In what follows, we will need the generating functions for the series $\{P^n\}$ and $\{{}_AP^n\}$. Let $|z| < 1$. Then the generating function for $\{P^n\}$ is

$$U^{(z)}(x, B) := \sum_{n=1}^{\infty} P^n(x, B) z^n$$

and the generating function for $\{{}_AP^n\}$ is

$$U_A^{(z)}(x, B) := \sum_{n=1}^{\infty} {}_AP^n(x, B) z^n.$$

By their definitions, it is easy to see that

$$U(x, B) = \sum_{n=1}^{\infty} P^n(x, B) = \lim_{z \uparrow 1} U^{(z)}(x, B).$$

The return probabilities $L(x, A) = P_x(\tau_A < \infty)$ satisfy

$$L(x, A) = \sum_{n=1}^{\infty} {}_AP^n(x, A) = \lim_{z \uparrow 1} U_A^{(z)}(x, A).$$

Now, multiply the first-entrance and last-exit decompositions by z^n and sum over n.

$$\begin{aligned} U^{(z)}(x, B) &= \sum_{n=1}^{\infty} P^n(x, B) z^n \\ &= U_A^{(z)}(x, B) + \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \int_A {}_AP^j(x, dy) P^{n-j}(y, B) z^n \\ &= U_A^{(z)}(x, B) + \int_A \sum_{n=1}^{\infty} {}_AP^n(x, dy) z^n \sum_{m=1}^{\infty} P^m(y, B) z^m \\ &= U_A^{(z)}(x, B) + \int_A U_A^{(z)}(x, dy) U^{(z)}(y, B) \qquad (9) \end{aligned}$$

and

$$U^{(z)}(x, B) = U_A^{(z)}(x, B) + \int_A U^{(z)}(x, dy) U_A^{(z)}(y, B). \qquad (10)$$

We now consider the case when the chain $\mathbf{\Phi}$ has an atom. We will show that the chain is either recurrent or transient, in the case when $\mathbf{\Phi}$ is ψ-irreducible. We then use the splitting techniques to classify general chains.

Lemma 3 *If $\mathbf{\Phi}$ is ψ-irreducible and has an atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathcal{X})$, then $U(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \infty$ if and only if $L(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 1$.*

Proof:

Theorem 8 *If* $\boldsymbol{\Phi}$ *is* ψ*-irreducible and has an atom* $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathcal{X})$ *and*

(i) if $\boldsymbol{\alpha}$ *is recurrent, then every set in* $\mathcal{B}^+(\mathcal{X})$ *is recurrent;*

(ii) if $\boldsymbol{\alpha}$ *is transient, then there is a countable covering of* $\mathcal{X}$ *by uniformly transient sets.*

Proof:

Definition 24 *If $A \in \mathcal{B}(\mathcal{X})$ can be covered by a countable number of uniformly transients sets, then we call A transient.*

Now we will consider chains $\mathbf{\Phi}$ that do not have an atom and use the split chain construction.

Definition 25 *The chain $\mathbf{\Phi}$ is called recurrent if it is ψ-irreducible and $U(x, A) \equiv \infty$ for every $x \in \mathcal{X}$ and every $A \in \mathcal{B}^+(\mathcal{X})$. The chain $\mathbf{\Phi}$ is called transient if it is ψ-irreducible and $\mathcal{X}$ is transient.*

Next we will show that the split and original chains have mutually consistent recurrent/transient classifications.

Proposition 15 *If $\mathbf{\Phi}$ is ψ-irreducible and strongly aperiodic, then $\mathbf{\Phi}$ and $\check{\mathbf{\Phi}}$ are either both recurrent or both transient.*

Proof:

Lemma 4 *For $\epsilon \in (0, 1)$ we have*

$$\sum_{n=1}^{\infty} K_{\epsilon}^{n}(x, A) = \frac{1-\epsilon}{\epsilon} \sum_{n=0}^{\infty} P^{n}(x, A).$$

Proof:

Now we can prove

Proposition 16 *If* $\mathbf{\Phi}$ *is* ψ*-irreducible, then*

(i) $\mathbf{\Phi}$ *is transient if and only if each* K_ϵ*-chain is transient.*

(ii) $\mathbf{\Phi}$ *is recurrent if and only if each* K_ϵ*-chain is recurrent.*

Proof: