(2)

See the handwritten solution on page 7.

(3)

Denote these parameters may be ordered from the smallest to the largest as follows $\theta_{(1)} \leq \theta_{(2)} \leq \theta_{(3)}$. The loss function may be expressed as

$$L((\theta_1, \theta_2, \theta_3), a_i) = 2\mathbb{1}(a_i = \theta_{(1)}) + \mathbb{1}(a_i = \theta_{(2)})$$

Without loss of generality, we may consider an order to the θ's: $\theta_1 \leq \theta_2 \leq \theta_3$. The risk function can be calculated as

$$\begin{aligned} R(\theta_1, \theta_2, \theta_3, \delta(x_i)) &= E\left[L((\theta_1, \theta_2, \theta_3), \delta(x_i))\right] \\ &= 2P\left(X_1 \geq \max\left\{X_2, X_3\right\}\right) \\ &\qquad + P\left(X_2 \geq \max\left\{X_1, X_3\right\}\right) \end{aligned}$$

We calculate components separately for $i = 1$:

$$\begin{aligned} P\left(X_1 \geq \max\left\{X_2, X_3\right\}\right) &= \int_0^{\theta_1} f_{X_1}(x) P(X_2 \leq x, X_3 \leq x)\, dx \\ &= \int_0^{\theta_1} \frac{1}{\theta_1} \frac{x^2}{\theta_2\theta_3}\, dx \\ &= \left.\frac{x^3}{3\theta_1\theta_2\theta_3}\right]_0^{\theta_1} = \frac{\theta_1^2}{3\theta_2\theta_3} \\ P\left(X_2 \ \geq \max\left\{X_1, X_3\right\}\right) &= \int_0^{\theta_1} f_{X_2}(x) P(X_1 \leq x, X_3 \leq x)\, dx + \int_{\theta_1}^{\theta_2} f_{X_2}(x) P(X_1 \leq \theta_1, X_3 \leq x)\, dx \\ &= \int_0^{\theta_1} \frac{1}{\theta_2} \frac{x^2}{\theta_1\theta_3}\, dx + \int_{\theta_1}^{\theta_2} \frac{1}{\theta_2} \frac{x}{\theta_3}\, dx \\ &= \left.\frac{x^3}{3\theta_1\theta_2\theta_3}\right]_0^{\theta_1} + \left.\frac{x^2}{2\theta_2\theta_3}\right]_{\theta_1}^{\theta_2} \\ &= \frac{\theta_1^2}{3\theta_2\theta_3} + \frac{\theta_2}{2\theta_3} - \frac{\theta_1^2}{2\theta_2\theta_3} \\ &= -\frac{\theta_1^2}{6\theta_2\theta_3} + \frac{\theta_2}{2\theta_3}. \end{aligned}$$

Then the risk function is

$$R(\theta_1, \theta_2, \theta_3, \delta(x_i)) = \frac{2\theta_1^2}{3\theta_2\theta_3} + \frac{\theta_1^2}{3\theta_2\theta_3} + \frac{\theta_2}{2\theta_3} - \frac{\theta_1^2}{2\theta_2\theta_3}$$

$$
\begin{aligned}
&= \frac{4\theta_1^2 + 2\theta_1^2 + 3\theta_2^2 - 3\theta_1^2}{6\theta_2\theta_3} \\
&= \frac{\theta_1^2 + \theta_2^2}{2\theta_2\theta_3}
\end{aligned}
$$

Because of symmetry among the three parameters, we can generalize the above result to the case of any permutation, say $\theta_i \leq \theta_j \leq \theta_k$, and the resulting risk is $\frac{\theta_i^2+\theta_j^2}{2\theta_j\theta_k}$.

(4)

The Bayes risk is

$$R_\Pi(\delta) = \int_\Theta R(\theta, \delta)\ d\Pi(\theta)$$

We have that $X|\theta \sim N(\theta, 1)$, $\theta \sim N(0, \tau^2)$. We can rewrite the loss function as $L(\theta, a) = \mathbb{1}(|\theta - a| > 1)$. We saw in class that the posterior distribution is

$$\theta|x \sim N\left(\frac{\tau^2 \bar{x}}{1+\tau^2}, \frac{\tau^2}{1+\tau^2}\right).$$

The posterior risk is

$$
\begin{aligned}
R_\Pi(a, x) &= \int_{-\infty}^{\infty} \mathbb{1}(|\theta - a| > 1) \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\theta - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} d\theta \\
&= \int_{-\infty}^{a-1} \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\theta - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} d\theta + \int_{a+1}^{\infty} \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\theta - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} d\theta
\end{aligned}
$$

We first find the Bayes solution.

