T-test

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Contents

1 Basic knowledge

$\phi(x)$ and $\Phi(x)$ are pdf and cdf of standard normal distribution, respectively. We use Z to represent a random variable that follows standard normal distribution and z_α the lower α quantile of standard normal distribution. Therefore

$$P(Z \leq z_\alpha) = \Phi(z_\alpha) = \alpha.$$

Theorem 1. *Let $x_1, \cdots, x_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then*

1. $\mathrm{E}\bar{x} = \mu$.

2. Var$\bar{x} = \sigma^2/n$.

3. E$S^2 = \sigma^2$, *where* $S^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i - \bar{x})^2$.

Theorem 2. *Let $x_1, \cdots, x_n$ be a random sample from $N(\mu, \sigma^2)$. Then*

1. $\bar{X} \sim N(\mu, \sigma^2/n)$.

2. $\bar{X}$ *is independent of* S^2.

3. $(n-1)S^2/\sigma^2$ *follows a chi-squared distribution with $n-1$ degree of freedom.*

2 One-sample test

Consider a random sample $x_1, \cdots, x_n$ from $N(\mu, \sigma^2)$. The likelihood is

$$\begin{aligned} f(x_1, \cdots, x_n) &= \prod_{i=1}^{n} \left(2\pi\sigma^2\right)^{-1/2} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \\ &= \left(2\pi\sigma^2\right)^{-n/2} \exp\left(-\frac{\sum\limits_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right). \end{aligned}$$

We propose the test

$$H_0: \ \mu = \mu_0 \quad \textbf{v.s} \quad H_1: \ \mu \neq \mu_0$$

2.1 variance known

Construct LRT

$$LR = \frac{\max\limits_{\mu \in H_0} f(x_1, \cdots, x_n|\mu)}{\max\limits_{\mu \in H_0 \cup H_1} f(x_1, \cdots, x_n|\mu)} = \frac{f(x_1, \cdots, x_n|\mu = \mu_0)}{f(x_1, \cdots, x_n|\mu = \bar{x})} = \exp\left(-\frac{(\bar{x}-\mu_0)^2}{2\sigma^2/n}\right)$$

Therefore rejecting H_0 when LR is smaller than some constant C is equivalent to rejecting H_0 when $|\bar{x} - \mu_0|$ is larger than some other constant C. Hence

$$\text{Reject Region: } \{\bar{x}: \quad |\bar{x} - \mu_0| > C\}$$

2.1.1 Decide C from α

From definition of α we know that C in the reject region is chosen such that

$$P(|\bar{x} - \mu_0| > C | H_0 \text{ is true }) \leq \alpha.$$

But to fully utilize the test, we choose to use equal sign instead of $\leq$. Therefore

$$P(|\bar{x} - \mu_0| > C | \mu = \mu_0) = \alpha.$$

Note that $\bar{x} \sim N(\mu, \sigma^2/n)$. Then under the condition $\mu = \mu_0$,

$$\frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0, 1).$$

Therefore we propose the reject region for H_0 being

$$\left|\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}\right| \geq z_{1-\alpha/2}.$$

Note: Here, even if the sample distribution is not normal, the result still holds due to CLT under large sample.

2.1.2 Power at given underlying μ

The power (the probability to reject H_0, when H_1 is true) of the proposed test procedure for any given underlying $\mu \neq \mu_0$ is computed as

$$\begin{aligned}
&P\left(\left|\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}\right| \geq z_{1-\alpha/2}\right) \\
=&P\left(\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2}\right) + P\left(\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2}\right) \\
=&P\left(\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right) + P\left(\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} \geq z_{1-\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right) \\
=&P\left(Z \leq z_{\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right) + P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right)
\end{aligned} \tag{1}$$

Here we use the fact that $\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$.

