A Supplementary Material: Proof Appendix

Lemma 5. *Let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be the ones in the definition of section 3.1. Then, for each $i \geq 0$,*

(i) $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$;

(ii) there is a (smallest) $k \geq 0$ such that $\mathcal{B}_{k+1} = \mathcal{B}_k$;

(iii) $\mathcal{B}_i$ is reflexive, symmetric and transitive.

Proof. (i) Base: it is clear for $i = 0$ by the above definition.

Step: suppose it holds for $i = n$, i.e., $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.
$(s, s') \in \mathcal{B}_{n+2}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s'_1) \in R'$ such that $(s_1, s'_1) \in \mathcal{B}_{n+1}$, and (c) for every $(s', s'_1) \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s'_1) \in \mathcal{B}_{n+1}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s'_1) \in R'$ such that $(s_1, s'_1) \in \mathcal{B}_n$ by inductive assumption, and (c) for every $(s', s'_1) \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s'_1) \in \mathcal{B}_n$ by inductive assumption
$\Rightarrow (s, s') \in \mathcal{B}_{n+1}$.

(ii) and (iii) are evident from (i) and the definition of $\mathcal{B}_i$. □

Lemma 1 The relation $\leftrightarrow_V$ is an equivalence relation.

Proof. It is clear from Lemma 5 (ii) such that there is a $k \geq 0$ where $\mathcal{B}_k = \mathcal{B}_{k+1}$ which is $\leftrightarrow_V$, and it is reflexive, symmetric and transitive by (iii). □

Proposition 1 Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, s'_is be two states, π'_is be two paths and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:

(i) $s'_1 \leftrightarrow_{V_i} s'_2$ $(i = 1, 2)$ implies $s'_1 \leftrightarrow_{V_1 \cup V_2} s'_2$;

(ii) $\pi'_1 \leftrightarrow_{V_i} \pi'_2$ $(i = 1, 2)$ implies $\pi'_1 \leftrightarrow_{V_1 \cup V_2} \pi'_2$;

(iii) for each path π_{s_1} of $\mathcal{M}_1$ there is a path π_{s_2} of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

Proof. In order to distinguish the relations $\mathcal{B}_0, \mathcal{B}_1, \dots$ for different set $V \subseteq \mathcal{A}$, by $\mathcal{B}_i^V$ we mean the relation $\mathcal{B}_1, \mathcal{B}_2, \dots$ for $V \subseteq \mathcal{A}$. Denote as $\mathcal{B}_0, \mathcal{B}_1, \dots$ when the underlying set V is clear from the context. Moreover, for the ease of notation, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$ (i.e., without subindex).

(i) Base: it is clear for $n = 0$.
Step: For $n > 0$, supposing if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1 \cup V_2}$ for all $0 \leq i \leq n$. We will show that if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1 \cup V_2}$.
(a) It is evident that $L_1(s_1) - (V_1 \cup V_2) = L_2(s_2) - (V_1 \cup V_2)$.
(b) We will show that for each $(s_1, s_1^1) \in R_1$ there is a $(s_2, s_2^1) \in R_2$ such that $(s_1^1, s_2^1) \in \mathcal{B}_n^{V_1 \cup V_2}$. There is $(\mathcal{K}_1^1, \mathcal{K}_2^1) \in \mathcal{B}_{n-1}^{V_1 \cup V_2}$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_1 \cup V_2}$ by inductive assumption. Then we only need to prove for each $(s_1^1, s_1^2) \in R_1$ there is a $(s_2^1, s_2^2) \in R_2$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$ and for each $(s_2^1, s_2^2) \in R_2$ there is a $(s_1^1, s_1^2) \in R_1$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$. Therefore, we only need to prove that for each $(s_1^n, s_1^{n+1}) \in R_1$ there is a $(s_2^n, s_2^{n+1}) \in R_2$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$ and for each $(s_2^n, s_2^{n+1}) \in R_2$ there is a $(s_1^n, s_1^{n+1}) \in R_1$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$. It is apparent that $L_1(s_1^{n+1}) - (V_1 \cup V_2) = L_1(s_2^{n+1}) - (V_1 \cup V_2)$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$. Where $\mathcal{K}_i^j = (\mathcal{M}_i, s_i^j)$ with $i \in \{1, 2\}$ and $0 < j \leq n + 1$.
(c) It is similar with (b).

(ii) It is clear from (i).

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s'_1) \in R_1$, there is $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$, and

(c) for every $(s_2, s'_2) \in R_2$, there is $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$,

where $\mathcal{K}'_i = (\mathcal{M}_i, s'_i)$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

($\Rightarrow$) (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s'_1) \in R_1$, there is a $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s'_1) - V = L_2(s'_2) - V$. Therefore, $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$. (c) This is similar with (b).

($\Leftarrow$) Apparently, $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) implies that for every $(s_1, s'_1) \in R_1$, there is $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_i$ for all $i \geq 0$; (c) implies that for every $(s_2, s'_2) \in R_2$, there is $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing (s_1, s_3) from the (a), (b) and (c) of the previous step (iii) of X-bisimulation (where X is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there exists $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and for all $q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and for all $q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have for all $r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then there exists $u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then there exists $u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then there exists $u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then there exists $u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(v) Let $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$ and $(s_{i,k}, s_{i,k+1}) \in R_i$ mean that $s_{i,k+1}$ is the $(k + 2)$-th node in the path

$(s_i, s_{i,1}, s_{i,2}, \ldots, s_{i,k+1}, \ldots)$ $(i = 1, 2)$. We will show that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_2}$ for all $n \geq 0$ inductively.

