Lecture 7. Leb int, meas, cond exp

Wednesday, September 27, 2017 9:43 AM

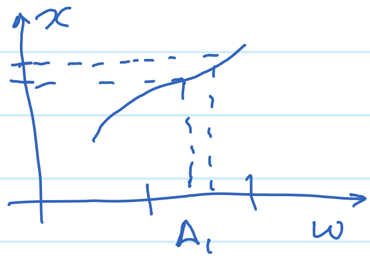
Lebesgue decomposition TH

Any CDF comes from one of 3 classes

1) Discrete
2) Abs continuous
3) Singular

Discrete: $\Omega = \bigcup_i A_i$ disjoint

r.v. X measurable wrt $\sigma(A_1, A_2, \dots)$



Abs. continuous wrt Lebesgue measure

$$\exists f : P(X \in B) = \int_B f(x) \underbrace{dx}$$

length, lebesgue measure

3) Singular: Support of μ has measure 0 wrt Lebesgue measure

Closed σ(subsets A: $\mu(A) > 0$)

μ is continuous

Cantor distribution

Simple functions $X(\omega)$

measurable wrt $\sigma(\{F_1, \dots, F_n\})$

sjoint

Measure μ (Not necessarily a prob. measure)

1) Countable additivity
$\{A_i\}$ disjoint $\quad \mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$

2) $\mu \geq 0 \quad \forall A \in \mathcal{F}$ $\qquad (\Omega, \mathcal{F})$ measurable space

3) $\mu(\emptyset) = 0$

DF

μ is abs. continuous wrt to measure λ

$\mu \prec \lambda$

if for any A : $\lambda(A) = 0 \Rightarrow \mu(A) = 0$

$\boxed{\text{TH}}$ Radon-Nikodým

$$\mu \prec \lambda \iff \exists f, \underbrace{\lambda\text{-equivalent}}_{\text{unique up to a set of } \lambda\text{-measure } 0}, :$$

$$\mu(A) = \int_A f \, d\lambda$$

w/o proof

Lebesgue integral

DF $\quad X_n \xrightarrow[a.e.]{} X$ wrt measure μ

$X_n(\omega) \to X(\omega)$ for all ω
except perhaps for
a subset of ωs that

have $\mu = 0$

DF Lebesgue integral for $X \geq 0$ where X is simple

$$\int_A X \, dP = \int X \cdot I_A \, dP = \sum_k x_k P(F_k)$$

DF for $X \geq 0$ not necessarily simple

For a sequence of simple $X_n \uparrow X$

$$\mathbb{E}(X;A) = \int_A X \, dP = \lim_{n \to \infty} \int_A X_n \, dP$$

DF Arbitrary X

$$X^{\pm} = \max(0, \pm X)$$

$$\mathbb{E}X = \mathbb{E}X^+ - \mathbb{E}X^-$$

$$\mathbb{E}(X;A) = \mathbb{E}(X^+;A) - \mathbb{E}(X^-;A)$$

$$|X| = X^+ + X^-$$

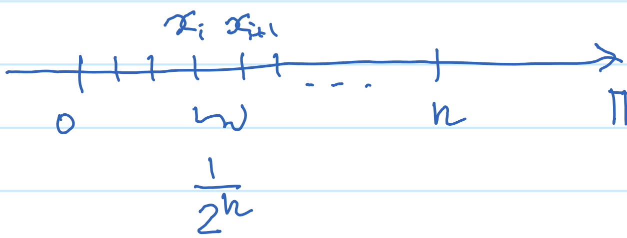
$$\mathbb{E}|X| = \mathbb{E}(X^+) + \mathbb{E}(X^-)$$

$\mathbb{E}X$ exists when $\mathbb{E}|X| < \infty$

Are simple $X_n \uparrow X$ approximations possible?

Yes Lemma

$\exists\, X \geq 0$



$\mathbb{R}$ = Range of X

$F_i = \{ \omega : X \in [x_i, x_{i+1}) \}$

$F_0 = \{ \{ \omega : X \in [0, x_i) \} \cup \{ \omega : X(\omega) \geq n \} \}$

$$X_n = \sum_{i=0}^{n \cdot 2^n - 1} x_i \cdot I_{F_i}(\omega) \leq X(\omega)$$

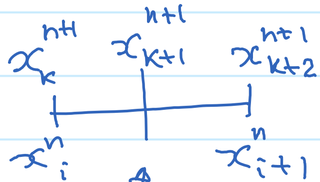
$\#$ intervals $= \dfrac{n}{\frac{1}{2^n}} = n \cdot 2^n$

$\downarrow$

b/c $X \geq x_i$ on F_i

$n \to n+1$: each previous interval is halfed by a mid-point

$X_{n+1} \leq X(\omega)$



$X_n \leq X_{n+1}$

$\searrow$ b/c $x_i < \dfrac{x_i + x_{i+1}}{2}$

$$X_n \leq X_{n+1} \leq X(\omega)$$

$\omega : X(\omega) < n \Rightarrow \quad 0 \leq X(\omega) - X_n(\omega) \leq \dfrac{1}{2^n}$

As $n \to \infty \Rightarrow X_n \uparrow X$ $\square$

Are limits of $\int X_n dP$ unique? Yes Lemma

$\exists$ Two sequences of simple functions X_n, Y_n:

$$X_n \uparrow X$$
$$Y_n \uparrow X$$

want to show

$$\lim_{n \to \infty} \int X_n dP = \lim_{n \to \infty} \int Y_n dP$$

Proof:

Simple functions are bounded

$X_m \le C_m$ - some constants

Choose $\forall\, \varepsilon > 0,\ n$

$A = \{\omega : X_m \geq Y_n + \varepsilon\}$

$\overline{A} = \{\omega : X_m < Y_n + \varepsilon\}$

$$X_m \le C_m \cdot I_A + Y_n + \varepsilon$$

$$\mathbb{E}(X_m) \le C_m \cdot \underbrace{\mathbb{P}(X_m \geq Y_n + \varepsilon)}_{\le \mathbb{P}(X \geq Y_n + \varepsilon) \xrightarrow[n\to\infty]{} 0} + \varepsilon + \mathbb{E}(Y_n)$$

$X \geq X_m \geq Y_n + \varepsilon \Rightarrow X \geq Y_n + \varepsilon$

b/c $X_m \uparrow X$

b/c $Y_n \uparrow X$

later will show

a.s. convergence $\Rightarrow$ conv. in p

$n \to \infty \quad \Downarrow$

$$\mathbb{E}X_m \le \lim_{n\to\infty} \mathbb{E}Y_n + \varepsilon$$

$m \to \infty \quad \Downarrow$

$$\lim \mathbb{E}\, X_n \le \lim \mathbb{E}\, Y_n$$

By symmetry $\lim \mathbb{E}\, Y_n \le \lim \mathbb{E}\, X_n$

limits are equal

$\square$

Next lecture: Conditional expectations

601:

for $B: P(B) > 0 \quad \mathbb{E}(X \mid B) = \dfrac{\mathbb{E}(X; B)}{P(B)}$

Continuous X, Y, $\quad X \perp Y$

$\mathbb{E}(\varphi(X,Y) \mid Y = y)$ - a number, $P(Y=y) = 0$

$\mathbb{E}(\varphi(X,Y) \mid Y)$ - random variable