Cox Proportional Hazard Model

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December 25, 2022

Contents

1 Introduction

In this note we will talk about the Cox's proportional hazards (Cox's PH) model. Suppose we observe some non-informativly right-censored data (U, δ) with covariate vector Z. That is, for subject i, the covariate vector is Z_i, survival time T_i and censoring time C_i. The observed data is (U_i, δ_i) where $U_i = \min(T_i, C_i)$ and $\delta_i = 1(T_i \leq C_i)$. Also $T_i \perp C_i | Z_i$.

And now we want to model the relationship between Z and T. One way to do that is to incorporate Z into the hazard function $h(\cdot)$, e.g.,

$$T \sim Exp(\lambda_Z) \implies h(t) = \lambda_Z \triangleq e^{\alpha + \beta Z} = \lambda_0 e^{\beta Z},$$

where $\lambda_0 = e^{\alpha}$ can be viewed as a baseline hazard. If $\beta = 0$ then Z is not associated with T.

We can generalize this idea as

$$h(t|Z) = h_0(t) \times g(Z).$$

So the hazard can be factorized and this model is sometimes called a "multiplicative intensive model" or "multiplicative hazard model" or "proportional hazard model" because this factorization implies that

$$\frac{h(t|Z = z_1)}{h(t|Z = z_2)} = \frac{g(z_1)}{g(z_2)}.$$

The hazard ratio is constant with respect to t, hence the (constant) proportional hazard. So in our previous model (the exponential survival time), the hazard ratio is

$$\frac{h\left(t|Z=z_1\right)}{h\left(t|Z=z_2\right)}=e^{\beta(z_1-z_2)}.$$

Also this exponential form of $g\left(Z\right)$

$$h\left(t|Z\right)=h_0\left(t\right)\cdot e^{\beta Z} \tag{1}$$

is the **Cox's PH** model.

2 Estimation

(1) implies that

$$\begin{aligned}
S\left(t|Z\right)=&\exp\left(-H\left(t|Z\right)\right)\\
=&\exp\left(-\int_0^t h\left(u|Z\right)\mathrm{d}u\right)\\
=&\exp\left(-\int_0^t h_0\left(t\right)\mathrm{d}u\cdot g\left(Z\right)\right)\\
=&\left(S_0\left(t\right)\right)^{g(Z)}=\left(S_0\left(t\right)\right)^{\exp(\beta Z)},
\end{aligned}$$

where $S_0\left(t\right)=\exp\left(-\int_0^t h_0\left(u\right)\mathrm{d}u\right)$, the survival function for $Z=0$, hence $S\left(t|Z=0\right)$. Also remember that $f\left(t|Z\right)=h\left(t|Z\right)S\left(t|Z\right)$. Thus, given n independent data $\left(u_i,\delta_i,z_i\right)$, the likelihood (one can refer to our previous notes about survival analysis.) is

$$\begin{aligned}
L\left(\beta,h_0\left(\cdot\right)\right)=&\prod_{i=1}^n\left(f\left(u_i|z_i\right)\right)^{\delta_i}\left(S\left(u_i|z_i\right)\right)^{1-\delta_i}=\prod_{i=1}^n h\left(u_i|z_i\right)^{\delta_i}S\left(u_i|z_i\right)\\
=&\prod_{i=1}^n\left(h_0\left(u_i\right)e^{\beta z_i}\right)^{\delta_i}\left(\exp\left(-\int_0^{u_i}h_0\left(t\right)\mathrm{d}t\right)\right)^{\exp(\beta z_i)}\\
=&\text{function}\left(data,h_0\left(\cdot\right),\beta\right).
\end{aligned} \tag{2}$$

If $h_0\left(\cdot\right)$ is allowed to be "arbitary", then the "parameter space " is

$$\mathcal{H}\times\mathcal{R}^p=\left\{\left(h\left(\cdot\right),\beta\right)\middle|h_0\left(\cdot\right)\geq 0,\int_0^\infty h_0\left(t\right)\mathrm{d}t=\infty,\beta\in\mathcal{R}^p\right\},$$

where $\int_0^\infty h_0\left(t\right)\mathrm{d}t=\infty$ ensures that $S_0\left(\infty\right)=0$.

In general this likelihood is hard to maximize. And Cox proposed this idea: to factor $L\left(\beta,h_0\left(\cdot\right)\right)$ as

$$L\left(\beta,h_0\left(\cdot\right)\right)=L_1\left(\beta\right)\times L_2\left(\beta,h_0\left(\cdot\right)\right),$$

where L_1 only depends on β and its maximization $(\hat{\beta})$ enjoys nice properties such as consistency and asymptotic normality while L_2 contains relatively little information about β. And this L_1 is called a **partial likelihood**.