Base: $L_1(s_1) - V_1 = L_2(s_2) - V_1$
$\Rightarrow$ for all $q \in \mathcal{A} - V_1$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$
$\Rightarrow$ for all $q \in \mathcal{A} - V_2$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$ due to $V_1 \subseteq V_2$
$\Rightarrow L_1(s_1) - V_2 = L_2(s_2) - V_2$, i.e., $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0^{V_2}$.

Step: Supposing that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ for all $0 \leq i \leq k$ $(k > 0)$, we will show $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{k+1}^{V_2}$.

(a) It is apparent that $L_1(s_1) - V_2 = L_2(s_2) - V_2$ by base.

(b) For all $(s_1, s_{1,1}) \in R_1$, we will show that there is a $(s_2, s_{2,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_{k-1}^{V_2}$ by inductive assumption, we need only to prove the following points:
(a) For all $(s_{1,k}, s_{1,k+1}) \in R_1$ there is a $(s_{2,k}, s_{2,k+1}) \in R_2$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. It is easy to see that $L_1(s_{1,k+1}) - V_1 = L_1(s_{2,k+1}) - V_1$, then there is $L_1(s_{1,k+1}) - V_2 = L_1(s_{2,k+1}) - V_2$. Therefore, $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$.
(b) For all $(s_{2,k}, s_{2,k+1}) \in R_1$ there is a $(s_{1,k}, s_{1,k+1}) \in R_1$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. This can be proved as (a).

(c) For all $(s_2, s_{2,1}) \in R_1$, we will show that there is a $(s_1, s_{1,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. This can be proved as (ii).

$\square$

Theorem1 Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and ϕ a formula with IR(ϕ, V). Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.

Proof. This theorem can be proved by inducting on the formula ϕ and supposing *Var*$(\phi) \cap V = \emptyset$. Let $\mathcal{K}_1 = (\mathcal{M}, s)$ and $\mathcal{K}_2 = (\mathcal{M}', s')$.

Case $\phi = p$ where $p \in \mathcal{A} - V$:
$(\mathcal{M}, s) \models \phi$ iff $p \in L(s)$ (by the definition of satisfiability)
$\Leftrightarrow p \in L'(s')$ $\quad (s \leftrightarrow_V s')$
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \neg\psi$:
$(\mathcal{M}, s) \models \phi$ iff $(\mathcal{M}, s) \not\models \psi$
$\Leftrightarrow (\mathcal{M}', s') \not\models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \psi_1 \vee \psi_2$:
$(\mathcal{M}, s) \models \phi$
$\Leftrightarrow (\mathcal{M}, s) \models \psi_1$ or $(\mathcal{M}, s) \models \psi_2$
$\Leftrightarrow (\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{EX}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s, s_1, ...)$ such that $\mathcal{M}, s_1 \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s', s_1', ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_1 \leftrightarrow_V s_1'$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s_1') \models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{EG}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that for each $i \geq 0$ there is $(\mathcal{M}, s_i) \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s' = s_0', s_1', ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_i \leftrightarrow_V s_i'$ for each $i \geq 0$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s_i') \models \psi$ for each $i \geq 0$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

Case $\phi = \mathrm{E}[\psi_1 \mathrm{U} \psi_2]$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that there is $i \geq 0$ such that $(\mathcal{M}, s_i) \models \psi_2$, and for all $0 \leq j < i$, $(\mathcal{M}, s_j) \models \psi_1$
$\Leftrightarrow$ There is a path $\pi' = (s = s_0', s_1', ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow (\mathcal{M}', s_i') \models \psi_2$, and for all $0 \leq j < i$ $(\mathcal{M}', s_j') \models \psi_1$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$ $\square$

Proposition 2 Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two K-structures. Then

$(s_1, s_2) \in \mathcal{B}_n$ iff $\mathrm{Tr}_j(s_1) \leftrightarrow_V \mathrm{Tr}_j(s_2)$ for every $0 \leq j \leq n$.

Proof. We will prove this from two aspects:

($\Rightarrow$) If $(s_1, s_2) \in \mathcal{B}_n$, then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$. $(s, s') \in \mathcal{B}_n$ implies both roots of $Tr_n(s_1)$ and $Tr_n(s_2)$ have the same atoms except those atoms in V. Besides, for any $s_{1,1}$ with $(s_1, s_{1,1}) \in R_1$, there is a $s_{2,1}$ with $(s_2, s_{2,1}) \in R_2$ s.t. $(s_{1,1}, s_{2,1}) \in \mathcal{B}_{n-1}$ and vice versa. Then we have $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$. Therefore, $Tr_n(s_1) \leftrightarrow_V Tr_n(s_2)$ by use such method recursively, and then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$.

($\Leftarrow$) If $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$, then $(s_1, s_2) \in \mathcal{B}_n$. $Tr_0(s_1) \leftrightarrow_V Tr_0(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and then $(s, s') \in \mathcal{B}_0$. $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and for every successors s of the root of one, it is possible to find a successor of the root of the other s' such that $(s, s') \in \mathcal{B}_0$. Therefore $(s_1, s_2) \in \mathcal{B}_1$, and then we will have $(s_1, s_2) \in \mathcal{B}_n$ by use such method recursively. $\square$

Proposition 3 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least k such that $\mathrm{Tr}_k(s)$ and $\mathrm{Tr}_k(s')$ are not V-bisimilar.

Proof. If $s \not\leftrightarrow_V s'$, then there exists a least constant c such that $(s_i, s_j) \notin \mathcal{B}_c$, and then there is a least constant m $(m \leq c)$ such that $\mathrm{Tr}_m(s_i)$ and $\mathrm{Tr}_m(s_j)$ are not V-bisimilar by Proposition 2. Let $k = m$, the lemma is proved. $\square$

Lemma2 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ and $\mathcal{M}'$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $\mathrm{Tr}_n(s) \leftrightarrow_{\overline{V}} \mathrm{Tr}_n(s')$, then $\mathcal{F}_V(\mathrm{Tr}_n(s)) \equiv \mathcal{F}_V(\mathrm{Tr}_n(s'))$.

