A Supplementary Material: Proof Appendix

Lemma 5. *Let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be the ones in the definition of section 3.1. Then, for each $i \geq 0$,*

(i) $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$;

(ii) there is a (smallest) $k \geq 0$ such that $\mathcal{B}_{k+1} = \mathcal{B}_k$;

(iii) $\mathcal{B}_i$ is reflexive, symmetric and transitive.

Proof. (i) Base: it is clear for $i = 0$ by the above definition.

Step: suppose it holds for $i = n$, i.e., $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.
$(s, s') \in \mathcal{B}_{n+2}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s'_1) \in R'$ such that $(s_1, s'_1) \in \mathcal{B}_{n+1}$, and (c) for every $(s', s'_1) \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s'_1) \in \mathcal{B}_{n+1}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s'_1) \in R'$ such that $(s_1, s'_1) \in \mathcal{B}_n$ by inductive assumption, and (c) for every $(s', s'_1) \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s'_1) \in \mathcal{B}_n$ by inductive assumption
$\Rightarrow (s, s') \in \mathcal{B}_{n+1}$.

(ii) and (iii) are evident from (i) and the definition of $\mathcal{B}_i$. □

Lemma 1 The relation $\leftrightarrow_V$ is an equivalence relation.

Proof. It is clear from Lemma 5 (ii) such that there is a $k \geq 0$ where $\mathcal{B}_k = \mathcal{B}_{k+1}$ which is $\leftrightarrow_V$, and it is reflexive, symmetric and transitive by (iii). □

Proposition 1 Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s'_is be two states and π'_is be two paths, and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:

(i) $s'_1 \leftrightarrow_{V_i} s'_2$ $(i = 1, 2)$ implies $s'_1 \leftrightarrow_{V_1 \cup V_2} s'_2$;

(ii) $\pi'_1 \leftrightarrow_{V_i} \pi'_2$ $(i = 1, 2)$ implies $\pi'_1 \leftrightarrow_{V_1 \cup V_2} \pi'_2$;

(iii) for each path π_{s_1} of $\mathcal{M}_1$ there is a path π_{s_2} of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Proof. In order to distinguish the relations $\mathcal{B}_0, \mathcal{B}_1, \ldots$ for different set $V \subseteq \mathcal{A}$, by $\mathcal{B}_i^V$ we mean the relation $\mathcal{B}_1, \mathcal{B}_2, \ldots$ for $V \subseteq \mathcal{A}$. Denote as $\mathcal{B}_0, \mathcal{B}_1, \ldots$ when the underlying set V is clear from the context. Moreover, for the ease of notation, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$ (i.e., without subindex).

(i) Base: it is clear for $n = 0$.
Step: For $n > 0$, supposing if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1 \cup V_2}$ for all $0 \leq i \leq n$. We will show that if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1 \cup V_2}$.
(a) It is evident that $L_1(s_1) - (V_1 \cup V_2) = L_2(s_2) - (V_1 \cup V_2)$.
(b) We will show that for each $(s_1, s_1^1) \in R_1$ there is a $(s_2, s_2^1) \in R_2$ such that $(s_1^1, s_2^1) \in \mathcal{B}_n^{V_1 \cup V_2}$. There is $(\mathcal{K}_1^1, \mathcal{K}_2^1) \in \mathcal{B}_{n-1}^{V_1 \cup V_2}$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_1 \cup V_2}$ by inductive assumption. Then we only need to prove for each $(s_1^1, s_1^2) \in R_1$ there is a $(s_2^1, s_2^2) \in R_2$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$ and for each $(s_2^1, s_2^2) \in R_2$ there is a $(s_1^1, s_1^2) \in R_1$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$. Therefore, we only need to prove that for each $(s_1^n, s_1^{n+1}) \in R_1$ there is a $(s_2^n, s_2^{n+1}) \in R_2$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$ and for each $(s_2^n, s_2^{n+1}) \in R_2$ there is a $(s_1^n, s_1^{n+1}) \in R_1$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$. It is apparent that $L_1(s_1^{n+1}) - (V_1 \cup V_2) = L_1(s_2^{n+1}) - (V_1 \cup V_2)$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$. Where $\mathcal{K}_i^j = (\mathcal{M}_i, s_i^j)$ with $i \in \{1, 2\}$ and $0 < j \leq n + 1$.
(c) It is similar with (b).

(ii) It is clear from (i).

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s'_1) \in R_1$, there is $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$, and

(c) for every $(s_2, s'_2) \in R_2$, there is $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$,

where $\mathcal{K}'_i = (\mathcal{M}_i, s'_i)$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

($\Rightarrow$) (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s'_1) \in R_1$, there is a $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s'_1) - V = L_2(s'_2) - V$. Therefore, $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$. (c) This is similar with (b).

($\Leftarrow$) (a) $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) Condition (ii) implies that for every $(s_1, s'_1) \in R_1$, there is $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_i$ for all $i \geq 0$; (c) Condition (iii) implies that for every $(s_2, s'_2) \in R_2$, there is $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing (s_1, s_3) from the (a), (b) and (c) of the previous step (iii) of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(v) Let $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$ and $(s_{i,k}, s_{i,k+1}) \in R_i$ mean that $s_{i,k+1}$ is the $(k + 2)$-th node in the path

$(s_i, s_{i,1}, s_{i,2}, \dots, s_{i,k+1}, \dots)$ $(i = 1, 2)$. We will show that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_2}$ for all $n \geq 0$ inductively.

