## A Supplementary Material: Proof Appendix

**Lemma 5.** *Let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be the ones in the definition of section 3.1. Then, for each $i \geq 0$,*

*(i)* $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$*;*

*(ii) there is a (smallest) $k \geq 0$ such that $\mathcal{B}_{k+1} = \mathcal{B}_k$;*

*(iii)* $\mathcal{B}_i$ *is reflexive, symmetric and transitive.*

*Proof.* (i) Base: it is clear for $i = 0$ by the above definition.

Step: suppose it holds for $i = n$, i.e., $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.
$(s, s') \in \mathcal{B}_{n+2}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s'_1) \in R'$ such that $(s_1, s'_1) \in \mathcal{B}_{n+1}$, and (c) for every $(s', s'_1) \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s'_1) \in \mathcal{B}_{n+1}$
$\Rightarrow$ (a) $(s, s') \in \mathcal{B}_0$, (b) for every $(s, s_1) \in R$, there is $(s', s'_1) \in R'$ such that $(s_1, s'_1) \in \mathcal{B}_n$ by inductive assumption, and (c) for every $(s', s'_1) \in R'$, there is $(s, s_1) \in R$ such that $(s_1, s'_1) \in \mathcal{B}_n$ by inductive assumption
$\Rightarrow (s, s') \in \mathcal{B}_{n+1}$.

(ii) and (iii) are evident from (i) and the definition of $\mathcal{B}_i$. □

**Lemma** 1 The relation $\leftrightarrow_V$ is an equivalence relation.

*Proof.* It is clear from Lemma 5 (ii) such that there is a $k \geq 0$ where $\mathcal{B}_k = \mathcal{B}_{k+1}$ which is $\leftrightarrow_V$, and it is reflexive, symmetric and transitive by (iii). □

**Proposition** 1 Let $i \in \{1, 2\}$, $V_1, V_2 \subseteq \mathcal{A}$, $s'_i$s be two states, $\pi'_i$s be two paths and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2, 3)$ be K-structures such that $\mathcal{K}_1 \leftrightarrow_{V_1} \mathcal{K}_2$ and $\mathcal{K}_2 \leftrightarrow_{V_2} \mathcal{K}_3$. Then:

(i) $s'_1 \leftrightarrow_{V_i} s'_2$ $(i = 1, 2)$ implies $s'_1 \leftrightarrow_{V_1 \cup V_2} s'_2$;

(ii) $\pi'_1 \leftrightarrow_{V_i} \pi'_2$ $(i = 1, 2)$ implies $\pi'_1 \leftrightarrow_{V_1 \cup V_2} \pi'_2$;

(iii) for each path $\pi_{s_1}$ of $\mathcal{M}_1$ there is a path $\pi_{s_2}$ of $\mathcal{M}_2$ such that $\pi_{s_1} \leftrightarrow_{V_1} \pi_{s_2}$, and vice versa;

(iv) $\mathcal{K}_1 \leftrightarrow_{V_1 \cup V_2} \mathcal{K}_3$;

(v) If $V_1 \subseteq V_2$ then $\mathcal{K}_1 \leftrightarrow_{V_2} \mathcal{K}_2$.

*Proof.* In order to distinguish the relations $\mathcal{B}_0, \mathcal{B}_1, \ldots$ for different set $V \subseteq \mathcal{A}$, by $\mathcal{B}_i^V$ we mean the relation $\mathcal{B}_1, \mathcal{B}_2, \ldots$ for $V \subseteq \mathcal{A}$. Denote as $\mathcal{B}_0, \mathcal{B}_1, \ldots$ when the underlying set $V$ is clear from the context. Moreover, for the ease of notation, we will refer to $\leftrightarrow_V$ by $\mathcal{B}$ (i.e., without subindex).

(i) Base: it is clear for $n = 0$.
Step: For $n > 0$, supposing if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_1 \cup V_2}$ for all $0 \leq i \leq n$. We will show that if $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$ then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1 \cup V_2}$.
(a) It is evident that $L_1(s_1) - (V_1 \cup V_2) = L_2(s_2) - (V_1 \cup V_2)$.
(b) We will show that for each $(s_1, s_1^1) \in R_1$ there is a $(s_2, s_2^1) \in R_2$ such that $(s_1^1, s_2^1) \in \mathcal{B}_n^{V_1 \cup V_2}$. There is $(\mathcal{K}_1^1, \mathcal{K}_2^1) \in \mathcal{B}_{n-1}^{V_1 \cup V_2}$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_1 \cup V_2}$ by inductive assumption. Then we only need to prove for each $(s_1^1, s_1^2) \in R_1$ there is a $(s_2^1, s_2^2) \in R_2$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$ and for each $(s_2^1, s_2^2) \in R_2$ there is a $(s_1^1, s_1^2) \in R_1$ such that $(\mathcal{K}_1^2, \mathcal{K}_2^2) \in \mathcal{B}_{n-2}^{V_1 \cup V_2}$. Therefore, we only need to prove that for each $(s_1^n, s_1^{n+1}) \in R_1$ there is a $(s_2^n, s_2^{n+1}) \in R_2$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$ and for each $(s_2^n, s_2^{n+1}) \in R_2$ there is a $(s_1^n, s_1^{n+1}) \in R_1$ such that $(\mathcal{K}_1^{n+1}, \mathcal{K}_2^{n+1}) \in \mathcal{B}_0^{V_1 \cup V_2}$. It is apparent that $L_1(s_1^{n+1}) - (V_1 \cup V_2) = L_1(s_2^{n+1}) - (V_1 \cup V_2)$ due to $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_1}$ and $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{n+1}^{V_2}$. Where $\mathcal{K}_i^j = (\mathcal{M}_i, s_i^j)$ with $i \in \{1, 2\}$ and $0 < j \leq n + 1$.
(c) It is similar with (b).

(ii) It is clear from (i).

(iii) The following property show our result directly. Let $V \subseteq \mathcal{A}$ and $\mathcal{K}_i = (\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be K-structures. Then $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ if and only if

(a) $L_1(s_1) - V = L_2(s_2) - V$,

(b) for every $(s_1, s'_1) \in R_1$, there is $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$, and

(c) for every $(s_2, s'_2) \in R_2$, there is $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$,

where $\mathcal{K}'_i = (\mathcal{M}_i, s'_i)$ with $i \in \{1, 2\}$.

We prove it from the following two aspects:

($\Rightarrow$) (a) It is apparent that $L_1(s_1) - V = L_2(s_2) - V$; (b) $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$ iff $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$, then for each $(s_1, s'_1) \in R_1$, there is a $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_{i-1}$ for all $i > 0$ and then $L_1(s'_1) - V = L_2(s'_2) - V$. Therefore, $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}$. (c) This is similar with (b).

($\Leftarrow$) Apparently, $L_1(s_1) - V = L_2(s_2) - V$ implies that $(s_1, s_2) \in \mathcal{B}_0$; (b) implies that for every $(s_1, s'_1) \in R_1$, there is $(s_2, s'_2) \in R_2$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_i$ for all $i \geq 0$; (c) implies that for every $(s_2, s'_2) \in R_2$, there is $(s_1, s'_1) \in R_1$ such that $(\mathcal{K}'_1, \mathcal{K}'_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i$ for all $i \geq 0$
$\Rightarrow (\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}$.

(iv) Let $\mathcal{M}_i = (S_i, R_i, L_i, s_i)$ $(i = 1, 2, 3)$, $s_1 \leftrightarrow_{V_1} s_2$ via a binary relation $\mathcal{B}$, and $s_2 \leftrightarrow_{V_2} s_3$ via a binary relation $\mathcal{B}''$. Let $\mathcal{B}' = \{(w_1, w_3) | (w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}_2\}$. It's apparent that $(s_1, s_3) \in \mathcal{B}'$. We prove $\mathcal{B}'$ is a $V_1 \cup V_2$-bisimulation containing $(s_1, s_3)$ from the (a), (b) and (c) of the previous step (iii) of $X$-bisimulation (where $X$ is a set of atoms). For all $(w_1, w_3) \in \mathcal{B}'$:

(a) there is $w_2 \in S_2$ such that $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$, and $\forall q \notin V_1$, $q \in L_1(w_1)$ iff $q \in L_2(w_2)$ by $w_1 \leftrightarrow_{V_1} w_2$ and $\forall q' \notin V_2$, $q' \in L_2(w_2)$ iff $q' \in L_3(w_3)$ by $w_2 \leftrightarrow_{V_2} w_3$. Then we have $\forall r \notin V_1 \cup V_2$, $r \in L_1(w_1)$ iff $r \in L_3(w_3)$.

