

# T-test

Chao Cheng

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## 1 Basic knowledge

$\phi(x)$  and  $\Phi(x)$  are pdf and cdf of standard normal distribution, respectively. We use  $Z$  to represent a random variable that follows standard normal distribution and  $z_\alpha$  the lower  $\alpha$  quantile of standard normal distribution. Therefore

$$P(Z \leq z_\alpha) = \Phi(z_\alpha) = \alpha.$$

**Theorem 1.** Let  $x_1, \dots, x_n$  be a random sample from a population with mean  $\mu$  and variance  $\sigma^2 < \infty$ . Then

1.  $E\bar{x} = \mu$ .
2.  $\text{Var}\bar{x} = \sigma^2/n$ .
3.  $ES^2 = \sigma^2$ , where  $S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ .

**Theorem 2.** Let  $x_1, \dots, x_n$  be a random sample from  $N(\mu, \sigma^2)$ . Then

1.  $\bar{X} \sim N(\mu, \sigma^2/n)$ .
2.  $\bar{X}$  is independent of  $S^2$ .
3.  $(n-1)S^2/\sigma^2$  follows a chi-squared distribution with  $n-1$  degree of freedom.

## 2 One-sample test

Consider a random sample  $x_1, \dots, x_n$  from  $N(\mu, \sigma^2)$ . The likelihood is

$$\begin{aligned} f(x_1, \dots, x_n) &= \prod_{i=1}^n (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2}\right). \end{aligned}$$

We propose the test

$$H_0 : \mu = \mu_0 \quad \text{v.s.} \quad H_1 : \mu \neq \mu_0$$

## 2.1 variance known

Construct LRT

$$LR = \frac{\max_{\mu \in H_0} f(x_1, \dots, x_n | \mu)}{\max_{\mu \in H_0 \cup H_1} f(x_1, \dots, x_n | \mu)} = \frac{f(x_1, \dots, x_n | \mu = \mu_0)}{f(x_1, \dots, x_n | \mu = \bar{x})} = \exp \left( -\frac{(\bar{x} - \mu_0)^2}{2\sigma^2/n} \right)$$

Therefore rejecting  $H_0$  when LR is smaller than some constant  $C$  is equivalent to rejecting  $H_0$  when  $|\bar{x} - \mu_0|$  is larger than some other constant  $C$ . Hence

$$\text{Reject Region: } \{\bar{x} : |\bar{x} - \mu_0| > C\}$$

### 2.1.1 Decide $C$ from $\alpha$

From definition of  $\alpha$  we know that  $C$  in the reject region is chosen such that

$$P(|\bar{x} - \mu_0| > C | H_0 \text{ is true}) \leq \alpha.$$

But to fully utilize the test, we choose to use equal sign instead of  $\leq$ . Therefore

$$P(|\bar{x} - \mu_0| > C | \mu = \mu_0) = \alpha.$$

Note that  $\bar{x} \sim N(\mu, \sigma^2/n)$ . Then under the condition  $\mu = \mu_0$ ,

$$\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0, 1).$$

Therefore we propose the reject region for  $H_0$  being

$$\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2}.$$

**Note:** Here, even if the sample distribution is not normal, the result still holds due to CLT under large sample.

### 2.1.2 Power at given underlying $\mu$

The power (the probability to reject  $H_0$ , when  $H_1$  is true) of the proposed test procedure for any given underlying  $\mu \neq \mu_0$  is computed as

$$\begin{aligned} & P \left( \left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2} \right) \\ &= P \left( \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} \right) + P \left( \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2} \right) \\ &= P \left( \frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) + P \left( \frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) \\ &= P \left( Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) + P \left( Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) \end{aligned} \tag{1}$$

Here we use the fact that  $\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sim N(0, 1)$ .

