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Properties and interrelationships of skeptical, weakly skeptical, and credulous inference induced by classes of minimal models

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ABSTRACT

There are multiple ways of defining nonmonotonic inference relations based on a conditional knowledge base. While the axiomatic system P is an important standard for such plausible nonmonotonic reasoning, inference relations obtained from system Z or from c-representations have been designed which go beyond system P by selecting preferred models for inference. For any class of models *M*, we propose the notion of weakly skeptical inference, first introduced in an ECAI conference paper this article revises and extends, that lies between skeptical and credulous inference with respect to *M*. Weakly skeptical c-inference properly extends skeptical c-inference, but avoids disadvantages of a too liberal credulous c-inference. We extend the concepts of skeptical, weakly skeptical, and credulous c-inference modes by taking models obtained from different minimality criteria into account. We illustrate the usefulness of the obtained inference relations and show that they fulfill various desirable properties put forward for nonmonotonic reasoning. Furthermore, we elaborate in detail the interrelationships among the inference relations when taking the different inference modes and various classes of minimal models into account.

1. Introduction

Conditionals are a key concept both in informal, natural language-based communication about pieces of knowledge and in formal, logic-based knowledge representation and reasoning. Everyday or *commonsense* reasoning is usually on sets of conditionals which we use to explain facts (“if it rains, then the street is wet”) or to express rules with exceptions (“if one studies hard for an exam, one usually passes”). Therefore, it is an established practice to use sets of conditionals to formalize and implement commonsense reasoning. Two prominent approaches in this area are Adams’s system P [2,33] and Pearl’s system Z [42]. The latter uses ordinal conditional functions (OCF [44]), also called ranking functions, as semantic for conditionals. Among many other possible semantics, e.g., Lewis’ system of spheres [36], Boolean intervals [21], or possibility distributions [14], ranking functions can also be used as models for system P; a comparison of different semantics of conditional logics and transformations among them can be found in [9].

Given a set $\mathcal{R}$ of conditionals, a major question is what inferences $\mathcal{R}$ entails, and there are many different ways to answer this question. Let $(B|A)$ denote the conditional *If A, then usually B*. Using system P, called the “industry standard” for

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qualitative nonmonotonic inference in [24], let us denote the inference relation based on $\mathcal{R}$ by $\mid\!\sim^{p}_{\mathcal{R}}$. This relation can be characterized by saying that $A \mid\!\sim^{p}_{\mathcal{R}} B$ holds if the conditional $(B|A)$ is accepted by *all* ranking models of $\mathcal{R}$. On the other hand, the system Z inference relation based on $\mathcal{R}$, denoted by $\mid\!\sim^{z}_{\mathcal{R}}$, is defined by saying $A \mid\!\sim^{z}_{\mathcal{R}} B$ if $(B|A)$ is accepted by the single unique system Z model of $\mathcal{R}$.

The skeptical nature of system P entailment leads to the so-called irrelevance problem, exemplified by the observation that knowing that birds usually fly and not knowing anything about being red, system P fails to infer that red birds usually fly. This pitfall is avoided in system Z, but both system P and system Z are known to suffer from the Drowning Problem [42,16], illustrated by their failure to infer that penguins, being exceptional birds with respect to flying, but not with respect to having wings, have wings. Here is another example for an inference we would reasonably expect.

Example 1. Let $\mathcal{R}_{bpf} = \{r_1, r_2, r_3, r_4\}$ be the following set of conditionals:

r_1 : $(f|b)$ birds fly
r_2 : $(\overline{f}|p)$ penguins don't fly
r_3 : $(\overline{f}|bp)$ penguin birds don't fly
r_4 : $(b|p)$ penguins are birds

Given these pieces of information, it can be argued that it is reasonable to infer that a flying penguin is still a bird. However, neither system P nor system Z support this inference.

While the axioms of system P are well-accepted as a baseline for nonmonotonic reasoning, selecting a single model for sanctioning inferences like it is done in system Z is only one of many different ways of escaping system P inference's skepticism. In this article, we elaborate such ways using three different methodological dimensions: (1) types of models, (2) modes of inference, and (3) selection of subclasses of models; combining methods from these three dimensions leads to a variety of inference relations, including inference relations supporting entailments as outlined in Example 1.

- **Types of models** As in system Z, we will use OCFs as models, but we focus on the specific type of ranking models called c-representations. While system Z inductively generates a unique ranking function for a knowledge base $\mathcal{R}$ that models all conditionals in $\mathcal{R}$, c-representations provide a schema for a set of OCFs that do so. The basic underlying feature of c-representations is the concept of conditional preservation, stating that the effect of conditionals on two different worlds should be the same if these worlds have the same verification and falsification behavior with respect to the conditionals [25,27]. A c-representation is obtained by assigning an individual impact to each conditional and by generating world ranks as the sum of impacts of falsified conditionals.
- **Modes of inference** Given a set of models M, skeptical and credulous inference with respect to M are well-established concepts [38]. While skeptical inference might be too cautious, credulous inference is often too bold, especially if it allows us to infer inconsistencies. For any set of models M, we therefore propose *weakly skeptical* inference as a novel inference mode extending skeptical inference with respect to M in that it allows us to infer new knowledge if it can be inferred credulously with respect to M, but its negation cannot be inferred by any model in M.
- **Subclasses of models** Another dimension of shaping an inference relation is to restrict the set of models to be taken into account. Guidelines for selecting a subclass of models might be obtained from notions of specificity or minimality criteria. Especially in combination with the other dimensions, different effects can be achieved. For instance, moving from a set of models to a proper subclass might result in a larger number of skeptical inferences and a lower number of credulous inferences, but for weakly skeptical inference, the situation is more intricate.

While each c-representation could be used for reasoning over $\mathcal{R}$, the notion of skeptical c-inference taking all c-representations for $\mathcal{R}$ into account has been introduced in [3]. Applying the notion of weakly skeptical inference to the set of all c-representations of a knowledge base $\mathcal{R}$ yields *weakly skeptical c-inference*. It allows us to infer new knowledge if it can be inferred by one c-representation of $\mathcal{R}$ and is not contradicted by any other c-representation. We show that weakly skeptical inference lies strictly between skeptical and credulous c-inference, illustrate its usefulness with several examples, and prove that it satisfies various desirable properties.

Regarding the dimension of selecting subclasses of models, several different minimality criteria have been proposed to select a set of preferred c-representations [8]. While the unique system Z ranking function is obtained by applying the criterion of pareto-minimality to the set of all ranking models, in the case of c-representations pareto-minimality as well as other notions of minimality proposed for c-representations yield, in general, non-singleton sets of minimal ranking functions.

For the different minimality criteria, we transfer the concept of credulous and weakly skeptical inference, study the properties of the resulting inference relations, and elaborate their interrelationships. We show that none of the minimality-based skeptical inference relations is able to capture skeptical c-inference over all c-representations. Furthermore, we argue that using both weakly skeptical inference and preferred minimal c-representations allows us to obtain desirable inferences that are not possible by weakly skeptical inference over all c-representations nor by skeptical inference over any of the considered sets of preferred models.

In summary, the main contributions of this article are:

- Introduction of skeptical c-inference over sets of minimal models;
- Notion of weakly skeptical inference mode, lying between skeptical and credulous inference;
- C-Inference relations obtained by taking both different inference modes and different classes of models and minimal models into account;
- Inference with respect to arbitrary sets of ranking models of a knowledge base;
- Elaboration of properties of the obtained inference relations regarding standard postulates proposed for nonmonotonic reasoning;
- Elaboration of interrelationships among the obtained inference relations.

This article is a fully revised and largely extended version of the conference contribution [4]. In particular, we added the study of nonmonotonic reasoning postulates for all considered inference relations, and completed the study of their interrelationships. The rest of this paper is organized as follows: We recall the basics of conditionals, ordinal conditional functions (OCF), plausible inference, system Z and c-representations which we need as formal background of this paper in Sections 2 and 3. In Section 4, we study skeptical c-inference and its properties, introduce the notion of weakly skeptical inference mode, and investigate the properties of weakly skeptical c-inference. Furthermore, we address c-inference relations with respect to different notions on preferred minimal models and taking the modes of skeptical, weakly skeptical, and credulous inference into account. Section 5 deals with inference with respect to arbitrary subsets of ranking models of a knowledge base. In Section 6, a full map of the interrelationships among the c-inference modes over the different model classes is developed, and in Section 7, we conclude and point out further work.

2. Background

We start with a propositional language $\mathcal{L}_\Sigma$, generated by a finite set Σ of atoms $a, b, c, \ldots$. Similarly as $\mathcal{L}_\Sigma$, many notions and definitions used in this article depend on the underlying signature Σ of propositional variables, e.g., the deductive closure of a set of propositional formulas or the nonmonotonic inference relation based on a conditional knowledge base. If Σ is clear from the context, we will often omit Σ, and make the dependence on Σ explicit only if it is particularly relevant. For instance, we will usually just write $\mathcal{L}$ instead of $\mathcal{L}_\Sigma$.

The formulas of $\mathcal{L}$ will be denoted by uppercase Roman letters $A, B, C, \ldots$. For conciseness of notation, we will omit the logical *and*-connective, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of possible worlds over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$ and can be identified with the set of all complete conjunctions over Σ. For $\omega \in \Omega$, $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world ω.

By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. $(B|A)$ formalizes "*if A then (normally) B*" and establishes a plausible, probable, possible etc. connection between the *antecedent* A and the *consequent* B. Here, conditionals are supposed not to be nested, that is, antecedent and consequent of a conditional will be propositional formulas.

A conditional $(B|A)$ is an object of a three-valued nature, partitioning the set of worlds Ω in three parts: those worlds satisfying AB, thus *verifying* the conditional, those worlds satisfying $A\overline{B}$, thus *falsifying* the conditional, and those worlds not fulfilling the premise A and to which the conditional may not be applied to at all. This allows us to represent $(B|A)$ as a *generalized indicator function* going back to [19] (where u stands for *unknown* or *indeterminate*):

$$(B|A)(\omega) = \begin{cases} 1 & \text{if} \quad \omega \models AB \\ 0 & \text{if} \quad \omega \models A\overline{B} \\ u & \text{if} \quad \omega \models \overline{A} \end{cases} \tag{1}$$

A conditional $(B|A)$ is *tolerated* [42] by a set $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ if and only if there is a world ω that verifies $(B|A)$ and does not falsify any conditional in $\mathcal{R}$.

To give appropriate semantics to conditionals, they are usually considered within richer structures such as *epistemic states*. Besides certain (logical) knowledge, epistemic states also allow for the representation of preferences, beliefs, assumptions of an intelligent agent. Basically, an epistemic state allows one to compare formulas or worlds with respect to plausibility, possibility, necessity, probability, etc.

Well-known qualitative, ordinal approaches to represent epistemic states are Spohn's *ordinal conditional functions, OCFs*, (also called *ranking functions*) [44], and *possibility distributions* [13], assigning degrees of plausibility, or of possibility, respectively, to formulas and possible worlds. In such qualitative frameworks, a conditional $(B|A)$ is valid (or *accepted*), if its confirmation, AB, is more plausible, possible, etc. than its refutation, $A\overline{B}$; a suitable degree of acceptance is calculated from the degrees associated with AB and $A\overline{B}$.

In this paper, we consider Spohn's OCFs [44]. An OCF is a function $\kappa : \Omega \to \mathbb{N}_0$ expressing degrees of plausibility of propositional formulas where a higher degree denotes "less plausible" or "more surprising". At least one world must be regarded as being normal (or maximally plausible or unsurprising); therefore, $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$. Each OCF can be taken as the representation of a full epistemic state of an agent. Each such κ uniquely extends to a function (also denoted by κ) mapping sentences $A \in \mathcal{L}$ to $\mathbb{N} \cup \{\infty\}$ by:

(REF)	*Reflexivity*	for all $A \in \mathcal{L}$ it holds that $A \mid\!\sim A$
(LLE)	*Left Logical Equivalence*	$A \equiv B$ and $B \mid\!\sim C$ imply $A \mid\!\sim C$
(RW)	*Right Weakening*	$B \models C$ and $A \mid\!\sim B$ imply $A \mid\!\sim C$
(CM)	*Cautious Monotony*	$A \mid\!\sim B$ and $A \mid\!\sim C$ imply $AB \mid\!\sim C$
(CUT)		$A \mid\!\sim B$ and $AB \mid\!\sim C$ imply $A \mid\!\sim C$
(OR)		$A \mid\!\sim C$ and $B \mid\!\sim C$ imply $(A \vee B) \mid\!\sim C$

Fig. 1. Axioms of system P [2].

$$\kappa(A) = \begin{cases} \min\{\kappa(\omega) \mid \omega \models A\} & \text{if } A \text{ is satisfiable,} \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

An OCF κ *accepts* a conditional $(B|A)$ (denoted by $\kappa \models (B|A)$) if and only if the verification of the conditional is less surprising than its falsification, i.e., if and only if $\kappa(AB) < \kappa(A\overline{B})$.

We call a conditional $(B|A)$ with $A \models B$ *self-fulfilling* since it can not be falsified by any world. Similarly, $(B|A)$ with $A \models \overline{B}$ is *contradictory* since it can not be verified by any world. Obviously, such conditionals are meaningless from a modeling point of view, and we will not consider them in the following in order to avoid many easy, but tedious case distinctions. A finite set $\mathcal{R} \subseteq (\mathcal{L}|\mathcal{L})$ of conditionals is called a *knowledge base* (over Σ) if it does not contain any self-fulfilling or contradictory conditional. An OCF κ accepts a knowledge base if and only if κ accepts all conditionals in $\mathcal{R}$ (i.e., κ is admissible with respect to $\mathcal{R}$ [22]); such an OCF is called a *(ranking) model* of $\mathcal{R}$. A knowledge base $\mathcal{R}$ is *consistent* if and only if a ranking model of $\mathcal{R}$ exists [42]. Note that in principle, $\mathcal{R}$ could be supplemented with a classical strict knowledge base, excluding worlds which violate the strict knowledge.

3. System P, system O, system Z, and c-representations

In this section, we recall established systems of nonmonotonic inference from conditional knowledge bases. Nonmonotonic inference relations are usually evaluated by means of properties. In particular, the axiom system P [2] provides an important standard for plausible, nonmonotonic inferences. With $\mid\!\sim$ being a generic nonmonotonic inference operator and A, B, C being formulas in $\mathfrak{L}$, the six axioms of system P are given in Fig. 1.

We begin by defining the inference relation induced by a single ranking function κ, written as $\mid\!\sim_\kappa$, as follows:

$$A \mid\!\sim_\kappa B \quad \text{iff} \quad A \equiv \bot \quad \text{or} \quad \kappa(AB) < \kappa(A\overline{B}) \tag{3}$$

Based on this, one of the most central notions in this field is *p-entailment* which has been defined in [22] as follows[1]:

Definition 2 *(p-entailment [22]).* Let $\mathcal{R}$ be a conditional knowledge base and let A, B be formulas. A *p-entails* B in the context of $\mathcal{R}$, written $A \mid\!\sim^p_{\mathcal{R}} B$, if $A \mid\!\sim_\kappa B$ for all $\kappa \models \mathcal{R}$.

This is a handy definition that serves the purposes of this paper well. Originally, p-entailment was introduced in [1] in a probabilistic framework, but can also be defined via preferential models [35] that generalize ranking functions.

P-entailment can be easily characterized:

Proposition 3 *([22]). Let $\mathcal{R}$ be a conditional knowledge base and let A, B be formulas. A* p-entails *B in the context of a knowledge base $\mathcal{R}$, if and only if $\mathcal{R} \cup \{(\overline{B}|A)\}$ is inconsistent.*

From these axioms it follows that if both B and C can be inferred from a premise A, also the conjunction BC can be inferred from A [38], formally

(AND) $\quad A \mid\!\sim B$ and $A \mid\!\sim C$ imply $A \mid\!\sim BC$.

Actually, p-entailment is sound and complete for inferences from conditional knowledge bases via system P, i.e., given a knowledge base $\mathcal{R}$, system P inference, denoted by $\mid\!\sim^p_{\mathcal{R}}$, is the same as p-entailment; this statement can be obtained by combining results from various papers:

Proposition 4 *([2,41,22]). Let A, B be formulas and let $\mathcal{R}$ be a consistent conditional knowledge base. B follows from A in the context of $\mathcal{R}$ with the rules of system P if and only if A p-entails B in the context of $\mathcal{R}$.*

[1] Note that in [22, Definition 7], inference based on a single κ does not explicitly handle the case of $A \equiv \bot$; instead [22, Definition 7] states that $A \mid\!\sim_\kappa B$ iff $\kappa(AB) < \kappa(A\overline{B})$. However, the definition of p-entailment based on this definition of $\mid\!\sim_\kappa$ does not satisfy system P. In particular, p-entailment as defined in [22, Definition 8] does not satisfy (REF), since $\kappa(\bot) \not< \kappa(\bot)$. Also in other publications, this special case of inference from $\bot$ via $\mid\!\sim_\kappa$ in compliance with system P has not been addressed properly, e.g., in our own work [4].

(REF)	*Reflexivity*	for all $A \in \mathcal{L}$ it holds that $A \mid\!\sim A$
(LLE)	*Left Logical Equivalence*	$A \equiv B$ and $B \mid\!\sim C$ imply $A \mid\!\sim C$
(RW)	*Right Weakening*	$B \models C$ and $A \mid\!\sim B$ imply $A \mid\!\sim C$
(VCM)	*Very Cautious Monotony*	$A \mid\!\sim BC$ implies $AB \mid\!\sim C$
(WAND)	*Weak And*	$A \mid\!\sim B$ and $A\overline{C} \mid\!\sim C$ implies $A \mid\!\sim BC$
(WOR)	*Weak Or*	$AB \mid\!\sim C$ and $A\overline{B} \mid\!\sim C$ implies $A \mid\!\sim C$

Fig. 2. Axioms of system O [33].

Proof. (Sketch) Due to [22,41], p-entailment is the same as Adams' ϵ-consequence, or Pearl's ϵ-entailment, which are both characterized by System P [2,41]. □

When using sets of conditionals as (knowledge) base of an inference relation, one would expect that the conditional rules in the knowledge base are valid in the respective inference relation. This is formalized in the following property [37].

(DI) *Direct Inference* if $(B|A) \in \mathcal{R}$ then $A \mid\!\sim_{\mathcal{R}} B$.

Note that (DI) is not derivable in system P because system P does not mention the conditional base explicitly; please see [29] for an inference relation that satisfies system P but not (DI).

An axiom system that is strictly weaker than system P is *system O* [33] which is given in Fig. 2.

Two inference relations which are defined by specific OCFs obtained inductively from a knowledge base $\mathcal{R}$ have received some attention: system Z and c-representations, or the induced inference relations, respectively, both show excellent inference properties. We recall both approaches briefly.

