Lecture 18. Slutsky, Char func

Wednesday, November 15, 2017 10:11 AM

Vector extension of $\rightsquigarrow$

Suppose X_n, X are vector-valued r.v.s $\mathbb{R}^m$

DF $X_n \rightsquigarrow X$ when $F_n \to F$ for every point of continuity of F

joint CDFs of r.v. vector

L $X_n \rightsquigarrow X$ r.v. in $\mathbb{R}^m$ $\Longleftrightarrow$

$\forall f: \mathbb{R}^m \xrightarrow{f} \mathbb{R}$, continuous and bounded

Portmanteau $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$

TH Continuous mapping (w/o proof)

$X_n \rightsquigarrow X$, f is continuous $\Rightarrow$

$$f(X_n) \rightsquigarrow f(X)$$

$\|x\|$ is the length of $x = \sqrt{x x^T}$

$x \in \mathbb{R}^m$

TH $X_n \rightsquigarrow X$, $\|X_n - Y_n\| \xrightarrow[P]{} 0$

$\Rightarrow Y_n \rightsquigarrow X$

Proof: Take f as arbitrary Lipschitz, bounded funct. $f \le C$

$$\|\mathbb{E}f(X_n) - \mathbb{E}f(Y_n)\| \le \mathbb{E}\|f(X_n) - f(Y_n)\| =$$

$$= \mathbb{E}\{\|\cdot\| ; \|\cdot\| \le \varepsilon\} + \mathbb{E}\{\|\cdot\| ; \|\cdot\| > \varepsilon\} \le$$

$$\le \varepsilon \cdot \Pr(\|\cdot\| \le \varepsilon) + C \cdot P(\|\cdot\| > \varepsilon)$$

$$\leq \underbrace{\varepsilon \cdot \Pr(\|\cdot\| \leq \varepsilon)}_{\leq \varepsilon,\ \text{can make arbitrary small}} + C_1 \cdot \underbrace{P(\|\cdot\| > \varepsilon)}_{\to 0 \text{ b/c}}$$

$$\lim \left(\mathbb{E} f(X_n) - \mathbb{E} f(Y_n) \right) = \mathbb{E} f(X) - \lim \mathbb{E} f(Y_n) = 0$$

$$\Rightarrow \quad \mathbb{E} f(Y_n) \to \mathbb{E} f(X) \Rightarrow Y_n \rightsquigarrow X$$

Portmanteau lemma (Lecture 14) $\square$

TH $X_n \rightsquigarrow X$, $Y_n \xrightarrow{P} c \Rightarrow \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ c \end{pmatrix}$

Take $f(\underbrace{x}_{\text{vector}}, y)$ as bounded and cont.

$$\left\| \begin{pmatrix} X_n \\ Y_n \end{pmatrix} - \begin{pmatrix} X_n \\ c \end{pmatrix} \right\| = \| Y_n - c \| \xrightarrow{P} 0$$

$\tilde{f}(x) = f(x, c)$ bounded and cont

$$\Rightarrow \mathbb{E} \tilde{f}(X_n) \to \mathbb{E} \tilde{f}(X)$$

$$\Rightarrow \mathbb{E} f(X_n, c) \to \mathbb{E} f(X, c) \Rightarrow$$

$$\Rightarrow \begin{pmatrix} X_n \\ c \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ c \end{pmatrix} \Rightarrow$$

By the previous Th

$\begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ and $\begin{pmatrix} X_n \\ c \end{pmatrix}$ have same lim in distr.

$$\Rightarrow \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ c \end{pmatrix} \quad \square$$

Corollary $X_n \rightsquigarrow X$, $Y_n \xrightarrow{P} c \Rightarrow F_{X_n Y_n} \to F_X$

Proof: $\Rightarrow \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ c \end{pmatrix}$

$F_{X_n | Y_n} \to F_{X|c}$ at every point of continuity

$\sigma(\text{non-random } c) = \{\emptyset, \Omega\}$

$P(X \in B \cap \emptyset \text{ or } \Omega) = P(X \in B) \cdot P(\emptyset \text{ or } \Omega) \Rightarrow$

