Part II: Statistical Decision Theory

Problem 3: Suppose X_1, X_2, X_3 are independent random variables with $X_i \sim \text{Unif}(0, \theta_i)$, $0 < \theta_i < \infty, i = 1, 2, 3$. Give data (x_1, x_2, x_3), one wants to decide which parameter among the three $\theta_1, \theta_2, \theta_3$ is the largest. The action space is given by $A = \{a_1, a_2, a_3\}$, where a_i is the action of identifying θ_i being the largest. Consider the following loss function:

$$L((\theta_1, \theta_2, \theta_3), a_i) = \begin{cases} 0, & \text{if } \theta_i \text{ is the largest;} \\ 2, & \text{if } \theta_i \text{ is the smallest;} \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that the decision rule $\delta(x_1, x_2, x_3)$ is to take action a_i when $x_i = \max\{x_1, x_2, x_3\}$. Calculate its risk function.

Problem 4: Suppose $X|\theta \sim N(\theta, 1), \theta \in R^1$ with prior distribution $\theta \sim N(0, \tau^2), \tau > 0$. Let the action space be $A = \{a : a \in R^1\}$. Consider the following loss function:

$$L(\theta, a) = \begin{cases} 0, & \text{if } |\theta - a| \leq 1; \\ 1, & \text{if } |\theta - a| > 1. \end{cases}$$

Find the Bayes solution and its Bayes risk.

Problem 5: Let $X_1, \ldots, X_n$ be *i.i.d.* with $N(\mu, \sigma^2), \mu \in (-\infty, \infty), \sigma > 0$. The objective is to obtain an esimator of μ under the squared error loss function $L((\mu, \sigma^2), a) = (\mu - a)^2$. Show that

(i) there does not exist a uniformlly superior estimator of μ;

(ii) estimator $\hat{\mu}(\mathbf{x}) = c$ (a constant) is an admissible estimator;

(iii) estimator $\hat{\mu}(\mathbf{x}) = c\bar{x}$ with $|c| > 1$ is not admissible, where $\bar{x}$ is the sample mean.