System Z [42] is based upon the ranking function κ^Z, which is the unique Pareto-minimal OCF that accepts $\mathcal{R}$. The system is set up by forming an ordered partition $(\mathcal{R}_0, ..., \mathcal{R}_m)$ of $\mathcal{R}$, where each $\mathcal{R}_i$ is the (with respect to set inclusion) maximal subset of $\bigcup_{j=i}^{m} \mathcal{R}_j$ that is tolerated by $\bigcup_{j=i}^{m} \mathcal{R}_j$. This partitioning is unique due to the maximality. The resulting OCF κ^Z is defined by assigning to each world ω a rank of 1 plus the maximal index $1 \leqslant i \leqslant m$ of the partition that contains conditionals falsified by ω, or 0 if ω does not falsify any conditional in $\mathcal{R}$. Formally, for all $(B|A) \in \mathcal{R}$ and for $Z(B|A) = i$ iff $(B|A) \in \mathcal{R}_i$, the OCF κ^Z is given by:

$$\kappa^Z(\omega) = \begin{cases} 0 & \text{iff } \omega \text{ does not falsify any conditional in } \mathcal{R} \\ \max\{Z(B|A) | (B|A) \in \mathcal{R}, \omega \models A\overline{B}\} + 1 & \text{otherwise.} \end{cases}$$

Other than system Z, the approach of c-representations does not use the most severe falsification of a conditional, but assigns an individual impact to each conditional and generates the world ranks as a sum of impacts of falsified conditionals (for a detailed introduction to c-representations see [25,27]).

Definition 5 *(c-Representation [25,27]).* A *c-representation* of a knowledge base $\mathcal{R}$ is a ranking function κ constructed from non-negative integer impacts $\eta_i \in \mathbb{N}_0$ assigned to each conditional $(B_i|A_i)$ such that κ accepts $\mathcal{R}$ and is given by:

$$\kappa(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{4}$$

Note that for the empty knowledge base $\mathcal{R} = \emptyset$, the unique c-representation accepting $\mathcal{R}$ assigns 0 to every world ω, thus considering every world to be equally unsurprising and therefore representing a state of complete ignorance.

Definition 6 *($CR(\mathcal{R})$ [8]).* Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. The constraint satisfaction problem for c-representations of $\mathcal{R}$, denoted by $CR(\mathcal{R})$, is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:

$$\eta_i \geqslant 0 \tag{5}$$

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \tag{6}$$

A solution of $CR(\mathcal{R})$ is an n-tuple $(\eta_1, \ldots, \eta_n)$ of natural numbers. For a constraint satisfaction problem CSP, the set of solutions is denoted by $Sol(CSP)$. Thus, with $Sol(CR(\mathcal{R}))$ we denote the set of all solutions of $CR(\mathcal{R})$.

Proposition 7 *(Soundness of $CR(\mathcal{R})$ [3]). For $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ let $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in Sol(CR(\mathcal{R}))$. Then the function κ defined by the equation system given by (4) is a c-representation that accepts $\mathcal{R}$.*

Definition 8 *($\kappa_{\vec{\eta}}$).* For $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in Sol(CR(\mathcal{R}))$ and κ as in Equation (4), κ is the *OCF induced by* $\vec{\eta}$ and is denoted by $\kappa_{\vec{\eta}}$.

Proposition 9 *(Completeness of $CR(\mathcal{R})$ [3]). Let κ be a c-representation for a knowledge base $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$, i.e., $\kappa \models \mathcal{R}$. Then there is a vector $\vec{\eta} \in Sol(CR(\mathcal{R}))$ such that $\kappa = \kappa_{\vec{\eta}}$.*

It has been shown that there is a c-representation for a knowledge base $\mathcal{R}$ if and only if $\mathcal{R}$ is consistent [25,27]. The soundness and completeness results in Propositions 7 and 9 give us that $CR(\mathcal{R})$is solvable if and only if there is a c-representation for $\mathcal{R}$. This gives us an additional criterion for the consistency of a knowledge base since $\mathcal{R}$ is consistent iff the constraint satisfaction problem $CR(\mathcal{R})$ is solvable. Applying a constraint satisfaction solver to $CR(\mathcal{R})$ and checking its solvability gives us an implementable alternative to the tolerance test algorithm that shows that $\mathcal{R}$ is consistent if and only if an ordered partition of $\mathcal{R}$ with respect to the tolerance criterion exists, which in turn is equivalent to the existence of a system Z ranking function for $\mathcal{R}$ [42].

4. Skeptical, credulous, and weakly skeptical inference over c-representations

Skeptical c-inference takes all c-representations of a knowledge base into account and is treated in Section 4.1. We then refine the notion of c-inference in two different dimensions. First, we define credulous and weakly skeptical c-inference as further inference modes in Section 4.2. Second, we define inference with respect to subclasses of minimal c-representations in Section 4.3.

4.1. Skeptical inference based on c-representations

Whereas Equation (3) defines an inference relation $\mid\!\sim_\kappa$ based on a single OCF κ, in [3] a skeptical c-inference relation $\mid\!\sim^{sk}_{\mathcal{R}}$ is introduced that takes all c-representations of a given knowledge base $\mathcal{R}$ into account.

Definition 10 *(skeptical c-inference, $\mid\!\sim^{sk}_{\mathcal{R}}$ [3]).* Let $\mathcal{R}$ be a knowledge base and let A, B be formulas. B is *a skeptical c-inference from A in the context of $\mathcal{R}$*, denoted by $A \mid\!\sim^{sk}_{\mathcal{R}} B$, if $A \mid\!\sim_\kappa B$ holds for all c-representations κ for $\mathcal{R}$.

Note that $\perp \mid\!\sim^{sk}_{\mathcal{R}} \perp$ holds for every knowledge base $\mathcal{R}$, as required by system P, due to our underlying definition of $\mid\!\sim_\kappa$ given in (3). With this underlying definition of $\mid\!\sim_\kappa$, skeptical c-inference satisfies system P, and it exceeds system P.

Proposition 11. *Skeptical c-inference satisfies system P.*

Proof. Let $\mathcal{R}$ be a knowledge base. We have to show for every property in system P, if all preconditions of the property hold for skeptical c-inference in the context of $\mathcal{R}$, then the conclusion of that property holds for skeptical c-inference in the context of $\mathcal{R}$. For each property Φ in system P it holds that, if all preconditions of Φ hold for skeptical c-inference in the context of $\mathcal{R}$, then by Definition 10 all preconditions Φ hold for all inference relations defined by any c-representation of $\mathcal{R}$. Since every inference relation $\mid\!\sim_\kappa$ defined by a ranking function κ satisfies all properties in system P, the conclusion of Φ holds in all inference relations defined by a c-representation of $\mathcal{R}$. Then, by Definition 10, the conclusion Φ holds for skeptical c-inference in the context of $\mathcal{R}$. Note that this argumentation holds trivially for (Ref), since (Ref) has no preconditions. □

Proposition 12 *([3]). Every system P entailment of a knowledge base $\mathcal{R}$ is also a skeptical c-inference of $\mathcal{R}$; the reverse is not true in general.*

An example of a preferential model which is not a c-representation is given in the following Example 13 where we present a knowledge base whose system Z ranking function is not a c-representation.

It is known that system P does not allow for any subclass inheritance, and system Z does license subclass inheritance for regular subclasses, but does not allow for subclass inheritance of exceptional subclasses (see, for instance, [45]), the latter is known as the *Drowning Problem* [16]. The following example illustrates that skeptical c-inference allows for subclass inheritance for exceptional subclasses.

Example 13. For illustration, we use the well known penguin example. Let $\Sigma = \{p, b, f, w\}$ be a set of atoms describing whether something is a penguin (p), bird (b), capable of flying (f) or has wings (w). From this set, we compose the following set $\mathcal{R}_{pen} = \{r_1, r_2, r_3, r_4\}$ of conditionals:

$$\mathcal{R}_{pen} = \left\{ \begin{array}{lll} r_1\colon & (f|b), & (\text{"birds usually fly"}) \\ r_2\colon & (\overline{f}|p), & (\text{"penguins usually do not fly"}) \\ r_3\colon & (b|p), & (\text{"penguins usually are birds"}) \\ r_4\colon & (w|b) & (\text{"birds usually have wings"}) \end{array} \right\} \tag{7}$$

Here, penguin is an exceptional subclass of bird, because penguins are birds that cannot fly, but penguins are not exceptional with respect to having wings. So we expect that it should not be possible to infer whether penguins have wings (or not) by either system P or system Z. Skeptical c-inference should license for the nonmonotonic entailment that penguins have wings and thus penguins should inherit the property of having wings from birds. We can extend $\mathcal{R}_{pen}$ consistently both with $(w|p)$ and $(\overline{w}|p)$, so according to Proposition 3 we can p-entail neither w nor $\overline{w}$ from p. Using Proposition 4, this means we have both $p \not\mid\!\sim^{p}_{\mathcal{R}_{pen}} w$ and $p \not\mid\!\sim^{p}_{\mathcal{R}_{pen}} \overline{w}$, as expected. The tolerance partitioning of $\mathcal{R}_{pen}$ is $\mathcal{R}_{pen_0} = \{(f|b), (w|b)\}$, $\mathcal{R}_{pen_1} = \{(\overline{f}|p), (b|p)\}$ which gives us the system Z ranking function $\kappa^Z_{\mathcal{R}_{pen}}$ given in Table 1. This OCF gives us

$$\begin{aligned} \kappa^Z_{\mathcal{R}_{pen}}(pw) &= \min\{\kappa^Z_{\mathcal{R}_{pen}}(pbfw), \kappa^Z_{\mathcal{R}_{pen}}(pb\overline{f}w), \kappa^Z_{\mathcal{R}_{pen}}(p\overline{b}fw), \kappa^Z_{\mathcal{R}_{pen}}(p\overline{b}\,\overline{f}w)\} \\ &= \min\{2, 1, 2, 2\} = 1 \\ \kappa^Z_{\mathcal{R}_{pen}}(p\overline{w}) &= \min\{\kappa^Z_{\mathcal{R}_{pen}}(pbf\overline{w}), \kappa^Z_{\mathcal{R}_{pen}}(pb\overline{f}\overline{w}), \kappa^Z_{\mathcal{R}_{pen}}(p\overline{b}f\overline{w}), \kappa^Z_{\mathcal{R}_{pen}}(p\overline{b}\,\overline{f}\overline{w})\} \\ &= \min\{2, 1, 2, 2\} = 1 \end{aligned}$$

and thus $\kappa^Z_{\mathcal{R}_{pen}}(pw) = \kappa^Z_{\mathcal{R}_{pen}}(p\overline{w})$ and therefore $p \not\mid\!\sim_{\kappa^Z_{\mathcal{R}_{pen}}} w$ and $p \not\mid\!\sim_{\kappa^Z_{\mathcal{R}_{pen}}} \overline{w}$, also as expected. For skeptical inference over all c-representations we have to inspect all solutions of $CR(\mathcal{R}_{pen})$. The system of inequalities that is solved by the solutions $\vec{\eta} \in Sol(\mathcal{R}_{pen})$ for this example is set up according to Definition 6 as follows:

$$\begin{aligned} \eta_1 &> \min\{\eta_2, \eta_2 + \eta_4, 0, \eta_4\} - \min\{0, \eta_4, 0, \eta_4\} \\ \eta_2 &> \min\{\eta_1, \eta_1 + \eta_4, \eta_3, \eta_3\} - \min\{0, \eta_4, \eta_3, \eta_3\} \\ \eta_3 &> \min\{\eta_2, \eta_2 + \eta_4, \eta_1, \eta_1 + \eta_4\} - \min\{\eta_2, \eta_2, 0, 0\} \\ \eta_4 &> \min\{\eta_2, \eta_1, 0, \eta_1\} - \min\{\eta_2, \eta_1, 0, \eta_1\} \end{aligned}$$

This system can be simplified to

$$\begin{array}{ll} \eta_1 > 0 & \eta_2 > \min\{\eta_1, \eta_3\} \\ \eta_3 > \min\{\eta_2, \eta_1\} & \eta_4 > 0; \end{array}$$

a case differentiation further yields $\eta_2 > \eta_1$ and $\eta_3 > \eta_1$. Table 1 lists two possible solution vectors, $\vec{\eta}_1 = (1, 2, 2, 1)$ and $\vec{\eta}_2 = (1, 2, 2, 3)$, of $Sol(\mathcal{R}_{pen})$; in fact, every vector $(1, 2, 2, x_4)$ with $x_4 \geqslant 1$ is also a solution of $Sol(\mathcal{R}_{pen})$. The sixth column in Table 1 schematically lists the ranks for every OCF $\kappa_{\vec{\eta}}$ induced by the solutions in $Sol(\mathcal{R}_{pen})$. So for all $\kappa_{\vec{\eta}}$ we have:

$$\begin{aligned} \kappa_{\vec{\eta}}(pw) &= \min\{\eta_2, \eta_1, \eta_2 + \eta_3, \eta_3\} = \eta_1 \\ \kappa_{\vec{\eta}}(p\overline{w}) &= \min\{\underbrace{\eta_2 + \eta_4}_{>\eta_1}, \underbrace{\eta_1 + \eta_4}_{>\eta_1}, \underbrace{\eta_2 + \eta_3}_{>\eta_1}, \underbrace{\eta_3}_{>\eta_1}\} \end{aligned}$$

and thus $\kappa_{\vec{\eta}}(pw) < \kappa_{\vec{\eta}}(p\overline{w})$ for all c-representations of $\mathcal{R}_{pen}$. This gives us $p \mid\!\sim^{sk}_{\mathcal{R}_{pen}} w$, and we see that using skeptical c-inference we can infer that penguins inherit the properties of their superclass birds of having wings.

Note that in the previous example, we have $\kappa^Z_{\mathcal{R}_{pen}}(pw) = \kappa^Z_{\mathcal{R}_{pen}}(p\overline{w})$ for the system Z ranking function $\kappa^Z_{\mathcal{R}_{pen}}$, but $\kappa(pw) \neq \kappa(p\overline{w})$ for every c-representation κ of $\mathcal{R}_{pen}$. Thus, Example 13 also illustrates this observation:

Observation 14. There are knowledge bases $\mathcal{R}$ such that the corresponding the system Z ranking function of $\mathcal{R}$ is not a c-representation of $\mathcal{R}$.

In the following, we study further properties of skeptical c-inference.

Definition 15 *(inference closure $Cl^{sk}_{\mathcal{R}}(A)$).* Let $\mathcal{R}$ be a knowledge base and let A be a formula. Then

$$Cl^{sk}_{\mathcal{R}}(A) = \{B \mid A \mid\!\sim^{sk}_{\mathcal{R}} B\} \tag{8}$$

is the *closure* of A under skeptical c-inference in the context of $\mathcal{R}$.

Table 1
System Z ranking function $\kappa^Z_{\mathcal{R}_{pen}}$ and schema of all c-representations $\kappa_{\vec{\eta}}$ for the penguin example $\mathcal{R}_{pen}$ in Example 13, and two solution vectors $\vec{\eta}_1, \vec{\eta}_2 \in Sol(\mathcal{R}_{pen})$ and their induced ranking functions $\kappa_{\vec{\eta}_1}$ and $\kappa_{\vec{\eta}_2}$.

ω	r_1: $(f\|b)$	r_2: $(\overline{f}\|p)$	r_3: $(b\|p)$	r_4: $(w\|b)$	impact on ω	$\kappa^Z_{\mathcal{R}_{pen}}(\omega)$	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$
$p\,b\,f\,w$	v	f	v	v	η_2	2	2	2
$p\,b\,f\,\overline{w}$	v	f	v	f	$\eta_2+\eta_4$	2	3	5
$p\,b\,\overline{f}\,w$	f	v	v	v	η_1	1	1	1
$p\,b\,\overline{f}\,\overline{w}$	f	v	v	f	$\eta_1+\eta_4$	1	2	4
$p\,\overline{b}\,f\,w$	–	f	f	–	$\eta_2+\eta_3$	2	4	4
$p\,\overline{b}\,f\,\overline{w}$	–	f	f	–	$\eta_2+\eta_3$	2	4	4
$p\,\overline{b}\,\overline{f}\,w$	–	v	f	–	η_3	2	2	2
$p\,\overline{b}\,\overline{f}\,\overline{w}$	–	v	f	–	η_3	2	2	2
$\overline{p}\,b\,f\,w$	v	–	–	v	0	0	0	0
$\overline{p}\,b\,f\,\overline{w}$	v	–	–	f	η_4	1	1	3
$\overline{p}\,b\,\overline{f}\,w$	f	–	–	v	η_1	1	1	1
$\overline{p}\,b\,\overline{f}\,\overline{w}$	f	–	–	f	$\eta_1+\eta_4$	1	2	4
$\overline{p}\,\overline{b}\,f\,w$	–	–	–	–	0	0	0	0
$\overline{p}\,\overline{b}\,f\,\overline{w}$	–	–	–	–	0	0	0	0
$\overline{p}\,\overline{b}\,\overline{f}\,w$	–	–	–	–	0	0	0	0
$\overline{p}\,\overline{b}\,\overline{f}\,\overline{w}$	–	–	–	–	0	0	0	0
	η_1	η_2	η_3	η_4				
$\vec{\eta}_1$	1	2	2	1				
$\vec{\eta}_2$	1	2	2	3				

Proposition 16 *($Cl^{sk}_{\mathcal{R}}(A)$ deductively closed). For every knowledge base $\mathcal{R}$ and every formula A, the set $Cl^{sk}_{\mathcal{R}}(A)$ is deductively closed.*

Proof. If $\mathcal{R}$ is not consistent, then $Cl^{sk}_{\mathcal{R}}(A)$ is the set of all formulas, which is trivially deductively closed. So let $\mathcal{R}$ be consistent. We first consider the closure of A with respect to a single c-representation κ that accepts $\mathcal{R}$, denoted by

$$Cl_\kappa(A) = \{B \mid A \mathrel{|\!\!\sim}_\kappa B\}, \tag{9}$$

and show that $Cl_\kappa(A)$ is deductively closed. Let $C \in \mathcal{L}$ be a formula such that $Cl_\kappa(A) \models C$. Due to the compactness of propositional logic, there is a finite set $\{B_1, \dots, B_m\} \subseteq Cl_\kappa(A)$ such that $\{B_1, \dots, B_m\} \models C$. The inference relation $\mathrel{|\!\!\sim}_\kappa$ as given in (3) satisfies system P (see, e.g., [22,30]) and therefore also the property (AND). Thus, from $A \mathrel{|\!\!\sim}_\kappa B_1$ and $A \mathrel{|\!\!\sim}_\kappa B_2$ we get $A \mathrel{|\!\!\sim}_\kappa B_1 B_2$. Applying this argument iteratively, we get

$$A \mathrel{|\!\!\sim}_\kappa B_1 \dots B_m. \tag{10}$$

Again using the observation that $\mathrel{|\!\!\sim}_\kappa$ satisfies system P, right weakening (RW) applied to $B_1 \dots B_n \models C$ and (10) yields $A \mathrel{|\!\!\sim}_\kappa C$, implying that $Cl_\kappa(A)$ is deductively closed. Since the intersection of deductively closed sets is again a deductively closed set and since

$$Cl^{sk}_{\mathcal{R}}(A) = \bigcap_{\text{c-representation } \kappa,\ \kappa \models \mathcal{R}} Cl_\kappa(A) \tag{11}$$

we conclude that $Cl^{sk}_{\mathcal{R}}(A)$ is deductively closed. $\square$

Note that if $\mathcal{R}$ is consistent and A is inconsistent, then $Cl_\kappa(A)$ is the set of all formulas and thus inconsistent for every κ accepting $\mathcal{R}$. The next two propositions state relationships between the consistency of $\mathcal{R}$ and of $Cl^{sk}_{\mathcal{R}}(A)$.