Base: $L_1(s_1) - V_1 = L_2(s_2) - V_1$
$\Rightarrow \forall q \in \mathcal{A} - V_1$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$
$\Rightarrow \forall q \in \mathcal{A} - V_2$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$ due to $V_1 \subseteq V_2$
$\Rightarrow L_1(s_1) - V_2 = L_2(s_2) - V_2$, i.e., $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0^{V_2}$.

Step: Supposing that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ for all $0 \leq i \leq k$ $(k > 0)$, we will show $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{k+1}^{V_2}$.

(a) It is apparent that $L_1(s_1) - V_2 = L_2(s_2) - V_2$ by base.

(b) $\forall (s_1, s_{1,1}) \in R_1$, we will show that there is a $(s_2, s_{2,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_{k-1}^{V_2}$ by inductive assumption, we need only to prove the following points:
(a) $\forall (s_{1,k}, s_{1,k+1}) \in R_1$ there is a $(s_{2,k}, s_{2,k+1}) \in R_2$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. It is easy to see that $L_1(s_{1,k+1}) - V_1 = L_1(s_{2,k+1}) - V_1$, then there is $L_1(s_{1,k+1}) - V_2 = L_1(s_{2,k+1}) - V_2$. Therefore, $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$.
(b) $\forall (s_{2,k}, s_{2,k+1}) \in R_1$ there is a $(s_{1,k}, s_{1,k+1}) \in R_1$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. This can be proved as (a).

(c) $\forall (s_2, s_{2,1}) \in R_1$, we will show that there is a $(s_1, s_{1,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. This can be proved as (ii).

$\square$

Theorem1 Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.

Proof. This theorem can be proved by inducting on the formula ϕ and supposing *Var*$(\phi) \cap V = \emptyset$. Let $\mathcal{K}_1 = (\mathcal{M}, s)$ and $\mathcal{K}_2 = (\mathcal{M}', s')$.

Case $\phi = p$ where $p \in \mathcal{A} - V$:
$(\mathcal{M}, s) \models \phi$ iff $p \in L(s)$ (by the definition of satisfiability)
$\Leftrightarrow p \in L'(s')$ $\quad (s \leftrightarrow_V s')$
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \neg\psi$:
$(\mathcal{M}, s) \models \phi$ iff $(\mathcal{M}, s) \not\models \psi$
$\Leftrightarrow (\mathcal{M}', s') \not\models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \psi_1 \vee \psi_2$:
$(\mathcal{M}, s) \models \phi$
$\Leftrightarrow (\mathcal{M}, s) \models \psi_1$ or $(\mathcal{M}, s) \models \psi_2$
$\Leftrightarrow (\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \text{EX}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s, s_1, ...)$ such that $\mathcal{M}, s_1 \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s', s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_1 \leftrightarrow_V s'_1$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s'_1) \models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \text{EG}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that for each $i \geq 0$ there is $(\mathcal{M}, s_i) \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s' = s'_0, s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_i \leftrightarrow_V s'_i$ for each $i \geq 0$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s'_i) \models \psi$ for each $i \geq 0$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \text{E}[\psi_1 \text{U} \psi_2]$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that there is $i \geq 0$ such that $(\mathcal{M}, s_i) \models \psi_2$, and for all $0 \leq j < i$, $(\mathcal{M}, s_j) \models \psi_1$
$\Leftrightarrow$ There is a path $\pi' = (s = s'_0, s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ ($s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow (\mathcal{M}', s'_i) \models \psi_2$, and for all $0 \leq j < i$ $(\mathcal{M}', s'_j) \models \psi_1$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$ $\square$

Proposition 2 Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two K-structures. Then

$(s_1, s_2) \in \mathcal{B}_n$ iff $\text{Tr}_j(s_1) \leftrightarrow_V \text{Tr}_j(s_2)$ for every $0 \leq j \leq n$.

Proof. We will prove this from two aspects:

($\Rightarrow$) If $(s_1, s_2) \in \mathcal{B}_n$, then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$. $(s, s') \in \mathcal{B}_n$ implies both roots of $Tr_n(s_1)$ and $Tr_n(s_2)$ have the same atoms except those atoms in V. Besides, for any $s_{1,1}$ with $(s_1, s_{1,1}) \in R_1$, there is a $s_{2,1}$ with $(s_2, s_{2,1}) \in R_2$ s.t. $(s_{1,1}, s_{2,1}) \in \mathcal{B}_{n-1}$ and vice versa. Then we have $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$. Therefore, $Tr_n(s_1) \leftrightarrow_V Tr_n(s_2)$ by use such method recursively, and then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$.

($\Leftarrow$) If $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$, then $(s_1, s_2) \in \mathcal{B}_n$. $Tr_0(s_1) \leftrightarrow_V Tr_0(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and then $(s, s') \in \mathcal{B}_0$. $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and for every successors s of the root of one, it is possible to find a successor of the root of the other s' such that $(s, s') \in \mathcal{B}_0$. Therefore $(s_1, s_2) \in \mathcal{B}_1$, and then we will have $(s_1, s_2) \in \mathcal{B}_n$ by use such method recursively. $\square$

Proposition 3 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least k such that $\text{Tr}_k(s)$ and $\text{Tr}_k(s')$ are not V-bisimilar.