2.1.3 Sample size at given α, β and underlying μ

W.l.o.g, assume that $\mu > \mu_0$, then in previous power equation (1)

$$P\left(Z \leq z_{\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right)$$

would be really close to zero and

$$P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right)$$

will offer most of the power. In order to guarantee a power of at least $1-\beta$, we could simply set

$$P\left(Z \geq z_{1-\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right) \geq 1-\beta,$$

which means

$$z_{1-\alpha/2} + \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}} \leq z_\beta.$$

Normally in test settings, $\alpha < 0.1$ and $\beta < 0.5$, which means $z_{1-\alpha/2}$ is positive and z_β is negative. Also $\mu_0 - \mu < 0$ in our assumption. This leads to

$$-z_{\alpha/2} - z_\beta \leq \frac{\sqrt{n}\,(\mu-\mu_0)}{\sigma}.$$

Hence the sample size requirement is

$$n \geq \frac{\sigma^2 \left(z_{\alpha/2} + z_\beta\right)^2}{\left(\mu - \mu_0\right)^2}. \tag{2}$$

Note: The sample size requirement can be deduced the same way when $\mu < \mu_0$. And the result is just the same as (2).

2.2 variance unknown

When σ^2 is unknown, the MLE under H_0 is

$$\mu_{(0)} = \mu_0, \quad \sigma^2_{(0)} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mu_0\right)^2.$$

And the MLE under $H_0 \cup H_1$ is

$$\mu_{(0\cup 1)} = \bar{x}, \quad \sigma^2_{(0\cup 1)} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2.$$

Note: MLE for σ^2 offers smaller MSE than S^2, but it's biased.

Then the likelihood ratio is

$$LR = \frac{f\left(x_1, \cdots, x_n \middle| \mu = \mu_{(0)}, \sigma^2 = \sigma^2_{(0)}\right)}{f\left(x_1, \cdots, x_n \middle| \mu = \mu_{(0\cup 1)}, \sigma^2 = \sigma^2_{(0\cup 1)}\right)} = \left(\frac{\sum\limits_{i=1}^{n}\left(x_i - \mu_0\right)^2}{\sum\limits_{i=1}^{n}\left(x_i - \bar{x}\right)^2}\right)^{-n/2} \propto \left(\frac{\sum\limits_{i=1}^{n}\left(\bar{x} - \mu_0\right)^2}{\sum\limits_{i=1}^{n}\left(x_i - \bar{x}\right)^2}\right)^{-n/2},$$

where for the last part we mainly focus on terms related to μ_0. So to reject H_0 when LR is small is equivalent to

$$\text{Reject Region:} \left\{ \bar{x} : \quad \frac{\left|\bar{x} - \mu_0\right|}{\sqrt{\sum\limits_{i=1}^{n}\left(x_i - \bar{x}\right)^2}} > C \right\}$$

The idea is similar to that in Section 2.1. But we replace σ^2 with S^2.

2.2.1 Decide C from α

First we can write

$$P\left(\frac{\left|\bar{x} - \mu_0\right|}{\sqrt{\sum\limits_{i=1}^{n}\left(x_i - \bar{x}\right)^2}} > C \middle| \mu = \mu_0\right) = P\left(\frac{\left|\bar{x} - \mu_0\right|}{\sqrt{\left(n-1\right)S^2}} > C \middle| \mu = \mu_0\right) = \alpha.$$

From Theorem 2 we know that

$$\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sim N\left(0, 1\right), \quad \left(n-1\right)S^2/\sigma^2 \sim \chi^2\left(n-1\right), \quad \bar{x} \perp S^2$$

Therefore

$$\frac{\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\bar{x}-\mu}{\sqrt{S^2/n}} \sim t(n-1).$$

Then we know the reject rejion is

$$\left|\frac{\bar{x}-\mu}{\sqrt{S^2/n}}\right| > t_{1-\alpha/2}\left(n-1\right).$$

Note: Here we need Theorem 2, which means the normal assumption of the sample is **necessary**. Though one might argue that without normal assumption, under large sample scenario, using Slutsky's theorem, asymptotically

$$\frac{\bar{x}-\mu}{\sqrt{S^2/n}} = \frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}}\sqrt{\frac{\sigma^2}{S^2}} \to N\left(0,1\right).$$

2.2.2 Power at given underlying μ and σ^2

Before any computation, we introduce the **non-central** t-distribution.

$$T = \frac{Z+\mu}{\sqrt{V/v}}, \tag{3}$$

where Z follows standard normal and V follows $\chi^2\left(v\right)$ and $Z \perp V$. Then T follows a non-central t-distribution with degree of freedom v and non-central parameter μ, denoted by $t\left(v,\mu\right)$.