Proposition 17 *(consistency of $Cl^{sk}_{\mathcal{R}}(A)$). For every knowledge base $\mathcal{R}$ and every formula A, the set $Cl^{sk}_{\mathcal{R}}(A)$ is consistent if and only if $\mathcal{R}$ is consistent and A is consistent.*

Proof. (i) We distinguish three cases and start with the case that $\mathcal{R}$ is consistent and A is consistent. Similarly as in the proof of Proposition 16, we first consider $Cl_\kappa(A)$ for a c-representation κ accepting $\mathcal{R}$. Since $Cl_\kappa(A)$ is deductively closed as shown in the proof of Proposition 16, $Cl_\kappa(A)$ is consistent if and only if $\bot \notin Cl_\kappa(A)$. Since A is consistent, we have $\kappa(A) < \infty$. If we had $\bot \in Cl_\kappa(A)$, then this would imply $A \mathrel{|\!\!\sim}_\kappa \bot$ and thus $\kappa(A\bot) < \kappa(A\top)$ which means $\kappa(\bot) = \infty < \kappa(A)$, leading to a contradiction. Therefore, $Cl_\kappa(A)$ is consistent. Using (11) and $\bot \notin Cl_\kappa(A)$ for every c-representation κ accepting $\mathcal{R}$, we get $\bot \notin Cl^{sk}_{\mathcal{R}}(A)$. Thus, since $Cl^{sk}_{\mathcal{R}}(A)$ is deductively closed, it is also consistent.

(ii) If $\mathcal{R}$ is consistent and A is inconsistent, then $Cl^{sk}_{\mathcal{R}}(A)$ is the set of all formulas which is trivially inconsistent.

(iii) If $\mathcal{R}$ is inconsistent, then for any formula A, $Cl^{sk}_{\mathcal{R}}(A)$ is again the set of all formulas and thus inconsistent. $\square$

Similarly as the inference relation $\mid\!\sim_\kappa$ satisfies (DI) where κ is any c-representation accepting $\mathcal{R}$, this direct inference property also holds for skeptical c-inference.

Proposition 18 *($\mid\!\sim^{sk}_{\mathcal{R}}$ satisfies (DI)). Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a knowledge base. Then for every $i \in \{1, \ldots, n\}$, $A_i \mid\!\sim^{sk}_{\mathcal{R}} B_i$.*

Proof. For every c-representation κ accepting $\mathcal{R}$, we have $A_i \mid\!\sim_\kappa B_i$ and thus $B_i \in Cl_\kappa(A_i)$. Using (11) yields $A_i \mid\!\sim^{sk}_{\mathcal{R}} B_i$. □

Since Proposition 12 gives us that c-inference includes and extends system P inferences, c-inference satisfies all properties included in system P. We here recall some major properties that can be derived from system P and hence also hold for c-inference ($A, B, C \in \mathfrak{L}$):

(SCL) Supraclassicality $\quad A \models B$ implies $A \mid\!\sim B$ [38]

states that the set of nonmonotonic inferences from a formula includes the classical consequences of this formula; (SCL) follows from (REF) and (RW).

(DED) Deduction $\quad AB \mid\!\sim C$ implies $A \mid\!\sim (B \Rightarrow C)$ [17]

introduces the so-called "tough half of the deduction theorem" to nonmonotonic inference relations; (DED) follows from (LLE), (OR), (SCL) and (RW).

An inference relation suffers from the *Drowning Problem* [42,16] if it does not allow us to infer properties of a superclass for a subclass that is exceptional with respect to another property because the respective conditional is "drowned" by others. E.g., if penguins are exceptional birds with respect to flying but not with respect to having wings, we would reasonably expect that penguins have wings. While system Z is known to suffer from the Drowning Problem, in [5] it is argued that skeptical c-inference does not suffer from the Drowning Problem.

Furthermore, it has been argued that skeptical c-inference properly deals with irrelevance in the sense that variables that do not appear in the knowledge base do not change the outcome of the inferences drawn with c-inference [5]. Different notions of irrelevance in the context of possibilistic semantics have been studied by Benferhat, Dubois and Prade [15]. Delgrande and Pelletier [20] investigate irrelevance for conditional logics and default reasoning, and they provide postulates expressing desirable properties for notions of irrelevance. Here, we provide a detailed formalization of irrelevance with respect to new or existing, but irrelevant symbols, and a proof that $\mid\!\sim^{sk}_{\mathcal{R}}$ satisfies irrelevance based on this formalization. For giving an axiomatic formalization of irrelevance with respect to symbols, we distinguish three different variants: Irrelevance with respect to new symbols (Irr-S1), irrelevance with respect to formulas involving already occurring symbols (Irr-S2), and irrelevance with respect to formulas involving new and already occurring symbols (Irr-S). Thus, the latter combines (Irr-S1) and (Irr-S2), so that (Irr-S) implies both (Irr-S1) and (Irr-S2). For defining the notions, we assume that $\mathcal{R}$ is a knowledge base over Σ, and that $\mid\!\sim_{\mathcal{R}}$ is an inference relation based on $\mathcal{R}$, as exemplified by system P inference, system Z, or c-representations. Furthermore, let A, B be arbitrary formulas over Σ, and let Σ_{new} be any signature such that $\Sigma \cap \Sigma_{new} = \emptyset$.

(Irr-S1) *Irrelevance with respect to new symbols*
If C is a formula over Σ_{new} then

$$A \mid\!\sim_{\mathcal{R}} B \text{ implies } AC \mid\!\sim_{\mathcal{R}'} B$$

where $\mathcal{R}'$ is the knowledge base over $\Sigma' = \Sigma \cup \Sigma_{new}$ containing exactly the same conditionals as $\mathcal{R}$. (Thus, no element from Σ_{new} is mentioned in $\mathcal{R}'$.)

(Irr-S2) *Irrelevance with respect to existing symbols*
If C is a formula over Σ such that there is a formula C' over Σ with $C \equiv C'$ and C' does not use any atom occurring in $\mathcal{R}$ then

$$A \mid\!\sim_{\mathcal{R}} B \text{ implies } AC \mid\!\sim_{\mathcal{R}} B.$$

(Irr-S) *Irrelevance with respect to new and existing symbols*
If C is a formula over $\Sigma \cup \Sigma_{new}$ such that there is a formula C' over $\Sigma \cup \Sigma_{new}$ with $C \equiv C'$ and C' does not use any atom occurring in $\mathcal{R}$ then

$$A \mid\!\sim_{\mathcal{R}} B \text{ implies } AC \mid\!\sim_{\mathcal{R}'} B$$

where $\mathcal{R}'$ is the knowledge base over $\Sigma' = \Sigma \cup \Sigma_{new}$ containing exactly the same conditionals as $\mathcal{R}$.

In the possibilistic setting used in [15], a special subclass of possibility distributions accepting a knowledge base $\mathcal{R}$ is defined, consisting of all distributions that coincide on two worlds ω, ω' if ω and ω' coincide on the interpretations of all symbols occurring in $\mathcal{R}$. These possibility distributions are said to cope with irrelevance[2] with respect to $\mathcal{R}$ [15, Definition 3]; thus, if ω and ω' only differ in the interpretations of symbols not occurring in $\mathcal{R}$, possibility distributions coping with irrelevance with respect to $\mathcal{R}$ must coincide on ω and ω'. The nonmonotonic inference relation based on $\mathcal{R}$ obtained by taking only possibility distributions that cope with irrelevance with respect to $\mathcal{R}$ satisfies (Irr-S1) according to [15, Proposition 2]; analogously, it can be shown that this inference relation also satisfies (Irr-S2) and (Irr-S). Without having to select a subclass of c-representations, the three irrelevance properties also hold for skeptical c-inference.

Proposition 19 *($\mid\!\sim^{sk}_{\mathcal{R}}$ and irrelevance). For every knowledge base $\mathcal{R}$, skeptical c-inference $\mid\!\sim^{sk}_{\mathcal{R}}$ satisfies (Irr-S1), (Irr-S2), and (Irr-S).*

Proof. Because (Irr-S) implies both (Irr-S1) and (Irr-S2), it suffices to show (Irr-S). For proving that $\mid\!\sim^{sk}_{\mathcal{R}}$ satisfies (Irr-S), we elaborate on the close correspondence between the set of c-representations of $\mathcal{R}$ and $\mathcal{R}'$.

Because Σ and Σ_{new} are disjoint, every world ω over $\Sigma' = \Sigma \cup \Sigma_{new}$ can uniquely be split into worlds ω_Σ and $\omega_{\Sigma_{new}}$ over Σ and Σ_{new}, respectively, such that $\omega \equiv \omega_\Sigma \wedge \omega_{\Sigma_{new}}$. Let κ be an OCF over the worlds over Σ. We uniquely extend κ to an OCF κ' over the worlds over $\Sigma' = \Sigma \cup \Sigma_{new}$ by defining $\kappa'(\omega) = \kappa(\omega_\Sigma)$ if $\omega \equiv \omega_\Sigma \wedge \omega_{\Sigma_{new}}$. Then we have:

$$\begin{array}{l}\kappa \text{ is a c-representation for } \mathcal{R} \text{ iff } \kappa' \text{ is a c-representation for } \mathcal{R}', \text{ and every c-representation } \kappa' \text{ for} \\ \mathcal{R}' \text{ can be obtained in this way.}\end{array} \tag{12}$$

The observation (12) holds due to

$$\kappa'(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \;=\; \sum_{\substack{1 \leqslant i \leqslant n \\ \omega_\Sigma \wedge \omega_{\Sigma_{new}} \models A_i \overline{B}_i}} \eta_i \;=\; \sum_{\substack{1 \leqslant i \leqslant n \\ \omega_\Sigma \models A_i \overline{B}_i}} \eta_i = \kappa(\omega_\Sigma) \tag{13}$$

where $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ because Σ_{new} does not contain any atom occurring in A_i or B_i and because the solutions of $CR(\mathcal{R})$ and $CR(\mathcal{R}')$ coincide since the symbols in Σ_{new} do not have any influence on the constraints for the impacts η_i. Thus, we conclude that for all formulas A_Σ and $A_{\Sigma_{new}}$ over Σ and Σ_{new}, respectively, we have $\kappa(A_\Sigma) = \kappa'(A_\Sigma A_{\Sigma_{new}})$ for every c-representation κ of $\mathcal{R}$. If A and B are formulas over Σ and C' is a formula over Σ_{new}, we get in particular $\kappa(AB) = \kappa'(AC'B)$ and $\kappa(A\overline{B}) = \kappa'(AC'\overline{B})$ for every c-representation κ of $\mathcal{R}$. Due to (12), from $A \mid\!\sim^{sk}_{\mathcal{R}} B$ we therefore get $AC' \mid\!\sim^{sk}_{\mathcal{R}'} B$ and thus $AC \mid\!\sim^{sk}_{\mathcal{R}'} B$ for every formula C over Σ' with $C \equiv C'$. □

Thus, while c-inference satisfies (Irr-S1), (Irr-S2), and (Irr-S), which is also true for system Z, it is well-known that system P-inference (or π-entailment) does not satisfy any of these notions of irrelevance. If a knowledge base $\mathcal{R}$ entails that birds usually fly, system P does not license to infer that red birds usually fly even if nothing is said about red objects in $\mathcal{R}$.

4.2. Credulous and weakly skeptical c-inference

Any single c-representation accepting $\mathcal{R}$ can be used for defining an inference relation inductively completing $\mathcal{R}$ that can be used as a prototype while exhibiting desirable properties [45]; this observation provides practical relevance for the following definition.

Definition 20 *(credulous c-inference, $\mid\!\sim^{cr}_{\mathcal{R}}$).* Let $\mathcal{R}$ be a knowledge base and let A, B be formulas. B is a *credulous c-inference from A in the context of* $\mathcal{R}$, denoted by $A \mid\!\sim^{cr}_{\mathcal{R}} B$, if there is a c-representation κ for $\mathcal{R}$ such that $A \mid\!\sim_\kappa B$ holds.

Credulous c-inference is a liberal extension of skeptical c-inference since $A \mid\!\sim^{sk}_{\mathcal{R}} B$ implies $A \mid\!\sim^{cr}_{\mathcal{R}} B$ for any consistent knowledge base $\mathcal{R}$ and any formulas A, B. However, credulous c-inference has the disadvantage that we might have both, $A \mid\!\sim^{cr}_{\mathcal{R}} B$ and $A \mid\!\sim^{cr}_{\mathcal{R}} \overline{B}$. The following example illustrates this.

Example 21. We recall Example 13 and select the two solutions $\vec{\eta}_1 = (1, 2, 2, 1)$ and $\vec{\eta}_2 = (1, 2, 2, 3)$ from $Sol(\mathcal{R}_{pen})$. These vectors induce the ranking functions $\kappa_{\vec{\eta}_1}$ and $\kappa_{\vec{\eta}_2}$ given in Table 1. From this table we obtain $\kappa_{\vec{\eta}_1}(pbf\overline{w}) = 3 < 4 = \kappa_{\vec{\eta}_1}(p\overline{b}f\overline{w})$ and thus $pf\overline{w} \mid\!\sim_{\kappa_{\vec{\eta}_1}} b$ whilst $\kappa_{\vec{\eta}_2}(pbf\overline{w}) = 5 > 4 = \kappa_{\vec{\eta}_2}(p\overline{b}f\overline{w})$ and thus also $pf\overline{w} \mid\!\sim_{\kappa_{\vec{\eta}_2}} \overline{b}$.

Therefore, we will introduce a new notion of inference lying between skeptical and credulous inference which we call *weakly skeptical*. It is strictly more liberal than skeptical inference, but less permissive than credulous inference. It coincides with credulous inference except that the inference of B from a formula A is not allowed if also $\overline{B}$ could be inferred credulously from A.

Note that just as skeptical and credulous inference are general concepts, also the notion of weakly skeptical inference is general in the sense that it can be defined with respect to any arbitrary set of models. Later on in Section 4.3, we

[2] In [15], a knowledge base may additionally contain a set of propositional formulas representing strict knowledge.

Table 2
Verification/falsification behavior for $\mathcal{R}_{bfa}$ from Example 23 and two solution vectors $\vec{\eta}_1, \vec{\eta}_2 \in Sol(\mathcal{R}_{bfa})$ and their induced ranking functions $\kappa_{\vec{\eta}_1}$ and $\kappa_{\vec{\eta}_2}$.

ω	r_1: $(f\|b)$	r_2: $(a\|b)$	r_3: $(a\|fb)$	impact on ω	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$
abf	v	v	v	0	0	0
$ab\overline{f}$	f	v	–	η_1	1	1
$a\overline{b}f$	–	–	–	0	0	0
$a\overline{b}\overline{f}$	–	–	–	0	0	0
$\overline{a}bf$	v	f	f	$\eta_2+\eta_3$	1	1
$\overline{a}b\overline{f}$	f	f	–	$\eta_1+\eta_2$	2	1
$\overline{a}\overline{b}f$	–	–	–	0	0	0
$\overline{a}\overline{b}\overline{f}$	–	–	–	0	0	0
	η_1	η_2	η_3			
$\vec{\eta}_1$	1	1	0			
$\vec{\eta}_2$	1	0	1			

will formally define this general notion; here, we start with weakly skeptical inference taking all c-representations of a knowledge base into account.

Definition 22 *(weakly skeptical c-inference, $\mid\!\sim^{ws}_{\mathcal{R}}$).* Let $\mathcal{R}$ be a knowledge base and let A, B be formulas. B is a *weakly skeptical c-inference from A in the context of $\mathcal{R}$*, denoted by $A \mid\!\sim^{ws}_{\mathcal{R}} B$, if $A \equiv \perp$,[3] or there is a c-representation κ for $\mathcal{R}$ such that $A \mid\!\sim_{\kappa} B$ holds and there is no c-representation κ' for $\mathcal{R}$ such that $A \mid\!\sim_{\kappa'} \overline{B}$.

The following example shows that weakly skeptical c-inference allows for some desirable inferences that are not possible under skeptical c-inference.

Example 23. Let $\mathcal{R}_{bfa} = \{r_1, r_2, r_3\}$ be the following set of conditionals:

r_1 :	$(f\|b)$	birds fly
r_2 :	$(a\|b)$	birds are animals
r_3 :	$(a\|fb)$	flying birds are animals

The verification/falsification behavior of $\mathcal{R}_{bfa}$ is given in Table 2, together with two solution vectors $\vec{\eta}_1 = (1, 1, 0)$ and $\vec{\eta}_2 = (1, 0, 1)$ for $CR(\mathcal{R}_{bfa})$ and their induced OCFs $\kappa_{\vec{\eta}_1}$ and $\kappa_{\vec{\eta}_2}$.

Consider a bird that lost its ability to fly ($b\overline{f}$). We would expect that this bird is still considered an animal (a). Yet, for skeptical c-inference it holds that $b\overline{f} \not\mid\!\sim^{sk}_{\mathcal{R}_{bfa}} a$, because, for instance, $\kappa_{\vec{\eta}_2}(ab\overline{f}) = 1 = \kappa_{\vec{\eta}_2}(\overline{a}b\overline{f})$ and thus $b\overline{f} \not\mid\!\sim_{\kappa_{\vec{\eta}_2}} a$. On the other hand, $b\overline{f} \mid\!\sim^{cr}_{\mathcal{R}_{bfa}} a$ due to $\kappa_{\vec{\eta}_1}(ab\overline{f}) = 1 < 2 = \kappa_{\vec{\eta}_1}(\overline{a}b\overline{f})$, and furthermore, $b\overline{f} \not\mid\!\sim^{cr}_{\mathcal{R}_{bfa}} \overline{a}$ holds. Therefore, from Definition 22 we get $b\overline{f} \mid\!\sim^{ws}_{\mathcal{R}_{bfa}} a$.

The next proposition states that weakly skeptical inference lies indeed strictly between skeptical and credulous inference.

Proposition 24. *For every consistent knowledge base $\mathcal{R}$ we have*

$$\mid\!\sim^{sk}_{\mathcal{R}} \;\subseteq\; \mid\!\sim^{ws}_{\mathcal{R}} \;\subseteq\; \mid\!\sim^{cr}_{\mathcal{R}} \tag{14}$$

and there are knowledge bases such that the inclusions in (14) are strict.

Proof. The set inclusions follow immediately from the definitions of the inference relations. For showing that the inclusions are strict, corresponding examples can be given. For instance, Example 23 shows that for the knowledge base $\mathcal{R} = \mathcal{R}_{bfa}$ the relation $\mid\!\sim^{sk}_{\mathcal{R}} \subsetneq \mid\!\sim^{ws}_{\mathcal{R}}$ is a proper inclusion. □

Weakly skeptical c-inference satisfies several of the crucial properties of nonmonotonic entailment relations discussed above.

