Solutions of A Probabilit Path

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Contents

1 Solutions to Chapter 1: Sets and Events

1.9.1 $\forall B \in \aleph$, since $\mathcal{C} \subset B$, we have $\{0\} \in B$, therefore $\Omega \setminus \{0\} = \{1\} \in B$. Also $\emptyset \in B$ and $\Omega \in B$. Therefore $\{\emptyset, \{0\}, \{1\}, \Omega\} \subset B$. Note that $\mathcal{P}(\Omega) = \{\emptyset, \{0\}, \{1\}, \Omega\}$. This means

$$\aleph = \{\mathcal{P}(\Omega)\}$$

1.9.2 Like in 1.9.1, we can conclude that

$$\forall B \in \aleph \quad \Rightarrow \{\emptyset, \{0\}, \{1,2\}, \Omega\} \subset B$$

Also note that $\{\emptyset, \{0\}, \{1,2\}, \Omega\}$ is a σ-field itself which means

$$\sigma(\mathcal{C}) = \{\emptyset, \{0\}, \{1,2\}, \Omega\}$$

Those subsets of Ω which are not included in $\sigma(\mathcal{C})$ are

$$\{1\}, \quad \{2\}, \quad \{0,1\}, \quad \{0,2\}$$

and it's easy to check that they are all included in B if any one of them is inclued. So to sum up, we have

$$\aleph = \{\sigma(\mathcal{C}), \mathcal{P}(\Omega)\}$$

1.9.3 Firstly

$$\begin{aligned}
\limsup_{n\to\infty} A_n \cup B_n &= \left\{ x \left| \sum_{n=1}^{\infty} 1_{A_n \cup B_n}(x) = \infty \right. \right\} \\
&= \left\{ x \left| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \quad \text{or} \quad \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right. \right\} \\
&= \left\{ x \left| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right. \right\} \cup \left\{ x \left| \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right. \right\} \\
&= \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n
\end{aligned}$$

Secondly, the statement

$$A_n \cup B_n \to A \cup B, \quad A_n \cap B_n \to A \cap B$$

is true if $A_n \to A$ and $B_n \to B$. Because we have

$$\begin{aligned}
\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \lim_{n\to\infty} A_n = A \\
\limsup_{n\to\infty} B_n = \liminf_{n\to\infty} B_n = \lim_{n\to\infty} B_n = B
\end{aligned}$$

Using the result of the first problem we can deduce that

$$\limsup_{n\to\infty} A_n \cup B_n = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n = A \cup B$$

We now have to show that

$$\liminf_{n\to\infty} A_n \cup B_n = \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n = A \cup B$$

Or equally

$$\limsup_{n\to\infty} A_n \cup B_n \subset \liminf_{n\to\infty} A_n \cup B_n$$

$$\begin{aligned}
&x \in \limsup_{n\to\infty} A_n \cup B_n \iff x \in A \cup B \iff \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n \\
\iff &\{x \notin A_n, \text{ finitely}\} \text{ or } \{x \notin B_n, \text{ finitely}\} \\
\implies &\{x \notin A_n \cup B_n, \text{ finitely}\} \iff x \in \liminf_{n\to\infty} A_n \cup B_n
\end{aligned}$$

This means $\forall x \in \limsup_{n\to\infty} A_n \cup B_n$, we have that $x \in \liminf_{n\to\infty} A_n \cup B_n$, therefore

$$\limsup_{n\to\infty} A_n \cup B_n \subset \liminf_{n\to\infty} A_n \cup B_n$$

which means

$$A_n \cup B_n \to A \cup B$$

and

$$A_n \cap B_n = (A_n^c \cup B_n^c)^c \to (A^c \cup B_c)^c = A \cap B$$

1.9.4

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{\frac{m}{k} : m \in \mathbb{N}\right\} \\
&= \bigcup_{n=1}^{\infty} \mathbb{N} = \mathbb{N} \\
\limsup_{n\to\infty} A_n &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
&= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{\frac{m}{k} : m \in \mathbb{N}\right\} \\
&= \bigcap_{n=1}^{\infty} \mathbb{Q}^+ = \mathbb{Q}^+
\end{aligned}$$

1.9.5

$$\begin{aligned}
&\{\omega : f_n(\omega) \nrightarrow f(\omega)\} \\
\iff &\{\omega : \exists\, \epsilon > 0, \text{ s.t. } \forall N, \exists\, n > N, \text{ s.t. } |f_n(\omega) - f(\omega)| > \epsilon\} \\
\iff &\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\omega : |f_n(\omega) - f(\omega)| > \frac{1}{k}\right\}
\end{aligned}$$

1.9.6 Use Lemma 1.3.1, we can conclude that

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = (0,\ 1]$$

1.9.7 1. Since $\theta = 1/8$, the period is $T = 8$. And there are actually 2 distinguished squares. Hence $\limsup_{n\to\infty} I_n$ is the star area covered by at least one squate and $\liminf_{n\to\infty} I_n$ is the area covered by both squares. Refer to Figure 1 as illustration.

