Gamma and Beta Distribution

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1 Introduction

The Gamma function is defined as

$$\Gamma\left(\alpha\right) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t. \tag{1}$$

And it's easy to verify that

$$\Gamma\left(\alpha+1\right) = \alpha\Gamma\left(\alpha\right), \quad \Gamma\left(n\right) = (n-1)!, \quad \Gamma\left(0.5\right) = \sqrt{\pi}.$$

Let $B\left(\alpha,\beta\right)$ denote the Beta function

$$B\left(\alpha,\beta\right) = \int_0^1 x^{\alpha-1}\left(1-x\right)^{\beta-1}\mathrm{d}x. \tag{2}$$

Here we point out that $B\left(\alpha,\beta\right)$ is related to Gamma function via

$$B\left(\alpha,\beta\right) = \frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}.$$

2 Gamma distribution

The pdf for a Gamma distribution is

$$f\left(x|\alpha,\beta\right) = \frac{1}{\Gamma\left(\alpha\right)\beta^\alpha}x^{\alpha-1}\exp\left(-x/\beta\right), \quad 0 \le x < \infty, \quad \alpha,\beta > 0. \tag{3}$$

3 Beta distribution

The pdf for a Beta distribution is

$$f\left(x|\alpha,\beta\right)=\frac{1}{B\left(\alpha,\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1}, \quad 0\leq x\leq 1, \quad \alpha,\beta>0. \tag{4}$$

4 Relationship with other distribution

- Gamma distribution with $\alpha=1$ is just exponential distribution.

$$gamma\left(1,\beta\right)=\exp\left(\beta\right).$$

- Gamma distribution with $\beta=2$ is just χ^2 distribution with degree of freedom $p=2\alpha$.

$$gamma\left(\alpha,2\right)=\chi^2\left(2\alpha\right).$$

- If X follows $gamma\left(\alpha,\beta\right)$ where α is an integer. Then for any x

$$P\left(X\leq x\right)=P\left(Y\geq\alpha\right).$$

  where Y follows $Poisson\left(x/\beta\right)$.

- If

$$X\sim gamma\left(\alpha_1,\beta\right), \quad Y\sim gamma\left(\alpha_2,\beta\right), \quad X\perp Y.$$

  Then

$$X+Y\sim gamma\left(\alpha_1+\alpha_2,\beta\right), \quad \frac{X}{X+Y}\sim beta\left(\alpha_1,\alpha_2\right), \quad \left(X+Y\right)\perp\frac{X}{X+Y}.$$

- $$beta\left(1,1\right)=Unif\left(0,1\right).$$

- $beta\left(\frac{1}{2},\frac{1}{2}\right)$ is the **non-informative** prior for binomial rate test.

References