Lecture 4. Prob prop, rv

Monday, September 18, 2017 9:49 AM

Doodle

Dec 14th Thursday Noon
15th Friday

Algebras and σ - Algebras

σ (Singletons) - not useful $\mathcal{F}$
too "primitive"

countable or
cocountable

σ (Power set) - problems with
all subsets of Ω measurability
too rich not usefull
w/o proof

Continuity for Probability

$\forall\ x_n \xrightarrow[n\to\infty]{} x \Rightarrow \underbrace{f(x_n)}_{f_n} \xrightarrow[n\to\infty]{} \underbrace{f(x)}_{f}$

"for any"

f is continuous

if f is monotonic then if $\exists$ "there exists" x_n
$x_n \to x \quad f_n \to f$
is enough for continuity

Limits of sets

$\bigcap_{n=1}^{\infty} B_n = B$ Sets

$B_n \to B$

∞ B_n are decreasing $B_{n+1} \subset B_n$

B_n are decreasing $B_{n+1} \subset B_n$

$$\bigcup_{n=1}^{\infty} B_n = B$$

$B_n \subset B_{n+1}$ increasing

lim inf , lim sup

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \quad , \quad \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

Probability
1) $P(A) \geq 0$
2) $P(\Omega) = 1$
3) Countable additivity
$\{A_i\}$ disjoint
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Example:

Suppose I have sequence of disjoint events $\{A_i\}$, $i = 1, 2, \ldots$

$\limsup_{i \to \infty} A_i = \emptyset$ b/c $\omega \in$ to at most one A.

$= \{$infinitely many A_is are happening$\}$
that ω belongs to
if $\omega \in$

Th: Axiom 3) of Prob. can be replaced by Finite additivity and (+) continuity (3')

↳ $\{A_i\}$ disjoint

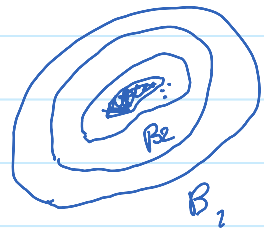
$$P\left(\bigcup_{n=1}^{m} A_n\right) = \sum_{n=1}^{m} P(A_n) \quad , \quad m < \infty$$

Continuity B_n – sequence of decreasing sets : $\bigcap_{n=1}^{\infty} B_n = B \in \mathcal{A}$

then $\mathbb{P}(B_n) \xrightarrow[n\to\infty]{} \mathbb{P}(B)$

<u>Proof.</u>

$3 \to 3'$



$C_k = B_k \cdot \overline{B}_{k+1}$

↓ Bagels

$$B_n = B \cup \left(\bigcup_{k=n}^{\infty} C_k \right)$$

3 (countable additivity) $\Rightarrow$

$$\mathbb{P}(B_n) = \mathbb{P}(B) + \sum_{k=n}^{\infty} \mathbb{P}(C_k)$$

This series converges $\Rightarrow \xrightarrow[n\to\infty]{} 0$

Series $\sum_{n=1}^{\infty} x_n$ converges

k- tail $\quad x_k + x_{k+1} + \dots$

$\lim_{k\to\infty}$ k-tail $= 0$

$$P(B_n) \xrightarrow[n\to\infty]{} P(B) \quad (3')$$

$3' \to 3$

↓ Finite additivity, Continuity $\overset{\text{want to show}}{\Rightarrow}$ Countable additivity

$$\bigcup_{n=1}^{\infty} \overset{\text{disjoint}}{A_n} = \bigcup_{n=1}^{m} A_n + \underbrace{\bigcup_{n=m}^{\infty} A_n}_{B_m \text{ - decreasing sets}}$$

$$\bigcap_{m=1}^{\infty} B_m = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \limsup_{n\to\infty} A_n = \emptyset$$

$$\bigcap_{m=1} B_m = \bigcap_{m=1} \bigcup_{n=m} A_n = \underbrace{\limsup_{n\to\infty} A_n}_{} = \emptyset$$

Take $\Omega(\cdot)$

By continuity of P

$$P(B_m) \xrightarrow[m\to\infty]{} P(B) = P(\emptyset) = 0$$

Ω

finite additivity

$$\underbrace{P\left(\bigcup_{n=1}^{\infty} A_k\right) = \sum_{k=1}^{m} P(A_k) +}_{} \qquad P(B_m) \quad \text{going to} \Rightarrow \lim_{m\to\infty}$$

$\downarrow m\to\infty$

0

$\xrightarrow[m\to\infty]{}$ countable additivity

$\square$

DF Probability space $(\Omega, \mathcal{F}, P)$

$\mathcal{F} \to$ σ-algebra

TH Carathéodory (measure extension)

$\exists$ unique measure Q on $\mathcal{F} = \sigma(\mathcal{A})$

such that

$$P \equiv Q \text{ on } \mathcal{A}$$

$\downarrow$ defined on $\mathcal{A}$

Example

$$(\mathbb{R}, \mathcal{B})$$

Can define $P((-\infty, x])$

Properties of P

1) $P(\emptyset) = 0 \Leftarrow \emptyset + \Omega = \Omega$, $P(\Omega) = 1$
2) $P(\bar{A}) = 1 - P(A) \Leftarrow A + \bar{A} = \Omega$
3) $A \subset B \Rightarrow P(A) \le P(B) \leftarrow$

$B = A + \bar{A}B$

$$P(B) = P(A) + \underbrace{P(A\bar{B})}_{\geq 0} \Big\} \Rightarrow$$

4) $P(A) \leq 1$

5) $P(A \cup B) = P(A) + P(B) - \underbrace{P(AB)}_{>0} \leq P(A) + P(B)$

6) $$P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{k=1}^{n} P(A_k) - \sum_{k<l} P(A_k A_l) +$$

$$+ \sum_{k<l<m} P(A_k A_l A_m) - \dots$$

$$\dots (-1)^{n-1} P(A_1 \times \dots \times A_n)$$

Random variable

$(\Omega, \mathcal{F}, P)$ probability space

Df r.v. X is a measurable function $X(\omega)$, $\omega \in \Omega$

$$\forall B \in \mathcal{B} \Rightarrow X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$$