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Axiomatic characterization of the AGM theory of belief revision in a temporal logic

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Abstract

Since belief revision deals with the interaction of belief and information over time, branching-time temporal logic seems a natural setting for a theory of belief change. We propose two extensions of a modal logic that, besides the next-time temporal operator, contains a belief operator and an information operator. The first logic is shown to provide an axiomatic characterization of the first six postulates of the AGM theory of belief revision, while the second, stronger, logic provides an axiomatic characterization of the full set of AGM postulates.

Keywords: Information; Belief; Branching time; Belief revision; AGM theory

1. Introduction

There is an unsatisfactory lack of uniformity in the literature between how static beliefs and changes in beliefs are modeled. Starting with Hintikka's [12] seminal contribution, the notion of static belief has been studied mainly within the context of modal logic. On the syntactic side a belief operator B is introduced, with the intended interpretation of $B\phi$ as "the individual believes that ϕ". Various properties of beliefs are then expressed by means of axioms (for example, the positive introspection axiom $B\phi \to BB\phi$, which says that if the individual believes ϕ then she believes that she believes ϕ). On the semantic side Kripke structures (Kripke [15]) are used, consisting of a set of states (or possible worlds) Ω together with a binary relation $\mathcal{B}$ on Ω, with the interpretation of $\alpha\mathcal{B}\beta$ as "at state α the individual considers state β possible". The connection between syntax and semantics is then obtained by means of a valuation that associates with every atomic proposition p the set of states at which p is true. Rules are given for determining the truth of an arbitrary formula at every state of a model; in particular, the formula $B\phi$ is true at state α if and only if ϕ is true at every ω such that $\alpha\mathcal{B}\omega$, that is, if ϕ is true at every state that the individual considers possible at α. Often one can show that there is a correspondence between a syntactic axiom and a property of the accessibility relation, in the sense that every instance of the axiom is true at every state of every model whose accessibility relation satisfies

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the property and *vice versa* (for example, the positive introspection axiom $B\phi \to BB\phi$ corresponds to transitivity of the relation $\mathcal{B}$).

On the other hand, in their seminal contribution on belief revision, Alchourrón, Gärdenfors and Makinson [1] model beliefs as sets of formulas in a given syntactic language and belief revision is construed as an operation that associates with every belief set K (thought of as the initial beliefs) and formula ϕ (thought of as new information) a new belief set K^*_ϕ representing the revised beliefs. Several requirements are imposed on this operator in order to capture the notion of "rational" belief change. Their approach has become known as the AGM theory of belief revision and has stimulated a large literature.

The purpose of this paper is to bridge the gap between these two strands of the literature, *by representing the AGM postulates as axioms in a modal language*. Since belief revision deals with the interaction of belief and information over time, temporal logic seems a natural starting point. Besides the next-time operator $\bigcirc$, our language contains a belief operator B and an information operator I. The information operator is not a normal operator and is formally similar to the "all I know" operator introduced by Levesque [16]. On the semantic side we consider branching-time frames to represent different possible evolutions of beliefs. For every date t, beliefs and information are represented by binary relations $\mathcal{B}_t$ and $\mathcal{I}_t$ on a set of states Ω. As in the static setting, the link between syntax and semantics is provided by the notion of valuation and model. The truth of a formula in a model is defined at a state-instant pair (ω, t).

The first logic that we propose provides an axiomatic characterization of the first six AGM postulates (the so-called "basic set"), in the following sense (Proposition 11):

(1) if K is the initial belief set, ϕ is a Boolean (i.e. non-modal) formula and K^*_ϕ is the revised belief set that satisfies the first six AGM postulates, then there is a model of the logic, a state $\alpha \in \Omega$ and instants t_1 and t_2 such that: (i) t_2 is an immediate successor of t_1, (ii) the set of Boolean formulas that the individual believes at (α, t_1) coincides with K, (iii) the individual at time t_2 and state α is informed that ϕ and (iv) the set of Boolean formulas that the individual believes at (α, t_2) coincides with K^*_ϕ, and

(2) for every model that validates the logic, every state α and every instants t_1 and t_2 such that t_2 is an immediate successor of t_1, if at time t_2 and state α the individual is informed that ϕ (where ϕ is a consistent Boolean formula, which is true at some state-instant pair) then K and K^*_ϕ defined as the sets of Boolean formulas that the individual believes at (α, t_1) and (α, t_2), respectively, satisfy the first six AGM postulates. (Furthermore, for every Boolean formula ϕ there exists a model and a state-instant pair where the individual is informed that ϕ.)

The remaining two AGM postulates deal with comparing how the individual revises his beliefs after learning first that ϕ and then that ψ with how he would revise his beliefs if he learned that $\phi \wedge \psi$. This is where the branching-time structure that we use becomes important, since two different evolutions of beliefs need to be compared. The second logic that we propose extends the first by adding two axioms, which correspond to the last two AGM postulates. We show (Proposition 12) that the stronger logic provides an axiomatic characterization of the full set of AGM axioms, in a sense analogous to the previous result.

The paper is organized as follows. In Section 2 we start with the semantics of temporal belief revision frames. In Section 3 we introduce the basic logic and two extensions of it, which—in Section 4—are proved to provide an axiomatic characterization of the first six and the full set of AGM postulates, respectively. Related literature is discussed in Section 5. Section 6 concludes.

2. The semantics

On the semantic side we consider branching-time structures with the addition of a belief relation and an information relation for every instant t.

Definition 1. A *next-time branching frame* is a pair $\langle T, \rightarrowtail \rangle$ where T is a (possibly infinite) set of instants or dates and $\rightarrowtail$ is a binary "precedence" relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$,

(1) uniqueness if $t_1 \rightarrowtail t_3$ and $t_2 \rightarrowtail t_3$ then $t_1 = t_2$,
(2) acyclicity if $\langle t_1, \dots, t_n \rangle$ is a sequence with $t_i \rightarrowtail t_{i+1}$ for every $i = 1, \dots, n-1$, then $t_n \neq t_1$.

The interpretation of $t_1 \rightarrowtail t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2: every instant has at most a unique immediate predecessor but can have several immediate successors.

Definition 2. A *temporal belief revision frame* is a quintuple $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t\}_{t\in T}, \{\mathcal{I}_t\}_{t\in T}\rangle$ where $\langle T, \rightarrowtail\rangle$ is a next-time branching frame, Ω is a set of states (or possible worlds) and, for every $t \in T$, $\mathcal{B}_t$ and $\mathcal{I}_t$ are binary relations on Ω.

The interpretation of $\omega\mathcal{I}_t\omega'$ is that at state ω and time t—according to the information received—it is possible that the true state is ω'. On the other hand, the interpretation of $\omega\mathcal{B}_t\omega'$ is that at state ω and time t—in light of the information received (if any)—the individual considers state ω' possible (an alternative expression is "ω' is a doxastic alternative to ω at time t"). We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega\colon \omega\mathcal{B}_t\omega'\} \quad \text{and, similarly,} \quad \mathcal{I}_t(\omega) = \{\omega' \in \Omega\colon \omega\mathcal{I}_t\omega'\}.$$

Thus $\mathcal{B}_t(\omega)$ is the set of states that are reachable from ω according to the relation $\mathcal{B}_t$ and similarly for $\mathcal{I}_t(\omega)$.

Temporal belief frames can be used to describe either a situation where the objective facts describing the world do not change—so that only the beliefs of the agent change over time—or a situation where both the facts and the doxastic state of the agent change. In the computer science literature the first situation is called belief revision, while the latter is called belief update (Katsuno and Mendelzon [13]). In this paper we restrict attention to belief revision.[2]

We consider a propositional language with five modal operators: the next-time operator $\bigcirc$ and it inverse $\bigcirc^{-1}$, the belief operator B, the information operator I and the "all state" operator A. The intended interpretation is as follows:

$\bigcirc\phi$: "at every next instant it will be the case that ϕ"
$\bigcirc^{-1}\phi$: "at every previous instant it was the case that ϕ"
$B\phi$: "the agent believes that ϕ"
$I\phi$: "the agent is informed that ϕ"
$A\phi$: "it is true at every state that ϕ".

The "all state" operator A is needed in order to capture the non-normality of the information operator I (see below). For a thorough discussion of the "all state" operator see Goranko and Passy [11].

Given a temporal belief revision frame $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t\}_{t\in T}, \{\mathcal{I}_t\}_{t\in T}\rangle$ one obtains a *model based on it* by adding a function $V : S \to 2^{\Omega}$ (where S is the set of atomic propositions and 2^{Ω} denotes the set of subsets of Ω) that associates with every atomic proposition q the set of states at which q is true. Note that defining a valuation this way is what frames the problem as one of belief revision, since the truth value of an atomic proposition depends only on the state and not on the time.[3] Given a model, a state ω, an instant t and a formula ϕ, we write $(\omega, t) \models \phi$ to denote that ϕ is true at state ω and time t. Let $\|\phi\|$ denote the truth set of ϕ, that is, $\|\phi\| = \{(\omega, t) \in \Omega \times T\colon (\omega, t) \models \phi\}$ and let $\lceil\phi\rceil_t \subseteq \Omega$ denote the set of states at which ϕ is true *at time* t, that is, $\lceil\phi\rceil_t = \{\omega \in \Omega\colon (\omega, t) \models \phi\}$. Truth at a pair (ω, t) is defined recursively as follows.

if $q \in S$,	$(\omega, t) \models q$ if and only if $\omega \in V(q)$.
$(\omega, t) \models \neg\phi$	if and only if $(\omega, t) \not\models \phi$.
$(\omega, t) \models \phi \vee \psi$	if and only if either $(\omega, t) \models \phi$ or $(\omega, t) \models \psi$ (or both).
$(\omega, t) \models \bigcirc\phi$	if and only if $(\omega, t') \models \phi$ for every t' such that $t \rightarrowtail t'$.
$(\omega, t) \models \bigcirc^{-1}\phi$	if and only if $(\omega, t'') \models \phi$ for every t'' such that $t'' \rightarrowtail t$.
$(\omega, t) \models B\phi$	if and only if $\mathcal{B}_t(\omega) \subseteq \lceil\phi\rceil_t$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{B}_t(\omega)$.
$(\omega, t) \models I\phi$	if and only if $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$, that is, if (1) $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{I}_t(\omega)$, and (2) if $(\omega', t) \models \phi$ then $\omega' \in \mathcal{I}_t(\omega)$.
$(\omega, t) \models A\phi$	if and only if $\lceil\phi\rceil_t = \Omega$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \Omega$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply requiring that $\mathcal{I}_t(\omega) \subseteq \lceil\phi\rceil_t$ we require equality: $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$. Thus our information operator is formally similar to the "all I know" operator introduced by Levesque [16], although the interpretation is different.

2 For example, our analysis would be appropriate to model the evolving beliefs of an archaeologist who is trying to learn what truly happened several thousand years ago. New archaeological discoveries provide clues and information about the past, which the scientist uses to update his beliefs. However, the facts he is trying to learn do not change: their truth value was fixed in the distant past.

3 Belief update would require a valuation to be defined as a function $V : S \to 2^{\Omega\times T}$.

A formula ϕ is *valid in a model* if $\|\phi\| = \Omega \times T$, that is, if ϕ is true at every state-instant pair (ω, t). A formula ϕ is *valid in a frame* if it is valid in every model based on it.

3. The basic logic and two extensions

The formal language is built in the usual way (see [4]) from a countable set of atomic propositions, the connectives $\neg$ and $\vee$ (from which the connectives $\wedge$, $\to$ and $\leftrightarrow$ are defined as usual) and the modal operators $\bigcirc$, $\bigcirc^{-1}$, B, I and A. Let $\Diamond\phi \stackrel{def}{=} \neg\bigcirc\neg\phi$, and $\Diamond^{-1}\phi \stackrel{def}{=} \neg\bigcirc^{-1}\neg\phi$. Thus the interpretation of $\Diamond\phi$ is "at *some* next instant it will be the case that ϕ" while the interpretation of $\Diamond^{-1}\phi$ is "at some previous instant it was the case that ϕ".

We denote by $\mathbb{L}_0$ the basic logic of belief revision defined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.
2. Axiom K for $\bigcirc$, $\bigcirc^{-1}$, B and A:
$$(\Box\phi \wedge \Box(\phi \to \psi)) \to \Box\psi \quad \text{for } \Box \in \{\bigcirc, \bigcirc^{-1}, B, A\} \quad (K)$$
3. Temporal axioms relating $\bigcirc$ and $\bigcirc^{-1}$:
$$\phi \to \bigcirc\Diamond^{-1}\phi \ (\mathrm{O}_1)$$
$$\phi \to \bigcirc^{-1}\Diamond\phi \ (\mathrm{O}_2)$$
4. Backward Uniqueness axiom:
$$\Diamond^{-1}\phi \to \bigcirc^{-1}\phi \quad (\mathrm{BU})$$
5. S5 axioms for A:
$$A\phi \to \phi \quad (\mathrm{T}_A)$$
$$\neg A\phi \to A\neg A\phi \quad (5_A)$$
6. Inclusion axiom for B (note the absence of an analogous axiom for I):
$$A\phi \to B\phi \quad (\mathrm{Incl}_B)$$
7. Axioms to capture the non-standard semantics for I:
$$(I\phi \wedge I\psi) \to A(\phi \leftrightarrow \psi) \quad (\mathrm{I}_1)$$
$$A(\phi \leftrightarrow \psi) \to (I\phi \leftrightarrow I\psi) \quad (\mathrm{I}_2)$$

RULES OF INFERENCE:

1. Modus Ponens:
$$\frac{\phi,\ \phi \to \psi}{\psi} \quad (MP)$$
2. Necessitation for A, $\bigcirc$ and $\bigcirc^{-1}$:
$$\frac{\phi}{\Box\phi} \quad \text{for } \Box \in \{\bigcirc, \bigcirc^{-1}, A\} \quad (Nec).$$

Note that from MP, $Incl_B$ and Necessitation for A one can derive necessitation for B ($\frac{\phi}{B\phi}$). On the other hand, Necessitation for I is *not* a rule of inference of this logic (indeed it is not validity preserving).

