Proof: Use the first-entrance decomposition. In the limit we have

$$\begin{aligned} U(x,A) &= U_A(x,A) + \int_A U_A(x,dy)U(y,A) \\ &\leq L(x,A) + \int_A U_A(x,dy) \sup_{y\in A} U(y,A) \\ &= L(x,A) + L(x,A) \sup_{y\in A} U(y,A) \\ &\leq 1 + M. \end{aligned}$$

Proposition 18 *If the chain $\mathbf{\Phi}$ is positive, then it is recurrent.*

Proof: Suppose that the chain is transient. Then there is a countable cover of $\mathcal{X}$ with uniformly transient sets A_j. Hence, there exists an M_j such that $U(x, A_j) \leq M_j$ by the previous lemma. Now for any j, k we have

$$\pi(A_j) = k^{-1} \sum_{n=1}^{k} \int_{\mathcal{X}} \pi(dx) P^n(x, A_j) \leq k^{-1} M_j$$

As $k \uparrow \infty$ we have $\pi(A_j) = 0$. Therefore π cannot be a probability measure and $\mathbf{\Phi}$ is null. $\Rightarrow\Leftarrow$. □

Positive chains are often referred to as positive recurrent to reinforce the fact that they are recurrent.

Definition 31 (Positive Harris chains) *If $\mathbf{\Phi}$ is Harris recurrent and positive, then $\mathbf{\Phi}$ is called a positive Harris (recurrent) chain.*

Now we set out to show that an invariant probability measure exists and that it is unique, up to a multiplicative constant, for certain chains. We will begin by showing that chains that admit atoms are positive, and then extend to strongly aperiodic chains and then to recurrent chains.

Definition 32 (Subinvariant measures) *if μ is σ-finite and satisfies*

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dw) P(w, A), \quad A \in \mathcal{B}(\mathcal{X}) \tag{13}$$

then μ is called subinvariant.

Iterating we get

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dx) P^n(x, A). \tag{14}$$

Multiplying by $a(n)$, where a is a sampling distribution on $\mathbb{N}_+$, and then summing we get

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dx) K_a(x, A). \tag{15}$$

Equations (14) and (15) tell us, respectively, that if μ is a subinvariant measure for $\mathbf{\Phi}$ is it also a subinvariant measure for any m-skeleton and for any sampled chain.

Proposition 19 *If $\mathbf{\Phi}$ is ψ-irreducible and if μ is any measure satisfying (13) with $\mu(A) < \infty$ for some $A \in \mathcal{B}^+(\mathcal{X})$, then*

(i) μ is σ-finite and thus μ is a subinvariant measure;

(ii) $\psi \prec \mu$;

(iii) if C is ν_a-petite, then $\mu(C) < \infty$;

(iv) if $\mu(\mathcal{X}) < \infty$, then μ is invariant.

Proof:
(i) Suppose $A \in \mathcal{B}^+(\mathcal{X})$ and $\mu(A) < \infty$. Consider the sets

$$A^*(j) = \{y : K_{1/2}(y, A) > j^{-1}\}.$$

Then

$$\begin{aligned} \infty > \mu(A) &\geq \int_{\mathcal{X}} \mu(dx) K_{1/2}(x, A) \\ &\geq \int_{A^*(j)} \mu(dx) K_{1/2}(x, A) \\ &> j^{-1} \mu(A^*(j)). \end{aligned}$$

So, each $A^*(j)$ has μ-finite measure. Furthermore since, $\lim_{j\uparrow\infty} A^*(j) = \cup_j A^*(j) = \mathcal{X}$, μ is σ-finite.

(ii) Let $A \in \mathcal{B}^+(\mathcal{X})$, i.e. $\psi(A) > 0$. Since $\mathbf{\Phi}$ is ψ-irreducible, $K_{1/2}(x, A) > 0$ which implies

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dx) K_{1/2}(x, A) > 0.$$

Hence, $\mu(A) > 0$ whenever $\psi(A) > 0$, or $\psi \prec \mu$.