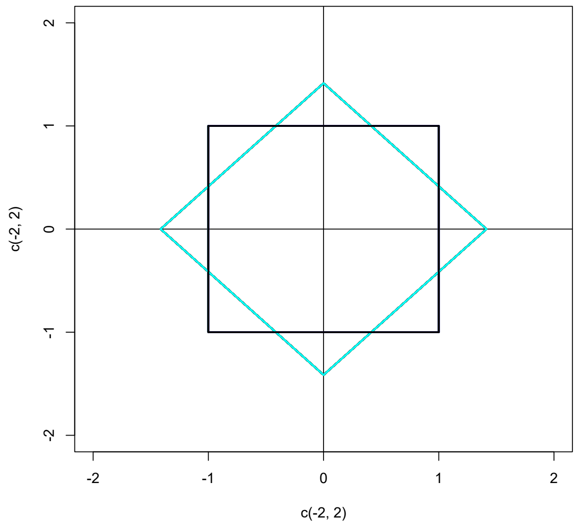

Figure 1: (a) $\theta = \frac{1}{8}$

2. If θ is rational, then it can be written in the form $\theta = \frac{m}{n}$ where both m and n are integers, which means there is a period in I_n. Hence like before, $\limsup\limits_{n\to\infty} I_n$ is the star area covered by at least one squate and $\liminf\limits_{n\to\infty} I_n$ is the area covered by all squares. Refer to Figure 2 as illustration.

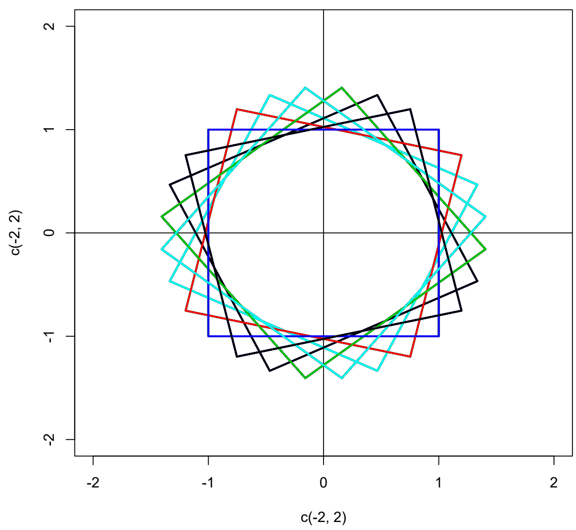

Figure 2: (b) $\theta = \frac{1}{7}$

3. If θ is irrational. These squares becomes dense and $\limsup\limits_{n\to\infty} I_n$ is the round area with radius $r_{\text{sup}} = \sqrt{2}$ and $\liminf\limits_{n\to\infty} I_n$ is the round area with radius $r_{\text{inf}} = 1$. Refer to Figure 3 as illustration.
4. Codes for drawing these figures are provided below:

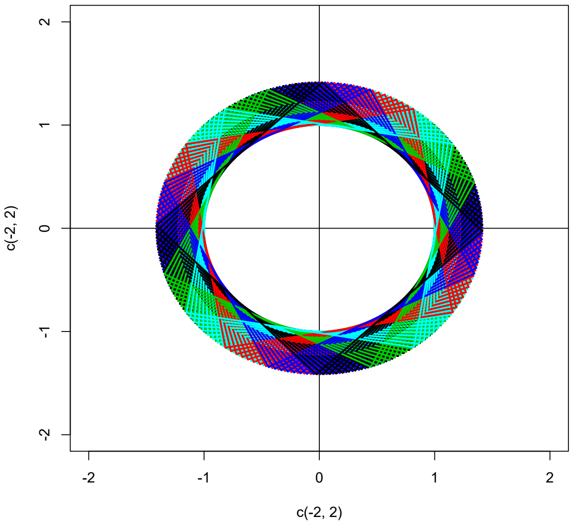

Figure 3: (c) $\theta = e^{1/2}$

```
x <- 1
y <- 1

###### sample codes, set theta as you wish. ######
theta <- 2/5
n <- 100
plot(c(-2,2),c(-2,2),type="n")
abline(v=0)
abline(h=0)

for(i in 0:n) {
  angle <- complex(real = cos(2*pi*theta*i),
                   imaginary = sin(2*pi*theta*i))
  point <- complex(real = x, imaginary = y)
  point0 <- point * angle
  x1 <- Re(point0)
  y1 <- Im(point0)
  segments(x1,y1,-y1,x1,col=(i %% 5)+1,lwd = 2)
  segments(-y1,x1,-x1,-y1,col=(i %% 5)+1,lwd = 2)
  segments(-x1,-y1,y1,-x1,col=(i %% 5)+1,lwd = 2)
  segments(y1,-x1,x1,y1,col=(i %% 5)+1,lwd = 2)
}

###### (a), (b) and (c) ######
theta.seq <- c(1/8, 1/7, exp(0.5))
n <- 100
for(ind in 1:3) {
  theta <- theta.seq[ind]
  fig.url <- paste("../Figures/1.9.7.",
```

```
                    letters[ind], ".pdf", sep = "")
    cairo_pdf(fig.url)
    plot(c(-2,2),c(-2,2),type="n")
    abline(v=0)
    abline(h=0)
    for(i in 0:n) {
      angle <- complex(real = cos(2*pi*theta*i),
                       imaginary = sin(2*pi*theta*i))
      point <- complex(real = x, imaginary = y)
      point0 <- point * angle
      x1 <- Re(point0)
      y1 <- Im(point0)
      segments(x1,y1,-y1,x1,col=(i %% 5)+1,lwd = 2)
      segments(-y1,x1,-x1,-y1,col=(i %% 5)+1,lwd = 2)
      segments(-x1,-y1,y1,-x1,col=(i %% 5)+1,lwd = 2)
      segments(y1,-x1,x1,y1,col=(i %% 5)+1,lwd = 2)
    }
    dev.off()
  }
```

1.9.8

$$\text{limsup} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = B \cup C$$

and

$$\text{liminf} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = B \cap C$$

1.9.9

$$\begin{aligned}
A \triangle B &= (A \setminus B) \cup (B \setminus A) \\
&= (A \cap B^c) \cup (B \cap A^c) \\
&= (B^c \cap (A^c)^c) \cup (A^c \cap (B^c)^c) \\
&= (B^c \setminus A^c) \cup (A^c \setminus B^c) \\
&= A^c \triangle B^c
\end{aligned}$$

1.9.10 $\Longrightarrow$:

Since $A_n \to A$, we have

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$$

If $w \in A \quad \Longrightarrow \quad w \in \liminf_{n\to\infty} A_n$, then

$$\begin{aligned}
&\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad w \in A_n \\
\Longrightarrow\, &\forall\, n \geq n_0, 1_{A_n}(w) = 1 \\
\Longrightarrow\, &\lim_{n\to\infty} 1_{A_n}(w) = 1 = 1_A(w)
\end{aligned}$$

And if $w \in A^c$, then $w \in \left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c$, which means

$$\begin{aligned}
&\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad w \in A_n^c \\
\Longrightarrow\ &\forall\, n \geq n_0, 1_{A_n}(w) = 0 \\
\Longrightarrow\ &\lim_{n\to\infty} 1_{A_n}(w) = 0 = 1_A(w)
\end{aligned}$$

Hence $A_n \to A \implies 1_{A_n} \to 1_A$.

$\Longleftarrow$: If $1_A(w) = 1$, then $\lim_{n\to\infty} 1_{A_n}(w) = 1$, which means

$$\begin{aligned}
&\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad 1_{A_n}(w) = 1 \\
\Longrightarrow\ &\forall\, n \geq n_0, \quad w \in A_n \\
\Longrightarrow\ &w \in \liminf_{n\to\infty} A_n \\
\Longrightarrow\ &A \subset \liminf_{n\to\infty} A_n
\end{aligned}$$

