Gamma and Beta Distribution

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1 Introduction

The Gamma function is defined as

$$\Gamma\left(\alpha\right)=\int_{0}^{\infty}t^{\alpha-1}e^{-t}\mathrm{d}t. \tag{1}$$

And it's easy to verify that

$$\Gamma\left(\alpha+1\right)=\alpha\Gamma\left(\alpha\right),\quad \Gamma\left(n\right)=\left(n-1\right)!,\quad \Gamma\left(0.5\right)=\sqrt{\pi}.$$

Let $B\left(\alpha,\beta\right)$ denote the Beta function

$$B\left(\alpha,\beta\right)=\int_{0}^{1}x^{\alpha-1}\left(1-x\right)^{\beta-1}\mathrm{d}x. \tag{2}$$

Here we point out that $B\left(\alpha,\beta\right)$ is related to Gamma function via

$$B\left(\alpha,\beta\right)=\frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}.$$

2 Gamma distribution

The pdf for a Gamma distribution is

$$f\left(x|\alpha,\beta\right)=\frac{1}{\Gamma\left(\alpha\right)\beta^{\alpha}}x^{\alpha-1}\exp\left(-x/\beta\right),\quad 0\le x<\infty,\quad \alpha,\beta>0. \tag{3}$$

3 Beta distribution

The pdf for a Beta distribution is

$$f\left(x|\alpha,\beta\right)=\frac{1}{B\left(\alpha,\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1},\quad 0\leq x\leq 1,\quad \alpha,\beta>0. \tag{4}$$

4 Relationship with other distribution

- Gamma distribution with $\alpha=1$ is just exponential distribution.
$$gamma\left(1,\beta\right)=\exp\left(\beta\right).$$
- Gamma distribution with $\beta=2$ is just χ^2 distribution with degree of freedom $p=2\alpha$.
$$gamma\left(\alpha,2\right)=\chi^2\left(2\alpha\right).$$
- If X follows $gamma\left(\alpha,\beta\right)$ where α is an integer. Then for any x
$$P\left(X\leq x\right)=P\left(Y\geq\alpha\right).$$
where Y follows $Poisson\left(x/\beta\right)$.
- If
$$X\sim gamma\left(\alpha_1,\beta\right),\quad Y\sim gamma\left(\alpha_2,\beta\right),\quad X\perp Y.$$
Then
$$X+Y\sim gamma\left(\alpha_1+\alpha_2,\beta\right),\quad \frac{X}{X+Y}\sim beta\left(\alpha_1,\alpha_2\right),\quad \left(X+Y\right)\perp\frac{X}{X+Y}.$$
- $$beta\left(1,1\right)=Unif\left(0,1\right).$$
- $beta\left(\frac{1}{2},\frac{1}{2}\right)$ is the **non-informative Jeffreys prior** for binomial rate test.
- Let $U_1,\cdots,U_n\overset{\text{i.i.d.}}{\sim}Unif(0,1)$. Then the order statistics
$$U_{(k)}\sim\ beta\left(k,n+1-k\right),\quad 1\leq k\leq n.$$
And
$$U_{(k)}-U_{(j)}\sim\ beta\left(k-j,n-(k-j)+1\right),\quad 1\leq j<k\leq n.$$
- If X follows a $Binomial(n,p)$, then the c.d.f of X satisfies
$$\begin{aligned}&P\left(X\leq k\right)\\=&\sum_{i=0}^{k}C_n^i p^i\left(1-p\right)^{n-i}\\=&\frac{\Gamma\left(n+1\right)}{\Gamma\left(n-k\right)\Gamma\left(k+1\right)}\int_0^{1-p}t^{n-k-1}\left(1-t\right)^k\mathrm{dt}=pbeta\left(1-p;n-k,k+1\right)\\=&P\left(Y\leq 1-p\right),\end{aligned}$$
where Y follows $beta\left(n-k,k+1\right)$. This can be verified using integration by parts.

References