Proof. If $s \not\leftrightarrow_V s'$, then there exists a least constant c such that $(s_i, s_j) \notin \mathcal{B}_c$, and then there is a least constant m $(m \leq c)$ such that $\text{Tr}_m(s_i)$ and $\text{Tr}_m(s_j)$ are not V-bisimilar by Proposition 2. Let $k = m$, the lemma is proved. $\square$

Lemma2 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ and $\mathcal{M}'$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $\text{Tr}_n(s) \leftrightarrow_{\overline{V}} \text{Tr}_n(s')$, then $\mathcal{F}_V(\text{Tr}_n(s)) \equiv \mathcal{F}_V(\text{Tr}_n(s'))$.

Proof. This result can be proved by inducting on n.

Base. It is apparent that for any $s_n \in S$ and $s'_n \in S'$, if $\text{Tr}_0(s_n) \leftrightarrow_{\overline{V}} \text{Tr}_0(s'_n)$ then $\mathcal{F}_V(\text{Tr}_0(s_n)) \equiv \mathcal{F}_V(\text{Tr}_0(s'_n))$ due to $L(s_n) - \overline{V} = L'(s'_n) - \overline{V}$ by known.

Step. Supposing that for $k = m$ $(0 < m \leq n)$ there is if $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ then $\mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k)) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))$, then we will show if $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$ then $\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1}))$. Apparent that:
$\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) = \left(\bigwedge_{(s_{k-1},s_k)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_{k-1},s_k)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k))\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s_{k-1}))$
$\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1})) = \left(\bigwedge_{(s'_{k-1},s'_k)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))\right) \wedge \mathrm{AX}\left(\bigvee_{(s'_{k-1},s'_k)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s'_{k-1}))$
by the definition of characterizing formula of the computation tree. Then we have for any $(s_{k-1}, s_k) \in R$ there is $(s'_{k-1}, s'_k) \in R'$ such that $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ by $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$. Besides, for any $(s'_{k-1}, s'_k) \in R'$ there is $(s_{k-1}, s_k) \in R$ such that $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ by $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$. Therefore, we have $\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1}))$ by induction hypothesis. $\square$

Theorem 2 Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures. Then,

(i) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s'_0)$;

(ii) $s_0 \leftrightarrow_{\overline{V}} s'_0$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s'_0)$.

In order to prove Theorem 2, we prove the following two lemmas at first.

Lemma 6. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s))$.

(ii) If $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s'))$ then $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$.

Proof. (i) It is apparent from the definition of $\mathcal{F}_V(\mathrm{Tr}_n(s))$.
Base. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s))$.
Step. For $k \geq 0$, supposing the result talked in (i) is correct in $k - 1$, we will show that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_{k+1}(s))$, i.e.,:

$$(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \mathrm{EX}T(s')\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s')\in R} T(s')\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s)).$$

Where $T(s') = \mathcal{F}_V(\mathrm{Tr}_k(s'))$. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s))$ by Base. It is apparent that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \mathcal{F}_V(\mathrm{Tr}_k(s'))$ by inductive assumption. Then we have $(\mathcal{M}, s) \models \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_k(s')$, and then $(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_k(s'))\right)$. Similarly, we have that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \bigvee_{(s,s'')\in R} \mathcal{F}_V(\mathrm{Tr}_k(s''))$. Therefore, $(\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s,s'')\in R} \mathcal{F}_V(\mathrm{Tr}_k(s''))\right)$.

(ii) **Base**. If $n = 0$, then $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s'))$ implies $L(s) - \overline{V} = L'(s') - \overline{V}$. Hence, $\mathrm{Tr}_0(s) \leftrightarrow_{\overline{V}} \mathrm{Tr}_0(s')$.
Step. Supposing $n > 0$ and the result talked in (ii) is correct in $n - 1$.
(a) It is easy to see that $L(s) - \overline{V} = L'(s') - \overline{V}$.
(b) We will show that for each $(s, s_1) \in R$, there is a $(s', s'_1) \in R'$ such that $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s',s'_1)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))\right)$. Therefore, for each $(s, s_1) \in R$ there is a $(s', s'_1) \in R'$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))$. Hence, $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$ by inductive hypothesis.
(c) We will show that for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $\mathrm{Tr}_{n-1}(s'_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \bigwedge_{(s',s'_1)\in R'} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))$. Therefore, for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1)$. Hence, $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$ by inductive hypothesis. $\square$

A consequence of the previous lemma is:

Lemma 7. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ a model structure, $k = ch(\mathcal{M}, V)$ and $s \in S$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_k(s))$, and

(ii) for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(Tr_k(s))$.

Proof. (i) It is apparent from the (i) of Lemma 6.
(ii) Let $\phi = \mathcal{F}_V(\mathrm{Tr}_k(s))$, where k is the V-characteristic number of $\mathcal{M}$. $(\mathcal{M}, s) \models \phi$ by the definition of $\mathcal{F}$, and then $\forall s' \in S$, if $s \leftrightarrow_{\overline{V}} s'$ there is $(\mathcal{M}, s') \models \phi$ by Theorem 1 due to $\mathrm{IR}(\phi, \mathcal{A} - V)$. Supposing $(\mathcal{M}, s') \models \phi$, if $s \not\leftrightarrow_{\overline{V}} s'$, then $\mathrm{Tr}_k(s) \not\leftrightarrow_{\overline{V}} \mathrm{Tr}_k(s')$, and then $(\mathcal{M}, s') \not\models \phi$ by Lemma 6, a contradiction. $\square$

Now we are in the position of proving Theorem 2.