(b) if $(w_1, u_1) \in \mathcal{R}_1$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_1, u_2) \in \mathcal{B}$ (due to $(w_1, w_2) \in \mathcal{B}$ and $(w_2, w_3) \in \mathcal{B}''$ by the definition of $\mathcal{B}'$); and then $\exists u_3 \in S_3$ such that $(w_3, u_3) \in \mathcal{R}_3$ and $(u_2, u_3) \in \mathcal{B}''$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(c) if $(w_3, u_3) \in \mathcal{R}_3$, then $\exists u_2 \in S_2$ such that $(w_2, u_2) \in \mathcal{R}_2$ and $(u_2, u_3) \in \mathcal{B}_2$; and then $\exists u_1 \in S_1$ such that $(w_1, u_1) \in \mathcal{R}_1$ and $(u_1, u_2) \in \mathcal{B}$, hence $(u_1, u_3) \in \mathcal{B}'$ by the definition of $\mathcal{B}'$.

(v) Let $\mathcal{K}_{i,j} = (\mathcal{M}_i, s_{i,j})$ and $(s_{i,k}, s_{i,k+1}) \in R_i$ mean that $s_{i,k+1}$ is the $(k+2)$-th node in the path

$(s_i, s_{i,1}, s_{i,2}, \dots, s_{i,k+1}, \dots)$ $(i = 1, 2)$. We will show that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_n^{V_2}$ for all $n \geq 0$ inductively.

Base: $L_1(s_1) - V_1 = L_2(s_2) - V_1$
$\Rightarrow \forall q \in \mathcal{A} - V_1$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$
$\Rightarrow \forall q \in \mathcal{A} - V_2$ there is $q \in L_1(s_1)$ iff $q \in L_2(s_2)$ due to $V_1 \subseteq V_2$
$\Rightarrow L_1(s_1) - V_2 = L_2(s_2) - V_2$, i.e., $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_0^{V_2}$.

Step: Supposing that $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_i^{V_2}$ for all $0 \leq i \leq k$ $(k > 0)$, we will show $(\mathcal{K}_1, \mathcal{K}_2) \in \mathcal{B}_{k+1}^{V_2}$.

(a) It is apparent that $L_1(s_1) - V_2 = L_2(s_2) - V_2$ by base.

(b) $\forall (s_1, s_{1,1}) \in R_1$, we will show that there is a $(s_2, s_{2,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_{k-1}^{V_2}$ by inductive assumption, we need only to prove the following points:
(a) $\forall (s_{1,k}, s_{1,k+1}) \in R_1$ there is a $(s_{2,k}, s_{2,k+1}) \in R_2$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. It is easy to see that $L_1(s_{1,k+1}) - V_1 = L_1(s_{2,k+1}) - V_1$, then there is $L_1(s_{1,k+1}) - V_2 = L_1(s_{2,k+1}) - V_2$. Therefore, $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$.
(b) $\forall (s_{2,k}, s_{2,k+1}) \in R_1$ there is a $(s_{1,k}, s_{1,k+1}) \in R_1$ s.t. $(\mathcal{K}_{1,k+1}, \mathcal{K}_{2,k+1}) \in \mathcal{B}_0^{V_2}$ due to $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_1}$. This can be proved as (a).

(c) $\forall (s_2, s_{2,1}) \in R_1$, we will show that there is a $(s_1, s_{1,1}) \in R_2$ s.t. $(\mathcal{K}_{1,1}, \mathcal{K}_{2,1}) \in \mathcal{B}_k^{V_2}$. This can be proved as (ii).

$\square$

**Theorem**1 Let $V \subseteq \mathcal{A}$, $\mathcal{K}_i$ $(i = 1, 2)$ be two K-structures such that $\mathcal{K}_1 \leftrightarrow_V \mathcal{K}_2$ and $\phi$ a formula with IR$(\phi, V)$. Then $\mathcal{K}_1 \models \phi$ if and only if $\mathcal{K}_2 \models \phi$.

*Proof.* This theorem can be proved by inducting on the formula $\phi$ and supposing *Var*$(\phi) \cap V = \emptyset$. Let $\mathcal{K}_1 = (\mathcal{M}, s)$ and $\mathcal{K}_2 = (\mathcal{M}', s')$.

**Case** $\phi = p$ where $p \in \mathcal{A} - V$:
$(\mathcal{M}, s) \models \phi$ iff $p \in L(s)$ (by the definition of satisfiability)
$\Leftrightarrow p \in L'(s')$ $\quad (s \leftrightarrow_V s')$
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

**Case** $\phi = \neg\psi$:
$(\mathcal{M}, s) \models \phi$ iff $(\mathcal{M}, s) \not\models \psi$
$\Leftrightarrow (\mathcal{M}', s') \not\models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

**Case** $\phi = \psi_1 \vee \psi_2$:
$(\mathcal{M}, s) \models \phi$
$\Leftrightarrow (\mathcal{M}, s) \models \psi_1$ or $(\mathcal{M}, s) \models \psi_2$
$\Leftrightarrow (\mathcal{M}', s') \models \psi_1$ or $(\mathcal{M}', s') \models \psi_2$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

**Case** $\phi = \text{EX}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s, s_1, ...)$ such that $\mathcal{M}, s_1 \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s', s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_1 \leftrightarrow_V s'_1$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s'_1) \models \psi$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

**Case** $\phi = \text{EG}\psi$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that for each $i \geq 0$ there is $(\mathcal{M}, s_i) \models \psi$
$\Leftrightarrow$ There is a path $\pi' = (s' = s'_0, s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow s_i \leftrightarrow_V s'_i$ for each $i \geq 0$ $\quad (\pi \leftrightarrow_V \pi')$
$\Leftrightarrow (\mathcal{M}', s'_i) \models \psi$ for each $i \geq 0$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$

**Case** $\phi = \text{E}[\psi_1 \text{U} \psi_2]$:
$\mathcal{M}, s \models \phi$
$\Leftrightarrow$ There is a path $\pi = (s = s_0, s_1, ...)$ such that there is $i \geq 0$ such that $(\mathcal{M}, s_i) \models \psi_2$, and for all $0 \leq j < i$, $(\mathcal{M}, s_j) \models \psi_1$
$\Leftrightarrow$ There is a path $\pi' = (s = s'_0, s'_1, ...)$ such that $\pi \leftrightarrow_V \pi'$ $(s \leftrightarrow_V s'$, Proposition 1)
$\Leftrightarrow (\mathcal{M}', s'_i) \models \psi_2$, and for all $0 \leq j < i$ $(\mathcal{M}', s'_j) \models \psi_1$ (induction hypothesis)
$\Leftrightarrow (\mathcal{M}', s') \models \phi$ $\square$

**Proposition** 2 Let $V \subseteq \mathcal{A}$ and $(\mathcal{M}_i, s_i)$ $(i = 1, 2)$ be two K-structures. Then

$(s_1, s_2) \in \mathcal{B}_n$ iff $\text{Tr}_j(s_1) \leftrightarrow_V \text{Tr}_j(s_2)$ for every $0 \leq j \leq n$.

*Proof.* We will prove this from two aspects:

$(\Rightarrow)$ If $(s_1, s_2) \in \mathcal{B}_n$, then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$. $(s, s') \in \mathcal{B}_n$ implies both roots of $Tr_n(s_1)$ and $Tr_n(s_2)$ have the same atoms except those atoms in $V$. Besides, for any $s_{1,1}$ with $(s_1, s_{1,1}) \in R_1$, there is a $s_{2,1}$ with $(s_2, s_{2,1}) \in R_2$ s.t. $(s_{1,1}, s_{2,1}) \in \mathcal{B}_{n-1}$ and vice versa. Then we have $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$. Therefore, $Tr_n(s_1) \leftrightarrow_V Tr_n(s_2)$ by use such method recursively, and then $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$.

