

T-test

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1 Basic knowledge

$\phi(x)$ and $\Phi(x)$ are pdf and cdf of standard normal distribution, respectively. We use Z to represent a random variable that follows standard normal distribution and z_α the lower α quantile of standard normal distribution. Therefore

$$P(Z \leq z_\alpha) = \Phi(z_\alpha) = \alpha.$$

Theorem 1. Let x_1, \dots, x_n be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then

1. $E\bar{x} = \mu$.
2. $\text{Var}\bar{x} = \sigma^2/n$.
3. $ES^2 = \sigma^2$, where $S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$.

Theorem 2. Let x_1, \dots, x_n be a random sample from $N(\mu, \sigma^2)$. Then

1. $\bar{X} \sim N(\mu, \sigma^2/n)$.
2. \bar{X} is independent of S^2 .
3. $(n-1)S^2/\sigma^2$ follows a chi-squared distribution with $n-1$ degree of freedom.

2 One-sample test

Consider a random sample x_1, \dots, x_n from $N(\mu, \sigma^2)$. The likelihood is

$$\begin{aligned} f(x_1, \dots, x_n) &= \prod_{i=1}^n (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2}\right). \end{aligned}$$

We propose the test

$$H_0 : \mu = \mu_0 \quad \text{v.s.} \quad H_1 : \mu \neq \mu_0$$

2.1 variance known

Construct LRT

$$LR = \frac{\max_{\mu \in H_0} f(x_1, \dots, x_n | \mu)}{\max_{\mu \in H_0 \cup H_1} f(x_1, \dots, x_n | \mu)} = \frac{f(x_1, \dots, x_n | \mu = \mu_0)}{f(x_1, \dots, x_n | \mu = \bar{x})} = \exp \left(-\frac{(\bar{x} - \mu_0)^2}{2\sigma^2/n} \right)$$

Therefore rejecting H_0 when LR is smaller than some constant C is equivalent to rejecting H_0 when $|\bar{x} - \mu_0|$ is larger than some other constant C . Hence

$$\text{Reject Region: } \{\bar{x} : |\bar{x} - \mu_0| > C\}$$

2.1.1 Decide C from α

From definition of α we know that C in the reject region is chosen such that

$$P(|\bar{x} - \mu_0| > C | H_0 \text{ is true}) \leq \alpha.$$

But to fully utilize the test, we choose to use equal sign instead of \leq . Therefore

$$P(|\bar{x} - \mu_0| > C | \mu = \mu_0) = \alpha.$$

Note that $\bar{x} \sim N(\mu, \sigma^2/n)$. Then under the condition $\mu = \mu_0$,

$$\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0, 1).$$

Therefore we propose the reject region for H_0 being

$$\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2}.$$

Note: Here, even if the sample distribution is not normal, the result still holds due to CLT under large sample.

2.1.2 Power at given underlying μ

The power (the probability to reject H_0 , when H_1 is true) of the proposed test procedure for any given underlying $\mu \neq \mu_0$ is computed as

$$\begin{aligned} & P \left(\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2} \right) \\ &= P \left(\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} \right) + P \left(\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2} \right) \\ &= P \left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) + P \left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) \\ &= P \left(Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) + P \left(Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \right) \end{aligned} \tag{1}$$

Here we use the fact that $\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sim N(0, 1)$.

2.1.3 Sample size at given α , β and underlying μ

W.l.o.g, assume that $\mu > \mu_0$, then in previous power equation (1)

$$P\left(Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)$$

would be really close to zero and

$$P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right)$$

will offer most of the power. In order to guarantee a power of at least $1 - \beta$, we could simply set

$$P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right) \geq 1 - \beta,$$

which means

$$z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} \leq z_\beta.$$

Normally in test settings, $\alpha < 0.1$ and $\beta < 0.5$, which means $z_{1-\alpha/2}$ is positive and z_β is negative. Also $\mu_0 - \mu < 0$ in our assumption. This leads to

$$-z_{\alpha/2} - z_\beta \leq \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}.$$

Hence the sample size requirement is

$$n \geq \frac{\sigma^2 (z_{\alpha/2} + z_\beta)^2}{(\mu - \mu_0)^2}. \quad (2)$$

Note: The sample size requirement can be deduced the same way when $\mu < \mu_0$. And the result is just the same as (2).