Then we know that

$$\frac{\bar{x}-\mu_0}{\sqrt{S^2/n}} = \frac{\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}}+\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} \sim t\left(n-1,\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right),$$

which means $\frac{\bar{x}-\mu_0}{\sqrt{S^2/n}}$ follows a non-central t-distribution $t\left(n-1,\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right)$. Therefore the power can be computed as

$$\begin{aligned}
&P\left(\left|\frac{\bar{x}-\mu_0}{\sqrt{S^2/n}}\right| \geq t_{1-\alpha/2}\left(n-1\right)\right)\\
=&P\left(\left|T\left(n-1,\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right)\right| \geq t_{1-\alpha/2}\left(n-1\right)\right)\\
=&P\left(T\left(n-1,\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \leq t_{\alpha/2}\left(n-1\right)\right)+P\left(T\left(n-1,\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}\left(n-1\right)\right).
\end{aligned} \tag{4}$$

2.2.3 Sample size at given α, β and underlying μ and σ^2

W.l.o.g, assume $\mu > \mu_0$, then in the previous power equation (4)

$$P\left(T\left(n-1,\frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \leq t_{\alpha/2}\left(n-1\right)\right)$$

would be close to zero and

$$P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}\left(n-1\right)\right)$$

will offer the most power. In order to guarantee a power of at least $1-\beta$, we could simply set

$$P\left(T\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right) \geq t_{1-\alpha/2}\left(n-1\right)\right) \geq 1-\beta,$$

which means

$$t_{1-\alpha/2}\left(n-1\right) \leq t_{\beta}\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right).$$

There's no close form for this inequality, we should use some numerical method to solve for n.

Note: If $\mu < \mu_0$, then similarly we can get the requirement as

$$t_{\alpha/2}\left(n-1\right) \geq t_{1-\beta}\left(n-1, \frac{\mu-\mu_0}{\sqrt{\sigma^2/n}}\right).$$

Use the fact that $t_{\alpha}\left(n, \mu\right) = -t_{1-\alpha}\left(n, -\mu\right)$, we can arrange the previous inequality as

$$t_{1-\alpha/2}\left(n-1\right) \leq t_{\beta}\left(n-1, \frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}\right).$$

Therefore in summary the sample size requirement is

$$t_{1-\alpha/2}\left(n-1\right) \leq t_{\beta}\left(n-1, \frac{|\mu_0-\mu|}{\sqrt{\sigma^2/n}}\right).$$

3 Two sample test

Consider two random samples $x_1, \cdots, x_{n_1} \sim N\left(\mu_1, \sigma_1^2\right)$ and $y_1, \cdots, y_{n_2} \sim N\left(\mu_2, \sigma_2^2\right)$. Then the likelihood of the data is

$$\begin{aligned} & f\left(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2} \middle| \mu_1, \mu_2, \sigma_1^2, \sigma_2^2\right) \\ = & \left(2\pi\sigma_1^2\right)^{-n_1/2}\left(2\pi\sigma_2^2\right)^{-n_2/2} \exp\left(-\frac{\sum_{i=1}^{n_1}\left(x_i-\mu_1\right)^2}{2\sigma_1^2} - \frac{\sum_{i=1}^{n_2}\left(y_i-\mu_2\right)^2}{2\sigma_2^2}\right) \\ = & \end{aligned}$$

We propose the test

$$H_0: \ \mu_1 = \mu_2 \quad \textbf{v.s.} \quad H_1: \ \mu_1 \neq \mu_2.$$

3.1 Two-sample, variance known

When σ_1^2 and σ_2^2 are known, the likelihood satisfies

$$f\left(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2} | \mu_1, \mu_2\right) \propto \exp\left(-\frac{n_1\left(\bar{x}-\mu_1\right)^2}{2\sigma_1^2} - \frac{n_2\left(\bar{y}-\mu_2\right)^2}{2\sigma_2^2}\right).$$

Therefore under H_0, the MLE for μ_1 and μ_2 is

$$\mu_{1(0)} = \mu_{2(0)} = \mu_{(0)} = \frac{\sigma_2^2 n_1 \bar{x} + \sigma_1^2 n_2 \bar{y}}{\sigma_2^2 n_1 + \sigma_1^2 n_2}.$$