### 2.1.3 Sample size at given $\alpha$ , $\beta$ and underlying $\mu$

W.l.o.g, assume that  $\mu > \mu_0$ , then in previous power equation (1)

$$P\left(Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)$$

would be really close to zero and

$$P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)$$

will offer most of the power. In order to guarantee a power of at least  $1 - \beta$ , we could simply set

$$P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right) \geq 1 - \beta,$$

which means

$$z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \leq z_\beta.$$

Normally in test settings,  $\alpha < 0.1$  and  $\beta < 0.5$ , which means  $z_{1-\alpha/2}$  is positive and  $z_\beta$  is negative. Also  $\mu_0 - \mu < 0$  in our assumption. This leads to

$$-z_{\alpha/2} - z_\beta \leq \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}.$$

Hence the sample size requirement is

$$n \geq \frac{\sigma^2 (z_{\alpha/2} + z_\beta)^2}{(\mu - \mu_0)^2}. \quad (2)$$

**Note:** The sample size requirement can be deduced the same way when  $\mu < \mu_0$ . And the result is just the same as (2).

## 2.2 variance unknown

When  $\sigma^2$  is unknown, the MLE under  $H_0$  is

$$\mu_{(0)} = \mu_0, \quad \sigma_{(0)}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2.$$

And the MLE under  $H_0 \cup H_1$  is

$$\mu_{(0 \cup 1)} = \bar{x}, \quad \sigma_{(0 \cup 1)}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2.$$

**Note:** MLE for  $\sigma^2$  offers smaller MSE than  $S^2$ , but it's biased.

Then the likelihood ratio is

$$LR = \frac{f(x_1, \dots, x_n | \mu = \mu_{(0)}, \sigma^2 = \sigma_{(0)}^2)}{f(x_1, \dots, x_n | \mu = \mu_{(0 \cup 1)}, \sigma^2 = \sigma_{(0 \cup 1)}^2)} = \left( \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)^{-n/2} \propto \left( \frac{\sum_{i=1}^n (\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)^{-n/2},$$

where for the last part we mainly focus on terms related to  $\mu_0$ . So to reject  $H_0$  when LR is small is equivalent to

$$\text{Reject Region: } \left\{ \bar{x} : \frac{|\bar{x} - \mu_0|}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} > C \right\}$$

The idea is similar to that in Section 2.1. But we replace  $\sigma^2$  with  $S^2$ .

### 2.2.1 Decide $C$ from $\alpha$

First we can write

$$P \left( \frac{|\bar{x} - \mu_0|}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} > C \middle| \mu = \mu_0 \right) = P \left( \frac{|\bar{x} - \mu_0|}{\sqrt{(n-1) S^2}} > C \middle| \mu = \mu_0 \right) = \alpha.$$

From Theorem 2 we know that

$$\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sim N(0, 1), \quad (n-1) S^2/\sigma^2 \sim \chi^2(n-1), \quad \bar{x} \perp S^2$$

Therefore

$$\frac{\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1) S^2}{(n-1) \sigma^2}}} = \frac{\bar{x} - \mu}{\sqrt{S^2/n}} \sim t(n-1).$$

Then we know the reject region is

$$\left| \frac{\bar{x} - \mu}{\sqrt{S^2/n}} \right| > t_{1-\alpha/2}(n-1).$$

**Note:** Here we need Theorem 2, which means the normal assumption of the sample is **necessary**. Though one might argue that without normal assumption, under large sample scenario, using Slutsky's theorem, asymptotically

$$\frac{\bar{x} - \mu}{\sqrt{S^2/n}} = \frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sqrt{\frac{\sigma^2}{S^2}} \rightarrow N(0, 1).$$

### 2.2.2 Power at given underlying $\mu$ and $\sigma^2$

Before any computation, we introduce the **non-central** t-distribution.

$$T = \frac{Z + \mu}{\sqrt{V/v}}, \tag{3}$$

where  $Z$  follows standard normal and  $V$  follows  $\chi^2(v)$  and  $Z \perp V$ . Then  $T$  follows a non-central t-distribution with degree of freedom  $v$  and non-central parameter  $\mu$ , denoted by  $t(v, \mu)$ .