If $1_A(w) = 0$, then $\lim_{n\to\infty} 1_{A_n}(w) = 0$, which means

$$\begin{aligned}
&\exists\, n_0, \quad \text{s.t. } \forall\, n \geq n_0, \quad 1_{A_n}(w) = 0 \\
\Longrightarrow\ &\forall\, n \geq n_0, \quad w \in A_n^c \\
\Longrightarrow\ &w \in \liminf_{n\to\infty} A_n^c = \left(\limsup_{n\to\infty} A_n\right)^c \\
\Longrightarrow\ &A^c \subset \left(\limsup_{n\to\infty} A_n\right)^c \\
\Longrightarrow\ &A \subset \limsup_{n\to\infty} A_n
\end{aligned}$$

Therefore $\limsup_{n\to\infty} A_n \subset \liminf_{n\to\infty} A_n$, which means

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$$

Therefore $1_{A_n} \to 1_A \implies A_n \to A$.

1.9.11

$$\begin{aligned}
&w \in \sup_{n\geq 1}\,[0,\ a_n) \\
\Longrightarrow\ &\exists\, n_0,\ \text{s.t. } w \in [0,\ a_{n_0}) \\
\Longrightarrow\ &w \in \left[0,\ \sup_{n\geq 1} a_n\right) \qquad (\text{since } \sup_{n\geq 1} a_n \geq a_{n_0}) \\
\Longrightarrow\ &\sup_{n\geq 1}\,[0,\ a_n) \subset \left[0,\ \sup_{n\geq 1} a_n\right)
\end{aligned}$$

Also note that

$$\begin{aligned}
&w \in \left[0,\ \sup_{n\geq 1} a_n\right) \implies w < \sup_{n\geq 1} a_n \\
\Longrightarrow\ &\exists\, a_{n_0},\ \text{s.t. } w < a_{n_0} \\
\Longrightarrow\ &w \in [0,\ a_{n_0}) \\
\Longrightarrow\ &w \in \sup_{n\geq 1}\,[0,\ a_n) \\
\Longrightarrow\ &\left[0,\ \sup_{n\geq 1} a_n\right) \subset \sup_{n\geq 1}\,[0,\ a_n)
\end{aligned}$$

Therefore

$$\sup_{n\geq 1}[0,\ a_n) = \left[0,\ \sup_{n\geq 1} a_n\right)$$

For the second part, the left hand side equals to

$$\sup_{n\geq 1}\left[0,\ \frac{n}{n+1}\right] = [0,\ 1)$$

while the right hand side equals to

$$\left[0,\ \sup_{n\geq 1}\frac{n}{n+1}\right] = [0,1]$$

Clearly, lhs $\neq$ rhs.

1.9.12

$$\mathcal{A}(\mathcal{C}) = \sigma(\mathcal{C}) = \left\{\begin{array}{l}\emptyset,\ \Omega,\ \{2,\ 4\},\ \{6\}\\ \{1,\ 3,\ 5,\ 6\},\ \{1,\ 2,\ 3,\ 4,\ 5\}\\ \{2,\ 4,\ 6\},\ \{1,\ 3,\ 5\}\end{array}\right\}$$

1.9.13 Check the definition of σ-field on $\mathcal{F}$.

1. $\hat{\mathcal{F}}$ is a σ-field, therefore $\hat{\Omega}\in\hat{\mathcal{F}}$, then

$$\bigcup_{C_t\in\hat{\Omega}} C_t = \bigcup_{t\in T} C_t = \Omega \in \mathcal{F}$$

2.

$$\forall\ A\in\mathcal{F} \quad\Longrightarrow\quad \exists\ \hat{A}\in\hat{\mathcal{F}} \text{ s.t. } A = \bigcup_{C_t\in\hat{A}} C_t$$

Note that $\Omega = \bigcup_{t\in T} C_t$ and $C_s\bigcap C_t = \emptyset$ for all $s,t\in T$. Then

$$A^c = \bigcup_{C_t\notin\hat{A}} C_t = \bigcup_{C_t\in\hat{A}^c} C_t \in \mathcal{F}$$

3. $\forall\ A_1,\ A_2,\ A_3,\dots\in\mathcal{F}$, we can write

$$\bigcup A_i = \bigcup\left\{\bigcup_{C_{i,t}\in\hat{A}_i} C_{i,t}\right\} = \bigcup_{C_t\in\bigcup\hat{A}_i} C_t \in \Omega$$

Therefore, $\mathcal{F}$ is a σ-field on Ω.
Also, note that $\forall\ \hat{A}_1\neq\hat{A}_2$:

$$\bigcup_{C_t\in\hat{A}_1} C_t \neq \bigcup_{C_t\in\hat{A}_2} C_t$$

Therefore, f is a 1-1 mapping from $\hat{\mathcal{F}}$ to $\mathcal{F}$.

