Practice Midterm Exam—Statistics 621

The midterm will be closed book exam, but you are allowed one formula sheet. Show your work for full or partial credit.

(1) Let $X_1, \ldots, X_n$ be random variables, and let M denote the minimum,
$$M = \min\{X_1, \ldots, X_n\}.$$
Show that if
$$\sum_{i=1}^{n} P(X_i > 0) > n - 1,$$
then $P(M > 0) > 0$.

(2) Let $\Omega = \{0, 1, 2, \ldots\}$, and let $\mathcal{C}$ be all subsets of Ω that are finite or have a finite complement,
$$\mathcal{C} = \bigl\{A \subset \Omega : \#A < \infty \text{ or } \#(\Omega \setminus A) < \infty\bigr\}.$$
Is $\mathcal{C}$ a field (or algebra)? Is $\mathcal{C}$ a σ-field? Explain your answers.

(3) Let $(\Omega, \mathcal{B}, P)$ be a probability space and define
$$\mathcal{C} = \bigl\{B \in \mathcal{B} : P(B) = 0 \text{ or } 1\bigr\}.$$
Is $\mathcal{C}$ a σ-field? Explain your answer.

(4) Let $(\Omega, \mathcal{B}) = \bigl(\mathbb{R}, \mathcal{B}(\mathbb{R})\bigr)$, define X by
$$X(\omega) = (\omega^+)^2,$$
and let P be a probability measure on $(\Omega, \mathcal{B})$ given by
$$P(A) = \int_A \frac{1}{2} e^{-|\omega|}\, d\omega, \qquad A \in \mathcal{B}.$$

(a) Which of the following sets: $S_1 = (-1, 1)$, $S_2 = [\pi, \infty)$, $S_3 = \{0\}$ and $S_4 = (-\infty, 7]$, lie in $\sigma(X)$?
(b) Let F denote the distribution of X. Find $F\bigl(\{0\}\bigr)$ and $F\bigl([1, \infty)\bigr)$.

(5) Let X_n, $n \geq 1$, be i.i.d. from a standard exponential distribution (so $P(X_n \leq x) = 1 - e^{-x}$, $x > 0$), and define $Y_n = X_n / \log n$. Find
$$\limsup_{n \to \infty} Y_n.$$
Hint: First compute $P(Y_n \geq c, \text{i.o.})$.

(6) Let X_i, $i \geq 1$, be i.i.d. and uniformly distributed on $(0, 1)$.

(a) Find $P(\inf_{n\geq 1} nX_n > 0)$.
(b) Find $P(\inf_{n\geq 1} n^2 X_n > 0)$.
(c) Find $\liminf_{n\to\infty} n^2 X_n$. (This variable should be almost surely constant, possibly $+\infty$, by Kolmogorov's zero-one law.)

(7) Let Z have a standard normal distribution with density $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$, and let h be a bounded differentiable function on $[0, \infty)$, vanishing at zero, $h(0) = 0$. Then

$$\int_0^\infty \bigl|h(1/x^2)\bigr|\, dx < \infty.$$

Use this fact and dominated convergence to find

$$\lim_{n\to\infty} nEh\bigl(1/(n^2 Z^2)\bigr).$$

Hint: The answer will naturally depend on h and should not be zero in general.

(8) Let X and Y be i.i.d. with common cumulative distribution function F. Express the integral

$$\int F^2(t) e^{-t}\, dt$$

as $E[h(X, Y)]$ for some specific function h.

(9) Suppose $X > 0$ almost surely. Find a function g so that

$$\int_0^\infty e^{-y} P(X < y)\, dy = E[g(X)].$$

(10) Show that $X_n \xrightarrow{p} 0$ if and only if

$$E\left[\frac{|X_n|}{1 + |X_n|}\right] \to 0.$$

(11) For $n \geq 1$, let X_n have a Bernoulli distribution with success probability p_n, and let c_n, $n \geq 1$, be a sequence of constants increasing to $+\infty$. When is the collection $\{c_n X_n, n \geq 1\}$ uniformly integrable?