Minimum Detectable Difference (MDD) in Hypothesis Testing

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1 Introduction

In this notes, we talk about minimum detectable difference (MDD) and other related concepts in hypothesis testing. First, we will use a test for normal mean when variance is known as an example. So for an i.i.d. sample $x_1, \cdots, x_n$ with $x_i \sim N(\mu, \sigma^2)$ with known σ, we want to test

$$H_0: \ \mu = \mu_0 \quad \text{v.s.} \quad H_1: \ \mu \neq \mu_0.$$

A two-sided test with significant level α can be constructed as to reject H_0 when

$$\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2},$$

since $\bar{x} \sim N(\mu_0, \sigma^2/n)$ under H_0. And for any given underlying μ that is not equal to μ_0, the probability to correctly reject H_0, i.e., the power is computed as

$$\begin{aligned}
&P\left(\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2}\right) \\
=&P\left(\bar{x} \geq \mu_0 + z_{1-\alpha/2} * \sqrt{\sigma^2/n}\right) + P\left(\bar{x} \leq \mu_0 - z_{1-\alpha/2} * \sqrt{\sigma^2/n}\right) \\
=&P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \geq \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} + z_{1-\alpha/2}\right) + P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \leq \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} - z_{1-\alpha/2}\right) \\
=&\left(1 - \Phi\left(\frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} + z_{1-\alpha/2}\right)\right) + \Phi\left(\frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} - z_{1-\alpha/2}\right).
\end{aligned}$$

And a confidence interval for μ is constructed as

$$\left[\bar{x} - z_{1-\alpha/2} * \sqrt{\sigma^2/n}, \quad \bar{x} + z_{1-\alpha/2} * \sqrt{\sigma^2/n}\right].$$

So here we can see that in order to reject H_0, the critical point is the normalized difference $\left|\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}\right|$ to be greater than $z_{1-\alpha/2}$. This critical point ($z_{1-\alpha/2}$) is determined by the type of test, which leads to the type of test statistics and the significant level. So in other words, as long as the observed difference $|\bar{x} - \mu_0|$ is greater than $z_{1-\alpha/2} * \sqrt{\sigma^2/n}$, H_0 would be rejected no matter what the underlying μ is. So here $z_{1-\alpha/2} * \sqrt{\sigma^2/n}$ is the minimum effect size (considering the sample size and variance) that would be just significant, hence the minimum detectable difference (MDD).

2 MDD and other concepts

References