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Propositional belief base update and minimal change

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Abstract

In this paper we examine ten concrete propositional update operations of the literature. We start by completely characterizing their relative strength and their computational complexity. Then we evaluate the competing update operations with respect to the postulates proposed by Katsuno and Mendelzon. It turns out that the majority violates most of the postulates. We argue that all violated postulates are undesirable except one. After that we evaluate the update operations with respect to another property which has been investigated extensively in the literature, viz. that disjunctive updates should not be identified with the exclusive disjunction. We argue that this is desirable, and show that the argument gives further support to the rejection of two of the postulates. Finally we study how the different approaches accommodate general laws governing the world, alias integrity constraints. Summing up our results, we conclude that only two of the update operations are satisfactory. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A database being a device to store and retrieve information, it has been proposed by Levesque [27] to view a database as being equipped with a querying function *ASK* and an update function *TELL*. *ASK(light-on)* is a query meaning that the database is asked whether it follows from the data contained in it that the light in question is on.

TELL(light-on) is an update meaning that *light-on* is a new piece of data which the database should take into account. This is a much fuzzier requirement than that for *ASK*.

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The easiest case seems to be when the input *light-on* is already in the database (or follows from it) and there is no need to act. [2] Things get more complex if *light-on* does not follow from the current database, and they get even harder if *light-on* contradicts it. In early database systems it had been considered that the new data should be systematically rejected in the latter case. Such a trivial *TELL*-function is unsatisfactory in most applications. Other approaches such as the introduction of so-called ‘null values’ in relational databases turned out to be problematic as well (cf. the discussion in [38, Section 1.1]). [3]

1.1. What changes?—updates versus revisions

Several authors coming from the database field such as Winslett, Katsuno, Mendelzon, Satoh and Grahne have linked the problematics of database updating to that of belief change as studied by philosophers in the field of formal epistemology. There, what is studied are operations $\diamond$ mapping a current belief base B and an input A to a new belief base $B \diamond A$. (We shall henceforth synonymously use the terms database and belief base.)

Alchourròn, Gärdenfors and Makinson had established in the eighties a set of rationality postulates that every reasonable belief revision operation should satisfy (the AGM postulates), and had proved characterization theorems.

It has been claimed that there is the following fundamental semantical difference between update operations and AGM revision operations.

- If the input does not correspond to a change in the real state of affairs then the belief base should be *revised*: change takes place only at the knowledge level, i.e., within the beliefs that are held about a fixed real state of affairs;
- If the input mirrors changes in the real state of affairs then the belief base should be *updated*.

The difference between these two operations has been pointed out first in [25], and has been taken up in [24]. In the case of an update, when the belief base is notified of a change occurring in the real state of affairs by a *TELL*-operation, the description of the possible states of affairs must be modified accordingly. The classical explanation as given in [18] is as follows.

> Since we are confined to our set of possibilities, we must make the change come true in all of our candidate worlds. Semantically, we change each of the possible worlds *‘as little as possible’* in order to make the new state of affairs hold. Our new syntactic description of the worlds of interest should now correctly reflect the outcome of this *set* of changes. The function that maps the old description to the new is called an *update*.

Katsuno and Mendelzon have based such a notion of minimal change on orderings of closeness (or similarity) between possible worlds, in much the same way as it has

[2] Note nevertheless that we might want to strengthen the degree of certainty (degree of belief, acceptance, ...) of the piece of data *light-on*.

[3] There is another function dual to *TELL* which has been discussed in the literature, that is called erasure or contraction, which retracts data from the database. We shall not treat it in this paper, one of the reasons being that it can be defined from the *TELL*-function via the so-called Harper identity [16,24].

been done by Lewis in conditional logics [28]. Paralleling the AGM postulates, Katsuno and Mendelzon have then characterized these models by a set of postulates that every reasonable belief update operation should satisfy (the KM postulates).

It can be shown [21, Theorem 22, p. 218] that update operations are not only different from AGM-revision operations, but even incompatible with them (in the sense that there is no operation that satisfies both the AGM and the KM postulates). Given that incompatibility, what remains is a practical problem: in a given situation, which change operation should we choose? Things are clear in the case, e.g., of fault diagnosis or detective stories, where the real world usually does not evolve any more: A murder or a fault in a circuit has occurred, and the detective revises his beliefs in the light of new information about that fixed picture. But in a lot of cases we ignore whether the input corresponds to an event in the real world or not.

It has been proposed, e.g., in [8,24] to view incoming information as time-stamped: the current belief base being labelled with t, when a belief labelled t' comes in then we choose revision if $t = t'$, and update else. In fact this amounts to explicitly say "revise" or "update". But in everyday situations (as well as, e.g., in robotics) we might hesitate over the choice of the change operation, and we cannot presuppose that incoming information is time-stamped.

1.2. What is minimal change?

The KM postulates give us the abstract properties of update operations with respect to the logical and metalogical operators of classical logic. One might expect that the set of postulates characterizes one single update operation: the 'right' one. [4] Is this the case, i.e., is there a unique operation such that for every description of the world and input computes the correct resulting belief base? First of all, we note that if such a $\diamond$ exists then our description of the world should not only contain the current belief base, but also something like a set of laws governing the world.

The answer is negative: the class of update operations admitted by the KM-postulates is not a singleton. (Worse, it contains the trivial update operation which retains the input (and destroys the belief base) when it does not follow from the belief base.) Semantically, this corresponds to the fact that there is more than one ordering of closeness between possible worlds. In the case of conditionals, Lewis [28, pp. 94–95] has stressed that this had been done deliberately, and that there are no context-independent closeness criteria. He predicts failure of "any humanly possible attempt at a precise definition of comparative similarity of worlds. Not only would we go wrong by giving a precise analysis of an imprecise concept; our precise concept would not fall within—or even near—the permissible range of variation of the ordinary concept". In the case of AGM revision, Gärdenfors [17, p. 11] remarks that "the postulates ... do not uniquely characterise the revision in terms of only the database and the input. This is, however, as it should be. I believe it would be a mistake to expect that only logical properties are sufficient to characterise the revision process".

[4] Compare this to the case of the deduction relation for classical logic: Gentzen's rules can be viewed as describing the interplay between the deduction relation $\vdash$ and the Boolean connectives, and they are sufficient to characterize a unique $\vdash$.

Lewis and Gärdenfors probably have in mind that we are not able to appropriately handle the description of the the world, and in particular the laws governing the world. Following Goodman, Lewis [28, p. 73] discusses the following solution: "whenever the laws prevailing at i are violated at a world k but not at a world j, j is closer than k to i". In other words, we should look for the closest worlds among those not violating the laws. This is a solution which has been widely used in artificial intelligence in order to implement integrity constraints, which are nothing else than non-logical laws. [5]

Lewis rejects such a solution because he thinks that laws should be defeasible. While such a position is satisfactory from the philosophical point of view, it is unsatisfactory for people in the database field and in AI, who are looking for explicit constructions such as algorithms or procedures. Fortunately, in many computer science applications the integrity constraints can be considered to be undefeasible, and Lewis' criticism does not apply.

1.3. How can we compare update operations?

In the AI literature several concrete, unique update operations have been proposed. They have been criticized, usually by means of counterexamples, leading to alternative approaches, and so on.

The different approaches as well as their criticisms are based on several hypotheses, which are more or less explicitly stated in the literature:

(1) A belief base describes what is believed by an agent about the 'real state of affairs' (or 'the actual world'). This description consists of a set of propositional formulas.
(2) The agent entertaining the belief base has sensors permitting him to acquire information about facts of the actual world. Such information consists of propositional logic formulas.
(3) The agent supposes that the input information describes an event that has happened in the actual world.
(4) The agent changes his beliefs in the belief base in such a way that the latter sticks as close as possible to what he believes to know about the actual world.
(5) The agent does not always know whether sensing is noisy or not, and whether there is misperception or not.

Now, our question is: how can we compare these operations, and how can we evaluate them with respect to both the KM postulates and other important requirements, such as the correct handling of integrity constraints? There are several ways to do this.

- One can study the computational properties of the update operations, in particular their theoretical complexity (complexity of the decision problem).
  This will be done in Section 3, where we present the operations that we study.
- One can compare their strength. We consider that an operation $\diamond_1$ is stronger than $\diamond_2$ if for every belief base the set of possible worlds obtained by updating with $\diamond_1$ contains those obtained with $\diamond_2$.
  We shall give such an ordering of strength in Section 4.
- One can evaluate the competing update operations with respect to the KM-postulates.

[5] But such a solution has been criticized as well, as discussed in Section 7.

We shall do that in Section 5. It will turn out that the majority violates most of the postulates. Based on our hypotheses, we shall argue that all violated postulates are undesirable except one.

- One can postulate other properties that update operations should satisfy, and then evaluate the update operations with respect to them.
  We shall examine in Section 6 a negative requirement that has been investigated extensively in the literature, viz. that disjunctive input should not be identified with the exclusive disjunction. We argue that this is desirable, giving further support to the rejection of some postulates.
- One can study how the different approaches accommodate with general laws governing the world, alias integrity constraints.
  This will be done in Section 7. Here the comparison is done by examples. It turns out that only two approaches can handle these examples correctly.

But first of all we recall some more or less standard classical logic notions and notations (Section 2). In the end we shall mention some other approaches to updates (Section 8).

2. Preliminaries

The language is built from a possibly infinite set of atoms $ATM = \{p, q, r, \ldots\}$ with the classical connectives $\wedge, \vee, \neg, \bot, \top$. $L, L_1, \ldots$ denote literals, LIT is the set of all literals. $c, c_1, \ldots$ denotes clauses, i.e., disjunctions of literals. $A, B, C, \ldots$ denote formulas. We confuse belief bases (that are finite sets) with the conjunction of their elements. As far as possible we shall use $B, B_1, \ldots$ for belief bases, and $A, A_1, \ldots$ for inputs (formulas to be added).

We stipulate that $\neg$ binds stronger than $\wedge$ and $\vee$, which bind stronger than the other connectives. We denote by $atm(A)$ the set of atoms appearing in the formula A. For example, $atm(p \wedge (p \vee q)) = \{p, q\}$, $atm(\bot) = atm(\top) = \emptyset$.

An atom p occurring in A is redundant if there is an equivalent formula A' where p does not occur. Hence p's truth value does not affect the truth value of A. For example p is redundant in $q \wedge (q \vee p)$. In order to establish that an atom is redundant one can use the following fact.

Fact 1. *An atom p is redundant in a formula A iff $A[p\backslash\top] \leftrightarrow A[p\backslash\bot]$.*

Fact 2. *To check redundancy of an atom is a coNP-complete problem.*

Fact 1 shows us that it is in coNP. The other way round, we can polynomially transform the problem of theoremhood of a formula A in the propositional calculus into that of the redundancy of p in $A[p_1\backslash p] \wedge \cdots \wedge A[p_n\backslash p]$, where $atm(A) = \{p_1, \ldots, p_n\}$ and $p \notin atm(A)$.

Fact 1 can be turned into an algorithm to get rid of redundant atoms. We note $A\!\downarrow$ the formula obtained by eliminating of all redundant atoms. For example, $(q \wedge (q \vee p))\!\downarrow = q$, $(p \vee \neg p)\!\downarrow = \top$, $((p \wedge \neg p) \wedge q)\!\downarrow = \bot$.

A clause c is an implicate of A iff $A \to c$. An implicate c is prime iff for all implicates c' of A such that $c' \to c$ we have $c \to c'$. A set D of prime implicates is a covering of A iff for every clause c such that $A \to c$ there is $c' \in D$ such that $c' \to c$.

Fact 3. *If D is a covering set of prime implicates of A then $D \leftrightarrow A$ and $atm(D) = atm(A \downarrow)$.*

Another name for a covering set of prime implicates is *Blake canonical form* [3]. Note that there may be several sets of prime implicates covering a given formula.

