Gamma and Beta Distribution

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September 8, 2022

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1 Introduction

The Gamma function is defined as

$$\Gamma\left(\alpha\right)=\int_0^{\infty}t^{\alpha-1}e^{-t}\mathrm{d}t. \tag{1}$$

And it's easy to verify that

$$\Gamma\left(\alpha+1\right)=\alpha\Gamma\left(\alpha\right),\quad \Gamma\left(n\right)=\left(n-1\right)!,\quad \Gamma\left(0.5\right)=\sqrt{\pi}.$$

Let $B\left(\alpha,\beta\right)$ denote the Beta function

$$B\left(\alpha,\beta\right)=\int_0^1 x^{\alpha-1}\left(1-x\right)^{\beta-1}\mathrm{d}x. \tag{2}$$

Here we point out that $B\left(\alpha,\beta\right)$ is related to Gamma function via

$$B\left(\alpha,\beta\right)=\frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}.$$

2 Gamma distribution

The pdf for a Gamma distribution is

$$f\left(x|\alpha,\beta\right)=\frac{1}{\Gamma\left(\alpha\right)\beta^{\alpha}}x^{\alpha-1}\exp\left(-x/\beta\right),\quad 0\le x<\infty,\quad \alpha,\beta>0, \tag{3}$$

with

$$\mathrm{E}X=\alpha\beta,\quad \mathrm{Var}X=\alpha\beta^2,$$

and m.g.f.

$$M_X\left(t\right)=\left(\frac{1}{1-\beta t}\right)^{\alpha},\quad t<\frac{1}{\beta}.$$

3 Beta distribution

The pdf for a Beta distribution is

$$f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1, \quad \alpha,\beta > 0, \tag{4}$$

with

$$\mathrm{E}X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

and m.g.f.

$$M_X(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}.$$

4 Relationship with other distribution

- Gamma distribution with $\alpha = 1$ is just exponential distribution.

$$gamma(1,\beta) = \exp(\beta).$$

- Gamma distribution with $\beta = 2$ is just χ^2 distribution with degree of freedom $p = 2\alpha$.

$$gamma(\alpha,2) = \chi^2(2\alpha).$$

- If X follows $gamma(\alpha,\beta)$ where α is an integer. Then for any x

$$P(X \le x) = P(Y \ge \alpha).$$

  where Y follows $Poisson(x/\beta)$.

- If

$$X \sim gamma(\alpha_1,\beta), \quad Y \sim gamma(\alpha_2,\beta), \quad X \perp Y.$$

  Then

$$X+Y \sim gamma(\alpha_1+\alpha_2,\beta), \quad \frac{X}{X+Y} \sim beta(\alpha_1,\alpha_2), \quad (X+Y) \perp \frac{X}{X+Y}.$$

- $$beta(1,1) = Unif(0,1).$$

- $beta\left(\frac{1}{2},\frac{1}{2}\right)$ is the **non-informative Jeffreys prior** for binomial rate test.

- Let $U_1,\cdots,U_n \overset{\text{i.i.d.}}{\sim} Unif(0,1)$. Then the order statistics

$$U_{(k)} \sim beta(k, n+1-k), \quad 1 \le k \le n.$$

  And

$$U_{(k)} - U_{(j)} \sim beta(k-j, n-(k-j)+1), \quad 1 \le j < k \le n.$$

- If X follows a $Binomial(n,p)$, then the c.d.f of X satisfies

$$
\begin{aligned}
&P\left(X \leq k\right) \\
=&\sum_{i=0}^{k} C_n^i p^i \left(1-p\right)^{n-i} \\
=&\frac{\Gamma\left(n+1\right)}{\Gamma\left(n-k\right)\Gamma\left(k+1\right)}\int_0^{1-p} t^{n-k-1}\left(1-t\right)^k \mathrm{dt} = pbeta\left(1-p; n-k, k+1\right) \\
=&P\left(Y \leq 1-p\right),
\end{aligned}
$$

where Y follows $beta\left(n-k, k+1\right)$. This can be verified using integration by parts.

References