Proof. (i) Let $\mathcal{F}_V(\mathcal{M}, s_0)$ be the characterizing formula of $(\mathcal{M}, s_0)$ on V. It is apparent that $\mathrm{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. We will show that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ at first.

It is apparent that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s_0))$ by Lemma 6. We must show that $(\mathcal{M}, s_0) \models \bigwedge_{s\in S} G(\mathcal{M}, s)$. Let $\mathcal{X} = \mathcal{F}_V(\mathrm{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$, we will show $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$. Where $G(\mathcal{M}, s) = \mathrm{AG}\mathcal{X}$. There are two cases we should consider:

- If $(\mathcal{M}, s_0) \not\models \mathcal{F}_V(\mathrm{Tr}_c(s))$, it is apparent that $(\mathcal{M}, s_0) \models \mathcal{X}$;
- If $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s))$:
  $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s))$
  $\Rightarrow s_0 \leftrightarrow_{\overline{V}} s$ by the definition of characteristic number and Lemma 7.
  For each $(s, s_1) \in R$ there is:
  $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_c(s_1))$ $\quad (s_1 \leftrightarrow_{\overline{V}} s_1)$
  $\Rightarrow (\mathcal{M}, s) \models \bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$
  $\Rightarrow (\mathcal{M}, s_0) \models \bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (by $\mathrm{IR}(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1)), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$).
  For each (s, s_1) there is:
  $\mathcal{M}, s_1 \models \bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))$

$\Rightarrow (\mathcal{M}, s) \models \text{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$

$\Rightarrow (\mathcal{M}, s_0) \models \text{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$ (by IR($\text{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right), \overline{V}$), $s_0 \leftrightarrow_{\overline{V}} s$)

$\Rightarrow (\mathcal{M}, s_0) \models \mathcal{X}$.

For any other states s' which can reach from s_0 can be proved similarly, i.e.,, $(\mathcal{M}, s') \models \mathcal{X}$. Therefore, $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$, and then $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$.

We will prove this theorem from the following two aspects:

($\Leftarrow$) If $s_0 \leftrightarrow_{\overline{V}} s_0'$, then $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$. Since $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and IR$(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$, hence $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

($\Rightarrow$) If $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$, then $s_0 \leftrightarrow_{\overline{V}} s_0'$. We will prove this by showing that $\forall n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s_0')$.

Base. It is apparent that $Tr_0(s_0) \equiv Tr_0(s_0')$.

Step. Supposing $\text{Tr}_k(s_0) \leftrightarrow_{\overline{V}} \text{Tr}_k(s_0')$ ($k > 0$), we will prove $\text{Tr}_{k+1}(s_0) \leftrightarrow_{\overline{V}} \text{Tr}_{k+1}(s_0')$. We should only show that $\text{Tr}_1(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_1(s_k')$. Where $(s_0, s_1), (s_1, s_2), \ldots, (s_{k-1}, s_k) \in R$ and $(s_0', s_1'), (s_1', s_2'), \ldots, (s_{k-1}', s_k') \in R'$, i.e., s_{i+1} (s_{i+1}') is an immediate successor of s_i (s_i') for all $0 \leq i \leq k-1$.

(a) It is apparent that $L(s_k) - \overline{V} = L'(s_k') - \overline{V}$ by inductive assumption.

Before talking about the other points, note the following fact that:

$(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$
$\Rightarrow \forall s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\text{Tr}_c(s)) \to \left(\bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ for any $s \in S$. **(fact)**

(I) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\text{Tr}_c(s_0)) \to \left(\bigwedge_{(s_0,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ **(fact)**

(II) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\text{Tr}_c(s_0)))$ (known)

(III) $(\mathcal{M}', s_0') \models \left(\bigwedge_{(s_0,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$ ((I),(II))

(b) We will show that for each $(s_k, s_{k+1}) \in R$ there is a $(s_k', s_{k+1}') \in R'$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.

(1) $(\mathcal{M}', s_0') \models \bigwedge_{(s_0,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))$ (III)

(2) $\forall (s_0, s_1) \in R$, $\exists (s_0', s_1') \in R'$ s.t. $(\mathcal{M}', s_1') \models \mathcal{F}_V(\text{Tr}_c(s_1))$ (2)

(3) $\text{Tr}_c(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_c(s_1')$ ((2), Lemma 6)

(4) $L(s_1) - \overline{V} = L'(s_1') - \overline{V}$ ((3), $c \geq 0$)

(5) $(\mathcal{M}', s_1') \models \mathcal{F}_V(\text{Tr}_c(s_1)) \to \left(\bigwedge_{(s_1,s_2)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_2))\right) \wedge \text{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$ **(fact)**

(6) $(\mathcal{M}', s_1') \models \left(\bigwedge_{(s_1,s_2)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_2))\right) \wedge \text{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))\right)$ ((2), (5))

(7) ......

(8) $(\mathcal{M}', s_k') \models \left(\bigwedge_{(s_k,s_{k+1})\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_{k+1}))\right) \wedge \text{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\text{Tr}_c(s_{k+1}))\right)$ (similar with (6))

(9) $\forall (s_k, s_{k+1}) \in R$, $\exists (s_k', s_{k+1}') \in R'$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\text{Tr}_c(s_{k+1}))$ (8)

(10) $\text{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \text{Tr}_c(s_{k+1}')$ ((9), Lemma 6)

(11) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((10), $c \geq 0$)

(c) We will show that for each $(s_k', s_{k+1}') \in R'$ there is a $(s_k, s_{k+1}) \in R$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.

