A Homomorphism between SL(2,C) and Lorentz Group

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A homomorphism exist between $SL(2,\mathbb{C})$ and Lorentz group. Let $M = \mathbb{R}^4$ be the 4-d Minkowski space with Lorentz metric:

$$\|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad \text{with} \quad x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1}$$

that is, M is the ordinary Minkowski space of special relativity, and we have chosen the speed of light to be unity. A Lorentz transformation, B, is a linear transformation of M into itself which preserves the Lorentz metric:

$$\|Bx\|^2 = \|x\|^2, \quad \text{for all } x \in M \tag{2}$$

We let L denote the group of all Lorentz transformations, and call it Lorentz group.

We now decribe a homomorphism from $SL(2,\mathbb{C})$ to L. For this purpose we shall identity every point x in M with a 2-by-2 self-adjoint matrix:

$$x := \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0\mathbb{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \tag{3}$$

which satisfies $x^\dagger = x$ and $\det(x) = \|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. In this notation, we have $x_0 = 1/2\,\mathrm{tr}(x)$, $x_3 = 1/2\tilde{\mathrm{tr}}(x)$ where $\tilde{tr}$ means the difference of the diagonal.

Now let A be any $2-by-2$ matrix. We define the action of the matrix A on the self-adjoint matrix x by

$$x \to AxA^\dagger \equiv \phi(A)x$$

where we denoted the corresponding concrete action on the vector x by $\phi(A)x$. The nicity can be seen from the fact

$$\left(AxA^\dagger\right)^\dagger = A^{\dagger\dagger}x^\dagger A^\dagger = AxA^\dagger.$$

so the new $AxA^\dagger$ is also self-adjoint. Notice also that

$$\|\phi(A)x\|^2 = \det\left(AxA^\dagger\right) = |\det(A)|^2 \det(x) \tag{4}$$

is $A \in SL(2,\mathbb{C})$, then

$$\|\phi(A)x\|^2 = \|x\|^2 \tag{5}$$

Therefore if A is in $SL(2,\mathbb{C})$, $\phi(A)$ represents a Lorentz transformation. Notice also that

$$ABx(AB)^\dagger = ABxB^\dagger A^\dagger = A(BxB^\dagger)A^\dagger.$$

so that

$$\phi(AB)x = \phi(A)\phi(B)x \tag{6}$$

Thus ϕ is a homomorphism! Also note that $\phi(-A) = \phi(A)$ so this map is not one-to-one, i.e. $\pm A$ shall be mapped to the same Lorentz transformation by ϕ.