Proof. This result can be proved by inducting on n.

Base. It is apparent that for any $s_n \in S$ and $s_n' \in S'$, if $\mathrm{Tr}_0(s_n) \leftrightarrow_{\overline{V}} \mathrm{Tr}_0(s_n')$ then $\mathcal{F}_V(\mathrm{Tr}_0(s_n)) \equiv \mathcal{F}_V(\mathrm{Tr}_0(s_n'))$ due to $L(s_n) - \overline{V} = L'(s_n') - \overline{V}$ by known.

Step. Supposing that for $k = m$ $(0 < m \leq n)$ there is if $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ then $\mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k)) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))$, then we will show if $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$ then $\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1}))$. Apparent that:
$\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) = \left(\bigwedge_{(s_{k-1},s_k)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_{k-1},s_k)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-k}(s_k))\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s_{k-1}))$
$\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1})) = \left(\bigwedge_{(s'_{k-1},s'_k)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))\right) \wedge \mathrm{AX}\left(\bigvee_{(s'_{k-1},s'_k)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-k}(s'_k))\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s'_{k-1}))$
by the definition of characterizing formula of the computation tree. Then we have for any $(s_{k-1}, s_k) \in R$ there is $(s'_{k-1}, s'_k) \in R'$ such that $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ by $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$. Besides, for any $(s'_{k-1}, s'_k) \in R'$ there is $(s_{k-1}, s_k) \in R$ such that $\mathrm{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k}(s'_k)$ by $\mathrm{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-k+1}(s'_{k-1})$. Therefore, we have $\mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\mathrm{Tr}_{n-k+1}(s'_{k-1}))$ by induction hypothesis. ☐

Theorem 2 Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures. Then,

(i) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s'_0)$;

(ii) $s_0 \leftrightarrow_{\overline{V}} s'_0$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s'_0)$.

In order to prove Theorem 2, we prove the following two lemmas at first.

Lemma 6. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s))$.

(ii) If $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s'))$ then $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$.

Proof. (i) It is apparent from the definition of $\mathcal{F}_V(\mathrm{Tr}_n(s))$.
Base. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s))$.
Step. For $k \geq 0$, supposing the result talked in (i) is correct in $k-1$, we will show that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_{k+1}(s))$, i.e.,:

$$(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \mathrm{EX}T(s')\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s')\in R} T(s')\right) \wedge \mathcal{F}_V(\mathrm{Tr}_0(s)).$$

Where $T(s') = \mathcal{F}_V(\mathrm{Tr}_k(s'))$. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s))$ by Base. It is apparent that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \mathcal{F}_V(\mathrm{Tr}_k(s'))$ by inductive assumption. Then we have $(\mathcal{M}, s) \models \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_k(s')$, and then $(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_k(s'))\right)$. Similarly, we have that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \bigvee_{(s,s'')\in R} \mathcal{F}_V(\mathrm{Tr}_k(s''))$. Therefore, $(\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s,s'')\in R} \mathcal{F}_V(\mathrm{Tr}_k(s''))\right)$.

(ii) **Base**. If $n = 0$, then $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_0(s'))$ implies $L(s) - \overline{V} = L'(s') - \overline{V}$. Hence, $\mathrm{Tr}_0(s) \leftrightarrow_{\overline{V}} \mathrm{Tr}_0(s')$.
Step. Supposing $n > 0$ and the result talked in (ii) is correct in $n-1$.
(a) It is easy to see that $L(s) - \overline{V} = L'(s') - \overline{V}$.
(b) We will show that for each $(s, s_1) \in R$, there is a $(s', s'_1) \in R'$ such that $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s',s'_1)\in R} \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))\right)$. Therefore, for each $(s, s_1) \in R$ there is a $(s', s'_1) \in R'$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))$. Hence, $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$ by inductive hypothesis.
(c) We will show that for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $\mathrm{Tr}_{n-1}(s'_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\mathrm{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \bigwedge_{(s',s'_1)\in R'} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1))$. Therefore, for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_{n-1}(s'_1)$. Hence, $\mathrm{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{n-1}(s'_1)$ by inductive hypothesis. ☐

A consequence of the previous lemma is:

Lemma 7. *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ a model structure, $k = ch(\mathcal{M}, V)$ and $s \in S$.*

(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_k(s))$, and

(ii) for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(Tr_k(s))$.

Proof. (i) It is apparent from the (i) of Lemma 6.
(ii) Let $\phi = \mathcal{F}_V(\mathrm{Tr}_k(s))$, where k is the V-characteristic number of $\mathcal{M}$. $(\mathcal{M}, s) \models \phi$ by the definition of $\mathcal{F}$, and then for all $s' \in S$, if $s \leftrightarrow_{\overline{V}} s'$ there is $(\mathcal{M}, s') \models \phi$ by Theorem 1 due to $\mathrm{IR}(\phi, \mathcal{A} - V)$. Supposing $(\mathcal{M}, s') \models \phi$, if $s \not\leftrightarrow_{\overline{V}} s'$, then $\mathrm{Tr}_k(s) \not\leftrightarrow_{\overline{V}} \mathrm{Tr}_k(s')$, and then $(\mathcal{M}, s') \not\models \phi$ by Lemma 6, a contradiction. ☐

Now we are in the position of proving Theorem 2.

Proof. (i) Let $\mathcal{F}_V(\mathcal{M}, s_0)$ be the characterizing formula of $(\mathcal{M}, s_0)$ on V. It is apparent that $\mathrm{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. We will show that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ at first.