(1) $(\mathcal{M}', s_k') \models \text{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\text{Tr}_c(s_{k+1}))\right)$ (by (8) talked above)

(2) $\forall (s_k', s_{k+1}') \in R'$, $\exists (s_k, s_{k+1}) \in R$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\text{Tr}_c(s_{k+1}'))$ (1)

(3) $\text{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \text{Tr}_c(s_{k+1}')$ ((2), Lemma 6)

(4) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((3), $c \geq 0$)

(ii) This is following Lemma 2 and the definition of the characterizing formula of initial K-structure $\mathcal{K}$ on V.

□

Lemma 3 Let φ be a formula. We have

$$\varphi \equiv \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \quad (3)$$

Proof. Let $(\mathcal{M}', s_0')$ be a model of φ. Then $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$ due to $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}', s_0')$. On the other hand, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. Then there is a $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. And then $(\mathcal{M}, s_0) \leftrightarrow_{\emptyset} (\mathcal{M}', s_0')$ by Theorem 2. Therefore, $(\mathcal{M}, s_0)$ is also a model of φ by Theorem 1. □

Theorem 3 **(Representation theorem)** Let φ, φ' and ϕ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

(i) $\varphi' \equiv \text{F}_{\text{CTL}}(\varphi, V)$,

(ii) $\varphi' \equiv \{\phi | \varphi \models \phi \text{ and } \text{IR}(\phi, V)\}$,

(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\text{F}_{\text{CTL}}(\varphi, V)) = Mod(\{\phi|\varphi \models \phi, \text{IR}(\phi, V)\})$$
$$= Mod(\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)).$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\text{F}_{\text{CTL}}(\varphi, V)$. Then there exists an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of φ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all ϕ that $\varphi \models \phi$ and $\text{IR}(\phi, V)$. Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi|\varphi \models \phi, \text{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s_0')$ is a models of $\{\phi|\varphi \models \phi, \text{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s_0')$

$\models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$ due to $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$ is irrelevant to V.

Finally, suppose that $(\mathcal{M}',s_0')$ is a model of $\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$. Then there exists $(\mathcal{M},s_0) \in Mod(\varphi)$ such that $(\mathcal{M}',s_0') \models \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$. Hence, $(\mathcal{M},s_0) \leftrightarrow_V (\mathcal{M}',s_0')$ by Theorem 2. Thus $(\mathcal{M}',s_0')$ is also a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. Suppose that all postulates hold. By Positive Persistence, we have $\varphi' \models \{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\}$. Now we show that $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\} \models \varphi'$. Otherwise, there exists formula ϕ' such that $\varphi' \models \phi'$ but $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\} \not\models \phi'$. There are three cases:

- ϕ' is relevant to V. Thus, φ' is also relevant to V, a contradiction to Irrelevance.
- ϕ' is irrelevant to V and $\varphi \models \phi'$. This contradicts to our assumption.
- ϕ' is irrelevant to V and $\varphi \not\models \phi'$. By Negative Persistence, $\varphi' \not\models \phi'$, a contradiction.

Thus, φ' is equivalent to $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\}$. □

Lemma 4 Let φ and α be two CTL formulae and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then $\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $(\mathcal{M},s)$ of $\mathrm{F}_{\mathrm{CTL}}(\varphi',q)$ there is an initial K-structure $(\mathcal{M}',s')$ s.t. $(\mathcal{M},s) \leftrightarrow_{\{q\}} (\mathcal{M}',s')$ and $(\mathcal{M}',s') \models \varphi'$. It's apparent that $(\mathcal{M}',s') \models \varphi$, and then $(\mathcal{M},s) \models \varphi$ since $\mathrm{IR}(\varphi,\{q\})$ and $(\mathcal{M},s) \leftrightarrow_{\{q\}} (\mathcal{M}',s')$ by Theorem 1.

Let $(\mathcal{M},s) \in Mod(\varphi)$ with $\mathcal{M} = (S,R,L,s)$. We construct $(\mathcal{M}',s)$ with $\mathcal{M}' = (S,R,L',s)$ as follows:

$L' : S \to \mathcal{A}$ *and* $\forall s^* \in S, L'(s^*) = L(s^*)$ *if* $(\mathcal{M},s^*) \not\models \alpha$, *else* $L'(s^*) = L(s^*) \cup \{q\}$,

$L'(s) = L(s) \cup \{q\}$ *if* $(\mathcal{M},s) \models \alpha$, *and* $L'(s) = L(s)$ *otherwise*.

