Gamma and Beta Distribution

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1 Introduction

The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t. \tag{1}$$

And it's easy to verify that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(n) = (n-1)!, \quad \Gamma(0.5) = \sqrt{\pi}.$$

Let $B(\alpha, \beta)$ denote the Beta function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \mathrm{d}x. \tag{2}$$

Here we point out that $B(\alpha, \beta)$ is related to Gamma function via

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

2 Gamma distribution

The pdf for a Gamma distribution is

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta), \quad 0 \le x < \infty, \quad \alpha, \beta > 0, \tag{3}$$

with

$$\mathrm{E}X = \alpha\beta, \quad \mathrm{Var}X = \alpha\beta^2,$$

and m.g.f.

$$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}.$$

3 Beta distribution

The pdf for a Beta distribution is

$$f\left(x|\alpha,\beta\right)=\frac{1}{B\left(\alpha,\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1},\quad 0\leq x\leq 1,\quad \alpha,\beta>0, \tag{4}$$

with

$$\mathrm{E}X=\frac{\alpha}{\alpha+\beta},\quad \mathrm{Var}X=\frac{\alpha\beta}{\left(\alpha+\beta\right)^2\left(\alpha+\beta+1\right)}$$

and m.g.f.

$$M_X\left(t\right)=1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}.$$

4 Relationship with other distribution

- Gamma distribution with $\alpha=1$ is just exponential distribution.

$$gamma\left(1,\beta\right)=\exp\left(\beta\right).$$

- Gamma distribution with $\beta=2$ is just χ^2 distribution with degree of freedom $p=2\alpha$.

$$gamma\left(\alpha,2\right)=\chi^2\left(2\alpha\right).$$

- If X follows $gamma\left(\alpha,\beta\right)$ where α is an integer. Then for any x

$$P\left(X\leq x\right)=P\left(Y\geq\alpha\right).$$

  where Y follows $Poisson\left(x/\beta\right)$.

- If

$$X\sim gamma\left(\alpha_1,\beta\right),\quad Y\sim gamma\left(\alpha_2,\beta\right),\quad X\perp Y.$$

  Then

$$X+Y\sim gamma\left(\alpha_1+\alpha_2,\beta\right),\quad \frac{X}{X+Y}\sim beta\left(\alpha_1,\alpha_2\right),\quad \left(X+Y\right)\perp\frac{X}{X+Y}.$$

- $$beta\left(1,1\right)=Unif\left(0,1\right).$$

- $beta\left(\frac{1}{2},\frac{1}{2}\right)$ is the **non-informative Jeffreys prior** for binomial rate test.

- Let $U_1,\cdots,U_n\overset{\text{i.i.d.}}{\sim}Unif(0,1)$. Then the order statistics

$$U_{(k)}\sim\ beta\left(k,n+1-k\right),\quad 1\leq k\leq n.$$

  And

$$U_{(k)}-U_{(j)}\sim\ beta\left(k-j,n-\left(k-j\right)+1\right),\quad 1\leq j<k\leq n.$$

- If X follows a $Binomial(n, p)$, then the c.d.f of X satisfies

$$\begin{aligned}
&P\left(X \leq k\right) \\
=&\sum_{i=0}^{k} C_n^i p^i \left(1-p\right)^{n-i} \\
=&\frac{\Gamma\left(n+1\right)}{\Gamma\left(n-k\right)\Gamma\left(k+1\right)} \int_0^{1-p} t^{n-k-1}\left(1-t\right)^k \mathrm{dt} = pbeta\left(1-p; n-k, k+1\right) \\
=&P\left(Y \leq 1-p\right),
\end{aligned}$$

where Y follows $beta\left(n-k, k+1\right)$. This can be verified using integration by parts.

References