Lecture 10. total prob

Monday, October 9, 2017 10:04 AM

Properties of $\mathbb{E}\{\cdot \mid \mathcal{U}\}$

Linearity

$$\mathbb{E}(aX_1 + bX_2 \mid \mathcal{U}) = a\,\mathbb{E}(X_1 \mid \mathcal{U}) + b\,\mathbb{E}(X_2 \mid \mathcal{U})$$

Ordering preservation

$$X_1 \le X_2 \text{ a.s.} \Rightarrow \mathbb{E}(X_1 \mid \mathcal{U}) \le \mathbb{E}(X_2 \mid \mathcal{U}) \text{ a.s.}$$

Conditional probability $P(A \mid \mathcal{U}) = \mathbb{E}\{I_A \mid \mathcal{U}\}$

Probability inequalities

$$X \geq 0,\ x > 0 \qquad P(X \geq x \mid \mathcal{U}) \le \frac{\mathbb{E}(X \mid \mathcal{U})}{x}$$

$$f \uparrow,\ f > 0 \qquad P(X \geq x \mid \mathcal{U}) \le \frac{\mathbb{E}(f(X) \mid \mathcal{U})}{f(x)}$$

$$P(|X - \mathbb{E}(X \mid \mathcal{U})| \geq x \mid \mathcal{U}) \le \frac{\mathrm{Var}(X \mid \mathcal{U})}{x^2}$$

Monotone convergence Th for $\mathbb{E}(\cdot \mid \mathcal{U})$

TH

$$0 \le X_n \uparrow X \text{ a.s.} \Rightarrow \mathbb{E}\{X_n \mid \mathcal{U}\} \uparrow \mathbb{E}\{X \mid \mathcal{U}\}$$

Proof.

$$X_{n+1} \geq X_n \text{ a.s.} \Rightarrow \mathbb{E}(X_{n+1} \mid \mathcal{U}) \geq \mathbb{E}(X_n \mid \mathcal{U})$$

ordering preservation of $\mathbb{E}(\cdot \mid \mathcal{U})$

Also $X_n \le X$ a.s. $\Rightarrow \mathbb{E}(X_n \mid \mathcal{U}) \le \mathbb{E}(X \mid \mathcal{U})$

$\Rightarrow \exists Z : Z$ is $\mathcal{U}$-measurable :

$$Y_n = \mathbb{E}(X_n \mid \mathcal{U}) \to Z \quad \text{a.s.}$$

by the bounded convergence Th from Calculus

Note: limits of measurable functions are measurable if they exist by the Bounded Convergence Th. from

Note: limits of measurable functions are measurable if they exist

Calculus for each ω except perhaps a set of measure zero

Using the following property of Lebesgue integrals from for example Shiryaev;
Proved using usual approximations by simple functions,

184 II. Mathematical Foundations of Probability Theory

Theorem 1 (On Monotone Convergence). *Let $\eta, \xi, \xi_1, \xi_2, \dots$ be random variables.*

(a) *If $\xi_n \geq \eta$ for all $n \geq 1$, $\mathsf{E}\eta > -\infty$, and $\xi_n \uparrow \xi$, then*

$$\mathsf{E}\xi_n \uparrow \mathsf{E}\xi.$$

(b) *If $\xi_n \leq \eta$ for all $n \geq 1$, $\mathsf{E}\eta < \infty$, and $\xi_n \downarrow \xi$, then*

$$\mathsf{E}\xi_n \downarrow \mathsf{E}\xi.$$

We have:

$$\mathbb{E}(Y_n ; A) \to \mathbb{E}(Z ; A)$$
$$\mathbb{E}(X_n ; A) \to \mathbb{E}(X ; A)$$

by properties of integrals

Now by definition of $\mathbb{E}(\cdot|\mathcal{U}) \Rightarrow$

$$\Rightarrow \mathbb{E}(Y_n ; A) = \mathbb{E}(X_n ; A)$$

$$\Downarrow$$

$$\mathbb{E}(Z ; A) = \mathbb{E}(X ; A)$$ by unique limit

by definition of $\mathbb{E}(\cdot|\mathcal{U}) \Rightarrow Z = \mathbb{E}\{X|\mathcal{U}\}$