$(\Leftarrow)$ If $Tr_j(s_1) \leftrightarrow_V Tr_j(s_2)$ for all $0 \leq j \leq n$, then $(s_1, s_2) \in \mathcal{B}_n$. $Tr_0(s_1) \leftrightarrow_V Tr_0(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and then $(s, s') \in \mathcal{B}_0$. $Tr_1(s_1) \leftrightarrow_V Tr_1(s_2)$ implies $L(s_1) - V = L'(s_2) - V$ and for every successors $s$ of the root of one, it is possible to find a successor of the root of the other $s'$ such that $(s, s') \in \mathcal{B}_0$. Therefore $(s_1, s_2) \in \mathcal{B}_1$, and then we will have $(s_1, s_2) \in \mathcal{B}_n$ by use such method recursively. $\square$

**Proposition** 3 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ be a model structure and $s, s' \in S$ such that $s \not\leftrightarrow_V s'$. There exists a least $k$ such that $\text{Tr}_k(s)$ and $\text{Tr}_k(s')$ are not $V$-bisimilar.

*Proof.* If $s \not\leftrightarrow_V s'$, then there exists a least constant $c$ such that $(s_i, s_j) \notin \mathcal{B}_c$, and then there is a least constant $m$ $(m \leq c)$ such that $\text{Tr}_m(s_i)$ and $\text{Tr}_m(s_j)$ are not V-bisimilar by Proposition 2. Let $k = m$, the lemma is proved. $\square$

**Lemma**2 Let $V \subseteq \mathcal{A}$, $\mathcal{M}$ and $\mathcal{M}'$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$. If $\text{Tr}_n(s) \leftrightarrow_{\overline{V}} \text{Tr}_n(s')$, then $\mathcal{F}_V(\text{Tr}_n(s)) \equiv \mathcal{F}_V(\text{Tr}_n(s'))$.

*Proof.* This result can be proved by inducting on $n$.

**Base.** It is apparent that for any $s_n \in S$ and $s'_n \in S'$, if $\text{Tr}_0(s_n) \leftrightarrow_{\overline{V}} \text{Tr}_0(s'_n)$ then $\mathcal{F}_V(\text{Tr}_0(s_n)) \equiv \mathcal{F}_V(\text{Tr}_0(s'_n))$ due to $L(s_n) - \overline{V} = L'(s'_n) - \overline{V}$ by known.

**Step.** Supposing that for $k = m$ $(0 < m \leq n)$ there is if $\text{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k}(s'_k)$ then $\mathcal{F}_V(\text{Tr}_{n-k}(s_k)) \equiv \mathcal{F}_V(\text{Tr}_{n-k}(s'_k))$, then we will show if $\text{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k+1}(s'_{k-1})$ then $\mathcal{F}_V(\text{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\text{Tr}_{n-k+1}(s'_{k-1}))$. Apparent that:
$\mathcal{F}_V(\text{Tr}_{n-k+1}(s_{k-1})) = \left(\bigwedge_{(s_{k-1},s_k)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_{n-k}(s_k))\right) \wedge \text{AX}\left(\bigvee_{(s_{k-1},s_k)\in R} \mathcal{F}_V(\text{Tr}_{n-k}(s_k))\right) \wedge \mathcal{F}_V(\text{Tr}_0(s_{k-1}))$
$\mathcal{F}_V(\text{Tr}_{n-k+1}(s'_{k-1})) = \left(\bigwedge_{(s'_{k-1},s'_k)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_{n-k}(s'_k))\right) \wedge \text{AX}\left(\bigvee_{(s'_{k-1},s'_k)\in R} \mathcal{F}_V(\text{Tr}_{n-k}(s'_k))\right) \wedge \mathcal{F}_V(\text{Tr}_0(s'_{k-1}))$
by the definition of characterizing formula of the computation tree. Then we have for any $(s_{k-1}, s_k) \in R$ there is $(s'_{k-1}, s'_k) \in R'$ such that $\text{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k}(s'_k)$ by $\text{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k+1}(s'_{k-1})$. Besides, for any $(s'_{k-1}, s'_k) \in R'$ there is $(s_{k-1}, s_k) \in R$ such that $\text{Tr}_{n-k}(s_k) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k}(s'_k)$ by $\text{Tr}_{n-k+1}(s_{k-1}) \leftrightarrow_{\overline{V}} \text{Tr}_{n-k+1}(s'_{k-1})$. Therefore, we have $\mathcal{F}_V(\text{Tr}_{n-k+1}(s_{k-1})) \equiv \mathcal{F}_V(\text{Tr}_{n-k+1}(s'_{k-1}))$ by induction hypothesis. $\square$

**Theorem** 2 Given $V \subseteq \mathcal{A}$, let $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures. Then,

(i) $(\mathcal{M}', s'_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ iff $(\mathcal{M}, s_0) \leftrightarrow_{\overline{V}} (\mathcal{M}', s'_0)$;

(ii) $s_0 \leftrightarrow_{\overline{V}} s'_0$ implies $\mathcal{F}_V(\mathcal{M}, s_0) \equiv \mathcal{F}_V(\mathcal{M}', s'_0)$.

In order to prove Theorem 2, we prove the following two lemmas at first.

**Lemma 6.** *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ and $\mathcal{M}' = (S', R', L', s'_0)$ be two model structures, $s \in S$, $s' \in S'$ and $n \geq 0$.*

*(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s))$.*

*(ii) If $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_n(s'))$ then $Tr_n(s) \leftrightarrow_{\overline{V}} Tr_n(s')$.*

*Proof.* (i) It is apparent from the definition of $\mathcal{F}_V(\text{Tr}_n(s))$.
Base. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_0(s))$.
Step. For $k \geq 0$, supposing the result talked in (i) is correct in $k-1$, we will show that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_{k+1}(s))$, i.e.,:

$$(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \text{EX}T(s')\right) \wedge \text{AX}\left(\bigvee_{(s,s')\in R} T(s')\right) \wedge \mathcal{F}_V(\text{Tr}_0(s)).$$

Where $T(s') = \mathcal{F}_V(\text{Tr}_k(s'))$. It is apparent that $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_0(s))$ by Base. It is apparent that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \mathcal{F}_V(\text{Tr}_k(s'))$ by inductive assumption. Then we have $(\mathcal{M}, s) \models \text{EX}\mathcal{F}_V(\text{Tr}_k(s')$, and then $(\mathcal{M}, s) \models \left(\bigwedge_{(s,s')\in R} \text{EX}\mathcal{F}_V(\text{Tr}_k(s'))\right)$. Similarly, we have that for any $(s, s') \in R$, there is $(\mathcal{M}, s') \models \bigvee_{(s,s'')\in R} \mathcal{F}_V(\text{Tr}_k(s''))$. Therefore, $(\mathcal{M}, s) \models \text{AX}\left(\bigvee_{(s,s'')\in R} \mathcal{F}_V(\text{Tr}_k(s''))\right)$.

(ii) **Base**. If $n = 0$, then $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_0(s'))$ implies $L(s) - \overline{V} = L'(s') - \overline{V}$. Hence, $\text{Tr}_0(s) \leftrightarrow_{\overline{V}} \text{Tr}_0(s')$.
**Step**. Supposing $n > 0$ and the result talked in (ii) is correct in $n-1$.
(a) It is easy to see that $L(s) - \overline{V} = L'(s') - \overline{V}$.
(b) We will show that for each $(s, s_1) \in R$, there is a $(s', s'_1) \in R'$ such that $\text{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s'_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \text{AX}\left(\bigvee_{(s',s'_1)\in R} \mathcal{F}_V(\text{Tr}_{n-1}(s'_1))\right)$. Therefore, for each $(s, s_1) \in R$ there is a $(s', s'_1) \in R'$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\text{Tr}_{n-1}(s'_1))$. Hence, $\text{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s'_1)$ by inductive hypothesis.
(c) We will show that for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $\text{Tr}_{n-1}(s'_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s_1)$. Since $(\mathcal{M}, s) \models \mathcal{F}_V(\text{Tr}_n(s'))$, then $(\mathcal{M}, s) \models \bigwedge_{(s',s'_1)\in R'} \text{EX}\mathcal{F}_V(\text{Tr}_{n-1}(s'_1))$. Therefore, for each $(s', s'_1) \in R'$ there is a $(s, s_1) \in R$ such that $(\mathcal{M}, s_1) \models \mathcal{F}_V(\text{Tr}_{n-1}(s'_1)$. Hence, $\text{Tr}_{n-1}(s_1) \leftrightarrow_{\overline{V}} \text{Tr}_{n-1}(s'_1)$ by inductive hypothesis. $\square$

A consequence of the previous lemma is:

**Lemma 7.** *Let $V \subseteq \mathcal{A}$, $\mathcal{M} = (S, R, L, s_0)$ a model structure, $k = ch(\mathcal{M}, V)$ and $s \in S$.*