$\Rightarrow X \perp c \Rightarrow F_{X|c} = F_X$ $\square$

TH $X_n \xrightarrow[p]{} X, \quad Y_n \xrightarrow[p]{} Y \Rightarrow$

$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow[p]{} \begin{pmatrix} X \\ Y \end{pmatrix}$

Proof:

$\sqrt{\underbrace{a^2}_{x_n - x} + \underbrace{b^2}_{y_n - y}} \le |a| + |b|$

$\varepsilon < \left\| \begin{pmatrix} X_n \\ Y_n \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} \right\| \le \|X_n - X\| + \|Y_n - Y\|$

$\Rightarrow P\left(\left\| \begin{pmatrix} X_n \\ Y_n \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} \right\| > \varepsilon \right) \le P\left(\|X_n - X\| + \|Y_n - Y\| > \varepsilon \right) \le$

$\le P\left(\overline{\|\cdot\| \le \frac{\varepsilon}{2} \cap \|\cdot\| \le \frac{\varepsilon}{2}} \right) \le \underbrace{P\left(\|\cdot\| > \frac{\varepsilon}{2} \right)}_{\to 0} + \underbrace{P\left(\|\cdot\| > \frac{\varepsilon}{2} \right)}_{\to 0}$ $\square$

Slutsky Th $X_n \rightsquigarrow X, \quad Y_n \xrightarrow[p]{} c$

$\Rightarrow$

$X_n + Y_n \rightsquigarrow X + c$

$X_n \cdot Y_n \rightsquigarrow c \cdot X$

Proof:

$\Rightarrow \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ c \end{pmatrix} \Rightarrow$

$\forall$ cont $f \Rightarrow f(X_n, Y_n) \rightsquigarrow f(X, c)$

by the cont. mapping Th $\square$

Example:

$\exists$ T_n is an estimator for θ

S_n^2 is an estimator for its variance σ^2

$$\sqrt{n}\,(T_n - \theta) \rightsquigarrow \mathcal{N}(0, \sigma^2)$$

$$S_n^2 \xrightarrow[P]{} \sigma^2$$

By Slutsky $\dfrac{\sqrt{n}\,(T_n - \theta)}{S_n^2} \rightsquigarrow \mathcal{N}(0,1)$

$$\begin{pmatrix} \sqrt{n}\,(T_n - \theta) \\ S_n^2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} X \\ \sigma^2 \end{pmatrix}, \quad X \sim \mathcal{N}(0, \sigma^2)$$

$$P\Big(\underbrace{T_n - z_{1-\frac{\alpha}{2}} \cdot \frac{S_n}{\sqrt{n}}} \le \theta \le \underbrace{T_n + z_{1-\frac{\alpha}{2}} \cdot \frac{S_n}{\sqrt{n}}}\Big) \to P\big(|X| \le z_{1-\frac{\alpha}{2}}\big) = \alpha$$

$(\downarrow \quad , \quad \downarrow)$ is $1-\alpha$ CI

Characteristic Function Properties

r.v. X

c.f. of X $\varphi(t) := \mathbb{E}\{e^{itX}\}$

Note: Fourier transform $\frac{1}{\sqrt{2\pi}} \cdot \varphi$

1. $\varphi(0) = 1$

2. $|\varphi| \le 1$

3. $\varphi_{a \cdot X + b} = \mathbb{E}\{e^{it \cdot (aX+b)}\} = e^{ibt}\underbrace{\mathbb{E}\{e^{itaX}\}}_{\varphi(ta)}$

4. $X \perp Y$

$$\varphi_{X+Y} = \mathbb{E}\{e^{i(X+Y)\cdot t}\} = \underbrace{\mathbb{E}\{e^{iXt}\}}_{\varphi_X} \cdot \underbrace{\mathbb{E}\{e^{iYt}\}}_{\varphi_Y}$$

5. Moments

$$\varphi^{(k)}(0) = i^k \; \mathbb{E}\{X^k e^{itX}\}\Big|_{t=0} =$$

$$= i^k \cdot \underbrace{\mathbb{E}(X^k)}_{:= M_k}$$

6. Taylor expansion

$$\varphi(t) = \sum_{k=1}^{n} \frac{(it)^k}{k!} \underbrace{\mathbb{E}\{X^k\}}_{M_k} + o\left(|t|^n\right)$$

if M_k exist