2.2 variance unknown

When σ^2 is unknown, the MLE under H_0 is

$$\mu_{(0)} = \mu_0, \quad \sigma_{(0)}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2.$$

And the MLE under $H_0 \cup H_1$ is

$$\mu_{(0 \cup 1)} = \bar{x}, \quad \sigma_{(0 \cup 1)}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2.$$

Note: MLE for σ^2 offers smaller MSE than S^2 , but it's biased.

Then the likelihood ratio is

$$LR = \frac{f(x_1, \dots, x_n | \mu = \mu_{(0)}, \sigma^2 = \sigma_{(0)}^2)}{f(x_1, \dots, x_n | \mu = \mu_{(0 \cup 1)}, \sigma^2 = \sigma_{(0 \cup 1)}^2)} = \left(\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)^{-n/2} \propto \left(\frac{\sum_{i=1}^n (\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right)^{-n/2},$$

where for the last part we mainly focus on terms related to μ_0 . So to reject H_0 when LR is small is equivalent to

$$\text{Reject Region: } \left\{ \bar{x} : \frac{|\bar{x} - \mu_0|}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} > C \right\}$$

The idea is similar to that in Section 2.1. But we replace σ^2 with S^2 .

2.3 Decide C from α

First we can write

$$P \left(\frac{|\bar{x} - \mu_0|}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} > C \middle| \mu = \mu_0 \right) = P \left(\frac{|\bar{x} - \mu_0|}{\sqrt{(n-1)S^2}} > C \middle| \mu = \mu_0 \right) = \alpha.$$

From Theorem 2 we know that

$$\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sim N(0, 1), \quad (n-1)S^2/\sigma^2 \sim \chi^2(n-1), \quad \bar{x} \perp S^2$$

Therefore

$$\frac{\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\bar{x} - \mu}{\sqrt{S^2/n}} \sim t(n-1).$$

Then we know the reject region is

$$\left| \frac{\bar{x} - \mu}{\sqrt{S^2/n}} \right| > t_{1-\alpha/2}(n-1).$$

Note: Here we need Theorem 2, which means the normal assumption of the sample is **necessary**. Though one might argue that without normal assumption, under large sample scenario, using Slutsky's theorem, asymptotically

$$\frac{\bar{x} - \mu}{\sqrt{S^2/n}} = \frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sqrt{\frac{\sigma^2}{S^2}} \rightarrow N(0, 1).$$

2.4 Power at given underlying μ and σ^2

Before any computation, we introduce the **non-central** t-distribution.

$$T = \frac{Z + \mu}{\sqrt{V/v}}, \tag{3}$$

where Z follows standard normal and V follows $\chi^2(v)$ and $Z \perp V$. Then T follows a non-central t-distribution with degree of freedom v and non-central parameter μ , denoted by $t(v, \mu)$.

Then we know that

$$\frac{\bar{x} - \mu_0}{\sqrt{S^2/n}} = \frac{\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} + \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} \sim t\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right),$$

which means $\frac{\bar{x} - \mu_0}{\sqrt{S^2/n}}$ follows a non-central t-distribution $t\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right)$. Therefore the power can be computed as

$$\begin{aligned} & P\left(\left|\frac{\bar{x} - \mu_0}{\sqrt{S^2/n}}\right| \geq t_{1-\alpha/2}(n-1)\right) \\ &= P\left(\left|T\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right)\right| \geq t_{1-\alpha/2}(n-1)\right) \\ &= P\left(T\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right) \leq t_{\alpha/2}(n-1)\right) + P\left(T\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}(n-1)\right). \end{aligned} \tag{4}$$

2.5 Sample size at given α , β and underlying μ and σ^2

W.l.o.g, assume $\mu > \mu_0$, then in the previous power equation (4)

$$P\left(T\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right) \leq t_{\alpha/2}(n-1)\right)$$

would be close to zero and

$$P\left(T\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}(n-1)\right)$$

will offer the most power. In order to guarantee a power of at least $1 - \beta$, we could simply set

$$P\left(T\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}(n-1)\right) \geq 1 - \beta,$$

which means

$$t_{1-\alpha/2}(n-1) \leq t_{\beta}\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right).$$

There's no close form for this inequality, we should use some numerical method to solve for n .