*(i) $(\mathcal{M}, s) \models \mathcal{F}_V(Tr_k(s))$, and*

*(ii) for each $s' \in S$, $(\mathcal{M}, s) \leftrightarrow_{\overline{V}} (\mathcal{M}, s')$ if and only if $(\mathcal{M}, s') \models \mathcal{F}_V(Tr_k(s))$.*

*Proof.* (i) It is apparent from the (i) of Lemma 6.
(ii) Let $\phi = \mathcal{F}_V(\text{Tr}_k(s))$, where $k$ is the V-characteristic number of $\mathcal{M}$. $(\mathcal{M}, s) \models \phi$ by the definition of $\mathcal{F}$, and then $\forall s' \in S$, if $s \leftrightarrow_{\overline{V}} s'$ there is $(\mathcal{M}, s') \models \phi$ by Theorem 1 due to $\text{IR}(\phi, \mathcal{A} - V)$. Supposing $(\mathcal{M}, s') \models \phi$, if $s \not\leftrightarrow_{\overline{V}} s'$, then $\text{Tr}_k(s) \not\leftrightarrow_{\overline{V}} \text{Tr}_k(s')$, and then $(\mathcal{M}, s') \not\models \phi$ by Lemma 6, a contradiction. $\square$

Now we are in the position of proving Theorem 2.

*Proof.* (i) Let $\mathcal{F}_V(\mathcal{M}, s_0)$ be the characterizing formula of $(\mathcal{M}, s_0)$ on $V$. It is apparent that $\text{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$. We will show that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ at first.
It is apparent that $(\mathcal{M}, s_0) \models \mathcal{F}_V(\text{Tr}_c(s_0))$ by Lemma 6. We must show that $(\mathcal{M}, s_0) \models \bigwedge_{s\in S} G(\mathcal{M}, s)$. Let $\mathcal{X} = \mathcal{F}_V(\text{Tr}_c(s)) \rightarrow \left(\bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))\right) \wedge \text{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\text{Tr}_c(s_1))\right)$, we will show $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$. Where $G(\mathcal{M}, s) = \text{AG}\mathcal{X}$. There are two cases we should consider:

- If $(\mathcal{M}, s_0) \not\models \mathcal{F}_V(\text{Tr}_c(s))$, it is apparent that $(\mathcal{M}, s_0) \models \mathcal{X}$;
- If $(\mathcal{M}, s_0) \models \mathcal{F}_V(\text{Tr}_c(s))$:
  $(\mathcal{M}, s_0) \models \mathcal{F}_V(\text{Tr}_c(s))$
  $\Rightarrow s_0 \leftrightarrow_{\overline{V}} s$ by the definition of characteristic number and Lemma 7.
  For each $(s, s_1) \in R$ there is:
  $(\mathcal{M}, s_1) \models \mathcal{F}_V(\text{Tr}_c(s_1))$ $\quad (s_1 \leftrightarrow_{\overline{V}} s_1)$
  $\Rightarrow (\mathcal{M}, s) \models \bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))$
  $\Rightarrow (\mathcal{M}, s_0) \models \bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1))$ (by $\text{IR}(\bigwedge_{(s,s_1)\in R} \text{EX}\mathcal{F}_V(\text{Tr}_c(s_1)), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$).
  For each $(s, s_1)$ there is:
  $\mathcal{M}, s_1 \models \bigvee_{(s,s_2)\in R} \mathcal{F}_V(\text{Tr}_c(s_2))$

$\Rightarrow (\mathcal{M}, s) \models \mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$

$\Rightarrow (\mathcal{M}, s_0) \models \mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ (by $\mathrm{IR}(\mathrm{AX}\left(\bigvee_{(s,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right), \overline{V})$, $s_0 \leftrightarrow_{\overline{V}} s$)

$\Rightarrow (\mathcal{M}, s_0) \models \mathcal{X}$.

For any other states $s'$ which can reach from $s_0$ can be proved similarly, i.e.,, $(\mathcal{M}, s') \models \mathcal{X}$. Therefore, $\forall s \in S$, $(\mathcal{M}, s_0) \models G(\mathcal{M}, s)$, and then $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$.

We will prove this theorem from the following two aspects:

($\Leftarrow$) If $s_0 \leftrightarrow_{\overline{V}} s_0'$, then $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$. Since $(\mathcal{M}, s_0) \models \mathcal{F}_V(\mathcal{M}, s_0)$ and $\mathrm{IR}(\mathcal{F}_V(\mathcal{M}, s_0), \overline{V})$, hence $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$ by Theorem 1.

($\Rightarrow$) If $(\mathcal{M}', s_0') \models \mathcal{F}_V(M, s_0)$, then $s_0 \leftrightarrow_{\overline{V}} s_0'$. We will prove this by showing that $\forall n \geq 0$, $Tr_n(s_0) \leftrightarrow_{\overline{V}} Tr_n(s_0')$.

**Base**. It is apparent that $Tr_0(s_0) \equiv Tr_0(s_0')$.

**Step**. Supposing $\mathrm{Tr}_k(s_0) \leftrightarrow_{\overline{V}} \mathrm{Tr}_k(s_0')$ ($k > 0$), we will prove $\mathrm{Tr}_{k+1}(s_0) \leftrightarrow_{\overline{V}} \mathrm{Tr}_{k+1}(s_0')$. We should only show that $\mathrm{Tr}_1(s_k) \leftrightarrow_{\overline{V}} \mathrm{Tr}_1(s_k')$. Where $(s_0, s_1), (s_1, s_2), \ldots, (s_{k-1}, s_k) \in R$ and $(s_0', s_1'), (s_1', s_2'), \ldots, (s_{k-1}', s_k') \in R'$, i.e., $s_{i+1}$ ($s_{i+1}'$) is an immediate successor of $s_i$ ($s_i'$) for all $0 \leq i \leq k-1$.

(a) It is apparent that $L(s_k) - \overline{V} = L'(s_k') - \overline{V}$ by inductive assumption.

Before talking about the other points, note the following fact that:

$(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathcal{M}, s_0)$

$\Rightarrow \forall s' \in S'$, $(\mathcal{M}', s') \models \mathcal{F}_V(\mathrm{Tr}_c(s)) \to \left(\bigwedge_{(s,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ for any $s \in S$. **(fact)**

(I) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathrm{Tr}_c(s_0)) \to \left(\bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ **(fact)**

(II) $(\mathcal{M}', s_0') \models \mathcal{F}_V(\mathrm{Tr}_c(s_0)))$ (known)

(III) $(\mathcal{M}', s_0') \models \left(\bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_0,s_1)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_1))\right)$ ((I),(II))

(b) We will show that for each $(s_k, s_{k+1}) \in R$ there is a $(s_k', s_{k+1}') \in R'$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.

(1) $(\mathcal{M}', s_0') \models \bigwedge_{(s_0,s_1)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (III)

(2) $\forall (s_0, s_1) \in R$, $\exists (s_0', s_1') \in R'$ s.t. $(\mathcal{M}', s_1') \models \mathcal{F}_V(\mathrm{Tr}_c(s_1))$ (2)

(3) $\mathrm{Tr}_c(s_1) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s_1')$ ((2), Lemma 6)

(4) $L(s_1) - \overline{V} = L'(s_1') - \overline{V}$ ((3), $c \geq 0$)

(5) $(\mathcal{M}', s_1') \models \mathcal{F}_V(\mathrm{Tr}_c(s_1)) \to \left(\bigwedge_{(s_1,s_2)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_2))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ **(fact)**

(6) $(\mathcal{M}', s_1') \models \left(\bigwedge_{(s_1,s_2)\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_2))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_1,s_2)\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_2))\right)$ ((2), (5))

(7) ......

(8) $(\mathcal{M}', s_k') \models \left(\bigwedge_{(s_k,s_{k+1})\in R} \mathrm{EX}\mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right) \wedge \mathrm{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right)$ (similar with (6))

(9) $\forall (s_k, s_{k+1}) \in R$, $\exists (s_k', s_{k+1}') \in R'$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))$ (8)

(10) $\mathrm{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s_{k+1}')$ ((9), Lemma 6)

(11) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((10), $c \geq 0$)

(c) We will show that for each $(s_k', s_{k+1}') \in R'$ there is a $(s_k, s_{k+1}) \in R$ such that $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$.

