Biostat 602 **Assignment 7** **Due: Tuesday, March 14**

This homework is a recap on finding best unbiased estimators based on Lehmann Scheffe theorem. The problems are a bit more involved than the usual homework problems. My recommendation would be to start early.

Homework Problems

1. Let $X_1, \cdots, X_n$ be *i.i.d.* observations from a gamma distribution Gamma(α, β) with the pdf

$$f_X(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad 0 < x < \infty,\ \alpha > 0,\ \beta > 0$$

   (a) Show that the sample arithmetic mean $T(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^n X_i$ and sample geometric mean $S(\mathbf{X}) = \left(\prod_{i=1}^n X_i\right)^{1/n}$ are jointly sufficient and complete for α, β.

   (b) Find the UMVUE for $(\alpha\beta)^n$.

   (c) Show that T and S/T are independent random variables.

2. Let $X_1, \cdots, X_n$ be *i.i.d.* observations from a Inverse Gaussian distribution IG(μ, λ) with the pdf

$$f_X(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\lambda(x-\mu)^2/(2\mu^2 x)\right], \quad 0 < x < \infty,\ \mu > 0,\ \lambda > 0$$

   (a) Show that the sample arithmetic mean $T(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^n X_i$ and sample harmonic mean $S(\mathbf{X}) = \frac{n}{\sum_{i=1}^n (1/X_i)}$ are jointly sufficient and complete for μ, λ.

   (b) Show that the MLEs for μ, λ are

$$\hat{\mu} = \overline{X}, \qquad \hat{\lambda} = \frac{n}{\sum_{i=1}^n \left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)}.$$

   You need not check for global optimality, but must verify that the MLEs are local maximizers and fall inside the parameter space.

   (c) Using the fact that $n\lambda/\hat{\lambda}$ has a χ^2_{n-1} distribution, find the bias and MSE of $1/\hat{\lambda}$ as an estimator of $1/\lambda$.

(d) Find the bias and MSE of $\hat{\lambda}$ as an estimator of λ for $n > 5$. You can use the fact that if $Y \sim \chi^2_k$, then

$$\begin{aligned} E(1/Y) &= \frac{1}{k-2}, \quad k > 2 \\ Var(1/Y) &= \frac{2}{(k-2)^2(k-4)}, \quad k > 4. \end{aligned}$$

3. The following are related to Poisson distribution.

   (a) Let X be a single observation from a $Poisson(\lambda)$ distribution. Show that the only unbiased estimator (and hence UMVUE) of $exp(-2\lambda)$ is $T(X) = (-1)^X$. Is this a reasonable estimator?

   (b) Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from a $Poisson(\lambda)$ distribution, where $n \geq 3$. Show that the only unbiased estimator (and hence UMVUE) of $exp(-2\lambda)$ is

$$T(\mathbf{X}) = \left(1 - \frac{2}{n}\right)^{\sum X_i}.$$

4. Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from a $Bernoulli(p)$ distribution, where $n \geq 3$. Find the UMVUE for p^3.