Remark 3. By MP, axiom K and Necessitation, the following is a derived rule of inference for the operators $\bigcirc$, $\bigcirc^{-1}$, B and A: $\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$ for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$. We call this rule RK. On the other hand, rule RK is not a valid rule of inference for the operator I (despite the fact that axiom K for I can be shown to be a theorem of $\mathbb{L}_0$).

The proof of the following proposition is standard and is relegated to Appendix A.[4]

Proposition 4. *Logic* $\mathbb{L}_0$ *is sound with respect to the class of temporal belief revision frames* (*see Definition* 2)*, that is, every theorem of* $\mathbb{L}_0$ *is valid in every model based on a temporal belief revision frame.*

Our purpose is to model how the *factual* beliefs of an individual change over time in response to *factual* information. Thus the axioms we introduce are restricted to *Boolean formulas*, which are formulas that do not contain any modal operators. That is, Boolean formulas are defined recursively as follows: (1) every atomic proposition is a Boolean formula, and (2) if ϕ and ψ are Boolean formulas then so are $\neg\phi$ and $(\phi \vee \psi)$. As the following proposition shows, the truth value of a Boolean formula does not change over time: it is only a function of the state. We denote by Φ^B the set of Boolean formulas.

Proposition 5. *Let* $\phi \in \Phi^B$*. Fix an arbitrary model. Then, for every* $\omega \in \Omega$ *and* $t, t' \in T$*,* $(\omega, t) \models \phi$ *if and only if* $(\omega, t') \models \phi$*. Hence, for all* $t, t' \in T$*,* $\lceil\phi\rceil_t = \lceil\phi\rceil_{t'}$*.*

Proof. Fix arbitrary $\omega \in \Omega$ and $t, t' \in T$. The proof is by induction on the complexity of ϕ. If $\phi = q$, where q is an atomic proposition, then $(\omega, t) \models q$ if and only if $\omega \in V(q)$ if and only if $(\omega, t') \models q$. Suppose now that the statement is true for ψ_1 and for ψ_2, that is, $(\omega, t) \models \psi_1$ if and only if $(\omega, t') \models \psi_1$, and similarly for ψ_2. We want to show that the statement is true for $\neg\psi_1$ and for $(\psi_1 \vee \psi_2)$. By definition, $(\omega, t) \models \neg\psi_1$ if and only if $(\omega, t) \not\models \psi_1$. By the induction hypothesis $(\omega, t) \not\models \psi_1$ if and only if $(\omega, t') \not\models \psi_1$. Hence $(\omega, t) \models \neg\psi_1$ if and only if $(\omega, t') \models \neg\psi_1$. By definition, $(\omega, t) \models \psi_1 \vee \psi_2$ if and only if either $(\omega, t) \models \psi_1$ or $(\omega, t) \models \psi_2$. By the induction hypothesis, $(\omega, t) \models \psi_1$ if and only if $(\omega, t') \models \psi_1$, and $(\omega, t) \models \psi_2$ if and only if $(\omega, t') \models \psi_2$. Thus $(\omega, t) \models \psi_1 \vee \psi_2$ if and only if $(\omega, t') \models \psi_1 \vee \psi_2$.

Fix an arbitrary $\omega' \in \Omega$. By definition of $\lceil\phi\rceil_t$, $\omega' \in \lceil\phi\rceil_t$ if and only if $(\omega', t) \models \phi$; by the result just proved, $(\omega', t) \models \phi$ if and only if $(\omega', t') \models \phi$ and, by definition of $\lceil\phi\rceil_{t'}$, $(\omega', t') \models \phi$ if and only if $\omega' \in \lceil\phi\rceil_{t'}$. Thus $\lceil\phi\rceil_t = \lceil\phi\rceil_{t'}$. □

We now introduce two sets of axioms that provide two extensions of logic $\mathbb{L}_0$, one of which will be shown to correspond to the basic set of AGM postulates and the other to the full set. *Note that all of the following axioms apply only to Boolean formulas.*

The first axiom says that factual information is believed. This is known in the literature as Success or Acceptance (‘A’ stands for ‘Acceptance’): if ϕ is a Boolean formula,

$$I\phi \to B\phi. \tag{A}$$

The second axiom requires the individual not to drop any of his current factual beliefs at any next instant where he is informed of some fact that he currently considers possible (‘ND’ stands for ‘Not Drop’): if ϕ and ψ are Boolean formulas

$$(\neg B\neg\phi \wedge B\psi) \to \bigcirc(I\phi \to B\psi). \tag{ND}$$

The third axiom requires that if the individual considers it possible that $(\phi \wedge \neg\psi)$, then at any next instant where he is informed that ϕ he does not believe that ψ, that is, he cannot add new factual beliefs, unless they are implied by the old beliefs and the information received (‘NA’ stands for ‘Not Add’):[5] if ϕ and ψ are Boolean formulas,

$$\neg B\neg(\phi \wedge \neg\psi) \to \bigcirc(I\phi \to \neg B\psi). \tag{NA}$$

[4] Completeness issues are not relevant for the results of this paper and are dealt with in a separate paper that studies several extensions of $\mathbb{L}_0$ besides the two considered here.

[5] Axiom *NA* can alternatively be written as $\neg B(\phi \to \psi) \to \bigcirc(I\phi \to \neg B\psi)$, which says that if the individual does not believe that whenever ϕ is the case then ψ is the case, then—at any next instant—if he is informed that ϕ then he cannot believe that ψ. Another, propositionally equivalent, formulation of *NA* is the following: $\Diamond(I\phi \wedge B\psi) \to B(\phi \to \psi)$, which says that if there is a next instant at which the individual is informed that ϕ and believes that ψ, then he must now believe that whenever ϕ is the case then ψ is the case.

The fourth axiom says that if the individual receives consistent information then his beliefs are consistent, in the sense that he does not simultaneously believe a formula and its negation ('WC' stands for 'Weak Consistency'): if ϕ is a Boolean formula,

$$(I\phi \wedge \neg A\neg\phi) \to (B\psi \to \neg B\neg\psi). \tag{WC}$$

Turning back to the semantics, we call the following property of temporal belief revision frames "Qualitative Bayes Rule" (QBR): $\forall\omega \in \Omega, \forall t_1, t_2 \in T$,

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \emptyset \quad \text{then } \mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega). \tag{QBR}$$

The expression "Qualitative Bayes Rule" is motivated by the following observation (see [5]). In a probabilistic setting, let P_{ω,t_1} be the probability measure over a set of states Ω representing the individual's beliefs at state ω and time t_1, let $F \subseteq \Omega$ be an event representing the information received by the individual at a later date t_2 and let P_{ω,t_2} be the posterior probability measure representing the revised beliefs at state ω and date t_2. Bayes' rule requires that, if $P_{\omega,t_1}(F) > 0$, then, for every event $E \subseteq \Omega$, $P_{\omega,t_2}(E) = \frac{P_{\omega,t_1}(E\cap F)}{P_{\omega,t_1}(F)}$. Bayes' rule thus implies the following (where $supp(P)$ denotes the support of the probability measure P):

$$\text{if } supp(P_{\omega,t_1}) \cap F \neq \emptyset, \quad \text{then } supp(P_{\omega,t_2}) = supp(P_{\omega,t_1}) \cap F.$$

If we set $\mathcal{B}_{t_1}(\omega) = supp(P_{\omega,t_1})$, $F = \mathcal{I}_{t_2}(\omega)$, with $t_1 \rightarrowtail t_2$, and $\mathcal{B}_{t_2}(\omega) = supp(P_{\omega,t_2})$ then we get the Qualitative Bayes Rule as stated above. Thus in a probabilistic setting the proposition "at date t the individual believes ϕ" would be interpreted as "the individual assigns probability 1 to the event $\lceil\phi\rceil_t \subseteq \Omega$".

Let $\mathbb{L}_b$ be the logic obtained by adding the above four axioms to the basic logic $\mathbb{L}_0$. We denote this by writing $\mathbb{L}_b = \mathbb{L}_0 + A + ND + NA + WC$ (the subscript 'b' was chosen because, as shown later, logic $\mathbb{L}_b$ provides an axiomatic characterization of the *basic* set of AGM postulates).

Definition 6. An $\mathbb{L}_b$*-frame* is a temporal belief revision frame that satisfies the following properties:

(1) the Qualitative Bayes Rule,
(2) $\forall\omega \in \Omega, \forall t \in T, \mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,
(3) $\forall\omega \in \Omega, \forall t \in T$, if $\mathcal{I}_t(\omega) \neq \emptyset$ then $\mathcal{B}_t(\omega) \neq \emptyset$.

An $\mathbb{L}_b$*-model* is a model based on an $\mathbb{L}_b$-frame.

Proposition 7. *Logic $\mathbb{L}_b$ is sound with respect to the class of $\mathbb{L}_b$-frames. That is, every theorem of $\mathbb{L}_b$ is valid in every $\mathbb{L}_b$-model.*

Proof. By Proposition 4 it is enough to show that the four axioms A, ND, NA and WC are valid in an arbitrary model based on a frame that satisfies the three properties of Definition 6. Fix an arbitrary such model.

Validity of A. Fix arbitrary $\alpha \in \Omega$, $t \in T$ and $\phi \in \Phi^B$ and suppose that $(\alpha, t) \models I\phi$. Then $\mathcal{I}_t(\alpha) = \lceil\phi\rceil_t$. Hence, by property (2) of Definition 6, $\mathcal{B}_t(\alpha) \subseteq \lceil\phi\rceil_t$, that is, $(\alpha, t) \models B\phi$.

Validity of ND. Fix arbitrary $\alpha \in \Omega$, $t_1 \in T$ and $\phi, \psi \in \Phi^B$ and suppose that $(\alpha, t_1) \models \neg B\neg\phi \wedge B\psi$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since $(\alpha, t_1) \models \neg B\neg\phi$, there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi$. Since ϕ is Boolean, by Proposition 5, $(\beta, t_2) \models \phi$ so that $\beta \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \emptyset$ and, by QBR, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_1}(\alpha)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$. Then $\omega \in \mathcal{B}_{t_1}(\alpha)$ and, since $(\alpha, t_1) \models B\psi$, $(\omega, t_1) \models \psi$. Since ψ is Boolean, by Proposition 5, $(\omega, t_2) \models \psi$. Hence, since $\omega \in \mathcal{B}_{t_2}(\alpha)$ was chosen arbitrarily, $(\alpha, t_2) \models B\psi$.

Validity of NA. Fix arbitrary $\alpha \in \Omega$, $t_1 \in T$ and $\phi, \psi \in \Phi^B$ and suppose that $(\alpha, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since $(\alpha, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$, there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi \wedge \neg\psi$. Since ϕ and ψ are Boolean, by Proposition 5, $(\beta, t_2) \models \phi \wedge \neg\psi$. Thus $\beta \in \mathcal{I}_{t_2}(\alpha)$, so that $\beta \in \mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha)$. By QBR , $\beta \in \mathcal{B}_{t_2}(\alpha)$. Thus, since $(\beta, t_2) \models \neg\psi$, $(\alpha, t_2) \models \neg B\psi$.

Validity of WC. Fix arbitrary $\alpha \in \Omega$, $t \in T$ and $\phi \in \Phi^B$ and suppose that $(\alpha, t) \models I\phi \wedge \neg A\neg\phi$. Then $\mathcal{I}_t(\alpha) = \lceil\phi\rceil_t$ and there exists a β such that $(\beta, t) \models \phi$. Thus $\mathcal{I}_t(\alpha) \neq \emptyset$ and, by property (3) of Definition 6, $\mathcal{B}_t(\alpha) \neq \emptyset$. Fix an arbitrary formula ψ and suppose that $(\alpha,t)\models B\psi$. Since $\mathcal{B}_t(\alpha)\neq\emptyset$, there exists a $\gamma\in\mathcal{B}_t(\alpha)$. Thus $(\gamma,t)\models\psi$ and hence $(\alpha,t)\models\neg B\neg\psi$. $\square$

We now strengthen logic $\mathbb{L}_b$ by adding two more axioms.