Then we know that

$$\frac{\bar{x} - \mu_0}{\sqrt{S^2/n}} = \frac{\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} + \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} \sim t \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right),$$

which means  $\frac{\bar{x} - \mu_0}{\sqrt{S^2/n}}$  follows a non-central t-distribution  $t \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right)$ . Therefore the power can be computed as

$$\begin{aligned} & P \left( \left| \frac{\bar{x} - \mu_0}{\sqrt{S^2/n}} \right| \geq t_{1-\alpha/2}(n-1) \right) \\ &= P \left( \left| T \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right) \right| \geq t_{1-\alpha/2}(n-1) \right) \\ &= P \left( T \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right) \leq t_{\alpha/2}(n-1) \right) + P \left( T \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right) \geq t_{1-\alpha/2}(n-1) \right). \end{aligned} \tag{4}$$

### 2.2.3 Sample size at given $\alpha$ , $\beta$ and underlying $\mu$ and $\sigma^2$

W.l.o.g, assume  $\mu > \mu_0$ , then in the previous power equation (4)

$$P \left( T \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right) \leq t_{\alpha/2}(n-1) \right)$$

would be close to zero and

$$P \left( T \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right) \geq t_{1-\alpha/2}(n-1) \right)$$

will offer the most power. In order to guarantee a power of at least  $1 - \beta$ , we could simply set

$$P \left( T \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right) \geq t_{1-\alpha/2}(n-1) \right) \geq 1 - \beta,$$

which means

$$t_{1-\alpha/2}(n-1) \leq t_{\beta} \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right).$$

There's no close form for this inequality, we should use some numerical method to solve for  $n$ .

**Note:** If  $\mu < \mu_0$ , then similarly we can get the requirement as

$$t_{\alpha/2}(n-1) \geq t_{1-\beta} \left( n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}} \right).$$

Use the fact that  $t_{\alpha}(n, \mu) = -t_{1-\alpha}(n, -\mu)$ , we can arrange the previous inequality as

$$t_{1-\alpha/2}(n-1) \leq t_{\beta} \left( n-1, \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right).$$

Therefore in summary the sample size requirement is

$$t_{1-\alpha/2}(n-1) \leq t_\beta \left( n-1, \frac{|\mu_0 - \mu|}{\sqrt{\sigma^2/n}} \right).$$

### 3 Two sample test

Consider two random samples  $x_1, \dots, x_{n_1} \sim N(\mu_1, \sigma_1^2)$  and  $y_1, \dots, y_{n_2} \sim N(\mu_2, \sigma_2^2)$ . Then the likelihood of the data is

$$\begin{aligned} & f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2} | \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \\ &= (2\pi\sigma_1^2)^{-n_1/2} (2\pi\sigma_2^2)^{-n_2/2} \exp \left( -\frac{\sum_{i=1}^{n_1} (x_i - \mu_1)^2}{2\sigma_1^2} - \frac{\sum_{i=1}^{n_2} (y_i - \mu_2)^2}{2\sigma_2^2} \right) \\ &= \end{aligned}$$

We propose the test

$$H_0 : \mu_1 = \mu_2 \quad \text{v.s.} \quad H_1 : \mu_1 \neq \mu_2.$$

#### 3.1 Two-sample, variance known

When  $\sigma_1^2$  and  $\sigma_2^2$  are known, the likelihood satisfies

$$f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2} | \mu_1, \mu_2) \propto \exp \left( -\frac{n_1 (\bar{x} - \mu_1)^2}{2\sigma_1^2} - \frac{n_2 (\bar{y} - \mu_2)^2}{2\sigma_2^2} \right).$$

Therefore under  $H_0$ , the MLE for  $\mu_1$  and  $\mu_2$  is

$$\mu_{1(0)} = \mu_{2(0)} = \mu_{(0)} = \frac{\sigma_2^2 n_1 \bar{x} + \sigma_1^2 n_2 \bar{y}}{\sigma_2^2 n_1 + \sigma_1^2 n_2}.$$

And under  $H_0 \cup H_1$ , the MLE for  $\mu_1$  and  $\mu_2$  is

$$\mu_{1(0 \cup 1)} = \bar{x}, \quad \mu_{2(0 \cup 1)} = \bar{y}.$$

Then the likelihood ratio is

$$\begin{aligned} LR &= \frac{\max_{H_0} f(\mathbf{x}, \mathbf{y} | \mu_1, \mu_2)}{\max_{H_0 \cup H_1} f(\mathbf{x}, \mathbf{y} | \mu_1, \mu_2)} \\ &= \frac{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_2 = \mu_{(0)})}{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_{1(0 \cup 1)}, \mu_2 = \mu_{2(0 \cup 1)})} \\ &\propto \exp \left( -\frac{1}{2} \left( \frac{n_1 (\bar{x} - \mu_{(0)})^2}{\sigma_1^2} + \frac{n_2 (\bar{y} - \mu_{(0)})^2}{\sigma_2^2} \right) \right) \\ &= \exp \left( -\frac{1}{2} \frac{n_1 n_2}{\sigma_2^2 n_1 + \sigma_1^2 n_2} (\bar{x} - \bar{y})^2 \right). \end{aligned}$$