Note: If $\mu < \mu_0$, then similarly we can get the requirement as

$$t_{\alpha/2}(n-1) \geq t_{1-\beta}\left(n-1, \frac{\mu - \mu_0}{\sqrt{\sigma^2/n}}\right).$$

Use the fact that $t_{\alpha}(n, \mu) = -t_{1-\alpha}(n, -\mu)$, we can arrange the previous inequality as

$$t_{1-\alpha/2}(n-1) \leq t_{\beta}\left(n-1, \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}}\right).$$

Therefore in summary the sample size requirement is

$$t_{1-\alpha/2}(n-1) \leq t_\beta \left(n-1, \frac{|\mu_0 - \mu|}{\sqrt{\sigma^2/n}} \right).$$

3 Two sample test

Consider two random samples $x_1, \dots, x_{n_1} \sim N(\mu_1, \sigma_1^2)$ and $y_1, \dots, y_{n_2} \sim N(\mu_2, \sigma_2^2)$. Then the likelihood of the data is

$$\begin{aligned} & f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2} | \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \\ &= (2\pi\sigma_1^2)^{-n_1/2} (2\pi\sigma_2^2)^{-n_2/2} \exp \left(-\frac{\sum_{i=1}^{n_1} (x_i - \mu_1)^2}{2\sigma_1^2} - \frac{\sum_{i=1}^{n_2} (y_i - \mu_2)^2}{2\sigma_2^2} \right) \\ &= \end{aligned}$$

We propose the test

$$H_0 : \mu_1 = \mu_2 \quad \text{v.s.} \quad H_1 : \mu_1 \neq \mu_2.$$

3.1 Two-sample, variance known

When σ_1^2 and σ_2^2 are known, the likelihood satisfies

$$f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2} | \mu_1, \mu_2) \propto \exp \left(-\frac{n_1 (\bar{x} - \mu_1)^2}{2\sigma_1^2} - \frac{n_2 (\bar{y} - \mu_2)^2}{2\sigma_2^2} \right).$$

Therefore under H_0 , the MLE for μ_1 and μ_2 is

$$\mu_{1(0)} = \mu_{2(0)} = \mu_{(0)} = \frac{\sigma_2^2 n_1 \bar{x} + \sigma_1^2 n_2 \bar{y}}{\sigma_2^2 n_1 + \sigma_1^2 n_2}.$$

And under $H_0 \cup H_1$, the MLE for μ_1 and μ_2 is

$$\mu_{1(0 \cup 1)} = \bar{x}, \quad \mu_{2(0 \cup 1)} = \bar{y}.$$

Then the likelihood ratio is

$$\begin{aligned} LR &= \frac{\max_{H_0} f(\mathbf{x}, \mathbf{y} | \mu_1, \mu_2)}{\max_{H_0 \cup H_1} f(\mathbf{x}, \mathbf{y} | \mu_1, \mu_2)} \\ &= \frac{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_2 = \mu_{(0)})}{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_{1(0 \cup 1)}, \mu_2 = \mu_{2(0 \cup 1)})} \\ &\propto \exp \left(-\frac{1}{2} \left(\frac{n_1 (\bar{x} - \mu_{(0)})^2}{\sigma_1^2} + \frac{n_2 (\bar{y} - \mu_{(0)})^2}{\sigma_2^2} \right) \right) \\ &= \exp \left(-\frac{1}{2} \frac{n_1 n_2}{\sigma_2^2 n_1 + \sigma_1^2 n_2} (\bar{x} - \bar{y})^2 \right). \end{aligned}$$

From the idea of LRT, H_0 is rejected when LR is small enough, which means the reject rule is

$$\text{Reject Region: } \{(\bar{x}, \bar{y}) | |\bar{x} - \bar{y}| > C\}$$

3.1.1 Decide C from α

From the definition of α we know that

$$P(|\bar{x} - \bar{y}| > C | H_0 \text{ is true}) \leq \alpha.$$

Note that $\bar{x} \sim N(\mu_1, \sigma_1^2/n_1)$, $\bar{y} \sim N(\mu_2, \sigma_2^2/n_2)$ and $\bar{x} \perp \bar{y}$. Therefore

$$\bar{x} - \bar{y} \sim N(\mu_1 - \mu_2, \sigma_1^2/n_1 + \sigma_2^2/n_2).$$

Then under H_0 , $\mu_1 = \mu_2$ and

$$\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1),$$

which means

$$P(|\bar{x} - \bar{y}| > C | \mu_1 = \mu_2) = P\left(|Z| > \frac{C}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \leq \alpha.$$

Here to fully utilize the test, we choose the equal sign. Hence

$$z_{1-\alpha/2} = \frac{C}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}.$$

Here the reject region is

$$\frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2}.$$

Note: Even the samples does not follow normal distribution, by CLT this test still holds true for large sample.