Fact 3 can also be used to get rid of redundant atoms (this has been proposed by [9]). Nevertheless, the number of prime implicates can be exponential in the length of the original formula. Therefore the algorithm based on Fact 1 is preferable.

Interpretations are sets of atoms. We shall often represent an interpretation by a maximally consistent set of literals. 2^{ATM} is the set of all interpretations (which might be viewed as possible worlds). Given a formula A, we note $[\![A]\!]$ the set of interpretations where A is true. A is valid if $[\![A]\!] = 2^{ATM}$.

The notion of distance between interpretations is a central device in update operations. The distance between w and v is the set of atoms whose truth value differs:

$$\begin{aligned} DIST(w, v) &= (w \setminus v) \cup (v \setminus w) \\ &= \big\{p\colon\ w \in [\![p]\!] \text{ and } v \notin [\![p]\!]\} \cup \{p\colon\ w \notin [\![p]\!] \text{ and } v \in [\![p]\!]\big\}. \end{aligned}$$

For example, suppose $ATM = \{p, q, r\}$, $w = \{p, q, \neg r\}$ and $v = \{p, \neg q, r\}$. Then $DIST(w, v) = \{q, r\}$.

3. Proposals for update operations

In this section, we present the proposals for concrete update operations that can be found in the literature that are not explicitly ordering-based.

Formally, an update operation is a function (noted $\diamond$) mapping a belief base B and an input A to a new belief base $B \diamond A$. According to the explanation of updates in Section 1.1, belief bases should be updated world by world. Therefore, let $w \in [\![B]\!]$, and let $w \cdot A$ be the set of interpretations resulting from the update of w by A. Then the models of $B \diamond A$ should be the collection of updates of each model of B by A. Formally, we should have

$$[\![B \diamond A]\!] = \bigcup_{w \in [\![B]\!]} w \cdot A.$$

Each operation is thus defined by associating to an interpretation and an input a set of interpretations. Basically there are two families of approaches. The first one works by minimizing distances between worlds, while the second constrains distances to be in some set of exceptions computed from the input.

We set the following parenthesis conventions: $\neg$ binds stronger than $\diamond$, which binds stronger than the others (although $\diamond$ is not in the object language).

3.1. PMA: Minimal change

The *Possible Models Approach* (*PMA*), was introduced by Winslett in [37] in the context of reasoning about action and change. It is based on minimization of the distance *DIST* between interpretations.

Let A be the formula representing the incoming information (the input). Then the update of w by $[\![A]\!]$ is defined as:

$$w \cdot_{pma} [\![A]\!] = \big\{u \in [\![A]\!] : \forall u' \in [\![A]\!],\ \mathit{DIST}(w,u') \not\subset \mathit{DIST}(w,u)\big\}.$$

In other terms, the set $w \cdot_{pma} [\![A]\!]$ contains all those elements of $[\![A]\!]$ that are minimal with respect to the closeness ordering $\leqslant_w$, where $\leqslant_w$ is defined by

$$u \leqslant_w v \quad \text{iff} \quad \mathit{DIST}(w,u) \subseteq \mathit{DIST}(w,v).$$

Every $\leqslant_w$ is a partial pre-order over interpretations.

Example 4. For $w = \{\neg p, \neg q\}$ and $A = p \vee q$, we get $[\![A]\!] = \{\{p,q\}, \{\neg p, q\},\ \{p, \neg q\}\}$, $\mathit{DIST}(w, \{p,q\}) = \{p,q\}$, $\mathit{DIST}(w, \{\neg p, q\}) = \{q\}$ and $\mathit{DIST}(w,\ \{p, \neg q\}) = \{p\}$. Thus the models of A which are minimal for distance set inclusion are $\{\{\neg p, q\}, \{p, \neg q\}\}$. Hence we get $w \cdot_{pma} [\![p \vee q]\!] = \{\{\neg p, q\}, \{p, \neg q\}\}$.

The computational complexity of the *PMA* is Π_2^{p}-complete [11].

3.2. FORBUS: Numeric minimal change

The operation proposed by Forbus in [15] is stronger than the *PMA*. It is the update counterpart of Dalal's semantics for belief revision [5]. There, the semantical update operation is defined not from the distance between interpretations $\mathit{DIST}(w,u)$, but from its cardinality $\mathit{card}(\mathit{DIST}(w,u))$:

$$\begin{aligned} & w \cdot_{Forbus} [\![A]\!] \\ &\quad = \big\{u \in [\![A]\!],\ \forall v \in [\![A]\!]\text{: } \mathit{card}(\mathit{DIST}(w,u)) \leqslant \mathit{card}(\mathit{DIST}(w,v))\big\}. \end{aligned}$$

The resulting set of interpretations contains those models of A that are minimal with respect to the closeness ordering $\leqslant_w$, where $\leqslant_w$ is defined by

$$u \leqslant_w v \quad \text{iff} \quad \mathit{card}(\mathit{DIST}(w,u)) \leqslant \mathit{card}(\mathit{DIST}(w,v)).$$

Every $\leqslant_w$ is a total pre-order over interpretations. On Example 4, Forbus' operation behaves just as the *PMA*.

Example 5. Let $w = \{\neg p, \neg q, \neg r\}$ and A be $(p \vee q) \wedge (p \vee r) \wedge (q \vee \neg r)$. Hence $[\![A]\!] = \{\{p, \neg q, \neg r\}, \{\neg p, q, r\}, \{p, q, r\},\ \{p, q, \neg r\}\}$. The cardinalities of the distances between w and the models of A are, respectively (1,2,3,2). Then we get $w \cdot_{Forbus} [\![A]\!] = \{\{p, \neg q, \neg r\}\}$.

This illustrates that *FORBUS* is different from the *PMA*. The computational complexity of *FORBUS* is Π_2^{p}-complete [11].

3.3. MCD: Going beyond the PMA

Both *PMA* and *FORBUS* have been criticized for their handling of disjunctive input. It has been argued that input such as $p \vee q$ in Example 4 is interpreted as if it was an exclusive disjunction $p \oplus q$ (see Section 6). Motivated by that, *Minimal Change with Maximal Disjunctive inclusion* (*MCD*), was introduced by Zhang and Foo in [39]. We give a reformulation which is more appropriate for our purposes, and which illustrates how *MCD* is built on top of the *PMA*.

Let U be the set of models of the input A, and let w be some interpretation. Let $V = w \cdot_{pma} U$ be the set of models resulting from *PMA*-updating w with U, and let $S = 2^V$. For $s \in S$, the 'cone' $C(s)$ is the set of those interpretations in U that are beyond all elements of s with respect to the *PMA* closeness ordering $\leqslant_w$:

$$C(s) = \{u \in U\colon\ \forall v \in s,\ v \leqslant_w u\}.$$

The set $\{C(s)\colon\ s \in S\}$ is a covering of U: $U = \bigcup\{C(s)\colon\ s \in S\}$. The key idea is that *PMA*-minimization in that set allows to obtain more interpretations than $w \cdot_{pma} U$ would give us:

$$w \cdot_{mcd} U = \bigcup_{s \in S} w \cdot_{pma} C(s).$$

Example 6. Let $w = \{\neg p, \neg q\}$ and let A be $p \vee q$, we have $V = w \cdot_{pma} [\![p \vee q]\!] = \{w_1, w_2\}$, where $w_1 = \{p, \neg q\}$ and $w_2 = \{\neg p, q\}$. Second, $S = \{\emptyset, \{w_1\}, \{w_2\}, \{w_1, w_2\}\}$. We construct $C(\emptyset) = \{w_1, w_2, \{p, q\}\}$, $C(\{w_1\}) = \{w_1, \{p, q\}\}$, $C(\{w_2\}) = \{w_2, \{p, q\}\}$, and $C(\{w_1, w_2\}) = \{\{p, q\}\}$.
Finally

$$\begin{aligned} w \cdot_{mcd} [\![p \vee q]\!] &= w \cdot_{pma} \{w_1, w_2, \{p, q\}\} \cup w \cdot_{pma} \{w_1, \{p, q\}\} \\ &\qquad \cup w \cdot_{pma} \{w_2, \{p, q\}\} \cup w \cdot_{pma} \{\{p, q\}\} \\ &= \{w_1, w_2\} \cup \{w_1\} \cup \{w_2\} \cup \{p, q\} = \{w_1, w_2\} \cup \{p, q\}. \end{aligned}$$

Together with Example 4 this illustrates that *MCD* is different from the *PMA*. The computational complexity of *MCD* is Π_2^{p}-complete [29].

3.4. MCD*: Iterating MCD

In [22] it has been shown that *MCD* fails to capture the intuitions that have been put forward in [39] (see Section 6), and an iterative version *MCD** of *MCD* is introduced as a correction for *MCD*. It behaves as *MCD* if for every subset of the set $w \cdot_{pma} U$ there is only one model of the input which is minimally beyond all its elements. This means that $w \cdot_{pma} C(s)$ is a singleton for all $s \subseteq w \cdot_{pma} U$. When there are two or more, contrarily to *MCD*, *MCD** continues to look for interpretations which are beyond, and so on.

Formally, let U be the set of models of the input A, $V_U = w \cdot_{pma} U$, $S_U = 2^{V_U}$, and $C_U(s)$ the subset of U defined as before (i.e., $C_U(s) = \{u \in U\colon\ \forall v \in s, v \leqslant_w u\}$). We use here a notation different from the above (i.e., we index S and C by U) because inputs are not the same at each step of our inductive definition. Then we define $\cdot_{mcd*}$ recursively as the smallest set such that:

$$w \cdot_{mcd*} U = \begin{cases} w \cdot_{pma} U & \text{if } card(w \cdot_{pma} U) \leqslant 1, \\ \bigcup_{s \in S_U} w \cdot_{mcd*} C_U(s) & \text{otherwise.} \end{cases}$$

Example 7. Let $w = \{\neg p, \neg q, \neg r, \neg t\}$, $U = [\![(p \vee q) \wedge ((p \wedge q) \rightarrow (r \vee t))]\!]$. First $w \cdot_{pma} U = \{w_1, w_2\}$ with $w_1 = \{p, \neg q, \neg r, \neg t\}$ and $w_2 = \{\neg p, q, \neg r, \neg t\}$. Second, we construct $C_U(s)$ for every $s \in S_U = \{\emptyset, \{w_1\}, \{w_2\}, \{w_1, w_2\}\}$. $U_1 = C_U(\{w_1\}) = \{w_1, \ldots\}$, $U_2 = C_U(\{w_2\}) = \{w_2, \ldots\}$, and $U_3 = C_U(\{w_1, w_2\}) = \{w_3, w_4, w_5\}$, where $w_3 = \{p, q, r, \neg t\}$, $w_4 = \{p, q, \neg r, t\}$, and $w_5 = \{p, q, r, t\}$. Now we compute $V_{U_1} = \{w_1\}$, $V_{U_2} = \{w_2\}$, and $V_{U_3} = \{w_3, w_4\}$. Again, we construct the sets $U_{31} = \{w_3, \ldots\}$, $U_{32} = \{w_4, \ldots\}$, and $U_{33} = \{w_5\}$. Then $V_{U_{31}} = \{w_3\}$, $V_{U_{32}} = \{w_4\}$ and $V_{U_{33}} = \{w_5\}$. Finally,

$$w \cdot_{mcd*} U = w \cdot_{mcd*} U1 \cup w \cdot_{mcd*} U2 \cup w \cdot_{mcd*} U3$$

where $w \cdot_{mcd*} U1 = V_{U_1} = \{w_1\}$, $w \cdot_{mcd*} U2 = V_{U_2} = \{w_2\}$, and $w \cdot_{mcd*} U3 = w \cdot_{mcd*} U31 \cup w \cdot_{mcd*} U32 \cup w \cdot_{mcd*} U33$. And $w \cdot_{mcd*} U31 = V_{U_{31}} = \{w_3\}$, $w \cdot_{mcd*} U32 = V_{U_{32}} = \{w_4\}$ and $w \cdot_{mcd*} U33 = V_{U_{33}} = \{w_5\}$. Hence

$$w \cdot_{mcd*} U = \{w_1, w_2, w_3, w_4, w_5\}.$$

Note that $w \cdot_{mcd} U = \{w_1, w_2, w_3, w_4\}$.