It is apparent that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s_0))$ by Lemma 6. We must show that $(\mathcal{M}, s_0) \models \bigwedge_{s\in S} G(\mathcal{M}, s)$. Let $\mathcal{X} = \mathcal{F}_V(\mathrm{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$, we will show for all $s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$. Where $G(\mathcal{M}, s) = \mathrm{AG}\mathcal{X}$. There are two cases we should consider:

- If $(\mathcal{M}, s_0) \not\models \mathcal{F}_V(\mathrm{Tr}_c(s))$, it is apparent that $(\mathcal{M}, s_0) \models \mathcal{X}$;
- If $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s))$:
  $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathrm{Tr}_c(s))$
  $\Rightarrow s_0 \leftrightarrow_{\overline{V}} s$ by the definition of characteristic number and Lemma 7.
  For each $(s, s_1) \in R$ there is:
  $(\mathcal{M}, s_1) \models \mathcal{F}_V(\mathrm{Tr}_c(s_1))$ $\qquad (s_1 \leftrightarrow_{\overline{V}} s_1)$
  $\Rightarrow (\mathcal{M}, s) \models \bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$
  $\Rightarrow (\mathcal{M}, s_0) \models \bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (by $\mathrm{IR}(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1)), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$).
  For each (s, s_1) there is:
  $\mathcal{M}, s_1 \models \bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))$

$\Rightarrow (\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$

$\Rightarrow (\mathcal{M}, s_0) \models \mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ (by $\mathrm{IR}(\mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$)

$\Rightarrow (\mathcal{M}, s_0) \models \mathcal{X}$.

For any other states s' which can reach from s_0 can be proved similarly, i.e.,, $(\mathcal{M}, s') \models \mathcal{X}$. Therefore, for all $s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$, and then $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$.

We will prove this theorem from the following two aspects:

($\Leftarrow$) If $s_0 \leftrightarrow_{\overline{V}} s_0'$, then $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$. Since $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and $\mathrm{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$, hence $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

($\Rightarrow$) If $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$, then $s_0 \leftrightarrow_{\overline{V}} s_0'$. We will prove this by showing that for all $n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s_0')$.

Base. It is apparent that $Tr_0(s_0) \equiv Tr_0(s_0')$.

Step. Supposing $\mathrm{Tr}_k(s_0) \leftrightarrow_{\overline{V}} \mathrm{Tr}_k(s_0')$ ($k > 0$), we will prove $\mathrm{Tr}_{k+1}(s_0) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{k+1}(s_0')$. We should only show that $\mathrm{Tr}_1(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_1(s_k')$. Where $(s_0, s_1), (s_1, s_2), \ldots, (s_{k-1}, s_k) \in R$ and $(s_0', s_1'), (s_1', s_2'), \ldots, (s_{k-1}', s_k') \in R'$, i.e., s_{i+1} (s_{i+1}') is an immediate successor of s_i (s_i') for all $0 \leq i \leq k-1$.

(a) It is apparent that $L(s_k) - \overline{V} = L'(s_k') - \overline{V}$ by inductive assumption.

Before talking about the other points, note the following fact that:

$(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$

$\Rightarrow$ For all $s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\mathrm{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ for any $s \in S$. **(fact)**

(I) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathrm{Tr}_c(s_0)) \rightarrow \left(\bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ **(fact)**

(II) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathrm{Tr}_c(s_0)))$ (known)

(III) $(\mathcal{M}', s_0') \models \left(\bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ ((I),(II))

(b) We will show that for each $(s_k, s_{k+1}) \in R$ there is a $(s_k', s_{k+1}') \in R'$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.

(1) $(\mathcal{M}', s_0') \models \bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (III)

(2) For all $(s_0, s_1) \in R$, there exists $(s_0', s_1') \in R'$ s.t. $(\mathcal{M}', s_1') \models \mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (2)

(3) $\mathrm{Tr}_c(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s_1')$ ((2), Lemma 6)

(4) $L(s_1) - \overline{V} = L'(s_1') - \overline{V}$ ((3), $c \geq 0$)

(5) $(\mathcal{M}', s_1') \models \mathcal{F}_V(\mathrm{Tr}_c(s_1)) \rightarrow \left(\bigwedge_{(s_1,s_2)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_2))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ **(fact)**

(6) $(\mathcal{M}', s_1') \models \left(\bigwedge_{(s_1,s_2)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_2))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ ((2), (5))

(7) ......

(8) $(\mathcal{M}', s_k') \models \left(\bigwedge_{(s_k,s_{k+1})\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right)$ (similar with (6))

(9) For all $(s_k, s_{k+1}) \in R$, there exists $(s_k', s_{k+1}') \in R'$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))$ (8)

(10) $\mathrm{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s_{k+1}')$ ((9), Lemma 6)

(11) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((10), $c \geq 0$)

(c) We will show that for each $(s_k', s_{k+1}') \in R'$ there is a $(s_k, s_{k+1}) \in R$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.

(1) $(\mathcal{M}', s_k') \models \mathrm{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right)$ (by (8) talked above)

(2) For all $(s_k', s_{k+1}') \in R'$, there exists $(s_k, s_{k+1}) \in R$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}'))$ (1)

(3) $\mathrm{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s_{k+1}')$ ((2), Lemma 6)

(4) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((3), $c \geq 0$)

(ii) This is following Lemma 2 and the definition of the characterizing formula of initial K-structure $\mathcal{K}$ on V.

