

Solutions of A Probabilit Path

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1 Solutions to Chapter 1: Sets and Events

1.9.1 $\forall B \in \mathfrak{N}$, since $\mathcal{C} \subset B$, we have $\{0\} \in B$, therefore $\Omega \setminus \{0\} = \{1\} \in B$. Also $\emptyset \in B$ and $\Omega \in B$. Therefore $\{\emptyset, \{0\}, \{1\}, \Omega\} \subset B$. Note that $\mathcal{P}(\Omega) = \{\emptyset, \{0\}, \{1\}, \Omega\}$. This means

$$\mathfrak{N} = \{\mathcal{P}(\Omega)\}$$

1.9.2 Like in 1.9.1, we can conclude that

$$\forall B \in \mathfrak{N} \Rightarrow \{\emptyset, \{0\}, \{1, 2\}, \Omega\} \subset B$$

Also note that $\{\emptyset, \{0\}, \{1, 2\}, \Omega\}$ is a σ -field itself which means

$$\sigma(\mathcal{C}) = \{\emptyset, \{0\}, \{1, 2\}, \Omega\}$$

Those subsets of Ω which are not included in $\sigma(\mathcal{C})$ are

$$\{1\}, \quad \{2\}, \quad \{0, 1\}, \quad \{0, 2\}$$

and it's easy to check that they are all included in B if any one of them is included. So to sum up, we have

$$\mathfrak{N} = \{\sigma(\mathcal{C}), \mathcal{P}(\Omega)\}$$

1.9.3 Firstly

$$\begin{aligned} \limsup_{n \rightarrow \infty} A_n \cup B_n &= \left\{ x \left| \sum_{n=1}^{\infty} 1_{A_n \cup B_n}(x) = \infty \right. \right\} \\ &= \left\{ x \left| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \quad \text{or} \quad \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right. \right\} \\ &= \left\{ x \left| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right. \right\} \cup \left\{ x \left| \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right. \right\} \\ &= \limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n \end{aligned}$$

Secondly, the statement

$$A_n \cup B_n \rightarrow A \cup B, \quad A_n \cap B_n \rightarrow A \cap B$$

is true if $A_n \rightarrow A$ and $B_n \rightarrow B$. Because we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} A_n &= \liminf_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} A_n = A \\ \limsup_{n \rightarrow \infty} B_n &= \liminf_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} B_n = B \end{aligned}$$

Using the result of the first problem we can deduce that

$$\limsup_{n \rightarrow \infty} A_n \cup B_n = \limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n = A \cup B$$

We now have to show that

$$\liminf_{n \rightarrow \infty} A_n \cup B_n = \liminf_{n \rightarrow \infty} A_n \cup \liminf_{n \rightarrow \infty} B_n = A \cup B$$

Or equally

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} A_n \cup B_n \subset \liminf_{n \rightarrow \infty} A_n \cup B_n \\
& x \in \limsup_{n \rightarrow \infty} A_n \cup B_n \iff x \in A \cup B \iff \liminf_{n \rightarrow \infty} A_n \cup \liminf_{n \rightarrow \infty} B_n \\
& \iff \{x \notin A_n, \text{ finitely}\} \text{ or } \{x \notin B_n, \text{ finitely}\} \\
& \implies \{x \notin A_n \cup B_n, \text{ finitely}\} \iff x \in \liminf_{n \rightarrow \infty} A_n \cup B_n
\end{aligned}$$

This means $\forall x \in \limsup_{n \rightarrow \infty} A_n \cup B_n$, we have that $x \in \liminf_{n \rightarrow \infty} A_n \cup B_n$, therefore

$$\limsup_{n \rightarrow \infty} A_n \cup B_n \subset \liminf_{n \rightarrow \infty} A_n \cup B_n$$

which means

$$A_n \cup B_n \rightarrow A \cup B$$

and

$$A_n \cap B_n = (A_n^c \cup B_n^c)^c \rightarrow (A^c \cup B^c)^c = A \cap B$$

1.9.4

$$\begin{aligned}
\liminf_{n \rightarrow \infty} A_n &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ \frac{m}{k} : m \in \mathbb{N} \right\} \\
&= \bigcup_{n=1}^{\infty} \mathbb{N} = \mathbb{N} \\
\limsup_{n \rightarrow \infty} A_n &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
&= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{ \frac{m}{k} : m \in \mathbb{N} \right\} \\
&= \bigcap_{n=1}^{\infty} \mathbb{Q}^+ = \mathbb{Q}^+
\end{aligned}$$

