Minimum Detectable Difference (MDD) in Hypothesis Testing

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1 Introduction

In this notes, we talk about minimum detectable difference (MDD) and other related concepts in hypothesis testing. First, we will use a test for normal mean when variance is known as an example. So for an i.i.d. sample $x_1, \cdots, x_n$ with $x_i \sim N(\mu, \sigma^2)$ with known σ, we want to test

$$H_0: \ \mu = \mu_0 \quad \text{v.s.} \quad H_1: \ \mu \neq \mu_0.$$

A two-sided test with significant level α can be constructed as to reject H_0 when

$$\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2},$$

since $\bar{x} \sim N(\mu_0, \sigma^2/n)$ under H_0. And for any given underlying μ that is not equal to μ_0, the probability to correctly reject H_0, i.e., the power is computed as

$$\begin{aligned}
&P\left(\left| \frac{\bar{x} - \mu_0}{\sqrt{\sigma^2/n}} \right| \geq z_{1-\alpha/2} \right) \\
=&P\left(\bar{x} \geq \mu_0 + z_{1-\alpha/2} * \sqrt{\sigma^2/n} \right) + P\left(\bar{x} \leq \mu_0 - z_{1-\alpha/2} * \sqrt{\sigma^2/n} \right) \\
=&P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \geq \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} + z_{1-\alpha/2} \right) + P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \leq \frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} - z_{1-\alpha/2} \right) \\
=&\left(1 - \Phi\left(\frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} + z_{1-\alpha/2} \right) \right) + \Phi\left(\frac{\mu_0 - \mu}{\sqrt{\sigma^2/n}} - z_{1-\alpha/2} \right).
\end{aligned} \tag{1}$$

And a confidence interval for μ is constructed as

$$\left[\bar{x} - z_{1-\alpha/2} * \sqrt{\sigma^2/n}, \quad \bar{x} + z_{1-\alpha/2} * \sqrt{\sigma^2/n} \right].$$

So here we can see that in order to reject H_0, the critical point is the normalized difference $\left|\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}\right|$ to be greater than $z_{1-\alpha/2}$. This critical point ($z_{1-\alpha/2}$) is determined by the type of test, which leads to the type of test statistics and the significant level. So in other words, as long as the observed difference $|\bar{x}-\mu_0|$ is greater than $z_{1-\alpha/2} * \sqrt{\sigma^2/n}$, H_0 would be rejected no matter what the underlying μ is. So here $z_{1-\alpha/2} * \sqrt{\sigma^2/n}$ is the minimum effect size (considering the sample size and variance) that would be just significant, hence the minimum detectable difference (MDD).

Also from (1) one can see that to achieve a pre-specified power $(1-\beta)$ at some given $\mu \neq \mu_0$, it must satisfy that

$$\left(1-\Phi\left(\frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}+z_{1-\alpha/2}\right)\right)+\Phi\left(\frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}-z_{1-\alpha/2}\right) \geq 1-\beta.$$

W.l.o.g., assume $\mu > \mu_0$, the approximately we have

$$1-\Phi\left(\frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}+z_{1-\alpha/2}\right) \geq 1-\beta.$$

Hence a minimum sample size should satisfy

$$\frac{\mu_0-\mu}{\sqrt{\sigma^2/n}}+z_{1-\alpha/2}=z_\beta,$$

which means

$$n=\left(z_{\alpha/2}+z_\beta\right)^2\sigma^2/\left(\mu_0-\mu\right)^2. \tag{2}$$

When designing a trial, we might choose the target improvemen $\mu-\mu_0$, then from (2) we can get the sample size at given α and β. Also from this we can connect MDD with the target improvement

$$MDD = z_{1-\alpha/2} * \sqrt{\sigma^2/n} = (\mu-\mu_0) * \frac{z_{1-\alpha/2}}{\left(z_{1-\alpha/2}+z_{1-\beta}\right)}. \tag{3}$$

Here $|\mu-\mu_0|$ is sometimes referred to as the minimum detectable effect (MDE) under given α, β and sample size n. And (3) is the relationship between MDD and MDE.

2 MDD and other concepts

Again, let's use the example in the introduction, in which we want to test

$$H_0: \ \mu=\mu_0 \quad \text{v.s.} \quad H_1: \ \mu \neq \mu_0.$$

The reject rule is

$$\left|\frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}}\right| \geq z_{1-\alpha/2}.$$

Therefore on the original scale, the reject rule is

$$\bar{x} \leq \mu_0 - z_{1-\alpha/2} * \sqrt{\sigma^2/n} \quad \text{or} \quad \bar{x} \geq \mu_0 + z_{1-\alpha/2} * \sqrt{\sigma^2/n}.$$

- The minimum detectable observation is $\bar{x}_{mdd} = \mu_0 \pm z_{1-\alpha/2} * \sqrt{\sigma^2/n}$ and the MDD is
$$|\bar{x}_{mdd} - \mu_0| = z_{1-\alpha/2} * \sqrt{\sigma^2/n}.$$
- The $1 - \alpha$ CI for μ is
$$\bar{x} - z_{1-\alpha/2} * \sqrt{\sigma^2/n}, \bar{x} + z_{1-\alpha/2} * \sqrt{\sigma^2/n},$$
where $\bar{x}$ is the observed effect on original scale in one test.
- MDE when designing the experiment, see equation (3) and the related paragraph.

For more details one can see Mair et al. [2020].

References

Magdalena M. Mair, Mira Kattwinkel, Oliver Jakoby, and Florian Hartig. The minimum detectable difference (MDD) concept for establishing trust in nonsignificant results: A critical review. *Environmental Toxicology and Chemistry*, 39(11):2109–2123, sep 2020. doi: 10.1002/etc.4847.