It is clear that $(\mathcal{M}',s) \models \varphi$, $(\mathcal{M}',s) \models q \leftrightarrow \alpha$ and $(\mathcal{M}',s) \leftrightarrow_{\{q\}} (\mathcal{M},s)$. Therefore $(\mathcal{M}',s) \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $(\mathcal{M}',s) \leftrightarrow_{\{q\}} (\mathcal{M},s)$. □

Proposition 4 Given a formula $\varphi \in$ CTL, V a set of atoms and p an atom such that $p \notin V$. Then,

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V).$$

Proof. Let $(\mathcal{M}_1,s_1)$ with $\mathcal{M}_1 = (S_1,R_1,L_1,s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,\{p\} \cup V)$. By the definition, there exists a model $(\mathcal{M},s)$ with $\mathcal{M} = (S,R,L,s)$ of φ, such that $(\mathcal{M}_1,s_1) \leftrightarrow_{\{p\}\cup V} (\mathcal{M},s)$. We construct an initial K-structure $(\mathcal{M}_2,s_2)$ with $\mathcal{M}_2 = (S_2,R_2,L_2,s_2)$ as follows:

(1) for s_2: let s_2 be the state such that:
- $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
- for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
- for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:
   (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\}\cup V} w_1$, let $w_2 \in S_2$ and
   - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
   - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
   - for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

   (ii) if $(w_1',w_1) \in R_1$, w_2 is constructed based on w_1 and $w_2' \in S_2$ is constructed based on w_1', then $(w_2',w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $(\mathcal{M},s) \leftrightarrow_{\{p\}} (\mathcal{M}_2,s_2)$ and $(\mathcal{M}_2,s_2) \leftrightarrow_V (\mathcal{M}_1,s_1)$. Thus, $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,p)$. And therefore $(\mathcal{M}_1,s_1) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V)$.

On the other hand, suppose that $(\mathcal{M}_1,s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V)$, then there exists an initial K-structure $(\mathcal{M}_2,s_2)$ such that $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,p)$ and $(\mathcal{M}_2,s_2) \leftrightarrow_V (\mathcal{M}_1,s_1)$, and there exists $(\mathcal{M},s)$ such that $(\mathcal{M},s) \models \varphi$ and $(\mathcal{M},s) \leftrightarrow_{\{p\}} (\mathcal{M}_2,s_2)$. Therefore, $(\mathcal{M},s) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}_1,s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1,s_1) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,\{p\} \cup V)$. □

Proposition 5 Let φ, φ_i, ψ_i $(i = 1,2)$ be formulas and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$ is satisfiable iff φ is;

(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1,V) \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi_2,V)$;

(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1,V) \models \mathrm{F}_{\mathrm{CTL}}(\varphi_2,V)$;

(iv) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V) \equiv \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$;

(v) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \wedge \psi_2,V) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \wedge \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$;

Proof. (i) $(\Rightarrow)$ Supposing $(\mathcal{M},s)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$, then there is a model $(\mathcal{M}',s')$ of φ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

$(\Leftarrow)$ Supposing $(\mathcal{M},s)$ is a model of φ, then there is an initial Kripke structure $(\mathcal{M}',s')$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$, and then $(\mathcal{M}',s') \models \mathrm{F}_{\mathrm{CTL}}(\varphi,V)$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

The (ii) and (iii) can be proved similarly.

(iv) $(\Rightarrow)$ $\forall(\mathcal{M},s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V))$, $\exists(\mathcal{M}',s') \in Mod(\psi_1 \vee \psi_2)$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$ and $(\mathcal{M}',s') \models \psi_1$ or $(\mathcal{M}',s') \models \psi_2$
$\Rightarrow$ $\exists(\mathcal{M}_1,s_1) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1,V))$ s.t. $(\mathcal{M}',s') \leftrightarrow_V (\mathcal{M}_1,s_1)$ or $\exists(\mathcal{M}_2,s_2) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_2,V))$ s.t. $(\mathcal{M}',s') \leftrightarrow_V (\mathcal{M}_2,s_2)$
$\Rightarrow (\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$ by Theorem 1.

$(\Leftarrow)$ $\forall(\mathcal{M},s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V))$
$\Rightarrow (\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V)$ or $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1,s_1)$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}_1,s_1)$ and $(\mathcal{M}_1,s_1) \models \psi_1$ or $(\mathcal{M}_1,s_1) \models \psi_2$
$\Rightarrow (\mathcal{M}_1,s_1) \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2,s_2)$ s.t. $(\mathcal{M}_1,s_1) \leftrightarrow_V (\mathcal{M}_2,s_2)$ and $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V)$
$\Rightarrow (\mathcal{M},s) \leftrightarrow_V (\mathcal{M}_2,s_2)$ and $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V)$.

The (v) can be proved as (iv). □

Proposition 6 **(Homogeneity)** Let $V \subseteq \mathcal{A}$ and $\phi \in$ CTL,

(i) $\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V) \equiv \text{AX}\mathrm{F}_{\text{CTL}}(\phi, V)$.

(ii) $\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V) \equiv \text{EX}\mathrm{F}_{\text{CTL}}(\phi, V)$.

(iii) $\mathrm{F}_{\text{CTL}}(\text{AF}\phi, V) \equiv \text{AF}\mathrm{F}_{\text{CTL}}(\phi, V)$.

(iv) $\mathrm{F}_{\text{CTL}}(\text{EF}\phi, V) \equiv \text{EF}\mathrm{F}_{\text{CTL}}(\phi, V)$.

Proof. Let $\mathcal{M} = (S, R, L, s_0)$ with initial state s_0 and $\mathcal{M}' = (S', R', L', s_0')$ with initial state s_0', then we call $\mathcal{M}', s_0'$ be a sub-structure of $\mathcal{M}, s_0$ if:

- $S' \subseteq S$ and $S' = \{s' | s'$ is reachable from $s_0'\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to 2^{\mathcal{A}}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- s_0' is s_0 or a state reachable from s_0.