Proposition 25 *([4]). For every consistent knowledge base $\mathcal{R}$, weakly skeptical c-inference satisfies the following properties:*

[3] Note that the handling of $A \equiv \perp$ as a special case corresponds directly to the handling of $A \equiv \perp$ in the definition of $\mid\!\sim_{\kappa}$ in (3).

1. *(REF)* $A \mid\!\sim A$
2. *(LLE)* $\dfrac{\models A \equiv B,\ A \mid\!\sim C}{B \mid\!\sim C}$
3. *(RW)* $\dfrac{A \mid\!\sim B,\ B \models C}{A \mid\!\sim C}$
4. *(VCM)* $\dfrac{A \mid\!\sim BC}{AB \mid\!\sim C}$
5. *(WAND)* $\dfrac{A \mid\!\sim B,\ A\overline{C} \mid\!\sim C}{A \mid\!\sim BC}$

Proposition 25 was shown directly in [4]. We will show these properties explicitly for the more general case of credulous and weakly skeptical inference over arbitrary sets of ranking models in Section 4.3.

Note that Proposition 25 covers all postulates of system O except for (WOR); it is still an open problem whether weakly skeptical c-inference satisfies (WOR).

The following definition generalizes the closure operator introduced in Definition 15.

Definition 26 *(closure).* Let $\mathcal{R}$ be a knowledge base over the propositional language $\mathcal{L}$, let $\mid\!\sim^{\diamond}_{\mathcal{R}} \subseteq \mathcal{L} \times \mathcal{L}$ be a binary relation, and $A \in \mathcal{L}$ a formula. Then

$$Cl^{\diamond}_{\mathcal{R}}(A) = \{B \mid A \mid\!\sim^{\diamond}_{\mathcal{R}} B\} \tag{15}$$

is the *closure* of A under $\mid\!\sim^{\diamond}_{\mathcal{R}}$ in the context of $\mathcal{R}$.

Thus, $Cl^{cr}_{\mathcal{R}}(A) = \{B \mid A \mid\!\sim^{cr}_{\mathcal{R}} B\}$ and $Cl^{ws}_{\mathcal{R}}(A) = \{B \mid A \mid\!\sim^{ws}_{\mathcal{R}} B\}$ are the closure of A in the context of $\mathcal{R}$ under credulous and weakly skeptical c-inference, respectively.

While the closure under skeptical c-inference is deductively closed and consistent (Propositions 16 and 17), Example 21 shows that the closure under credulous c-inference may be inconsistent. In the following we will investigate this question for the closure also with respect to weakly skeptical c-inference.

Proposition 27. *Let $\mathcal{R}$ be a knowledge base, A a formula, and $\tau \in \{ws, cr\}$.*

1. *If $\mathcal{R}$ is inconsistent, then $Cl^{\tau}_{\mathcal{R}}(A) = \emptyset$.*
2. *If $\mathcal{R}$ is consistent and A is consistent, then $\perp \notin Cl^{\tau}_{\mathcal{R}}(A)$.*

Proof.
1. If $\mathcal{R}$ is inconsistent, it does not have any model which is, however, a prerequisite for every credulous and every weakly skeptical inference. Furthermore, there is no formula B together with an OCF κ such that $\perp \mid\!\sim_{\kappa} B$ holds since $\kappa(\perp B) < \kappa(\perp \overline{B})$ can not hold.
2. If $\mathcal{R}$ is consistent and $\perp \in Cl^{\tau}_{\mathcal{R}}(A)$ then there would be a κ accepting $\mathcal{R}$ such that $A \mid\!\sim_{\kappa} \perp$ which is only possible if $A \equiv \perp$ (cf. the proof of Proposition 17). $\square$

Weakly skeptical c-inference also satisfies (DED) and (SCL) if $\mathcal{R}$ is consistent. Before proving this, we first state an auxiliary proposition dealing with $\mid\!\sim^{ws}_{\mathcal{R}}$ and inconsistency.

Proposition 28 *($\mid\!\sim^{ws}_{\mathcal{R}}$ and inconsistency). If $\mathcal{R}$ is inconsistent then $A \mid\!\sim^{ws}_{\mathcal{R}} B$ iff $A \equiv \perp$. If $\mathcal{R}$ is consistent then $A \mid\!\sim^{ws}_{\mathcal{R}} B$ iff $A \equiv \perp$, or $A \not\equiv \perp$ and there is a c-representation κ for $\mathcal{R}$ such that $\kappa(AB) < \kappa(A\overline{B})$, and for all c-representations κ', it holds that $\kappa'(AB) \leqslant \kappa'(A\overline{B})$.*

Proof. If $\mathcal{R}$ is inconsistent then there can be no c-representation of $\mathcal{R}$ so $A \mid\!\sim^{ws}_{\mathcal{R}} B$ can only hold for the trivial case $A \equiv \perp$. In case that $\mathcal{R}$ is consistent and $A \not\equiv \perp$, the statement is a straightforward rephrasing of Definition 22 using (3). $\square$

Proposition 29 *($\mid\!\sim^{ws}_{\mathcal{R}}$ and (DED), (SCL)). Let $\mathcal{R}$ be consistent. Then the weakly skeptical c-inference relation $\mid\!\sim^{ws}_{\mathcal{R}}$ satisfies (DED) and (SCL). If $\mathcal{R}$ is inconsistent, neither of the properties hold.*

Proof. Let us first consider the case that $\mathcal{R}$ is inconsistent. Then for (DED), $AB \mid\!\sim^{ws}_{\mathcal{R}} C$ implies $AB \equiv \perp$ according to Proposition 28, but $A \not\equiv \perp$ would still be possible, in which case $A \mid\!\sim^{ws}_{\mathcal{R}} B \Rightarrow C$ would not hold. For (SCL), $A \models B$ usually does not imply $A \equiv \perp$, so also $A \mid\!\sim^{ws}_{\mathcal{R}} B$ would not hold in general.

In the following, we presuppose that $\mathcal{R}$ is consistent, so there must be at least one c-representation of $\mathcal{R}$.

In order to prove (DED), let $AB \mid\!\sim^{ws}_{\mathcal{R}} C$ hold. We have to show that $A \mid\!\sim^{ws}_{\mathcal{R}} B \Rightarrow C$ also holds. If $A \equiv \bot$, both statements hold trivially. If $A \not\equiv \bot$ but $AB \equiv \bot$, we have $A \models \overline{B}$ which implies $A(B \Rightarrow C) \equiv A(\overline{B} \vee C) \equiv A\overline{B} \vee AC \equiv A \vee AC \equiv A$, and $AB\overline{C} \equiv \bot$. Hence $\kappa(A(B \Rightarrow C)) = \kappa(A) < \infty = \kappa(AB\overline{C})$ for all c-representations of $\mathcal{R}$. This shows $A \mid\!\sim^{ws}_{\mathcal{R}} B \Rightarrow C$. Let us now assume that $AB \not\equiv \bot$. Due to the presupposition $AB \mid\!\sim^{ws}_{\mathcal{R}} C$ and Proposition 28, there is a c-representation κ of $\mathcal{R}$ such that $\kappa(ABC) < \kappa(AB\overline{C})$, and for all c-representations κ', it holds that $\kappa'(ABC) \leqslant \kappa'(AB\overline{C})$. Since $A(B \Rightarrow C) \equiv A(\overline{B} \vee C) \equiv A\overline{B} \vee ABC$, $\kappa''(A(B \Rightarrow C)) \leqslant \kappa''(ABC)$ for all c-representations κ'' of $\mathcal{R}$. Moreover, $A \wedge \neg(B \Rightarrow C) \equiv AB\overline{C}$. Therefore, with the same κ resp. κ' as above, we obtain $\kappa(A(B \Rightarrow C)) \leqslant \kappa(ABC) < \kappa(AB\overline{C}) = \kappa(A \wedge \neg(B \Rightarrow C))$, and $\kappa'(A(B \Rightarrow C)) \leqslant \kappa'(ABC) \leqslant \kappa'(AB\overline{C}) = \kappa'(A \wedge \neg(B \Rightarrow C))$. Hence the conclusion follows.

We will now prove (SCL), so let $A \models B$, i.e., $AB \equiv A$ and $A\overline{B} \equiv \bot$. If $A \equiv \bot$, $A \mid\!\sim^{ws}_{\mathcal{R}} B$ holds trivially. In case that $A \not\equiv \bot$, we have $\kappa(AB) = \kappa(A) < \infty = \kappa(A\overline{B})$ for all c-representations of $\mathcal{R}$, and hence $A \mid\!\sim^{ws}_{\mathcal{R}} B$ holds. $\square$

Just as skeptical c-inference, also the credulous and weakly skeptical inference modes properly cope with irrelevance with respect to symbols as formalized in Section 4.1.

Proposition 30 *($\mid\!\sim^{cr}_{\mathcal{R}}$, $\mid\!\sim^{ws}_{\mathcal{R}}$ and irrelevance). For every knowledge base $\mathcal{R}$, credulous c-inference $\mid\!\sim^{cr}_{\mathcal{R}}$ and weakly skeptical c-inference $\mid\!\sim^{ws}_{\mathcal{R}}$ satisfy (Irr-S1), (Irr-S2), and (Irr-S).*

Proof. The proof is along the lines of the proof of Proposition 19, exploiting the fact that the conjunction of a formula A with any formula C over symbols not occurring in $\mathcal{R}$ is evaluated to the same value as A by every c-representation of $\mathcal{R}$. $\square$

4.3. Inference with respect to minimal c-representations

While c-representations provide an excellent basis for model-based inference [26,25], from the point of view of minimal specificity (see e.g. [13]), those c-representations yielding minimal degrees of implausibility are most interesting.

Different orderings on $Sol(CR(\mathcal{R}))$ leading to different minimality notions can be used.

Definition 31 *($\preccurlyeq_+$, $\preccurlyeq_{cw}$, $\preccurlyeq_O$, sum-, cw-, ind-minimal).* Let $\mathcal{R}$ be a knowledge base and $\vec{\eta}$, $\vec{\eta}' \in Sol(CR(\mathcal{R}))$.

1. The relation $\preccurlyeq_+$ on $Sol(CR(\mathcal{R}))$ is given by:

$$(\eta_1, \ldots, \eta_n) \preccurlyeq_+ (\eta'_1, \ldots, \eta'_n) \quad \text{iff} \quad \sum_{1 \leqslant i \leqslant n} \eta_i \leqslant \sum_{1 \leqslant i \leqslant n} \eta'_i \tag{16}$$

   A vector $\vec{\eta}$ is *sum-minimal* iff $\vec{\eta} \preccurlyeq_+ \vec{\eta}'$ for all $\vec{\eta}' \in Sol(CR(\mathcal{R}))$. We write $\vec{\eta} \prec_+ \vec{\eta}'$ iff $\vec{\eta} \preccurlyeq_+ \vec{\eta}'$ and $\vec{\eta}' \not\preccurlyeq_+ \vec{\eta}$.
2. The relation $\preccurlyeq_{cw}$ on $Sol(CR(\mathcal{R}))$ is given by:

$$(\eta_1, \ldots, \eta_n) \preccurlyeq_{cw} (\eta'_1, \ldots, \eta'_n) \quad \text{iff} \quad \eta_i \leqslant \eta'_i \text{ for all } i \in \{1, \ldots, n\} \tag{17}$$

   A vector $\vec{\eta}$ is *componentwise minimal* or *cw-minimal* iff there is no vector $\vec{\eta}' \in Sol(CR(\mathcal{R}))$ such that $\vec{\eta}' \preccurlyeq_{cw} \vec{\eta}$ and $\vec{\eta} \not\preccurlyeq_{cw} \vec{\eta}'$.
3. The relation $\preccurlyeq_O$ on $Sol(CR(\mathcal{R}))$ is given by:

$$(\eta_1, \ldots, \eta_n) \preccurlyeq_O (\eta'_1, \ldots, \eta'_n) \quad \text{iff} \quad \kappa_{\vec{\eta}}(\omega) \leqslant \kappa_{\vec{\eta}'}(\omega) \text{ for all } \omega \in \Omega \tag{18}$$

   A vector $\vec{\eta}$ is *induced minimal* or *ind-minimal* iff there is no vector $\vec{\eta}' \in Sol(CR(\mathcal{R}))$ such that $\vec{\eta}' \preccurlyeq_O \vec{\eta}$ and $\vec{\eta} \not\preccurlyeq_O \vec{\eta}'$.

Thus, while sum-minimal and cw-minimal are defined by just taking the components of the solution vectors $\vec{\eta}$ into account, ind-minimality refers to the ranking function induced by a solution vector.

Example 32. Consider $\mathcal{R}_{bfa}$ from Example 23. From (6), we get

$$\begin{aligned}
&\eta_1 > 0 \\
&\eta_2 > 0 - min\{\eta_1, \eta_3\} \\
&\eta_3 > 0 - \eta_2
\end{aligned}$$

and since $\eta_i \geqslant 0$ according to (5), the two vectors

$$\begin{aligned}
\vec{\eta}_1 &= (\eta_1, \eta_2, \eta_3) = (1, 1, 0) && \text{(cw-, sum-minimal)} \\
\vec{\eta}_2 &= (\eta_1, \eta_2, \eta_3) = (1, 0, 1) && \text{(cw-, sum-, ind-minimal)}
\end{aligned}$$

are two different solutions of $CR(\mathcal{R}_{bfa})$ that are both sum-minimal and cw-minimal in $Sol(\mathcal{R}_{bfa})$ with respect to $\preccurlyeq_+$; moreover, there are no other sum-minimal or cw-minimal solutions. Table 2 shows the ranking functions induced by $\vec{\eta}_1$ and $\vec{\eta}_2$. Only $\vec{\eta}_2$ is ind-minimal because $\kappa_{\vec{\eta}_2}(\overline{a}b\overline{f}) = 1 < 2 = \kappa_{\vec{\eta}_1}(\overline{a}b\overline{f})$ and $\kappa_{\vec{\eta}_2}(\omega) = \kappa_{\vec{\eta}_1}(\omega)$ for all ω with $\omega \neq \overline{a}b\overline{f}$.

Each of the ordering relations $\preccurlyeq_\bullet$ with $\bullet \in \{+, cw, O\}$ induces a set of minimal solutions of $CR(\mathcal{R})$, denoted by

$$Sol^{min}_{\preccurlyeq_\bullet}(CR(\mathcal{R})) = \{\vec{\eta} \mid \vec{\eta} \in Sol(CR(\mathcal{R})) \text{ and } \vec{\eta} \text{ is } \bullet\text{-minimal}\} \tag{19}$$

that are minimal with respect to $\preccurlyeq_\bullet$. These minimal models can be viewed as preferred models and, in the following, we will define nonmonotonic inference relations based on these preferred models. Skeptical, credulous, and weakly skeptical inference versions are obtained from the definitions of $\mathrel{|\!\!\sim}^{sk}_{\mathcal{R}}$, $\mathrel{|\!\!\sim}^{cr}_{\mathcal{R}}$, and $\mathrel{|\!\!\sim}^{ws}_{\mathcal{R}}$ by replacing the set of all solutions of $CR(\mathcal{R})$ by the respective set of minimal solutions. The following definition formally introduces the resulting inference relations.

Definition 33 *(min-inference).* Let $\mathcal{R}$ be a knowledge base, let A, B be formulas, and let $\bullet \in \{+, cw, O\}$.

1. B is *a skeptical* $\bullet$*-min-inference from* A *in the context of* $\mathcal{R}$, denoted by $A \mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} B$, if $A \mathrel{|\!\!\sim}_{\kappa_{\vec{\eta}}} B$ holds for all $\vec{\eta} \in Sol^{min}_{\preccurlyeq_\bullet}(CR(\mathcal{R}))$.
2. B is a *credulous* $\bullet$*-min-inference from* A *in the context of* $\mathcal{R}$, denoted by $A \mathrel{|\!\!\sim}^{cr,\bullet}_{\mathcal{R}} B$, if there is a $\vec{\eta} \in Sol^{min}_{\preccurlyeq_\bullet}(CR(\mathcal{R}))$ such that $A \mathrel{|\!\!\sim}_{\kappa_{\vec{\eta}}} B$ holds.
3. B is a *weakly skeptical* $\bullet$*-min-inference from* A *in the context of* $\mathcal{R}$, denoted by $A \mathrel{|\!\!\sim}^{ws,\bullet}_{\mathcal{R}} B$, if there is a $\vec{\eta} \in Sol^{min}_{\preccurlyeq_\bullet}(CR(\mathcal{R}))$ such that $A \mathrel{|\!\!\sim}_{\kappa_{\vec{\eta}}} B$ holds and there is no $\vec{\eta}\,' \in Sol^{min}_{\preccurlyeq_\bullet}(CR(\mathcal{R}))$ such that $A \mathrel{|\!\!\sim}_{\kappa_{\vec{\eta}'}} \overline{B}$.

The different notions of min-inference may lead to distinct sets of inferences from a knowledge base $\mathcal{R}$; this aspect will be illustrated by elaborating several properties of min-inference in the remaining parts of this article. On the other hand, there are knowledge bases where all three modes of min-inference coincide. Consider again the knowledge base $\mathcal{R}_{pen}$ from Example 13. Here,

$$\vec{\eta}_1 = (1, 2, 2, 1) \qquad \text{(cw-, sum-, ind-minimal)}$$

is the only solution vector of $CR(\mathcal{R}_{pen})$ that is minimal with respect to any of the three cw-, sum-, and ind-minimality criteria. Thus, we have $\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}} = \mathrel{|\!\!\sim}^{cr,\bullet}_{\mathcal{R}} = \mathrel{|\!\!\sim}^{ws,\bullet}_{\mathcal{R}}$ for every $\bullet \in \{+, cw, O\}$.

The system Z ranking function $\kappa^Z_{\mathcal{R}_{pen}}$ for $\mathcal{R}_{pen}$, which is not a c-representation, concludes that penguins without wings are normally birds, i.e., $p\overline{w} \mathrel{|\!\!\sim}_{\kappa^Z_{\mathcal{R}_{pen}}} b$ (cf. Table 1). While this entailment is supported by the argument of specificity because penguins are more specific than birds, it does not hold for any of the three min-inference relations. An inference relation extending c-inference that also deals properly with specificity like system Z, is system W that fully captures both skeptical c-inference and system Z [31].

Let $\tau \in \{sk, cr, ws\}$ and let $\bullet \in \{+, cw, O\}$. Using the construction of Definition 26, the closure for each of the nine min-inference relations $\mathrel{|\!\!\sim}^{\tau,\bullet}_{\mathcal{R}}$ is denoted by $Cl^{\tau,\bullet}_{\mathcal{R}}(A) = \{B \mid A \mathrel{|\!\!\sim}^{\tau,\bullet}_{\mathcal{R}} B\}$.