1.9.14 Check the definition of field on $\cup_n\mathcal{A}_n$:

1. $\forall i,\ \Omega\in\mathcal{A}_i \implies \Omega\in\cup_n\mathcal{A}_n$.

2. $\forall\ A \in \cup_n \mathcal{A}_n \quad \Longrightarrow \quad \exists\ i, \quad \text{s.t. } A \in \mathcal{A}_i \quad \Longrightarrow \quad A^c \in \mathcal{A}_i \quad \Longrightarrow \quad A^c \in \cup_n \mathcal{A}_n$
3. $\forall A, B \in \cup_n \mathcal{A}_n \quad \Longrightarrow \quad \exists\ i,\ j$ s.t. $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$.
   Note that $\mathcal{A}_n \in \mathcal{A}_{n+1}$. Therefore
   $\exists\ i_0 = \max\{i, j\}, \quad \text{s.t. } A \in \mathcal{A}_{i_0} \quad \text{and} \quad B \in \mathcal{A}_{i_0}$
   Then
   $A \cup B \in \mathcal{A}_{i_0} \quad \Longrightarrow \quad A \cup B \in \cup_n \mathcal{A}_n$

Hence $\cup_n \mathcal{A}_n$ is a field.

1.9.15 To be added. You can refer to the old solutions online.

1.9.16 Let $\Omega = \{1, 2, 3, 4\}$ and

$$\mathcal{A} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{2, 3\}, \{1, 4\}\}$$

But $\mathcal{A}$ is not a field since

$$\{1, 2\} \cup \{2, 3\} \notin \mathcal{A}$$

1.9.17

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \{\omega : \ \omega \in A_n \text{ for all } n \text{ except a finite number}\} \\
&= \left\{\omega : \ \sum_n 1_{A_n^c}(\omega) < \infty\right\} \\
&= \left\{\omega : \ \lim_{n\to\infty} 1_{A_n}(\omega) = 1\right\}
\end{aligned}$$

1.9.18 Check the definition of field on $\mathcal{A}$:

1. $\Omega \in \mathcal{A}$ by definition.
2. $\forall\ A \in \mathcal{A}$, we have $A^c = \Omega \setminus A \in \mathcal{A}$ since $\Omega \in \mathcal{A}$.
3. $\forall\ A, B \in \mathcal{A}$, we can write

$$A, B \in \mathcal{A} \implies A, B^c \in \mathcal{A} \implies A \setminus B^c = A \cap B \in \mathcal{A}$$

Therefore $\mathcal{A}$ is a field.

1.9.19

$$\begin{aligned}
1_{A\cup B}(\omega) = 1 &\iff \omega \in A \cup B \iff 1_A(\omega) \vee 1_B(\omega) = 1 \\
1_{A\cup B}(\omega) = 0 &\iff \omega \notin A \cup B \iff 1_A(\omega) \vee 1_B(\omega) = 0 \\
1_{A\cap B}(\omega) = 1 &\iff \omega \in A \cap B \iff 1_A(\omega) \wedge 1_B(\omega) = 1 \\
1_{A\cap B}(\omega) = 0 &\iff \omega \notin A \cap B \iff 1_A(\omega) \wedge 1_B(\omega) = 0
\end{aligned}$$

Therefore

$$1_{A\cup B} = 1_A \vee 1_B \qquad 1_{A\cap B} = 1_A \wedge 1_B$$

2 Solutions to Chapter 2: Probability Spaces

2.6.1 (a) Check the definition of filed on $\mathcal{F}_0$:

(1) $\Omega \in \mathcal{F}_0$ since $|\Omega^c| = |\emptyset| = 0$

(2) $\forall A \in \mathcal{F}_0$, by definition of $\mathcal{F}_0$ we know that either $|A|$ or $|A^c|$ is finite, then clearly $A^c \in \mathcal{F}_0$.