From the idea of LRT,  $H_0$  is rejected when  $LR$  is small enough, which means the reject rule is

$$\text{Reject Region: } \{(\bar{x}, \bar{y}) | |\bar{x} - \bar{y}| > C\}$$

### 3.1.1 Decide $C$ from $\alpha$

From the definition of  $\alpha$  we know that

$$P(|\bar{x} - \bar{y}| > C | H_0 \text{ is true}) \leq \alpha.$$

Note that  $\bar{x} \sim N(\mu_1, \sigma_1^2/n_1)$ ,  $\bar{y} \sim N(\mu_2, \sigma_2^2/n_2)$  and  $\bar{x} \perp \bar{y}$ . Therefore

$$\bar{x} - \bar{y} \sim N(\mu_1 - \mu_2, \sigma_1^2/n_1 + \sigma_2^2/n_2).$$

Then under  $H_0$ ,  $\mu_1 = \mu_2$  and

$$\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1),$$

which means

$$P(|\bar{x} - \bar{y}| > C | \mu_1 = \mu_2) = P\left(|Z| > \frac{C}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \leq \alpha.$$

Here to fully utilize the test, we choose the equal sign. Hence

$$z_{1-\alpha/2} = \frac{C}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}.$$

Here the reject region is

$$\frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2}.$$

**Note:** Even the samples does not follow normal distribution, by CLT this test still holds true for large sample.

### 3.1.2 Power at given $\Delta = \mu_1 - \mu_2$

The power at a given  $\Delta = \mu_1 - \mu_2$  is

$$\begin{aligned} & P\left(\frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2} \middle| \Delta = \mu_1 - \mu_2\right) \\ &= P\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2}\right) + P\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z_{\alpha/2}\right) \\ &= P\left(\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) + P\left(\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z_{\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \\ &= P\left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) + P\left(Z < z_{\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \end{aligned}$$

### 3.1.3 Sample size at given $\alpha$ , $\beta$ , $\Delta$ and $k = \frac{n_1}{n_2}$

W.l.o.g, assume  $\Delta > 0$ , then the power of the test comes mostly from

$$P \left( Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right).$$

So to achieve the power, we can set

$$P \left( Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right) \geq 1 - \beta.$$

And this means

$$z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq z_\beta.$$

Rearrange this inequality we have

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \leq \frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2}.$$

1. When  $n_1$  is fixed and given, we need

$$n_2 \geq \frac{\sigma_2^2}{\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} - \frac{\sigma_1^2}{n_1}}.$$

Also this fixed and given  $n_1$  must satisfy

$$\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} > \frac{\sigma_1^2}{n_1}$$

for the test to be feasible.

2. Similarly, when  $n_2$  is given and fixed, we need

$$n_1 \geq \frac{\sigma_1^2}{\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} - \frac{\sigma_2^2}{n_2}}.$$

Also this fixed and given  $n_2$  must satisfy

$$\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} > \frac{\sigma_2^2}{n_2}$$

for the test to be feasible.

3. For a fixed and given sample size ratio  $k = n_2/n_1$ , we need

$$n_1 \geq \frac{(\sigma_1^2 + \sigma_2^2/k) (z_{\alpha/2} + z_\beta)^2}{\Delta^2}$$

**Note:** These results hold the same form when  $\Delta < 0$ .



### 3.2 Two-sample, variance unknown but equal

When  $\sigma_1$  and  $\sigma_2$  are both unknown but equal, denoted by  $\sigma$ . The likelihood of the data becomes

$$\begin{aligned} & f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2} | \mu_1, \mu_2, \sigma^2) \\ &= (2\pi\sigma^2)^{-n_1/2-n_2/2} \exp \left( -\frac{\sum_{i=1}^{n_1} (x_i - \mu_1)^2 + \sum_{i=1}^{n_2} (y_i - \mu_2)^2}{2\sigma^2} \right). \end{aligned}$$

