Solutions of A Probabilit Path

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1 Solutions to Chapter 1: Sets and Events

1.9.1 $\forall B \in \aleph$, since $\mathcal{C} \subset B$, we have $\{0\} \in B$, therefore $\Omega \setminus \{0\} = \{1\} \in B$. Also $\emptyset \in B$ and $\Omega \in B$. Therefore $\{\emptyset, \{0\}, \{1\}, \Omega\} \subset B$. Note that $\mathcal{P}(\Omega) = \{\emptyset, \{0\}, \{1\}, \Omega\}$. This means

$$\aleph = \{\mathcal{P}(\Omega)\}$$

1.9.2 Like in 1.9.1, we can conclude that

$$\forall B \in \aleph \quad \Rightarrow \{\emptyset, \{0\}, \{1,2\}, \Omega\} \subset B$$

Also note that $\{\emptyset, \{0\}, \{1,2\}, \Omega\}$ is a σ-field itself which means

$$\sigma(\mathcal{C}) = \{\emptyset, \{0\}, \{1,2\}, \Omega\}$$

Those subsets of Ω which are not included in $\sigma(\mathcal{C})$ are

$$\{1\}, \quad \{2\}, \quad \{0,1\}, \quad \{0,2\}$$

and it's easy to check that they are all included in B if any one of them is inclued. So to sum up, we have

$$\aleph = \{\sigma(\mathcal{C}), \mathcal{P}(\Omega)\}$$

1.9.3 Firstly

$$\begin{aligned}
\limsup_{n\to\infty} A_n \cup B_n &= \left\{ x \middle| \sum_{n=1}^{\infty} 1_{A_n \cup B_n}(x) = \infty \right\} \\
&= \left\{ x \middle| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \quad \text{or} \quad \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right\} \\
&= \left\{ x \middle| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right\} \cup \left\{ x \middle| \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right\} \\
&= \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n
\end{aligned}$$

Secondly, the statement

$$A_n \cup B_n \to A \cup B, \quad A_n \cap B_n \to A \cap B$$

is true if $A_n \to A$ and $B_n \to B$. Because we have

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \lim_{n\to\infty} A_n = A$$
$$\limsup_{n\to\infty} B_n = \liminf_{n\to\infty} B_n = \lim_{n\to\infty} B_n = B$$

Using the result of the first problem we can deduce that

$$\limsup_{n\to\infty} A_n \cup B_n = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n = A \cup B$$

We now have to show that

$$\liminf_{n\to\infty} A_n \cup B_n = \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n = A \cup B$$

Or equally

$$\limsup_{n\to\infty} A_n \cup B_n \subset \liminf_{n\to\infty} A_n \cup B_n$$

$$\begin{aligned} x \in \limsup_{n\to\infty} A_n \cup B_n \iff x \in A \cup B \iff \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n \\ \iff \{x \notin A_n, \text{finitely}\} \text{ or } \{x \notin B_n, \text{finitely}\} \\ \implies \{x \notin A_n \cup B_n, \text{finitely}\} \iff x \in \liminf_{n\to\infty} A_n \cup B_n \end{aligned}$$

This means $\forall x \in \limsup_{n\to\infty} A_n \cup B_n$, we have that $x \in \liminf_{n\to\infty} A_n \cup B_n$, therefore

$$\limsup_{n\to\infty} A_n \cup B_n \subset \liminf_{n\to\infty} A_n \cup B_n$$

which means

$$A_n \cup B_n \to A \cup B$$

and

$$A_n \cap B_n = (A_n^c \cup B_n^c)^c \to (A^c \cup B_c)^c = A \cap B$$

1.9.4

$$\begin{aligned} \liminf_{n\to\infty} A_n &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\ &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ \frac{m}{k} : m \in \mathbb{N} \right\} \\ &= \bigcup_{n=1}^{\infty} \mathbb{N} = \mathbb{N} \end{aligned}$$

$$\begin{aligned} \limsup_{n\to\infty} A_n &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\ &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{ \frac{m}{k} : m \in \mathbb{N} \right\} \\ &= \bigcap_{n=1}^{\infty} \mathbb{Q}^+ = \mathbb{Q}^+ \end{aligned}$$

