Pearson's Chi-square Test

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There are mainly two types of situations that's suitable for a Pearson's Chi-square test. The first is to test one sample against a given vector, the so-called goodness-of-fit test. And the second is to test the existence of correlation between two samples, the so-called contingency/independence/association test.

1 Effect size index w

The Effect size index w from Chapter 7 in Cohen [2013] is

$$w = \sqrt{\sum_{i=1}^{m} \frac{\left(P_{1i} - P_{0i}\right)^2}{P_{0i}}}, \tag{1}$$

where

- m is the number of cell.
- P_{0i} is the **propotion** in cell i proposed by the null hypothesis.
- P_{1i} is the **propotion** in cell i proposed by the alternative hypothesis and refects the effect for that cell.

2 Test statistics χ^2

The test statistic is just

$$\chi_T^2 = nw^2 = \sum_{i=1}^{m} \frac{\left(nP_{1i} - nP_{0i}\right)^2}{nP_{0i}}. \tag{2}$$

3 Goodness of fit test

Let $\boldsymbol{x} \in \mathcal{R}^m$ be a sample from $multinomial(n, \boldsymbol{p})$ where n is the number of trials and $\boldsymbol{p} = (p_1, \cdots, p_m)^T$ and $\sum\limits_{i=1}^{m} p_i = 1$. Here we assume $p_i > 0$ for all i to eliminate some edge cases where some nomial is utterly impossible to happen. Then the probability of any given $\boldsymbol{x} = (x_1, \cdots, x_m)^T$ is

$$P\left(\boldsymbol{x} = (x_1, \cdots, x_m)^T\right) = \prod_{i=1}^{m} p_i^{x_i},$$

where $\sum_{i=1}^{m} x_i = n$. This $\boldsymbol{x}$ can also be seen as the summation of n samples $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ where each $\boldsymbol{x}_i \in \mathcal{R}^m$ follows $multinomial(1, \boldsymbol{p})$. And one and only one entry in each $\boldsymbol{x}_i$ is a single one while others $m-1$ entries all remain zero.

Based on this observed $\boldsymbol{x}$, we want to test its underlying distribution $\boldsymbol{p}$ against a given vector $\boldsymbol{p}_0 = (p_{01}, \cdots, p_{0m})^T$. And from (2) we know that the test statistic is

$$\chi_T^2 = \sum_{i=1}^{m} \frac{(x_i - np_{0i})^2}{np_{0i}}. \tag{3}$$

3.1 Reject rule

Under null hypothesis, this test statistics follows a χ^2 distribution with degree of freedom being $m-1$. Proof for this statement can be found in Chapter 9 Pearson's chi-square test in David R. Hunter's **Notes for a graduate-level course in asymptotics for statisticians** [Hunter, 2014]. And we reject H_0 when this test statistic χ_T^2 is large enough.

3.2 Power analysis

Under alternative hypothesis, i.e. $\boldsymbol{p} = (p_1, \cdots, p_m)^T \neq \boldsymbol{p}_0$. Denote $\boldsymbol{\delta} = \sqrt{n}\,(\boldsymbol{p} - \boldsymbol{p}_0)$ and $\boldsymbol{\Gamma} = \mathrm{diag}\,(\boldsymbol{p}_0)$. Then the test statistic now follows a **non-central chi-square distribution** with non-central parameter

$$\lambda = \boldsymbol{\delta}^T \boldsymbol{\Gamma}^{-1} \boldsymbol{\delta}.$$

Non-central chi-square distribution: Let $x_1, \cdots, x_n$ be independent normal distribution with means $\mu_1, \cdots, \mu_n$ and unit variance. Then $\sum_{i=1}^{n} x_i^2$ follows a non-central chi-square distribution with non-central parameter being

$$\lambda = \sum_{i=1}^{n} \mu_i^2$$

and degree of freedom being n. And the pdf of $X = \sum x_i$ is given by

$$f\,(x; n, \lambda) = \exp\,(-\lambda/2) \sum_{i=0}^{\infty} \frac{(\lambda/2)^i}{i!} f_{n+2i}\,(x)\,,$$

where $f_n\,(x)$ stands for the pdf of a ordinary chi-square distribution with n degree of freedom. This result can also be found in Hunter's **Notes for a graduate-level course in asymptotics for statisticians** [Hunter, 2014]. Also Guenther [1977] and Meng and Chapman [1966] offers the same results.

4 Contingency test

The same idea as that in Section 3 for the goodness of fit test except for that $\boldsymbol{p}_0$ is not now given, but rather computed based on **marginal proportion** of the data. So consider a $r \times c$ contingency table in Table 1 and Table 2.

The null hypothesis is that these two types of categories (arranged in row and column, respectively) is independent. Therefore the underlying distribution satisfies

$$p_{ij} = p_{i\cdot} p_{\cdot j}, \quad 1 \leq i \leq r, \quad 1 \leq j \leq c. \tag{4}$$

	col_1	$\cdots$	col_c	Total
row_1	x_{11}	$\cdots$	x_{1c}	$x_{1\cdot} = \sum_{j=1}^{c} x_{1j}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
row_r	x_{r1}	$\cdots$	x_{rc}	$x_{r\cdot} = \sum_{j=1}^{c} x_{rj}$
Total	$x_{\cdot 1} = \sum_{i=1}^{r} x_{i1}$	$\cdots$	$x_{\cdot c} = \sum_{i=1}^{r} x_{ic}$	$n = \sum_{i=1}^{r}\sum_{j=1}^{c} x_{ij}$

Table 1: A contingency table, counts in cell

	col_1	$\cdots$	col_c	Total
row_1	$p_{11} = x_{11}/n$	$\cdots$	$p_{1c} = x_{1c}/n$	$p_{1\cdot} = x_{1\cdot}/n$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
row_r	$p_{r1} = x_{r1}/n$	$\cdots$	$p_{rc} = x_{rc}/n$	$p_{r\cdot} = x_{r\cdot}/n$
Total	$p_{\cdot 1} = x_{\cdot 1}/n$	$\cdots$	$p_{\cdot c} = x_{\cdot c}/n$	1

Table 2: A contingency table, proportion in cell

Then the alternative hypothesis is that there exists at least one (i, j) such that (4) does not hold.

4.1 Reject rule

Here the test statistic is

$$\chi_T^2 = n \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(P_{1,ij} - P_{0,ij})^2}{P_{0,ij}} = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(x_{ij} - np_{i\cdot}p_{\cdot j})^2}{np_{i\cdot}p_{\cdot j}},$$

where $P_{1,ij}$ is just the observed proportion in cell (i, j) and $P_{0,ij} = p_{i\cdot}p_{\cdot j}$ is the expected proportion computed based on marginal data.

Under null hypothesis, χ_T^2 follows a χ^2 distribution with degree of freedom being $(r-1)(c-1)$. And H_0 is rejected for large value of χ_T^2.

4.2 Power analysis

The same as that in Section 3.2. Just now the $\boldsymbol{p}$ is length $r \times c$ instead of m, and the degree of freedom of the chi-square distribution is $(r-1)(c-1)$.

5 Other related tests

- For 2×2 table with small sample size, a Fisher's exact test can be considered.
- For 2×1 table, a binomial test can be considered: Clopper-Pearson's test is an exact one, while the chi-square test or a normal test is a continuous approximation here.

References

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