And under $H_0 \cup H_1$, the MLE for μ_1 and μ_2 is

$$\mu_{1(0\cup1)} = \bar{x}, \quad \mu_{2(0\cup1)} = \bar{y}.$$

Then the likelihood ratio is

$$\begin{aligned}
LR =& \frac{\max_{H_0} f\left(\boldsymbol{x}, \boldsymbol{y} | \mu_1, \mu_2\right)}{\max_{H_0\cup H_1} f\left(\boldsymbol{x}, \boldsymbol{y} | \mu_1, \mu_2\right)} \\
=& \frac{f\left(\boldsymbol{x}, \boldsymbol{y} \middle| \mu_1 = \mu_2 = \mu_{(0)}\right)}{f\left(\boldsymbol{x}, \boldsymbol{y} \middle| \mu_1 = \mu_{1(0\cup1)}, \mu_2 = \mu_{2(0\cup1)}\right)} \\
\propto& \exp\left(-\frac{1}{2}\left(\frac{n_1\left(\bar{x}-\mu_{(0)}\right)^2}{\sigma_1^2} + \frac{n_2\left(\bar{y}-\mu_{(0)}\right)^2}{\sigma_2^2}\right)\right) \\
=& \exp\left(-\frac{1}{2}\frac{n_1 n_2}{\sigma_2^2 n_1 + \sigma_1^2 n_2}\left(\bar{x}-\bar{y}\right)^2\right).
\end{aligned}$$

From the idea of LRT, H_0 is rejected when LR is small enough, which means the reject rule is

$$\text{Reject Region: } \left\{(\bar{x}, \bar{y}) \middle| |\bar{x}-\bar{y}| > C\right\}$$

3.1.1 Decide C from α

From the definition of α we know that

$$P\left(|\bar{x}-\bar{y}| > C | H_0 \text{ is true}\right) \leq \alpha.$$

Note that $\bar{x} \sim N\left(\mu_1, \sigma_1^2/n_1\right)$, $\bar{y} \sim N\left(\mu_2, \sigma_2^2/n_2\right)$ and $\bar{x} \perp \bar{y}$. Therefore

$$\bar{x}-\bar{y} \sim N\left(\mu_1-\mu_2, \sigma_1^2/n_1 + \sigma_2^2/n_2\right).$$

Then under H_0, $\mu_1 = \mu_2$ and

$$\frac{\bar{x}-\bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1),$$

which means

$$P\left(|\bar{x}-\bar{y}| > C | \mu_1 = \mu_2\right) = P\left(|Z| > \frac{C}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \leq \alpha.$$

Here to fully utilize the test, we choose the equal sign. Hence

$$z_{1-\alpha/2} = \frac{C}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}.$$

Here the reject region is

$$\frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2}.$$

Note: Even the samples does not follow normal distribution, by CLT this test still holds true for large sample.

3.1.2 Power at given $\Delta = \mu_1 - \mu_2$

The power at a given $\Delta = \mu_1 - \mu_2$ is

$$\begin{aligned}
&P\left(\frac{|\bar{x} - \bar{y}|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2} \middle| \Delta = \mu_1 - \mu_2\right) \\
=&P\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2}\right) + P\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z_{\alpha/2}\right) \\
=&P\left(\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) + P\left(\frac{\bar{x} - \bar{y} - \Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} < z_{\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \\
=&P\left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) + P\left(Z < z_{\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right)
\end{aligned}$$

3.1.3 Sample size at given α, β, Δ and $k = \frac{n_1}{n_2}$

W.l.o.g, assume $\Delta > 0$, then the power of the test comes mostly from

$$P\left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right).$$

So to achieve the power, we can set

$$P\left(Z > z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}\right) \geq 1 - \beta.$$

And this means

$$z_{1-\alpha/2} - \frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq z_\beta.$$

Rearrange this inequality we have

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \leq \frac{\Delta^2}{\left(z_{\alpha/2} + z_\beta\right)^2}.$$

1. When n_1 is fixed and given, we need
$$n_2 \geq \frac{\sigma_2^2}{\frac{\Delta^2}{\left(z_{\alpha/2}+z_\beta\right)^2} - \frac{\sigma_1^2}{n_1}}.$$
Also this fixed and given n_1 must satisfiy
$$\frac{\Delta^2}{\left(z_{\alpha/2}+z_\beta\right)^2} > \frac{\sigma_1^2}{n_1}$$
for the test to be feasible.

