Solutions of A Probabilit Path

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1 Solutions to Chapter 1: Sets and Events

1.9.1 $\forall B \in \aleph$, since $\mathcal{C} \subset B$, we have $\{0\} \in B$, therefore $\Omega \setminus \{0\} = \{1\} \in B$. Also $\emptyset \in B$ and $\Omega \in B$. Therefore $\{\emptyset, \{0\}, \{1\}, \Omega\} \subset B$. Note that $\mathcal{P}(\Omega) = \{\emptyset, \{0\}, \{1\}, \Omega\}$. This means

$$\aleph = \{\mathcal{P}(\Omega)\}$$

1.9.2 Like in 1.9.1, we can conclude that

$$\forall B \in \aleph \quad \Rightarrow \{\emptyset, \{0\}, \{1,2\}, \Omega\} \subset B$$

Also note that $\{\emptyset, \{0\}, \{1,2\}, \Omega\}$ is a σ-field itself which means

$$\sigma(\mathcal{C}) = \{\emptyset, \{0\}, \{1,2\}, \Omega\}$$

Those subsets of Ω which are not included in $\sigma(\mathcal{C})$ are

$$\{1\}, \quad \{2\}, \quad \{0,1\}, \quad \{0,2\}$$

and it's easy to check that they are all included in B if any one of them is inclued. So to sum up, we have

$$\aleph = \{\sigma(\mathcal{C}), \mathcal{P}(\Omega)\}$$

1.9.3 Firstly

$$\begin{aligned}
\limsup_{n\to\infty} A_n \cup B_n &= \left\{ x \middle| \sum_{n=1}^{\infty} 1_{A_n \cup B_n}(x) = \infty \right\} \\
&= \left\{ x \middle| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \quad \text{or} \quad \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right\} \\
&= \left\{ x \middle| \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right\} \cup \left\{ x \middle| \sum_{n=1}^{\infty} 1_{B_n}(x) = \infty \right\} \\
&= \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n
\end{aligned}$$

Secondly, the statement

$$A_n \cup B_n \to A \cup B, \quad A_n \cap B_n \to A \cap B$$

is true if $A_n \to A$ and $B_n \to B$. Because we have

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \lim_{n\to\infty} A_n = A$$
$$\limsup_{n\to\infty} B_n = \liminf_{n\to\infty} B_n = \lim_{n\to\infty} B_n = B$$

Using the result of the first problem we can deduce that

$$\limsup_{n\to\infty} A_n \cup B_n = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n = A \cup B$$

We now have to show that

$$\liminf_{n\to\infty} A_n \cup B_n = \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n = A \cup B$$

Or equally

$$\limsup_{n\to\infty} A_n \cup B_n \subset \liminf_{n\to\infty} A_n \cup B_n$$

$$x \in \limsup_{n\to\infty} A_n \cup B_n \iff x \in A \cup B \iff \liminf_{n\to\infty} A_n \cup \liminf_{n\to\infty} B_n$$
$$\iff \{x \notin A_n, \text{ finitely}\} \text{ or } \{x \notin B_n, \text{ finitely}\}$$
$$\implies \{x \notin A_n \cup B_n, \text{ finitely}\} \iff x \in \liminf_{n\to\infty} A_n \cup B_n$$

This means $\forall x \in \limsup_{n\to\infty} A_n \cup B_n$, we have that $x \in \liminf_{n\to\infty} A_n \cup B_n$, therefore

$$\limsup_{n\to\infty} A_n \cup B_n \subset \liminf_{n\to\infty} A_n \cup B_n$$

which means

$$A_n \cup B_n \to A \cup B$$

and

$$A_n \cap B_n = (A_n^c \cup B_n^c)^c \to (A^c \cup B_c)^c = A \cap B$$

1.9.4