Gamma and Beta Distribution

Chao Cheng

September 7, 2022

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1 Introduction

The Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t. \tag{1}$$

And it's easy to verify that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(n) = (n-1)!, \quad \Gamma(0.5) = \sqrt{\pi}.$$

Let $B(\alpha, \beta)$ denote the Beta function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \mathrm{d}x. \tag{2}$$

Here we point out that $B(\alpha, \beta)$ is related to Gamma function via

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

2 Gamma distribution

The pdf for a Gamma distribution is

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta), \quad 0 \leq x < \infty, \quad \alpha, \beta > 0. \tag{3}$$

3 Beta distribution

The pdf for a Beta distribution is

$$f\left(x|\alpha,\beta\right)=\frac{1}{B\left(\alpha,\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1},\quad 0\leq x\leq 1,\quad \alpha,\beta>0. \tag{4}$$

4 Relationship with other distribution

- If
$$X\sim gamma\left(\alpha_1,\beta\right),\quad Y\sim gamma\left(\alpha_2,\beta\right),\quad X\perp Y.$$
  Then
$$X+Y\sim gamma\left(\alpha_1+\alpha_2,\beta\right),\quad \frac{X}{X+Y}\sim beta\left(\alpha_1,\alpha_2\right),\quad \left(X+Y\right)\perp\frac{X}{X+Y}.$$
- $$beta\left(1,1\right)=Unif\left(0,1\right).$$

References