1.9.5

$$\begin{aligned}
& \{\omega : f_n(\omega) \not\rightarrow f(\omega)\} \\
& \iff \{\omega : \exists \epsilon > 0, \text{ s.t. } \forall N, \exists n > N, \text{ s.t. } |f_n(\omega) - f(\omega)| > \epsilon\} \\
& \iff \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \omega : |f_n(\omega) - f(\omega)| > \frac{1}{k} \right\}
\end{aligned}$$

1.9.6 Use Lemma 1.3.1, we can conclude that

$$\limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = (0, 1]$$

1.9.7 1. Since $\theta = 1/8$, the period is $T = 8$. And there are actually 2 distinguished squares. Hence $\limsup_{n \rightarrow \infty} I_n$ is the star area covered by at least one square and $\liminf_{n \rightarrow \infty} I_n$ is the area covered by both squares. Refer to Figure 1 as illustration.

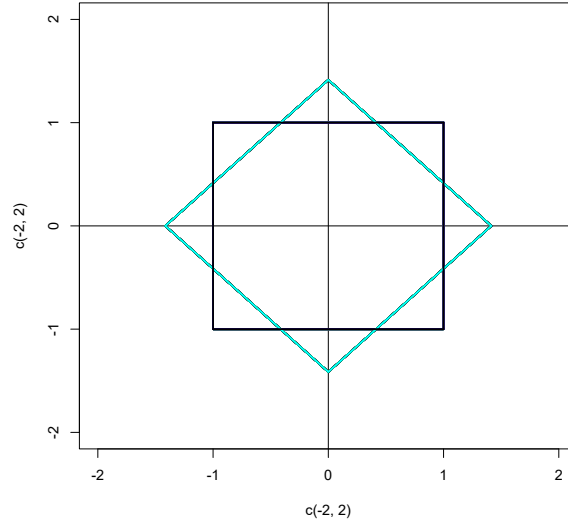


Figure 1: (a)

2. If θ is rational, then it can be written in the form $\theta = \frac{m}{n}$ where both m and n are integers, which means there is a period in I_n . Hence like before, $\limsup_{n \rightarrow \infty} I_n$ is the star area covered by at least one square and $\liminf_{n \rightarrow \infty} I_n$ is the area covered by all squares. Refer to Figure 2 as illustration.

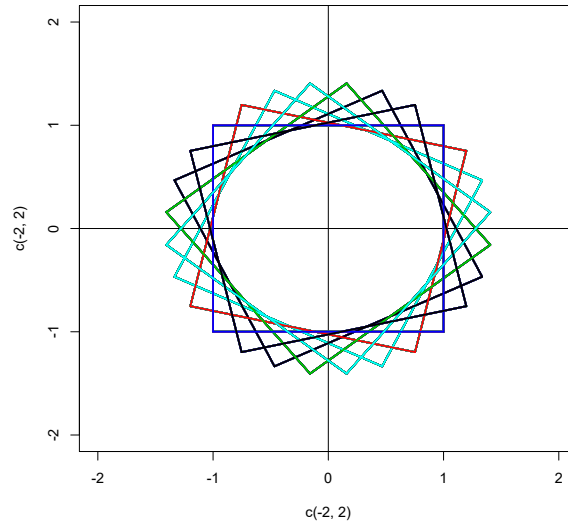


Figure 2: (b) $\theta = \frac{1}{7}$

3. If θ is irrational. These squares become dense and $\limsup_{n \rightarrow \infty} I_n$ is the round area with radius $r_{\text{sup}} = \sqrt{2}$ and $\liminf_{n \rightarrow \infty} I_n$ is the round area with radius $r_{\text{inf}} = 1$. Refer to Figure as illustration.
4. Codes for drawing these figures are provided below:

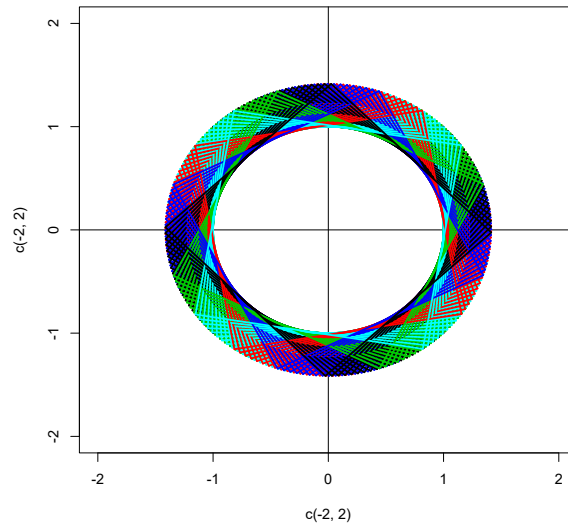


Figure 3: (c) $\theta = e^{1/2}$

```
x <- 1
y <- 1
```

```
##### sample codes, set theta as you wish. #####
```

```
theta <- 2/5
```

```
n <- 100
```

```
plot(c(-2,2),c(-2,2),type="n")
```

```
abline(v=0)
```

```
abline(h=0)
```

```
for(i in 0:n) {
```

```
  angle <- complex(real = cos(2*pi*theta*i),
                    imaginary = sin(2*pi*theta*i))
```

```
  point <- complex(real = x, imaginary = y)
```

```
  point0 <- point * angle
```

```
  x1 <- Re(point0)
```

```
  y1 <- Im(point0)
```

```
  segments(x1,y1,-y1,x1,col=(i %% 5)+1,lwd = 2)
```

```
  segments(-y1,x1,-x1,-y1,col=(i %% 5)+1,lwd = 2)
```

```
  segments(-x1,-y1,y1,-x1,col=(i %% 5)+1,lwd = 2)
```

```
  segments(y1,-x1,x1,y1,col=(i %% 5)+1,lwd = 2)
```

```
}
```

```
##### (a), (b) and (c) #####
```

```
theta.seq <- c(1/8, 1/7, exp(0.5))
```

```
n <- 100
```

```
for(ind in 1:3) {
```

```
  theta <- theta.seq[ind]
```

```
  fig.url <- paste("../Figures/1.9.7.",
```

```

                                letters[ind], ".pdf", sep = "")
cairo_pdf(fig.url)
plot(c(-2,2),c(-2,2),type="n")
abline(v=0)
abline(h=0)
for(i in 0:n) {
  angle <- complex(real = cos(2*pi*theta*i),
                    imaginary = sin(2*pi*theta*i))
  point <- complex(real = x, imaginary = y)
  point0 <- point * angle
  x1 <- Re(point0)
  y1 <- Im(point0)
  segments(x1,y1,-y1,x1,col=(i%%5)+1,lwd = 2)
  segments(-y1,x1,-x1,-y1,col=(i%%5)+1,lwd = 2)
  segments(-x1,-y1,y1,-x1,col=(i%%5)+1,lwd = 2)
  segments(y1,-x1,x1,y1,col=(i%%5)+1,lwd = 2)
}
dev.off()
}

```

1.9.8

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = B \cup C$$

and

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = B \cap C$$

1.9.9

$$\begin{aligned}
A \triangle B &= (A \setminus B) \cup (B \setminus A) \\
&= (A \cap B^c) \cup (B \cap A^c) \\
&= (B^c \cap (A^c)^c) \cup (A^c \cap (B^c)^c) \\
&= (B^c \setminus A^c) \cup (A^c \setminus B^c) \\
&= A^c \triangle B^c
\end{aligned}$$

1.9.10 \implies :