(3) $\forall A,\ B \in \mathcal{F}_0$, then either $|A|$ or $|A^c|$ is finite. Same holds for either $|B|$ and $|B^c|$. Then we know:

3.1 If $|A|$ and $|B|$ are both finite, then $|A \cup B|$ is finite.

3.2 If $|A|$ and $|B^c|$ are both finite, then $|(A \cup B)^c| = |A^c \cap B^c| \leq |B^c|$ is finite.

3.3 If $|A^c|$ and $|B|$ are both finite, then same as before, we know $|A^c \cap B^c|$ is finite.

3.4 If $|A^c|$ and $|B^c|$ are both finite, then $|(A \cup B)^c| = |A^c \cap B^c|$ is finite.

In summary, either $|A \cup B|$ or $|A^c \cap B^c|$ is finite, hence $A \cup B \in \mathcal{F}_0$.

Therefore we have shown that $\mathcal{F}_0$ is a field.

(b) Since Ω is countably infinite, then $\forall A \in \mathcal{F}_0$, one and only one of A and A^c is finite. Now check the finite additivity of P:
$\forall A_1, A_2, \cdots, A_n \in \mathcal{F}_0$ where $\forall\ i \neq j,\ A_i \cap A_j = \emptyset$.

(1) If A_i is finite for all $i = 1, 2, \cdots, n$, then so is $\bigcup\limits_{i=1}^{n} A_i$. Therefore $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 0 = \sum_{i=1}^{n} 0 = \sum_{i=1}^{n} P(A_i)$$

(2) If one(and only one because $\{A_i\}$ are disjoint) of $\{A_i\}$ is infinite, say A_{i_0}. Then $\left(\bigcup\limits_{i=1}^{n} A_i\right)^c$ is finite, therefore $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 = 1 + \sum_{i \neq i_0} 0 = P(A_{i_0}) + \sum_{i \neq i_0} P(A_i) = \sum_{i=1}^{n} P(A_i)$$

By this point we have shown the finite additivity of P.
But P is not σ-additive. To show this, just pick finite disjoint set series $A_1, A_2, \cdots$ from $\mathcal{F}_0$ such that $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}_0$. Then clearly $\bigcup\limits_{i=1}^{\infty} A_i$ is inifite, which means $\left(\bigcup\limits_{i=1}^{\infty} A_i\right)^c$ is finite. Then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \neq 0 = \sum_{i=1}^{\infty} P(A_i)$$

(c) Like before, since Ω is uncountably infinite, $\forall A \in \mathcal{F}_0$, one and only one of A and A^c is finite. Its counterpart is uncountably infinite.

Now pick any disjoint set series $A_1, A_2, \cdots$ from $\mathcal{F}_0$ such that $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}_0$. Clearly, $\bigcup\limits_{i=1}^{\infty} A_i$ is infinite, then $\left(\bigcup\limits_{i=1}^{\infty} A_i\right)^c$ has to be finite in order to make it belong to $\mathcal{F}_0$. This means $\bigcup\limits_{i=1}^{\infty} A_i$ is uncountably infite, which indicates one(and only one because $\{A_i\}$ are disjoint) of $\{A_i\}$, say A_{i_0}, is uncountably infite while the others are all finite. Then we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 = 1 + \sum_{i \neq i_0} 0 = P\left(A_{i_0}\right) + \sum_{i \neq i_0} P\left(A_i\right) = \sum_{i=1}^{n} P\left(A_i\right)$$

Therefore, P is σ-additive.

3 Solutions to Chapter 3: Random Variables, Elements, and Measurable Maps

3.4.1 Note that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$, where $\mathcal{C} = \{(-\infty, a]\,,\ a \in \mathbb{R}\}$. Then we can show that

$$1_A^{-1}((-\infty,\ a]) = \begin{cases} \emptyset & a < 0 \\ A^c & 0 \leq a < 1 \\ \Omega & 1 \leq a \end{cases}$$

Clearly, $A \in \mathcal{B} \iff 1_A \in \mathcal{B}$.

3.4.2 Note that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$, where $\mathcal{C} = \{(-\infty, a]\,,\ a \in \mathbb{R}\}$. Therefore for X_1:

$$X_1^{-1}((-\infty,\ a]) = \begin{cases} \emptyset & a < 0 \\ \Omega & 0 \leq a \end{cases}$$

Hence

$$\sigma(X_1) = \sigma(\{\emptyset,\ \Omega\}) = \{\emptyset,\ \Omega\}$$

Similarly

$$X_2^{-1}((-\infty,\ a]) = \begin{cases} \emptyset & a < 0 \\ (0,\ 1/2) \cup (1/2,\ 1] & 0 \leq a < 1 \\ \Omega & 1 \leq a \end{cases}$$