1.9.5

$$
\begin{aligned}
& \{\omega : f_n(\omega) \nrightarrow f(\omega)\} \\
\iff & \{\omega : \exists\, \epsilon > 0,\ \text{s.t. } \forall N,\ \exists\, n > N,\ \text{s.t. } |f_n(\omega) - f(\omega)| > \epsilon\} \\
\iff & \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\omega : |f_n(\omega) - f(\omega)| > \frac{1}{k}\right\}
\end{aligned}
$$

1.9.6 Use Lemma 1.3.1, we can conclude that

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = (0,\ 1]$$

1.9.7

1. Since $\theta = 1/8$, the period is $T = 8$. And there are actually 2 distinguished squares. Hence $\limsup\limits_{n\to\infty} I_n$ is the star area covered by at least one squate and $\liminf\limits_{n\to\infty} I_n$ is the area covered by both squares. Refer to Figure 1 as illustration.

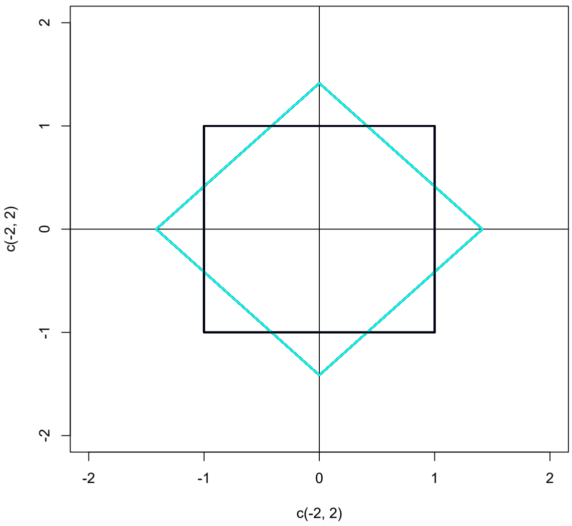

Figure 1: (a)

2. If θ is rational, then it can be written in the form $\theta = \frac{m}{n}$ where both m and n are integers, which means there is a period in I_n. Hence like before, $\limsup\limits_{n\to\infty} I_n$ is the star area covered by at least one squate and $\liminf\limits_{n\to\infty} I_n$ is the area covered by all squares. Refer to Figure 2 as illustration.
3. If θ is irrational. These squares becomes dense and $\limsup\limits_{n\to\infty} I_n$ is the round area with radius $r_{\text{sup}} = \sqrt{2}$ and $\liminf\limits_{n\to\infty} I_n$ is the round area with radius $r_{\text{inf}} = 1$. Refer to Figure as illustration.
4. Codes for drawing these figures are provided below:

```
x <- 1
y <- 1
```