Since $A_n \rightarrow A$, we have

$$\liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n = A$$

If $w \in A \implies w \in \liminf_{n \rightarrow \infty} A_n$, then

$$\begin{aligned}
&\exists n_0, \quad \text{s.t. } \forall n \geq n_0, \quad w \in A_n \\
&\implies \forall n \geq n_0, 1_{A_n}(w) = 1 \\
&\implies \lim_{n \rightarrow \infty} 1_{A_n}(w) = 1 = 1_A(w)
\end{aligned}$$

And if $w \in A^c$, then $w \in \left(\limsup_{n \rightarrow \infty} A_n \right)^c = \liminf_{n \rightarrow \infty} A_n^c$, which means

$$\begin{aligned} & \exists n_0, \quad \text{s.t. } \forall n \geq n_0, \quad w \in A_n^c \\ \implies & \forall n \geq n_0, 1_{A_n}(w) = 0 \\ \implies & \lim_{n \rightarrow \infty} 1_{A_n}(w) = 0 = 1_A(w) \end{aligned}$$

Hence $A_n \rightarrow A \implies 1_{A_n} \rightarrow 1_A$.

\Leftarrow : If $1_A(w) = 1$, then $\lim_{n \rightarrow \infty} 1_{A_n}(w) = 1$, which means

$$\begin{aligned} & \exists n_0, \quad \text{s.t. } \forall n \geq n_0, \quad 1_{A_n}(w) = 1 \\ \implies & \forall n \geq n_0, \quad w \in A_n \\ \implies & w \in \liminf_{n \rightarrow \infty} A_n \\ \implies & A \subset \liminf_{n \rightarrow \infty} A_n \end{aligned}$$

If $1_A(w) = 0$, then $\lim_{n \rightarrow \infty} 1_{A_n}(w) = 0$, which means

$$\begin{aligned} & \exists n_0, \quad \text{s.t. } \forall n \geq n_0, \quad 1_{A_n}(w) = 0 \\ \implies & \forall n \geq n_0, \quad w \in A_n^c \\ \implies & w \in \liminf_{n \rightarrow \infty} A_n^c = \left(\limsup_{n \rightarrow \infty} A_n \right)^c \\ \implies & A^c \subset \left(\limsup_{n \rightarrow \infty} A_n \right)^c \\ \implies & A \subset \limsup_{n \rightarrow \infty} A_n \end{aligned}$$

Therefore $\limsup_{n \rightarrow \infty} A_n \subset \liminf_{n \rightarrow \infty} A_n$, which means

$$\liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n = A$$

Therefore $1_{A_n} \rightarrow 1_A \implies A_n \rightarrow A$.

1.9.11

$$\begin{aligned} & w \in \sup_{n \geq 1} [0, a_n) \\ \implies & \exists n_0, \text{ s.t. } w \in [0, a_{n_0}) \\ \implies & w \in \left[0, \sup_{n \geq 1} a_n \right) \quad (\text{since } \sup_{n \geq 1} a_n \geq a_{n_0}) \\ \implies & \sup_{n \geq 1} [0, a_n) \subset \left[0, \sup_{n \geq 1} a_n \right) \end{aligned}$$

Also note that

$$\begin{aligned} & w \in \left[0, \sup_{n \geq 1} a_n \right) \implies w < \sup_{n \geq 1} a_n \\ \implies & \exists a_{n_0}, \text{ s.t. } w < a_{n_0} \\ \implies & w \in [0, a_{n_0}) \\ \implies & w \in \sup_{n \geq 1} [0, a_n) \\ \implies & \left[0, \sup_{n \geq 1} a_n \right) \subset \sup_{n \geq 1} [0, a_n) \end{aligned}$$

Therefore

$$\sup_{n \geq 1} [0, a_n) = \left[0, \sup_{n \geq 1} a_n \right)$$

For the second part, the left hand side equals to

$$\sup_{n \geq 1} \left[0, \frac{n}{n+1} \right] = [0, 1)$$

while the right hand side equals to

$$\left[0, \sup_{n \geq 1} \frac{n}{n+1} \right] = [0, 1]$$

Clearly, lhs \neq rhs.