This illustrates that *MCD** is different from *MCD*.

We conjecture that *MCD** is PSPACE-complete. (We were not able to find a proof, which seems to be involved.)

3.5. WSS: Minimal change with exceptions

Winslett's standard semantics (*WSS*) [38] has received only little attention until recently. It was intended to be the weakest operation deserving the name of update. It is attractive because its definition is simple, and because it handles disjunctions correctly.

$w \cdot_{wss} A$ is the set of those models of A which preserve the truth value of atoms not occurring in A. Formally:

$$w \cdot_{wss} A = \big\{u \in [\![A]\!]\colon DIST(w, u) \subseteq atm(A)\big\}.$$

Example 8. Let $w = \{\neg p, \neg q, \neg r\}$. Then $w \cdot_{wss} p \vee q = [\![(p \vee q) \wedge \neg r]\!]$. And $w \cdot_{wss} p \vee \neg p = [\![\neg q \wedge \neg r]\!]$. The last example illustrates that *WSS* is sensitive to syntax: although $p \vee \neg p$ is logically equivalent to $\top$, we have $w \cdot_{wss} \top \neq w \cdot_{wss} p \vee \neg p$.

WSS is coNP-complete [29].

3.6. *WSS*↓: *Making WSS syntax-insensitive*

In fact *WSS* is syntax-sensitive in the input only, in the sense that the elimination of redundant input atoms changes the result of the update. Therefore it is interesting to combine *WSS* with preprocessing of the input to eliminate redundant atoms.

This has been proposed in [22]. Noting *WSS*↓ the resulting operation we have

$$w \cdot_{wss\downarrow} A = w \cdot_{wss} A\downarrow.$$

We shall show that this is equivalent to Hegner's semantics for updates [19]. Hegner's original presentation is much more complex. He starts by eliminating redundant atoms in a different manner. First, he defines the following notions: For a formula A, the set of partial models of A, denoted $LB(A)$ (called the literal base in [19]), is the set $\{u \subseteq LIT$ such that u is consistent and $u \in [\![A]\!]\}$. For $L \in LIT$, L is irrelevant for A if for every $u \in LB(A)$, $L \in u$ implies that $u \backslash L \in LB(A)$. [6] u is *minimal* if it contains no irrelevant elements. $u \in LB(A)$ is *complete* if it is minimal, and for any other minimal $v \in LB(A)$, $u \subseteq v$ implies that $u = v$. In fact the set $\{u \in LB(A), u \textit{ is complete}\}$ is nothing but the set of prime implicants of A. Finally, if w is the actual state of the world then we get

$$w \cdot_{Hegner} A = \bigcup_{u \in LB(A),\ u \textit{ is complete}} \textit{insert}[u][w]$$

where $\textit{insert}[u][w] = u \cup \{L \in w, \neg L \notin u\}$.

Example 9. Suppose $w = \{\neg p, \neg q\}$, and the input A is $(p \wedge q) \vee (p \wedge \neg q)$. Then $LB(A) = \{\{p\}, \{p, q\}, \{p, \neg q\}\}$. The only complete set is $\{p\}$, and $w \cdot_{Hegner} A = \textit{insert}[\{p\}][\{\neg p, \neg q\}] = \{p, \neg q\}$.

Doherty et al. [9] have proposed a similar operation as a generalization of the *PMA*. It is called the modified *PMA* (*MPMA*) and is defined as follows:

$$w \cdot_{mpma} A = w \cdot^{P}_{pma} [\![A]\!]$$

and

$$w \cdot^{P}_{pma} [\![A]\!] = \{u \in [\![A]\!]:\ DIST(w, u) \subseteq P\}$$

where $P = atom(A\downarrow)$ is the set of non redundant atoms. [7]

Example 10. Suppose $w = \{\neg p, \neg q\}$, and the input A is $(p \wedge q) \vee (p \wedge \neg q)$. Then $P = \{p\}$. $DIST(w, \{p, q\}) \not\subseteq \{p\}$, and $DIST(w, \{p, \neg q\}) \subseteq \{p\}$. Thus $w \cdot_{mpma} A = \{p, \neg q\}$.

WSS↓, *HEGNER* and *MPMA* are equivalent.

[6] In the original definition: if for every $u \in LB(A)$, $L \in u$ implies that both $u\backslash\{L\}$ and $u\backslash\{\neg L\}$ are also in $LB(A)$.

[7] To compute P, they propose to compute *Blake canonical form* of A (cf. Section 2). We note in passing that this might cause exponential growth of A, and that it is preferable to use the algorithm derived from Fact 1.

Lemma 11. $w \cdot_{wss\downarrow} A = w \cdot_{Hegner} A$.

Proof. Suppose that there is no redundant atom in A. Then the set of $u \in LB(A)$ which are *complete*, is $[\![A]\!]$. Hence $w \cdot_{Hegner} A = [\![A]\!]$. And $w \cdot_{wss\downarrow} A = [\![A]\!]$.

Suppose now that there is one redundant atom $p \in atm(A)$, then

$$w \cdot_{wss\downarrow} A = [\![A[p := \top] \wedge L]\!]$$

where $atm(L) = p$ and $w \in [\![L]\!]$. The $u \in LB(A)$ such that u is *complete*, are those partial models of A such that neither p nor $\neg p \in u$ and for all other atoms q of A either q or $\neg q$ $\in u$. Let L be the literal such that $w \in [\![L]\!]$ and $atm(L) = p$, then

$$w \cdot_{Hegner} A = \bigcup_{\substack{u \in LB(A) \\ u \text{ is complete}}} u \cup L$$

which is equal to $w \cdot_{wss\downarrow} A$. $\square$

Lemma 12. $w \cdot_{mpma} A = w \cdot_{wss\downarrow} A$.

Proof. In the two operations, A is put in normal form without redundant atoms $A\downarrow$. Furthermore, $v \in w \cdot_{mpma} A$ is equivalent to $DIST(w, v) \subseteq P$ which is equivalent to $v \in w \cdot_{wss\downarrow} A$. $\square$

It follows from the complexity of *WSS* together with Fact 2 that *WSS*↓ is coNP-complete.

3.7. MCE: Making WSS↓ conservative

Minimal change with exceptions (*MCE*) has also been proposed by Zhang and Foo in [39], based on the same motivation as *MCD*. *MCE* has a more syntactical flavour than *MCD*. Just as in *WSS* and *WSS*↓, the basic idea is that some atoms occurring in the input should be exempted from minimization of change, i.e., they should not count when distance between interpretations is computed. Such a principle being *a priori* very liberal, it has two restrictions: first, redundant input atoms should be eliminated in a preprocessing step, avoiding thus syntax-sensitivity. Second, all those consequences of the input that are not already true in the current interpretation w should not be taken into account. Both restrictions are achieved by computing prime implicates.

More formally, suppose that A is the input. Let D be a covering set of prime implicates of A, and D' the subset of D which contains all clauses not inferred by w:

$$D' = \{c \in D\colon\ w \notin [\![c]\!]\}.$$

Then the set exceptions $EXC(A) = atm(D')$ is exempted from minimization:

$$w \cdot_{mce} A = \big\{u \in [\![A]\!] : \ \forall u' \in [\![A]\!],\ DIST(w, u') \backslash EXC(A) \not\subset DIST(w, u) \backslash EXC(A)\big\}.$$

Note that by Theorem 22 of Section 4 we can simplify:

$$w \cdot_{mce} A = \{u \in [\![A]\!]: DIST(w,u) \subseteq EXC(A)\}.$$

$\diamond_{MCE}$ is the update version of Weber's revision operation [36].

Example 13. Suppose $w = \{\neg p, \neg q, \neg r\}$, and the input A is $(\neg p \vee q) \wedge (p \vee r)$. A covering set of prime implicates for A is $D = \{\neg p \vee q, p \vee r, q \vee r\}$. The set D' is $\{p \vee r, q \vee r\}$. Thus $EXC(A) = \{p, q, r\}$. Hence for all models v of the input, $DIST(w,v) \backslash EXC(A) = \emptyset$. Then $w \cdot_{mce} A = [\![A]\!]$.

MCE is Π^p_2-complete [29].

3.8. WSS^{dep} and $WSS{\downarrow}^{dep}$: Enhancing WSS by dependence information

All the proposals up to now have problems with the handling of integrity constraints. Motivated by that, Herzig defines in [20] a dependence function between atoms in order to correctly handle such constraints. Formally, we suppose given a *dependence function dep* mapping atoms to sets of atoms such that $p \in dep(p)$ for all atoms p. Dependence is extended to general formulas by stipulating $dep(A) = \bigcup_{p \in atm(A)} dep(p)$.

The idea of using dependence information is as follows: suppose p does not depend on A. Then p should 'survive' an update by A: $w \in [\![p]\!]$ should imply $w \cdot_{wss^{dep}} A \subseteq [\![p]\!]$. On the contrary, if p depends on A then it should *a priori* be retracted: we neither expect $w \cdot_{wss^{dep}} A \subseteq [\![p]\!]$ nor $w \cdot_{wss^{dep}} A \subseteq [\![\neg p]\!]$. Then WSS^{dep} is the operation obtained by generalizing the condition $DIST(w,u) \subseteq atm(A)$ of *WSS* to $DIST(w,u) \subseteq dep(A)$:

$$w \cdot_{wss^{dep}} A = \{u \in [\![A]\!]: DIST(w,u) \subseteq dep(A)\}.$$

Indeed, if $dep(p) = \{p\}$ for all atoms p then we obtain the *WSS*.

Example 14. If we have $dep(p) = \{p, r\}$ then we get

$$\{\neg p, \neg q, \neg r\} \cdot_{wss^{dep}} p = \{\{p, \neg q, r\}, \{p, \neg q, \neg r\}\}.$$

It is then straightforward to define also $WSS{\downarrow}^{dep}$ by combining WSS^{dep} and input preprocessing in order to eliminate redundant atoms:

$$w \cdot_{wss\downarrow^{dep}} A = w \cdot_{wss^{dep}} A\downarrow.$$

Liberatore [29] shows that WSS^{dep} is coNP-complete. With Fact 2, we deduce that $WSS{\downarrow}^{dep}$ is coNP-complete, too.

3.9. $MPMA^{\gg}$: Enhancing WSS↓ by causality information

In a similar but different approach, Doherty et al. [9] used causal connections between formulas. Formally, there is a relation $A \gg C$, where A and C are formulas, respectively, referred to as an antecedent and a consequent of the causal rule. This rule has the following intuitive interpretation:

- The formula $A \rightarrow C$ holds in both the initial and the updated knowledge base.

- If $\neg A$ held in the initial knowledge base and A holds in the updated one, then there is a cause for C to hold in the updated knowledge base.

A causal rule $A \gg C$ is said to be active with respect to a pair of interpretations $\langle w, u \rangle$ iff $w \notin [\![A]\!]$ and $u \in [\![A]\!]$, i.e., A is false in w and true in u.