2.1 What is $L_1(\beta)$

In this section we introduce the L_1 proposed by Cox. First let's assume there are **NO tied** nor censoring observations. And define the distinct times of failure $\tau_1 < \tau_2 < \cdots$. Denote

$$R_j = \{i | U_i \geq \tau_j\} = \text{risk set at } \tau_j,$$

and

$$Z_{(j)} = \text{value of Z for the subject who fails at } \tau_j.$$

we can reconstruct the data from $\{\tau_j\}$, $\{R_j\}$ and $\{Z_{(j)}\}$. And L_1 is defined as

$$L_1(\beta) \triangleq \prod_j \left\{ \frac{e^{\beta Z_{(j)}}}{\sum_{l \in R_j} e^{\beta Z_l}} \right\} \tag{3}$$

Note that under this setting (no tie, no censor), the full likelihood (2) becomes

$$L(\beta, h_0(\cdot)) = \prod_{i=1}^{n} h_0(u_i) e^{\beta z_i} \left(\exp\left(- \int_0^{u_i} h_0(t) \, \mathrm{d}t \right) \right)^{\exp(\beta z_i)}.$$

Furthermore, we can assume $u_i = \tau_i$, i.e. the data has been sorted based on survival time. And use the KM idea, i.e. assume the survival function is **discrete** with baseline hazard value h_j at u_j. Then this likelihood becomes

$$L(\beta, h_1, \cdots, h_n) = \prod_{i=1}^{n} h_i e^{\beta z_i} \exp\left(- \sum_{j=1}^{i} h_j \right)^{\exp(\beta z_i)}. \tag{4}$$

Note that, in previous notes we have deduct that in discrete case, for any $t \in [v_j, v_{j+1})$:

$$H(t) = \sum_{i=1}^{j} h_i \qquad S(t) = \prod_{i=1}^{j} (1 - h_i).$$

Here in (4) we use the approximation that $e^{-h_j} \approx 1 - h_j$ when h_j is close to 0.

2.1.1 Intuition: profile likelihood perspective

We can use the method of profile likelihood: That is, for any given β, we maximize L (or equivalently, $\log L$) over h_js so the result is a function of β. Taking derivative, we have

$$\frac{\partial \log L}{\partial h_j} = \frac{1}{h_j} - \sum_{i \leq j} \exp(\beta z_i), \qquad j = 1, \cdots, n.$$

Set them to 0 we have $\hat{h}_j = 1/\sum\limits_{i \leq j} \exp(\beta z_i)$. And the log profile likelihood of β is

$$\begin{aligned}
\log L_{profile}(\beta) =& \log \left\{ \prod_{i=1}^{n} \frac{\exp(\beta z_i)}{\sum\limits_{k \leq i} \exp(\beta z_k)} \exp \left(-\sum_{j=1}^{i} \frac{1}{\sum\limits_{k \leq j} \exp(\beta z_k)} \right)^{\exp(\beta z_i)} \right\} \\
=& \sum_{i=1}^{n} \left\{ \log \left\{ \frac{\exp(\beta z_i)}{\sum\limits_{k \leq i} \exp(\beta z_k)} \exp \left(-\sum_{j=1}^{i} \frac{1}{\sum\limits_{k \leq j} \exp(\beta z_k)} \right)^{\exp(\beta z_i)} \right\} \right\} \\
=& \sum_{i=1}^{n} \left\{ \log \left(\frac{\exp(\beta z_i)}{\sum\limits_{k \leq i} \exp(\beta z_k)} \right) - \exp(\beta z_i) \cdot \left(\sum_{j=1}^{i} \frac{1}{\sum\limits_{k \leq j} \exp(\beta z_k)} \right) \right\} \\
=& \sum_{i=1}^{n} \left\{ \log \left(\frac{\exp(\beta z_i)}{\sum\limits_{k \leq i} \exp(\beta z_k)} \right) \right\} - \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{\exp(\beta z_i)}{\sum\limits_{k \leq j} \exp(\beta z_k)},
\end{aligned} \tag{5}$$

where the second part of last equation can be reduced to $-n$?, which means

$$L_{profile}(\beta) \propto \prod_{i=1}^{n} \frac{\exp(\beta z_i)}{\sum\limits_{k \leq i} \exp(\beta z_k)}.$$

And this is what Cox uses as $L_1(\beta)$.

2.1.2 Intuition: conditional distribution perspective

To be added.

2.1.3 What if there is censoring?

Then (3) is still used.

2.1.4 What if there are tied event times?

Exact partial likelihood

Breslow's approximation

Efron's approximation

3 Inference

References