The first axiom says that if there is a next instant where the individual is informed that $\phi\wedge\psi$ and believes that χ, then at every next instant it must be the case that if the individual is informed that ϕ then he must believe that $(\phi\wedge\psi)\to\chi$ (we call this axiom $K7$ because, as we will show later, it corresponds to AGM postulate (K*7)): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I(\phi\wedge\psi)\wedge B\chi)\to\bigcirc(I\phi\to B((\phi\wedge\psi)\to\chi)). \tag{$K7$}$$

The second axiom says that if there is a next instant where the individual is informed that ϕ, considers $\phi\wedge\psi$ possible and believes that $\psi\to\chi$, then at every next instant it must be the case that if he is informed that $\phi\wedge\psi$ then he believes that χ (we call this axiom $K8$ because it corresponds to AGM postulate (K*8)): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I\phi\wedge\neg B\neg(\phi\wedge\psi)\wedge B(\psi\to\chi))\to\bigcirc(I(\phi\wedge\psi)\to B\chi). \tag{$K8$}$$

Let $\mathbb{L}_{AGM}$ be the logic obtained by adding the above two axioms to $\mathbb{L}_b$. Thus $\mathbb{L}_{AGM}=\mathbb{L}_0+A+ND+NA+WC+K7+K8$ (the subscript 'AGM' was chosen because, as shown later, logic $\mathbb{L}_{AGM}$ provides an axiomatic characterization of the full set of AGM postulates).

Definition 8. An $\mathbb{L}_{AGM}$*-frame* is an $\mathbb{L}_b$-frame (see Definition 6) that satisfies the following additional property: $\forall\omega\in\Omega,\ \forall t_1,t_2,t_3\in T$,

$$\begin{array}{l}\text{if } t_1\rightarrowtail t_2,\ t_1\rightarrowtail t_3,\ \mathcal{I}_{t_3}(\omega)\subseteq\mathcal{I}_{t_2}(\omega)\text{ and }\mathcal{I}_{t_3}(\omega)\cap\mathcal{B}_{t_2}(\omega)\neq\emptyset\\ \text{then }\mathcal{B}_{t_3}(\omega)=\mathcal{I}_{t_3}(\omega)\cap\mathcal{B}_{t_2}(\omega).\end{array} \tag{CAB}$$

An $\mathbb{L}_{AGM}$*-model* is a model based on an $\mathbb{L}_{AGM}$-frame.[6]

Proposition 9. *Logic $\mathbb{L}_{AGM}$ is sound with respect to the class of $\mathbb{L}_{AGM}$-frames. That is, every theorem of $\mathbb{L}_{AGM}$ is valid in every $\mathbb{L}_{AGM}$-model.*

Proof. By Proposition 7 and Definition 8 it is sufficient to show that axioms $K7$ and $K8$ are valid in an arbitrary model based on an $\mathbb{L}_b$ frame that satisfies *CAB*. Fix an arbitrary such model.

Validity of $K7$. Fix arbitrary $\alpha\in\Omega$ and $t_1\in T$ and suppose that $(a,t_1)\models\Diamond(I(\phi\wedge\psi)\wedge B\chi)$, where ϕ, ψ and χ are Boolean formulas. Then there exists a t_3 such that $t_1\rightarrowtail t_3$ and $(\alpha,t_3)\models I(\phi\wedge\psi)\wedge B\chi$. Then $\mathcal{I}_{t_3}(\alpha)=\lceil\phi\wedge\psi\rceil_{t_3}$. Fix an arbitrary t_2 such that $t_1\rightarrowtail t_2$ and suppose that $(\alpha,t_2)\models I\phi$. Then $\mathcal{I}_{t_2}(\alpha)=\lceil\phi\rceil_{t_2}$. Since ϕ and ψ are Boolean, by Proposition 5 $\lceil\phi\wedge\psi\rceil_{t_3}=\lceil\phi\wedge\psi\rceil_{t_2}$. Thus, since $\lceil\phi\wedge\psi\rceil_{t_2}\subseteq\lceil\phi\rceil_{t_2}$, $\mathcal{I}_{t_3}(\alpha)\subseteq\mathcal{I}_{t_2}(\alpha)$. If $\mathcal{I}_{t_3}(\alpha)\cap\mathcal{B}_{t_2}(\alpha)=\emptyset$, then, for every $\omega\in\mathcal{B}_{t_2}(\alpha)$, $(\omega,t_2)\models\neg(\phi\wedge\psi)$ and thus $(\omega,t_2)\models(\phi\wedge\psi)\to\chi$, so that $(\alpha,t_2)\models B((\phi\wedge\psi)\to\chi)$. If, on the other hand, $\mathcal{I}_{t_3}(\alpha)\cap\mathcal{B}_{t_2}(\alpha)\neq\emptyset$, then, by *CAB*, $\mathcal{B}_{t_3}(\alpha)=\mathcal{I}_{t_3}(\alpha)\cap\mathcal{B}_{t_2}(\alpha)$. Fix an arbitrary $\beta\in\mathcal{B}_{t_2}(\alpha)$. If $(\beta,t_2)\models\neg(\phi\wedge\psi)$ then $(\beta,t_2)\models(\phi\wedge\psi)\to\chi$. If $(\beta,t_2)\models\phi\wedge\psi$, then, by Proposition 5, $(\beta,t_3)\models\phi\wedge\psi$ and, therefore, $\beta\in\mathcal{I}_{t_3}(\alpha)$. Hence $\beta\in\mathcal{B}_{t_3}(\alpha)$. Since $(\alpha,t_3)\models B\chi$, $(\beta,t_3)\models\chi$ and, therefore, $(\beta,t_3)\models(\phi\wedge\psi)\to\chi$. Since $(\phi\wedge\psi\to\chi)$ is Boolean (because ϕ, ψ and χ are), by Proposition 5, $(\beta,t_2)\models(\phi\wedge\psi)\to\chi$. Thus, since $\beta\in\mathcal{B}_{t_2}(\alpha)$ was chosen arbitrarily, $(\alpha,t_2)\models B((\phi\wedge\psi)\to\chi)$.

Validity of $K8$. Fix arbitrary $\alpha\in\Omega$ and $t_1\in T$ and suppose that $(\alpha,t_1)\models\Diamond(I\phi\wedge\neg B\neg(\phi\wedge\psi)\wedge B(\psi\to\chi))$, where ϕ, ψ and χ are Boolean formulas. Then there exists a t_2 such that $t_1\rightarrowtail t_2$ and $(\alpha,t_2)\models I\phi\wedge\neg B\neg(\phi\wedge\psi)\wedge B(\psi\to\chi)$. Thus $\mathcal{I}_{t_2}(\alpha)=\lceil\phi\rceil_{t_2}$ and there exists a $\beta\in\mathcal{B}_{t_2}(\alpha)$ such that $(\beta,t_2)\models\phi\wedge\psi$. Fix an arbitrary t_3 such that $t_1\rightarrowtail t_3$ and suppose that $(\alpha,t_3)\models I(\phi\wedge\psi)$. Then $\mathcal{I}_{t_3}(\alpha)=\lceil\phi\wedge\psi\rceil_{t_3}$. Since $\phi\wedge\psi$ is a Boolean formula and $(\beta,t_2)\models\phi\wedge\psi$, by Proposition 5 $(\beta,t_3)\models\phi\wedge\psi$ and therefore $\beta\in\mathcal{I}_{t_3}(\alpha)$. Hence $\mathcal{I}_{t_3}(\alpha)\cap\mathcal{B}_{t_2}(\alpha)\neq\emptyset$. Furthermore, since

[6] 'CAB' stands for 'Comparison Across Branches'. This property says that if t_2 and t_3 are immediate successors of t_1 and the set of states that are possible according to the information received at (state ω and) time t_3 is a subset of the set of states that are possible according to the information received at (state ω and) time t_2 and, furthermore, the information received at time t_3 is compatible with the beliefs held at time t_2, then the beliefs at time t_3 must coincide with the intersection of the information at time t_3 and the beliefs at time t_2.

ϕ is Boolean, by Proposition 5 $\lceil\phi\rceil_{t_3} = \lceil\phi\rceil_{t_2}$. Thus, since $\lceil\phi \wedge \psi\rceil_{t_3} \subseteq \lceil\phi\rceil_{t_3}$ it follows that $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence, by *CAB*, $\mathcal{B}_{t_3}(\alpha) = \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Fix an arbitrary $\gamma \in \mathcal{B}_{t_3}(\alpha)$. Then $\gamma \in \mathcal{B}_{t_2}(\alpha)$ and, since $(\alpha, t_2) \models B(\psi \to \chi)$, $(\gamma, t_2) \models (\psi \to \chi)$. Since $(\psi \to \chi)$ is a Boolean formula, by Proposition 5 $(\gamma, t_3) \models (\psi \to \chi)$. Since $\mathcal{B}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_3}(\alpha)$ (by definition of $\mathbb{L}_b$-frame) and $\mathcal{I}_{t_3}(\alpha) = \lceil\phi \wedge \psi\rceil_{t_3}$, $(\gamma, t_3) \models \psi$. Thus $(\gamma, t_3) \models \chi$. Hence, since $\gamma \in \mathcal{B}_{t_3}(\alpha)$ was chosen arbitrarily, $(\alpha, t_3) \models B\chi$. □

We end this section with a lemma that will be used later.

Lemma 10. *In any logic where B is a normal operator* (*that is, it satisfies axiom K and the rule of necessitation*) *the following is a theorem*:

$$(B\phi \wedge \neg B\neg\psi) \to \neg B\neg(\phi \wedge \psi).$$

Proof. (‘PL’ stands for ‘Propositional Logic’)

1.	$B\phi \wedge B(\phi \to \neg\psi) \to B\neg\psi$	axiom K
2.	$B\phi \to (B(\phi \to \neg\psi) \to B\neg\psi)$	1, PL
3.	$(B(\phi \to \neg\psi) \to B\neg\psi) \to (\neg B\neg\psi \to \neg B(\phi \to \neg\psi))$	tautology
4.	$B\phi \to (\neg B\neg\psi \to \neg B(\phi \to \neg\psi))$	2, 3, PL
5.	$(B\phi \wedge \neg B\neg\psi) \to \neg B(\phi \to \neg\psi)$	4, PL
6.	$\neg(\phi \wedge \psi) \to (\phi \to \neg\psi)$	tautology
7.	$B\neg(\phi \wedge \psi) \to B(\phi \to \neg\psi)$	6, RK (see Remark 3)
8.	$\neg B(\phi \to \neg\psi) \to \neg B\neg(\phi \wedge \psi)$	7, PL
9.	$(B\phi \wedge \neg B\neg\psi) \to \neg B\neg(\phi \wedge \psi)$	5, 8, PL. □

4. Axiomatic characterization of AGM

The AGM theory of belief revision was developed within the framework of belief sets. Let Γ be the set of formulas in a propositional language. Given a subset $F \subseteq \Gamma$, its PL-deductive closure $[F]^{PL}$ is defined as follows: $\psi \in [F]^{PL}$ if and only if there exist $\phi_1, \ldots, \phi_n \in F$ such that $(\phi_1 \wedge \cdots \wedge \phi_n) \to \psi$ is a tautology (that is, a theorem of Propositional Logic). A set $F \subseteq \Gamma$ is *consistent* if $[F]^{PL} \neq \Gamma$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[F]^{PL}$). A set $F \subseteq \Gamma$ is *deductively closed* if $F = [F]^{PL}$. Given a consistent and deductively closed set K (thought of as the initial beliefs of the individual) and a formula ϕ (thought of as a new piece of information), the *revision of K by ϕ*, denoted by K^*_ϕ, is a subset of Γ that satisfies the following conditions, known as the AGM postulates:

(K*1) K^*_ϕ is deductively closed
(K*2) $\phi \in K^*_\phi$
(K*3) $K^*_\phi \subseteq [K \cup \{\phi\}]^{PL}$
(K*4) if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K^*_\phi$
(K*5) $K^*_\phi = \Gamma$ if and only if ϕ is a contradiction
(K*6) if $\phi \leftrightarrow \psi$ is a tautology then $K^*_\phi = K^*_\psi$
(K*7) $K^*_{\phi\wedge\psi} \subseteq [K^*_\phi \cup \{\psi\}]^{PL}$
(K*8) if $\neg\psi \notin K^*_\phi$, then $[K^*_\phi \cup \{\psi\}]^{PL} \subseteq K^*_{\phi\wedge\psi}$.

(K*1) requires the revised belief set to be deductively closed. In our framework this corresponds to requiring the B operator to be a normal operator, that is, to satisfy axiom K and the inference rule Necessitation.

(K*2) requires that the information be believed. In our framework, this corresponds to the Acceptance axiom (for Boolean ϕ): $I\phi \to B\phi$.

(K*3) says that beliefs should be revised minimally, in the sense that no new belief should be added unless it can be deduced from the information received and the initial beliefs. As shown below, this requirement corresponds to our axiom *NA* (for Boolean ϕ and ψ): $\neg B\neg(\phi \wedge \neg\psi) \to \bigcirc(I\phi \to \neg B\psi)$.

(K*4) says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs. As shown below, this requirement corresponds to our axiom *ND* (for Boolean ϕ and ψ): $(\neg B\neg\phi \wedge B\psi) \to \bigcirc(I\phi \to B\psi)$.

(K*5) requires the revised beliefs to be consistent, unless the information ϕ is contradictory (that is, $\neg\phi$ is a tautology). This corresponds to our axiom *WC* (for Boolean ϕ): $(I\phi \wedge \neg A\neg\phi) \to (B\psi \to \neg B\neg\psi)$.