$\square$

Lemma 3 Let φ be a formula. We have

$$\varphi \equiv \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \quad (3)$$

Proof. Let $(\mathcal{M}', s_0')$ be a model of φ. Then $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$ due to $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}', s_0')$. On the other hand, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. Then there is a $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. And then $(\mathcal{M}, s_0) \leftrightarrow_{\emptyset} (\mathcal{M}', s_0')$ by Theorem 2. Therefore, $(\mathcal{M}, s_0)$ is also a model of φ by Theorem 1. $\square$

Theorem 3 **(Representation theorem)** Let φ, φ' and ϕ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

(i) $\varphi' \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$,

(ii) $\varphi' \equiv \{\phi | \varphi \models \phi \text{ and } \mathrm{IR}(\phi, V)\}$,

(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.

Proof. $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\mathrm{F}_{\mathrm{CTL}}(\varphi, V)) = Mod(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\})$$
$$= Mod(\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)).$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. Then there exists an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of φ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all ϕ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s'_0)$ is a models of $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s'_0) \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$ due to $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$ is irrelevant to V.

Finally, suppose that $(\mathcal{M}', s'_0)$ is a model of $\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$. Then there exists $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s'_0) \models \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)$. Hence, $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$ by Theorem 2. Thus $(\mathcal{M}', s'_0)$ is also a model of $\mathrm{F}_{\text{CTL}}(\varphi, V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. Suppose that all postulates hold. By Positive Persistence, we have $\varphi' \models \{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Now we show that $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\} \models \varphi'$. Otherwise, there exists formula ϕ' such that $\varphi' \models \phi'$ but $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\} \not\models \phi'$. There are three cases:

- ϕ' is relevant to V. Thus, φ' is also relevant to V, a contradiction to Irrelevance.
- ϕ' is irrelevant to V and $\varphi \models \phi'$. This contradicts to our assumption.
- ϕ' is irrelevant to V and $\varphi \not\models \phi'$. By Negative Persistence, $\varphi' \not\models \phi'$, a contradiction.

Thus, φ' is equivalent to $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. □

Lemma 4 Let φ and α be two CTL formulae and $q \in Var(\varphi) \cup Var(\alpha)$. Then $\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

Proof. Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $(\mathcal{M}, s)$ of $\mathrm{F}_{\text{CTL}}(\varphi', q)$ there is an initial K-structure $(\mathcal{M}', s')$ s.t. $(\mathcal{M}, s) \leftrightarrow_{\{q\}} (\mathcal{M}', s')$ and $(\mathcal{M}', s') \models \varphi'$. It's apparent that $(\mathcal{M}', s') \models \varphi$, and then $(\mathcal{M}, s) \models \varphi$ since $\mathrm{IR}(\varphi, \{q\})$ and $(\mathcal{M}, s) \leftrightarrow_{\{q\}} (\mathcal{M}', s')$ by Theorem 1.

Let $(\mathcal{M}, s) \in Mod(\varphi)$ with $\mathcal{M} = (S, R, L, s)$. We construct $(\mathcal{M}', s)$ with $\mathcal{M}' = (S, R, L', s)$ as follows:

$$L' : S \to \mathcal{A} \; and \; \forall s^* \in S, L'(s^*) = L(s^*) \; if \; (\mathcal{M}, s^*) \not\models \alpha,$$
$$else \; L'(s^*) = L(s^*) \cup \{q\},$$
$$L'(s) = L(s) \cup \{q\} \; if \; (\mathcal{M}, s) \models \alpha, \; and \; L'(s) = L(s)$$
$$otherwise.$$

It is clear that $(\mathcal{M}', s) \models \varphi$, $(\mathcal{M}', s) \models q \leftrightarrow \alpha$ and $(\mathcal{M}', s) \leftrightarrow_{\{q\}} (\mathcal{M}, s)$. Therefore $(\mathcal{M}', s) \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $(\mathcal{M}, s) \models \mathrm{F}_{\text{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $(\mathcal{M}', s) \leftrightarrow_{\{q\}} (\mathcal{M}, s)$. □

Proposition 4 Given a formula $\varphi \in$ CTL, V a set of atoms and p an atom such that $p \notin V$. Then,

$$\mathrm{F}_{\text{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\text{CTL}}(\mathrm{F}_{\text{CTL}}(\varphi, p), V).$$

Proof. Let $(\mathcal{M}_1, s_1)$ with $\mathcal{M}_1 = (S_1, R_1, L_1, s_1)$ be a model of $\mathrm{F}_{\text{CTL}}(\varphi, \{p\} \cup V)$. By the definition, there exists a model $(\mathcal{M}, s)$ with $\mathcal{M} = (S, R, L, s)$ of φ, such that $(\mathcal{M}_1, s_1) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}, s)$. We construct an initial K-structure $(\mathcal{M}_2, s_2)$ with $\mathcal{M}_2 = (S_2, R_2, L_2, s_2)$ as follows:

(1) for s_2: let s_2 be the state such that:
- $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
- for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
- for all other atoms q', $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:
  (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\}\cup V} w_1$, let $w_2 \in S_2$ and
  - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
  - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
  - for all other atoms q', $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

  (ii) if $(w'_1, w_1) \in R_1$, w_2 is constructed based on w_1 and $w'_2 \in S_2$ is constructed based on w'_1, then $(w'_2, w_2) \in R_2$.

(3) delete duplicated states in S_2 and pairs in R_2.

Then we have $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$. Thus, $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\varphi, p)$. And therefore $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\text{CTL}}(\mathrm{F}_{\text{CTL}}(\varphi, p), V)$.

