Poisson and Negative Binomial Distribution

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Contents

1 Introduction

This notes is about poisson and negative binomial distribution, and their application in modeling count data.

2 Poisson Distribution

2.1 Basic Information

The p.m.f of a random variable X that follows Poisson distribution is

$$P\left(X=x\right)=\frac{\mathrm{e}^{-\lambda}\lambda^{x}}{x!},\quad x=0,1,\cdots.$$

The expectation and variance of X is

$$\mathrm{E}X=\mathrm{Var}X=\lambda.$$

2.2 Some Additoinal Properties

1. Summation of independent poisson distributed variables is still poisson distributed. That is to say,

   if
   $$X_1 \sim Poisson(\lambda_1), \quad X_2 \sim Poisson(\lambda_2), \quad X_1 \perp X_2,$$
   then
   $$X_1 + X_2 \sim Poisson(\lambda_1 + \lambda_2).$$

2. Poisson distribution can be related to Gamma distribution, which is also mentioned in "gamma_and_beta.pdf" notes.

3. The conjugate prioir for a Poisson distribution parameter λ is Gamma distribution.

4. A poisson distributed variable can be seen as counts of customer to a store or occurence of some events. And the **time interval between each count** follows exponential distribution.

3 Negative Binomial Distribution

3.1 Basic Information

Denote X the number of failing attempts just before reaching a total of r times of success and p the probability of success. If each attempt is independent of each other, then the distribution of X is called a negative binomial distribution. And the p.m.f of X is

$$P\left(X = x\right) = C_{x+r-1}^{x} p^r \left(1-p\right)^x = \frac{(x+r-1)!}{(r-1)!x!} p^r \left(1-p\right)^x, \quad x = 0, 1, \cdots .$$

In this notation, X is the number of failing attempts. In some cases, one will model the distribution of number of total attempts X', which is just $X' = X + r$.

Previously, r is the number of required success, which is naturally an positive integer. But one can actually generalize r to positive real number using Gamma function. Then the p.m.f of X is written as

$$P\left(X = x\right) = \frac{\Gamma(x+r)}{\Gamma(r)\Gamma(x+1)} p^r \left(1-p\right)^x, \quad x = 0, 1, \cdots ,$$

which is also called **Polya** distribution.

The expectation of a negative binomial/Polya distribution is

$$\mathrm{E}X = \frac{r\left(1-p\right)}{p}.$$

The variance of a negative binomial/Polya distribution is

$$\mathrm{Var}X = \frac{r\left(1-p\right)}{p^2}.$$

3.2 Some Additional Properties

1. Summation of independent negative binomial distributed variables. That is to say, if $X_1 \sim NB(r_1, p)$, $X_2 \sim NB(r_2, p)$ and $X_1 \perp X_2$, then

$$X_1 + X_2 \sim NB(r_1 + r_2, p).$$

2. Relationship to Poisson and Gamma distribution.

   - Related to poisson when $r \to \infty$.
   - Related to poisson and gamma distribution in a hierarchical model:

$$\lambda \sim Gamma(\alpha, \beta), \quad f(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} \exp(-\lambda/\beta)$$
$$X|\lambda \sim Poisson(\lambda), \quad P(X = x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

   Then the marginal distribution of X is

$$\begin{aligned} P(X = x) &= \int_0^\infty P(X = x|\lambda) f(\lambda) \mathrm{d}\lambda \\ &= \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} \exp(-\lambda/\beta) \mathrm{d}\lambda \\ &= \frac{1}{x!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{x+\alpha-1} \exp\left(-\frac{\lambda}{\beta/(1+\beta)}\right) \mathrm{d}\lambda \quad \text{The kernel of Gamma}(x+\alpha, \frac{\beta}{1+\beta}) \\ &= \frac{\Gamma(x+\alpha)\left(\frac{\beta}{1+\beta}\right)^{x+\alpha}}{x!\Gamma(\alpha)\beta^\alpha} \\ &= \frac{\Gamma(x+\alpha)}{\Gamma(\alpha)\Gamma(x+1)} \left(\frac{1}{1+\beta}\right)^\alpha \left(1 - \frac{1}{1+\beta}\right)^x, \end{aligned}$$

   which is a negative binomial distribution with parameters $r = \alpha$ and $p = \frac{1}{1+\beta}$. In this way,

$$\mathrm{E}X = \frac{r(1-p)}{p} = \alpha\beta$$
$$\mathrm{Var}X = \frac{r(1-p)}{p^2} = \alpha\beta(1+\beta)$$

   Note that here β is actually the **failure odds**, which is mentioned in later properties.

3. Dispersion parameter and other parametrization of negative binomial distribution.

   - In some literature, expectation and variance are used to describe negative binomial distribution:

$$\mathrm{E}X = \mu$$
$$\mathrm{Var}X = \sigma^2$$
$$r = \frac{\mu^2}{\sigma^2 - \mu}$$
$$p = \frac{\mu}{\sigma^2}$$

- In some literature, $\alpha = \frac{1}{r}$ is called the dispersion parameter while others call r the dispersion parameter. It also has other names such as "shape parameter", "clustering coefficient", "heterogeneity" and "aggregation parameter". Note that the expectation and the variance of a negative binomial distributed variable satisfies
$$\sigma^2 = \mu + \alpha\mu^2 = \mu + \frac{\mu^2}{r}.$$
- Denote β the **failure odds**, then
$$\begin{aligned} \beta &= \frac{1-p}{p} \\ P\left(X = x\right) &= \frac{\Gamma\left(x+r\right)}{\Gamma\left(r\right)\Gamma\left(x+1\right)}\left(\frac{\beta}{1+\beta}\right)^r\left(\frac{1}{1+\beta}\right)^x, \quad x = 0, 1, \cdots, \\ \mathrm{E}X &= r\beta \\ \mathrm{Var}X &= r\beta\left(1+\beta\right). \end{aligned}$$

4. Geometric distribution is a special case of negative binomial distribution.

4 Modeling Counts Data

4.1 Basic Modeling

4.2 Zero Inflation

References