So the MLE under  $H_0$  is

$$\mu_{1(0)} = \mu_{2(0)} = \mu_{(0)} = \frac{n_1\bar{x} + n_2\bar{y}}{n_1 + n_2}, \quad \sigma_{(0)}^2 = \frac{\sum_{i=1}^{n_1} (x_i - \mu_{(0)})^2 + \sum_{i=1}^{n_2} (y_i - \mu_{(0)})^2}{n_1 + n_2}$$

And the MLE under  $H_0 \cup H_1$  is

$$\mu_{1(0 \cup 1)} = \bar{x}, \quad \mu_{2(0 \cup 1)} = \bar{y}, \quad \sigma_{(0 \cup 1)}^2 = \frac{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2}{n_1 + n_2}.$$

Then the likelihood ratio is

$$\begin{aligned} LR &= \frac{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_2 = \mu_{(0)}, \sigma^2 = \sigma_{(0)}^2)}{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_{1(0 \cup 1)}, \mu_2 = \mu_{2(0 \cup 1)}, \sigma^2 = \sigma_{(0 \cup 1)}^2)} \\ &= \left( \frac{\sigma_{(0)}^2}{\sigma_{(0 \cup 1)}^2} \right)^{-n_1/2-n_2/2} \\ &= \left( \frac{\sum_{i=1}^{n_1} (x_i - \bar{x} + \bar{x} - \mu_{(0)})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y} + \bar{y} - \mu_{(0)})^2}{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2} \right)^{-n_1/2-n_2/2} \\ &= \left( 1 + \frac{n_1 (\bar{x} - \mu_{(0)})^2 + n_2 (\bar{y} - \mu_{(0)})^2}{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2} \right)^{-n_1/2-n_2/2} \\ &= \left( 1 + \frac{n_1 n_2}{n_1 + n_2} \cdot \frac{(\bar{x} - \bar{y})^2}{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2} \right)^{-n_1/2-n_2/2}. \end{aligned}$$

So to reject  $H_0$  when the likelihood ratio is small enough implies that the reject region is

$$\text{Reject region: } \left\{ (\mathbf{x}, \mathbf{y}) \left| \frac{|\bar{x} - \bar{y}|}{\sqrt{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2}} > C \right. \right\}.$$

### 3.2.1 Decide $C$ from $\alpha$

Like before, we know that

$$\bar{x} \sim N\left(\mu_1, \frac{\sigma^2}{n_1}\right), \quad (n_1 - 1) S_x^2 / \sigma^2 \sim \chi^2(n_1 - 1), \quad \bar{x} \perp S_x^2,$$

and

$$\bar{y} \sim N\left(\mu_2, \frac{\sigma^2}{n_2}\right), \quad (n_2 - 1) S_y^2 / \sigma^2 \sim \chi^2(n_2 - 1), \quad \bar{y} \perp S_y^2.$$

Since these two samples  $\mathbf{x}$  and  $\mathbf{y}$  are independent, we have

$$\bar{x} - \bar{y} \sim N\left(\mu_1 - \mu_2, \frac{n_1 + n_2}{n_1 n_2} \sigma^2\right),$$

which implies

$$\frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{\frac{n_1 + n_2}{n_1 n_2} \sigma^2}} \sim N(0, 1).$$

And more importantly (the summation of independent  $\chi^2$  variables)

$$\frac{(n_1 - 1) S_x^2 + (n_2 - 1) S_y^2}{\sigma^2} \sim \chi^2(n_1 + n_2 - 2)$$

and

$$\frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{\frac{n_1 + n_2}{n_1 n_2} \sigma^2}} \perp \frac{(n_1 - 1) S_x^2 + (n_2 - 1) S_y^2}{\sigma^2}.$$

This leads us to

$$\frac{\frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{\frac{n_1 + n_2}{n_1 n_2} \sigma^2}}}{\sqrt{\frac{1}{n_1 + n_2 - 2} \frac{(n_1 - 1) S_x^2 + (n_2 - 1) S_y^2}{\sigma^2}}} = \frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \cdot \frac{(n_1 - 1) S_x^2 + (n_2 - 1) S_y^2}{n_1 + n_2 - 2}}} \sim t(n_1 + n_2 - 2).$$