On the other hand, suppose that $(\mathcal{M}_1, s_1)$ is a model of $\mathrm{F}_{\text{CTL}}(\mathrm{F}_{\text{CTL}}(\varphi, p), V)$, then there exists an initial K-structure $(\mathcal{M}_2, s_2)$ such that $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\varphi, p)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$, and there exists $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \varphi$ and $(\mathcal{M}, s) \leftrightarrow_{\{p\}} (\mathcal{M}_2, s_2)$. Therefore, $(\mathcal{M}, s) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}_1, s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1, s_1) \models \mathrm{F}_{\text{CTL}}(\varphi, \{p\} \cup V)$. □

Proposition 5 Let φ, φ_i, ψ_i $(i = 1, 2)$ be formulas and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_{\text{CTL}}(\varphi, V)$ is satisfiable iff φ is;

(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\text{CTL}}(\varphi_1, V) \equiv \mathrm{F}_{\text{CTL}}(\varphi_2, V)$;

(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\text{CTL}}(\varphi_1, V) \models \mathrm{F}_{\text{CTL}}(\varphi_2, V)$;

(iv) $\mathrm{F}_{\text{CTL}}(\psi_1 \vee \psi_2, V) \equiv \mathrm{F}_{\text{CTL}}(\psi_1, V) \vee \mathrm{F}_{\text{CTL}}(\psi_2, V)$;

(v) $\mathrm{F}_{\text{CTL}}(\psi_1 \wedge \psi_2, V) \models \mathrm{F}_{\text{CTL}}(\psi_1, V) \wedge \mathrm{F}_{\text{CTL}}(\psi_2, V)$;

Proof. (i) $(\Rightarrow)$ Supposing $(\mathcal{M}, s)$ is a model of $\mathrm{F}_{\text{CTL}}(\varphi, V)$, then there is a model $(\mathcal{M}', s')$ of φ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$ by the definition of F_{CTL}.

$(\Leftarrow)$ Supposing $(\mathcal{M}, s)$ is a model of φ, then there is an initial Kripke structure $(\mathcal{M}', s')$ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$, and then $(\mathcal{M}', s') \models \mathrm{F}_{\text{CTL}}(\varphi, V)$ by the definition of F_{CTL}.

The (ii) and (iii) can be proved similarly.

(iv) $(\Rightarrow)$ For all$(\mathcal{M}, s) \in Mod(\mathrm{F}_{\text{CTL}}(\psi_1 \vee \psi_2, V))$, there exists $(\mathcal{M}', s') \in Mod(\psi_1 \vee \psi_2)$ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$ and $(\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$
$\Rightarrow$ there exists $(\mathcal{M}_1, s_1) \in Mod(\mathrm{F}_{\text{CTL}}(\psi_1, V))$ s.t. $(\mathcal{M}', s') \leftrightarrow_V (\mathcal{M}_1, s_1)$ or there exists $(\mathcal{M}_2, s_2) \in Mod(\mathrm{F}_{\text{CTL}}(\psi_2, V))$ s.t. $(\mathcal{M}', s') \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{F}_{\text{CTL}}(\psi_1, V) \vee \mathrm{F}_{\text{CTL}}(\psi_2, V)$ by Theorem 1.

$(\Leftarrow)$ for all $(\mathcal{M}, s) \in Mod(\mathrm{F}_{\text{CTL}}(\psi_1, V) \vee \mathrm{F}_{\text{CTL}}(\psi_2, V))$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{F}_{\text{CTL}}(\psi_1, V)$ or $(\mathcal{M}, s) \models \mathrm{F}_{\text{CTL}}(\psi_2, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_1, s_1)$ and $(\mathcal{M}_1, s_1) \models \psi_1$ or $(\mathcal{M}_1, s_1) \models \psi_2$
$\Rightarrow (\mathcal{M}_1, s_1) \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$ and $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\text{CTL}}(\psi_1 \vee \psi_2, V)$
$\Rightarrow (\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ and $(\mathcal{M}, s) \models \mathrm{F}_{\text{CTL}}(\psi_1 \vee \psi_2, V)$.

The (v) can be proved as (iv). □

Proposition 6 **(Homogeneity)** Let $V \subseteq \mathcal{A}$ and $\phi \in$ CTL,

(i) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AXF}_{\mathrm{CTL}}(\phi, V)$.

(ii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EXF}_{\mathrm{CTL}}(\phi, V)$.

(iii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AF}\phi, V) \equiv \mathrm{AFF}_{\mathrm{CTL}}(\phi, V)$.

(iv) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EF}\phi, V) \equiv \mathrm{EFF}_{\mathrm{CTL}}(\phi, V)$.

Proof. Let $\mathcal{M} = (S, R, L, s_0)$ with initial state s_0 and $\mathcal{M}' = (S', R', L', s'_0)$ with initial state s'_0, then we call $\mathcal{M}', s'_0$ be a sub-structure of $\mathcal{M}, s_0$ if:

- $S' \subseteq S$ and $S' = \{s' | s'$ is reachable from $s'_0\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to 2^{\mathcal{A}}$ and for all $s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- s'_0 is s_0 or a state reachable from s_0.

(i) In order to prove $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$, we only need to prove *Mod*$(\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)) =$ *Mod*$(\mathrm{AXF}_{\mathrm{CTL}}(\phi, V))$:

($\Rightarrow$) For all $(\mathcal{M}', s') \in$ *Mod*$(\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \mathrm{AX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ for any sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ there is $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ with s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow$ $(\mathcal{M}_3, s_3) \models \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$ and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow (\mathcal{M}', s') \models \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$.

($\Leftarrow$) For all $(\mathcal{M}_3, s_3) \in$ *Mod*$(\mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V)))$, then for any sub-structure $(\mathcal{M}_2, s_2)$ with s_2 is a directed successor of s_3 there is $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$
$\Rightarrow$ for any $(\mathcal{M}_2, s_2)$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ with s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{AX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EXF}_{\mathrm{CTL}}(\phi, V)$, we only need to prove *Mod* $(\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi,\ V)) =$ *Mod*$(\mathrm{EXF}_{\mathrm{CTL}}(\phi, V))$:

($\Rightarrow$) For all $(\mathcal{M}', s') \in$ *Mod*$(\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \mathrm{EX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ there is a sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$, where s_1 is a directed successor of s
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ that s_2 is a direct successor of s_3 and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow (\mathcal{M}_3, s_3) \models \mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$
$\Rightarrow (\mathcal{M}', s') \models \mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$.