2. Similarly, when n_2 is given and fixed, we need
$$n_1 \geq \frac{\sigma_1^2}{\frac{\Delta^2}{\left(z_{\alpha/2}+z_\beta\right)^2} - \frac{\sigma_2^2}{n_2}}.$$
Also this fixed and given n_2 must satisfy
$$\frac{\Delta^2}{\left(z_{\alpha/2}+z_\beta\right)^2} > \frac{\sigma_2^2}{n_2}$$
for the test to be feasible.

3. For a fixed and given sample size ratio $k = n_2/n_1$, we need
$$n_1 \geq \frac{\left(\sigma_1^2 + \sigma_2^2/k\right)\left(z_{\alpha/2}+z_\beta\right)^2}{\Delta^2}$$

Note: These results hold the same form when $\Delta < 0$.

3.2 Two-sample, variance unknown but equal

When σ_1 and σ_2 are both unknown but equal, denoted by σ. The likelihood of the data becomes
$$\begin{aligned} & f\left(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2} \middle| \mu_1, \mu_2, \sigma^2\right) \\ = & \left(2\pi\sigma^2\right)^{-n_1/2-n_2/2} \exp\left(-\frac{\sum_{i=1}^{n_1}\left(x_i-\mu_1\right)^2 + \sum_{i=1}^{n_2}\left(y_i-\mu_2\right)^2}{2\sigma^2}\right). \end{aligned}$$

So the MLE under H_0 is
$$\mu_{1(0)} = \mu_{2(0)} = \mu_{(0)} = \frac{n_1\bar{x} + n_2\bar{y}}{n_1+n_2}, \quad \sigma_{(0)}^2 = \frac{\sum_{i=1}^{n_1}\left(x_i-\mu_{(0)}\right)^2 + \sum_{i=1}^{n_2}\left(y_i-\mu_{(0)}\right)^2}{n_1+n_2}$$

And the MLE under $H_0 \cup H_1$ is
$$\mu_{1(0\cup1)} = \bar{x}, \quad \mu_{2(0\cup1)} = \bar{y}, \quad \sigma_{(0\cup1)}^2 = \frac{\sum_{i=1}^{n_1}\left(x_i-\bar{x}\right)^2 + \sum_{i=1}^{n_2}\left(y_i-\bar{y}\right)^2}{n_1+n_2}.$$

Then the likelihood ratio is

$$
\begin{aligned}
LR =& \frac{f\left(\boldsymbol{x}, \boldsymbol{y} \middle| \mu_1 = \mu_2 = \mu_{(0)}, \sigma^2 = \sigma^2_{(0)}\right)}{f\left(\boldsymbol{x}, \boldsymbol{y} \middle| \mu_1 = \mu_{1(0\cup 1)}, \mu_2 = \mu_{2(0\cup 1)}, \sigma^2 = \sigma^2_{(0\cup 1)}\right)} \\
=& \left(\frac{\sigma^2_{(0)}}{\sigma^2_{(0\cup 1)}}\right)^{-n_1/2-n_2/2} \\
=& \left(\frac{\sum_{i=1}^{n_1}\left(x_i - \bar{x} + \bar{x} - \mu_{(0)}\right) + \sum_{i=1}^{n_2}\left(y_i - \bar{y} + \bar{y} - \mu_{(0)}\right)}{\sum_{i=1}^{n_1}\left(x_i - \bar{x}\right)^2 + \sum_{i=1}^{n_2}\left(y_i - \bar{y}\right)^2}\right)^{-n_1/2-n_2/2} \\
=& \left(1 + \frac{n_1\left(\bar{x} - \mu_{(0)}\right)^2 + n_2\left(\bar{y} - \mu_{(0)}\right)^2}{\sum_{i=1}^{n_1}\left(x_i - \bar{x}\right)^2 + \sum_{i=1}^{n_2}\left(y_i - \bar{y}\right)^2}\right)^{-n_1/2-n_2/2} \\
=& \left(1 + \frac{n_1 n_2}{n_1 + n_2} \cdot \frac{\left(\bar{x} - \bar{y}\right)^2}{\sum_{i=1}^{n_1}\left(x_i - \bar{x}\right)^2 + \sum_{i=1}^{n_2}\left(y_i - \bar{y}\right)^2}\right)^{-n_1/2-n_2/2}.
\end{aligned}
$$

