathit{Mod}(\phi)$ by Theorem 9, hence $\Gamma \diamond_\mu \phi \models \phi$.

For (U2), we will prove $\Gamma \diamond_\mu \phi \models \Gamma$ at first. For any model $\mathcal{M}$ of $\Gamma \diamond_\mu \phi$ there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma)$ and $V_{min}$ such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Then we have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Similarly, for any model $\mathcal{M}$ of $\Gamma$, there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma \diamond_\mu \phi)$ and $V_{min}$ such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. We have $V_{min} = \emptyset$ since $\Gamma \models \phi$. Hence $\Gamma \models \Gamma \diamond_\mu \phi$.

It is easy to show $\diamond_\mu$ satisfies (U3) and (U4). We now prove (U5). For any model $\mathcal{M}$ of $(\Gamma \diamond_\mu \phi) \wedge \psi$ there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma)$ and $V_{min}$ such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Besides, we can see that $\mathcal{M} \models \phi \wedge \psi$. Therefore, we have $\mathcal{M} \models \Gamma \diamond_\mu (\phi \wedge \psi)$.

For (U6), we will prove $\Gamma \diamond_\mu \phi \models \Gamma \diamond_\mu \psi$, and the other direction can be proved similarly. For any model $\mathcal{M}$ of $\Gamma \diamond_\mu \phi$, $\mathcal{M}$ is also a model of $\psi$. There is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma)$ and $V_{min}$ such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore $\mathcal{M}$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \psi$. This shows that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Moreover, $V_{min}$ is also the minimal set such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \psi$ is consistent. Otherwise, suppose that $V \subset V_{min}$ such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V) \wedge \psi$ is consistent as well. Then, $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V) \wedge$

$\phi$ should also be consistent by $\Gamma \diamond_\mu \psi \models \phi$, which contradicts to the fact that $V_{min}$ is the minimal set of atoms such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}_1), V_{min}) \wedge \phi$ is consistent. Hence, $\mathcal{M}$ is also a model of $\Gamma \diamond_\mu \psi \models \psi$.

Now we prove (U7). Suppose that $\Gamma$ has the unique model $\mathcal{M}$. For each $\mathcal{M}_1 \in \mathit{Mod}((\Gamma \diamond_\mu \phi) \wedge (\Gamma \diamond_\mu \psi))$ there exists $V_1$ and $V_2$ which are minimal such that $\mathcal{M} \leftrightarrow_{V_1} \mathcal{M}_1$ and $\mathcal{M} \leftrightarrow_{V_2} \mathcal{M}_1$, i.e. $\mathcal{M}_1$ is a model of both $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge \phi$ and $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_2) \wedge \psi$. Therefore $\mathcal{M}_1 \leftrightarrow_{V_1 \cap V_2} \mathcal{M}$. Thus, $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1 \cap V_2)$. Then we have $V_1 = V_2$, otherwise $V_1$ (or $V_2$) is not the minimal set. $\mathcal{M}_1$ is a model of $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ as well. Moreover, $V_1$ is the minimal set such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_1) \wedge (\phi \vee \psi)$ is satisfiable. Otherwise, suppose that $V_3 \subset V_1$ such that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge (\phi \vee \psi)$ is satisfiable. Then $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \phi$ or $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \psi$ is satisfiable. Without loss of generality, suppose that $\mathrm{F}_\mu(\mathcal{F}_\mathcal{A}(\mathcal{M}), V_3) \wedge \phi$ is satisfiable, $V_1$ is not the minimal set, a contradiction. Therefore $\mathcal{M}_1$ is also a model of $\Gamma \diamond_\mu (\phi \vee \psi)$.

For (U8), we will prove $(\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi \models (\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi)$ at first. For each $\mathcal{M} \in \mathit{Mod}((\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi)$, there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma_1)$ (or $\mathcal{M}_1 \in \mathit{Mod}(\Gamma_2)$) and $V_{min}$ such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Therefore, we have $\mathcal{M} \models (\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi)$. Similarly, for each $\mathcal{M} \in \mathit{Mod}((\Gamma_1 \diamond_\mu \phi) \vee (\Gamma_2 \diamond_\mu \phi))$ there is a $\mathcal{M}_1 \in \mathit{Mod}(\Gamma_1)$ (or $\mathcal{M}_1 \in \mathit{Mod}(\Gamma_2)$) and $V_{min}$ such that $\mathcal{M} \leftrightarrow_{V_{min}} \mathcal{M}_1$. Hence, $\mathcal{M} \models (\Gamma_1 \vee \Gamma_2) \diamond_\mu \phi$.

## 6 Conclusion and Future work

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