T-test

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August 3, 2022

1 Basic knowledge

$\phi(x)$ and $\Phi(x)$ are pdf and cdf of standard normal distribution, respectively. We use Z to represent a random variable that follows standard normal distribution and z_α the lower α quantile of standard normal distribution. Therefore

$$P(Z \leq z_\alpha) = \Phi(z_\alpha) = \alpha.$$

Theorem 1. *Let $x_1, \cdots, x_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then*

1. $\mathrm{E}\bar{x} = \mu$.
2. $\mathrm{Var}\bar{x} = \sigma^2/n$.
3. $\mathrm{E}S^2 = \sigma^2$, *where* $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Theorem 2. *Let $x_1, \cdots, x_n$ be a random sample from $N(\mu, \sigma^2)$. Then*

1. $\bar{X} \sim N(\mu, \sigma^2/n)$.
2. $\bar{X}$ *is independent of* S^2.
3. $(n-1)S^2/\sigma^2$ *follows a chi-squared distribution with* $n-1$ *degree of freedom.*

2 One-sample test

Consider a random sample $x_1, \cdots, x_n$ from $N(\mu, \sigma^2)$. The likelihood is

$$\begin{aligned} f(x_1, \cdots, x_n) &= \prod_{i=1}^{n} \left(2\pi\sigma^2\right)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \\ &= \left(2\pi\sigma^2\right)^{-n/2} \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right). \end{aligned}$$

We propose the test

$$H_0: \ \mu = \mu_0 \quad \textbf{v.s} \quad H_1: \ \mu \neq \mu_0$$

2.1 variance known

Construct LRT

$$LR = \frac{\max_{\mu \in H_0} f(x_1, \cdots, x_n|\mu)}{\max_{\mu \in H_0 \cup H_1} f(x_1, \cdots, x_n|\mu)} = \frac{f(x_1, \cdots, x_n|\mu = \mu_0)}{f(x_1, \cdots, x_n|\mu = \bar{x})} = \exp\left(-\frac{(\bar{x} - \mu_0)^2}{2\sigma^2/n}\right)$$

Therefore rejecting H_0 when LR is smaller than some constant C is equivalent to rejecting H_0 when $|\bar{x} - \mu_0|$ is larger than some other constant C. Hence

$$\text{Reject Region: } \{\bar{x} : \quad |\bar{x} - \mu_0| > C\}$$

2.1.1 Decide C from α

From definition of α we know that C in the reject region is chosen such that

$$P(|\bar{x} - \mu_0| > C | H_0 \text{ is true }) \leq \alpha.$$

But to fully utilize the test, we choose to use equal sign instead of $\leq$. Therefore

$$P(|\bar{x} - \mu_0| > C | \mu = \mu_0) = \alpha.$$

Note that $\bar{x} \sim N(\mu, \sigma^2/n)$. Then under the condition $\mu = \mu_0$,

$$\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0, 1).$$

Therefore we propose the reject region for H_0 being

$$\left|\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}}\right| \geq z_{1-\alpha/2}.$$

Note: Here, even if the sample distribution is not normal, the result still holds due to CLT under large sample.

2.1.2 Power at given underlying μ

The power (the probability to reject H_0, when H_1 is true) of the proposed test procedure for any given underlying $\mu \neq \mu_0$ is computed as

$$\begin{aligned}
&P\left(\left|\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}}\right| \geq z_{1-\alpha/2}\right) \\
=&P\left(\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2}\right) + P\left(\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2}\right) \\
=&P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right) + P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right) \\
=&P\left(Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right) + P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)
\end{aligned} \tag{1}$$

Here we use the fact that $\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} \sim N(0, 1)$.

2.1.3 Sample size at given α, β and underlying μ

W.l.o.g, assume that $\mu > \mu_0$, then in previous power equation (1)

$$P\left(Z \le z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)$$

would be really close to zero and

$$P\left(Z \ge z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)$$

will offer most of the power. In order to guarantee a power of at least $1-\beta$, we could simply set

$$P\left(Z \ge z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right) \ge 1-\beta,$$

which means

$$z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \le z_\beta.$$

Normally in test settings, $\alpha < 0.1$ and $\beta < 0.5$, which means $z_{1-\alpha/2}$ is positive and z_β is negative. Also $\mu_0 - \mu < 0$ in our assumption. This leads to

$$-z_{\alpha/2} - z_\beta \le \frac{\sqrt{n}\,(\mu - \mu_0)}{\sigma}.$$

Hence the sample size requirement is

$$n \ge \frac{\sigma^2 \left(z_{\alpha/2} + z_\beta\right)^2}{(\mu - \mu_0)^2}. \tag{2}$$

Note: The sample size requirement can be deduced the same way when $\mu < \mu_0$. And the result is just the same as (2).