So to reject H_0 when the likelihood ratio is small enough implies that the reject region is

$$
\text{Reject region:} \quad \left\{ (\boldsymbol{x}, \boldsymbol{y}) \middle| \frac{|\bar{x} - \bar{y}|}{\sqrt{\sum_{i=1}^{n_1}\left(x_i - \bar{x}\right)^2 + \sum_{i=1}^{n_2}\left(y_i - \bar{y}\right)^2}} > C \right\}.
$$

3.2.1 Decide C from α

Like before, we know that

$$
\bar{x} \sim N\left(\mu_1, \frac{\sigma^2}{n_1}\right), \quad (n_1 - 1) S_x^2/\sigma^2 \sim \chi^2(n_1 - 1), \quad \bar{x} \perp S_x^2,
$$

and

$$
\bar{y} \sim N\left(\mu_2, \frac{\sigma^2}{n_2}\right), \quad (n_2 - 1) S_y^2/\sigma^2 \sim \chi^2(n_2 - 1), \quad \bar{y} \perp S_y^2.
$$

Since these two samples $\boldsymbol{x}$ and $\boldsymbol{y}$ are independent, we have

$$
\bar{x} - \bar{y} \sim N\left(\mu_1 - \mu_2, \frac{n_1 + n_2}{n_1 n_2}\sigma^2\right),
$$

which implies

$$
\frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{\frac{n_1+n_2}{n_1 n_2}\sigma^2}} \sim N(0, 1).
$$

And more importantly (the summation of independent χ^2 variables)

$$
\frac{(n_1 - 1) S_x^2 + (n_2 - 1) S_y^2}{\sigma_2} \sim \chi^2(n_1 + n_2 - 2)
$$

and

$$\frac{\bar{x}-\bar{y}-(\mu_1-\mu_2)}{\sqrt{\frac{n_1+n_2}{n_1 n_2}\sigma^2}} \perp \frac{(n_1-1)\,S_x^2+(n_2-1)\,S_y^2}{\sigma_2}.$$

This leads us to

$$\frac{\frac{\bar{x}-\bar{y}-(\mu_1-\mu_2)}{\sqrt{\frac{n_1+n_2}{n_1 n_2}\sigma^2}}}{\sqrt{\frac{1}{n_1+n_2-2}\frac{(n_1-1)S_x^2+(n_2-1)S_y^2}{\sigma_2}}} = \frac{\bar{x}-\bar{y}-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\cdot\frac{(n_1-1)S_x^2+(n_2-1)S_y^2}{n_1+n_2-2}}} \sim t(n_1+n_2-2).$$

Here we use S_p to represent the pooled standard deviation of the data, i.e.

$$S_p = \sqrt{\frac{(n_1-1)\,S_x^2+(n_2-1)\,S_y^2}{n_1+n_2-2}}.$$

Under H_0, the type-I error is controlled as

$$P\left(\frac{|\bar{x}-\bar{y}|}{\sqrt{\sum_{i=1}^{n_1}(x_i-\bar{x})^2+\sum_{i=1}^{n_2}(y_i-\bar{y})^2}} > C \middle| \mu_1=\mu_2\right) \le \alpha.$$

Therefore we can write

$$P\left(\frac{|\bar{x}-\bar{y}|}{\sqrt{\sum_{i=1}^{n_1}(x_i-\bar{x})^2+\sum_{i=1}^{n_2}(y_i-\bar{y})^2}} > C \middle| \mu_1=\mu_2\right) = P\left(\left|T_{(n_1+n_2-2)}\right| > \frac{C}{\sqrt{\frac{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}{n_1+n_2-2}}}\right) = \alpha.$$

Here in the last part we use the equal sign instead of $\le$ for fully utilize the test. So we can construct the test to reject H_0 when

$$\frac{|\bar{x}-\bar{y}|}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)S_p^2}} > t_{1-\alpha/2}\,(n_1+n_2-2)\,.$$