(K*6) is automatically satisfied in our framework, since if $\phi \leftrightarrow \psi$ is a tautology then $\|\phi\| = \|\psi\|$ in every model and therefore the formula $I\phi \leftrightarrow I\psi$ is valid. Hence revision based on ϕ must coincide with revision based on ψ.

(K*7) and (K*8) are a generalization of (K*3) and (K*4) that

> "applies to *iterated* changes of belief. The idea is that if K^*_ϕ is a revision of K and K^*_ϕ is to be changed by adding further sentences, such a change should be made by using expansions of K^*_ϕ whenever possible. More generally, the minimal change of K to include both ϕ and ψ (that is, $K^*_{\phi\wedge\psi}$) ought to be the same as the expansion of K^*_ϕ by ψ, so long as ψ does not contradict the beliefs in K^*_ϕ" (Gärdenfors [10, p. 55]).[7]

We will show below that (K*7) corresponds to our axiom $K7$ and (K*8) to axiom $K8$.

The set of postulates (K*1) through (K*6) is called the *basic set* of postulates for belief revision (Gärdenfors [10, p. 55]). The following proposition shows that logic $\mathbb{L}_b$ characterizes this basic set.

Proposition 11. *Logic $\mathbb{L}_b$ provides an axiomatic characterization of the set of basic AGM postulates* (K*1)–(K*6)*, in the sense that both* (A) *and* (B) *below hold* (*recall that Φ^B denotes the subset of Boolean formulas*):

(A) *Let $K \subseteq \Phi^B$ be a consistent and deductively closed set and $\phi \in \Phi^B$. If $K^*_\phi \subseteq \Phi^B$ satisfies AGM postulates* (K*1)–(K*6) *then there exist an $\mathbb{L}_b$-model, $t_1, t_2 \in T$ and $\alpha \in \Omega$ such that*

(A.1) $t_1 \rightarrowtail t_2$,
(A.2) $K = \{\psi \in \Phi^B\colon (\alpha, t_1) \models B\psi\}$,
(A.3) $(\alpha, t_2) \models I\phi$,
(A.4) $K^*_\phi = \{\psi \in \Phi^B\colon (\alpha, t_2) \models B\psi\}$,
(A.5) *if ϕ is consistent then $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$.*

(B) *Fix an $\mathbb{L}_b$-model such that* (1) *for some $t_1, t_2 \in T$, $\alpha \in \Omega$ and $\phi \in \Phi^B$, $t_1 \rightarrowtail t_2$ and $(\alpha, t_2) \models I\phi$ and* (2) *if ϕ is not a contradiction, then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$.*[8] *Define $K = \{\psi \in \Phi^B\colon (\alpha, t_1) \models B\psi\}$ and $K^*_\phi = \{\psi \in \Phi^B\colon (\alpha, t_2) \models B\psi\}$. Then K^*_ϕ satisfies AGM postulates* (K*1)–(K*6). *Furthermore, for every $\phi \in \Phi^B$, there exists an $\mathbb{L}_b$-model such that* (1) *$(\alpha, t_0) \models I\phi$, for some $\alpha \in \Omega$ and $t_0 \in T$, and* (2) *if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$.*

Proof. (A) First we prove that if $K \subseteq \Phi^B$ is a consistent and deductively closed set, $\phi \in \Phi^B$ and $K^*_\phi \subseteq \Phi^B$ satisfies AGM postulates (K*1)–(K*6) then there is an $\mathbb{L}_b$-model, $t_1, t_2 \in T$ and $\alpha \in \Omega$ such that (A.1)–(A.5) are satisfied. Let $\mathbb{M}^{PL}_B$ be the set of maximally consistent sets of formulas for a propositional logic whose set of formulas is Φ^B. For any $F \subseteq \Phi^B$ let $\mathbb{M}_F = \{\omega \in \mathbb{M}^{PL}_B\colon F \subseteq \omega\}$. By Lindenbaum's lemma, $\mathbb{M}_F \neq \emptyset$ if and only if F is a consistent set, that is, $[F]^{PL} \neq \Phi^B$. To simplify the notation, for $\psi \in \Phi^B$ we write $\mathbb{M}_\psi$ rather than $\mathbb{M}_{\{\psi\}}$.

Define the following belief revision frame: $T = \{t_1, t_2\}$, $\rightarrowtail = \{(t_1, t_2)\}$, $\Omega = \mathbb{M}^{PL}_B$ and, for every $\omega \in \Omega$,

$$\mathcal{B}_{t_1}(\omega) = \mathcal{I}_{t_1}(\omega) = \mathbb{M}_K$$

$$\mathcal{I}_{t_2}(\omega) = \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi & \text{otherwise} \end{cases}$$

$$\mathcal{B}_{t_2}(\omega) = \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K^*_\phi} & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset. \end{cases}$$

[7] The expansion of K^*_ϕ by ψ is $[K^*_\phi \cup \{\psi\}]^{PL}$.

[8] In an arbitrary model, if ϕ is not a contradiction, there is not guarantee that $(\beta, t) \models \phi$ for some (β, t). However, as shown below, for every consistent Boolean formula, there exists an $\mathbb{L}_b$-model where the formula is true at some state-instant pair (β, t). Given a consistent Boolean formula ϕ, let $\mathcal{M}_\phi$ be such a model. Let $\mathcal{M}$ be an arbitrary $\mathbb{L}_b$-model. By taking the union of $\mathcal{M}$ and $\mathcal{M}_\phi$ one can transform the former into a model that satisfies the hypothesis that ϕ is true at some state-instant pair (β, t).

First we show that this frame is an $\mathbb{L}_b$-frame (see Definition 6).

The Qualitative Bayes Rule is clearly satisfied, since $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \emptyset$ if and only if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$, in which case $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K = \mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_1}(\omega)$.

The property that $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$ (for every ω and t) is also satisfied: the only case where, possibly, $\mathcal{B}_t(\omega) \neq \mathcal{I}_t(\omega)$ is when $t = t_2$ and ϕ is a consistent formula. In this case, there are two possibilities: (1) $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$ and (2) $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$. In case (1) $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K \subseteq \mathbb{M}_\phi = \mathcal{I}_{t_2}(\omega)$. In case (2) $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_{K^*_\phi}$ and $\mathcal{I}_{t_2}(\omega) = \mathbb{M}_\phi$. Now, if $\omega' \in \mathbb{M}_{K^*_\phi}$ then $K^*_\phi \subseteq \omega'$ and, since by AGM postulate (K*2), $\phi \in K^*_\phi$, it follows that $\phi \in \omega'$, that is, $\omega' \in \mathbb{M}_\phi$. Hence $\mathbb{M}_{K^*_\phi} \subseteq \mathbb{M}_\phi$.

Finally, the property that, for every ω and t, $\mathcal{B}_t(\omega) \neq \emptyset$ whenever $\mathcal{I}_t(\omega) \neq \emptyset$ is also satisfied. If $t = t_1$, trivially because $\mathcal{B}_{t_1}(\omega) = \mathcal{I}_{t_1}(\omega)$. If $t = t_2$, $\mathcal{I}_{t_2}(\omega) \neq \emptyset$ if and only if ϕ is a consistent formula; in this case either $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K$, if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$, or $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_{K^*_\phi}$, in which case by AGM postulate (K*5) K^*_ϕ is a consistent set and therefore, by Lindenbaum's lemma, $\mathbb{M}_{K^*_\phi} \neq \emptyset$.

Now define the following model based on this frame: for every atomic proposition q, for every $\omega \in \Omega$ and for every $t \in T$, $(\omega, t) \models q$ if and only if $q \in \omega$. First we prove that, for every $\psi \in \Phi^B$,

$$\forall t \in T,\ \lceil \psi \rceil_t = \mathbb{M}_\psi, \quad \text{that is,} \quad \forall \omega \in \Omega,\ (\omega, t) \models \psi \quad \text{if and only if} \quad \psi \in \omega. \tag{1}$$

The proof is by induction on the complexity of ψ. If $\psi = q$, for some atomic proposition q, then the statement is true by construction. Now suppose that the statement is true of $\psi_1, \psi_2 \in \Phi^B$; we want to show that it is true for $\neg\psi_1$ and for $(\psi_1 \vee \psi_2)$. By definition, $(\omega, t) \models \neg\psi_1$ if and only if $(\omega, t) \not\models \psi_1$ if and only if (by the induction hypothesis) $\psi_1 \notin \omega$ if and only if (by definition of MCS) $\neg\psi_1 \in \omega$. By definition, $(\omega, t) \models (\psi_1 \vee \psi_2)$ if and only if either $(\omega, t) \models \psi_1$, in which case, by the induction hypothesis, $\psi_1 \in \omega$, or $(\omega, t) \models \psi_2$, in which case, by the induction hypothesis, $\psi_2 \in \omega$. By definition of MCS, $(\psi_1 \vee \psi_2) \in \omega$ if and only if either $\psi_1 \in \omega$ or $\psi_2 \in \omega$.

Note also the following (see Theorem 2.20 in Chellas [6, p. 57]): $\forall F \subseteq \Phi^B, \forall \psi \in \Phi^B$,

$$\psi \in [F]^{PL} \quad \text{if and only if} \quad \psi \in \omega,\ \forall \omega \in \mathbb{M}_F. \tag{2}$$

Now, fix an arbitrary $\alpha \in \Omega$. We want to show that properties (A.1)–(A.5) are satisfied.

(A.1): $t_1 \rightarrowtail t_2$ by construction.

(A.2): we need to show that $K = \{\psi \in \Phi^B\colon (\alpha, t_1) \models B\psi\}$. First we show that $K \subseteq \{\psi \in \Phi^B\colon (\alpha, t_1) \models B\psi\}$. Let $\psi \in K$. Then $\psi \in \omega$ for every $\omega \in \mathbb{M}_K$, that is, $\mathbb{M}_K \subseteq \mathbb{M}_\psi$. Thus, since, by construction, $\mathcal{B}_{t_1}(\alpha) = \mathbb{M}_K$ and, by (1), $\mathbb{M}_\psi = \lceil \psi \rceil_{t_1}$ it follows that $\mathcal{B}_{t_1}(\alpha) \subseteq \lceil \psi \rceil_{t_1}$, that is, $(\alpha, t_1) \models B\psi$. Next we show that $\{\psi \in \Phi^B\colon (\alpha, t_1) \models B\psi\} \subseteq K$. Let $\psi \in \Phi^B$ be such that $(\alpha, t_1) \models B\psi$. Then $\mathcal{B}_{t_1}(\alpha) \subseteq \lceil \psi \rceil_{t_1}$. Since, by construction, $\mathcal{B}_{t_1}(\alpha) = \mathbb{M}_K$, and, by (1), $\mathbb{M}_\psi = \lceil \psi \rceil_{t_1}$ it follows that $\mathbb{M}_K \subseteq \mathbb{M}_\psi$, that is, $\psi \in \omega$ for every $\omega \in \mathbb{M}_K$; hence, by (2), $\psi \in [K]^{PL}$. By hypothesis, K is deductively closed, that is, $K = [K]^{PL}$. Hence $\psi \in K$.

(A.3): we need to show that $(\alpha, t_2) \models I\phi$. By (1) $\lceil \phi \rceil_{t_2} = \mathbb{M}_\phi$. Since, by construction, $\mathcal{I}_{t_2}(\alpha) = \mathbb{M}_\phi$, it follows that $(\alpha, t_2) \models I\phi$.

(A.4): we need to show that $K^*_\phi = \{\psi \in \Phi^B\colon (\alpha, t_2) \models B\psi\}$. There are several cases to be considered.

(4.i) ϕ is a contradiction. Then, by AGM postulate (K*5), $K^*_\phi = \Phi^B$ and, by construction, $\mathcal{B}_{t_2}(\alpha) = \emptyset$, so that $(\alpha, t_2) \models B\psi$ for every formula ψ. Hence $\{\psi \in \Phi^B\colon (\alpha, t_2) \models B\psi\} = \Phi^B = K^*_\phi$.

(4.ii) ϕ is consistent and $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$. In this case, by construction, $\mathcal{B}_{t_2}(\alpha) = \mathbb{M}_{K^*_\phi}$. If $\psi \in K^*_\phi$ then $\mathbb{M}_{K^*_\phi} \subseteq \mathbb{M}_\psi$. By (1) $\mathbb{M}_\psi = \lceil \psi \rceil_{t_2}$. Thus $\mathcal{B}_{t_2}(\alpha) \subseteq \lceil \psi \rceil_{t_2}$, that is, $(\alpha, t_2) \models B\psi$. Conversely, if $(\alpha, t_2) \models B\psi$ then $\mathcal{B}_{t_2}(\alpha) \subseteq \lceil \psi \rceil_{t_2}$, and, since $\mathcal{B}_{t_2}(\alpha) = \mathbb{M}_{K^*_\phi}$ and, by (1), $\lceil \psi \rceil_{t_2} = \mathbb{M}_\psi$, it follows that $\mathbb{M}_{K^*_\phi} \subseteq \mathbb{M}_\psi$, that is, $\psi \in \omega$ for all $\omega \in \mathbb{M}_{K^*_\phi}$, so that, by (2), $\psi \in [K^*_\phi]^{PL}$. By AGM postulate (K*1), $K^*_\phi = [K^*_\phi]^{PL}$. Thus $\psi \in K^*_\phi$.