(1) $(\mathcal{M}', s_k') \models \mathrm{AX}\left(\bigvee_{(s_k,s_{k+1})\in R} \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}))\right)$ (by (8) talked above)

(2) $\forall (s_k', s_{k+1}') \in R'$, $\exists (s_k, s_{k+1}) \in R$ s.t. $(\mathcal{M}', s_{k+1}') \models \mathcal{F}_V(\mathrm{Tr}_c(s_{k+1}'))$ (1)

(3) $\mathrm{Tr}_c(s_{k+1}) \leftrightarrow_{\overline{V}} \mathrm{Tr}_c(s_{k+1}')$ ((2), Lemma 6)

(4) $L(s_{k+1}) - \overline{V} = L'(s_{k+1}') - \overline{V}$ ((3), $c \geq 0$)

(ii) This is following Lemma 2 and the definition of the characterizing formula of initial K-structure $\mathcal{K}$ on $V$.

□

**Lemma** 3 Let $\varphi$ be a formula. We have

$$\varphi \equiv \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0). \tag{3}$$

*Proof.* Let $(\mathcal{M}', s_0')$ be a model of $\varphi$. Then $(\mathcal{M}', s_0') \models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$ due to $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}', s_0')$. On the other hand, suppose that $(\mathcal{M}', s_0')$ is a model of $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. Then there is a $(\mathcal{M}, s_0) \in Mod(\varphi)$ such that $(\mathcal{M}', s_0') \models \mathcal{F}_{\mathcal{A}}(\mathcal{M}, s_0)$. And then $(\mathcal{M}, s_0) \leftrightarrow_{\emptyset} (\mathcal{M}', s_0')$ by Theorem 2. Therefore, $(\mathcal{M}, s_0)$ is also a model of $\varphi$ by Theorem 1. □

**Theorem** 3 **(Representation theorem)** Let $\varphi$, $\varphi'$ and $\phi$ be CTL formulas and $V \subseteq \mathcal{A}$. Then the following statements are equivalent:

(i) $\varphi' \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$,

(ii) $\varphi' \equiv \{\phi | \varphi \models \phi \text{ and } \mathrm{IR}(\phi, V)\}$,

(iii) Postulates (**W**), (**PP**), (**NP**) and (**IR**) hold.

*Proof.* $(i) \Leftrightarrow (ii)$. To prove this, we will show that:

$$Mod(\mathrm{F}_{\mathrm{CTL}}(\varphi, V)) = Mod(\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\})$$
$$= Mod(\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M}, s_0)).$$

Firstly, suppose that $(\mathcal{M}', s_0')$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. Then there exists an initial K-structure $(\mathcal{M}, s_0)$ such that $(\mathcal{M}, s_0)$ is a model of $\varphi$ and $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. By Theorem 1, we have $(\mathcal{M}', s_0') \models \phi$ for all $\phi$ that $\varphi \models \phi$ and $\mathrm{IR}(\phi, V)$. Thus, $(\mathcal{M}', s_0')$ is a model of $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$.

Secondly, suppose that $(\mathcal{M}', s_0')$ is a models of $\{\phi | \varphi \models \phi, \mathrm{IR}(\phi, V)\}$. Thus, $(\mathcal{M}', s_0')$

$\models \bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$ due to $\bigvee_{(\mathcal{M},s_0)\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$ is irrelevant to $V$.

Finally, suppose that $(\mathcal{M}',s_0')$ is a model of $\bigvee_{\mathcal{M},s_0\in Mod(\varphi)} \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$. Then there exists $(\mathcal{M},s_0) \in Mod(\varphi)$ such that $(\mathcal{M}',s_0') \models \mathcal{F}_{\mathcal{A}-V}(\mathcal{M},s_0)$. Hence, $(\mathcal{M},s_0) \leftrightarrow_V (\mathcal{M}',s_0')$ by Theorem 2. Thus $(\mathcal{M}',s_0')$ is also a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$.

$(ii) \Rightarrow (iii)$. It is not difficult to prove it.

$(iii) \Rightarrow (ii)$. Suppose that all postulates hold. By Positive Persistence, we have $\varphi' \models \{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\}$. Now we show that $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\} \models \varphi'$. Otherwise, there exists formula $\phi'$ such that $\varphi' \models \phi'$ but $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\} \not\models \phi'$. There are three cases:

- $\phi'$ is relevant to $V$. Thus, $\varphi'$ is also relevant to $V$, a contradiction to Irrelevance.
- $\phi'$ is irrelevant to $V$ and $\varphi \models \phi'$. This contradicts to our assumption.
- $\phi'$ is irrelevant to $V$ and $\varphi \not\models \phi'$. By Negative Persistence, $\varphi' \not\models \phi'$, a contradiction.

Thus, $\varphi'$ is equivalent to $\{\phi|\varphi \models \phi, \mathrm{IR}(\phi,V)\}$. □

**Lemma** 4 Let $\varphi$ and $\alpha$ be two CTL formulae and $q \in \overline{Var(\varphi) \cup Var(\alpha)}$. Then $\mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q) \equiv \varphi$.

*Proof.* Let $\varphi' = \varphi \wedge (q \leftrightarrow \alpha)$. For any model $(\mathcal{M},s)$ of $\mathrm{F}_{\mathrm{CTL}}(\varphi',q)$ there is an initial K-structure $(\mathcal{M}',s')$ s.t. $(\mathcal{M},s) \leftrightarrow_{\{q\}} (\mathcal{M}',s')$ and $(\mathcal{M}',s') \models \varphi'$. It's apparent that $(\mathcal{M}',s') \models \varphi$, and then $(\mathcal{M},s) \models \varphi$ since $\mathrm{IR}(\varphi,\{q\})$ and $(\mathcal{M},s) \leftrightarrow_{\{q\}} (\mathcal{M}',s')$ by Theorem 1.

Let $(\mathcal{M},s) \in Mod(\varphi)$ with $\mathcal{M} = (S,R,L,s)$. We construct $(\mathcal{M}',s)$ with $\mathcal{M}' = (S,R,L',s)$ as follows:

$L' : S \to \mathcal{A}$ *and* $\forall s^* \in S, L'(s^*) = L(s^*)$ *if* $(\mathcal{M},s^*) \not\models \alpha$, *else* $L'(s^*) = L(s^*) \cup \{q\}$,

$L'(s) = L(s) \cup \{q\}$ *if* $(\mathcal{M},s) \models \alpha$, *and* $L'(s) = L(s)$ *otherwise.*

It is clear that $(\mathcal{M}',s) \models \varphi$, $(\mathcal{M}',s) \models q \leftrightarrow \alpha$ and $(\mathcal{M}',s) \leftrightarrow_{\{q\}} (\mathcal{M},s)$. Therefore $(\mathcal{M}',s) \models \varphi \wedge (q \leftrightarrow \alpha)$, and then $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi \wedge (q \leftrightarrow \alpha), q)$ by $(\mathcal{M}',s) \leftrightarrow_{\{q\}} (\mathcal{M},s)$. □

**Proposition** 4 Given a formula $\varphi \in$ CTL, $V$ a set of atoms and $p$ an atom such that $p \notin V$. Then,

$$\mathrm{F}_{\mathrm{CTL}}(\varphi, \{p\} \cup V) \equiv \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p), V).$$

*Proof.* Let $(\mathcal{M}_1,s_1)$ with $\mathcal{M}_1 = (S_1,R_1,L_1,s_1)$ be a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,\{p\} \cup V)$. By the definition, there exists a model $(\mathcal{M},s)$ with $\mathcal{M} = (S,R,L,s)$ of $\varphi$, such that $(\mathcal{M}_1,s_1) \leftrightarrow_{\{p\}\cup V} (\mathcal{M},s)$. We construct an initial K-structure $(\mathcal{M}_2,s_2)$ with $\mathcal{M}_2 = (S_2,R_2,L_2,s_2)$ as follows:

(1) for $s_2$: let $s_2$ be the state such that:
  - $p \in L_2(s_2)$ iff $p \in L_1(s_1)$,
  - for all $q \in V$, $q \in L_2(s_2)$ iff $q \in L(s)$,
  - for all other atoms $q'$, $q' \in L_2(s_2)$ iff $q' \in L_1(s_1)$ iff $q' \in L(s)$.