(i) In order to prove $\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V) \equiv \text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V))$, we only need to prove $\mathit{Mod}(\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V)) = \mathit{Mod}(\text{AX}\mathrm{F}_{\text{CTL}}(\phi, V))$:

$(\Rightarrow)$ $\forall (\mathcal{M}', s') \in \mathit{Mod}(\mathrm{F}_{\text{CTL}}(\text{AX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \text{AX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ for any sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ there is $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ with s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow$ $(\mathcal{M}_3, s_3) \models \text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V))$ and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow (\mathcal{M}', s') \models \text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V))$.

$(\Leftarrow)$ $\forall$ $(\mathcal{M}_3, s_3) \in \mathit{Mod}(\text{AX}(\mathrm{F}_{\text{CTL}}(\phi, V)))$, then for any sub-structure $(\mathcal{M}_2, s_2)$ with s_2 is a directed successor of s_3 there is $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$
$\Rightarrow$ for any $(\mathcal{M}_2, s_2)$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ with s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \text{AX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\text{CTL}}(\text{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V) \equiv \text{EX}\mathrm{F}_{\text{CTL}}(\phi, V)$, we only need to prove Mod $(\mathrm{F}_{\text{CTL}}(\text{EX}\phi, \ V)) = \mathit{Mod}(\text{EX}\mathrm{F}_{\text{CTL}}(\phi, V))$:

$(\Rightarrow)$ $\forall \mathcal{M}', s' \in \mathit{Mod}(\mathrm{F}_{\text{CTL}}(\text{EX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \text{EX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ there is a sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ that s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow (\mathcal{M}_3, s_3) \models \text{EX}(\mathrm{F}_{\text{CTL}}(\phi, V))$
$\Rightarrow (\mathcal{M}', s') \models \text{EX}(\mathrm{F}_{\text{CTL}}(\phi, V))$.

$(\Leftarrow)$ $\forall$ $(\mathcal{M}_3, s_3) \in \mathit{Mod}(\text{EX}(\mathrm{F}_{\text{CTL}}(\phi, V)))$, then there exists a sub-structure $(\mathcal{M}_2, s_2)$ of $(\mathcal{M}_3, s_3)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\phi, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ that s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \text{EX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\text{CTL}}(\text{EX}\phi, V)$.

(iii) and (iV) can be proved as (i) and (ii) respectively. $\square$

Proposition7 **(Model Checking on Forgetting)** Let $(\mathcal{M}, s_0)$ be an initial K-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is NP-complete.

Proof. The problem can be determined by the following two things: (1) guessing an initial K-structure $(\mathcal{M}', s_0')$ satisfying φ; and (2) checking if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Both two steps can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard (Zhang and Zhou 2008). $\square$

Theorem 5 **(Entailment on Forgetting)** Let φ and ψ be two CTL_{AF} formulas and V a set of atoms. Then, results:

(i) deciding $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models^? \psi$ is co-NP-complete,

(ii) deciding $\psi \models^? \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is Π_2^P-complete,

(iii) deciding $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models^? \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ is Π_2^P-complete.

Proof. (1) It is proved that deciding whether ψ is satisfiable is NP-Complete (Meier et al. 2015). The hardness is easy to see by setting $\mathrm{F}_{\mathrm{CTL}}(\varphi, \mathit{Var}(\varphi)) \equiv \top$, i.e., deciding whether ψ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $IR(\psi, V)$. Clearly, in CTL_{AF}, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $IR(\psi, V)$ is also in co-NP. Without loss of generality, we assume that ψ is satisfiable. We consider the complement of the problem: deciding whether ψ is not irrelevant to V. It is easy to see that ψ is not irrelevant to V iff there exist a model $(\mathcal{M}, s_0)$ of ψ and an initial K-structure $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ and $(\mathcal{M}', s_0') \not\models \psi$. So checking whether ψ is not irrelevant to V can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s_0')$, (2) check if $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s_0') \not\models \psi$, and (3) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Obviously (1) can be done in polynomial time and also (2) and (3) can be done in polynomial time.

(2) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ and check whether $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}, s_0) \not\models \mathrm{F}_{\mathrm{CTL}}(\ \varphi, V)$. From Proposition 7, we know that this is in Σ_2^P. So the original problem is in Π_2^P. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case temporal forgetting, the hardness is directly followed from the proof of Proposition 24 in (Lang, Liberatore, and Marquis 2003).

(3) Membership. If $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$, i.e.,, there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models$

ψ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M}, s)$, $(\mathcal{M}_1, s_1)$ with $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ and checking $(\mathcal{M}_1, s_1) \models \varphi$ are feasible while checking $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ can be done in polynomial time. Thus the problem is in Π_2^P.

Hardness. It follows from (2) due to the fact that $\mathrm{F_{CTL}}(\varphi, V) \models \mathrm{F_{CTL}}(\psi, V)$ iff $\varphi \models \mathrm{F_{CTL}}(\psi, V)$ thanks to $IR(\mathrm{F_{CTL}}(\psi, V), V)$. □

Proposition 8 **(dual)** Let V, q, φ and ψ are like in Definition 4. The ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \rightarrow \psi$. Thus $\varphi \models \neg\psi \rightarrow \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \rightarrow \neg q$. Then $\varphi \models q \rightarrow \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \rightarrow \neg\psi'$ by assumption. So $\varphi \models \psi' \rightarrow \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case. □

Proposition 9 Let Γ and α be two formulas, $V \subseteq Var(\alpha) \cup Var(\Gamma)$ and q is a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

Proof. We prove this for SNC. The case for WSC is similar. Let $SNC(\varphi, \alpha, V, \Gamma)$ denote that φ is the SNC of α on V under Γ, and $NC(\varphi, \alpha, V, \Gamma)$ denote that φ is the NC of α on V under Γ.