($\Leftarrow$) For all $(\mathcal{M}_3, s_3) \in$ *Mod*$(\mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V)))$, there exists a sub-structure $(\mathcal{M}_2, s_2)$ of $(\mathcal{M}_3, s_3)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ that s_1 is a direct successor of s and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{EX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V)$.

(iii) and (iV) can be proved as (i) and (ii) respectively. $\square$

Proposition7 **(Model Checking on Forgetting)** Let $(\mathcal{M}, s_0)$ be an initial K-structure, φ be a CTL formula and V a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is NP-complete.

Proof. The problem can be determined by the following two things: (1) guessing an initial K-structure $(\mathcal{M}', s'_0)$ satisfying φ; and (2) checking if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$. Both two steps can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard (Zhang and Zhou 2008). $\square$

Theorem 5 **(Entailment on Forgetting)** Let φ and ψ be two $\mathrm{CTL}_{\mathrm{AF}}$ formulas and V a set of atoms. Then, results:

(i) deciding $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models^? \psi$ is co-NP-complete,

(ii) deciding $\psi \models^? \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is Π_2^P-complete,

(iii) deciding $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models^? \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ is Π_2^P-complete.

Proof. (i) It is known that deciding whether ψ is satisfiable is NP-Complete (Meier et al. 2015). The hardness is easy to see by setting $\mathrm{F}_{\mathrm{CTL}}(\varphi, \mathit{Var}(\varphi)) \equiv \top$, i.e., deciding whether ψ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $\mathrm{IR}(\psi, V)$. Clearly, in $\mathrm{CTL}_{\mathrm{AF}}$, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $\mathrm{IR}(\psi, V)$ is also in co-NP. Without loss of generality, we assume that ψ is satisfiable. We consider the complement of the problem: deciding whether ψ is not irrelevant to V. It is easy to see that ψ is not irrelevant to V iff there exist a model $(\mathcal{M}, s_0)$ of ψ and an initial K-structure $(\mathcal{M}', s'_0)$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$ and $(\mathcal{M}', s'_0) \not\models \psi$. So checking whether ψ is not irrelevant to V can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s'_0)$ such that $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s'_0) \not\models \psi$, and (2) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s'_0)$. Obviously, both (1) and (2) can be done in polynomial time.

(ii) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ satisfying ψ and check whether $(\mathcal{M}, s_0) \not\models \mathrm{F}_{\mathrm{CTL}}(\ \varphi, V)$. From Proposition 7, we know that this is in Σ_2^P. So the original problem is in Π_2^P. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case of temporal forgetting, the hardness is directly followed from the proof of Proposition 24 in (Lang, Liberatore, and Marquis 2003).

(iii) Membership. If $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$, i.e.,, there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models$

ψ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M}, s)$, $(\mathcal{M}_1, s_1)$ with $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ and checking $(\mathcal{M}_1, s_1) \models \varphi$ are feasible while checking $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$ can be done in polynomial time. Thus the problem is in Π_2^P.

Hardness. It follows from (ii) due to the fact that $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ iff $\varphi \models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ by $\mathrm{IR}(\mathrm{F}_{\mathrm{CTL}}(\psi, V), V)$. □

Proposition 8 **(dual)** Let V, q, φ and ψ are like in Definition 4. The ψ is a SNC (WSC) of q on V under φ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on V under φ.

Proof. (i) Suppose ψ is the SNC of q. Then $\varphi \models q \to \psi$. Thus $\varphi \models \neg\psi \to \neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose ψ' is any other SC of $\neg q$: $\varphi \models \psi' \to \neg q$. Then $\varphi \models q \to \neg\psi'$, this means $\neg\psi'$ is a NC of q on P under φ. Thus $\varphi \models \psi \to \neg\psi'$ by assumption. So $\varphi \models \psi' \to \neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case. □

Proposition 9 Let Γ and α be two formulas, $V \subseteq \mathit{Var}(\alpha) \cup \mathit{Var}(\Gamma)$ and q be a new proposition not in Γ and α. Then, a formula φ of V is the SNC (WSC) of α on V under Γ iff it is the SNC (WSC) of q on V under $\Gamma' = \Gamma \cup \{q \leftrightarrow \alpha\}$.

Proof. We prove this for SNC. The case for WSC is similar. Let $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ denote that φ is the SNC of α on V under Γ, and $\mathit{NC}(\varphi, \alpha, V, \Gamma)$ denote that φ is the NC of α on V under Γ.

($\Rightarrow$) We will show that if $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds, then $\mathit{SNC}(\varphi, q, V, \Gamma')$ will be true. According to $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ and $\alpha \equiv q$, we have $\Gamma' \models q \to \varphi$, which means φ is a NC of q on V under Γ'. Suppose φ' is any NC of q on V under Γ', then $\mathrm{F}_{\mathrm{CTL}}(\Gamma', q) \models \alpha \to \varphi'$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \to \varphi', \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \to \varphi'$ by Lemma 4, this means $\mathit{NC}(\varphi', \alpha, V, \Gamma)$. Therefore, $\Gamma \models \varphi \to \varphi'$ by the definition of SNC and $\Gamma' \models \varphi \to \varphi'$. Hence, $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds.