Proposition 34 *(skeptical min-inference). Let $\mathcal{R}$ be a knowledge base, let A be a formula, and let $\bullet \in \{+, cw, O\}$.*

1. $\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}}$ *satisfies system P.*
2. $Cl^{sk,\bullet}_{\mathcal{R}}(A)$ *is deductively closed.*
3. $Cl^{sk,\bullet}_{\mathcal{R}}(A)$ *is consistent iff $\mathcal{R}$ is consistent and A is consistent.*
4. $\mathrel{|\!\!\sim}^{sk,\bullet}_{\mathcal{R}}$ *satisfies (DI).*

Proof. The first property can be shown along the lines of the proof that $\mathrel{|\!\!\sim}^{sk}_{\mathcal{R}}$ satisfies system P (Proposition 11). The proof of (2) is analogous to the proof of Proposition 16, the proof of (3) is analogous to the proof of Proposition 17, and the proof of (4) is analogous to the proof of Proposition 18. □

Before investigating further properties for credulous and weakly skeptical c-inference, we will generalize these inference modes to any set of ranking models.

5. Modes of inference over arbitrary sets of ranking models

In this section, we will investigate the modes of skeptical, weakly skeptical, and credulous inference over sets of ranking models in a more general way. In the previous sections, we investigated inference relations defined over sets of c-representations, either by taking all c-representations into account, or by only considering minimal c-representations. Here, we will define and examine skeptical, weakly skeptical, and credulous inference defined over arbitrary sets of ranking models.

5.1. Modes of inference

We first define skeptical, credulous and weakly skeptical inference over arbitrary sets of ranking functions accepting a knowledge base.

Definition 35 *(Modes of inference $\mathrel{|\!\sim}^{sk,M}_{\mathcal{R}}$, $\mathrel{|\!\sim}^{ws,M}_{\mathcal{R}}$, $\mathrel{|\!\sim}^{cr,M}_{\mathcal{R}}$).* Let M be a subset of all ranking functions accepting a knowledge base $\mathcal{R}$, and let A and B be formulas. Then

1. $A \mathrel{|\!\sim}^{sk,M}_{\mathcal{R}} B$ if $A \mathrel{|\!\sim}_{\kappa} B$ for all $\kappa \in M$
2. $A \mathrel{|\!\sim}^{ws,M}_{\mathcal{R}} B$ if $A \equiv \bot$, or there is a $\kappa \in M$ such that $A \mathrel{|\!\sim}_{\kappa} B$ and there is no $\kappa' \in M$ such that $A \mathrel{|\!\sim}_{\kappa'} \overline{B}$
3. $A \mathrel{|\!\sim}^{cr,M}_{\mathcal{R}} B$ if there is a $\kappa \in M$ such that $A \mathrel{|\!\sim}_{\kappa} B$

Note that the c-inference relations defined earlier are special cases of the general inference relations defined in Definition 35. System P inference from a knowledge base $\mathcal{R}$ is also a special case of Definition 35, since it can be characterized as skeptical inference over all ranking functions accepting the knowledge base $\mathcal{R}$. For the special case of $|M| = 1$, all three inference modes yield the same inference relation.

Proposition 36 *(Inference over singleton sets). Let $\mathcal{R}$ be a knowledge base and let M be a set of ranking functions accepting $\mathcal{R}$ with $|M| = 1$. Then it holds that*

$$\mathrel{|\!\sim}^{sk,M}_{\mathcal{R}} = \mathrel{|\!\sim}^{ws,M}_{\mathcal{R}} = \mathrel{|\!\sim}^{cr,M}_{\mathcal{R}}.$$

The proof of Proposition 36 follows immediately from Definition 35.

By fixing a set of ranking functions M accepting a knowledge base $\mathcal{R}$, we can describe a hierarchy of the three inference modes, which generalizes Proposition 24.

Proposition 37 *(Hierarchy of inference modes). Let $\mathcal{R}$ be a knowledge base and let M be a set of ranking functions accepting $\mathcal{R}$. Then it holds that*

$$\mathrel{|\!\sim}^{sk,M}_{\mathcal{R}} \subseteqq \mathrel{|\!\sim}^{ws,M}_{\mathcal{R}} \subseteqq \mathrel{|\!\sim}^{cr,M}_{\mathcal{R}}.$$

On the other hand, by fixing either the skeptical or credulous inference mode, we can make the following general observations for inference over subsets M' of M.

Proposition 38 *(Inference over subsets). Let $\mathcal{R}$ be a knowledge base. Let M be a set of ranking functions accepting $\mathcal{R}$ and let M' be a subset of M, i.e. $M' \subseteqq M$. Then it holds that*

$$\mathrel{|\!\sim}^{sk,M}_{\mathcal{R}} \subseteqq \mathrel{|\!\sim}^{sk,M'}_{\mathcal{R}}$$
$$\mathrel{|\!\sim}^{cr,M}_{\mathcal{R}} \supseteqq \mathrel{|\!\sim}^{cr,M'}_{\mathcal{R}}.$$

For weakly skeptical inference, the relationship between inference relations defined over a set M and a subset of M is more intricate, as we will illustrate by using c-representations in Section 6.

5.2. Properties of credulous inference

We will now show that the properties (REF), (LLE), (RW), (VCM) and (WAND) hold for credulous inference over any set of ranking functions. In particular, this implies that these properties hold for credulous inference over any of the three sets of minimal c-representations, as well as credulous inference over all c-representations of a knowledge base $\mathcal{R}$. For all the following proofs we assume that for any antecedent A in a statement of the form $A \mathrel{|\!\sim} B$ it holds that $A \not\equiv \bot$, since, due to (3), all of the following propositions hold trivially for $A \equiv \bot$.

Proposition 39 *(credulous inference, $\mathrel{|\!\sim}^{cr,M}_{\mathcal{R}}$). Let $\mathcal{R}$ be a knowledge base, and M be a set of ranking functions accepting $\mathcal{R}$. Credulous inference over M, i.e., the inference relation $\mathrel{|\!\sim}^{cr,M}_{\mathcal{R}}$, satisfies:*

1. *(REF)* $\quad A \mathrel{|\!\sim} A$
2. *(LLE)* $\quad \dfrac{\models A \equiv B,\ A \mathrel{|\!\sim} C}{B \mathrel{|\!\sim} C}$

3. *(RW)* $\dfrac{A \mid\!\sim B,\ B \models C}{A \mid\!\sim C}$
4. *(VCM)* $\dfrac{A \mid\!\sim BC}{AB \mid\!\sim C}$
5. *(WAND)* $\dfrac{A \mid\!\sim B,\ A\overline{C} \mid\!\sim C}{A \mid\!\sim BC}$

Proof. 1. We need to show that for all $A \in \mathcal{L}$ there is a $\kappa \in M$ such that $A \mid\!\sim_\kappa A$. Since $A \not\equiv \bot$, we have $\kappa(A) < \kappa(A\overline{A}) = \kappa(\bot) = \infty$, and therefore $\kappa(AA) < \kappa(A\overline{A})$ for all $\kappa \models \mathcal{R}$.
2. We need to show that from $A \equiv B$ and $B \mid\!\sim^{cr,M}_{\mathcal{R}} C$ it follows that $A \mid\!\sim^{cr,M}_{\mathcal{R}} C$. Because $B \mid\!\sim^{cr,M}_{\mathcal{R}} C$, there is a $\kappa \in M$ such that $\kappa(BC) < \kappa(B\overline{C})$. Therefore, there is also a $\kappa \in M$ such that $\kappa(AC) < \kappa(A\overline{C})$ holds, implying $A \mid\!\sim^{cr,M}_{\mathcal{R}} C$.
3. We have to show that from $B \models C$ and $A \mid\!\sim^{cr,M}_{\mathcal{R}} B$ it follows that $A \mid\!\sim^{cr,M}_{\mathcal{R}} C$. Because $B \models C$, we also have $AB \models AC$ and $A\overline{C} \models A\overline{B}$, therefore for all ranking functions κ it holds that $\kappa(AB) \geqslant \kappa(AC)$ and $\kappa(A\overline{B}) \leqslant \kappa(A\overline{C})$. From $A \mid\!\sim^{cr,M}_{\mathcal{R}} B$ we get that there is a $\kappa \in M$ such that $\kappa(AB) < \kappa(A\overline{B})$. Let $\hat{\kappa}$ be the ranking function in M such that $\hat{\kappa}(AB) < \hat{\kappa}(A\overline{B})$. Then we have $\hat{\kappa}(AC) \leqslant \hat{\kappa}(AB) < \hat{\kappa}(A\overline{B}) \leqslant \hat{\kappa}(A\overline{C})$. Therefore there is a $\kappa \in M$ such that $\kappa(AC) < \kappa(A\overline{C})$.
4. We have to show that from $A \mid\!\sim^{cr,M}_{\mathcal{R}} BC$ it follows that $AB \mid\!\sim^{cr,M}_{\mathcal{R}} C$. Let $\hat{\kappa} \in M$ be the ranking function such that $\hat{\kappa}(ABC) < \hat{\kappa}(A\overline{BC})$. Then $\hat{\kappa}(ABC) < min\left\{\hat{\kappa}(AB\overline{C}), \hat{\kappa}(A\overline{B}C), \hat{\kappa}(A\overline{B}\,\overline{C})\right\}$ and thus $\hat{\kappa}(ABC) < \hat{\kappa}(AB\overline{C})$.
5. We have to show that from $A \mid\!\sim^{cr,M}_{\mathcal{R}} B$ and $A\overline{C} \mid\!\sim^{cr,M}_{\mathcal{R}} C$ it follows that $A \mid\!\sim^{cr,M}_{\mathcal{R}} BC$. Since $A\overline{C} \mid\!\sim^{cr,M}_{\mathcal{R}} C$ can not hold because $A\overline{C}C \equiv \bot$ and there is no κ such that $\kappa(\bot) < \kappa(A\overline{C})$, (WAND) holds. $\square$

5.3. Properties of weakly skeptical inference

We will now investigate weakly skeptical inference relations over arbitrary sets of ranking functions with respect to the properties (REF), (LLE), (RW), (VCM), and (WAND). In particular, this implies that these properties hold for weakly skeptical inference over any of the three sets of minimal c-representations, as well as weakly skeptical inference over all c-representations of a knowledge base $\mathcal{R}$, and therefore serves as a proof of Proposition 25. Again, for all the following proofs we assume that for any antecedent A in a statement of the form $A \mid\!\sim B$ it holds that $A \not\equiv \bot$, since, due to (3), all of the following propositions hold trivially for $A \equiv \bot$. Note that Proposition 28 gives us, that in order to show a statement of the form $A \mid\!\sim^{ws,M}_{\mathcal{R}} B$, we need to show that $A \mid\!\sim^{cr,M}_{\mathcal{R}} B$ and that $A \not\mid\!\sim^{cr,M}_{\mathcal{R}} \overline{B}$. Since we have shown that credulous inference over sets of ranking functions satisfies the properties (REF), (LLE), (RW), (VCM), and (WAND), we focus on the second part of showing $A \mid\!\sim^{ws,M}_{\mathcal{R}} B$, i.e. that from the preconditions it follows that $\kappa(AB) \leqslant \kappa(A\overline{B})$ for all $\kappa \in M$.

Proposition 40 *(weakly skeptical inference, $\mid\!\sim^{ws,M}_{\mathcal{R}}$). Let $\mathcal{R}$ be a knowledge base, and M be a set of ranking functions accepting $\mathcal{R}$. Weakly skeptical inference over M, i.e., the inference relation $\mid\!\sim^{ws,M}_{\mathcal{R}}$, satisfies:*

1. *(REF)* $A \mid\!\sim A$
2. *(LLE)* $\dfrac{\models A \equiv B,\ A \mid\!\sim C}{B \mid\!\sim C}$
3. *(RW)* $\dfrac{A \mid\!\sim B,\ B \models C}{A \mid\!\sim C}$
4. *(VCM)* $\dfrac{A \mid\!\sim BC}{AB \mid\!\sim C}$
5. *(WAND)* $\dfrac{A \mid\!\sim B,\ A\overline{C} \mid\!\sim C}{A \mid\!\sim BC}$

Proof. 1. We have to show that for each $A \in \mathcal{L}$ we have $A \mid\!\sim^{ws,M}_{\mathcal{R}} A$. For $A \not\equiv \bot$, since $\kappa(\bot) = \infty$ for each OCF κ, we have $\kappa(AA) < \kappa(A\overline{A}) = \infty$ for each OCF and thus also for each ranking function κ in M, implying $A \mid\!\sim^{ws,M}_{\mathcal{R}} A$.
2. We have to show that if $A \equiv B$ and $B \mid\!\sim^{ws,M}_{\mathcal{R}} C$ hold, then for all $\kappa \in M$ it holds that $\kappa(AC) \leqslant \kappa(A\overline{C})$. Since $B \mid\!\sim^{ws,M}_{\mathcal{R}} C$ holds, we get that for all $\kappa \in M$ it holds that $\kappa(BC) \leqslant \kappa(B\overline{C})$. Because $A \equiv B$, it holds that $\kappa(AC) \leqslant \kappa(A\overline{C})$ for all $\kappa \in M$.
3. We have to show that if $B \models C$ and $A \mid\!\sim^{ws,M}_{\mathcal{R}} B$ hold, then for all $\kappa \in M$ it holds that $\kappa(AC) \leqslant \kappa(A\overline{C})$. From $B \models C$ we also have $AB \models AC$ and $A\overline{C} \models A\overline{B}$, therefore for all ranking functions κ it holds that $\kappa(AB) \geqslant \kappa(AC)$ and $\kappa(A\overline{B}) \leqslant \kappa(A\overline{C})$. Since $A \mid\!\sim^{ws,M}_{\mathcal{R}} B$, we get $\kappa(AB) \leqslant \kappa(A\overline{B})$ for all $\kappa \in M$. Therefore, it then also holds that $\kappa(AC) \leqslant \kappa(AB) \leqslant \kappa(A\overline{B}) \leqslant \kappa(A\overline{C})$ for all $\kappa \in M$.

4. We have to show that from $A \mid\!\sim^{ws,M}_{\mathcal{R}} BC$ it follows that for all $\kappa \in M$ it holds that $\kappa(ABC) \leqslant \kappa(AB\overline{C})$. From $A \mid\!\sim^{ws,M}_{\mathcal{R}} BC$ we get that for all $\kappa \in M$ it holds that $\kappa(ABC) \leqslant \kappa(A\overline{BC})$, which implies $\kappa(ABC) \leqslant min\left\{\kappa(AB\overline{C}), \kappa(A\overline{B}C), \kappa(A\overline{B}\,\overline{C})\right\}$, and therefore also $\kappa(ABC) \leqslant \kappa(AB\overline{C})$ holds for all $\kappa \in M$.
5. Since $A\overline{C} \mid\!\sim^{cr,M}_{\mathcal{R}} C$ is never satisfied, because there is no κ such that $\kappa(\perp) < \kappa(A\overline{C})$, $A\overline{C} \mid\!\sim^{ws,M}_{\mathcal{R}} C$ is never satisfied, and thus (WAND) holds. □

In addition to the properties studied in this section, other properties of the three different inference modes carry over to inference with respect to any subset of models. In the following section, we will consider the statements in Proposition 37 and 38 about the hierarchy of inference modes and the relationship among inference relations defined over sets and subsets of ranking functions in the special case of c-inference relations.

6. Interrelationships of c-inference modes over minimal models

The combinations of the three c-inference modes – skeptical, weakly skeptical, and credulous – with the four classes of preferred models – all, sum-minimal, cw-minimal, and ind-minimal – lead to twelve inference relations. In this section, we investigate their interrelationships. In addition to the interrelationships already established in [4], we also present answers to several questions regarding these interrelationships stated as open problems in [4]. We first compare the three inference modes across the different notions of preferred models (Section 6.1), and then study the interrelations among skeptical (Section 6.2), credulous (Section 6.3), and weakly skeptical (Section 6.4) inference relations induced by the different preferred model classes. In Section 6.5, we present a summary of all elaborated interrelationships among the inference relations.

6.1. From skeptical to weakly skeptical to credulous c-inference over minimal models

When introducing weakly skeptical c-inference with respect to all models being c-representations of a knowledge base $\mathcal{R}$ in Section 4, we already showed that this inference mode is different from both skeptical c-inference and from credulous c-inference, but lies strictly between these two modes of inference (Proposition 24). Considering inference with respect to the preferred model classes defined in Section 4.3, we observe that the strict inclusions given in (14) in Proposition 24 carry over to all three kinds of min-inference because of Proposition 37.

Proposition 41. *For every consistent knowledge base $\mathcal{R}$ and each notion of minimal c-representations, skeptical inference is subsumed by weakly skeptical inference, which in turn is subsumed by credulous inference, i.e., for $\bullet \in \{+, cw, O\}$ we have*

$$\mid\!\sim^{sk,\bullet}_{\mathcal{R}} \;\subseteq\; \mid\!\sim^{ws,\bullet}_{\mathcal{R}} \;\subseteq\; \mid\!\sim^{cr,\bullet}_{\mathcal{R}} \tag{20}$$

and there are knowledge bases such that the inclusions in (20) are strict.

Proof. The set inclusions follow as in Proposition 24. Example 23 also holds if only sum- or cw-minimal c-representations are taken into account, thus showing that the inclusions $\mid\!\sim^{sk,+}_{\mathcal{R}} \subsetneq \mid\!\sim^{ws,+}_{\mathcal{R}}$ and $\mid\!\sim^{sk,cw}_{\mathcal{R}} \subsetneq \mid\!\sim^{ws,cw}_{\mathcal{R}}$ are strict. Similarly, examples can be constructed showing that the other four inclusions covered by (20) are proper inclusions. □

Thus, Propositions 24 and 41 clarify the interrelationships of the respective c-inference relations across the different inference modes skeptical, weakly skeptical, and credulous as being strict subset relations for each of the four classes of preferred models.

6.2. Skeptical c-inference over minimal model classes

For investigating the interrelationships among the skeptical inference relations induced by the various classes of preferred models, we start with the observation that each type of skeptical min-inference generalizes skeptical c-inference since the set of models to be taken into account is a subset (cf. Proposition 38), yielding the following:

Proposition 42. *For every consistent knowledge base $\mathcal{R}$, skeptical inference over all c-representations of $\mathcal{R}$ is subsumed by skeptical inference over each of the introduced sets of minimal c-representations, i.e., for $\bullet \in \{+, cw, O\}$ we have:*

$$\mid\!\sim^{sk}_{\mathcal{R}} \;\subseteq\; \mid\!\sim^{sk,\bullet}_{\mathcal{R}} \tag{21}$$

Looking at the different min-inferences, we can show that skeptical cw-minimal inference is generalized by both sum-minimal inference and ind-minimal inference.