The set of atoms P is now indexed by interpretations and augmented by atoms occurring in the consequents of active causal rules. Suppose that A is the input without redundant atoms and $CR = \{A_i \gg C_i\}$ is the set of causal rules, where A_i and C_i are without redundant atoms. Then

$$P(A, w, u) = atm(A) \cup \bigcup \bigl\{atm(C_i\colon\ A_i \gg C_i \text{ is active with respect to } \langle w, u \rangle\bigr\}.$$

Finally [8]

$$w \cdot_{mpma\gg} A = \bigl\{u \in [\![A]\!],\ DIST(w, u) \subseteq P(A, w, u)\bigr\}.$$

Example 15. Suppose $w = \{\neg p, \neg q, \neg r\}$, the input is $p \wedge q$, and the causal rule is $p \wedge q \gg r$. There are two models of the input: $u_1 = \{p, q, \neg r\}$ and $u_2 = \{p, q, r\}$. The causal rule is active with respect to u_1 and u_2. Then $P_{u_1} = P_{u_2} = \{p, q, r\}$, and $DIST(w, u_1) \subseteq P_{u_1}$ and $DIST(w, u_2) \subseteq P_{u_2}$. Hence $w \cdot_{mpma\gg} p \wedge q = \{u_1, u_2\}$.

It is straightforward to establish that $MPMA^{\gg}$ is coNP-complete (assuming that computing $P(A, w, u)$ is in polynomial time).

4. Putting things in order

What is the relation between all theses updates operations? We give some theorems and lemmas to situate each of them with respect to the others.

Lemma 16. *$w \cdot_{Forbus} [\![A]\!] \subseteq w \cdot_{pma} [\![A]\!]$ for all w and A, and there are w and A such that $w \cdot_{Forbus} [\![A]\!] \neq w \cdot_{pma} [\![A]\!]$.*

Proof. Let $u \in w \cdot_{Forbus} A$, and suppose now that $u \notin w \cdot_{pma} [\![A]\!]$. Then there exists $v \in w \cdot_{pma} [\![A]\!]$ such that $DIST(w, v) \subset DIST(w, u)$. This implies that $card(DIST(w, v)) < card(DIST(w, u))$ which contradicts the hypothesis. That *PMA* and *FORBUS* are different is shown by Example 5. □

Lemma 17. *$w \cdot_{pma} [\![A]\!] \subseteq w \cdot_{mcd} [\![A]\!]$ for all w and A, and there are w and A such that $w \cdot_{pma} [\![A]\!] \neq w \cdot_{mcd} [\![A]\!]$.*

Proof. By definition of *MCD*, and by Example 6. □

Lemma 18. *$w \cdot_{mcd} A \subseteq w \cdot_{mcd*} A$ for all w and A, and there are w and A such that $w \cdot_{mcd} A \neq w \cdot_{mcd*} A$.*

[8] The original definition involves handling of integrity constraints. We have preferred to separate this issue, which is investigate in Section 7.

Proof. By definition of *MCD**, and by Example 7. $\square$

Lemma 19. *There are w and A such that $w \cdot_{mce} A \subset w \cdot_{mcd} [\![A]\!]$ and there are w and A such that $w \cdot_{mcd} [\![A]\!] \subset w \cdot_{mce} A$.* [9]

Proof. In Example 13 given for *MCE*, the model $\{\neg p, q, r\}$ of the input is a model of the base updated under *MCE*, but it is not under *MCD*. In this example we have therefore $w \cdot_{mcd} [\![A]\!] \subset w \cdot_{mce} A$. The other way round, take $w = \{\neg p, \neg q, \neg r\}$ and $[\![A]\!] = \{\{\neg p, q, \neg r\}, \{p, \neg q, \neg r\}, \{p, q, r\}\}$. Then we get $w \cdot_{mce} A = \{\{\neg p, q, \neg r\}, \{p, \neg q, \neg r\}\}$ and $w \cdot_{mcd} [\![A]\!] = [\![A]\!]$. $\square$

Together with Lemma 18, Lemma 19 shows that we cannot compare *MCE* with *MCD* and *MCD**. In order to give a relation between *PMA* and *MCE*, we give a new update operation based on models which is equivalent to *MCE*.

Lemma 20. *If a literal L occurs in a prime implicate of A then there is $w \in [\![A]\!]$ such that $w \in [\![L]\!]$.*

Proof. Suppose that $L \vee c$ is a prime implicate of A, where L is a literal and c is a disjunction of literals. Suppose now that $\forall w \in [\![A]\!], w \in [\![\neg L]\!]$, then $A \to \neg L$. This implies that $A \to c$. And so $L \vee c$ cannot be prime. $\square$

Theorem 21. *Let $PI(A)$ be a covering set of prime implicates of A. Let C_1 be the set of clauses of $PI(A)$ true in w, and $C_2 = PI(A) \backslash C_1$. Then*

$$w \cdot_{pma} [\![C_1 \cup C_2]\!] = w \cdot_{pma} [\![C_2]\!].$$

Proof. Let $L_1 \vee \cdots \vee L_n$ be a clause of C_1, $C = C_1 \backslash (L_1 \vee \cdots \vee L_n) \cup C_2$. We will prove by induction that $w \cdot_{pma} [\![(L_1 \vee \cdots \vee L_n) \cup C]\!] = w \cdot_{pma} [\![C]\!]$.

From left to right. Let $v \in w \cdot_{pma} [\![(L_1 \vee \cdots \vee L_n) \cup C]\!]$. Suppose that $v \notin w \cdot_{pma} [\![C]\!]$ which implies that $\exists\, v'$ such that $v' \in w \cdot_{pma} [\![C]\!]$ and $DIST(w, v') \subset DIST(w, v)$. We have also $v' \notin [\![L_1 \vee \cdots \vee L_n]\!]$, thus $v' \in [\![\neg L_1 \wedge \cdots \wedge \neg L_n]\!]$. By hypothesis, we have $w \in [\![L_1 \vee \cdots \vee L_n]\!]$. In particular, we can consider that $w \in [\![L_1]\!]$. So $atm(L_1) \in DIST(w, v')$. Let's take now the interpretation v'' such that $DIST(v', v'') = atm(L_1)$. In other words, v' and v'' differ only in L_1 and $DIST(w, v'') \subset DIST(w, v')$. We will prove now that $v'' \in [\![C]\!]$ which permits to conclude that $v' \notin w \cdot_{pma} [\![C]\!]$. For every clause in C, if it does not contain $atm(L_1)$ then v'' is a model for it, it is also a model if the clause is of the form $L_1 \vee \cdots$. If the clause is of the form $\neg L_1 \vee R$, where R is a disjunction of literals not containing L_1 then because of the structure of C, the clauses $(L_2 \vee \cdots \vee L_n \vee R) \in C$. This implies that $v' \in [\![R]\!]$. v' and v'' differ only by L_1, and R does not contain L_1, hence $v'' \in [\![R]\!]$, too. Thus $v'' \in [\![C]\!]$.

From right to left. Let $v \in w \cdot_{pma} [\![C]\!]$. If $v \in [\![L_1 \vee \cdots \vee L_n]\!]$ then $v \in w \cdot_{pma} [\![(L_1 \vee \cdots \vee L_n) \cup C]\!]$. Suppose that $v \notin [\![L_1 \vee \cdots \vee L_n]\!]$. Let v' be the interpretation

[9] Contrary to what is said in [39], where it is stated that $\diamond_{MCE}$ and $\diamond_{MCD}$ are the same operation.

such that $DIST(v, v') = atm(L_1)$, thus we have $v' \in [\![L_1 \vee \cdots \vee L_n]\!]$ and $DIST(w, v') \subset DIST(w, v)$. For every clause in C, if it doesn't contain $atm(L_1)$ or it is of the form $L_1 \vee \cdots$ then v' is a model for it. If it is of the form $\neg L_1 \vee R$ then like above C contains clauses of the form $L_2 \vee \cdots \vee L_n \vee R$. By hypothesis $v \in [\![C]\!]$. Hence $v \in [\![R]\!]$ and because R does not contain L_1 we have $v' \in [\![R]\!]$. Thus $v' \in [\![C]\!]$, which implies that $v \notin w \cdot_{pma} [\![C]\!]$. □

Theorem 22. *Let $PI(A)$ be a covering set of prime implicates of A. Let C_1 be the set of clauses of $PI(A)$ true in w, and let $C_2 = PI(A) \backslash C_1$. Then*

$$atm(C_2) = \bigcup_{v \in w \cdot_{pma} [\![A]\!]} DIST(w, v).$$

Proof. Note that $EXC(A) = atm(C_2)$.

From left to right. Let $p \in atm(C_2)$ then $\exists\, c \in C_2$ such that

$$p \in atm(c), \quad \text{and by hypothesis} \quad w \in [\![\neg c]\!]. \tag{1}$$

We must prove that $\exists\, v \in w \cdot_{pma} [\![A]\!]$ such that $p \in DIST(w, v)$. Suppose the contrary, then

$$\forall v \in w \cdot_{pma} [\![A]\!], \quad p \notin DIST(w, v). \tag{2}$$

(1) implies that $c \leftrightarrow (p \vee c')$, $w \in [\![\neg p]\!]$ and $w \in [\![\neg c']\!]$.
(2) implies that

$$\forall v \in w \cdot_{pma} [\![A]\!],\ v \in [\![\neg p]\!] \quad \text{and} \quad v \in [\![c']\!]. \tag{3}$$

Let $u \in [\![A]\!]$ such that $u \in [\![p]\!]$ (u exists, Lemma 20) then (3) implies that $\exists v, DIST(w, v) \subset DIST(w, u)$, $w \in [\![\neg c']\!]$ and $v \in [\![c']\!]$, thus $\exists q \in DIST(w, v)$ such that q causes that $v \in [\![c']\!]$ and $q \in DIST(w, u)$, which implies that $u \in [\![c']\!]$. Finally we conclude that $\forall u \in [\![A]\!]$ such that $u \in [\![p]\!]$ we have $u \in [\![c']\!]$. And for all $u \in [\![A]\!]$ such that $u \in [\![\neg p]\!]$ we have $u \in [\![c']\!]$. Thus $\forall u \in [\![A]\!], u \in [\![c']\!]$ which implies that $c' \in C_2$. Hence by definition of C_2, $p \vee c'$ could not be in C_2.

From right to left. Let

$$p \in \bigcup_{v \in w \cdot_{pma} [\![A]\!]} DIST(w, v),$$

then $\exists v \in w \cdot_{pma} [\![A]\!]$ such that $p \in DIST(w, v)$. We suppose that $w \in [\![\neg p]\!]$, and $v \in [\![p]\!]$. We must prove that $\exists c \in PI(A)$ such that $p \in atm(c)$ and $w \in [\![c]\!]$ (i.e., $c \in C_2$).

Suppose that there is no clause in $PI(A)$ containing p, so the truth value of A is independent of that of p, thus for every $u \in [\![A]\!]$ such that $u \in [\![p]\!]$ there exists $u' \in [\![A]\!]$ such that $u \in [\![\neg p]\!]$ and $DIST(u, u') = \{p\}$. Which implies that there exists $v' \in [\![A]\!]$ such that $DIST(v, v') = \{p\}$, and $DIST(w, v') \subset DIST(w, v)$ which implies that $v \notin w \cdot_{pma} [\![A]\!]$. Thus there exists $c \in PI(A)$ such that $p \in atm(c)$.

It remains to prove that for some $c \in PI(A)$ containing p, $w \in [\![\neg c]\!]$. Suppose the contrary, i.e., for all $c \in PI(A)$ containing p, $w \in [\![c]\!]$. Let c_2 be the conjunction of clauses of $PI(A)$ that are false in w, and let c_1 be the conjunction of the rest. By Theorem 21, we have $w \cdot_{pma} (c_1 \wedge c_2) = w \cdot_{pma} c_2$. Thus $v \in [\![w \cdot_{pma} c_2]\!]$. Like above, there exists $v' \in [\![w \cdot_{pma} c_2]\!]$, such that $DIST(v, v') = \{p\}$, and $DIST(w, v') \subset DIST(w, v)$, which contradicts the hypothesis. □

Theorem 23. *$w \cdot_{pma} [\![A]\!] \subseteq w \cdot_{mce} A$ for all w and A, and there are w and A such that $w \cdot_{pma} [\![A]\!] \neq w \cdot_{mce} A$.*

Proof. Let $v \in w \cdot_{pma} [\![A]\!]$. Then by Theorem 22 $DIST(w, v) \subseteq EXC(A)$, so $DIST(w, v) \setminus EXC(A) = \emptyset$ which implies that $v \in w \cdot_{mce} A$. That *PMA* and *MCE* are different is shown by Example 13. (It follows also from the inclusion in Lemma 17 and 18 and the non-inclusion in Lemma 19.) □

Theorem 24. *$w \cdot_{mce} A \subseteq w \cdot_{wss} A$ for all w and A, and there are w and A such that $w \cdot_{mce} A \neq w \cdot_{wss} A$.*

Proof. Let $u \in w \cdot_{mce} A$ then $DIST(w, u) \subseteq EXC(A)$ and we have $EXC(A) \subseteq atm(A)$. Thus $u \in w \cdot_{wss} A$. The following example shows the difference.