(4.iii) ϕ is consistent and $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$, in which case $\mathcal{B}_{t_2}(\alpha) = \mathbb{M}_\phi \cap \mathbb{M}_K$. First of all, note that $\mathbb{M}_\phi \cap \mathbb{M}_K = \mathbb{M}_{K \cup \{\phi\}}$. Secondly, it must be that $\neg\phi \notin K$ (if $\neg\phi \in K$ then $\neg\phi \in \omega$ for every $\omega \in \mathbb{M}_K$ and therefore $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$). Hence, by AGM postulates (K*3) and (K*4), $K^*_\phi = [K \cup \{\phi\}]^{PL}$. By (2), for every Boolean formula ψ, $\psi \in [K \cup \{\phi\}]^{PL}$ if and only if $\psi \in \omega$, for all $\omega \in \mathbb{M}_{K \cup \{\phi\}}$. Thus $\psi \in K^*_\phi = [K \cup \{\phi\}]^{PL}$ if and only if $\psi \in \omega$ for all $\omega \in \mathbb{M}_{K \cup \{\phi\}} = \mathbb{M}_\phi \cap \mathbb{M}_K = \mathcal{B}_{t_2}(\alpha)$. By (1), for every $\omega \in \Omega$, $\psi \in \omega$ if and only if $(\omega, t_2) \models \psi$. Hence $K^*_\phi = \{\psi \in \Phi^B\colon (\alpha, t_2) \models B\psi\}$.

(A.5): we need to show that, if ϕ is consistent, then $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$. If ϕ is consistent, then, by Lindenbaum's lemma, there exists a $\beta \in \mathbb{M}^{PL}_B$ such that $\phi \in \beta$. By (1), $(\beta, t) \models \phi$ for all $t \in T$.

(B) Fix an $\mathbb{L}_b$-model such that (1) for some $t_1, t_2 \in T$, $\alpha \in \Omega$ and $\phi \in \Phi^B$, $t_1 \rightarrowtail t_2$ and $(\alpha, t_2) \models I\phi$ and (2) if ϕ is not a contradiction, then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$. Let $K = \{\psi \in \Phi^B\colon (\alpha, t_1) \models B\psi\}$ and $K^*_\phi = \{\psi \in \Phi^B\colon (\alpha, t_2) \models B\psi\}$. We need to prove that AGM postulates (K*1)–(K*6) are satisfied.

(K*1): we need to show that K^*_ϕ is deductively closed, that is, $K^*_\phi = [K^*_\phi]^{PL}$. If $\psi \in K^*_\phi$ then $\psi \in [K^*_\phi]^{PL}$, because $\psi \to \psi$ is a tautology. Now let $\psi \in [K^*_\phi]^{PL}$. Then there exist $\phi_1, \ldots, \phi_n \in K^*_\phi$ such that $(\phi_1 \wedge \cdots \wedge \phi_n) \to \psi$ is a tautology, hence a theorem of $\mathbb{L}_b$. Then, by necessitation for B and Proposition 7, $(\alpha, t_2) \models B((\phi_1 \wedge \cdots \wedge \phi_n) \to \psi)$. By definition of K^*_ϕ, since $\phi_1, \ldots, \phi_n \in K^*_\phi$, $(\alpha, t_2) \models B(\phi_1 \wedge \cdots \wedge \phi_n)$. By axiom K for B and Proposition 7, $(\alpha, t_2) \models B((\phi_1 \wedge \cdots \wedge \phi_n) \to \psi) \wedge B(\phi_1 \wedge \cdots \wedge \phi_n) \to B\psi$. Thus $(\alpha, t_2) \models B\psi$, that is, $\psi \in K^*_\phi$.

(K*2): we need to show that $\phi \in K^*_\phi$, that is, $(\alpha, t_2) \models B\phi$. By axiom A and Proposition 7, $(\alpha, t_2) \models I\phi \to B\phi$ and by hypothesis $(\alpha, t_2) \models I\phi$. Thus $(\alpha, t_2) \models B\phi$.

(K*3): we need to show that $K^*_\phi \subseteq [K \cup \{\phi\}]^{PL}$. Let $\psi \in K^*_\phi$, i.e. $(\alpha, t_2) \models B\psi$. First of all, note that axiom *NA* is propositionally equivalent to $\Diamond(I\phi \wedge B\psi) \to B(\phi \to \psi)$. Thus, by Proposition 7, $(\alpha, t_1) \models \Diamond(I\phi \wedge B\psi) \to B(\phi \to \psi)$. By hypothesis, $t_1 \rightarrowtail t_2$ and $(\alpha, t_2) \models I\phi \wedge B\psi$. Thus $(\alpha, t_1) \models \Diamond(I\phi \wedge B\psi)$ and, therefore, $(\alpha, t_1) \models B(\phi \to \psi)$, that is $(\phi \to \psi) \in K$. Hence $\{\phi, (\phi \to \psi)\} \subseteq K \cup \{\phi\}$ so that, since $(\phi \wedge (\phi \to \psi)) \to \psi$ is a tautology, $\psi \in [K \cup \{\phi\}]^{PL}$.

(K*4): we need to show that if $\neg\phi \notin K$ then $[K \cup \{\phi\}]^{PL} \subseteq K^*_\phi$. Suppose that $\neg\phi \notin K$, that is, $(\alpha, t_1) \models \neg B\neg\phi$. First of all, note that axiom *ND* is propositionally equivalent to $\neg B\neg\phi \to (B\psi \to \bigcirc(I\phi \to B\psi))$. Thus, by Proposition 7, $(\alpha, t_1) \models \neg B\neg\phi \to (B\psi \to \bigcirc(I\phi \to B\psi))$. Hence

$$(\alpha, t_1) \models B\psi \to \bigcirc(I\phi \to B\psi), \quad \text{for every Boolean formula } \psi. \tag{3}$$

Let $\chi \in [K \cup \{\phi\}]^{PL}$, that is, there exist $\phi_1, \ldots, \phi_n \in K \cup \{\phi\}$ such that $(\phi_1 \wedge \cdots \wedge \phi_n) \to \chi$ is a tautology. We want to show that $\chi \in K^*_\phi$, i.e. $(\alpha, t_2) \models B\chi$. Since $(\phi_1 \wedge \cdots \wedge \phi_n) \to \chi$ is a tautology, by necessitation for B and Proposition 7, $(\alpha, t_1) \models B((\phi_1 \wedge \cdots \wedge \phi_n) \to \chi)$. If $\phi_i \in K$ for every $i = 1, \ldots, n$, then $(\alpha, t_1) \models B(\phi_1 \wedge \cdots \wedge \phi_n)$ and therefore (using axiom K for B and Proposition 7) $(\alpha, t_1) \models B\chi$. Thus, by (3), $(\alpha, t_1) \models \bigcirc(I\phi \to B\chi)$ so that, since $t_1 \rightarrowtail t_2$, $(\alpha, t_2) \models I\phi \to B\chi$. Since, by hypothesis, $(\alpha, t_2) \models I\phi$, it follows that $(\alpha, t_2) \models B\chi$, i.e. $\chi \in K^*_\phi$. If $\phi_i \notin K$, for some $i = 1, \ldots, n$ then we can assume (renumbering the formulas, if necessary) that $\phi_n \notin K$, which implies (since $\phi_i \in K \cup \{\phi\}$ for all $i = 1, \ldots, n$) that $\phi_n = \phi$, and $\phi_1, \ldots, \phi_{n-1} \in K$, so that $(\alpha, t_1) \models B(\phi_1 \wedge \cdots \wedge \phi_{n-1})$. Since, by hypothesis, $(\phi_1 \wedge \cdots \wedge \phi_{n-1} \wedge \phi) \to \chi$ is a tautology and is propositionally equivalent to $(\phi_1 \wedge \cdots \wedge \phi_{n-1}) \to (\phi \to \chi)$, by necessitation for B and Proposition 7 $(\alpha, t_1) \models B((\phi_1 \wedge \cdots \wedge \phi_{n-1}) \to (\phi \to \chi))$. Thus $(\alpha, t_1) \models B(\phi \to \chi)$ (appealing, once again, to axiom K for B and Proposition 7). Hence, by (3) (with $\psi = (\phi \to \chi)$), $(\alpha, t_1) \models \bigcirc(I\phi \to B(\phi \to \chi))$. Since $t_1 \rightarrowtail t_2$, $(\alpha, t_2) \models I\phi \to B(\phi \to \chi)$. By hypothesis, $\alpha \models I\phi$ and by (K*2) (proved above), $(\alpha, t_2) \models B\phi$. Thus $(\alpha, t_2) \models B(\phi \to \chi) \wedge B\phi$. By axiom K and Proposition 7, $(\alpha, t_2) \models (B(\phi \to \chi) \wedge B\phi) \to B\chi$. Hence $(\alpha, t_2) \models B\chi$, i.e. $\chi \in K^*_\phi$.

(K*5): we have to show that $K^*_\phi \neq \Phi^B$ unless ϕ is a contradiction (that is, $\neg\phi$ is a tautology). If ϕ is a contradiction, then $\|\phi\| = \emptyset$ and therefore, since, by hypothesis, $(\alpha, t_2) \models I\phi$, $\mathcal{I}_{t_2}(\alpha) = \emptyset$. By definition of $\mathbb{L}_b$-model, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_2}(\alpha) = \emptyset$ so that $(\alpha, t_2) \models B\psi$ for every formula ψ. Hence $K^*_\phi = \Phi^B$. If ϕ is not a contradiction, then by hypothesis, $(\beta, t) \models \phi$, for some (β, t). Since ϕ is Boolean, by Proposition 5, $(\beta, t_2) \models \phi$. Thus $(\alpha, t_2) \models \neg A\neg\phi$. By hypothesis, $(\alpha, t_2) \models I\phi$. Thus $(\alpha, t_2) \models I\phi \wedge \neg A\neg\phi$. By axiom *WC* and Proposition 7, $(\alpha, t_2) \models (I\phi \wedge \neg A\neg\phi) \to (B\psi \to \neg B\neg\psi)$. Thus $(\alpha, t_2) \models B\psi \to \neg B\neg\psi$ for every formula ψ, that is, if $\psi \in K^*_\phi$ then $\neg\psi \notin K^*_\phi$. Since, by (K*2), $\phi \in K^*_\phi$, it follows that $\neg\phi \notin K^*_\phi$ and therefore $K^*_\phi \neq \Phi^B$.

(K*6): we have to show that if $\phi \leftrightarrow \psi$ is a tautology then $K^*_\phi = K^*_\psi$. If $\phi \leftrightarrow \psi$ is a tautology, then $\|\phi \leftrightarrow \psi\| = \Omega \times T$, so that $\lceil\phi\rceil_{t_2} = \lceil\psi\rceil_{t_2}$. Thus $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$ if and only if $\mathcal{I}_{t_2}(\alpha) = \lceil\psi\rceil_{t_2}$, that is, $(\alpha, t_2) \models I\phi$ if and only if $(\alpha, t_2) \models I\psi$. Hence, by definition, $K^*_\phi = K^*_\psi$.[9]

It remains to show that, for every $\phi \in \Phi^B$, there exists an $\mathbb{L}_b$-model such that (1) $(\alpha, t_0) \models I\phi$, for some $\alpha \in \Omega$ and $t_0 \in T$ and (2) if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$. Fix an arbitrary $\phi \in \Phi^B$. Define the following belief revision frame: $T = \{t_0\}$, $\rightarrowtail = \emptyset$, $\Omega = \mathbb{M}^{PL}_B$ and, for every $\omega \in \Omega$, $\mathcal{B}_{t_0}(\omega) = \mathcal{I}_{t_0}(\omega) = \mathbb{M}_\phi$. This is an $\mathbb{L}_b$ frame, since the properties of Definition 6 are trivially satisfied. Define the following model based on this frame: for every atomic proposition q and for every $\omega \in \Omega$, $(\omega, t_0) \models q$ if and only if $q \in \omega$. As shown in part (A)

[9] For every Boolean formula χ, K^*_χ is the set of Boolean formulas believed at (α, t_2) if $(\alpha, t_2) \models I\chi$.

(see (1)), $\lceil\phi\rceil_{t_0} = \mathbb{M}_\phi$. Fix an arbitrary $\alpha \in \Omega$. By construction, $\mathcal{I}_{t_0}(\alpha) = \mathbb{M}_\phi$. Thus $(\alpha, t_0) \models I\phi$. Furthermore, if ϕ is not a contradiction, then, by Lindenbaum's lemma, $\mathbb{M}_\phi \neq \emptyset$ so that there exists a $\beta \in \mathbb{M}_B^{PL}$ with $\phi \in \beta$. Thus, by (1), $(\beta, t_0) \models \phi$. $\square$

The following proposition shows that logic $\mathbb{L}_{AGM}$ characterizes the full set of AGM postulates.