(2) for another:
  (i) for all pairs $w \in S$ and $w_1 \in S_1$ such that $w \leftrightarrow_{\{p\}\cup V} w_1$, let $w_2 \in S_2$ and
  - $p \in L_2(w_2)$ iff $p \in L_1(w_1)$,
  - for all $q \in V$, $q \in L_2(w_2)$ iff $q \in L(w)$,
  - for all other atoms $q'$, $q' \in L_2(w_2)$ iff $q' \in L_1(w_1)$ iff $q' \in L(w)$.

  (ii) if $(w_1',w_1) \in R_1$, $w_2$ is constructed based on $w_1$ and $w_2' \in S_2$ is constructed based on $w_1'$, then $(w_2',w_2) \in R_2$.

(3) delete duplicated states in $S_2$ and pairs in $R_2$.

Then we have $(\mathcal{M},s) \leftrightarrow_{\{p\}} (\mathcal{M}_2,s_2)$ and $(\mathcal{M}_2,s_2) \leftrightarrow_V (\mathcal{M}_1,s_1)$. Thus, $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,p)$. And therefore $(\mathcal{M}_1,s_1) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V)$.

On the other hand, suppose that $(\mathcal{M}_1,s_1)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\mathrm{F}_{\mathrm{CTL}}(\varphi,p),V)$, then there exists an initial K-structure $(\mathcal{M}_2,s_2)$ such that $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,p)$ and $(\mathcal{M}_2,s_2) \leftrightarrow_V (\mathcal{M}_1,s_1)$, and there exists $(\mathcal{M},s)$ such that $(\mathcal{M},s) \models \varphi$ and $(\mathcal{M},s) \leftrightarrow_{\{p\}} (\mathcal{M}_2,s_2)$. Therefore, $(\mathcal{M},s) \leftrightarrow_{\{p\}\cup V} (\mathcal{M}_1,s_1)$ by Proposition 1, and consequently, $(\mathcal{M}_1,s_1) \models \mathrm{F}_{\mathrm{CTL}}(\varphi,\{p\} \cup V)$. □

**Proposition** 5 Let $\varphi$, $\varphi_i$, $\psi_i$ $(i = 1,2)$ be formulas and $V \subseteq \mathcal{A}$. We have

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$ is satisfiable iff $\varphi$ is;

(ii) If $\varphi_1 \equiv \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1,V) \equiv \mathrm{F}_{\mathrm{CTL}}(\varphi_2,V)$;

(iii) If $\varphi_1 \models \varphi_2$, then $\mathrm{F}_{\mathrm{CTL}}(\varphi_1,V) \models \mathrm{F}_{\mathrm{CTL}}(\varphi_2,V)$;

(iv) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V) \equiv \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$;

(v) $\mathrm{F}_{\mathrm{CTL}}(\psi_1 \wedge \psi_2,V) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \wedge \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$;

*Proof.* (i) $(\Rightarrow)$ Supposing $(\mathcal{M},s)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)$, then there is a model $(\mathcal{M}',s')$ of $\varphi$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

$(\Leftarrow)$ Supposing $(\mathcal{M},s)$ is a model of $\varphi$, then there is an initial Kripke structure $(\mathcal{M}',s')$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$, and then $(\mathcal{M}',s') \models \mathrm{F}_{\mathrm{CTL}}(\varphi,V)$ by the definition of $\mathrm{F}_{\mathrm{CTL}}$.

The (ii) and (iii) can be proved similarly.

(iv) $(\Rightarrow)$ $\forall(\mathcal{M},s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V))$, $\exists(\mathcal{M}',s') \in Mod(\psi_1 \vee \psi_2)$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}',s')$ and $(\mathcal{M}',s') \models \psi_1$ or $(\mathcal{M}',s') \models \psi_2$
$\Rightarrow \exists(\mathcal{M}_1,s_1) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1,V))$ s.t. $(\mathcal{M}',s') \leftrightarrow_V (\mathcal{M}_1,s_1)$ or $\exists(\mathcal{M}_2,s_2) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_2,V))$ s.t. $(\mathcal{M}',s') \leftrightarrow_V (\mathcal{M}_2,s_2)$
$\Rightarrow (\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$ by Theorem 1.

$(\Leftarrow)$ $\forall(\mathcal{M},s) \in Mod(\mathrm{F}_{\mathrm{CTL}}(\psi_1,V) \vee \mathrm{F}_{\mathrm{CTL}}(\psi_2,V))$
$\Rightarrow (\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1,V)$ or $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_2,V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1,s_1)$ s.t. $(\mathcal{M},s) \leftrightarrow_V (\mathcal{M}_1,s_1)$ and $(\mathcal{M}_1,s_1) \models \psi_1$ or $(\mathcal{M}_1,s_1) \models \psi_2$
$\Rightarrow (\mathcal{M}_1,s_1) \models \psi_1 \vee \psi_2$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2,s_2)$ s.t. $(\mathcal{M}_1,s_1) \leftrightarrow_V (\mathcal{M}_2,s_2)$ and $(\mathcal{M}_2,s_2) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V)$
$\Rightarrow (\mathcal{M},s) \leftrightarrow_V (\mathcal{M}_2,s_2)$ and $(\mathcal{M},s) \models \mathrm{F}_{\mathrm{CTL}}(\psi_1 \vee \psi_2,V)$.

The (v) can be proved as (iv). □

**Proposition** 6 **(Homogeneity)** Let $V \subseteq \mathcal{A}$ and $\phi \in$ CTL,

(i) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(ii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(iii) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AF}\phi, V) \equiv \mathrm{AF}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

(iv) $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EF}\phi, V) \equiv \mathrm{EF}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$.

*Proof.* Let $\mathcal{M} = (S, R, L, s_0)$ with initial state $s_0$ and $\mathcal{M}' = (S', R', L', s_0')$ with initial state $s_0'$, then we call $\mathcal{M}', s_0'$ be a sub-structure of $\mathcal{M}, s_0$ if:

- $S' \subseteq S$ and $S' = \{s' | s'$ is reachable from $s_0'\}$,
- $R' = \{(s_1, s_2) | s_1, s_2 \in S'$ and $(s_1, s_2) \in R\}$,
- $L' : S' \to 2^{\mathcal{A}}$ and $\forall s_1 \in S'$ there is $L'(s_1) = L(s_1)$, and
- $s_0'$ is $s_0$ or a state reachable from $s_0$.

(i) In order to prove $\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V) \equiv \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$, we only need to prove $\mathit{Mod}(\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)) = \mathit{Mod}(\mathrm{AX}\mathrm{F}_{\mathrm{CTL}}(\phi, V))$:

$(\Rightarrow)$ $\forall (\mathcal{M}', s') \in \mathit{Mod}(\mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \mathrm{AX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ for any sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ there is $(\mathcal{M}_1, s_1) \models \phi$, where $s_1$ is a directed successor of $s$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ with $s_2$ is a direct successor of $s_3$ and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow$ $(\mathcal{M}_3, s_3) \models \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$ and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow (\mathcal{M}', s') \models \mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$.

$(\Leftarrow)$ $\forall$ $(\mathcal{M}_3, s_3) \in \mathit{Mod}(\mathrm{AX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V)))$, then for any sub-structure $(\mathcal{M}_2, s_2)$ with $s_2$ is a directed successor of $s_3$ there is $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$
$\Rightarrow$ for any $(\mathcal{M}_2, s_2)$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ with $s_1$ is a direct successor of $s$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{AX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{AX}\phi, V)$.

(ii) In order to prove $\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V) \equiv \mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V)$, we only need to prove $\mathit{Mod}$ $(\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V)) = \mathit{Mod}(\mathrm{EX}\mathrm{F}_{\mathrm{CTL}}(\phi, V))$:

$(\Rightarrow)$ $\forall \mathcal{M}', s' \in \mathit{Mod}(\mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V))$ there exists an initial K-structure $(\mathcal{M}, s)$ s.t. $(\mathcal{M}, s) \models \mathrm{EX}\phi$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}', s')$
$\Rightarrow$ there is a sub-structure $(\mathcal{M}_1, s_1)$ of $(\mathcal{M}, s)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$, where $s_1$ is a directed successor of $s$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$ and $(\mathcal{M}_2, s_2) \leftrightarrow_V (\mathcal{M}_1, s_1)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}_3, s_3)$ by $(\mathcal{M}_2, s_2)$ s.t. $(\mathcal{M}_2, s_2)$ is a sub-structure of $(\mathcal{M}_3, s_3)$ that $s_2$ is a direct successor of $s_3$ and $(\mathcal{M}_3, s_3) \leftrightarrow_V (\mathcal{M}, s)$
$\Rightarrow (\mathcal{M}_3, s_3) \models \mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$
$\Rightarrow (\mathcal{M}', s') \models \mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V))$.