($\Leftarrow$) We will show that if $\mathit{SNC}(\varphi, q, V, \Gamma')$ holds, then $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ will be true. According to $\mathit{SNC}(\varphi, q, V, \Gamma')$, it's not difficult to know that $\mathrm{F}_{\mathrm{CTL}}(\Gamma', \{q\}) \models \alpha \to \varphi$ due to $\alpha \equiv q$, $\mathit{IR}(\alpha \to \varphi, \{q\})$ and (**PP**), i.e., $\Gamma \models \alpha \to \varphi$ by Lemma 4, this means $\mathit{NC}(\varphi, \alpha, V, \Gamma)$. Suppose φ' is any NC of α on V under Γ. Then $\Gamma' \models q \to \varphi'$ since $\alpha \equiv q$ and $\Gamma' = \Gamma \cup \{q \equiv \alpha\}$, which means $\mathit{NC}(\varphi', q, V, \Gamma')$. According to $\mathit{SNC}(\varphi, q, V, \Gamma')$, $\mathit{IR}(\varphi \to \varphi', \{q\})$ and (**PP**), we have $\mathrm{F}_{\mathrm{CTL}}(\Gamma', \{q\}) \models \varphi \to \varphi'$, and $\Gamma \models \varphi \to \varphi'$ by Lemma 4. Hence, $\mathit{SNC}(\varphi, \alpha, V, \Gamma)$ holds. □

Theorem 7 Let φ be a formula, $V \subseteq \mathit{Var}(\varphi)$ and $q \in \mathit{Var}(\varphi) - V$.

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ is a SNC of q on V under φ.

(ii) $\neg\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge \neg q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$ is a WSC of q on V under φ.

Proof. We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F} = \mathrm{F}_{\mathrm{CTL}}(\varphi \wedge q, (\mathit{Var}(\varphi) \cup \{q\}) - V)$.

The "NC" part: It's easy to see that $\varphi \wedge q \models \mathcal{F}$ by **(W)**. Hence, $\varphi \models q \to \mathcal{F}$, this means $\mathcal{F}$ is a NC of q on P under φ.

The "SNC" part: for all ψ', ψ' is the NC of q on V under φ, s.t. $\varphi \models \mathcal{F} \to \psi'$. Suppose that there is a NC ψ of q on V under φ and ψ is not logic equivalence with $\mathcal{F}$ under φ, s.t. $\varphi \models \psi \to \mathcal{F}$. We know that $\varphi \wedge q \models \psi$ iff $\mathcal{F} \models \psi$ by **(PP)**, since $\mathit{IR}(\psi, (\mathit{Var}(\varphi) \cup \{q\}) - V)$. Hence, $\varphi \wedge \mathcal{F} \models \psi$ by $\varphi \wedge q \models \psi$ (by suppose). We can see that $\varphi \wedge \psi \models \mathcal{F}$ by suppose. Therefore, $\varphi \models \psi \leftrightarrow \mathcal{F}$, which means ψ is logic equivalence with $\mathcal{F}$ under φ. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of q on P under φ. □

Theorem 8 Let $\mathcal{K} = (\mathcal{M}, s)$ be an initial K-structure with $\mathcal{M} = (S, R, L, s_0)$ on the set $\mathcal{A}$ of atoms, $V \subseteq \mathcal{A}$ and $q \in V' = \mathcal{A} - V$. Then:

(i) the SNC of q on V under $\mathcal{K}$ is $\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, V')$.

(ii) the WSC of q on V under $\mathcal{K}$ is $\neg\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge \neg q, V')$.

Proof. (i) As we know that any initial K-structure $\mathcal{K}$ can be described as a characterizing formula $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$, then the SNC of q on V under $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$ is $\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K}) \wedge q, \mathcal{A} - V)$.

(ii) This is proved by the dual property. □

Proposition10 Let φ be a CTL formula and $V \subseteq \mathcal{A}$ with $|\mathcal{S}| = m$, $|\mathcal{A}| = n$ and $|V| = x$. The the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity of Algorithm 1 is at least the same as the space.

Proof. Supposing each state or atom occupy one byte, then a state pair (s, s') occupy two bytes. For any $B \subseteq \mathcal{S}$ with $B \neq \emptyset$ and $s_0 \in B$, we can construct an initial K-structure $(\mathcal{M}, s_0)$ with $\mathcal{M} = (B, R, L, s_0)$, in which there is at most $\frac{|B|^2}{2}$ state pairs in R and $|B| * n$ (s, A) $(A \subseteq \mathcal{A})$ in L. Hence, the $(\mathcal{M}, s_0)$ occupy at most $|B| + |B|^2 + |B| * n$ bytes. Besides, there is at most $|B|^{|B|+1} * 2^{nm}$ number of initial K-structures. Therefore, there is at most $m^{m+2} * 2^{nm}$ number of initial K-structures, hence it will at most cost $m^{m+2} * 2^{nm} * (m + m^2 + nm)$ bytes.

Let $k = n - x$, for any initial K-structure $\mathcal{K} = (\mathcal{M}, s_0)$ with $i \geq 1$ nodes, in the worst, i.e., $ch(\mathcal{M}, V) = i$, we will spend $N(i)$ space to store the characterizing formula.

$$\begin{aligned} N(i) &= (k + (\cdots + (k + 2ik) * (2i)) \cdots * (2i)) \\ &= (2i)^0 k + 2ik + (2i)^2 k + \cdots + (2i)^{(i-1)} k \\ &= \frac{(2i)^i - 1}{2i - 1} k. \end{aligned}$$

In the worst case, i.e., there is $m^{m+2} * 2^{nm}$ initial K-structures with m nodes, we will spent $m^{m+2} * 2^{nm} * N(m)$ bytes to store the result of forgetting.

Therefore, the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity is at least the same as the space. $\square$

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