Screening of Coulomb interaction

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To what extent is the perturbation theory based on Fermi gas a valid theory? This amounts to determining a threshold whereby Columb interaction between electrons becomes effectively negligible. This can be done by simply estimating the ratio between two length scales: (i) the interparticle spacing associated with Kinetic energy or Fermi momentum, and (ii) the effective Bohr radius associated with Columb potential energy. The length scale of the kinetic energy of an electron, i.e. the interparticle spacing, is set by Fermi momentum k_F according to:

$$r_0 \sim k_F^{-1} \tag{1}$$

It is readily to check this by counting the states inside the Fermi surface of a 3D Fermi gas: $k_F = \left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{1/3}$, where g counts the spin degeneracy, and the interparticle spacing defined by $r_0 = (V/N)^{\frac{1}{3}}$ scales linearly with k_F^{-1}. One the other hand, the length scale of the potential energy of an electron is set by the effective Bohr radius a_0:

$$\frac{\hbar^2}{ma_0^2} \sim \frac{e^2}{a_0} \;\Rightarrow\; a_0 \sim \frac{\hbar^2}{me^2} \tag{2}$$

where m is the effective mass of an electron. Define the dimensionless ratio r_s:

$$r_s \equiv \frac{r_0}{a_0} \sim \frac{1}{k_F a_0} \tag{3}$$

which compares the interparicle spacing against the effective Bohr radius, thus it can be perceived as an effective spacing between Bloch electrons. From this we immediately discern two regimes:

$$\begin{aligned} \text{small } r_s \;&\leftrightarrow\; \text{high density} \rightarrow \text{ Potential } \ll \text{ Kinetic} \\ \text{large } r_s \;&\leftrightarrow\; \text{low density} \rightarrow \text{ Potential } \gg \text{ Kinetic} \end{aligned}$$

Let us now discuss how are these two regimes manifasted in the Hamiltonian:

$$H \sim \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i\neq j}\frac{e^2}{x_{ij}} \tag{4}$$

where for convenience we set $\hbar = 1$. Let us switch to a dimenson where aforementioned length scales become explicit. We define

$$\tilde{x} = \frac{x}{r_s a_0}, \quad \tilde{p} = (r_s a_0)p \tag{5}$$

that is, they are renormalized by the Fermi momentum $k_F^{-1} = r_s a_0$, whereby the canonical commutator $[\tilde{x}, \tilde{p}] = i$ retains the same form. With this, the Hamiltonian can be rewritten as

$$H \sim \frac{1}{r_s^2} \underbrace{\left(\frac{e^2}{a_0}\right)}_{1 \text{ Rydberg}} \left(\sum_{i=1}^{N}\frac{\tilde{p}_i^2}{2} + r_s \sum_{i\neq j}\frac{1}{\tilde{x}_{ij}}\right) \tag{6}$$

where we used $e^2 \sim \frac{1}{ma_0}$ from Eq. 2 with $\hbar = 1$. It is now clear that for small $r_s \ll 1$, i.e. high density, the potential energy due to Coulumb interaction can be ignored. In contrast, for $r_s \gg 1$, i.e. low density, the Coulumb interaction leads to the formation of a crystal of electrons (Wigner crystal).

Beyond this estimation of length scales, the detailed discussion of Coulumb interaction in Fermion gas can be found in [1]. It is shown that the energy of interaction scales with density according to $P \sim \rho^{1/3}$, while the kinetic energy scales faster as a function of density $K \sim \rho^{2/3}$. At the critical point $P = K$ we have

$$a_0 \rho_c^{1/3} \sim 0.2 \tag{7}$$

which is called the Mott-Wigner criterion for the electron gas instability – the estimate for the electron localization threshold of the Fermi gas.

* This brief note is based on Prof. Mohit Randeria's discussion in PHYS8820, OSU.

References

[1] Honig, J. & Spałek, J. Chapter 16 - an elementary examination of quantum phase transitions involving fermions∗. In Honig, J. & Spałek, J. (eds.) *A Primer to the Theory of Critical Phenomena*, 203–220 (Elsevier, Amsterdam, 2018). URL https://www.sciencedirect.com/science/article/pii/B978012804685200022X.