Proposition 43. *For every consistent knowledge base $\mathcal{R}$, skeptical inference over all cw-minimal c-representations of $\mathcal{R}$ is subsumed by skeptical inference over both sum- and ind-minimal c-representations of $\mathcal{R}$, i.e., for $\circ \in \{+, O\}$ we have:*

$$\mid\!\sim^{sk,cw}_{\mathcal{R}} \;\subseteq\; \mid\!\sim^{sk,\circ}_{\mathcal{R}} \tag{22}$$

Proof. We first prove that the following inclusions hold:

$$Sol^{min}_{\preccurlyeq_+}(CR(\mathcal{R})) \subseteq Sol^{min}_{\preccurlyeq_{cw}}(CR(\mathcal{R})) \tag{23}$$

$$Sol^{min}_{\preccurlyeq_O}(CR(\mathcal{R})) \subseteq Sol^{min}_{\preccurlyeq_{cw}}(CR(\mathcal{R})) \tag{24}$$

For proving (23), assume there is a $\vec{\eta} \in Sol^{min}_{\preccurlyeq_+}(CR(\mathcal{R}))$ with $\vec{\eta} \notin Sol^{min}_{\preccurlyeq_{cw}}(CR(\mathcal{R}))$. Then there is a $\vec{\eta}' \in Sol^{min}_{\preccurlyeq_{cw}}(CR(\mathcal{R}))$ with $\vec{\eta}' \preccurlyeq_{cw} \vec{\eta}$ and $\vec{\eta}' \neq \vec{\eta}$. From (17) we get $\eta'_i \leqslant \eta_i$ for all $i \in \{1, \ldots, n\}$ and $\eta'_s < \eta_s$ for some $s \in \{1, \ldots, n\}$, and thus:

$$\sum_{i=1}^{n} \eta'_i < \sum_{i=1}^{n} \eta_i$$

Therefore, $\vec{\eta}' \prec_+ \vec{\eta}$ and hence $\vec{\eta} \notin Sol^{min}_{\preccurlyeq_+}(CR(\mathcal{R}))$, contradicting the assumption and thus implying (23).

For proving (24), assume there is a $\vec{\eta} \in Sol^{min}_{\preccurlyeq_O}(CR(\mathcal{R}))$ with $\vec{\eta} \notin Sol^{min}_{\preccurlyeq_{cw}}(CR(\mathcal{R}))$. Then there is a $\vec{\eta}' \in Sol^{min}_{\preccurlyeq_{cw}}(CR(\mathcal{R}))$ with $\vec{\eta}' \preccurlyeq_{cw} \vec{\eta}$ and $\vec{\eta}' \neq \vec{\eta}$. From (17) we get $\eta'_i \leqslant \eta_i$ for all $i \in \{1, \ldots, n\}$ and $\eta'_s < \eta_s$ for some $s \in \{1, \ldots, n\}$. Since $\mathcal{R}$ does not contain any self-fulfilling conditional $(B|A)$ with $A \models B$, there is at least one world $\omega_s \in \Omega$ with $\omega_s \models A_s\overline{B_s}$. From (5) we get:

$$\kappa_{\vec{\eta}'}(\omega) \leqslant \kappa_{\vec{\eta}}(\omega) \qquad \text{for all } \omega \in \Omega$$
$$\kappa_{\vec{\eta}'}(\omega_s) < \kappa_{\vec{\eta}}(\omega_s)$$

That means that $\vec{\eta}' \preccurlyeq_O \vec{\eta}$ and $\vec{\eta} \not\preccurlyeq_O \vec{\eta}'$; hence, $\vec{\eta} \notin Sol^{min}_{\preccurlyeq_O}(CR(\mathcal{R}))$, implying (24). Since skeptical inference over a subset of models is a generalization, (23) and (24) imply (22). □

The next proposition shows that the inclusion in (22) is strict for $\circ$ standing for sum-min inference, and that there is a sum-min inference that is not an ind-min inference.

Proposition 44. *There is a knowledge base $\mathcal{R}$ such that skeptical inference over all sum-minimal c-representations of $\mathcal{R}$ is not contained in either skeptical inference over cw- or ind-minimal c-representations of $\mathcal{R}$, i.e., we have:*

$$\mid\!\sim^{sk,+}_{\mathcal{R}} \;\not\subseteq\; \mid\!\sim^{sk,cw}_{\mathcal{R}} \tag{25}$$
$$\mid\!\sim^{sk,+}_{\mathcal{R}} \;\not\subseteq\; \mid\!\sim^{sk,O}_{\mathcal{R}} \tag{26}$$

Proof. For $\mathcal{R}$, choose the knowledge base $R_{str} = \{r_1, r_2, r_3, r_4, r_5\}$ regarding a zoo situation with some strange birds and consisting of the following set of conditionals:

$r_1 :\ (b|p)$ *penguins are birds*
$r_2 :\ (b|\overline{p})$ *non-penguins are birds*
$r_3 :\ (b|\overline{p}sf)$ *flying strange non-penguins are birds*
$r_4 :\ (s|\overline{b}f)$ *flying things that aren't birds are strange*
$r_5 :\ (p|\overline{f})$ *things that don't fly are penguins*

For $CR(R_{str})$, the impact vector $\vec{\eta}_1 = (1, 0, 1, 2, 1)$ is both cw- and ind-minimal while $\vec{\eta}_2 = (1, 1, 0, 1, 1)$ is cw-, ind- and sum-minimal; there are no other minimal solutions (cf. Table 3).

For the conditional $(b|\overline{p}\overline{f})$ observe that $\kappa_{\vec{\eta}_1} \not\models (b|\overline{p}\overline{f})$ since $\kappa_{\vec{\eta}_1}(\overline{p}b\overline{f}) = \kappa_{\vec{\eta}_1}(\overline{p}\overline{b}\,\overline{f})$, but $\kappa_{\vec{\eta}_2} \models (b|\overline{p}\overline{f})$ since $\kappa_{\vec{\eta}_2}(\overline{p}b\overline{f}) = 1 < 2 = \kappa_{\vec{\eta}_2}(\overline{p}\overline{b}\,\overline{f})$. Thus, $\overline{p}\overline{f} \not\mid\!\sim^{sk,cw}_{R_{str}} b$ and $\overline{p}\overline{f} \not\mid\!\sim^{sk,O}_{R_{str}} b$, but $\overline{p}\overline{f} \mid\!\sim^{sk,+}_{R_{str}} b$. This shows that skeptical inference over all c-representations induced by sum-minimal impact vectors differs both from inference over cw-minimal and over ind-minimal models in general. □

While the relations in Propositions 42–44 were already given in [4], it had been stated as an open question in [4] whether the opposite direction of (22) also holds for $\circ = O$, i.e., whether every skeptical cw-min inference is also an ind-min inference. The following proposition, which was first proven by Obergrusberger [39], shows that this is not the case.

Table 3
Verification (v), falsification (f), impacts (η_i), and solution vectors $\vec{\eta}_1$, $\vec{\eta}_2$, and their induced OCFs for knowledge base R_{str} used in the proof of Proposition 44; for illustration, the system Z ranking function $\kappa^Z_{R_{str}}$ is also given.

ω	r_1: $(b\|p)$	r_2: $(b\|\overline{p})$	r_3: $(b\|\overline{p}sf)$	r_4: $(s\|\overline{b}f)$	r_5: $(p\|\overline{f})$	impact on ω	$\kappa^Z_{R_{str}}(\omega)$	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$
$pbsf$	v	–	–	–	–	0	0	0	0
$pbs\overline{f}$	v	–	–	–	v	0	0	0	0
$pb\overline{s}f$	v	–	–	–	–	0	0	0	0
$pb\overline{s}\overline{f}$	v	–	–	–	v	0	0	0	0
$p\overline{b}sf$	f	–	–	v	–	η_1	1	1	1
$p\overline{b}s\overline{f}$	f	–	–	–	v	η_1	1	1	1
$p\overline{b}\overline{s}f$	f	–	–	f	–	$\eta_1+\eta_4$	2	3	2
$p\overline{b}\overline{s}\overline{f}$	f	–	–	–	v	η_1	1	1	1
$\overline{p}bsf$	–	v	v	–	–	0	0	0	0
$\overline{p}bs\overline{f}$	–	v	–	–	f	η_5	1	1	1
$\overline{p}b\overline{s}f$	–	v	–	–	–	0	0	0	0
$\overline{p}b\overline{s}\overline{f}$	–	v	–	–	f	η_5	1	1	1
$\overline{p}\overline{b}sf$	–	f	f	v	–	$\eta_2+\eta_3$	1	1	1
$\overline{p}\overline{b}s\overline{f}$	–	f	–	–	f	$\eta_2+\eta_5$	1	1	2
$\overline{p}\overline{b}\overline{s}f$	–	f	–	f	–	$\eta_2+\eta_4$	2	2	2
$\overline{p}\overline{b}\overline{s}\overline{f}$	–	f	–	–	f	$\eta_2+\eta_5$	1	1	2
	η_1	η_2	η_3	η_4	η_5				
$\vec{\eta}_1$	1	0	1	2	1	*cw-, ind-min.*			
$\vec{\eta}_2$	1	1	0	1	1	*cw-, sum-, ind-min.*			

Table 4
Verification (v), falsification (f), impacts (η_i), and solution vectors $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\eta}_3$ and their induced c-representations for knowledge base $\mathcal{R}$ used in the proof of Proposition 45.

ω	r_1: $(b\|a)$	r_2: $(\overline{b}\|c)$	r_3: $(c\|d)$	r_4: $(\overline{c}\|ad)$	impact on ω	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$	$\kappa_{\vec{\eta}_3}(\omega)$
$abcd$	v	f	v	f	$\eta_2+\eta_4$	2	2	3
$abc\overline{d}$	v	f	–	–	η_2	1	2	1
$ab\overline{c}d$	v	–	f	v	η_3	1	1	1
$ab\overline{c}\overline{d}$	v	–	–	–	0	0	0	0
$a\overline{b}cd$	f	v	v	f	$\eta_1+\eta_4$	2	2	3
$a\overline{b}c\overline{d}$	f	v	–	–	η_1	1	2	1
$a\overline{b}\overline{c}d$	f	–	f	v	$\eta_1+\eta_3$	2	3	2
$a\overline{b}\overline{c}\overline{d}$	f	–	–	–	η_1	1	2	1
$\overline{a}bcd$	–	f	v	–	η_2	1	2	1
$\overline{a}bc\overline{d}$	–	f	–	–	η_2	1	2	1
$\overline{a}b\overline{c}d$	–	–	f	–	η_3	1	1	1
$\overline{a}b\overline{c}\overline{d}$	–	–	–	–	0	0	0	0
$\overline{a}\overline{b}cd$	–	v	v	–	0	0	0	0
$\overline{a}\overline{b}c\overline{d}$	–	v	–	–	0	0	0	0
$\overline{a}\overline{b}\overline{c}d$	–	–	f	–	η_3	1	1	1
$\overline{a}\overline{b}\overline{c}\overline{d}$	–	–	–	–	0	0	0	0
	η_1	η_2	η_3	η_4				
$\vec{\eta}_1$	1	1	1	1	*cw-, sum-, ind-min.*			
$\vec{\eta}_2$	2	2	1	0	*cw-min.*			
$\vec{\eta}_3$	1	1	1	2				

Proposition 45 *([11,39]). There is a knowledge base $\mathcal{R}$ such that skeptical inference over all ind-minimal c-representations of $\mathcal{R}$ is not contained in skeptical inference over all cw-minimal c-representations of $\mathcal{R}$, i.e., it holds that:*

$$\mathrel{|\!\!\sim}^{sk,O}_{\mathcal{R}} \not\subseteq \mathrel{|\!\!\sim}^{sk,cw}_{\mathcal{R}} \tag{27}$$

Proof. Consider $\mathcal{R} = \{r_1 : (b|a), r_2 : (\overline{b}|c), r_3 : (c|d), r_4 : (\overline{c}|ad)\}$ over the atoms $\Sigma = \{a, b, c, d\}$. Table 4 shows the verification and falsification behavior of all worlds with respect to the four conditionals in $\mathcal{R}$ and the induced impacts for each world (for the logical interdependencies in $\mathcal{R}$).

$CR(\mathcal{R})$ is given by the conjunction of the following constraints:

$$\eta_i \geqslant 0 \quad \text{for } 1 \leqslant i \leqslant 4 \tag{28}$$

$$\eta_1 > 0 \tag{29}$$

$$\eta_2 > 0 \tag{30}$$

$$\eta_3 > 0 \tag{31}$$

$$\eta_4 > \eta_3 - min\{\eta_1, \eta_2\} \tag{32}$$

Both $\vec{\eta}_1 = (\eta_{1,1}, ..., \eta_{1,4}) = (1, 1, 1, 1)$ and $\vec{\eta}_2 = (\eta_{2,1}, ..., \eta_{2,4}) = (2, 2, 1, 0)$ are solutions of $CR(\mathcal{R})$ as can easily be checked by substituting the respective values in (28)-(32). Their induced OCFs are also given in Table 4.

We will now prove that $\{\vec{\eta}_1, \vec{\eta}_2\}$ is the set of all cw-minimal solutions. To see that, assume that there were $\vec{\eta}\,' = (\eta'_1, \eta'_2, \eta'_3, \eta'_4) \in Sol(CR(\mathcal{R}))$ with $\vec{\eta}_1 \not\preccurlyeq_{cw} \vec{\eta}\,'$ and $\vec{\eta}_2 \not\preccurlyeq_{cw} \vec{\eta}\,'$. It cannot be the case that $\eta'_1 < \eta_{1,i}$ for $i \in \{1, 2, 3\}$ because of (29), (30), and (31), leaving $\eta'_4 < \eta_{1,4}$ as the only possibility. From $\eta'_4 < \eta_{1,4}$ and (28), we get

$$0 \leqslant \eta'_4 = 0 < \eta_{1,4} = 1. \tag{33}$$

From (32) we then get $0 > \eta'_3 - min\{\eta'_1, \eta'_2\}$ and therefore $\eta'_3 < min\{\eta'_1, \eta'_2\}$. This leads to $0 < \eta'_3 < min\{\eta'_1, \eta'_2\}$ due to (31). In particular, this implies $min\{\eta_1, \eta_2\} > 1$ and consequently, both

$$\eta'_1 > 1 \tag{34}$$

$$\eta'_2 > 1 \tag{35}$$

must hold. Because of $\vec{\eta}_2 \not\preccurlyeq_{cw} \vec{\eta}\,'$, it is required that for one $i \in \{1, 2, 3, 4\}$ it is the case that $\eta'_i < \eta_{2,i}$. Therefore, one of the following four conditions

$$0 < \eta'_1 < \eta_{2,1} = 2 \tag{36}$$

$$0 < \eta'_2 < \eta_{2,2} = 2 \tag{37}$$

$$0 < \eta'_3 < \eta_{2,3} = 1 \tag{38}$$

$$0 = \eta'_4 < \eta_{2,4} = 0 \tag{39}$$

must be true. Conditions (38) and (39) are, on their own, not satisfiable and (36) and (37) contradict (34) and (35), respectively. This contradicts the assumption and shows that $\vec{\eta}_1$ and $\vec{\eta}_2$ are the only cw-minimal solutions.

We observe that $\kappa_{\vec{\eta}_1}(\omega) \leqslant \kappa_{\vec{\eta}_2}(\omega)$ for all ω, and that, e.g., $\kappa_{\vec{\eta}_1}(a\bar{b}\bar{c}d) = 2 < 3 = \kappa_{\vec{\eta}_2}(a\bar{b}\bar{c}d)$ (cf. Table 4). Thus, since every ind-minimal solution is also cw-minimal, we conclude that $\vec{\eta}_1$ is the only ind-minimal solution of $Sol(CR(\mathcal{R}))$.

Now consider the conditional $(\bar{d}|ac)$. Since

$$\kappa_{\vec{\eta}_1}(\bar{d}ac) = 1 < 2 = \kappa_{\vec{\eta}_1}(dac)$$

$(\bar{d}|ac)$ is accepted by the c-representation induced by the only ind-minimal solution $\vec{\eta}_1$. On the other hand, since

$$\kappa_{\vec{\eta}_2}(\bar{d}ac) = 2 = \kappa_{\vec{\eta}_2}(dac)$$

the conditional is rejected by the c-representation induced by the cw-minimal c-representation $\vec{\eta}_2$. Therefore, we have

$$ac \mathrel{\substack{\sim}{}}\!\!\!\!\mid\;\;^{sk,O}_{\mathcal{R}}\; \bar{d} \quad \text{and} \quad ac \not\mathrel{|\!\sim}^{sk,cw}_{\mathcal{R}} \bar{d}$$

which completes the proof. □

Note that for illustration, Table 4 also displays a non-minimal impact vector $\vec{\eta}_3$ and its induced OCF.

Another open problem stated in [4] is the question whether skeptical c-inference can be modeled by skeptical cw-min inference. Since skeptical c-inference is defined with respect to the set $Sol(CR(\mathcal{R}))$ and since the set of cw-minimal solutions is a subset thereof, we trivially have

$$\mathrel{|\!\sim}^{sk}_{\mathcal{R}} \;\subseteq\; \mathrel{|\!\sim}^{sk,cw}_{\mathcal{R}} \tag{40}$$

for every knowledge base $\mathcal{R}$ (cf. Proposition 42). The next proposition, first proven in [39], shows that the inverse direction of (40) does not hold.

Proposition 46 *([11,39]). There is a knowledge base $\mathcal{R}$ such that skeptical inference over all cw-minimal c-representations of $\mathcal{R}$ is not contained in skeptical inference over all c-representations of $\mathcal{R}$, i.e., it holds that:*

$$\mathrel{|\!\sim}^{sk,cw}_{\mathcal{R}} \;\not\subseteq\; \mathrel{|\!\sim}^{sk}_{\mathcal{R}} \tag{41}$$

Table 5
Verification (v), falsification (f), impacts (η_i), and the solution vectors $\vec{\eta}_1$, $\vec{\eta}_2$ and their induced OCFs for knowledge base $\mathcal{R}$ used in the proof of Proposition 46.

ω	r_1: $(b\|a)$	r_2: $(b\|ac)$	r_3: $(\bar{a}\|b\bar{c})$	r_4: $(\bar{a}\|bc)$	impact on ω	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$
abc	v	v	–	f	η_4	1	1
$ab\bar{c}$	v	–	f	–	η_3	1	3
$a\bar{b}c$	f	f	–	–	$\eta_1+\eta_2$	2	2
$a\bar{b}\bar{c}$	f	–	–	–	η_1	2	2
$\bar{a}bc$	–	–	–	v	0	0	0
$\bar{a}b\bar{c}$	–	–	v	–	0	0	0
$\bar{a}\bar{b}c$	–	–	–	–	0	0	0
$\bar{a}\bar{b}\bar{c}$	–	–	–	–	0	0	0
	η_1	η_2	η_3	η_4			
$\vec{\eta}_1$	2	0	1	1	*cw-, sum-, ind-min.*		
$\vec{\eta}_2$	2	0	3	1			

Proof. Consider $\mathcal{R} = \{(b|a), (b|ac), (\bar{a}|b\bar{c}), (\bar{a}|bc)\}$ over the atoms $\Sigma = \{a, b, c\}$. Table 5 shows the verification and falsification behavior of all worlds with respect to the four conditionals in $\mathcal{R}$ and the induced impacts for each world.