($\Rightarrow$) We will show that if $SNC(\varphi, \alpha, V, \Gamma)$ holds, then $SNC(\varphi, q, V, \Gamma')$ will be true. According to $SNC(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \rightarrow \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F_{CTL}}(\Gamma', q) \models \alpha \rightarrow \varphi'$ due to $\alpha \equiv q$, $IR(\alpha \rightarrow \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi'$ by Lemma 4, this means $NC(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \rightarrow \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \rightarrow \varphi'$. Hence, $SNC(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $SNC(\varphi, q, V, \Gamma')$ holds, then $SNC(\varphi, \alpha, V, \Gamma)$ will be true. According to $SNC(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F_{CTL}}(\Gamma', \{q\}) \models \alpha \rightarrow \varphi$ due to $\alpha \equiv q$, $IR(\alpha \rightarrow \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \rightarrow \varphi$ by Lemma 4, this means $NC(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \rightarrow \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $NC(\varphi', q, V, \Gamma')$. According to $SNC(\varphi, q, V, \Gamma')$, $IR(\varphi \rightarrow \varphi', \{q\})$ and (**PP**), we have $\mathrm{F_{CTL}}(\Gamma', \{q\}) \models \varphi \rightarrow \varphi'$, and $\Gamma \models \varphi \rightarrow \varphi'$ by Lemma 4. Hence, $SNC(\varphi, \alpha, V, \Gamma)$ holds. □

Theorem 7 Let φ be a formula, $V \subseteq Var(\varphi)$ and $q \in Var(\varphi) - V$.

(i) $\mathrm{F_{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ.

(ii) $\neg\mathrm{F_{CTL}}(\varphi \wedge \neg q, (Var(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ.

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F_{CTL}}(\varphi \wedge q, (Var(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \rightarrow \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \rightarrow \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \rightarrow \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $IR(\psi, (Var(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ. □

Theorem 8 Let $\mathcal{K} = (\mathcal{M}, s)$ be an initial K-structure with $\mathcal{M} = (S, R, L, s_0)$ on the set $\mathcal{A}$ of atoms, $V \subseteq \mathcal{A}$ and $q \in V' = \mathcal{A} - V$. Then:

(i) the SNC of q on V under $\mathcal{K}$ is $\mathrm{F_{CTL}}(\mathcal{F_A}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q on V under $\mathcal{K}$ is $\neg\mathrm{F_{CTL}}(\mathcal{F_A}(\mathcal{K}) \wedge \neg q, V')$.

Proof. (i) As we know that any initial K-structure $\mathcal{K}$ can be described as a characterizing formula $\mathcal{F_A}(\mathcal{K})$, then the SNC of q on V under $\mathcal{F_A}(\mathcal{K})$ is $\mathrm{F_{CTL}}(\mathcal{F_A}(\mathcal{K}) \wedge q, \mathcal{A} - V)$.

(ii) This is proved by the dual property. □

Proposition10 Let φ be a CTL formula and $V \subseteq \mathcal{A}$ with $|\mathcal{S}| = m$, $|\mathcal{A}| = n$ and $|V| = x$. The the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity of Algorithm 1 is at least the same as the space.

Proof. Supposing each state or atom occupy one byte, then a state pair (s, s') occupy two bytes. For any $B \subseteq \mathcal{S}$ with $B \neq \emptyset$ and $s_0 \in B$, we can construct an initial K-structure $(\mathcal{M}, s_0)$ with $\mathcal{M} = (B, R, L, s_0)$, in which there is at most $\frac{|B|^2}{2}$ state pairs in R and $|B| * n$ (s, A) $(A \subseteq \mathcal{A})$ in L. Hence, the $(\mathcal{M}, s_0)$ occupy at most $|B| + |B|^2 + |B| * n$ bytes. Besides, there is at most $|B|^{|B|+1} * 2^{nm}$ number of initial K-structures. Therefore, there is at most $m^{m+2} * 2^{nm}$ number of initial K-structures, hence it will at most cost $m^{m+2} * 2^{nm} * (m + m^2 + nm)$ bytes.

Let $k = n - x$, for any initial K-structure $\mathcal{K} = (\mathcal{M}, s_0)$ with $i \geq 1$ nodes, in the worst, i.e., $ch(\mathcal{M}, V) = i$, we will spend $N(i)$ space to store the characterizing formula.

$$
\begin{aligned}
N(i) &= (k + (\cdots + (k + 2ik) * (2i)) \cdots * (2i)) \\
&= (2i)^0 k + 2ik + (2i)^2 k + \cdots + (2i)^{(i-1)} k \\
&= \frac{(2i)^i - 1}{2i - 1} k.
\end{aligned}
$$

In the worst case, i.e., there is $m^{m+2} * 2^{nm}$ initial K-structures with m nodes, we will spent $m^{m+2} * 2^{nm} * N(m)$ bytes to store the result of forgetting.

Therefore, the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity is at least the same as the space. $\square$