BIOS653 HW1 Solutions (1-4)

Fall 2017

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Wald approach:
We have a total of $N = \sum_{i=1}^{r} n_i$ observations and rp different β's to estimate.
$\hat{\beta}_i = (X_i^T X_i)^{-1} X_i^T Y_i$ using either OLS or MLE methods,
$\hat{V}(\beta_i) = \sigma^2 (X_i^T X_i)^{-1}$ using either OLS or MLE methods,
$\hat{\sigma} = \frac{1}{N-rp} \sum_{i=1}^{r} (Y_i - X_i \hat{\beta}_i)^T (Y_i - X_i \hat{\beta}_i)$ is the UMVUE, other consistent estimators acceptable.
Let $\beta = (\beta_1^T, \beta_2^T, ..., \beta_r^T)^T$ so we have:

$$\hat{\beta} \sim MVN_{rp}(\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix}, \begin{pmatrix} \hat{V}(\beta_1) & ... & 0 \\ \vdots & \ddots & \vdots \\ 0 & ... & \hat{V}(\beta_r) \end{pmatrix})$$

and contrast matrix:

$$L = \begin{bmatrix} 1 & -1 & 0 & ... & 0 & 0 \\ 0 & 1 & -1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & -1 \end{bmatrix}_{(r-1)\times r} \otimes I_p$$

[Kronecker product, $p \times p$ identity matrix, so dimension of L is $(r-1)p \times rp$, when multiplying out, check that $L\beta = (\beta_1^T - \beta_2^T, \beta_2^T - \beta_3^T, ..., \beta_{r-1}^T - \beta_r^T)^T = 0$ is our desired null hypothesis]
Then you can use the Wald test formula: $W^2 = (L\beta)^T (L\hat{V}(\beta)L^T)^{-1}(L\beta) \sim \chi^2_{(r-1)p}$

LRT approach:
$L(\beta) = \prod_{i=1}^{r} (2\pi\sigma^2)^{-n_i/2} \exp(-\frac{1}{2\sigma^2}(Y_i - X_i\beta_i)^T (Y_i - X_i\beta_i))$
Estimate $\hat{\beta}$ under H_1 as above, estimate $\tilde{\beta}$ under H_0 (total p of them) as:
$\tilde{\beta} = (\sum_{i=1}^{r} X_i^T X_i)^{-1} \sum_{i=1}^{r} (X_i^T Y_i)$
Then you can use LRT formula: $L = -2\log(L(\tilde{\beta})/L(\hat{\beta})) \sim \chi^2_{df}$
Going from rp to parameters to p parameters, so $df = (r-1)p$

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(1)
From OLS or MLE, $\hat{\beta} = (X^TX)^{-1}(X^TY) \approx (-1.05, 0.06)^T$,
$\hat{\sigma}^2 = \frac{1}{13}(Y-\hat{Y})^T(Y-\hat{Y}) = \frac{1}{13}Y^T(I-H)Y = \frac{1}{13}(Y^TY - Y^TX(X^TX)^{-1}X^TY) \approx 0.012$
If you divided by 15 instead, it's still a valid estimate, just note that it's not the best.
(2) We have $\hat{\beta} \sim MVN_2(\beta, \sigma^2(X^TX)^{-1}), \sigma^2(X^TX)^{-1} \approx \begin{pmatrix} 0.058 & -2.30\times10^{-3} \\ -2.30\times10^{-3} & 9.22\times10^{-5} \end{pmatrix}$,
so $\hat{\beta_2} \sim N(\beta_2, 9.22\times10^{-5})$, and the CI is $0.06 \pm t_{13,0.025}\sqrt{9.22\times10^{-5}} \approx (0.037, 0.079)$.
(Note: while I do say Normal, it's important to know where the t-dist would come from. Since we are dividing by an unknown σ^2, we are effectively dividing by a χ^2_{13} distribution.)
Let $c = (-1, 1)$ so $c\hat{\beta} \sim N(\beta_2 - \beta_1, 0.063)$ and the CI is:
$(0.06 + 1.05) \pm t_{13,0.025}\sqrt{0.063} \approx (0.56, 1.65)$
(3)
Under $H_0, \beta_1 \sim N(0.5, 0.058)$, so the test statistic is $t = \frac{\hat{\beta}_1[=-1.05]-0.5}{\sqrt{0.058}} \approx -6.40 < -t_{13,0.025}$, reject H_0.
(4)
Under $H_0, \beta_1 + \beta_2 \sim N(0, 0.054)$, so $t = \frac{\hat{\beta}_1+\hat{\beta_2}[=-0.99]}{\sqrt{0.058}} \approx -1.77 \not> t_{13,0.05}$. DNR H_0.

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(1)
After a bunch of algebra, you should get $c_1^2\sigma_1^2 + 2c_1c_2\sigma_{12} + c_2^2\sigma_2^2$
(2) $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$, so you get $c^T\Sigma c$ equal to what you got in part 1.

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(1)
$var(Y_{ij}) = var(\mu + b_i + \epsilon_{ij}) = var(b_i) + var(\epsilon_{ij}) = \sigma_b^2 + \sigma_e^2$
(2)
$cov(Y_{ij}, Y_{ik}) = cov(\mu + b_i + \epsilon_{ij}, \mu + b_i + \epsilon_{ik}) = cov(b_i, b_i) = \sigma_b^2$
(3)
$var(Y_i) = \mathbf{I_m}\sigma_e^2 + \mathbf{1_m}\mathbf{1_m}^T\sigma_b^2$ [$\mathbf{1_m}$ is a $m \times 1$ vector of 1's]