Proposition 12. *Logic $\mathbb{L}_{AGM}$ provides an axiomatic characterization of the full set of AGM postulates* (K*1)–(K*8)*, in the sense that both* (A) *and* (B) *below hold*:

(A) *Let $K \subseteq \Phi^B$ be a consistent and deductively closed set and $\phi, \psi \in \Phi^B$. If $K^*_\phi, K^*_{\phi\wedge\psi} \subseteq \Phi^B$ satisfy AGM postulates* (K*1)–(K*8) *then there is an $\mathbb{L}_{AGM}$-model, $t_1, t_2, t_3 \in T$ and $\alpha \in \Omega$ such that*

(A.1) $t_1 \rightarrowtail t_2$,
(A.2) $K = \{\chi \in \Phi^B\colon (\alpha, t_1) \models B\chi\}$,
(A.3) $(\alpha, t_2) \models I\phi$,
(A.4) $K^*_\phi = \{\chi \in \Phi^B\colon (\alpha, t_2) \models B\chi\}$,
(A.5) *if ϕ is consistent then $(\beta, t) \models \phi$ for some $\beta \in \Omega$ and $t \in T$,*
(A.6) $t_1 \rightarrowtail t_3$,
(A.7) $(\alpha, t_3) \models I(\phi \wedge \psi)$,
(A.8) $K^*_{\phi\wedge\psi} = \{\chi \in \Phi^B\colon (\alpha, t_3) \models B\chi\}$,
(A.9) *if $(\phi \wedge \psi)$ is consistent then $(\gamma, t') \models (\phi \wedge \psi)$ for some $\gamma \in \Omega$ and $t' \in T$.*

(B) *Fix an $\mathbb{L}_{AGM}$-model such that* (1) *for some $t_1, t_2, t_3 \in T$, $\alpha \in \Omega$ and $\phi, \psi \in \Phi^B$, $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $(\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$,* (2) *if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$ and* (3) *if $(\phi \wedge \psi)$ is not a contradiction then $(\gamma, t') \models (\phi \wedge \psi)$, for some $\gamma \in \Omega$ and $t' \in T$. Define $K = \{\chi \in \Phi^B\colon (\alpha, t_1) \models B\chi\}$, $K^*_\phi = \{\chi \in \Phi^B\colon (\alpha, t_2) \models B\chi\}$ and $K^*_{\phi\wedge\psi} = \{\chi \in \Phi^B\colon (\alpha, t_3) \models B\chi\}$. Then K^*_ϕ and $K^*_{\phi\wedge\psi}$ satisfy AGM postulates* (K*1)–(K*8)*. Furthermore, for every $\phi, \psi \in \Phi^B$, there exists an $\mathbb{L}_{AGM}$-model such that, for some $\alpha \in \Omega$ and $t_2, t_3 \in T$,* (1) *$(\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$,* (2) *if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$ and* (3) *if $(\phi \wedge \psi)$ is not a contradiction then $(\gamma, t') \models (\phi \wedge \psi)$, for some $\gamma \in \Omega$ and $t' \in T$.*

Proof. (A) First we prove that if $K \subseteq \Phi^B$ is consistent and deductively closed, $\phi, \psi \in \Phi^B$ and $K^*_\phi, K^*_{\phi\wedge\psi} \subseteq \Phi^B$ satisfy AGM postulates (K*1)–(K*8) then there is an $\mathbb{L}_{AGM}$-model, $t_1, t_2, t_3 \in T$ and $\alpha \in \Omega$ such that (A.1)–(A.9) are satisfied. We proceed as in the proof of Proposition 11. Thus $\mathbb{M}_B^{PL}$ denotes the set of maximally consistent sets of formulas for a propositional logic whose set of formulas is Φ^B and, for $F \subseteq \Phi^B$, let $\mathbb{M}_F = \{\omega \in \mathbb{M}_B^{PL}\colon F \subseteq \omega\}$.

Define the following belief revision frame: $T = \{t_1, t_2, t_3\}$, $\rightarrowtail = \{(t_1, t_2), (t_1, t_3)\}$, $\Omega = \mathbb{M}_B^{PL}$ and, for every $\omega \in \Omega$,

$$\mathcal{B}_{t_1}(\omega) = \mathcal{I}_{t_1}(\omega) = \mathbb{M}_K$$

$$\mathcal{I}_{t_2}(\omega) = \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi & \text{otherwise} \end{cases}$$

$$\mathcal{B}_{t_2}(\omega) = \begin{cases} \emptyset & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K^*_\phi} & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset \end{cases}$$

$$\mathcal{I}_{t_3}(\omega) = \begin{cases} \emptyset & \text{if } \phi \wedge \psi \text{ is a contradiction} \\ \mathbb{M}_{\phi\wedge\psi} & \text{otherwise} \end{cases}$$

$$\mathcal{B}_{t_3}(\omega) = \begin{cases} \emptyset & \text{if } \phi \wedge \psi \text{ is a contradiction} \\ \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K & \text{if } \phi \wedge \psi \text{ is consistent and } \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K^*_{\phi\wedge\psi}} & \text{if } \phi \wedge \psi \text{ is consistent and } \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset. \end{cases}$$

First we show that this frame is an $\mathbb{L}_{AGM}$-frame (see Definition 8). Note that $\mathcal{B}_{t_1}$, $\mathcal{I}_{t_1}$, $\mathcal{B}_{t_2}$ and $\mathcal{I}_{t_2}$ are the same as in the $\mathbb{L}_b$-frame defined in the proof of Proposition 11. Thus we only need to focus on the additional elements.

The Qualitative Bayes Rule is satisfied, since $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_3}(\omega) \neq \emptyset$ if and only if $\mathbb{M}_K \cap \mathbb{M}_{\phi\wedge\psi} \neq \emptyset$, in which case $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_K \cap \mathbb{M}_{\phi\wedge\psi} = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_3}(\omega)$.

The property that, for every ω and t, $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$ is also satisfied. The only case left to examine is the case where $t = t_3$ and $\phi \wedge \psi$ is a consistent formula. If $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \emptyset$, then $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \subseteq \mathbb{M}_{\phi\wedge\psi} = \mathcal{I}_{t_3}(\omega)$. If $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset$ then $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{K^*_{\phi\wedge\psi}}$ and $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi}$. Now, if $\omega \in \mathbb{M}_{K^*_{\phi\wedge\psi}}$ then $K^*_{\phi\wedge\psi} \subseteq \omega$ and, since by AGM postulate (K*2), $\phi \wedge \psi \in K^*_{\phi\wedge\psi}$, it follows that $\phi \wedge \psi \in \omega$, that is, $\omega \in \mathbb{M}_{\phi\wedge\psi}$. Hence $\mathbb{M}_{K^*_{\phi\wedge\psi}} \subseteq \mathbb{M}_{\phi\wedge\psi}$.

The property that, for every ω and t, $\mathcal{B}_t(\omega) \neq \emptyset$ whenever $\mathcal{I}_t(\omega) \neq \emptyset$ is also satisfied. The only case left to examine is the case where $t = t_3$. Now, $\mathcal{I}_{t_3}(\omega) \neq \emptyset$ if and only if $\phi \wedge \psi$ is a consistent formula; in this case either $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K$, if $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \emptyset$, or $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{K^*_{\phi\wedge\psi}}$, in which case, by AGM postulate (K*5), $K^*_{\phi\wedge\psi}$ is a consistent set and therefore, by Lindenbaum's lemma, $\mathbb{M}_{K^*_{\phi\wedge\psi}} \neq \emptyset$.

Next we have to show that the $\mathbb{L}_{AGM}$-specific property *CAB* is satisfied, namely that if t_1, t, t' and ω are such that $t_1 \rightarrowtail t$, $t_1 \rightarrowtail t'$, $\mathcal{I}_{t'}(\omega) \subseteq \mathcal{I}_t(\omega)$ and $\mathcal{I}_{t'}(\omega) \cap \mathcal{B}_t(\omega) \neq \emptyset$ then $\mathcal{B}_{t'}(\omega) = \mathcal{I}_{t'}(\omega) \cap \mathcal{B}_t(\omega)$.

We start with $t = t_3$ and $t' = t_2$. In this case the joint condition $\mathcal{I}_{t_2}(\omega) \subseteq \mathcal{I}_{t_3}(\omega)$ and $\mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_3}(\omega) \neq \emptyset$ holds only if $(\phi \wedge \psi)$ is consistent (implying that ϕ is consistent) and $\mathbb{M}_\phi \subseteq \mathbb{M}_{\phi\wedge\psi}$, which implies that $\mathbb{M}_{\phi\wedge\psi} = \mathbb{M}_\phi$. This, in turn, implies that $(\phi \wedge \psi) \leftrightarrow \phi$ is a tautology, so that, by AGM postulate (K*6), $K^*_{\phi\wedge\psi} = K^*_\phi$. Then

$$\mathcal{B}_{t_2}(\omega) = \begin{cases} \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K^*_\phi} & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset \end{cases}$$

and

$$\mathcal{B}_{t_3}(\omega) = \begin{cases} \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset \\ \mathbb{M}_{K^*_{\phi\wedge\psi}} = \mathbb{M}_{K^*_\phi} & \text{if } \mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset. \end{cases}$$

Thus $\mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_3}(\omega)$. Hence, since $\mathcal{B}_{t_2}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ (proved above for all t), it follows that $\mathcal{B}_{t_2}(\omega) = \mathcal{I}_{t_2}(\omega) \cap \mathcal{B}_{t_3}(\omega)$.

Next we consider the case where $t = t_2$ and $t' = t_3$. In this case we do have that $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ (in fact, $\mathcal{I}_{t_3}(\omega) \neq \emptyset$ if and only if $\phi \wedge \psi$ is consistent, in which case ϕ must be consistent and then $\mathcal{I}_{t_2}(\omega) = \mathbb{M}_\phi$ and $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi}$ and $\mathbb{M}_{\phi\wedge\psi} \subseteq \mathbb{M}_\phi$). Now, $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \emptyset$ only if $\phi \wedge \psi$ is consistent in which case $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi}$. Assume, therefore, that $\phi \wedge \psi$ is consistent (which implies that ϕ is consistent). We need to consider several cases.

(i) $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \emptyset$. Then, since $\mathbb{M}_{\phi\wedge\psi} \subseteq \mathbb{M}_\phi$, $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$; it follows, by construction, that $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K$ and $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K$ so that (since $\mathcal{I}_{t_3}(\omega) = \mathbb{M}_{\phi\wedge\psi}$ and $\mathbb{M}_{\phi\wedge\psi} \subseteq \mathbb{M}_\phi$) $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$.

(ii) $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset$ but $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \emptyset$. In this case $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_\phi \cap \mathbb{M}_K$ and thus $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset$ and therefore there is nothing to prove, since the requirement that $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$ only holds if $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \emptyset$.

(iii) $\mathbb{M}_\phi \cap \mathbb{M}_K = \emptyset$, which implies that $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset$. In this case $\mathcal{B}_{t_2}(\omega) = \mathbb{M}_{K^*_\phi}$ and $\mathcal{B}_{t_3}(\omega) = \mathbb{M}_{K^*_{\phi\wedge\psi}}$, so that $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \emptyset$ if and only if $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_{K^*_\phi} \neq \emptyset$. Assume this. Then it must be that $\neg\psi \notin K^*_\phi$ (if it were the case that $\neg\psi \in K^*_\phi$, then we would have that $\neg\psi \in \omega$ for every $\omega \in \mathbb{M}_{K^*_\phi}$, contradicting the fact that $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_{K^*_\phi} \neq \emptyset$). Thus, by AGM postulates (K*7) and (K*8), $K^*_{\phi\wedge\psi} = [K^*_\phi \cup \{\psi\}]^{PL}$. We need to show that $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$, that is, that $\mathbb{M}_{K^*_{\phi\wedge\psi}} = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_{K^*_\phi}$. Let $\omega \in \mathbb{M}_{K^*_{\phi\wedge\psi}}$. Then $\omega \supseteq K^*_{\phi\wedge\psi} = [K^*_\phi \cup \{\psi\}]^{PL} \supseteq K^*_\phi$. Thus $\omega \in \mathbb{M}_{K^*_\phi}$. Furthermore, by AGM postulate (K*2), $(\phi \wedge \psi) \in K^*_{\phi\wedge\psi}$, so that $\mathbb{M}_{K^*_{\phi\wedge\psi}} \subseteq \mathbb{M}_{\phi\wedge\psi}$. Thus $\mathbb{M}_{K^*_{\phi\wedge\psi}} \subseteq \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_{K^*_\phi}$. Next we prove that $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_{K^*_\phi} \subseteq \mathbb{M}_{K^*_{\phi\wedge\psi}}$. Since, by AGM postulate (K*2), $\phi \in K^*_\phi$, $[K^*_\phi \cup \{\psi\}]^{PL} = [K^*_\phi \cup \{\phi \wedge \psi\}]^{PL}$. Thus, if $\omega \in \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_{K^*_\phi}$ then $\omega \supseteq [K^*_\phi \cup \{\phi \wedge \psi\}]^{PL} = [K^*_\phi \cup \{\psi\}]^{PL} = K^*_{\phi\wedge\psi}$, that is, $\omega \in \mathbb{M}_{K^*_{\phi\wedge\psi}}$.