$\square$

TH $\mathcal{U}$-measurable r.v.s behave like Const inside $\mathbb{E}(\cdot|\mathcal{U})$

$\exists$

$X \sim \mathcal{F}$-measurable $\quad \mathcal{U} \subset \mathcal{F}$

$Y \sim \mathcal{U}$-measurable

$\mathbb{E}(|X|) < \infty$

$\mathbb{E}(|X \cdot Y|) < \infty$

Then

$$\mathbb{E}(X \cdot Y | \mathcal{U}) = Y \cdot \mathbb{E}(X|\mathcal{U})$$

Proof.

$\exists\ Y = I_B, \quad B \in \mathcal{U}$

Take $\forall A \in \mathcal{U}$

$$\mathbb{E}(X \cdot I_B ; A) = \mathbb{E}(X \cdot \underbrace{I_A I_B}_{= I_{AB}}) =$$

$$\underline{\mathbb{E}(X\cdot I_B ; A)} = \mathbb{E}(X\cdot I_B \cdot I_A) =$$

$$= \mathbb{E}(X ; AB) := \mathbb{E}\{\mathbb{E}(X|\mathcal{U}); \underset{\in \mathcal{U}}{AB}\} =$$

$$= \underline{\mathbb{E}\{I_B \cdot \mathbb{E}(X|\mathcal{U}); A\}} \implies$$

$$\underbrace{I_B \mathbb{E}(X|\mathcal{U})}_{\mathcal{U}-\text{meas. r.v.}} = \mathbb{E}(X\cdot I_B|\mathcal{U}) \quad \text{by def of } \mathbb{E}(\cdot|\mathcal{U})$$

Suppose all variables are ≥ 0

Construct Y_n - simple and : $0 \leq Y_n \uparrow Y$, $X \geq 0$

The fact we are trying to prove is valid for simple functions b/c

$$\mathbb{E}(X\cdot Y_n|\mathcal{U}) = Y_n \mathbb{E}(X|\mathcal{U})$$

Going to $\lim_{n\to\infty}$ in $\uparrow$ and monotone conv. Th for $\mathbb{E}(\cdot|\mathcal{U})$

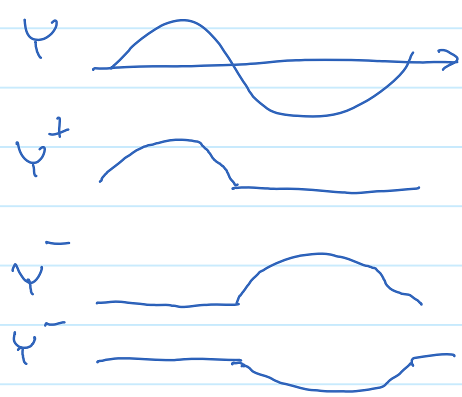
we will have the result for ≥ 0 r.v.s.

Any $\quad Y = \underset{\geq 0}{Y^+} - \underset{\geq 0}{Y^-} \implies$ get the validity of the Th for arbitrary Y

$$X = \underset{\geq 0}{X^+} - \underset{\geq 0}{X^-}$$

$\square$

$$Y^{\pm} = \max(0, \pm Y)$$



Formula of total probability for $\mathbb{E}$

TH $\quad \mathbb{E}(X) = \mathbb{E}\{\underbrace{\mathbb{E}\{X|\mathcal{U}\}}_{Y}\}$

$\mathbb{E}(X;A) = \mathbb{E}(Y;A)$ ...

$$\mathbb{E}(\overset{Y}{Y}; A) := \mathbb{E}(X; A) \quad , \forall A \in \mathcal{U}$$

