Biostat 801 Hw 5

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1. Problem 1
   Proof:

   (a) ($\Rightarrow$) Let $A \in \mathcal{F}$. Then $A \perp A^C$, so $P(A)P(A^C) = P(AA^C) = P(\phi) = 0$. Thus $P(A) = 0$ or 1.

   (b) ($\Leftarrow$): Let $A, B \in \mathcal{F}$. Then $P(AB) = 0$ or 1.

      i. If $P(AB) = 0$, then $P(A) = 0$ or $P(B) = 0$, so $P(A)P(B) = 0$.

      ii. If $P(AB) = 1$, then $P(A) = 1$ and $P(B) = 1$, so $P(A)P(B) = 1$.

      In both cases, $P(AB) = P(A)P(B)$, so $A \perp B$.

2. Problem 2
   Let $(\Omega, \mathcal{B}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$, where λ is the Lebesgue measure. Define

   $$X(\omega) := \begin{cases} -1, & \text{if } \omega \in [0, \frac{1}{2}] \\ 1, & \text{if } \omega \in (\frac{1}{2}, 1] \end{cases}$$

   and $Y := -X$. Then clearly $Y \not\perp X$. However, $Y^2 = X^2 = 1$, so $Y^2 \perp X^2$ (see Problem 1).

3. Problem 3
   Proof:

   (a) ($\Leftarrow$): Suppose $P[X = a] = 1$. Let $A, B \in \mathcal{B}$.

      i. If $a \in AB$, then $P[X \in AB] = 1$, and at the same time $P[X \in A] = P[X \in B] = 1$, since $a \in A$ and $a \in B$.

      ii. If $a \notin AB$, then $P[X \in AB] = 0$, and at the same time $P[X \in A] = 0$ or $P[X \in B] = 0$, since $a \notin A$ or $a \notin B$.

      In both cases, $P[X \in AB] = P[X \in A]P[X \in B]$, so $X \perp X$.

   (b) ($\Rightarrow$): Suppose $X \perp X$. Let $A \in \mathcal{B}$. Then

   $$0 = P(\phi) = P(AA^C) = P(A)P(A^C),$$

   so $P(A) = 0$ or 1 for all $A \in \mathcal{B}$. Let $F(x) = P[X \leq x]$. Then $F(x) = 0$ or 1 for all x. Define $S := \{x : F(x) = 1\}$ and $b := \inf S$. Since

$F(\infty) = 1$ and $F(-\infty) = 0$, $b \neq -\infty$ or ∞. Moreover, F is non-decreasing and right-continuous, so $b \in S$ and thus $P[X \leq b] = 1$. Further more,

$$P[X < b] = P(\bigcup_{n=1}^{\infty}[X \leq b - \frac{1}{n}]) = \lim_{n\to\infty} P[X \leq b - \frac{1}{n}] = 0,$$

so

$$P[X = b] = P[X \leq b] - P[X < b] = 1 - 0 = 1$$

4. Problem 4
   To show:
   $$E[\cdots E[X|\mathcal{B}_1]\cdots|\mathcal{B}_n] = E[X|\mathcal{B}_n] \quad a.s.$$
   Proof:
   First consider $E[E[X|\mathcal{B}_1]|\mathcal{B}_2]$. Let $B_2 \in \mathcal{B}_2$. Then
   $$\int_{B_2} E[E[X|\mathcal{B}_1]|\mathcal{B}_2]dP = \int_{B_2} E[X|\mathcal{B}_1]dP = \int_{B_2} XdP = \int_{B_2} E[X|\mathcal{B}_2]dP,$$
   since $\mathcal{B}_2 \subset \mathcal{B}_1$. By the Integral Comparison Lemma,
   $$E[E[X|\mathcal{B}_1]|B_2] = E[X|\mathcal{B}_2] \quad a.s.$$
   Then by mathematical induction,
   $$E[\cdots E[X|\mathcal{B}_1]\cdots|\mathcal{B}_n] = E[X|\mathcal{B}_n] \quad a.s.$$