Biostat 801 Homework 7

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1. Since $\mathcal{B}([0,1])$ is generated by $\{[0,a] : a \in [0,1]\}$, we just need to show $P_n([0,a]) \to P([0,a]) \quad \forall a \in [0,1]$. For every $n \geq 1$, there exists a unique $0 \leq m \leq n$ such that $m/n \leq a \leq (m+1)/n$, so $|a - m/n| \leq 1/n$ and $|a - (m+1)/n| \leq 1/n$. Then

$$|P_n([0,a]) - P([0,a])| = |m/n - a \text{ or } (m+1)/n - a| \leq 1/n \to 0.$$

Thus $P_n \to P$.

2. Let X_n, Y, and (X_n, Y) be r.v.'s on $(\Omega_{X_n}, \mathcal{B}_{X_n}, P_{X_n})$ and $(\Omega_Y, \mathcal{B}_Y, P_Y)$, and $(\Omega_{(X_n,Y)}, \mathcal{B}_{(X_n,Y)}, P_{(X_n,Y)})$, respectively, for $n \geq 0$. (Treat X as X_0 to simplify notations.) Since $X_n \perp Y$, we have $\Omega_{(X_n,Y)} = \Omega_{X_n} \times \Omega_Y$ and $P_{(X_n,Y)} = P_{X_n} P_Y$. Then

$$\begin{aligned} E[F(X_n)] &= \int P_Y(Y \leq X_n) dP_{X_n} = \int \int 1_{[Y \leq X_n]} dP_Y dP_{X_n} \\ &= \int 1_{[Y \leq X_n]} dP_{(X_n,Y)} = P_{(X_n,Y)}[Y \leq X_n] \end{aligned}$$

by Fubini. The result follows trivially.

3. Since $p_n(x) \to p(x)$ uniformly on $[-a, a]$ for all $0 < a < \infty$, the uniform convergence holds on any interval with finite endpoints. Then

$$\int_a^b p_n(x) dx \to \int_a^b p(x) dx \quad \forall -\infty < a \leq b < \infty.$$

We show that

$$\int_{-\infty}^b p_n(x) dx \to \int_{-\infty}^b p(x) dx \quad \forall -\infty < b < \infty.$$

   (a) To show: $\liminf \int_{-\infty}^b p_n(x) dx \geq \int_{-\infty}^b p(x) dx$.

Proof: We have

$$\begin{aligned}
\int_{-\infty}^{b} p_n(x)dx \geq & \int_a^b p_n(x)dx && \forall a \in (-\infty, \infty) \\
\to & \int_a^b p(x)dx && \text{as } n \to \infty \\
\to & \int_{-\infty}^b p(x)dx && \text{as } a \to -\infty,
\end{aligned}$$

so

$$\liminf \int_{-\infty}^{b} p_n(x)dx \geq \int_{-\infty}^{b} p(x)dx$$

(b) To show: $\limsup \int_{-\infty}^{b} p_n(x)dx \leq \int_{-\infty}^{b} p(x)dx$. Let $\epsilon > 0$ and let c be small enough so that $\int_c^b p(x)dx > \int_{-\infty}^b p(x)dx - \epsilon$, so

$$\int_c^b p_n(x)dx > \int_{-\infty}^b p(x)dx - \epsilon \quad \forall n \geq N.$$

Then

$$\int_{-\infty}^{c} p_n(x)dx < \epsilon \quad \forall n \geq N.$$

Hence

$$\begin{aligned}
\int_{-\infty}^{b} p_n(x)dx =& \int_{-\infty}^{c} p_n(x)dx + \int_c^b p_n(x)dx \\
\leq& \epsilon + \int_c^b p_n(x)dx \\
\to& \epsilon + \int_c^b p(x)dx && \text{as } n \to \infty \\
\to& \int_c^b p(x)dx && \text{as } \epsilon \to 0 \\
\to& \int_{-\infty}^b p(x)dx && \text{as } c \to -\infty
\end{aligned}$$

Thus

$$\limsup \int_{-\infty}^{b} p_n(x)dx \leq \int_{-\infty}^{b} p(x)dx.$$

Therefore, $\int_{-\infty}^{b} p_n(x)dx \to \int_{-\infty}^{b} p(x)dx$, so $P_n \xrightarrow{d} P$.

4. Since $X_n \xrightarrow{L_2} X$, we have $X_n \xrightarrow{L_1} X$ and therefore $E|X_n - X| \to 0$. Then

$$E|X| \leq E|X - X_n| + E|X_n| \leq \epsilon + E|X_n| < \infty$$

for all $n \geq N$. Moreover,

$$|EX_n - EX| = |E(X_n - X)| \leq E|X_n - X| \to 0,$$

so $EX_n \to EX$.