1.9.12

$$\mathcal{A}(\mathcal{C}) = \sigma(\mathcal{C}) = \left\{ \begin{array}{l} \emptyset, \Omega, \{2, 4\}, \{6\} \\ \{1, 3, 5, 6\}, \{1, 2, 3, 4, 5\} \\ \{2, 4, 6\}, \{1, 3, 5\} \end{array} \right\}$$

1.9.13 Check the definition of σ -field on \mathcal{F} .

1. $\hat{\mathcal{F}}$ is a σ -field, therefore $\hat{\Omega} \in \hat{\mathcal{F}}$, then

$$\bigcup_{C_t \in \hat{\Omega}} C_t = \bigcup_{t \in T} C_t = \Omega \in \mathcal{F}$$

2.

$$\forall A \in \mathcal{F} \implies \exists \hat{A} \in \hat{\mathcal{F}} \text{ s.t. } A = \bigcup_{C_t \in \hat{A}} C_t$$

Note that $\Omega = \bigcup_{t \in T} C_t$ and $C_s \cap C_t = \emptyset$ for all $s, t \in T$. Then

$$A^c = \bigcup_{C_t \notin \hat{A}} C_t = \bigcup_{C_t \in \hat{A}^c} C_t \in \mathcal{F}$$

3. $\forall A_1, A_2, A_3, \dots \in \mathcal{F}$, we can write

$$\bigcup A_i = \bigcup \left\{ \bigcup_{C_{i,t} \in \hat{A}_i} C_{i,t} \right\} = \bigcup_{C_t \in \bigcup \hat{A}_i} C_t \in \Omega$$

Therefore, \mathcal{F} is a σ -field on Ω .

Also, note that $\forall \hat{A}_1 \neq \hat{A}_2$:

$$\bigcup_{C_t \in \hat{A}_1} C_t \neq \bigcup_{C_t \in \hat{A}_2} C_t$$

Therefore, f is a 1-1 mapping from $\hat{\mathcal{F}}$ to \mathcal{F} .

1.9.14 Check the definition of field on $\cup_n \mathcal{A}_n$:

1. $\forall i, \Omega \in \mathcal{A}_i \implies \Omega \in \cup_n \mathcal{A}_n$.

$$2. \forall A \in \cup_n \mathcal{A}_n \implies \exists i, \text{ s.t. } A \in \mathcal{A}_i \implies A^c \in \mathcal{A}_i \implies A^c \in \cup_n \mathcal{A}_n$$

$$3. \forall A, B \in \cup_n \mathcal{A}_n \implies \exists i, j \text{ s.t. } A \in \mathcal{A}_i \text{ and } B \in \mathcal{A}_j.$$

Note that $\mathcal{A}_n \in \mathcal{A}_{n+1}$. Therefore

$$\exists i_0 = \max\{i, j\}, \text{ s.t. } A \in \mathcal{A}_{i_0} \text{ and } B \in \mathcal{A}_{i_0}$$

Then

$$A \cup B \in \mathcal{A}_{i_0} \implies A \cup B \in \cup_n \mathcal{A}_n$$

Hence $\cup_n \mathcal{A}_n$ is a field.

2 Solutions to Chapter 2: Probability Spaces

2.6.1 To be added.

3 Solutions to Chapter 3: Random Variables, Elements, and Measurable Maps

3.4.1 To be added.

4 Solutions to Chapter 4: Independence

4.6.1

$$P\left(\bigcup_{i=1}^n B_i\right) = 1 - P\left(\bigcap_{i=1}^n B_i^c\right) = 1 - \prod_{i=1}^n P(B_i^c) = 1 - \prod_{i=1}^n (1 - P(B_i))$$

4.6.2 I guess the answer is 2^n . But a proof is needed.

4.6.3

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\lim_{m \rightarrow \infty} \bigcap_{n=1}^m A_n\right) = \lim_{m \rightarrow \infty} P\left(\bigcap_{n=1}^m A_n\right) = \lim_{m \rightarrow \infty} \prod_{n=1}^m P(A_n) = \prod_{n=1}^{\infty} P(A_n)$$

The second equation follows the fact that probability measure P is continuous on monotone sequence $\left\{\bigcap_{n=1}^m A_n, m \geq 1\right\}$.