Example 25. Suppose $w = \{\neg p, \neg q, \neg r\}$ and the input A is $(\neg p \vee q) \wedge (q \vee r)$. $w \cdot_{wss} A = [\![A]\!]$ and $w \cdot_{mce} A = [\![A]\!] \setminus \{\{p, q, \neg r\}, \{p, q, r\}\}$.

This ends the proof of the theorem. □

5. The status of the KM-postulates

Katsuno and Mendelzon [24] have proposed eight postulates that they claim every rational update operation should satisfy. They have established that the *PMA* and *FORBUS* satisfy all the KM-postulates. In this section we check each of these postulates with respect to the update operations that we have introduced. For completeness we also prove the results for the *PMA* and *FORBUS* that were already in [24].

We also discuss the plausibility of those postulates that are controversial with respect to our official reading of updates. [10]

In the end of the section we give a characterization result for WSS^{dep} in terms of a set of postulates.

5.1. (U1) $B \diamond A \rightarrow A$

(U1) stipulates input priority: whatever the content of B is, every update satisfies the input. ((U1) has also been called the success postulate.)

This is an uncontroversial postulate. [11]

Theorem 26. (U1) *is satisfied by every update operation.*

Proof. Straightforward. □

[10] There is a ninth postulate that is only satisfied by *FORBUS* and Boutilier's approach [1]. We do not discuss it here because it requires total orders, while in the sequel we shall criticise already the partial order semantical base of the KM framework.

[11] There are criticisms arguing that input has not necessarily priority over the base. But this means that we are rather speaking about belief base fusion. Clearly such operations obey different postulates [26].

5.2. (U2) *If* $B \to A$ *then* $B \diamond A \leftrightarrow B$

(U2) says that nothing needs to be changed if the input is vacuous in the sense that it is already among the consequences of the base. (U2) can be decomposed into two postulates "If $B \to A$ then $B \to B \diamond A$" and "If $B \to A$ then $B \diamond A \to B$".

Lemma 27. (U2) *is equivalent to*

(U2.1) $B \wedge A \to B \diamond A$
(U2.2) $(B \wedge A) \diamond A \to B$.

Proof. If we replace B by $B \wedge A$ in the first postulate then we get $B \wedge A \to B \diamond A$. The other way round (U2.1) and $B \to A$ imply that $B \to B \diamond A$.

To prove (U2.2) from the second postulate, it is sufficient to replace B by $B \wedge A$ in the latter. The other way round, (U2.2) and $B \to A$ imply that $B \diamond A \to B$. □

Theorem 28. (U2.1) *is satisfied by every update operation.*

Proof. It suffices to prove that if $w \in [\![A]\!]$ then $w \in w \cdot A$. Due to the ordering strength on the operations that we have established in the previous section, it is sufficient to prove that *FORBUS* satisfies (U2.1). Let $w \in [\![A]\!]$. Then $card(DIST(w, w)) = 0$ and $w \cdot_{Forbus} [\![A]\!] = \{w\}$. □

Theorem 29. (U2.2) *is satisfied exactly by FORBUS, PMA, MCD, MCD*, MCE.*

Proof. Due to the structure of the update operations in order to prove that *FORBUS*, *PMA*, *MCD*, *MCD**, *MCE* satisfy (U2.2), it is sufficient to prove that *MCD** and *MCE* satisfy (U2.2).

Let $v \in [\![(B \wedge A) \diamond_{MCD^*} A]\!]$, then $\exists w \in [\![B \wedge A]\!]$ such that $v \in w \cdot_{mcd^*} [\![A]\!]$. As $w \cdot_{pma} [\![A]\!] = \{w\}$ (because $DIST(w, w) = \emptyset$), we must have $w \cdot_{mcd^*} [\![A]\!] = \{w\}$. Thus $v = w$ and $v \in [\![B]\!]$.

Now, let $v \in [\![(B \wedge A) \diamond_{MCE} A]\!]$, then $\exists w \in [\![B \wedge A]\!]$ such that $v \in w \cdot_{mce} A$. As $w \in [\![A]\!]$, we must have $EXC(A) = \emptyset$, and hence $w \cdot_{mce} A = \{w\}$. Thus $v = w$ and $v \in [\![B]\!]$.

To see that the other operations do not satisfy (U2.2), consider the base p and the input $p \vee q$. We have $p \diamond (p \vee q) = p \vee q$, for $\diamond = \diamond_{WSS\downarrow}$ or any weaker operation (in particular, for every dependence relation). □

Corollary 30. (U2) *is satisfied exactly by FORBUS, PMA, MCD, MCD*, MCE.*

The instance of (U2) where A is tautologous is uncontroversial:

Theorem 31. *Every update operator satisfies (U2$^\top$)* $B \diamond \top \leftrightarrow B$.

Discussion

So it is (U2.2) that is controversial, while (U2.1) is not. We agree with the viewpoints of [1,2,9,10] and others, who have argued that (U2) should be abandoned. Consider the example in the proof of Theorem 29 discriminating violation of (U2.2). If we take seriously our hypotheses of Section 1.3 then $p \diamond (p \vee q)$ cannot be equivalent to p. Indeed, suppose an agent believes p. Suppose later on he gets sensor information expressed by the formula $p \vee q$. Given that time has passed, maintaining belief in p means that the agent prefers to consider that sensing got noisy in an otherwise unchanged world, instead of correct sensing in an evolving world. But it is at least unjustified to always make such a hypothesis.

To witness, suppose with [2] that p means that a certain coin shows heads, and q that it shows tails, and that the agent perceives that another agent grasps the coin and tosses it (but without perceiving the outcome). Then clearly $p \diamond (p \vee q)$ should be $p \vee q$, and not p.

Our rejection is supported by the discussion on the conditional logic principles (MP) and (CS), that are, respectively, equivalent to (U2.1) and (U2.2) [21,34]. (MP) is generally accepted, while (CS) is rejected by most of the authors [28,31,32].

5.3. (U3) *If B and A are consistent, then $B \diamond A$ is consistent*

The consistency postulate says that inconsistency of the updated base comes from either inconsistency of the original base, or inconsistency of the input.

(U3) is uncontroversial.

Theorem 32. (U3) *is satisfied by every update operation.*

Proof. The proof is straightforward. Note that for *WSS*^{dep} and *WSS*$\downarrow^{dep}$ it is essential that $p \in dep(p)$. □

5.4. (U4) *If $B_1 \leftrightarrow B_2$ and $A_1 \leftrightarrow A_2$ then $B_1 \diamond A_1 \leftrightarrow B_2 \diamond A_2$*

(U4) stipulates that update operations should not be sensitive to the syntactical formulation of the belief base and the input. This has been called the principle of irrelevance of syntax in [5].

It would make no sense to accept that principle for belief bases without accepting it for the input. We nevertheless decompose (U4) into

(U4.1) If $B_1 \leftrightarrow B_2$ then $B_1 \diamond A \leftrightarrow B_2 \diamond A$
(U4.2) If $A_1 \leftrightarrow A_2$ then $B \diamond A_1 \leftrightarrow B \diamond A_2$.

Theorem 33. (U4.1) *is satisfied by every update operation.*

Proof. Suppose $B_1 \leftrightarrow B_2$. For all update operations $\diamond$, belief bases are updated modelwise. Therefore $[\![B_1 \diamond A]\!] = \bigcup_{w \in [\![B_1]\!]} w \cdot A$ and $[\![B_2 \diamond A]\!] = \bigcup_{w \in [\![B_2]\!]} w \cdot A$. As $[\![B_1]\!] = [\![B_2]\!]$, $[\![B_1 \diamond A]\!]$ and $[\![B_2 \diamond A]\!]$ are equal. □

Theorem 34. (U4.2) *is satisfied exactly by FORBUS, PMA, MCD, MCD*, MCE, WSS*$\downarrow$, *WSS*$\downarrow^{dep}$, *MPMA, MPMA*$^{\gg}$.

Proof. For *FORBUS*, *PMA*, *MCD*, *MCD**, this follows from the fact that not the input formula is considered but its set of models. For the rest, this follows from the fact that redundant atoms are eliminated from inputs (cf. Section 3.6). □

Discussion

We think that (U4) is a desirable property. Without such a requirement the result of an update is "extremely dependent on the syntactic form of the belief base. Even apparently meaningless distinctions ... have an effect" [6]. For example, the belief bases $p \wedge q$, $p \wedge p \wedge q$, and $q \wedge p$ could be updated differently. We refer to, e.g., [6] for a detailed critique. The same arguments support syntax-independence in the input.

We stress that the syntax-sensitive update operations are so in a weak way only, and can be 'saved'. Indeed, all approaches satisfy the weaker

(U4.2′) If $A_1 \leftrightarrow A_2$ and $atm(A_1) = atm(A_2)$ then $B \diamond A_1 \leftrightarrow B \diamond A_2$.

Now there is a simple way to enforce (U4.2): just put to work Fact 1 and eliminate redundant atoms from the input. This exactly what is done in *WSS*↓, *WSS*$\downarrow^{dep}$, *MPMA*$^{\gg}$.

5.5. (U5) $(B \diamond A) \wedge C \to B \diamond (A \wedge C)$

The postulate says that an update by $A \wedge C$ is weaker than just adding C to the update by A. In particular suppose B and $A \wedge C$ are both consistent. Then $B \diamond (A \wedge C)$ is consistent under (U3), while there is no guarantee that $(B \diamond A) \wedge C$ is consistent. (For example take p for B, $\top$ for A, and $\neg p$ for C.)

Lemma 35. (U5) *is equivalent in the presence of* (U1) *and* (U4) *to*

(U5′) $B \diamond (A_1 \vee A_2) \to (B \diamond A_1) \vee (B \diamond A_2)$. [12]

Proof ([16, Formula 3.14]). First, (U5′) entails (U5): $B \diamond A_1 \leftrightarrow B \diamond ((A_1 \wedge A_2) \vee (A_1 \wedge \neg A_2))$ by (U4). By (U5′),

$$B \diamond ((A_1 \wedge A_2) \vee (A_1 \wedge \neg A_2)) \to B \diamond (A_1 \wedge A_2) \vee B \diamond (A_1 \wedge \neg A_2).$$

Hence

$$B \diamond A_1 \wedge A_2 \to (B \diamond (A_1 \wedge A_2) \wedge A_2) \vee (B \diamond (A_1 \wedge \neg A_2) \wedge A_2)$$

by classical logic. Then (U5) follows because $B \diamond (A_1 \wedge \neg A_2) \wedge A_2$ is inconsistent due to (U1).

In the other sense, we start with a consequence of (U1):

$$B \diamond (A_1 \vee A_2) \to B \diamond (A_1 \vee A_2) \wedge (A_1 \vee A_2).$$

By classical logic,

$$B \diamond (A_1 \vee A_2) \to (B \diamond (A_1 \vee A_2) \wedge A_1) \vee (B \diamond (A_1 \vee A_2) \wedge A_2).$$

[12] (U5′) is the counterpart of the conditional logic axiom "conjunction in the antecedent" (CA) [21,34].