3.2.2 Power at given underlying $\Delta=\mu_1-\mu_2$ and σ^2

The distribution of the test statistics is derived as

$$\begin{aligned}
&\frac{\bar{x}-\bar{y}}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)S_p^2}} \\
=&\frac{\bar{x}-\bar{y}-\Delta+\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)S_p^2}} \\
=&\frac{\frac{\bar{x}-\bar{y}-\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}+\frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}}{\sqrt{\frac{(n_1+n_2-2)S_p^2}{\sigma^2}\cdot\frac{1}{n_1+n_2-2}}} \sim t(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}).
\end{aligned}$$

So the test statistic follows a non-central t-distribution with degree of freedom n_1+n_2-2 and non-central parameter $\frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}$. Then the power of the test

$$
\begin{aligned}
&P\left(\frac{|\bar{x}-\bar{y}|}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)S_p^2}} > t_{1-\alpha/2}\left(n_1+n_2-2\right)\right)\\
=&P\left(\left|T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right)\right| > t_{1-\alpha/2}\left(n_1+n_2-2\right)\right)\\
=&P\left(T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right) > t_{1-\alpha/2}\left(n_1+n_2-2\right)\right)\\
&+P\left(T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right) < t_{\alpha/2}\left(n_1+n_2-2\right)\right)
\end{aligned} \tag{5}
$$

3.2.3 Sample size at given α, β, Δ and σ^2

W.l.o.g, assume $\Delta = \mu_1 - \mu_2 > 0$. Then in the previous power equation (5),

$$
P\left(T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right) < t_{\alpha/2}\left(n_1+n_2-2\right)\right)
$$

will be close to zero and

$$
P\left(T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right) > t_{1-\alpha/2}\left(n_1+n_2-2\right)\right)
$$

will offer most of the power. So to guarantee a $1-\beta$ power we can simply set

$$
P\left(T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right) > t_{1-\alpha/2}\left(n_1+n_2-2\right)\right) \geq 1-\beta.
$$

Therefore

$$
t_{1-\alpha/2}\left(n_1+n_2-2\right) \leq t_\beta\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right).
$$

If in another way around $\Delta < 0$, then we set the power inequality

$$P\left(T\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right) < t_{\alpha/2}\left(n_1+n_2-2\right)\right) \geq 1-\beta,$$

which means

$$t_{\alpha/2}\left(n_1+n_2-2\right) \geq t_{1-\beta}\left(n_1+n_2-2, \frac{\Delta}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right).$$

So in summary (using $t_\alpha(v,\delta)+t_{1-\alpha}(v,-\delta)=0$),

$$0 \leq t_{\alpha/2}\left(n_1+n_2-2\right)+t_\beta\left(n_1+n_2-2, \frac{|\Delta|}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sigma^2}}\right).$$

3.3 Two-sample, variance unknown and unequal

For this we refer to the "Welch's unequal variance t-test"[WELCH, 1947]. The test statistic is

$$t=\frac{\bar{x}-\bar{y}}{s_{\bar{\Delta}}},$$

where

$$s_{\bar{\Delta}}=\sqrt{\frac{s_x^2}{n_1}+\frac{s_y^2}{n_2}}.$$

Here $s_x^2=\frac{1}{n_1-1}\sum_{i=1}^{n_1}\left(x_i-\bar{x}\right)^2$ and $s_y^2=\frac{1}{n_2-1}\sum_{i=1}^{n_2}\left(y_i-\bar{y}\right)^2$ are the unbiased estimator for σ_1^2 and σ_2^2. The test statistic approximately follows a t-distribution with degree of freedom being

$$\mathbf{d.f.}=\frac{\left(\frac{s_x^2}{n_1}+\frac{s_y^2}{n_2}\right)^2}{\frac{\left(s_x^2/n_1\right)^2}{n_1-1}+\frac{\left(s_y^2/n_2\right)^2}{n_2-1}}.$$

References

B. L. WELCH. THE GENERALIZATION OF 'STUDENT's' PROBLEM WHEN SEVERAL DIFFERENT POPULATION VARLANCES ARE INVOLVED. *Biometrika*, 34 (1-2):28–35, 1947. doi: 10.1093/biomet/34.1-2.28.