$(\Leftarrow)$ $\forall$ $(\mathcal{M}_3, s_3) \in \mathit{Mod}(\mathrm{EX}(\mathrm{F}_{\mathrm{CTL}}(\phi, V)))$, then there exists a sub-structure $(\mathcal{M}_2, s_2)$ of $(\mathcal{M}_3, s_3)$ s.t. $(\mathcal{M}_2, s_2) \models \mathrm{F}_{\mathrm{CTL}}(\phi, V)$
$\Rightarrow$ there is an initial K-structure $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1) \models \phi$ and $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}_2, s_2)$
$\Rightarrow$ it is easy to construct an initial K-structure $(\mathcal{M}, s)$ by $(\mathcal{M}_1, s_1)$ s.t. $(\mathcal{M}_1, s_1)$ is a sub-structure of $(\mathcal{M}, s)$ that $s_1$ is a direct successor of $s$ and $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_3, s_3)$
$\Rightarrow (\mathcal{M}, s) \models \mathrm{EX}\phi$ and then $(\mathcal{M}_3, s_3) \models \mathrm{F}_{\mathrm{CTL}}(\mathrm{EX}\phi, V)$.

(iii) and (iV) can be proved as (i) and (ii) respectively. □

**Proposition**7 **(Model Checking on Forgetting)** Let $(\mathcal{M}, s_0)$ be an initial K-structure, $\varphi$ be a CTL formula and $V$ a set of atoms. Deciding whether $(\mathcal{M}, s_0)$ is a model of $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is NP-complete.

*Proof.* The problem can be determined by the following two things: (1) guessing an initial K-structure $(\mathcal{M}', s_0')$ satisfying $\varphi$; and (2) checking if $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Both two steps can be done in polynomial time. Hence, the problem is in NP. The hardness follows that the model checking for propositional variable forgetting is NP-hard (Zhang and Zhou 2008). □

**Theorem** 5 **(Entailment on Forgetting)** Let $\varphi$ and $\psi$ be two $\mathrm{CTL}_{\mathrm{AF}}$ formulas and $V$ a set of atoms. Then, results:

(i) deciding $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models^? \psi$ is co-NP-complete,

(ii) deciding $\psi \models^? \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ is $\Pi_2^P$-complete,

(iii) deciding $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models^? \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ is $\Pi_2^P$-complete.

*Proof.* (i) It is known that deciding whether $\psi$ is satisfiable is NP-Complete (Meier et al. 2015). The hardness is easy to see by setting $\mathrm{F}_{\mathrm{CTL}}(\varphi, \mathit{Var}(\varphi)) \equiv \top$, i.e., deciding whether $\psi$ is valid. For membership, from Theorem 3, we have $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \models \psi$ iff $\varphi \models \psi$ and $\mathrm{IR}(\psi, V)$. Clearly, in $\mathrm{CTL}_{\mathrm{AF}}$, deciding $\varphi \models \psi$ is in co-NP. We show that deciding whether $\mathrm{IR}(\psi, V)$ is also in co-NP. Without loss of generality, we assume that $\psi$ is satisfiable. We consider the complement of the problem: deciding whether $\psi$ is not irrelevant to $V$. It is easy to see that $\psi$ is not irrelevant to $V$ iff there exist a model $(\mathcal{M}, s_0)$ of $\psi$ and an initial K-structure $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$ and $(\mathcal{M}', s_0') \not\models \psi$. So checking whether $\psi$ is not irrelevant to $V$ can be achieved in the following steps: (1) guess two initial K-structures $(\mathcal{M}, s_0)$ and $(\mathcal{M}', s_0')$ such that $(\mathcal{M}, s_0) \models \psi$ and $(\mathcal{M}', s_0') \not\models \psi$, and (2) check $(\mathcal{M}, s_0) \leftrightarrow_V (\mathcal{M}', s_0')$. Obviously, both (1) and (2) can be done in polynomial time.

(ii) Membership. We consider the complement of the problem. We may guess an initial K-structure $(\mathcal{M}, s_0)$ satisfying $\psi$ and check whether $(\mathcal{M}, s_0) \not\models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$. From Proposition 7, we know that this is in $\Sigma_2^P$. So the original problem is in $\Pi_2^P$. Hardness. Let $\psi \equiv \top$. Then the problem is reduced to decide $\mathrm{F}_{\mathrm{CTL}}(\varphi, V)$'s validity. Since a propositional variable forgetting is a special case of temporal forgetting, the hardness is directly followed from the proof of Proposition 24 in (Lang, Liberatore, and Marquis 2003).

(iii) Membership. If $\mathrm{F}_{\mathrm{CTL}}(\varphi, V) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$ then there exist an initial K-structure $(\mathcal{M}, s)$ such that $(\mathcal{M}, s) \models \mathrm{F}_{\mathrm{CTL}}(\varphi, V)$ but $(\mathcal{M}, s) \not\models \mathrm{F}_{\mathrm{CTL}}(\psi, V)$, i.e.,, there is $(\mathcal{M}_1, s_1) \leftrightarrow_V (\mathcal{M}, s)$ with $(\mathcal{M}_1, s_1) \models \varphi$ but $(\mathcal{M}_2, s_2) \not\models \psi$ for every $(\mathcal{M}_2, s_2)$ with $(\mathcal{M}, s) \leftrightarrow_V (\mathcal{M}_2, s_2)$. It is evident that guessing such $(\mathcal{M},s)$, $(\mathcal{M}_1,s_1)$ with $(\mathcal{M}_1,s_1)\leftrightarrow_V(\mathcal{M},s)$ and checking $(\mathcal{M}_1,s_1)\models\varphi$ are feasible while checking $(\mathcal{M}_2,s_2)\not\models\psi$ for every $(\mathcal{M},s)\leftrightarrow_V(\mathcal{M}_2,s_2)$ can be done in polynomial time. Thus the problem is in $\Pi_2^P$.

Hardness. It follows from (ii) due to the fact that $\mathrm{F}_{\mathrm{CTL}}(\varphi,V)\models\mathrm{F}_{\mathrm{CTL}}(\psi,V)$ iff $\varphi\models\mathrm{F}_{\mathrm{CTL}}(\psi,V)$ by $\mathrm{IR}(\mathrm{F}_{\mathrm{CTL}}(\psi,V),V)$. $\square$

**Proposition** 8 **(dual)** Let $V,q,\varphi$ and $\psi$ are like in Definition 4. The $\psi$ is a SNC (WSC) of $q$ on $V$ under $\varphi$ iff $\neg\psi$ is a WSC (SNC) of $\neg q$ on $V$ under $\varphi$.

*Proof.* (i) Suppose $\psi$ is the SNC of $q$. Then $\varphi\models q\rightarrow\psi$. Thus $\varphi\models\neg\psi\rightarrow\neg q$. So $\neg\psi$ is a SC of $\neg q$. Suppose $\psi'$ is any other SC of $\neg q$: $\varphi\models\psi'\rightarrow\neg q$. Then $\varphi\models q\rightarrow\neg\psi'$, this means $\neg\psi'$ is a NC of $q$ on $P$ under $\varphi$. Thus $\varphi\models\psi\rightarrow\neg\psi'$ by assumption. So $\varphi\models\psi'\rightarrow\neg\psi$. This proves that $\neg\psi$ is the WSC of $\neg q$. The proof of the other part of the proposition is similar.

(ii) The WSC case can be proved similarly with SNC case. $\square$

**Proposition** 9 Let $\Gamma$ and $\alpha$ be two formulas, $V\subseteq \mathit{Var}(\alpha)\cup\mathit{Var}(\Gamma)$ and $q$ is a new proposition not in $\Gamma$ and $\alpha$. Then, a formula $\varphi$ of $V$ is the SNC (WSC) of $\alpha$ on $V$ under $\Gamma$ iff it is the SNC (WSC) of $q$ on $V$ under $\Gamma'=\Gamma\cup\{q\leftrightarrow\alpha\}$.

*Proof.* We prove this for SNC. The case for WSC is similar. Let $\mathit{SNC}(\varphi,\alpha,V,\Gamma)$ denote that $\varphi$ is the SNC of $\alpha$ on $V$ under $\Gamma$, and $\mathit{NC}(\varphi,\alpha,V,\Gamma)$ denote that $\varphi$ is the NC of $\alpha$ on $V$ under $\Gamma$.

