Definition 21 (Petite sets) *A set $C \in \mathcal{B}(\mathcal{X})$ is ν_a-petite if the sample chain satisfies the bound*

$$K_a(x, B) \geq \nu_a(B),$$

for all $x \in C$, $B \in \mathcal{B}(\mathcal{X})$, where ν_a is a non-trivial measure on $\mathcal{B}(\mathcal{X})$.

The following shows that every small set is petite.

Proposition 12 *If $C \in \mathcal{B}(\mathcal{X})$ is ν_m-small, then C is ν_{δ_m}-petite.*

Proof:

Proposition 13

(i) If $A \in \mathcal{B}(\mathcal{X})$ is ν_a-petite and $D \overset{d}{\rightsquigarrow} A$, then D is ν_{d*a}-petite, where ν_{d*a} can be chosen as a multiple of ν_a.

(ii) If $\mathbf{\Phi}$ is ψ-irreducible and if $A \in \mathcal{B}^+(\mathcal{X})$ is ν_a-petite, then ν_a is an irreducibility measure for $\mathbf{\Phi}$.

Proof:

So, the above proposition tells us if we have a petite set, then we can generate an irreducibility measure for $\mathbf{\Phi}$. There are other useful properties of petite sets that distinguish them from small sets.

Proposition 14 *Suppose $\mathbf{\Phi}$ is ψ-irreducible.*

(i) If A is ν_a-petite, then there exists a sampling distribution b such that A is also ψ_b-petite where ψ_b is a maximal irreducibility measure.

(ii) The union of two petite sets is petite.

(iii) There exists a sampling distribution c, an everywhere strictly positive, measurable function $s : \mathcal{X} \to \mathbb{R}$, and a maximal irreducibility measure ψ_c such that

$$K_c(x, B) \geq s(x)\psi_c(B), \quad x \in \mathcal{X},\ B \in \mathcal{B}(\mathcal{X}).$$

Thus, there is an increasing sequence $\{C_i\}$ of ψ_c-petite sets, all with the same sampling distribution c and minorizing measure equivalent to ψ with $\cup C_i = \mathcal{X}$.

Proof:

We have already seen that every small set is petite. We now show that if $\mathbf{\Phi}$ is also aperiodic, then every petite set is small.

Theorem 7 *If $\mathbf{\Phi}$ is irreducible and aperiodic, then every petite set is small.*

Proof:

1.6 Transience and Recurrence/Harris Recurrence

Definition 22 (Uniform Transience and Recurrence) *A set A is called uniformly transient if there exists a real valued number M such that for all $x \in \mathcal{A}$, $\mathbb{E}_x(\eta_A) \leq M$.*

A is called recurrent if for all $x \in A$, $\mathbb{E}_x(\eta_A) = \infty$.

Note that in this definition, it is not $U(x, A) := \mathbb{E}_x(\eta_A) = \sum_{n=1}^{\infty} P^n(x, A)$, $x \in \mathcal{X}$. We need another definition here to aid in the discussion.

Definition 23 (Taboo Probabilities) *The n-step taboo probability is*

$$_AP^n(x, B) := P_x(\Phi_n \in B, \tau_A \geq n), \quad x \in \mathcal{X};\ A, B \in \mathcal{B}(\mathcal{X}).$$

$_AP^n(x, B)$ denotes the probability of a transition to B in n steps of the chain, "avoiding" the set A. The taboo probabilities satisfy the iterative relation

$$_AP^1(x, B) = P(x, B)$$

and for $n \geq 1$

$$_AP^n(x, B) = \int_{A^c} P(x, dy)\,_AP^{n-1}(y, B), \quad x \in \mathcal{X};\ A, B \in \mathcal{B}(\mathcal{X}).$$

Define

$$U_A(x, B) := \sum_{n=1}^{\infty} {_AP^n}(x, B), \quad x \in \mathcal{X};\ A, B \in \mathcal{B}(\mathcal{X}).$$

Note also that

$$L(x, A) = U_A(x, A), \quad x \in \mathcal{X};\ A \in \mathcal{B}(\mathcal{X}).$$

By convention $_AP^0(x, A) = 0$.

Now, for $B \in \mathcal{B}(\mathcal{X})$ consider the event

$$\{\Phi_n \in B\} = \cup_{j=1}^{n-1}\{\Phi_n \in B, \tau_A = j\} \cup \{\Phi_n \in B, \tau_A \geq n\}, \quad A \in \mathcal{B}(\mathcal{X}),\ A \neq B.$$

The sets on the r.h.s. are mutually exclusive. Based on this decomposition we have the *first-entrance decomposition*

$$P^n(x, B) = {_AP^n}(x, B) + \sum_{j=1}^{n-1} \int_A {_AP^j}(x, dy) P^{n-j}(y, B).$$

Similarly, there exists a decomposition of $\{\Phi_n \in B\}$ into mutually exclusive sets that results in the *last-exit decomposition*

$$P^n(x, B) = {_AP^n}(x, B) + \sum_{j=1}^{n-1} \int_A P^j(x, dy)\,_AP^{n-j}(y, B).$$