Now (U5) gives us

$$B \diamond (A_1 \vee A_2) \wedge A_1 \to (B \diamond ((A_1 \vee A_2) \wedge A_1),$$

and

$$B \diamond (A_1 \vee A_2) \wedge A_2 \to (B \diamond ((A_1 \vee A_2) \wedge A_2).$$

By (U4), the consequences of the first material implication becomes $B \diamond A_1$, and that of the second $B \diamond A_2$. Putting things together, we obtain $B \diamond (A_1 \vee A_2) \to B \diamond A_1 \vee B \diamond A_2$. □

Although it is claimed in [39] that *MCD* and *MCE* satisfy (U5), this is not the case. Indeed, the following example shows that *MCD* violates (U5′).

Example 36. Suppose *MCD* satisfies (U5′). Then we should have

$$\{\neg p, \neg q\} \cdot_{mcd} [\![(p \vee q)]\!] \subseteq (\{\neg p, \neg q\} \cdot_{mcd} [\![p]\!]) \cup (\{\neg p, \neg q\} \cdot_{mcd} [\![q]\!]).$$

We have seen that $\{\neg p, \neg q\} \cdot_{mcd} [\![(p \vee q)]\!] = [\![p \vee q]\!]$, and it is easy to check that

$$(\{\neg p, \neg q\} \cdot_{mcd} [\![p]\!]) = [\![p \wedge \neg q]\!] \text{ and } (\{\neg p, \neg q\} \cdot_{mcd} [\![q]\!]) = [\![\neg p \wedge q]\!].$$

Then by the above we obtain $p \vee q \to (p \wedge \neg q) \vee (\neg p \wedge q)$, which is not valid.

The same example shows that *MCE* also violates (U5). The same is the case for *WSS*, *WSS*$\downarrow$, *WSS*^{dep}, *WSS*$\downarrow^{dep}$, *MPMA*, *MPMA*$^{\gg}$.

Theorem 37. *Only FORBUS and PMA satisfy* (U5).

Proof. The above example has already demonstrated that the other approaches do not satisfy (U5).

We prove that the *PMA* satisfies (U5). Let $v \in [\![B \diamond_{PMA} A \wedge C]\!]$ then there exists $w \in [\![B]\!]$ such that $v \in w \cdot_{pma} [\![A]\!] \cap [\![C]\!]$. Suppose that $v \notin w \cdot_{pma} [\![A \wedge C]\!]$ then there exists $v' \in w \cdot_{pma} [\![A \wedge C]\!]$ such that $DIST(w, v') \subset DIST(w, v)$. This contradicts that $v \in w \cdot_{pma} [\![A]\!]$.

A similar proof can be sketched for *FORBUS*. □

Discussion

We start with an innocent-looking consequence of (U5).[13]

Lemma 38 [22]. (U1) *and* (U5) *entail*

(Exor) If $B \diamond A_1 \to \neg A_2$ *and* $B \diamond A_2 \to \neg A_1$ then
$B \diamond (A_1 \vee A_2) \to A_1 \oplus A_2$.

Proof. Suppose that $B \diamond A_1 \to \neg A_2$, and $B \diamond A_2 \to \neg A_1$. By (U5′) (which follows from (U1) and (U5)), $B \diamond (A_1 \vee A_2) \to B \diamond A_1 \vee B \diamond A_2$. By (U1), $B \diamond A_1 \to A_1$ and

[13] The link between (U5) and the property (Exor) of the lemma had not been noticed before [22], and, e.g., [39] claim that *MCD* satisfies (U1) and (U5) while avoiding an exclusive interpretation of the inclusive or.

by hypothesis $B \diamond A_1 \to \neg A_2$. Thus $B \diamond A_1 \to A_1 \wedge \neg A_2$. Symmetrically $B \diamond A_2 \to A_2 \wedge \neg A_1$. Finally we have

$$B \diamond (A_1 \vee A_2) \to (A_1 \wedge \neg A_2) \vee (A_2 \wedge \neg A_1),$$

which is equivalent to $B \diamond (A_1 \vee A_2) \to A_1 \oplus A_2$. □

The condition of (Exor) expresses that A_1 and A_2 are in some sense independent of each other.

(Exor) is undesirable under the hypotheses of Section 1. To witness, consider the case where $B \to \neg A_2$ and $B \to \neg A_1$. Then independence corresponds to the preservation of $\neg A_2$ under update by A_1 (and vice versa), i.e., A_1 does not undermine the belief that $\neg A_2$. According to (Exor), an update by the inclusive disjunction always leads to the exclusive disjunction. But why should new information about an event such that $A_1 \vee A_2$ make us exclude that $A_1 \wedge A_2$? There seems to be no reason for that. This is illustrated by an adaptation of Reiter's example against the *PMA* (which is an instance of Example 36).

Example 39. Suppose you throw a coin on a chessboard. Suppose *Black* means "the coin falls on a black field", and *White* "the coin falls on a white field". The coin might fall either on a black field, or on a white field, or on both of them. Initially we hold the coin, i.e., B is $\neg Black \wedge \neg White$. If we observe that the coin falls on a white field we should keep on believing that $\neg Black$, and vice versa. This means that $B \diamond Black \to \neg White$, and $B \diamond White \to \neg Black$. Now suppose that (due to our distance to the chessboard or the observation angle) we can only see that the coin fell down, without perceiving its position. This corresponds to an update with $Black \vee White$. (Exor) tells us that we should have $B \diamond (Black \vee White) \to Black \oplus White$, i.e., the coin cannot touch a black and a white field at the same time. This is clearly unintuitive.

As (U1) is uncontroversial, (U5) is the culprit and must be declared undesirable.

Remark 40. Circumscription and other minimization-based approaches behave in the same way [12]. We note that this does not mean that (U5) and the exclusive interpretation of disjunctions must be abandoned in the context of belief base revision and also nonmonotonic reasoning. [14]

5.6. (U6) *If* $(B \diamond A_1) \to A_2$ *and* $(B \diamond A_2) \to A_1$ *then* $B \diamond A_1 \leftrightarrow B \diamond A_2$

This means that if A_1 and A_2 are 'equivalent under B' then they lead to the same update. [15]

Lemma 41 [21]. (U6) *is equivalent in the presence of* (U1) *and* (U5) *to*

(U6′) *If* $B \diamond A_1 \to A_2$ *then* $B \diamond (A_1 \wedge A_2) \to B \diamond A_1$.

[14] Thanks to David Makinson for an enlightening discussion on that point.

[15] (U6) corresponds to the well-known conditional logic principle (CSO) [21,34].

Proof. First (U6′) entails (U6): suppose we have $B \diamond A_1 \to A_2$. By (U6′) we have $B \diamond (A_1 \wedge A_2) \to B \diamond A_1$. Suppose now that we have $B \diamond A_2 \to A_1$. By (U5) we obtain $(B \diamond A_2) \wedge A_1 \to B \diamond (A_1 \wedge A_2)$. Hence $B \diamond A_2 \to B \diamond (A_1 \wedge A_2)$. Thus $(B \diamond A_2) \to B \diamond A_1$. By symmetry we obtain $B \diamond A_1 \to B \diamond A_2$.

In the other sense, by (U1) we have $B \diamond (A_1 \wedge A_2) \to A_1 \wedge A_2$. Hence $B \diamond (A_1 \wedge A_2) \to A_1$. By hypothesis, we have $B \diamond A_1 \to A_2$. Hence $B \diamond A_1 \to A_1 \wedge A_2$. Then (U6) implies $B \diamond A_1 \leftrightarrow B \diamond (A_1 \wedge A_2)$. □

The preceding proof that (U6) entails (U6′) shows that (U6) entails (U2) in the presence of (U2$^\top$): $B \diamond \top \leftrightarrow B$ and (U1). As well (U6) implies (U4.2) in the presence of (U1).

Theorem 42. *Only FORBUS, PMA, MCD, MCD* and MCE satisfy* (U6). [16]

Proof. Suppose that the *PMA* does not satisfy the postulate, and suppose then there exists $v \in [\![B \diamond_{PMA} A_1]\!]$ such that $v \notin [\![B \diamond_{PMA} A_2]\!]$. But by hypothesis $v \in [\![A_2]\!]$. This implies that $\exists w \in [\![B]\!]$ such that $v \in w \cdot_{pma} [\![A_1]\!]$, and in particular $v \notin w \cdot_{pma} [\![A_2]\!]$ so there exists $v' \in w \cdot_{pma} [\![A_2]\!]$, $DIST(w, v') \subset DIST(w, v)$ and we have $v' \in [\![A_1]\!]$ by hypothesis, which contradicts that $v \in w \cdot_{pma} [\![A_1]\!]$.

For *MCD*, suppose idem that $v \in w \cdot_{mcd} [\![A_1]\!]$, and $v \notin w \cdot_{mcd} [\![A_2]\!]$. There are two cases. For $v \in w \cdot_{pma} [\![A_1]\!]$, we have $v \in w \cdot_{pma} [\![A_2]\!]$ and so $v \in w \cdot_{mcd} [\![A_2]\!]$. If it is not the case then there exists $v_1, \ldots, v_n \in w \cdot_{pma} [\![A_1]\!]$ such that $DIST(w, v_i) \subset DIST(w, v)$ for all i and for all $v' \in [\![A_1]\!]$ such that $DIST(w, v_i) \subset DIST(w, v')$ for all i, we have $DIST(w, v') \not\subset DIST(w, v)$. Now, by hypothesis $v_1, \ldots, v_n, v \in [\![A_2]\!]$, $v_1, \ldots, v_n \in w \cdot_{pma} [\![A_2]\!]$ (because the *PMA* satisfies (U6)), and $v \notin w \cdot_{mcd} [\![A_2]\!]$. Then $\exists v' \in w \cdot_{mcd} [\![A_2]\!]$ as v above except that $DIST(w, v') \subset DIST(w, v)$. As $v' \in [\![A_1]\!]$ by hypothesis, this implies that $v \notin w \cdot_{mcd} [\![A_1]\!]$ [39].

For *MCD**, we prove that by induction on the number of steps in the recursive definition of $w \cdot_{mcd*} [\![A]\!]$. The base is proved by *MCD*. Suppose now that until step p, (U6) is satisfied. Hence, we have $v \in w \cdot_{mcd*} [\![A_1]\!]$ at step $p + 1$, and there exists $v_1, \ldots, v_n \in w \cdot_{mcd*} [\![A_1]\!]$ at step p such that $DIST(w, v_i) \subset DIST(w, v)$ for all i and for all v' such that $DIST(w, v_i) \subset DIST(w, v')$ for all i, we have $DIST(w, v') \not\subset DIST(w, v)$. Now, by hypothesis $v_1, \ldots, v_n, v \in [\![A_2]\!]$, $v_1, \ldots, v_n \in w \cdot_{mcd*} [\![A_2]\!]$ (because *MCD** satisfies (U6) at step p). Then $v \in w \cdot_{mcd*} [\![A_2]\!]$ (because, if it is not the case then there exists $v' \in w \cdot_{mcd*} [\![A_2]\!]$ as v above except that $DIST(w, v') \subset DIST(w, v)$. As $v' \in [\![A_1]\!]$ by hypothesis, this implies that $v \notin w \cdot_{mcd*} [\![A_1]\!]$).