$$\delta_\Pi = \arg\min_{a \in A} R_\Pi(a, x)$$

To find the minimum, we take derivatives with respect to a, set equal to 0 and solve for a:

$$
\begin{aligned}
0 &= \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(a - 1 - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} - \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(a + 1 - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} \\
&\Rightarrow \left(a - 1 - \frac{\tau^2\bar{x}}{1+\tau^2}\right)^2 = \left(a + 1 - \frac{\tau^2\bar{x}}{1+\tau^2}\right)^2 \\
&\Rightarrow a + 1 - \frac{\tau^2\bar{x}}{1+\tau^2} = -a + 1 + \frac{\tau^2\bar{x}}{1+\tau^2}
\end{aligned}
$$

$$\Rightarrow a = \frac{\tau^2 \bar{x}}{1+\tau^2}$$

Thus the Bayes solution is $\delta_\Pi = \frac{\tau^2\bar{x}}{1+\tau^2}$. First we find the posterior risk:

$$\begin{aligned}
R_\Pi(\delta_\Pi, x) &= \int_{-\infty}^{\frac{\tau^2\bar{x}}{1+\tau^2}-1} \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\theta - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} d\theta + \int_{\frac{\tau^2\bar{x}}{1+\tau^2}+1}^{\infty} \frac{\sqrt{1+\tau^2}}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\theta - \frac{\tau^2\bar{x}}{1+\tau^2})^2}{2\frac{\tau^2}{1+\tau^2}}\right\} d\theta \\
&= \int_{-\infty}^{-\sqrt{\frac{1+\tau^2}{\tau^2}}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx + \int_{\sqrt{\frac{1+\tau^2}{\tau^2}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \\
&= \Phi\left(-\sqrt{\frac{1+\tau^2}{\tau^2}}\right) + 1 - \Phi\left(\sqrt{\frac{1+\tau^2}{\tau^2}}\right) \\
&= 2\Phi\left(-\sqrt{\frac{1+\tau^2}{\tau^2}}\right).
\end{aligned}$$

Now we find its Bayes risk. Because the risk function is a constant in x, we have

$$\begin{aligned}
R_\Pi(\delta_\Pi) &= \int_{-\infty}^{\infty} R_\Pi(\delta_\Pi, x)\, dP(x) \\
&= R_\Pi(\delta_\Pi, x) \\
&= 2\Phi\left(-\sqrt{\frac{1+\tau^2}{\tau^2}}\right).
\end{aligned}$$

(5)

(a)

If there exists a uniformly superior estimator of μ, denoted by $\delta(x)$, then its risk function $R(\theta, \delta) = E_\theta(\delta(X) - \mu)^2$ should satisfy

$$R(\theta, \delta) < R(\theta, a), \quad \text{for all actions } a \in D \text{ over } \theta. \tag{1}$$

Note that

$$R(\theta, \delta) \;=\; Var_\theta(\delta(X)) + (E_\theta\delta(X) - \mu)^2.$$

If the second term $(E_\theta\delta(X) - \mu)^2 = 0$, then $\delta(x)$ will be an unbiased estimator of μ. In this case, since $\bar{x}$ is the UMVUE, we have

$$0 < \frac{\sigma^2}{n} = Var_\theta(\bar{X}) \le Var_\theta(\delta(X)) = R(\theta, \delta), \text{ for all } \theta \in \Theta.$$

On the other hand, for an estimator $\tilde{\delta}(x) \equiv 1$, whose risk function is $R(\theta, \tilde{\delta}) = (1-\mu)^2$, which will be zero at $\mu = 1$. In other words, there exists one action above at one parameter value, the statement in (1) is invalid.
If the second term $(E_\theta \delta(X) - \mu)^2 > 0$ for all μ, then risk function $R(\theta, \delta) > 0$ for all μ. Using the same constant estimator $\tilde{\delta}(x) \equiv 1$, we again see that the statement in (1) is invalid at one value $\mu = 1$.
In conclusion, there does not exist a uniformly superior estimator of μ.

(b)

Let $\delta(X) = c$ be an estimator. Then its risk is

$$\begin{aligned} E\left[(\mu - \delta(X))^2\right] &= E\left((c-\mu)^2\right) \\ &= (c-\mu)^2, \end{aligned}$$

i.e. the risk equals to zero when $\mu = c$. This is the smallest risk value that can not be beat by any other estimators. Thus, a constant estimator is admissible.

(c)

Let $\delta(X) = c\bar{X}$ with $|c| > 1$. We calculate the risk of $\delta(X)$:

$$\begin{aligned} R(\theta, \delta) = E\left[(\mu - \delta(X))^2\right] &= E\left[\left(\mu - c\bar{X}\right)^2\right] \\ &= E\left[\left(\mu - c\mu + c\mu - c\bar{X}\right)^2\right] \\ &= c^2 Var(\bar{X}) + (1-c)^2\mu^2 \\ &\geq c^2 Var(\bar{X}) \\ &> Var(\bar{X}) \;\text{ if } |c| > 1 = R(\theta, \bar{x}). \end{aligned}$$

Since $\bar{x}$ dominates $\hat{\mu}(x)$, so $\hat{\mu}(x)$ is not admissible.