Therefore

$$\sigma(X_2) = \sigma(\{\emptyset,\ (0,\ 1/2) \cup (1/2,\ 1]\,,\ \Omega\}) = \{\emptyset,\ \{1/2\}\,,\ (0,\ 1/2) \cup (1/2,\ 1]\,,\ \Omega\}$$

And

$$X_3^{-1}((-\infty, a]) = \begin{cases} \emptyset & a < 0 \\ \mathbb{Q}^c & 0 \leq a < 1 \\ \Omega & 1 \leq a \end{cases}$$

Hence

$$\sigma(X_3) = \sigma(\{\emptyset,\ \mathbb{Q},\ \Omega\}) = \{\emptyset,\ \mathbb{Q},\ \mathbb{Q}^c,\ \Omega\}$$

4 Solutions to Chapter 4: Independence

4.6.1

$$P\left(\bigcup_{i=1}^{n} B_i\right) = 1 - P\left(\bigcap_{i=1}^{n} B_i^c\right) = 1 - \prod_{i=1}^{n} P\left(B_i^c\right) = 1 - \prod_{i=1}^{n} \left(1 - P\left(B_i\right)\right)$$

4.6.2 I guess the answer is 2^n. But a proof is needed.

4.6.3

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\lim_{m\to\infty} \bigcap_{n=1}^{m} A_n\right) = \lim_{m\to\infty} P\left(\bigcap_{n=1}^{m} A_n\right) = \lim_{m\to\infty} \prod_{n=1}^{m} P\left(A_n\right) = \prod_{n=1}^{\infty} P\left(A_n\right)$$

The second equation follows the fact that probability measure P is continuous on monotone sequence $\left\{\bigcap_{n=1}^{m} A_n,\ m \geq 1\right\}$.

5 Solutions to Chapter 5: Integration and Expectation

5.10.1

$$\mathrm{E}\left(X_1+X_2\right)=\mathrm{E}\left(X_1\right)+\mathrm{E}\left(X_2\right)=0+0.5=0.5$$

5.10.2

$$\operatorname{Var}(c)=\mathrm{E}\left(c^2\right)-\mathrm{E}^2(c)=c^2-c^2=0$$

and

$$\begin{aligned}
\operatorname{Var}(X+c) &= \mathrm{E}(X+c)^2-\mathrm{E}^2(X+c) \\
&= \mathrm{E}\left(X^2+2cX+c^2\right)-\left(\mathrm{E}^2(X)+2c\mathrm{E}(X)+c^2\right) \\
&= \mathrm{E}\left(X^2\right)-\mathrm{E}^2(X) \\
&= \operatorname{Var}(X)
\end{aligned}$$

6 Solutions to Chapter 6: Convergence Concepts

6.7.1 To be added.

7 Solutions to Chapter 7: Laws of Large Numbers and Sums of Independent Random Variables

7.7.1 $\sum_n 1/n = \infty$, not convergent. $\sum_n (-1)^n /n < \infty$, convergent. Now that $\{X_n\}$ are iid with

$$P(X_n = \pm 1) = \frac{1}{2}$$

Then $\mathrm{E}X_n = 0$ and $\mathrm{Var}X = 1$. Therefore

$$\sum_{n=1}^{\infty} \mathrm{Var}\left(\frac{X_n}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

By Kolmogorov Convergence Criterion, $\sum\limits_{n=1}^{\infty} \left(\frac{X_n}{n} - \mathrm{E}\frac{X_n}{n}\right) = \sum\limits_{n=1}^{\infty} \frac{X_n}{n}$ convergence almost surely.

7.7.2

$$\frac{1}{n}\sum_{i=1}^{n} \left(X_i - \bar{X}\right) = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

Note that

$$\begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{i.p.}} \mathrm{E}X^2 \\ \bar{X} \xrightarrow{\text{i.p.}} \mathrm{E}X = \mu \implies \bar{X}^2 \xrightarrow{\text{i.p.}} \mu^2 \end{cases}$$

Therefore $\frac{1}{n}\sum\limits_{i=1}^{n} \left(X_i - \bar{X}\right)^2 \xrightarrow{\text{i.p.}} \sigma^2$.

8 Solutions to Chapter 8: Convergence in Distribution

8.8.1 To be added.

9 Solutions to Chapter 9: Characteristic Functions and the Central Limit Theorem

9.9.1 To be added.