$CR(\mathcal{R})$ is given by the conjunction of the following constraints:

$$\eta_i \geqslant 0 \quad \text{for } 1 \leqslant i \leqslant 4 \tag{42}$$

$$\eta_1 > min\{\eta_3, \eta_4\} - min\{\eta_2, 0\} = min\{\eta_3, \eta_4\} \tag{43}$$

$$\eta_2 > \eta_4 - \eta_1 \tag{44}$$

$$\eta_3 > 0 \tag{45}$$

$$\eta_4 > 0 \tag{46}$$

The impact vector $\vec{\eta}_1 = (\eta_1, ..., \eta_4) = (2, 0, 1, 1)$ is a solution of $CR(\mathcal{R})$ as can easily be checked by substituting the values in (42) - (46). The induced OCF $\kappa_{\vec{\eta}_1}$ is given in the seventh column of Table 5.

In the following, we will prove that $\vec{\eta}_1$ is the only cw-minimal solution of $CR(\mathcal{R})$. In order to show this, let us assume that there were $\vec{\eta}\,'_1 \in Sol(CR(\mathcal{R}))$ that is also cw-minimal, but different from $\vec{\eta}_1$. Then there must be a non-empty set I of indices, $I \subseteq \{1, 2, 3, 4\}$ for which it is the case that $\eta'_i < \eta_i$ for all $i \in I$. First we note that $2 \notin I$ because $\eta'_2 < \eta_2 = 0$ violates (42). Furthermore, $3 \notin I$ because $\eta'_3 < \eta_3 = 1$ violates (45), and $4 \notin I$ because $\eta'_4 < \eta_4 = 1$ violates (46). Finally, $\eta'_1 < \eta_1 = 2$ would imply $\eta'_1 \leqslant 1$ which together with (43) implies $1 > min\{\eta'_3, \eta'_4\}$ and therefore $\eta'_3 = 0$ or $\eta'_4 = 0$, contradicting (45) or (46); hence $1 \notin I$. These observations imply $I = \emptyset$, contradicting the assumption that $\vec{\eta}\,'_1$ is cw-minimal and establishing that $\vec{\eta}_1$ is the only cw-minimal element of $Sol(CR(\mathcal{R}))$.

Now consider the conditional $(b|a\bar{c})$. For the single cw-minimal c-representation $\vec{\eta}_1$ we have

$$\kappa_{\vec{\eta}_1}(a\bar{c}b) = 1 < 2 = \kappa_{\vec{\eta}_1}(a\bar{c}\bar{b})$$

and therefore

$$a\bar{c} \;\; \mid\!\sim^{sk,cw}_{\mathcal{R}} \;\; b. \tag{47}$$

The impact vector $\vec{\eta}_2 = (2, 0, 3, 1)$ is also a solution of $CR(\mathcal{R})$ given in (42)-(46). The induced OCF $\kappa_{\vec{\eta}_2}$ is given in the last column of Table 5. For $\kappa_{\vec{\eta}_2}$ we have

$$\kappa_{\vec{\eta}_2}(a\bar{c}b) = 3 \not< 2 = \kappa_{\vec{\eta}_2}(a\bar{c}\bar{b})$$

and therefore, $\kappa_{\vec{\eta}_2}$ does not accept the conditional $(b|a\bar{c})$. Since $\mid\!\sim^{sk}_{\mathcal{R}}$ is defined skeptically taking all OCFs accepting $\mathcal{R}$ into account, we get

$$a\bar{c} \;\; \mid\!\not\sim^{sk}_{\mathcal{R}} \;\; b \tag{48}$$

which together with (47) completes the proof. □

While the knowledge base $\mathcal{R}_{pen}$ given in Example 13 is an example for a knowledge base having a system Z ranking function that is not a c-representation (cf. Table 1 and Observation 14), the situation is different for the knowledge base $\mathcal{R}$ used in the proof of Proposition 46. Here, $\vec{\eta}_1 = (2, 0, 1, 1)$ is the only solution vector of $CR(\mathcal{R})$ that is minimal with respect to any of the three cw-, sum-, and ind-minimality criteria (cf. Table 5), and $\kappa_{\vec{\eta}_1}$ is the same as the system Z ranking function for $\mathcal{R}$.

6.3. Credulous c-inference over minimal model classes

As Proposition 38 shows for two sets of OCFs M, M' with $M \subseteq M'$, skeptical inference over M generalizes skeptical inference over M', credulous inference over M' generalizes skeptical inference over M. Thus, corresponding results for skeptical inference from the previous section dually hold for credulous inference.

Proposition 47. *For every consistent knowledge base $\mathcal{R}$, credulous inference over all c-representations of $\mathcal{R}$ subsumes credulous inference over cw-minimal c-representations of $\mathcal{R}$, which in turn subsumes credulous inference over both sum- and ind-minimal c-representations of $\mathcal{R}$, i.e., for $\bullet \in \{+, cw, O\}$ we have:*

$$\mid\!\sim^{cr,\bullet}_{\mathcal{R}} \ \subseteq\ \mid\!\sim^{cr}_{\mathcal{R}} \tag{49}$$

$$\mid\!\sim^{cr,\bullet}_{\mathcal{R}} \ \subseteq\ \mid\!\sim^{cr,cw}_{\mathcal{R}} \tag{50}$$

Proof. Since every set of minimal models is a subset of the models of $\mathcal{R}$, (49) holds, and the subset relations (23) and (24) shown in the proof of Proposition 43 imply (50). □

In the following, the inverse relations of (49) and (50) will be investigated. We first show that not every credulous cw-minimal inference is also a sum- or ind-minimal inference.

Proposition 48. *There is a knowledge base $\mathcal{R}$ such that credulous inference over all cw-minimal c-representations of $\mathcal{R}$ is not contained in credulous inference over either sum- or ind-minimal c-representations of $\mathcal{R}$, i.e., it holds that:*

$$\mid\!\sim^{cr,cw}_{\mathcal{R}} \ \not\subseteq\ \mid\!\sim^{cr,+}_{\mathcal{R}} \tag{51}$$

$$\mid\!\sim^{cr,cw}_{\mathcal{R}} \ \not\subseteq\ \mid\!\sim^{cr,O}_{\mathcal{R}} \tag{52}$$

Proof. Let $\mathcal{R}$ be the knowledge base used in the proof of Proposition 45 (see Table 4). For the conditional $(c|a\overline{b}d)$ we have $\kappa_{\vec{\eta}_1}(a\overline{b}cd) = 2 \not< 2 = \kappa_{\vec{\eta}_1}(a\overline{b}\overline{c}d)$, and since $\vec{\eta}_1$ is the only sum- and the only ind-minimal solution of $CR(\mathcal{R})$ we get

$$a\overline{b}d \ \mid\!\not\sim^{cr,+}_{\mathcal{R}}\ c \ \text{ and } \ a\overline{b}d \ \mid\!\not\sim^{cr,O}_{\mathcal{R}}\ c. \tag{53}$$

But for the cw-minimal $\vec{\eta}_2$ we have $\kappa_{\vec{\eta}_2}(a\overline{b}cd) = 2 < 3 = \kappa_{\vec{\eta}_2}(a\overline{b}\overline{c}d)$ and therefore

$$a\overline{b}d \ \mid\!\sim^{cr,cw}_{\mathcal{R}}\ c \tag{54}$$

which together with (53) completes the proof. □

Finally, we show that there are strictly less credulous cw-min-inferences than inferences over all c-representations.

Proposition 49. *There is a knowledge base $\mathcal{R}$ such that credulous inference over all c-representations of $\mathcal{R}$ is not contained in credulous inference over all cw-minimal c-representations of $\mathcal{R}$, i.e., it holds that:*

$$\mid\!\sim^{cr}_{\mathcal{R}} \ \not\subseteq\ \mid\!\sim^{cr,cw}_{\mathcal{R}} \tag{55}$$

Proof. Let $\mathcal{R}$ be as in the proof of Proposition 46 and consider the conditional $(\overline{b}|a\overline{c})$. For $\vec{\eta}_2$ (cf. Table 5) we have $\kappa_{\vec{\eta}_2}(a\overline{c}\overline{b}) = 2 < 3 = \kappa_{\vec{\eta}_2}(a\overline{c}b)$ and therefore

$$a\overline{c} \ \mid\!\sim^{cr}_{\mathcal{R}}\ \overline{b}. \tag{56}$$

For $\vec{\eta}_1$, the only cw-minimal solution of $CR(\mathcal{R})$ (cf. the proof of Proposition 46) we have $\kappa_{\vec{\eta}_1}(a\overline{c}\overline{b}) = 2 \not< 1 = \kappa_{\vec{\eta}_1}(a\overline{c}b)$ and therefore

$$a\overline{c} \ \mid\!\not\sim^{cr,cw}_{\mathcal{R}}\ \overline{b} \tag{57}$$

which together with (56) completes the proof. □

Table 6
Verification (v), falsification (f), impacts (η_i), the solution vectors $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\eta}_3$, $\vec{\eta}_4$ and their induced c-representations $\kappa_{\vec{\eta}_1}$, $\kappa_{\vec{\eta}_2}$, $\kappa_{\vec{\eta}_3}$, $\kappa_{\vec{\eta}_4}$, and the system Z ranking function $\kappa^Z_{\mathcal{R}_{bpf}}$ for knowledge base $\mathcal{R}_{bpf}$ from Example 50.

ω	r_1: $(f\|b)$	r_2: $(\overline{f}\|p)$	r_3: $(\overline{f}\|bp)$	r_4: $(b\|p)$	impact on ω	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$	$\kappa_{\vec{\eta}_3}(\omega)$	$\kappa_{\vec{\eta}_4}(\omega)$	$\kappa^Z_{\mathcal{R}_{bpf}}(\omega)$
bpf	v	f	f	v	$\eta_2+\eta_3$	2	2	2	5	2
$bp\overline{f}$	f	v	v	v	η_1	1	1	1	1	1
$b\overline{p}f$	v	–	–	–	0	0	0	0	0	0
$b\overline{p}\overline{f}$	f	–	–	–	η_1	1	1	1	1	1
$\overline{b}pf$	–	f	–	f	$\eta_2+\eta_4$	2	4	3	4	2
$\overline{b}p\overline{f}$	–	v	–	f	η_4	2	2	2	3	2
$\overline{b}\overline{p}f$	–	–	–	–	0	0	0	0	0	0
$\overline{b}\overline{p}\overline{f}$	–	–	–	–	0	0	0	0	0	0
	η_1	η_2	η_3	η_4						
$\vec{\eta}_1$	1	0	2	2	*cw-*, *sum-*, *ind-min.*					
$\vec{\eta}_2$	1	2	0	2	*cw-*, *sum-min.*					
$\vec{\eta}_3$	1	1	1	2	*cw-*, *sum-min.*					
$\vec{\eta}_4$	1	1	4	3						

6.4. Weakly skeptical c-inference over minimal model classes

While skeptical inference over all c-representations is subsumed by skeptical inference over any set of minimal c-representations (Proposition 42), credulous inference over all c-representations subsumes credulous inference over any set of minimal c-representations (Proposition 47). Obviously, this pattern for skeptical and credulous inference, respectively, holds in every situation when moving from a larger set of models to be taken into account to a subset of these models. The following example illustrates that the relationships are more intricate in the case of weakly skeptical inference.

Example 50. Consider again the knowledge base $\mathcal{R}_{bpf}$ from Example 1 which we repeat here for convenience:

r_1 : $(f|b)$ birds fly
r_2 : $(\overline{f}|p)$ penguins don't fly
r_3 : $(\overline{f}|bp)$ penguin birds don't fly
r_4 : $(b|p)$ penguins are birds

Table 6 shows the verification/falsification behavior of all worlds over $\Sigma_{bpf} = \{b, p, f\}$ with respect to the four conditionals in $\mathcal{R}_{bpf}$, and it lists all cw-minimal ($\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3$), sum-minimal ($\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3$), and ind-minimal ($\vec{\eta}_1$) solutions of $CR(\mathcal{R}_{bpf})$, as well as a further non-minimal solution ($\vec{\eta}_4$), and their induced ranking functions. The system Z ranking function $\kappa^Z_{\mathcal{R}_{bpf}}$ is given in the last column of Table 6; note that in this example, the system Z ranking function $\kappa^Z_{\mathcal{R}_{bpf}}$ coincides with the, in this case single, ind-minimal c-representation $\kappa_{\vec{\eta}_1}$.

The entailment that a flying penguin is a bird, i.e. $fp \mid\!\sim b$, is sanctioned neither by system P inference nor by system Z. Furthermore, this entailment does not hold under skeptical c-inference over all or over any of the minimal sets of c-representations, i.e., we have $fp \not\mid\!\approx^{sk}_{\mathcal{R}_{bpf}} b$ and $fp \not\mid\!\approx^{sk,\bullet}_{\mathcal{R}_{bpf}} b$ for every $\bullet \in \{cw, +, O\}$. Under weakly skeptical inference, the entailment does not hold with respect to all c-representations, i.e., $fp \not\mid\!\approx^{ws}_{\mathcal{R}_{bpf}} b$, because for instance $fp \mid\!\sim_{\kappa_{\vec{\eta}_4}} \overline{b}$. Also for weakly skeptical inference with respect to ind-minimal c-representations the entailment does not hold, i.e., $fp \not\mid\!\approx^{ws,O}_{\mathcal{R}_{bpf}} b$ because $\kappa_{\vec{\eta}_1}(bpf) = \kappa_{\vec{\eta}_1}(\overline{b}pf)$ for the single ind-minimal OCF $\kappa_{\vec{\eta}_1}$. However, evaluating the three ranking models $\kappa_{\vec{\eta}_1}, \kappa_{\vec{\eta}_2}, \kappa_{\vec{\eta}_3}$ shows that weakly skeptical inference with respect to cw-minimal models as well as with respect to sum-minimal models license the inference that a flying penguin is a bird, i.e., $fp \mid\!\approx^{ws,cw}_{\mathcal{R}_{bpf}} b$ and $fp \mid\!\approx^{ws,+}_{\mathcal{R}_{bpf}} b$.

Because the set of cw-minimal models of $\mathcal{R}_{bpf}$ is both a strict subset of all c-representations and also a proper superset of all ind-minimal c-representations, this shows that the number of ranking functions taken into account to determine weakly skeptical inference in general does not lead to a more general or a more restricted inference relation.

Example 50 illustrates that the restriction to minimal c-representations in combination with weakly skeptical c-inference yields an inference relation that allows us to obtain desirable entailments that are not possible by weakly skeptical inference over all c-representations nor by skeptical inference over any of the considered sets of preferred models, while still avoiding any directly contradictory inferences that could be possible in credulous inference. Thus, while the subset relationships in Proposition 38 in general and Section 6.2 and 6.3 in particular reflect the fact that over a subset of models skeptical inference yields a stricter and credulous inference yields a more tolerant inference relation, the interrelationships are not governed by this principle in the case of weakly skeptical inference. As Proposition 28 points out, weakly skeptical inference is akin to a restricted credulous inference, where the inclusion of a particular ranking model can 'promote' and 'restrict' inferences.

This suggests that there might well be situations where, compared to inference over all c-representations, inference over cw-minimal solutions allows for weakly skeptical inferences that are not contained in any other weakly skeptical inference relation, because it does not take models into account that allow for contradicting consequences. When comparing to inference over sum- or ind-minimal c-representations, inference over cw-minimal c-representations might allow for more inferences, because it takes more models into account that allow for desirable inferences. While in Example 50 cw-minimal and sum-minimal weakly skeptical inferences coincide, the following proposition shows that there are in fact situations where weakly skeptical inference over cw-minimal models goes beyond all other variants of weakly skeptical inference.

Proposition 51. *There is a knowledge base $\mathcal{R}$ such that weakly skeptical inference over all cw-minimal c-representation goes beyond weakly skeptical inference over any other introduced set of c-representations (i.e., all, sum-minimal, or ind-minimal) of $\mathcal{R}$, i.e. it holds that:*

$$\mid\!\sim_{\mathcal{R}}^{ws,cw} \;\not\subseteq\; \mid\!\sim_{\mathcal{R}}^{ws} \tag{58}$$

$$\mid\!\sim_{\mathcal{R}}^{ws,cw} \;\not\subseteq\; \mid\!\sim_{\mathcal{R}}^{ws,+} \tag{59}$$

$$\mid\!\sim_{\mathcal{R}}^{ws,cw} \;\not\subseteq\; \mid\!\sim_{\mathcal{R}}^{ws,O} \tag{60}$$

Proof. Consider $\mathcal{R} = \{(b|a), (\bar{b}|c), (c|d), (\bar{c}|ad)\}$ used in the proofs of Proposition 45 and 48. Table 4 shows all minimal solutions ($\vec{\eta}_1$ and $\vec{\eta}_2$) as well as the solution

$$\vec{\eta}_3 = (1, 1, 1, 2)$$

to $CR(\mathcal{R})$ that is not minimal with respect to any of the minimality criteria. From the proof of Proposition 48 we already know that for the conditional $(c|a\bar{b}d)$

$$a\bar{b}d \;\mid\!\sim_{\mathcal{R}}^{cr,cw}\; c. \tag{61}$$

Since we have both

$$\kappa_{\vec{\eta}_1}(a\bar{b}\bar{c}d) = 2 \not< 2 = \kappa_{\vec{\eta}_1}(a\bar{b}cd) \text{ and}$$
$$\kappa_{\vec{\eta}_2}(a\bar{b}\bar{c}d) = 3 \not< 2 = \kappa_{\vec{\eta}_2}(a\bar{b}cd)$$

we know that $a\bar{b}d \not\mid\!\sim_{\mathcal{R}}^{cr,cw} \bar{c}$ which together with (61) shows that

$$a\bar{b}d \;\mid\!\sim_{\mathcal{R}}^{ws,cw}\; c. \tag{62}$$

For proving (58), we show that the inference in (62) does not hold for $\mid\!\sim_{\mathcal{R}}^{ws}$, since $\mid\!\sim_{\mathcal{R}}^{cr}$ infers both c and $\bar{c}$ from $a\bar{b}d$. In Table 4 we can see that $\kappa_{\vec{\eta}_3}(a\bar{b}\bar{c}d) = 2 < 3 = \kappa_{\vec{\eta}_3}(a\bar{b}cd)$ showing that $a\bar{b}d \mid\!\sim_{\mathcal{R}}^{cr} \bar{c}$ which means that $a\bar{b}d \not\mid\!\sim_{\mathcal{R}}^{ws} c$ which together with (62) shows (58). For proving (59) and (60), we show that the inference in (62) does not hold for $\mid\!\sim_{\mathcal{R}}^{ws,+}$ or $\mid\!\sim_{\mathcal{R}}^{ws,O}$, since neither $\mid\!\sim_{\mathcal{R}}^{cr,+}$ nor $\mid\!\sim_{\mathcal{R}}^{cr,O}$ infer c from $a\bar{b}d$. Since $\vec{\eta}_1$ is the only sum-minimal and the only ind-minimal solution to $CR(\mathcal{R})$, $\mid\!\sim_{\mathcal{R}}^{ws,+}$ and $\mid\!\sim_{\mathcal{R}}^{ws,O}$ coincide. We already know from (53) in the proof of Proposition 48 that

$$a\bar{b}d \;\not\mid\!\sim_{\mathcal{R}}^{cr,+}\; c$$

so we also have

$$a\bar{b}d \;\not\mid\!\sim_{\mathcal{R}}^{ws,+}\; c \text{ and } a\bar{b}d \;\not\mid\!\sim_{\mathcal{R}}^{ws,O}\; c \tag{63}$$

which together with (62) shows (59) and (60), respectively. □

This shows that even with the same knowledge base and query, weakly skeptical inference both over a subset or over a superset of models can lead to a more tolerant inference relation. The exact relationships still need to be investigated.