Now define the following model based on this frame: for every atomic proposition q, for every $\omega \in \Omega$ and for every $t \in T$, $(\omega, t) \models q$ if and only if $q \in \omega$. As in the proof of Proposition 11 (see (1)) it can be shown that, $\forall \chi \in \Phi^B$,

$$\forall t \in T,\ \lceil \chi \rceil_t = \mathbb{M}_\chi, \quad \text{that is,} \quad \forall \omega \in \Omega,\ (\omega, t) \models \chi \quad \text{if and only if} \quad \chi \in \omega. \tag{4}$$

Recall also (see (2)) that, $\forall F \subseteq \Phi^B, \forall \chi \in \Phi^B$,

$$\chi \in [F]^{PL} \quad \text{if and only if} \quad \chi \in \omega,\ \forall \omega \in \mathbb{M}_F. \tag{5}$$

We need to show that properties (A.1)–(A.9) are satisfied. The proof of (A.1)–(A.5) is identical to the proof given for Proposition 11 (since the current frame restricted to $\{t_1, t_2\}$ coincides with the frame considered there). (A.6) is true by construction. Thus we only need to prove (A.7)–(A.9).

(A.7): If $(\phi \wedge \psi)$ is a contradiction then $\lceil \phi \wedge \psi \rceil_{t_3} = \emptyset$ and, by construction, $\mathcal{I}_{t_3}(\alpha) = \emptyset$. If $(\phi \wedge \psi)$ is consistent, by construction $\mathcal{I}_{t_3}(\alpha) = \mathbb{M}_{\phi\wedge\psi}$ and by (4) $\lceil \phi \wedge \psi \rceil_{t_3} = \mathbb{M}_{\phi\wedge\psi}$. Thus, in either case, $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$, that is, $(\alpha, t_3) \models I(\phi \wedge \psi)$.

(A.8): we need to show that $K^*_{\phi\wedge\psi} = \{\chi \in \Phi^B\colon (\alpha, t_3) \models B\chi\}$. There are several cases to be considered.

(8.i) $(\phi \wedge \psi)$ is a contradiction. Then, by AGM postulate (K*5), $K^*_{\phi\wedge\psi} = \Phi^B$ and, by construction, $\mathcal{B}_{t_3}(\alpha) = \emptyset$, so that $(\alpha, t_3) \models B\chi$ for every formula χ. Hence $\{\chi \in \Phi^B\colon (\alpha, t_3) \models B\chi\} = \Phi^B = K^*_{\phi\wedge\psi}$.

(8.ii) $(\phi \wedge \psi)$ is consistent and $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset$. In this case $\mathcal{B}_{t_3}(\alpha) = \mathbb{M}_{K^*_{\phi\wedge\psi}}$. If $\chi \in K^*_{\phi\wedge\psi}$ then $\mathbb{M}_{K^*_{\phi\wedge\psi}} \subseteq \mathbb{M}_\chi$ and, by (4), $\mathbb{M}_\chi = \lceil\chi\rceil_{t_3}$. Thus $\mathcal{B}_{t_3}(\alpha) \subseteq \lceil\chi\rceil_{t_3}$, that is, $(\alpha, t_3) \models B\chi$. Conversely, if $(\alpha, t_3) \models B\chi$ then $\mathcal{B}_{t_3}(\alpha) \subseteq \lceil\chi\rceil_{t_3}$ and, since $\mathcal{B}_{t_3}(\alpha) = \mathbb{M}_{K^*_{\phi\wedge\psi}}$ and, by (4), $\lceil\chi\rceil_{t_3} = \mathbb{M}_\chi$, it follows that $\mathbb{M}_{K^*_{\phi\wedge\psi}} \subseteq \mathbb{M}_\chi$, that is, $\chi \in \omega$ for all $\omega \in \mathbb{M}_{K^*_{\phi\wedge\psi}}$. It follows from (5) that $\chi \in [K^*_{\phi\wedge\psi}]^{PL}$. By AGM postulate (K*1), $K^*_{\phi\wedge\psi} = [K^*_{\phi\wedge\psi}]^{PL}$. Thus $\chi \in K^*_{\phi\wedge\psi}$.

(8.iii) $(\phi \wedge \psi)$ is consistent and $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \emptyset$. In this case $\mathcal{B}_{t_3}(\alpha) = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K$. First of all, it must be that $\neg\phi \notin K$ (if $\neg\phi \in K$ then $\neg\phi \in \omega$ for every $\omega \in \mathbb{M}_K$, which would imply $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \emptyset$, since $\phi \in \omega$ for every $\omega \in \mathbb{M}_{\phi\wedge\psi}$). Hence, by AGM postulates (K*3) and (K*4).

$$K^*_\phi = [K \cup \{\phi\}]^{PL}. \tag{6}$$

Secondly, it must be that

$$\neg\psi \notin [K \cup \{\phi\}]^{PL}. \tag{7}$$

In fact, if $\neg\psi \in [K \cup \{\phi\}]^{PL}$ then, since $[K \cup \{\phi\}]^{PL} \subseteq [K \cup \{\phi \wedge \psi\}]^{PL}$, $\neg\psi \in [K \cup \{\phi \wedge \psi\}]^{PL}$ which, by (5), implies that $\neg\psi \in \omega$ for every $\omega \in \mathbb{M}_{K\cup\{\phi\wedge\psi\}}$. Since $\psi \in \omega$, for every $\omega \in \mathbb{M}_{K\cup\{\phi\wedge\psi\}}$, this would imply that $\mathbb{M}_{K\cup\{\phi\wedge\psi\}} = \emptyset$; however, since

$$\mathbb{M}_{K\cup\{\phi\wedge\psi\}} = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \tag{8}$$

this contradicts the hypothesis that $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \emptyset$.

It follows from (6) and (7) that $\neg\psi \notin K^*_\phi$. Hence by AGM postulates (K*7) and (K*8),

$$K^*_{\phi\wedge\psi} = [K^*_\phi \cup \{\psi\}]^{PL}. \tag{9}$$

Next we note that

$$[K \cup \{\phi \wedge \psi\}]^{PL} = [[K \cup \{\phi\}]^{PL} \cup \{\psi\}]^{PL}. \tag{10}$$

In fact, since $\phi \to (\psi \to (\phi \wedge \psi))$ is a tautology, $(\psi \to (\phi \wedge \psi)) \in [K \cup \{\phi\}]^{PL}$. Thus $(\phi \wedge \psi) \in [[K \cup \{\phi\}]^{PL} \cup \{\psi\}]^{PL}$. Hence $[K \cup \{\phi \wedge \psi\}]^{PL} \subseteq [[K \cup \{\phi\}]^{PL} \cup \{\psi\}]^{PL}$. To prove the converse, first note that, since $\phi \in [K \cup \{\phi \wedge \psi\}]^{PL}$, $[K \cup \{\phi\}]^{PL} \subseteq [K \cup \{\phi \wedge \psi\}]^{PL}$. Hence, since $\psi \in [K \cup \{\phi \wedge \psi\}]^{PL}$, it follows that $[[K \cup \{\phi\}]^{PL} \cup \{\psi\}]^{PL} \subseteq [K \cup \{\phi \wedge \psi\}]^{PL}$.

By (6), $[[K \cup \{\phi\}]^{PL} \cup \{\psi\}]^{PL} = [K^*_\phi \cup \{\psi\}]^{PL}$ and, by (9), $[K^*_\phi \cup \{\psi\}]^{PL} = K^*_{\phi\wedge\psi}$. Thus, by (10),

$$[K \cup \{\phi \wedge \psi\}]^{PL} = K^*_{\phi\wedge\psi}. \tag{11}$$

By (5) for every $\chi \in \Phi^B$, $\chi \in [K \cup \{\phi \wedge \psi\}]^{PL}$ if and only if $\chi \in \omega$, for every $\omega \in \mathbb{M}_{K\cup\{\phi\wedge\psi\}}$. It follows from this, (8) and (11) that, for every $\chi \in \Phi^B$, $\chi \in K^*_{\phi\wedge\psi}$ if and only if $\chi \in \omega$, for every $\omega \in \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \mathcal{B}_{t_3}(\alpha)$. Since, by (4), for every $\chi \in \Phi^B$ and $\omega \in \Omega$, $\chi \in \omega$ if and only if $(\omega, t_3) \models \chi$, it follows that, for every $\chi \in \Phi^B$, $\chi \in K^*_{\phi\wedge\psi}$ if and only if, for every $\omega \in \mathcal{B}_{t_3}(\alpha)$, $(\omega, t_3) \models \chi$, that is, if and only if $(\alpha, t_3) \models B\chi$.

(A.9): we need to show that, if $(\phi \wedge \psi)$ is consistent, then $(\gamma, t') \models (\phi \wedge \psi)$ for some $\gamma \in \Omega$ and $t' \in T$. If $(\phi \wedge \psi)$ is consistent, then by Lindenbaum's lemma, there exists a $\gamma \in \mathbb{M}^{PL}_B$ such that $(\phi \wedge \psi) \in \beta$. By (4), $(\gamma, t') \models \phi \wedge \psi$ for all $t' \in T$.

(B) Fix an $\mathbb{L}_{AGM}$-model such that (1) for some $t_1, t_2, t_3 \in T$, $\alpha \in \Omega$ and $\phi, \psi \in \Phi^B$, $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $(\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$, (2) if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$ and (3) if $(\phi \wedge \psi)$ is not a contradiction then $(\gamma, t') \models (\phi \wedge \psi)$, for some $\gamma \in \Omega$ and $t' \in T$. Define $K = \{\chi \in \Phi^B\colon (\alpha, t_1) \models B\chi\}$, $K^*_\phi = \{\chi \in \Phi^B\colon (\alpha, t_2) \models B\chi\}$ and $K^*_{\phi\wedge\psi} = \{\chi \in \Phi^B\colon (\alpha, t_3) \models B\chi\}$. We need to show that K^*_ϕ and $K^*_{\phi\wedge\psi}$ satisfy AGM postulates (K*1)–(K*8). The proof that AGM postulates (K*1)–(K*6) are satisfied is the same as in Proposition 11 (every $\mathbb{L}_{AGM}$-model is an $\mathbb{L}_b$-model). Thus we shall only prove that AGM postulates (K*7) and (K*8) are satisfied.

First we show that (K*7) is satisfied, that is, that $K^*_{\phi\wedge\psi} \subseteq [K^*_\phi \cup \{\psi\}]^{PL}$. Fix an arbitrary $\chi \in K^*_{\phi\wedge\psi}$, that is, $(\alpha, t_3) \models B\chi$. By hypothesis, $(\alpha, t_3) \models I(\phi \wedge \psi)$. Thus $(\alpha, t_3) \models I(\phi \wedge \psi) \wedge B\chi$ and, since $t_1 \rightarrowtail t_3$, $(\alpha, t_1) \models \Diamond(I(\phi \wedge \psi) \wedge B\chi)$. By axiom $K7$ and Proposition 9, $(\alpha, t_1) \models \Diamond(I(\phi \wedge \psi) \wedge B\chi) \to \bigcirc(I\phi \to B((\phi \wedge \psi) \to \chi))$. Hence

$(\alpha,t_1) \models \bigcirc(I\phi \to B((\phi\wedge\psi)\to\chi))$, from which it follows, since $t_1 \rightarrowtail t_2$, that $(\alpha,t_2) \models I\phi \to B((\phi\wedge\psi)\to\chi)$. By hypothesis, $(\alpha,t_2)\models I\phi$. Hence $(\alpha,t_2)\models B(\phi\to(\psi\to\chi))$ [since $(\phi\wedge\psi)\to\chi$ is tautologically equivalent to $\phi\to(\psi\to\chi)$]. By axiom A and Proposition 9, $(\alpha,t_2)\models I\phi\to B\phi$ and by hypothesis $(\alpha,t_2)\models I\phi$. Thus $(\alpha,t_2)\models B\phi$. By axiom K and Proposition 9, $(\alpha,t_2)\models (B(\phi\to(\psi\to\chi))\wedge B\phi)\to B(\psi\to\chi)$. Thus $(\alpha,t_2)\models B(\psi\to\chi)$, that is, $(\psi\to\chi)\in K^*_\phi$. Hence $\chi\in[K^*_\phi\cup\{\psi\}]^{PL}$.