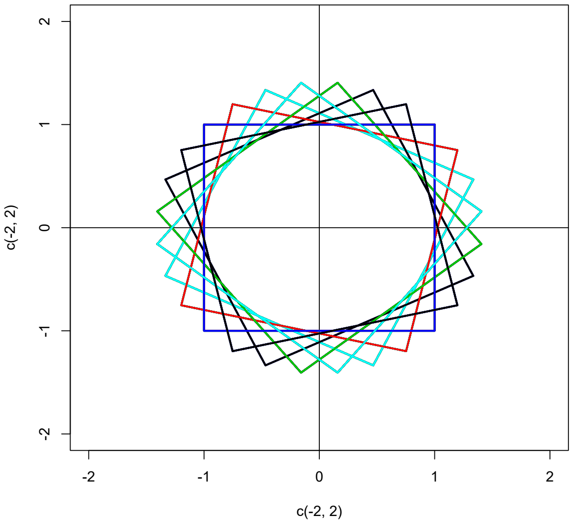

Figure 2: (b) $\theta = \frac{1}{7}$

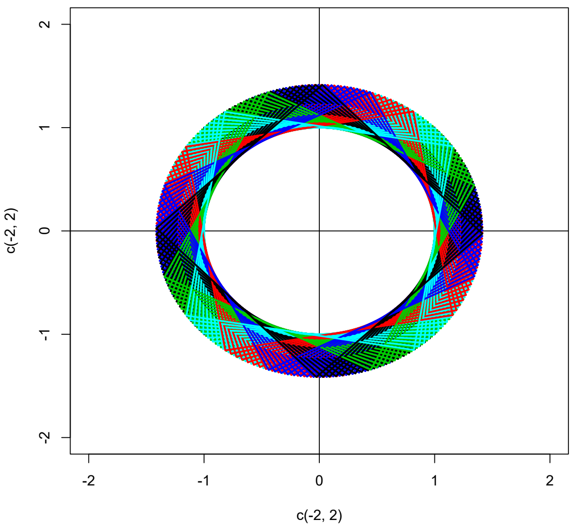

Figure 3: (c) $\theta = e^{1/2}$

```r
###### sample codes, set theta as you wish. ######
theta <- 2/5
n <- 100
plot(c(-2,2),c(-2,2),type="n")
abline(v=0)
abline(h=0)

for(i in 0:n) {
  angle <- complex(real = cos(2*pi*theta*i),
                   imaginary = sin(2*pi*theta*i))
  point <- complex(real = x, imaginary = y)
  point0 <- point * angle
  x1 <- Re(point0)
  y1 <- Im(point0)
  segments(x1,y1,-y1,x1,col=(i %% 5)+1,lwd = 2)
  segments(-y1,x1,-x1,-y1,col=(i %% 5)+1,lwd = 2)
  segments(-x1,-y1,y1,-x1,col=(i %% 5)+1,lwd = 2)
  segments(y1,-x1,x1,y1,col=(i %% 5)+1,lwd = 2)
}

###### (a), (b) and (c) ######
theta.seq <- c(1/8, 1/7, exp(0.5))
n <- 100
for(ind in 1:3) {
  theta <- theta.seq[ind]
  fig.url <- paste("../Figures/1.9.7.",
                   letters[ind], ".pdf", sep = "")
  cairo_pdf(fig.url)
  plot(c(-2,2),c(-2,2),type="n")
  abline(v=0)
  abline(h=0)
  for(i in 0:n) {
    angle <- complex(real = cos(2*pi*theta*i),
                     imaginary = sin(2*pi*theta*i))
    point <- complex(real = x, imaginary = y)
    point0 <- point * angle
    x1 <- Re(point0)
    y1 <- Im(point0)
    segments(x1,y1,-y1,x1,col=(i %% 5)+1,lwd = 2)
    segments(-y1,x1,-x1,-y1,col=(i %% 5)+1,lwd = 2)
    segments(-x1,-y1,y1,-x1,col=(i %% 5)+1,lwd = 2)
    segments(y1,-x1,x1,y1,col=(i %% 5)+1,lwd = 2)
  }
  dev.off()
}
```

1.9.8
$$\text{limsup}A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = B \cup C$$

and
$$\text{liminf}A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = B \cap C$$

1.9.9
$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (A \cap B^c) \cup (B \cap A^c) \\ &= (B^c \cap (A^c)^c) \cup (A^c \cap (B^c)^c) \\ &= (B^c \setminus A^c) \cup (A^c \setminus B^c) \\ &= A^c \triangle B^c \end{aligned}$$

1.9.10 $\Longrightarrow$:

Since $A_n \to A$, we have
$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$$

If $w \in A \quad \Longrightarrow \quad w \in \liminf_{n\to\infty} A_n$, then

$$\begin{aligned} &\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad w \in A_n \\ \Longrightarrow\ &\forall\, n \geq n_0, 1_{A_n}(w) = 1 \\ \Longrightarrow\ &\lim_{n\to\infty} 1_{A_n}(w) = 1 = 1_A(w) \end{aligned}$$

And if $w \in A^c$, then $w \in \left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c$, which means

$$\begin{aligned} &\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad w \in A_n^c \\ \Longrightarrow\ &\forall\, n \geq n_0, 1_{A_n}(w) = 0 \\ \Longrightarrow\ &\lim_{n\to\infty} 1_{A_n}(w) = 0 = 1_A(w) \end{aligned}$$

Hence $A_n \to A \implies 1_{A_n} \to 1_A$.