For *MCE*, the proof follows from Theorem 22: let $v \in [\![B \diamond_{MCE} A_1]\!]$. This implies that there is $w \in [\![B]\!]$ such that $v \in w \cdot_{mce} A_1$. Hence $DIST(w, v) \backslash EXC(A_1) = \emptyset$. where $EXC(A_1)$ represents the set of exceptions with respect to w and A_1. Suppose $w \cdot_{mce} A_1 \subseteq [\![A_2]\!]$, and $w \cdot_{mce} A_2 \subseteq [\![A_1]\!]$. Because $w \cdot_{pma} [\![A_1]\!] \subseteq w \cdot_{mce} [\![A_1]\!]$, $w \cdot_{pma} [\![A_2]\!] \subseteq w \cdot_{mce} A_2$ and the *PMA* satisfies (U6) then $w \cdot_{pma} [\![A_1]\!] = w \cdot_{pma} [\![A_2]\!]$. By Theorem 22,

$$EXC(A_1) = \bigcup_{v \in w \cdot_{pma} [\![A_1]\!]} DIST(w, v).$$

[16] In [40] only a weaker variant of (U6) is proved to hold for *MCE*.

This implies that $EXC(A_1) = EXC(A_2)$. By hypothesis $v \in [\![A_2]\!]$, and

$$DIST(w, v) \backslash EXC(A_2) = \emptyset$$

which implies that $v \in w \cdot_{mce} A_2$. Hence $v \in [\![B \diamond_{MCE} A_2]\!]$.

WSS, *WSS*↓, WSS^{dep}, $WSS{\downarrow}^{dep}$, *MPMA* and $MPMA^{\gg}$ do not satisfy (U6) because they do not satisfy (U2). (We recall that all these operations satisfy (U1) and ($U2^{\top}$), and that we have shown in the begining of the section that an update operator satisfying (U1), ($U2^{\top}$) and (U6) also satisfies (U2).) □

Discussion

(U6) entails (U2) in the presence of ($U2^{\top}$) [21]. As ($U2^{\top}$) is uncontroversial, our criticism of (U2) transfers to (U6).

5.7. (U7) *If B is complete then* $(B \diamond A_1) \wedge (B \diamond A_2) \to B \diamond (A_1 \vee A_2)$

A formula B is complete iff for every formula A, either $B \to A$ or $B \to \neg A$. Therefore (U7) makes sense only if the language is finite.

Without (U7) the postulates would not completely characterize the models given by Katsuno and Mendelzon. This is probably the *raison d'être* of this postulate, which is difficult to explain intuitively.

Theorem 43. (U7) *is satisfied exactly by FORBUS, PMA, WSS and* WSS^{dep}.

Proof. For *FORBUS*, let $u \in w \cdot_{Forbus} [\![A_1]\!] \cap w \cdot_{Forbus} [\![A_2]\!]$. This implies for every $v \in [\![A_1]\!]$,

$$card(DIST(w, v)) \geqslant card(DIST(w, u))$$

and for every $v \in [\![A_2]\!]$,

$$card(DIST(w, v)) \geqslant card(DIST(w, u)).$$

Thus for every $v \in [\![A_2]\!] \cup [\![A_2]\!] = [\![A_1 \vee A_2]\!]$, $card(DIST(w, v)) \geqslant card(DIST(w, u))$, which implies that $u \in w \cdot_{Forbus} [\![A_1 \vee A_2]\!]$.

For *PMA*, let $u \in w \cdot_{pma} [\![A_1]\!] \cap w \cdot_{pma} [\![A_2]\!]$ which implies for every $v \in [\![A_1]\!]$,

$$DIST(w, v) \not\subseteq DIST(w, u)$$

and for every $v \in [\![A_2]\!]$, $DIST(w, v) \not\subseteq DIST(w, u)$. Thus for every $v \in [\![A_2]\!] \cup [\![A_2]\!] = [\![A_1 \vee A_2]\!]$, $DIST(w, v) \not\subseteq DIST(w, u)$, which implies that $u \in w \cdot_{pma} [\![A_1 \vee A_2]\!]$.

For *WSS* and WSS^{dep} the proofs follows from the fact that

$$atm(A_1 \vee A_2) = atm(A_1) \cup atm(A_2).$$

The rest of the update operations does not satisfy (U7). The following example is a counterexample for *MCD* and *MCD**.

Example 44. Suppose that $[\![A_1]\!] = \{\{p, q, \neg r\}, \{p, \neg q, r\}, \{p, q, r\}\}$, $[\![A_2]\!] = \{\{p, \neg q, \neg r\}, \{\neg p, q, \neg r\}, \{p, q, r\}\}$ and

$$w = \{\neg p, \neg q, \neg r\}. \ \{p, q, r\} \in w \cdot_{mcd} [\![A_1]\!] \cap w \cdot_{mcd} [\![A_2]\!]$$

and

$$w \cdot_{mcd} [\![A_1 \vee A_2]\!] = \{\{p, \neg q, \neg r\}, \{\neg p, q, \neg r\}, \{p, q, \neg r\}\}.$$

We have the same result for *MCD**.

The following example is a counterexample for *MCE*. [17]

Example 45. Suppose that $[\![A_1]\!] = \{\{p, \neg q, \neg r\}, \{\neg p, q, r\}, \{p, q, r\}\}$, $[\![A_2]\!] = \{\{\neg p, q, \neg r\}, \{p, \neg q, r\}, \{p, q, r\}\}$ and

$$w = \{\neg p, \neg q, \neg r\}. \ \{p, q, r\} \in w \cdot_{mce} [\![A_1]\!] \cap w \cdot_{mce} [\![A_2]\!]$$

and $w \cdot_{mce} [\![A_1 \vee A_2]\!] = \{\{p, \neg q, \neg r\}, \{\neg p, q, \neg r\}\}$.

The following example is a counterexample for *WSS*$\downarrow$, *WSS*$\downarrow^{dep}$, *MPMA*$^{\gg}$.

Example 46. Suppose that

$$\begin{aligned} w = \{\neg p, \neg q\}. \ & w \cdot_{wss\downarrow} (p \vee q) \cap w \cdot_{wss\downarrow} (\neg p \vee \neg q) \\ &= \{\{\neg p, q\}, \{p, \neg q\}\}. \ w \cdot_{wss\downarrow} (p \vee q \vee \neg p \vee \neg q) = w \cdot_{wss\downarrow} \top = w. \end{aligned}$$

This ends the proof of the theorem. □

Discussion

We claim that (U7) is almost meaningless: to consider complete belief bases does not 'give' us very much, because even for finite languages belief bases are in general incomplete, in the sense that they do not completely describe the actual world, but rather several possible ones.

5.8. (U8) $(B_1 \vee B_2) \diamond A \leftrightarrow (B_1 \diamond A) \vee (B_2 \diamond A)$

(U8) is the postulate corresponding to modelwise updating, which is at the base of the standard explanation of updates.

(U8) is uncontroversial.

Theorem 47. (U8) *is satisfied by every update operation.*

Proof. By definition of an update. □

[17] This contradicts [39], where it is said that *MCE* satisfies (U7).

5.9. From postulates to characterization theorems

Beyond property checking, one might ask whether it is possible to exactly characterize the different operations by identifying further postulates. The only operations for which this has been done are those of WSS^{dep} (of which *WSS* is a particular case). [18]

Theorem 48 (Characterization of WSS^{dep} [22]).[19] *Suppose given a dependence relation dep and an update operation $\diamond$. Then $\diamond$ is the WSS^{dep}operator iff $\diamond$ satisfies* (U1), (U2.1), (U4.1), (U8), *and*

(NI1) $(B_1 \wedge B_2) \diamond A \leftrightarrow B_1 \wedge (B_2 \diamond A)$ *if* $dep(A) \cap atm(B_1) = \emptyset$

(NI2) $A \rightarrow (B \diamond A)$ *if* $atm(B) \subseteq dep(A)$ *and* B *is consistent.*

(NI1) and (NI2) are new postulates that are parametrized by dependence. We note that language finiteness is not required for our axiomatization. Note also that ($U2^{\top}$) follows from (NI1) and (NI2).

It has been shown that the above characterization can be turned in a decision procedure [22].

6. The problem of disjunctive input

In the preceding section we have already pointed out that (U5) is closely related to an undesirable exclusive interpretation of the inclusive disjunction. [20]

As the *PMA* and *FORBUS* satisfy all the KM-postulates they are immediately subject to our criticism. This can be illustrated by Example 39.

The other approaches have all been designed in a way such that (Exor) is avoided. In the rest of the section we show that failure of (U5) and (Exor) is not enough to guarantee correct handling of disjunctive input, and that *MCD*, *MCD** and *MCE* fail to capture the intuitions that have been put forward. The arguments are in terms of toy examples. *WSS* and its relatives $WSS{\downarrow}$, $WSS{\downarrow}^{dep}$ and $MPMA^{\gg}$ give the intuitive result at least for these examples.

6.1. A counterexample against MCD

MCD handles Reiter's Example 6 correctly. But consider Example 7 where $w = \{\neg p, \neg q, \neg r, \neg s\}$ and $[\![A]\!] = [\![(p \vee q) \wedge ((p \wedge q) \rightarrow (r \vee s))]\!]$. Suppose that you hold a coin with two hands, and you read p as 'my left hand is open', q as 'my right hand is open',

[18] There is a characterization of the *PMA* in [14], but only in terms of conditional logic.

[19] We have corrected an error in [22]: the equivalence in (NI1) is an implication there.

[20] We can even strengthen Lemma 38, establishing that under its independence hypotheses and (U1), (U5) and (U6), updates by inclusive disjunctions behave just as updates by exclusive disjunctions, in the sense that $B \diamond (A_1 \vee A_2) \leftrightarrow B \diamond (A_1 \oplus A_2)$. (From Lemma 38 we obtain $B \diamond (A_1 \vee A_2) \rightarrow A_1 \oplus A_2$. Now by (U1) $B \diamond (A_1 \oplus A_2) \rightarrow (A_1 \oplus A_2)$. By classical logic, $B \diamond (A_1 \oplus A_2) \rightarrow (A_1 \vee A_2)$. Then (U6) says that $B \diamond (A_1 \vee A_2) \leftrightarrow B \diamond (A_1 \oplus A_2)$.)

r as 'the coin falls in a white field', and s as 'the coin falls in a black field'. According to the *MCD*, the result is equivalent to

$$((p \oplus q) \wedge \neg r \wedge \neg s) \vee (p \wedge q \wedge (r \oplus s)).$$

The interpretation $\{p, q, r, s\}$ is not a model of the updated belief base. Hence the coin cannot touch a black and a white field at the same time, which is clearly counterintuitive. Thus the *MCD* interprets the nested inclusive disjunction $r \vee s$ by the exclusive one $r \oplus s$.

This example motivated the introduction of *MCD** in [22].

6.2. A counterexample against MCD* and MCE

Suppose we have to move a box around in a warehouse with two floors. Suppose that p means "the box is in the bottom of the warehouse", q means "the box is in the second floor", and r means "the box is on the right side". Suppose initially the box is on the left side of the front region, on the first floor, i.e., the belief base is $\{\neg p, \neg q, \neg r\}$. Now suppose we ask a robot to put the box somewhere else in the warehouse, but not in the bottom of the first floor. This means that the box should be moved either on the second floor or in the front of the right first floor side, i.e., the input is $(\neg p \wedge r) \vee q$. Then we would expect the result $(\neg p \wedge r) \vee q$. Example 25 shows that with respect to *MCE* it is prohibited that robot puts the box in the bottom of the second floor, while he could put it on the right side of the second floor! In the two cases the robot will make the same 'steps' in the sense of area change. Why one is preferred over the other? In the case of the *PMA* and *FORBUS*, the robot would go right or go up. This corresponds to the fact that in these approaches the 'steps' the robot makes are minimized. *MCD* and *MCD** behave as *MCE*.

7. The problem of handling integrity constraints

Integrity constraints are formulas that have a particular status: they must be guaranteed to hold after every update. Update operations should take into account such constraints.

We view integrity constraints as a finite set of formulas *IC* which can be confused with the conjunction of its elements, just as in the case of belief bases. Formally, what we are interested in are update operations $\diamond_{IC}$ such that $B \diamond_{IC} A \to IC$.

