Cox Proportional Hazard Model

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Contents

1 Introduction

In this note we will talk about the Cox's proportional hazards (Cox's PH) model. Suppose we observe some non-informativly right-censored data (U, δ) with covariate vector Z. That is, for subject i, the covariate vector is Z_i, survival time T_i and censoring time C_i. The observed data is (U_i, δ_i) where $U_i = \min(T_i, C_i)$ and $\delta_i = 1(T_i \leq C_i)$. Also $T_i \perp C_i | Z_i$.

And now we want to model the relationship between Z and T. One way to do that is to incorporate Z into the hazard function $h(\cdot)$, e.g.,

$$T \sim Exp(\lambda_Z) \implies h(t) = \lambda_Z \triangleq e^{\alpha+\beta Z} = \lambda_0 e^{\beta Z},$$

where $\lambda_0 = e^{\alpha}$ can be viewed as a baseline hazard. If $\beta = 0$ then Z is not associated with T.

We can generalize this idea as

$$h(t|Z) = h_0(t) \times g(Z).$$

So the hazard can be factorized and this model is sometimes called a "multiplicative intensive model" or "multiplicative hazard model" or "proportional hazard model" because this factorization implies that

$$\frac{h(t|Z = z_1)}{h(t|Z = z_2)} = \frac{g(z_1)}{g(z_2)}.$$

The hazard ratio is constant with respect to t, hence the (constant) proportional hazard. So in our previous model (the exponential survival time), the hazard ratio is

$$\frac{h(t|Z = z_1)}{h(t|Z = z_2)} = e^{\beta(z_1 - z_2)}.$$

Also this exponential form of $g(Z)$

$$h(t|Z) = h_0(t) \cdot e^{\beta Z} \tag{1}$$

is the **Cox's PH** model.

2 Estimation

(1) implies that

$$\begin{aligned}S(t|Z) =&\exp\left(-H(t|Z)\right)\\ =&\exp\left(-\int_0^t h(u|Z)\,\mathrm{d}u\right)\\ =&\exp\left(-\int_0^t h_0(t)\,\mathrm{d}u\cdot g(Z)\right)\\ =&\left(S_0(t)\right)^{g(Z)}=\left(S_0(t)\right)^{\exp(\beta Z)},\end{aligned}$$

where $S_0(t)=\exp\left(-\int_0^t h_0(t)\,\mathrm{d}u\right)$, the survival function for $Z=0$, hence $S(t|Z=0)$. Also remember that $f(t|Z)=h(t|Z)\,S(t|Z)$. Thus, given n independent data (u_i,δ_i,z_i), the likelihood (one can refer to our previous notes about survival analysis.) is

$$\begin{aligned}L(\beta,h_0(\cdot)) =&\prod_{i=1}^n\left(f(u_i|z_i)\right)^{\delta_i}\left(S(u_i|z_i)\right)^{1-\delta_i}=\prod_{i=1}^n h(u_i|z_i)^{\delta_i}\,S(u_i|z_i)\\ =&\prod_{i=1}^n\left(h_0(u_i|z_i)\,e^{\beta z_i}\right)^{\delta_i}\left(\exp\left(-\int_0^{u_i}h_0(t)\,\mathrm{d}t\right)\right)^{\exp(\beta z_i)}\\ =&\text{function}\left(data,h_0(\cdot),\beta\right).\end{aligned}$$

If $h_0(\cdot)$ is allowed to be "arbitary", then the "parameter space " is

$$\mathcal{H}\times\mathcal{R}^p=\left\{(h(\cdot),\beta)\middle|h_0(\cdot)\geq 0,\int_0^\infty h_0(t)\,\mathrm{d}t=\infty,\beta\in\mathcal{R}^p\right\},$$

where $\int_0^\infty h_0(t)\,\mathrm{d}t=\infty$ ensures that $S_0(\infty)=0$.

In general this likelihood is hard to maximize. And Cox proposed this idea: to factor $L(\beta,h_0(\cdot))$ as

$$L(\beta,h_0(\cdot))=L_1(\beta)\times L_2(\beta,h_0(\cdot)),$$

where L_1 only depends on β and its maximization $(\hat{\beta})$ enjoys nice properties such as consistency and asymptotic normality while L_2 contains relatively little information about β. And this L_1 is called a **partial likelihood**.

3 Inference

References