Poisson and Negative Binomial Distribution

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February 12, 2024

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1 Introduction

This notes is about poisson and negative binomial distribution, and their application in modeling count data.

2 Poisson Distribution

2.1 Basic Information

The p.m.f of a random variable X that follows Poisson distribution is

$$P\left(X = x\right) = \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \cdots.$$

The expectation and variance of X is

$$\mathrm{E}X = \mathrm{Var}X = \lambda.$$

2.2 Some Additoinal Properties

1. Summation of independent poisson distributed variables is still poisson distributed. That is to say,

   if
   $$X_1 \sim Poisson(\lambda_1), \quad X_2 \sim Poisson(\lambda_2), \quad X_1 \perp X_2,$$

   then
   $$X_1 + X_2 \sim Poisson(\lambda_1 + \lambda_2).$$

2. Poisson distribution can be related to Gamma distribution, which is also mentioned in "gamma_and_beta.pdf" notes.

3. A poisson distributed variable can be seen as counts of customer to a store or occurence of some events. And the **time interval between each count** follows exponential distribution.

3 Negative Binomial Distribution

3.1 Basic Information

Denote X the number of failing attempts just before reaching a total of r times of success and p the probability of success. If each attempt is independent of each other, then the distribution of X is called a negative binomial distribution. And the p.m.f of X is

$$P(X = x) = C_{x+r-1}^x p^r (1-p)^x = \frac{(x+r-1)!}{(r-1)!x!} p^r (1-p)^x, \quad x = 0, 1, \cdots.$$

In this notation, X is the number of failing attempts. In some cases, one will model the distribution of number of total attempts X', which is just $X' = X + r$.

Previously, r is the number of required success, which is naturally an positive integer. But one can actually generalize r to positive real number using Gamma function. Then the p.m.f of X is written as

$$P(X = x) = \frac{\Gamma(x+r)}{\Gamma(r)\Gamma(x+1)} p^r (1-p)^x, \quad x = 0, 1, \cdots,$$

which is also called **Polya** distribution.

The expectation of a negative binomial/Polya distribution is

$$\mathrm{E}X = \frac{r(1-p)}{p}.$$

The variance of a negative binomial/Polya distribution is

$$\mathrm{Var}X = \frac{r(1-p)}{p^2}.$$

3.2 Some Additional Properties

1. Summation of independent negative binomial distributed variables.

2. Relationship to Poisson and Gamma distribution.

3. Dispersion parameter and other parameterization of negative binomial distribution.

4   Modeling Counts Data

4.1   Basic Modeling

4.2   Zero Inflation

References