It has often been proposed (see, e.g., [23,37]) to define the update $B \diamond_{IC} A$ as $B \diamond (A \wedge IC)$, for an appropriate update operation $\diamond$. This amounts to selecting the closest models of $A \wedge IC$. By means of the following counterexample it has been shown by Ginsberg (see, e.g., [30]) that such a proposal is problematic in particular if $\diamond$ is the *PMA* update operator: let Up_1 mean 'switch 1 is up', Up_2 'switch 2 is up', and *Light* 'the light is on'. Suppose there is a circuit such that the light is on exactly when both switches are in the same position. Hence the integrity constraint is $IC = (Up_1 \leftrightarrow Up_2) \leftrightarrow Light$. Let B be $Up_1 \wedge Up_2 \wedge Light$. Then one would expect that $B \diamond (\neg Up_1 \wedge IC) \to \neg Light$. As well, we would expect $B \diamond (\neg Up_1 \wedge IC) \to Up_2$, i.e., the second switch does not move. It turns out that neither is the case in the *PMA*, *MCE*, *MCD**, *MCD**, *WSS*, *WSS*$\downarrow$.

In this section we concentrate on approaches that have been designed to correctly handle such constraints, viz. WSS^{dep} and $MPMA^{\gg}$. Both of them resort to causal notions. Indeed, one story to learn from Lifschitz' counterexample above is that there are aspects of the domain structure which cannot be expressed by classical integrity constraints. In [30], Lifschitz proposes to distinguish between frame and non-frame atoms. We can recast his solution in terms of a boolean dependence function. For example, the atom *Light* is dependent, and this means that it is a secondary atom whose value is defined by primary atoms such as Up_1 and Up_2.

Thielscher [35] has given a counterexample showing that such a categorization is too simple. Basically, his argument is that in particular circumstances, every atom can be forced to be dependent. He proposes a solution in the framework of reasoning about actions. It is in terms of a dependence function mapping atoms to sets of atoms. (He uses the term influence relation.) In the example, $Light \in dep(Up_1)$, but $Up_2 \notin dep(Up_1)$. Hence *Light* depends on Up_1, but Up_2 does not depend on Up_1: the update of any belief base by Up_1 or $\neg Up_1$ can never change the truth value of Up_2.

$MPMA^{\gg}$ is close to Thielscher's approach. Here the integrity constraints are not conjoined with the input, and thus no minimization is applied to them. Instead, they are used to restrict the models of the update:

$$B \diamond_{MPMA^{\gg},IC} A = (B \diamond_{MPMA^{\gg}} A) \wedge IC \wedge T(CR)$$

where $T(CR)$ denotes the formula $\bigwedge_{i=1}^{n}(A_i \rightarrow C_i)$ where $A_i \gg C_i$ are the causal rules. [21]

Example 49. Let $(Up_1 \leftrightarrow Up_2) \gg Light$ and $\neg(Up_1 \leftrightarrow Up_2) \gg \neg Light$ be the causal rules. $\{Up_1, Up_2, Light\} \cdot_{mpma^{\gg}} \neg Up_1 = \{\{\neg Up_1, Up_2, Light\}, \{\neg Up_1, Up_2, \neg Light\}\}$.

The handling of integrity constraints in $WSS{\downarrow}^{dep}$ is similar to that in $MPMA^{\gg}$. It has been sketched in the framework of conditionals in [13,20]. Formally, given a set of integrity constraints *IC* and a dependence function *dep*, we have

$$B \diamond_{WSS{\downarrow}^{dep},IC} A = (B \diamond_{WSS{\downarrow}^{dep}} A) \wedge IC.$$

Example 50. Let $dep(Up_1) = \{Up_1, Light\}$, $dep(Up_2) = \{Up_2, Light\}$, and $dep(Light) = \{Light, Up_1, Up_2\}$. We have now

$$\{Up_1, Up_2, Light\} \cdot_{wss\downarrow^{dep}} \neg Up_1 = \{\{\neg Up_1, Up_2, Light\}, \{\neg Up_1, Up_2, \neg Light\}\}.$$

Integrity constraints are then taken into account by dropping from $\cdot_{wss\downarrow^{dep}}$ and $\cdot_{mpma^{\gg}}$ those interpretations which violate *IC*. In this way *Light* follows from the updated belief base of the two examples if we simply drop the interpretation $\{\neg Up_1, Up_2, Light\}$.

It remains to illustrate the differences between $WSS{\downarrow}^{dep}$, and $MPMA^{\gg}$: first, the dependence function is a sort of weak causal connection, while in $MPMA^{\gg}$ the causal connection is a strong one. Dependence is a function which maps an atom to a set of atoms, while in $MPMA^{\gg}$ the causal connection is between formulas. We can also remark that the causal connection in $MPMA^{\gg}$ is transitive in the sense that the causal rules $\{A \gg B, B \gg C\}$ and $\{A \gg B, B \gg C, A \gg C\}$ lead to the same update operation. This is not the case in $WSS{\downarrow}^{dep}$, due to the weak character of the causal connection.

All these differences make a formal comparison difficult.

[21] The original definition in [9] is slightly different but equivalent.

8. Other approaches

The approaches presented in this section are not in the tradition of those presented in Section 3. We discuss them now in order to complete the picture and conclude that our criticisms of the KM-postulates also apply to them.

The first approach is that of Boutilier in [1] in the framework of propositional knowledge base update. The basic idea of his work is to combine revision and update, providing a more realistic characterization of belief change. An incoming information about a change in the world can lead an agent to revise his prior beliefs before updating it. Boutilier presents a general model for update taking into account such considerations. The framework is based on the notions of ranking, event and transition.

What we are interested in is the relationship between his model and that of KM. In fact, his operator satisfies in the general case (U1), (U4), (U6), (U7) and (U9). But we have rejected (U6): we have seen in Section 5 that (U6) implies (U6′) under (U1), which in turn implies (U2) under ($U2^\top$). As Boutilier rejects (U2) he thus rejects ($U2^\top$). This is due to his hypothesis that the input do not correspond necessarily to event. But this contradicts our hypotheses of Section 1.3.

The same criticisms apply to Del Val and Shoham in [7]. They propose a theory of update using the situation calculus in which frame and ramification problems can be solved in a systematic way by default persistence of facts. They provide a relation between their approach and the KM-postulates, and prove that their construction satisfies (U1), (U3), (U5), (U6) and (U8). The undesirability of their operator is enforced by the presence of (U5) and (U6).

Another framework far from the ones we have described was proposed by Reiter in [33], using the situation calculus and theories of actions to perform update.

9. Summary and conclusion

Here we collect the main results of the paper.

(1) We have given an exhaustive analysis of the comparative strength of update operators that have been proposed in the literature (Section 4). The graph of Fig. 1 illustrates that, where the upper operations are the stronger ones.

(2) We have checked satisfaction of the KM postulates. The results of Section 5 are put together in Table 1.
The postulates (U1), (U3), and (U8) are uncontroversial. [22] The other postulates are violated by most of the operations.

(3) We have carefully discussed the plausibility of the controversial postulates. We have argued that (U4) is desirable, while (U2), (U5), and (U6) are not, and that (U7) is without importance. We have also shown how (U4) can be enforced in a simple way. In the KM framework, (U5) and (U6) are crucial if we want to give semantics to update operations in terms of partial preorders on interpretations. Given our critical

[22] We note that there are criticisms of (U3) as in [1,10]. They have mainly to do with the fact that logically consistent inputs can be inconsistent with the domain laws—see our reformulation of (U3) below.

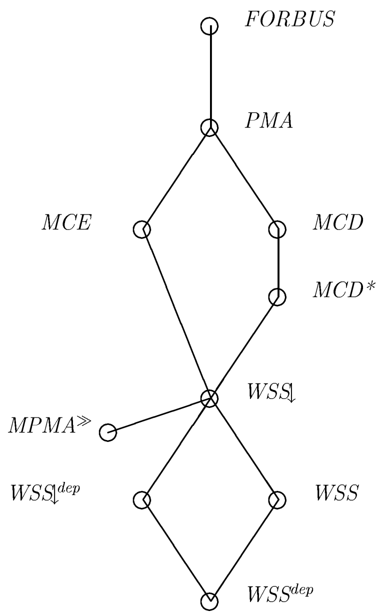

Fig. 1. Ordering of strength (Section 4).

Table 1
Satisfaction of the KM-postulates (Section 5)

	(U1)	(U2)	(U3)	(U4)	(U5)	(U6)	(U7)	(U8)
FORBUS	y	y	y	y	y	y	y	y
PMA	y	y	y	y	y	y	y	y
MCD	y	y	y	y	n	y	n	y
*MCD**	y	y	y	y	n	y	n	y
WSS	y	n	y	n	n	n	y	y
WSS$\downarrow$	y	n	y	y	n	n	n	y
MCE	y	y	y	y	n	y	n	y
WSS^{dep}	y	n	y	n	n	n	y	y
$WSS{\downarrow}^{dep}$	y	n	y	y	n	n	n	y
$MPMA^{\gg}$	y	n	y	y	n	n	n	y

examination of these postulates, this means the other way round that we cannot resort to such devices any longer when we look for semantics of update operations.

(4) We have thoroughly investigated updates with disjunctive input. We have shown in Section 5 that it is (U5) which unintuitively makes inclusive disjunctions behave just as exclusive ones. We have gone beyond that in Section 6 and have shown by counterexamples that the operations *MCD*, *MCD**, and *MCE*, while not satisfying

(U5) still have that unintuitive feature, and therefore fail to do the job they were designed for. Hence only $WSS{\downarrow}$, $WSS{\downarrow}^{dep}$, $MPMA$, and $MPMA^{\gg}$ correctly handle disjunctive input.

(5) We have characterized one of the approaches $WSS{\downarrow}^{dep}$ by a complete set of postulates.

(6) Our last contribution is an account of integrity constraints handling. Only $WSS{\downarrow}^{dep}$ and WSS^{dep} of [22] and $MPMA^{\gg}$ of [9] resist to Lifschitz' toy example. These two approaches are based on dependence or causality information, and it seems that one cannot do without such devices.

Let us add here that a realistic set of update postulates should take into account integrity constraints. This can be done in a straightforward way by adding a further postulate ($U0_{IC}$) and adapting those postulates appealing to consistence.

Putting together our results on the plausibility of the KM postulates and our remark on integrity constraints we obtain our official set of postulates.

($U0_{IC}$) $B \diamond A \to IC$
(U1) $B \diamond A \to A$
(U2.1) $B \wedge A \to B \diamond A$
($U2^{\top}$) $B \diamond \top \leftrightarrow B$
($U3_{IC}$) If B and A are consistent with IC, then $B \diamond A$ is consistent with IC
(U4.1) If $B_1 \leftrightarrow B_2$ then $B_1 \diamond A \leftrightarrow B_2 \diamond A$
(U4.2) If $A_1 \leftrightarrow A_2$ then $B \diamond A_1 \leftrightarrow B \diamond A_2$
(U8) $(B_1 \vee B_2) \diamond A \leftrightarrow (B_1 \diamond A) \vee (B_2 \diamond A)$.

It is difficult to give the form of postulates to the other requirement that we advocate, viz. that inclusive input should not be interpreted exclusively.

Among the ten update operations that we have presented, only $WSS{\downarrow}^{dep}$ and $MPMA^{\gg}$ have shown to fulfil the requirements and resist to all the counterexamples. The simplicity of $WSS{\downarrow}^{dep}$ and the fact that it is coNP-complete make us think that it is the most promising approach for further investigations.

Finally, it remains to notice that the laws we consider are static in the sense that they rule only one state of affairs. We agree with Reiter [33] that dynamic laws which rule at least two states of affairs are an important issue. We have proposed in previous work to handle such constraints by using a conditional operator [13]. Recent results on the interaction between updates and conditionals support such an approach [4,21,34]. But this is beyond the scope of the present paper.

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