2.2 variance unknown

When σ^2 is unknown, the MLE under H_0 is

$$\mu_{(0)} = \mu_0, \quad \sigma^2_{(0)} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2.$$

And the MLE under $H_0 \cup H_1$ is

$$\mu_{(0\cup 1)} = \bar{x}, \quad \sigma^2_{(0\cup 1)} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Note: MLE for σ^2 offers smaller MSE than S^2, but it's biased.

Then the likelihood ratio is

$$LR = \frac{f\left(x_1,\cdots,x_n \middle| \mu = \mu_{(0)}, \sigma^2 = \sigma^2_{(0)}\right)}{f\left(x_1,\cdots,x_n \middle| \mu = \mu_{(0\cup 1)}, \sigma^2 = \sigma^2_{(0\cup 1)}\right)} = \left(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-n/2} \propto \left(\frac{\sum_{i=1}^{n}(\bar{x}-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-n/2},$$

where for the last part we mainly focus on terms related to μ_0. So to reject H_0 when LR is small is equivalent to

$$\text{Reject Region: } \left\{ \bar{x} : \quad \frac{|\bar{x}-\mu_0|}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}} > C \right\}$$

The idea is similar to that in Section 2.1. But we replace σ^2 with S^2.

2.3 Decide C from α

First we can write

$$P\left(\frac{|\bar{x}-\mu_0|}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}} > C \middle| \mu=\mu_0 \right) = P\left(\frac{|\bar{x}-\mu_0|}{\sqrt{(n-1)\,S^2}} > C \middle| \mu=\mu_0 \right) = \alpha.$$

From Theorem 2 we know that

$$\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1), \quad (n-1)\,S^2/\sigma^2 \sim \chi^2(n-1), \quad \bar{x} \perp S^2$$

Therefore

$$\frac{\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\bar{x}-\mu}{\sqrt{S^2/n}} \sim t(n-1).$$

Then we know the reject rejion is

$$\left| \frac{\bar{x}-\mu}{\sqrt{S^2/n}} \right| > t_{1-\alpha/2}(n-1).$$

Note: Here we need Theorem 2, which means the normal assumption of the sample is **necessary**. Though one might argue that without normal assumption, under large sample scenario, using Slutsky's theorem, asymptotically

$$\frac{\bar{x}-\mu}{\sqrt{S^2/n}} = \frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}}\sqrt{\frac{\sigma^2}{S^2}} \to N(0,1).$$

2.4 Power at given underlying μ and σ^2

Before any computation, we introduce the **non-central** t-distribution.

$$T = \frac{Z+\mu}{\sqrt{V/v}}, \tag{3}$$

where Z follows standard normal and V follows $\chi^2(v)$ and $Z \perp V$. Then T follows a non-central t-distribution with degree of freedom v and non-central parameter μ, denoted by $t(v,\mu)$.

Then we know that

$$\frac{\bar{x}-\mu_0}{\sqrt{S^2/n}} = \frac{\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} + \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} \sim t\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right),$$

which means $\frac{\bar{x}-\mu_0}{\sqrt{S^2/n}}$ follows a non-central t-distribution $t\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right)$. Therefore the power can be computed as

$$\begin{aligned}
&P\left(\left|\frac{\bar{x}-\mu_0}{\sqrt{S^2/n}}\right| \geq t_{1-\alpha/2}(n-1)\right) \\
=&P\left(\left|T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right)\right| \geq t_{1-\alpha/2}(n-1)\right) \\
=&P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \leq t_{\alpha/2}(n-1)\right) + P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}(n-1)\right).
\end{aligned} \tag{4}$$

2.5 Sample size at given α, β and underlying μ and σ^2

W.l.o.g, assume $\mu > \mu_0$, then in the previous power equation (4)

$$P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \leq t_{\alpha/2}(n-1)\right)$$

would be close to zero and

$$P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}(n-1)\right)$$

will offer the most power. In order to guarantee a power of at least $1-\beta$, we could simply set

$$P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}(n-1)\right) \geq 1-\beta,$$

which means

$$t_{1-\alpha/2}(n-1) \leq t_\beta\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right).$$

There's no close form for this inequality, we should use some numerical method to solve for n.

Note: If $\mu < \mu_0$, then similarly we can get the requirement as

$$t_{\alpha/2}(n-1) \geq t_{1-\beta}\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right).$$

Use the fact that $t_\alpha(n,\mu) = -t_{1-\alpha}(n,-\mu)$, we can arrange the previous inequality as

$$t_{1-\alpha/2}(n-1) \leq t_\beta\left(n-1, \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right).$$

Therefore in summary the sample size requirement is

$$t_{1-\alpha/2}\left(n-1\right) \leq t_{\beta}\left(n-1, \frac{|\mu_0-\mu|}{\sqrt{\sigma^2/n}}\right).$$

3 Two sample test

3.1 Two-sample, variance known

3.2 Two-sample, variance unknown but equal

3.3 Two-sample, variance unknown and unequal