$\Longleftarrow$: If $1_A(w) = 1$, then $\lim_{n\to\infty} 1_{A_n}(w) = 1$, which means

$$\begin{aligned} &\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad 1_{A_n}(w) = 1 \\ \Longrightarrow\ &\forall\, n \geq n_0, \quad w \in A_n \\ \Longrightarrow\ &w \in \liminf_{n\to\infty} A_n \\ \Longrightarrow\ &A \subset \liminf_{n\to\infty} A_n \end{aligned}$$

If $1_A(w) = 0$, then $\lim_{n\to\infty} 1_{A_n}(w) = 0$, which means

$$
\begin{aligned}
&\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad 1_{A_n}(w) = 0 \\
\Longrightarrow\; &\forall\, n \geq n_0, \quad w \in A_n^c \\
\Longrightarrow\; &w \in \liminf_{n\to\infty} A_n^c = \left(\limsup_{n\to\infty} A_n\right)^c \\
\Longrightarrow\; &A^c \subset \left(\limsup_{n\to\infty} A_n\right)^c \\
\Longrightarrow\; &A \subset \limsup_{n\to\infty} A_n
\end{aligned}
$$

Therefore $\limsup_{n\to\infty} A_n \subset \liminf_{n\to\infty} A_n$, which means

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$$

Therefore $1_{A_n} \to 1_A \implies A_n \to A$.

1.9.11

$$
\begin{aligned}
&w \in \sup_{n\geq 1}\, [0,\ a_n) \\
\Longrightarrow\; &\exists\, n_0,\ \text{s.t. } w \in [0,\ a_{n_0}) \\
\Longrightarrow\; &w \in \left[0,\ \sup_{n\geq 1} a_n\right) \qquad (\text{since } \sup_{n\geq 1} a_n \geq a_{n_0}) \\
\Longrightarrow\; &\sup_{n\geq 1}\, [0,\ a_n) \subset \left[0,\ \sup_{n\geq 1} a_n\right)
\end{aligned}
$$

Also note that

$$
\begin{aligned}
&w \in \left[0,\ \sup_{n\geq 1} a_n\right) \implies w < \sup_{n\geq 1} a_n \\
\Longrightarrow\; &\exists\, a_{n_0},\ \text{s.t. } w < a_{n_0} \\
\Longrightarrow\; &w \in [0,\ a_{n_0}) \\
\Longrightarrow\; &w \in \sup_{n\geq 1}\, [0,\ a_n) \\
\Longrightarrow\; &\left[0,\ \sup_{n\geq 1} a_n\right) \subset \sup_{n\geq 1}\, [0,\ a_n)
\end{aligned}
$$

Therefore

$$\sup_{n\geq 1}\, [0,\ a_n) = \left[0,\ \sup_{n\geq 1} a_n\right)$$

For the second part, the left hand side equals to

$$\sup_{n\geq 1} \left[0,\ \frac{n}{n+1}\right] = [0,\ 1)$$

while the right hand side equals to

$$\left[0,\ \frac{n}{n+1}_{n\geq 1}\right] = [0, 1]$$

Clearly, lhs $\neq$ rhs.

1.9.12

$$\mathcal{A}(\mathcal{C}) = \sigma(\mathcal{C}) = \left\{ \begin{array}{l} \emptyset,\ \Omega,\ \{2,\ 4\},\ \{6\} \\ \{1,\ 3,\ 5,\ 6\},\ \{1,\ 2,\ 3,\ 4,\ 5\} \\ \{2,\ 4,\ 6\},\ \{1,\ 3,\ 5\} \end{array} \right\}$$

1.9.13 To be added.