Here we use  $S_p$  to represent the pooled standard deviation of the data, i.e.

$$S_p = \sqrt{\frac{(n_1 - 1) S_x^2 + (n_2 - 1) S_y^2}{n_1 + n_2 - 2}}.$$

Under  $H_0$ , the type-I error is controlled as

$$P\left(\frac{|\bar{x} - \bar{y}|}{\sqrt{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2}} > C \middle| \mu_1 = \mu_2\right) \leq \alpha.$$

Therefore we can write

$$P\left(\frac{|\bar{x} - \bar{y}|}{\sqrt{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2}} > C \middle| \mu_1 = \mu_2\right) = P\left(|T_{(n_1 + n_2 - 2)}| > \frac{C}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \frac{1}{n_1 + n_2 - 2}}}\right) = \alpha.$$

Here in the last part we use the equal sign instead of  $\leq$  for fully utilize the test. So we can construct the test to reject  $H_0$  when

$$\frac{|\bar{x} - \bar{y}|}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_p^2}} > t_{1-\alpha/2}(n_1 + n_2 - 2).$$

### 3.2.2 Power at given underlying $\Delta = \mu_1 - \mu_2$ and $\sigma^2$

The distribution of the test statistics is derived as

$$\begin{aligned}
& \frac{\bar{x} - \bar{y}}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_p^2}} \\
&= \frac{\bar{x} - \bar{y} - \Delta + \Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_p^2}} \\
&= \frac{\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}} + \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}}{\sqrt{\frac{(n_1 + n_2 - 2) S_p^2}{\sigma^2}} \cdot \frac{1}{n_1 + n_2 - 2}} \sim t(n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}).
\end{aligned}$$

So the test statistic follows a non-central t-distribution with degree of freedom  $n_1 + n_2 - 2$  and non-central parameter  $\frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}$ . Then the power of the test

$$\begin{aligned}
& P\left(\frac{|\bar{x} - \bar{y}|}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_p^2}} > t_{1-\alpha/2}(n_1 + n_2 - 2)\right) \\
&= P\left(\left|T\left(n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}\right)\right| > t_{1-\alpha/2}(n_1 + n_2 - 2)\right) \\
&= P\left(T\left(n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}\right) > t_{1-\alpha/2}(n_1 + n_2 - 2)\right) \\
&\quad + P\left(T\left(n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}\right) < -t_{\alpha/2}(n_1 + n_2 - 2)\right)
\end{aligned} \tag{5}$$

### 3.2.3 Sample size at given $\alpha$ , $\beta$ , $\Delta$ and $\sigma^2$

W.l.o.g, assume  $\Delta = \mu_1 - \mu_2 > 0$ . Then in the previous power equation (5),

$$P\left(T\left(n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}\right) < -t_{\alpha/2}(n_1 + n_2 - 2)\right)$$

will be close to zero and

$$P\left(T\left(n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}}\right) > t_{1-\alpha/2}(n_1 + n_2 - 2)\right)$$

will offer most of the power. So to guarantee a  $1 - \beta$  power we can simply set

$$P \left( T \left( n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}} \right) > t_{1-\alpha/2}(n_1 + n_2 - 2) \right) \geq 1 - \beta.$$

Therefore

$$t_{1-\alpha/2}(n_1 + n_2 - 2) \leq t_\beta \left( n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}} \right).$$

If in another way around  $\Delta < 0$ , then we set the power inequality

$$P \left( T \left( n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}} \right) < t_{\alpha/2}(n_1 + n_2 - 2) \right) \geq 1 - \beta,$$

which means

$$t_{\alpha/2}(n_1 + n_2 - 2) \geq t_{1-\beta} \left( n_1 + n_2 - 2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}} \right).$$

So in summary (using  $t_\alpha(v, \delta) + t_{1-\alpha}(v, -\delta) = 0$ ),

$$0 \leq t_{\alpha/2}(n_1 + n_2 - 2) + t_\beta \left( n_1 + n_2 - 2, \frac{|\Delta|}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \sigma^2}} \right).$$

### 3.3 Two-sample, variance unknown and unequal