($\Rightarrow$) We will show that if $\mathit{SNC}(\varphi,\alpha,V,\Gamma)$ holds, then $\mathit{SNC}(\varphi,q,V,\Gamma')$ will be true. According to $\mathit{SNC}(\varphi,\alpha,V,\Gamma)$ and $\alpha\equiv q$, we have $\Gamma'\models q\rightarrow\varphi$, which means $\varphi$ is a NC of $q$ on $V$ under $\Gamma'$. Suppose $\varphi'$ is any NC of $q$ on $V$ under $\Gamma'$, then $\mathrm{F}_{\mathrm{CTL}}(\Gamma',q)\models\alpha\rightarrow\varphi'$ due to $\alpha\equiv q$, $\mathit{IR}(\alpha\rightarrow\varphi',\{q\})$ and (**PP**), i.e., $\Gamma\models\alpha\rightarrow\varphi'$ by Lemma 4, this means $\mathit{NC}(\varphi',\alpha,V,\Gamma)$. Therefore, $\Gamma\models\varphi\rightarrow\varphi'$ by the definition of SNC and $\Gamma'\models\varphi\rightarrow\varphi'$. Hence, $\mathit{SNC}(\varphi,q,V,\Gamma')$ holds.

($\Leftarrow$) We will show that if $\mathit{SNC}(\varphi,q,V,\Gamma')$ holds, then $\mathit{SNC}(\varphi,\alpha,V,\Gamma)$ will be true. According to $\mathit{SNC}(\varphi,q,V,\Gamma')$, it's not difficult to know that $\mathrm{F}_{\mathrm{CTL}}(\Gamma',\{q\})\models\alpha\rightarrow\varphi$ due to $\alpha\equiv q$, $\mathit{IR}(\alpha\rightarrow\varphi,\{q\})$ and (**PP**), i.e., $\Gamma\models\alpha\rightarrow\varphi$ by Lemma 4, this means $\mathit{NC}(\varphi,\alpha,V,\Gamma)$. Suppose $\varphi'$ is any NC of $\alpha$ on $V$ under $\Gamma$. Then $\Gamma'\models q\rightarrow\varphi'$ since $\alpha\equiv q$ and $\Gamma'=\Gamma\cup\{q\equiv\alpha\}$, which means $\mathit{NC}(\varphi',q,V,\Gamma')$. According to $\mathit{SNC}(\varphi,q,V,\Gamma')$, $\mathit{IR}(\varphi\rightarrow\varphi',\{q\})$ and (**PP**), we have $\mathrm{F}_{\mathrm{CTL}}(\Gamma',\{q\})\models\varphi\rightarrow\varphi'$, and $\Gamma\models\varphi\rightarrow\varphi'$ by Lemma 4. Hence, $\mathit{SNC}(\varphi,\alpha,V,\Gamma)$ holds. $\square$

**Theorem** 7 Let $\varphi$ be a formula, $V\subseteq\mathit{Var}(\varphi)$ and $q\in\mathit{Var}(\varphi)-V$.

(i) $\mathrm{F}_{\mathrm{CTL}}(\varphi\wedge q,(\mathit{Var}(\varphi)\cup\{q\})-V)$ is a SNC of $q$ on $V$ under $\varphi$.

(ii) $\neg\mathrm{F}_{\mathrm{CTL}}(\varphi\wedge\neg q,(\mathit{Var}(\varphi)\cup\{q\})-V)$ is a WSC of $q$ on $V$ under $\varphi$.

*Proof.* We will prove the SNC part, while it is not difficult to prove the WSC part according to Proposition 8. Let $\mathcal{F}=\mathrm{F}_{\mathrm{CTL}}(\varphi\wedge q,(\mathit{Var}(\varphi)\cup\{q\})-V)$.

The "NC" part: It's easy to see that $\varphi\wedge q\models\mathcal{F}$ by **(W)**. Hence, $\varphi\models q\rightarrow\mathcal{F}$, this means $\mathcal{F}$ is a NC of $q$ on $P$ under $\varphi$.

The "SNC" part: for all $\psi'$, $\psi'$ is the NC of $q$ on $V$ under $\varphi$, s.t. $\varphi\models\mathcal{F}\rightarrow\psi'$. Suppose that there is a NC $\psi$ of $q$ on $V$ under $\varphi$ and $\psi$ is not logic equivalence with $\mathcal{F}$ under $\varphi$, s.t. $\varphi\models\psi\rightarrow\mathcal{F}$. We know that $\varphi\wedge q\models\psi$ iff $\mathcal{F}\models\psi$ by **(PP)**, since $\mathit{IR}(\psi,(\mathit{Var}(\varphi)\cup\{q\})-V)$. Hence, $\varphi\wedge\mathcal{F}\models\psi$ by $\varphi\wedge q\models\psi$ (by suppose). We can see that $\varphi\wedge\psi\models\mathcal{F}$ by suppose. Therefore, $\varphi\models\psi\leftrightarrow\mathcal{F}$, which means $\psi$ is logic equivalence with $\mathcal{F}$ under $\varphi$. This is contradict with the suppose. Then $\mathcal{F}$ is the SNC of $q$ on $P$ under $\varphi$. $\square$

**Theorem** 8 Let $\mathcal{K}=(\mathcal{M},s)$ be an initial K-structure with $\mathcal{M}=(S,R,L,s_0)$ on the set $\mathcal{A}$ of atoms, $V\subseteq\mathcal{A}$ and $q\in V'=\mathcal{A}-V$. Then:

(i) the SNC of $q$ on $V$ under $\mathcal{K}$ is $\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K})\wedge q,V')$.

(ii) the WSC of $q$ on $V$ under $\mathcal{K}$ is $\neg\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K})\wedge\neg q,V')$.

*Proof.* (i) As we know that any initial K-structure $\mathcal{K}$ can be described as a characterizing formula $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$, then the SNC of $q$ on $V$ under $\mathcal{F}_{\mathcal{A}}(\mathcal{K})$ is $\mathrm{F}_{\mathrm{CTL}}(\mathcal{F}_{\mathcal{A}}(\mathcal{K})\wedge q,\mathcal{A}-V)$.

(ii) This is proved by the dual property. $\square$

**Proposition**10 Let $\varphi$ be a CTL formula and $V\subseteq\mathcal{A}$ with $|\mathcal{S}|=m$, $|\mathcal{A}|=n$ and $|V|=x$. The the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity of Algorithm 1 is at least the same as the space.

*Proof.* Supposing each state or atom occupy one byte, then a state pair $(s,s')$ occupy two bytes. For any $B\subseteq\mathcal{S}$ with $B\neq\emptyset$ and $s_0\in B$, we can construct an initial K-structure $(\mathcal{M},s_0)$ with $\mathcal{M}=(B,R,L,s_0)$, in which there is at most $\frac{|B|^2}{2}$ state pairs in $R$ and $|B|{*}n$ $(s,A)$ $(A\subseteq\mathcal{A})$ in $L$. Hence, the $(\mathcal{M},s_0)$ occupy at most $|B|+|B|^2+|B|*n$ bytes. Besides, there is at most $|B|^{|B|+1}*2^{nm}$ number of initial K-structures. Therefore, there is at most $m^{m+2}*2^{nm}$ number of initial K-structures, hence it will at most cost $m^{m+2}*2^{nm}*(m+m^2+nm)$ bytes.

Let $k=n-x$, for any initial K-structure $\mathcal{K}=(\mathcal{M},s_0)$ with $i\geq 1$ nodes, in the worst, i.e., $ch(\mathcal{M},V)=i$, we will spend $N(i)$ space to store the characterizing formula.

$$
\begin{aligned}
N(i)&=(k+(\cdots+(k+2ik)*(2i))\cdots*(2i))\\
&=(2i)^0k+2ik+(2i)^2k+\cdots+(2i)^{(i-1)}k\\
&=\frac{(2i)^i-1}{2i-1}k.
\end{aligned}
$$

In the worst case, i.e., there is $m^{m+2}*2^{nm}$ initial K-structures with $m$ nodes, we will spent $m^{m+2}*2^{nm}*N(m)$ bytes to store the result of forgetting.

Therefore, the space complexity is $O((n-x)m^{2(m+1)}2^{nm})$ and the time complexity is at least the same as the space. $\square$