Next we prove that (K*8) is satisfied, that is, that if $\neg\psi\notin K^*_\phi$ then $[K^*_\phi\cup\{\psi\}]^{PL}\subseteq K^*_{\phi\wedge\psi}$. Fix an arbitrary $\chi\in[K^*_\phi\cup\{\psi\}]^{PL}$. Then there exist $\phi_1,\dots,\phi_n\in K^*_\phi\cup\{\psi\}$ such that $(\phi_1\wedge\cdots\wedge\phi_n)\to\chi$ is a tautology. If $\phi_i\in K^*_\phi$ for every $i=1,\dots,n$ then, since by AGM postulate (K*1) K^*_ϕ is deductively closed (that is, $K^*_\phi=[K^*_\phi]^{PL}$), $\chi\in K^*_\phi$ and thus $(\psi\to\chi)\in K^*_\phi$ (since $\chi\to(\psi\to\chi)$ is a tautology). If $\phi_i\notin K^*_\phi$ for some i then we can assume (renumbering the formulas, if necessary) that $\phi_n\notin K^*_\phi$, from which it follows (since $\phi_i\in K^*_\phi\cup\{\psi\}$ for all $i=1,\dots,n$) that $\phi_n=\psi$. Since, by hypothesis, $(\phi_1\wedge\cdots\wedge\phi_n)\to\chi$ is a tautology and it is tautologically equivalent to $(\phi_1\wedge\cdots\wedge\phi_{n-1})\to(\phi_n\to\chi)$ and $\phi_n=\psi$, it follows that $(\psi\to\chi)\in[K^*_\phi]^{PL}=K^*_\phi$. Thus

$$(\psi\to\chi)\in K^*_\phi,\quad\text{that is,}\quad(\alpha,t_2)\models B(\psi\to\chi).\tag{12}$$

By hypothesis, $\neg\psi\notin K^*_\phi$, that is, $(\alpha,t_2)\models\neg B\neg\psi$. By axiom A and Proposition 9, $(\alpha,t_2)\models I\phi\to B\phi$ and by hypothesis $(\alpha,t_2)\models I\phi$. Thus $(\alpha,t_2)\models B\phi$ and, therefore, $(\alpha,t_2)\models B\phi\wedge\neg B\neg\psi$. By Lemma 10 and Proposition 9, $(\alpha,t_2)\models(B\phi\wedge\neg B\neg\psi)\to\neg B\neg(\phi\wedge\psi)$. Thus

$$(\alpha,t_2)\models\neg B\neg(\phi\wedge\psi).\tag{13}$$

By hypothesis, $(\alpha,t_2)\models I\phi$. This, together with (12) and (13) yields $(\alpha,t_2)\models I\phi\wedge\neg B\neg(\phi\wedge\psi)\wedge B(\psi\to\chi)$. Hence, since $t_1\rightarrowtail t_2$, $(\alpha,t_1)\models\Diamond(I\phi\wedge\neg B\neg(\phi\wedge\psi)\wedge B(\psi\to\chi))$. By axiom $K8$ and Proposition 9, $(\alpha,t_1)\models\Diamond(I\phi\wedge\neg B\neg(\phi\wedge\psi)\wedge B(\psi\to\chi))\to\bigcirc(I(\phi\wedge\psi)\to B\chi)$. Thus $(\alpha,t_1)\models\bigcirc(I(\phi\wedge\psi)\to B\chi)$ from which it follows, since $t_1\rightarrowtail t_3$, that $(\alpha,t_3)\models I(\phi\wedge\psi)\to B\chi$. By hypothesis, $(\alpha,t_3)\models I(\phi\wedge\psi)$. Hence $(\alpha,t_3)\models B\chi$, that is, $\chi\in K^*_{\phi\wedge\psi}$.

It remains to show that for every $\phi,\psi\in\Phi^B$, there exists an $\mathbb{L}_{AGM}$-model such that, for some $\alpha\in\Omega$ and $t_2,t_3\in T$, (1) $(\alpha,t_2)\models I\phi$ and $(\alpha,t_3)\models I(\phi\wedge\psi)$, (2) if ϕ is not a contradiction then $(\beta,t)\models\phi$, for some $\beta\in\Omega$ and $t\in T$ and (3) if $(\phi\wedge\psi)$ is not a contradiction then $(\gamma,t')\models(\phi\wedge\psi)$, for some $\gamma\in\Omega$ and $t'\in T$. Fix arbitrary $\phi,\psi\in\Phi^B$. Define the following belief revision frame: $T=\{t_2,t_3\}$, $\rightarrowtail=\emptyset$, $\Omega=\mathbb{M}^{PL}_B$ and, for every $\omega\in\Omega$, $\mathcal{B}_{t_2}(\omega)=\mathcal{I}_{t_2}(\omega)=\mathbb{M}_\phi$ and $\mathcal{B}_{t_3}(\omega)=\mathcal{I}_{t_3}(\omega)=\mathbb{M}_{\phi\wedge\psi}$. This is an $\mathbb{L}_{AGM}$ frame, since the properties of Definition 8 are trivially satisfied. Define the following model based on this frame: for every atomic proposition q, for every $\omega\in\Omega$ and $t\in T$, $(\omega,t)\models q$ if and only if $q\in\omega$. Fix an arbitrary $\alpha\in\Omega$. By (4), $\lceil\phi\rceil_{t_2}=\mathbb{M}_\phi$ and $\lceil\phi\rceil_{t_3}=\mathbb{M}_{\phi\wedge\psi}$. Thus $(\alpha,t_2)\models I\phi$ and $(\alpha,t_3)\models I(\phi\wedge\psi)$. Furthermore, if ϕ is not a contradiction, then, by Lindenbaum's lemma, $\mathbb{M}_\phi\neq\emptyset$ so that there exists a $\beta\in\mathbb{M}^{PL}_B$ with $\phi\in\beta$. Thus, by (4), $(\beta,t_2)\models\phi$. Similarly, if $(\phi\wedge\psi)$ is not a contradiction, then, by Lindenbaum's lemma, $\mathbb{M}_{\phi\wedge\psi}\neq\emptyset$ so that there exists a $\gamma\in\mathbb{M}^{PL}_B$ with $(\phi\wedge\psi)\in\gamma$. Thus, by (4), $(\gamma,t_3)\models\phi\wedge\psi$. □

For lack of a better expression, we referred to the results of Propositions 11 and 12 as "axiomatic characterizations". As pointed out by a reviewer, this expression is not entirely appropriate.[10] Perhaps alternative expressions could be "axiomatic representation" or "axiomatic counterpart".

5. Related literature

Some of the ideas contained in this paper (in particular the modeling of information by means of a non-normal modal operator) were first put forward in [5]. The framework in that paper was different, however, since it was not based on branching-time structures and only two dates were considered with two associated belief operators, B_0 (representing initial beliefs) and B_1 (representing revised beliefs). The main contribution of that paper was a soundness and completeness result for the proposed logic with respect to the class of frames that satisfy the Qualitative Bayes Rule.

The interaction of a belief operator and a next-time operator is briefly discussed by Kraus and Lehmann [14]. They propose three "plausible" axioms and state that "an open problem is to find a natural family of models for which

[10] According to the reviewer, the expression "axiomatic characterization" is typically used to refer to a result that allows one to say "whatever is valid in this formal system can be derived from this finite list of principles".

the systems considered are complete". The logic they consider contains also a knowledge operator K with standard interaction properties between knowledge and beliefs (e.g. $B\phi \to KB\phi$: if the agent believes ϕ then he knows that he believes ϕ). The interaction of knowledge and belief over time is further studied in Battigalli and Bonanno [2], where, instead of introducing a temporal modality, they define a different belief and knowledge modality for each instant t: $B_t\phi$ reads "the individual believes ϕ at time t". Within this framework they provide a characterization of a property similar to the Qualitative Bayes Rule in terms of the axiom $B_t\phi \leftrightarrow B_t B_{t+1}\phi$. For a discussion of this axiom and its relationship to the Qualitative Bayes Rule see [5].

An approach related to the one suggested in this paper, but carried out in the situation calculus (extended to include a belief operator), can be found in Shapiro et al. [21]. The authors discuss a variety of topics, including belief revision, belief update and iterated belief change.

Instead of temporal logic, a number of authors have used dynamic modal logic to model belief revision (see Fuhrmann [9], de Rijke [18], Segerberg [19,20], van Ditmarsch [7,8]). This approach is known as *dynamic doxastic logic*. Despite some differences in the proposed logics, the common idea is to think of revision as a dynamic action. Besides the standard belief operator B, these authors introduce, for every (Boolean) formula ϕ, a revision operator $[*\phi]$ with the intended interpretation of $[*\phi]\chi$ as "after performing the action of revising by ϕ the individual believes that χ" (some authors also discuss the expansion operator $[+\phi]$ and the contraction operator $[-\phi]$). These logics are considerably more complex than ours: besides requiring the extra apparatus of dynamic logic, they involves an *infinite* number of modal operators (one for each formula ϕ), while our logic uses only three operators.

A modal logic analysis of belief revision was recently proposed by Board [3]. His approach also uses an infinite number of modal operators: for every formula ϕ, an operator B^{ϕ} is introduced, representing the hypothetical beliefs of the individual in the case where she learns that ϕ. Thus the interpretation of $B^{\phi}\psi$ is "upon learning that ϕ, the individual believes that ψ". On the semantic side, Board considers a set of states and a collection of binary relations, one for each state, representing the plausibility ordering of the individual at that state. The truth condition for the formula $B^{\phi}\psi$ at a state expresses the idea that the individual believes that ψ on learning that ϕ if and only if ψ is true in all the most plausible worlds in which ϕ is true. The author gives a list of axioms which is sound and complete with respect to the semantics. The infinite collection of belief operators in Board's framework is what makes it possible for him to compare revisions based on different hypothetical pieces of information. Time does not enter his analysis. Instead we use an information operator to model the information *actually* received by the individual at any instant and the comparison of revisions based on different pieces of information is made possible by the branching-time structure and the associated temporal operator.

For further discussion of literature that is somewhat related to the general approach proposed in this paper, the reader is referred to [5].

6. Conclusion

We proposed a temporal logic where information and beliefs are modeled explicitly by means of two modal operators I and B, respectively. A branching-time structure with the associated next-time operator makes it possible to compare different belief revisions following the receipt of different pieces of information. The proposed logic provides an axiomatic system that corresponds to the AGM postulates for belief revision.

One of the advantages of modeling belief revision in modal logic is that properties of beliefs can be stated in a clear and transparent way by means of syntactic axioms. Another advantage of the approach proposed in this paper is that it offers a uniform treatment of static and dynamic beliefs, thus providing a unified framework for both. Static beliefs would correspond to the case where the set of instants T is a singleton. All the properties of beliefs studied in the static approach (see Hintikka [12]), such as consistency ($B\phi \to \neg B\neg\phi$), positive introspection ($B\phi \to BB\phi$) and negative introspection ($\neg B\phi \to B\neg B\phi$), can be added to our list of axioms to provide stronger logics, which we intend to study in future work.

It is also worth noting that the branching-time structure considered here provides a natural framework for studying iterated belief revision, a topic that has received considerable attention in recent years (see, for example, Nayak et al. [17]). Since the framework allows for a sequence of revisions based on a sequence of pieces of information, an interesting topic for future research is whether the principles for iterated revision that have been proposed in the literature can be translated into syntactic axioms.

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Appendix A

Proof of Proposition 4. The proof that $\mathbb{L}_0$ is sound with respect to the class of temporal belief revision frames is along the usual lines (see [4] or [6]). We need to show that (1) the rules of inference are validity preserving and (2) the axioms of $\mathbb{L}_0$ are valid in an arbitrary temporal belief revision frame. The proof of (1) is entirely standard and is omitted. The proof of validity of axiom K for $\bigcirc$, $\bigcirc^{-1}$ and A and for the temporal axioms (O_1) and (O_2) is also standard and is omitted.

Validity of the backward uniqueness axiom (BU) is an immediate consequence of the fact that in a belief revision frame every instant t has at most a unique immediate predecessor: if $(\omega, t_2) \models \Diamond^{-1}\phi$ then there exists a t_1 such that $t_1 \rightarrowtail t_2$ and $(\omega, t_1) \models \phi$. Since, for every $t \in T$, $t \rightarrowtail t_2$ if and only if $t = t_1$, it follows that $(\omega, t_2) \models \bigcirc^{-1}\phi$.

Validity of the S5 axioms for A is also straightforward. Suppose that $(\alpha, t) \models A\phi$. Then $(\omega, t) \models \phi$ for every $\omega \in \Omega$, thus in particular for $\omega = \alpha$. Similarly, if $(\alpha, t) \models \neg A\phi$ then there exists a $\beta \in \Omega$ such that $(\beta, t) \models \neg\phi$. Hence $(\omega, t) \models \neg A\phi$ for every $\omega \in \Omega$ and therefore $(\alpha, t) \models A\neg A\phi$.

The proof that the inclusion axiom for B (Incl_B) is valid is straightforward and is omitted.

Validity of axiom I_1: $I\phi \wedge I\psi \rightarrow A(\phi \leftrightarrow \psi)$. Suppose that $(\alpha, t) \models I\phi \wedge I\psi$. Then $\mathcal{I}_t(\alpha) = \lceil\phi\rceil_t$ and $\mathcal{I}_t(\alpha) = \lceil\psi\rceil_t$. Thus $\lceil\phi\rceil_t = \lceil\psi\rceil_t$ and hence $\lceil\phi \leftrightarrow \psi\rceil_t = \Omega$, yielding $(\alpha, t) \models A(\phi \leftrightarrow \psi)$.

Validity of axiom I_2: $A(\phi \leftrightarrow \psi) \rightarrow (I\phi \leftrightarrow I\psi)$. Suppose that $(\alpha, t) \models A(\phi \leftrightarrow \psi)$. Then $\lceil\phi \leftrightarrow \psi\rceil_t = \Omega$ and, therefore, $\lceil\phi\rceil_t = \lceil\psi\rceil_t$. Thus, $(\alpha, t) \models I\phi$ if and only if $\mathcal{I}_t(\alpha) = \lceil\phi\rceil_t$, if and only if $\mathcal{I}_t(\alpha) = \lceil\psi\rceil_t$, if and only if $(\alpha, t) \models I\psi$. Hence $(\alpha, t) \models I\phi \leftrightarrow I\psi$. □

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