Multivariate Normal Distirbution

Chao Cheng

July 23, 2022

For a multivariate normal distribution denoted by

$$\boldsymbol{x} \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right),$$

where $\boldsymbol{x} = \left(x_1, \cdots, x_n\right)^T \in \mathcal{R}^n$, $\boldsymbol{\mu} = \left(\mu_1, \cdots, \mu_n\right)^T$ is the mean vector and

$$\boldsymbol{\Sigma} = \left(\rho_{ij}\sigma_i\sigma_j\right)_{n\times n} = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \rho_{ij}\sigma_i\sigma_j & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \cdots & \sigma_n^2 \end{pmatrix}$$

is the $n \times n$ covariance matrix. Here $\sigma_1^2, \cdots, \sigma_n^2$ is the variance of $x_1, \cdots, x_n$ and ρ_{ij} is the correlation between x_i and x_j. Since it's symmetric, $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 1$ for $i = 1, \cdots, n$.

The density function is

$$f\left(\boldsymbol{x}\right) = \left(2\pi\right)^{-n/2} \left|\boldsymbol{\Sigma}\right|^{-1/2} \exp\left(-\frac{1}{2}\left(\boldsymbol{x} - \boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{x} - \boldsymbol{\mu}\right)\right).$$

1 Conditional distribution

2 Bivariate normal distribution

For a bivarite normal variable (x, y) following

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right),$$

and the density function is

$$\begin{aligned} f_{X,Y}\left(x, y\right) =& \frac{1}{2\pi}\left(\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2\right)^{-1/2} \exp\left(-\frac{\begin{pmatrix} x - \mu_1 & y - \mu_2 \end{pmatrix}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}\begin{pmatrix} x - \mu_1 \\ y - \mu_2 \end{pmatrix}}{2\left(\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2\right)}\right) \\ =& \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{\sigma_2^2\left(x - \mu_1\right)^2 - 2\rho\sigma_1\sigma_2\left(x - \mu_1\right)\left(y - \mu_2\right) + \sigma_1^2\left(y - \mu_2\right)^2}{2\sigma_1^2\sigma_2^2\left(1 - \rho^2\right)}\right). \end{aligned}$$

The probability function $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$ is

$$
\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,\mathrm{d}s\mathrm{d}t \\
&= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{Y|X}(s|t)\, f_X(t)\,\mathrm{d}s\mathrm{d}t \\
&= \int_{-\infty}^{x} f_X(t)\left(\int_{-\infty}^{y} f_{Y|X}(s|t)\,\mathrm{d}s\right)\mathrm{d}t.
\end{aligned}
$$

For the conditional distribution $Y|X$, the density is

$$
\begin{aligned}
f_{Y|X}(y|x) =& \frac{f_{X,Y}(x,y)}{f_X(x)} \\
\overset{\text{focus on } y}{\propto}&
\end{aligned}
$$

A R codes

```
R codes
```