6.5. Summary of interrelationships

Fig. 3 illustrates the interrelationships among c-inference relations when assuming a fixed knowledge base $\mathcal{R}$.

The illustration is separated into three layers, representing the three modes of inference. Between the layers we have the inclusions stated by Proposition 41. The arrows point downwards, indicating that, when fixing a set of models of $\mathcal{R}$, skeptical inference allows for less inferences than weakly skeptical inference, and weakly skeptical inference allows for less inferences than credulous inference.

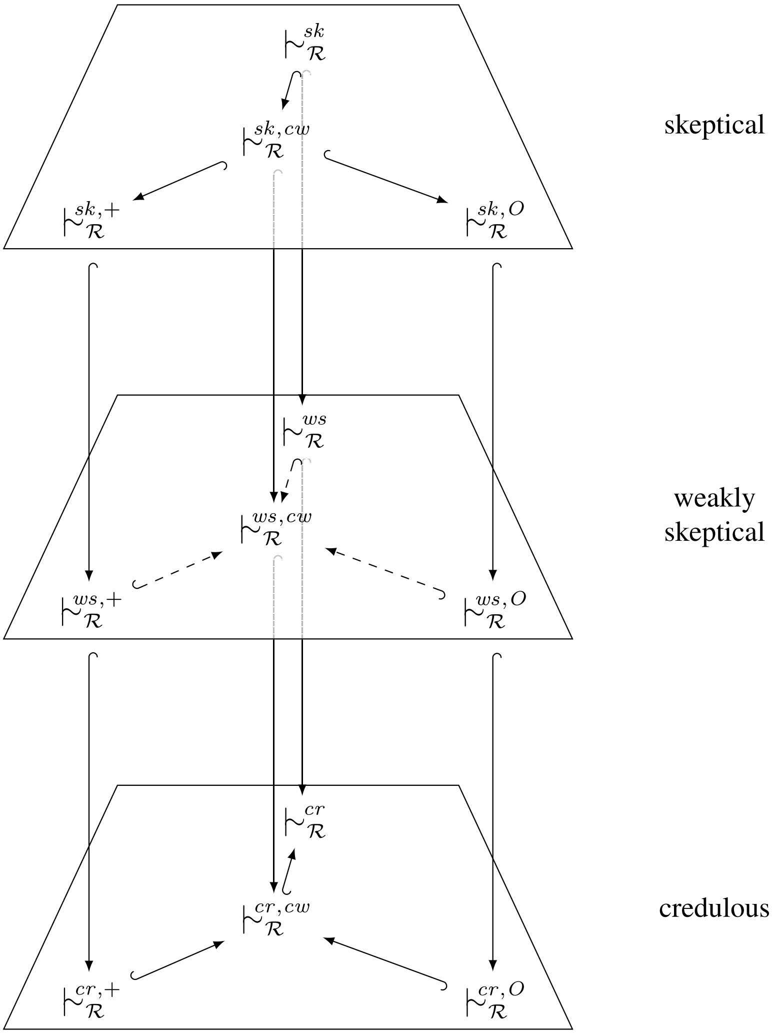

Fig. 3. Overview of the twelve c-inference relations induced by a knowledge base $\mathcal{R}$ and the three inference modes (skeptical, weakly skeptical, credulous) and four model classes (all, sum-minimal, cw-minimal, ind-minimal) and the relationships among them. A solid arrow $\hookrightarrow$ indicates a proper subset relationship. The relations for weakly skeptical inference indicated by a dashed arrow $\hookrightarrow$ are illustrated in Example 50 and in Proposition 51, but are not yet generally known.

In each layer, we have the four inference relations obtained from the four classes (i.e. all, sum-minimal, cw-minimal, ind-minimal) of c-representations of $\mathcal{R}$, employing the respective inference mode. As stated in Proposition 38, the inclusions among the obtained inference relations for each of the three inference modes are quite clear in the case of skeptical and credulous inference, represented in the top and bottom layer, respectively. In the top layer, the subset arrows point to the inference relation defined over the smaller set of models, as they allow for more inferences under skeptical inference mode. The arrows are reversed in the bottom layer, as the larger sets of models allow for more inferences in credulous inference mode.

As discussed in Section 6.4 and shown in Proposition 51, the inclusions are not straightforward in the case of weakly skeptical inference, represented in the central layer; the general interrelationships among weakly skeptical c-inference relations are still an open problem. The dashed arrows pointing towards the weakly skeptical inference relation defined over cw-minimal c-representations are used to illustrate the fact, as exemplified by Example 50 and Proposition 51, that there are knowledge bases where under weakly skeptical inference mode, the class of cw-minimal c-representations allows for more reasonable inferences than both the larger set of all c-representations and also the smaller sets of sum- and ind-minimal c-representations.

7. Conclusions and future work

In this paper, we refined c-inference over all c-representations by taking only minimal models into account. We introduced the novel concept of weakly skeptical c-inference. Weakly skeptical c-inference strictly extends skeptical c-inference over all c-representations of a knowledge base $\mathcal{R}$, while avoiding disadvantages of permissive credulous c-inference. Transferring these modes of inference to models obtained by applying different minimality criteria, yields several variants of inference relations. These inference relations and their interrelationships elaborated here are summarized in Fig. 3. We illustrated that weakly skeptical c-inference allows for desirable entailments that are not possible with skeptical inference, especially in combination with preferred minimal models. We also extended the inference modes such that arbitrary subsets of ranking models of a knowledge base are considered to be relevant for entailment. For the different inference relations, we showed that they fulfill various desirable properties that have been proposed for nonmonotonic reasoning.

As suggested by our findings, c-representations and inference over minimal c-representations provide promising opportunities for modeling plausible relationships and default reasoning. While we mainly used standard example scenarios, like birds and penguins, for illustration here, reasoning based on the inference relations developed and presented in this article has been applied successfully to biomedical knowledge modeled with qualitative conditionals [23]. We are currently evaluating our approach empirically in several other applications, as well as with respect to systematically generated knowledge bases and the closures of the inference relations induced by them [7,34].

For realizing inference based on c-representations, we implemented a constraint logic programming system for solving the CSPs characterizing c-representations and c-inference queries [6]. In order to be able to employ a finite domain constraint solver as it is available in SICStus Prolog [18], the arising CSPs can be transformed to CSPs over finite domains by introducing an upper bound for the impact values η_i [10,32]. Scaling up this approach to knowledge bases with increasing numbers of propositional variables and conditionals is subject to current and future work; for this, compilation techniques for c-representations [12] and syntax splitting [40,43], which has recently been shown to be fully compatible with skeptical c-inference [28], will be helpful.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare that they do not have any Conflicts of Interest regarding the work reported in this paper.

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References

[1] Ernest W. Adams, Probability and the logic of conditionals, in: J. Hintikka, P. Suppes (Eds.), Aspects of inductive logic, North-Holland, Amsterdam, 1966, pp. 265–316.
[2] Ernest W. Adams, The Logic of Conditionals: An Application of Probability to Deductive Logic, Synthese Library, Springer Science+Business Media, Dordrecht, NL, 1975.
[3] Christoph Beierle, Christian Eichhorn, Gabriele Kern-Isberner, Skeptical inference based on c-representations and its characterization as a constraint satisfaction problem, in: M. Gyssens, G.R. Simari (Eds.), Foundations of Information and Knowledge Systems – Proceedings of the 9th International Symposium, FoIKS 2016, Linz, Austria, March 7–11, 2016, in: LNCS, vol. 9616, Springer, 2016, pp. 65–82.
[4] Christoph Beierle, Christian Eichhorn, Gabriele Kern-Isberner, Steven Kutsch, Skeptical, weakly skeptical, and credulous inference based on preferred ranking functions, in: Gal A. Kaminka, Maria Fox, Paolo Bouquet, Eyke Hüllermeier, Virginia Dignum, Frank Dignum, Frank van Harmelen (Eds.), Proceedings 22nd European Conference on Artificial Intelligence, ECAI-2016, in: Frontiers in Artificial Intelligence and Applications, vol. 285, IOS Press, 2016, pp. 1149–1157.
[5] Christoph Beierle, Christian Eichhorn, Gabriele Kern-Isberner, Steven Kutsch, Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem, Ann. Math. Artif. Intell. 83 (3–4) (2018) 247–275.
[6] Christoph Beierle, Christian Eichhorn, Steven Kutsch, A practical comparison of qualitative inferences with preferred ranking models, Künstl. Intell. 31 (1) (2017) 41–52.
[7] Christoph Beierle, Jonas Haldimann, Normal forms of conditional knowledge bases respecting system P-entailments, in: Andreas Herzig, Juha Kontinen (Eds.), Foundations of Information and Knowledge Systems – Proceedings of the 11th International Symposium, FoIKS 2020, Dortmund, Germany, February 17-21, 2020, in: LNCS, vol. 12012, Springer, 2020, pp. 22–41.
[8] Christoph Beierle, G. Kern-Isberner, K. Södler, A declarative approach for computing ordinal conditional functions using constraint logic programming, in: 19th International Conference on Applications of Declarative Programming and Knowledge Management, INAP 2011, and 25th Workshop on Logic Programming, WLP 2011, Wien, Austria, in: LNAI, vol. 7773, Springer, 2013, pp. 175–192, Revised Selected Papers.
[9] Christoph Beierle, Gabriele Kern-Isberner, Semantical investigations into nonmonotonic and probabilistic logics, Ann. Math. Artif. Intell. 65 (2–3) (2012) 123–158.
[10] Christoph Beierle, Steven Kutsch, Computation and comparison of nonmonotonic skeptical inference relations induced by sets of ranking models for the realization of intelligent agents, Appl. Intell. 49 (1) (Jan. 2019) 28–43.
[11] Christoph Beierle, Steven Kutsch, Andreas Obergrusberger, On the interrelationships among c-inference relations based on preferred models for sets of default rules, in: Vasile Rus, Zdravko Markov (Eds.), Proceedings of the Thirtieth International Florida Artificial Intelligence Research Society Conference, FLAIRS 2017, Marco Island, Florida, USA, May 22-24, 2017, AAAI Press, 2017, pp. 724–729.

[12] Christoph Beierle, Steven Kutsch, Kai Sauerwald, Compilation of static and evolving conditional knowledge bases for computing induced nonmonotonic inference relations, Ann. Math. Artif. Intell. 87 (1–2) (2019) 5–41.
[13] S. Benferhat, D. Dubois, H. Prade, Representing default rules in possibilistic logic, in: Proceedings 3rd International Conference on Principles of Knowledge Representation and Reasoning, KR'92, 1992, pp. 673–684.
[14] S. Benferhat, D. Dubois, H. Prade, Possibilistic and standard probabilistic semantics of conditional knowledge bases, J. Log. Comput. 9 (6) (1999) 873–895.
[15] S. Benferhat, D. Dubois, H. Prade, The possibilistic handling of irrelevance in exception-tolerant reasoning, Ann. Math. Artif. Intell. 35 (2002) 29–61.
[16] Salem Benferhat, Claudette Cayrol, Didier Dubois, Jérôme Lang, Henri Prade, Inconsistency management and prioritized syntax-based entailment, in: Proceedings of the Thirteenth International Joint Conference on Artificial Intelligence, vol. 1, IJCAI'93, San Francisco, CA, USA, Morgan Kaufmann Publishers, 1993, pp. 640–647.
[17] A. Bochman, A Logical Theory of Nonmonotonic Inference and Belief Change, Artificial Intelligence, Springer, Berlin, Heidelberg, New York, 2001.
[18] M. Carlsson, G. Ottosson, B. Carlson, An open-ended finite domain constraint solver, in: H. Glaser, P.H. Hartel, H. Kuchen (Eds.), Programming Languages: Implementations, Logics, and Programs, PLILP'97, in: LNCS, vol. 1292, Springer, 1997, pp. 191–206.
[19] Bruno de Finetti, La prévision, ses lois logiques et ses sources subjectives, Ann. Inst. Henri Poincaré 7 (1) (1937), English translation in: H. Kyburg, H.E. Smokler (Eds.), Studies in Subjective Probability, Wiley & Sons, New York, 1974, pp. 93–158.
[20] J. Delgrande, F.J. Pelletier, Formal senses of relevance in default reasoning, Erkenntnis 49 (1998) 137–173.
[21] D. Dubois, H. Prade, Conditional objects as nonmonotonic consequence relationships, in: Special Issue on Conditional Event Algebra, IEEE Trans. Syst. Man Cybern. Syst. 24 (12) (1994) 1724–1740.
[22] Moisés Goldszmidt, Judea Pearl, Qualitative probabilities for default reasoning, belief revision, and causal modeling, Artif. Intell. 84 (1–2) (1996) 57–112.
[23] Jonas Haldimann, Anna Osiak, Christoph Beierle, Modelling and reasoning in biomedical applications with qualitative conditional logic, in: Ute Schmid, Franziska Klügl, Diedrich Wolter (Eds.), KI 2020: Advances in Artificial Intelligence – Proceedings of the 43rd German Conference on AI, Bamberg, Germany, September 21-25, 2020, in: LNCS, vol. 12325, Springer, 2020, pp. 283–289.
[24] J. Hawthorne, D. Makinson, The quantitative/qualitative watershed for rules of uncertain inference, Stud. Log. 86 (2) (2007) 247–297.
[25] Gabriele Kern-Isberner, Conditionals in Nonmonotonic Reasoning and Belief Revision – Considering Conditionals as Agents, Lecture Notes in Computer Science, vol. 2087, Springer Science+Business Media, Berlin, DE, 2001.
[26] Gabriele Kern-Isberner, Handling conditionals adequately in uncertain reasoning and belief revision, J. Appl. Non-Class. Log. 12 (2) (2002) 215–237.
[27] Gabriele Kern-Isberner, A thorough axiomatization of a principle of conditional preservation in belief revision, Ann. Math. Artif. Intell. 40 (2004) 127–164.
[28] Gabriele Kern-Isberner, Christoph Beierle, Gerhard Brewka, Syntax splitting = relevance + independence: new postulates for nonmonotonic reasoning from conditional belief bases, in: Diego Calvanese, Esra Erdem, Michael Thielscher (Eds.), Principles of Knowledge Representation and Reasoning: Proceedings of the 17th International Conference, KR 2020, IJCAI Organization, 2020, pp. 560–571.
[29] Gabriele Kern-Isberner, Christian Eichhorn, A structural base for conditional reasoning, in: Thomas Barkowsky, Marco Ragni, Frieder Stolzenburg (Eds.), Human Reasoning and Automated Deduction, Workshop at the 35th German Conference on Artificial Intelligence, KI-2012, 2012.
[30] Gabriele Kern-Isberner, Christian Eichhorn, Structural inference from conditional knowledge bases, in: Matthias Unterhuber, Gerhard Schurz (Eds.), Logic and Probability: Reasoning in Uncertain Environments, Stud. Log. 102 (4) (2014) 751–769, Springer Science+Business Media, Dordrecht, NL.
[31] Christian Komo, Christoph Beierle, Nonmonotonic inferences with qualitative conditionals based on preferred structures on worlds, in: Ute Schmid, Franziska Klügl, Diedrich Wolter (Eds.), KI 2020: Advances in Artificial Intelligence - 43rd German Conference on AI, Bamberg, Germany, September 21-25, 2020, Proceedings, in: LNCS, vol. 12325, Springer, 2020, pp. 102–115.
[32] Christian Komo, Christoph Beierle, Upper and lower bounds for finite domain constraints to realize skeptical c-inference over conditional knowledge bases, in: International Symposium on Artificial Intelligence and Mathematics, ISAIM 2020, Fort Lauderdale, FL, USA, January 6-8, 2020.
[33] Sarit Kraus, Daniel J. Lehmann, Menachem Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, Artif. Intell. 44 (1–2) (1990) 167–207.
[34] Steven Kutsch, Christoph Beierle, Semantic classification of qualitative conditionals and calculating closures of nonmonotonic inference relations, Int. J. Approx. Reason. 130 (2021) 297–313.
[35] D. Lehmann, M. Magidor, What does a conditional knowledge base entail?, Artif. Intell. 55 (1992) 1–60.
[36] D. Lewis, Counterfactuals, Harvard University Press, Cambridge, Mass, 1973.
[37] Thomas Lukasiewicz, Weak nonmonotonic probabilistic logics, Artif. Intell. 168 (1–2) (2005) 119–161.
[38] David Makinson, General patterns in nonmonotonic reasoning, in: Dov M. Gabbay, C.J. Hogger, J.A. Robinson (Eds.), Handbook of Logic in Artificial Intelligence and Logic Programming, vol. 3, Oxford University Press, 1994, pp. 35–110.
[39] Andreas Obergrusberger, A Constraint-Logic Approach to Inference from Qualitative Conditionals Using Minimal Ranking Functions, M.Sc. Thesis, Dept. of Computer Science, Fern Universität in Hagen, Germany, 2017.
[40] Rohit Parikh, Beliefs, belief revision, and splitting languages, in: Logic, Language, and Computation, vol. 2, 1999, pp. 266–278.
[41] Judea Pearl, Probabilistic Reasoning in Intelligent Systems, Morgan Kaufmann, San Mateo, CA, 1988.
[42] Judea Pearl, System Z: a natural ordering of defaults with tractable applications to nonmonotonic reasoning, in: Rohit Parikh (Ed.), Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning About Knowledge, TARK1990, San Francisco, CA, USA, Morgan Kaufmann Publishers Inc., 1990, pp. 121–135.
[43] Pavlos Peppas, Mary-Anne Williams, Samir Chopra, Norman Y. Foo, Relevance in belief revision, Artif. Intell. 229 (1–2) (2015) 126–138.
[44] Wolfgang Spohn, Ordinal conditional functions: a dynamic theory of epistemic states, in: W.L. Harper, B. Skyrms (Eds.), Causation in Decision, Belief Change, and Statistics, II, Kluwer Academic Publishers, 1988, pp. 105–134.
[45] Paul D. Thorn, Christian Eichhorn, Gabriele Kern-Isberner, Gerhard Schurz, Qualitative probabilistic inference with default inheritance for exceptional subclasses, in: Christoph Beierle, Gabriele Kern-Isberner, Marco Ragni, Frieder Stolzenburg (Eds.), Proceedings of the 5th Workshop on Dynamics of Knowledge and Belief (DKB-2015) and the 4th Workshop KI & Kognition (KIK-2015) Co-located with 38th German Conference on Artificial Intelligence (KI-2015), in: CEUR Workshop Proceedings, vol. 1444, 2015.