3.1.2 Power at given $\Delta = \mu_1 - \mu_2$

The power at a given $\Delta = \mu_1 - \mu_2$ is

$$\begin{aligned} & P\left(\frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2} \middle| \Delta = \mu_1 - \mu_2\right) \\ &= P\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2}\right) + P\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z_{\alpha/2}\right) \\ &= P\left(\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) + P\left(\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z_{\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \\ &= P\left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) + P\left(Z < z_{\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \end{aligned}$$

3.1.3 Sample size at given α , β , Δ and $k = \frac{n_1}{n_2}$

W.l.o.g, assume $\Delta > 0$, then the power of the test comes mostly from

$$P \left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right).$$

So to achieve the power, we can set

$$P \left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \right) \geq 1 - \beta.$$

And this means

$$z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq z_\beta.$$

Rearrange this inequality we have

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \leq \frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2}.$$

1. When n_1 is fixed and given, we need

$$n_2 \geq \frac{\sigma_2^2}{\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} - \frac{\sigma_1^2}{n_1}}.$$

Also this fixed and given n_1 must satisfy

$$\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} > \frac{\sigma_1^2}{n_1}$$

for the test to be feasible.

2. Similarly, when n_2 is given and fixed, we need

$$n_1 \geq \frac{\sigma_1^2}{\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} - \frac{\sigma_2^2}{n_2}}.$$

Also this fixed and given n_2 must satisfy

$$\frac{\Delta^2}{(z_{\alpha/2} + z_\beta)^2} > \frac{\sigma_2^2}{n_2}$$

for the test to be feasible.

3. For a fixed and given sample size ratio $k = n_2/n_1$, we need

$$n_1 \geq \frac{(\sigma_1^2 + \sigma_2^2/k) (z_{\alpha/2} + z_\beta)^2}{\Delta^2}$$

Note: These results hold the same form when $\Delta < 0$.

3.2 Two-sample, variance unknown but equal

When σ_1 and σ_2 are both unknown but equal, denoted by σ . The likelihood of the data becomes

$$\begin{aligned} & f(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2} | \mu_1, \mu_2, \sigma^2) \\ &= (2\pi\sigma^2)^{-n_1/2-n_2/2} \exp \left(-\frac{\sum_{i=1}^{n_1} (x_i - \mu_1)^2 + \sum_{i=1}^{n_2} (y_i - \mu_2)^2}{2\sigma^2} \right). \end{aligned}$$

So the MLE under H_0 is

$$\mu_{1(0)} = \mu_{2(0)} = \mu_{(0)} = \frac{n_1\bar{x} + n_2\bar{y}}{n_1 + n_2}, \quad \sigma_{(0)}^2 = \frac{\sum_{i=1}^{n_1} (x_i - \mu_{(0)})^2 + \sum_{i=1}^{n_2} (y_i - \mu_{(0)})^2}{n_1 + n_2}$$

And the MLE under $H_0 \cup H_1$ is

$$\mu_{1(0 \cup 1)} = \bar{x}, \quad \mu_{2(0 \cup 1)} = \bar{y}, \quad \sigma_{(0 \cup 1)}^2 = \frac{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2}{n_1 + n_2}.$$

Then the likelihood ratio is

$$\begin{aligned} LR &= \frac{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_2 = \mu_{(0)}, \sigma^2 = \sigma_{(0)}^2)}{f(\mathbf{x}, \mathbf{y} | \mu_1 = \mu_{1(0 \cup 1)}, \mu_2 = \mu_{2(0 \cup 1)}, \sigma^2 = \sigma_{(0 \cup 1)}^2)} \\ &= \left(\frac{\sigma_{(0)}^2}{\sigma_{(0 \cup 1)}^2} \right)^{-n_1/2-n_2/2} \end{aligned}$$

3.3 Two-sample, variance unknown and unequal