Solution to Assignment 6

6.1: **(b)** Define

$$Y_n \stackrel{\text{def}}{=} \sup_{k \geq n} |X_k - X|, \qquad n \geq 1.$$

Since $X_n \to X$ iff $Y_n \to 0$, we have $X_n \stackrel{\text{a.s.}}{\to} X$ iff $Y_n \stackrel{\text{a.s.}}{\to} 0$. So $X_n \stackrel{\text{a.s.}}{\to} X$ implies $Y_n \stackrel{p}{\to} 0$. For the converse, if $Y_n \stackrel{p}{\to} 0$, then for any $\epsilon > 0$ and any $k \geq 1$,

$$P(\limsup Y_n > \epsilon) \leq P(Y_k > \epsilon) \to 0 \qquad \text{as } k \to \infty.$$

So $P(\limsup Y_n > \epsilon) = 0$ and as $\epsilon > 0$ is arbitrary $\limsup Y_n = 0$ almost surely. Then $X_n \stackrel{\text{a.s.}}{\to} X$. (The converse can also be established using part (a).) **(d)** Let $M_n = \max\{X_1, \ldots, X_n\}$ and $M_\infty = \liminf M_n$, and note that $P(M_\infty > x_0) = 0$ because $P(X_n \leq x_0) = F(x_0) = 1$ for all $n \geq 1$. Next, since $M_n \leq M_\infty$, for any $\epsilon > 0$ and any $n \geq 1$ we have

$$P(M_\infty < x_0 - \epsilon) \leq P(M_n < x_0 - \epsilon) = \big[F(x_0 - \epsilon)\big]^n.$$

Since $F(x_0 - \epsilon) < 1$ this tends to zero as $n \to \infty$. So $P(M_\infty < x_0 - \epsilon) = 0$, and as $\epsilon > 0$ is arbitrary, $M_\infty = x_0$ almost surely. Thus $M_n \stackrel{\text{a.s.}}{\to} x_0$.

6.5: Since a single L_1 variable is ui and the X_i are identically distributed, $\{X_k, k \geq 1\}$ is ui. By Theorem 6.5.1, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$E|X_k|1_B < \epsilon, \qquad \text{whenever } P(B) < \delta.$$

But then if $P(B) < \delta$ we have

$$E|n^{-1}S_n|1_B = \frac{1}{n}E\left|\sum_{k=1}^n X_k 1_B\right| \leq \frac{1}{n}E\sum_{k=1}^n |X_k|1_B < \epsilon.$$

Thus Part A of Theorem 6.5.1 holds for $\{n^{-1}S_n, n \geq 1\}$. Part B also holds for this family because

$$E|n^{-1}S_n| \leq \frac{1}{n}E\sum_{k=1}^n |X_k| = E|X_1| < \infty,$$

so the family must be uniformly integrable.

6.13: By dominated convergence,

$$nP\big[|X_1 \geq \epsilon\sqrt{n}\big] = \int_{\left[|X_1 \geq \epsilon\sqrt{n}\right]} n\,dP \leq \frac{1}{\epsilon^2}\int_{\left[|X_1 \geq \epsilon\sqrt{n}\right]} X_1^2\,dP \to 0.$$

So

$$P\left[\frac{\bigvee_{j=1}^{n}|X_j|}{\sqrt{n}} \geq \epsilon\right] = P\big[|X_j| \geq \epsilon\sqrt{n}, \exists j = 1, \dots, n\big]$$
$$\leq \sum_{j=1}^{n} P\big[|X_j| \geq \epsilon\sqrt{n}\big]$$
$$= nP\big[|X_1| \geq \epsilon\sqrt{n}\big] \to 0.$$

6.14: Since

$$\sum_k P(X_k = k^2) = \sum_k \frac{1}{k^2} = \frac{\pi^2}{6} < \infty,$$

by Borel-Cantelli $P(X_k = k^2, \text{i.o.}) = 0$, and on $[X_k = k^2, \text{i.o.}]^c$, $\sum_{k=1}^{n} X_k \to -\infty$.

6.15: Since $x \rightsquigarrow x^+$ is continuous, $(X_0 - X_n)^+ \xrightarrow{p} 0$. These variables are also dominated by $X_0 \in L_1$, so by Lebesgue dominated convergence (Corollary 6.3.2), $E(X_0 - X_n)^+ \to 0$. So

$$E|X_n - X_0| = E\big[X_n - X_0 + 2(X_0 - X_n)^+\big]$$
$$= EX_n - EX_0 + 2E(X_0 - X_n)^+ \to 0.$$

6.20: Note first that since

$$\mathrm{Var}\big(|X|\big) = 1 - \big(E|X|\big)^2 \geq 0,$$

we have $a \leq 1$. Next, define $Y = |X| - \lambda a$ and note that

$$EY = E|X| - \lambda a \geq (1-\lambda)a \geq 0.$$

If $t > 0$, then $(1 - tY)^2 < 1$ on $Y \leq 0$, and so

$$P(Y < 0) \leq E(1 - tY)^2 = 1 - 2tEY + t^2 EY^2.$$

Taking $t = EY/EY^2$,

$$P(Y \leq 0) \leq 1 - \frac{(EY)^2}{EY^2}.$$

Since

$$EY^2 = 1 - 2\lambda a EX + \lambda^2 a^2 \leq (1 - \lambda a)^2 \leq 1,$$

we have

$$P\big[|X| \geq \lambda a\big] = P(Y > 0) \geq (EY)^2 \geq (1-\lambda)^2 a^2.$$