Take $A = \Omega \Rightarrow \mathbb{E}(Y) = \mathbb{E}(X)$

$\square$

TH $\exists\ \mathbb{E}X^2 < \infty$ $\qquad \mathcal{U} \subset \mathcal{F} = \sigma(X)$

$$\arg \min_{Y:\ \mathcal{U}\text{-measurable}} \underbrace{\mathbb{E}\{(X-Y)^2\}}_{\text{distance}^2 \text{ between } X \,\&\, Y} = \mathbb{E}(X|\mathcal{U})$$

Proof.

$$\underbrace{\mathbb{E}((X-Y)^2) = \mathbb{E}(\mathbb{E}(}_{\text{formula of total prob. for } \mathbb{E}}\underbrace{[X-Y]^2}_{X^2 - 2XY + Y^2}|\mathcal{U})) =$$

Y, Y^2: $\mathcal{U}$-measurable

$$= \mathbb{E}\{\mathbb{E}(X^2|\mathcal{U}) - 2Y\mathbb{E}(X|\mathcal{U}) + Y^2\}$$ achieves

min of zero when $Y = \underbrace{\mathbb{E}(X|\mathcal{U})}_{\mathcal{U}\text{-meas. by def.}}$

$\square$

Formula of total probability for Var

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\mathcal{U})) + \mathrm{Var}(\mathbb{E}(X|\mathcal{U}))$$

Model $(a, b) \Rightarrow \hat{a}, \hat{b}$ MLE

$\mathrm{Var}\,\hat{a} = \mathrm{Var}\,\hat{\hat{a}} +$ Var due to b being estimated

$\hat{\hat{a}}$: Model $(a,b)|_{b \text{ as known}}$

Proof

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) =$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) =$$

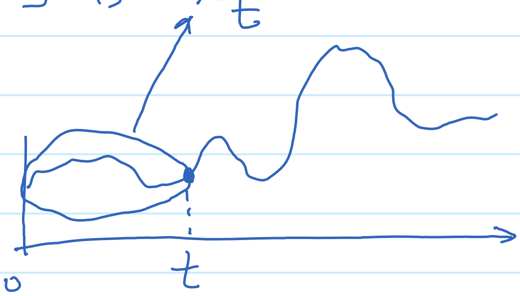
$$= \mathbb{E}\left(\mathbb{E}(X^2|\mathcal{U})\right) - \left[\mathbb{E}\big(\overbrace{\mathbb{E}(X|\mathcal{U})}^{Y}\big)\right]^2$$

$$- \mathbb{E}\left(\mathbb{E}^2(X|\mathcal{U})\right) + \mathbb{E}\big(\underbrace{\mathbb{E}^2(X|\mathcal{U})}_{Y^2}\big)$$

$$= \mathbb{E}\left(\mathrm{Var}(X|\mathcal{U})\right) + \mathrm{Var}\left(\mathbb{E}(X|\mathcal{U})\right) \quad \square$$

Random process / Stochastic is a collection of r.v.s X_t

Trajectory of X on $[0,t]$ is $\overline{X}_t$



$X(t,\omega)$, $\omega \in \mathcal{F}$

Filtration is a sequence of increasing σ-algebras indexed by t

$$\mathcal{F}_\tau \subset \mathcal{F}_t \subset \mathcal{F}$$
$$\tau < t$$

Increment

$$dX_t = X_{t+dt} - X_t$$

DF Take $\mathcal{F}_t = \sigma(\overline{X}_t)$ $\Rightarrow$ The process X_t is adapted to filtration

DF Martingale is a process : $\mathbb{E}\{dX_t \mid \mathcal{F}_{t-}\} = 0$

(TH) Increments of a Martingale are uncorrelated

$$\mathbb{E}(dX_t \cdot dX_\tau) = \mathbb{E}\Big(dX_\tau \cdot \underbrace{\mathbb{E}(dX_t \mid \mathcal{F}_{t-})}_{:=0}\Big) = 0$$

$$\tau < t$$

$X_t = \int_0^t dX_x$ is a "sum" of uncorrelated increments