(iii) Suppose C is ν_a-petite. Then ν_a is a non-trivial measure and

$$K_a(x, B) \geq \nu_a(B)$$

for all $B \in \mathcal{B}(\mathcal{X})$ and $x \in \mathcal{C}$. Hence, there exists a set $A \in \mathcal{B}(\mathcal{X})$ with $\nu_a(A) > 0$ and, by assumption, $\mu(A) < \infty$. So, by (i),

$$\infty > \mu(A) \geq \int_{\mathcal{X}} \mu(dx) K_a(x, A) \geq \int_C \mu(dx) K_a(x, A) \geq \mu(C)\nu_a(A)$$

so that $\mu(C) < \infty$.

(iv) Suppose not. Suppose $\mu(\mathcal{X}) < \infty$ and μ is not invariant. Then there exists an A such that $\mu(A) > \int_{\mathcal{X}} \mu(dx) P(x, A)$.

$$\begin{aligned}
\mu(\mathcal{X}) = \mu(A) + \mu(A^c) &> \int_{\mathcal{X}} \mu(dx) P(x, A) + \int_{\mathcal{X}} \mu(dx) P(x, A^c) \\
&= \int_{\mathcal{X}} \mu(dx) P(x, \mathcal{X}) \\
&= \mu(\mathcal{X}).
\end{aligned}$$

This implies that $\mu(\mathcal{X}) = \infty$. $\Rightarrow\Leftarrow$. Hence, μ must be invariant. □

1.6.2 The existence of an invariant measure—chains with atoms

We are interested in Harris recurrent ψ-irreducible chains for MCMC theory. However, to show the existence of an invariant measure for recurrent ψ-irreducible chain, we will first proof the existence for chains with atoms (not necessarily recurrent) and then use Nummelin's splitting technique to extend the results to recurrent chains.

Lemma 9 *Suppose Φ is a Markov chain. Let $A \in \mathcal{B}(\mathcal{X})$. If $L(x, A) = 1$ for all $x \in A$, then A is a recurrent set.*

Proof: Suppose $L(x, A) = 1$ for all $x \in A$. Use the last-exit decomposition to get

$$U^{(z)}(x, A) = U_A^{(z)}(x, A) + \int_A U^{(z)}(x, dy) U_A^{(z)}(y, A).$$

Now take the limit as $z \uparrow 1$. Then

$$U(x, A) = L(x, A) + L(x, A)U(x, A) = 1 + U(x, A).$$

Therefore $U(x, A) = \infty$ for all $x \in A$ and A is recurrent by definition. □

Theorem 15 *Let $\mathbf{\Phi}$ be ψ-irreducible and suppose $\mathcal{X}$ contains an accessible atom $\boldsymbol{\alpha}$.*

(i) There exists a subinvariant measure $\mu^o_{\boldsymbol{\alpha}}$ for $\mathbf{\Phi}$ given by

$$\mu^o_{\boldsymbol{\alpha}}(A) = U_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}, A) = \sum_{n=1}^{\infty} {}_{\boldsymbol{\alpha}}P^n(\boldsymbol{\alpha}, A), \quad \forall\, A \in \mathcal{B}(\mathcal{X}),$$

where $\mu^o_{\boldsymbol{\alpha}}$ is invariant if and only if $\mathbf{\Phi}$ is recurrent.

(ii) The measure $\mu^o_{\boldsymbol{\alpha}}$ is minimal in the sense that if μ is subinvariant with $\mu(\boldsymbol{\alpha}) = 1$ then

$$\mu(A) \geq \mu^o_{\boldsymbol{\alpha}}(A), \quad \forall\, A \in \mathcal{B}(\mathcal{X}).$$

When $\mathbf{\Phi}$ is recurrent, $\mu^o_{\boldsymbol{\alpha}}$ is the unique (sub)invariant measure with $\mu(\boldsymbol{\alpha}) = 1$.

(iii) The subinvariant measure $\mu^o_{\boldsymbol{\alpha}}$ is a finite measure if and only if

$$\mathbb{E}_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}}) < \infty,$$

in which case $\mu^o_{\boldsymbol{\alpha}}$ is invariant.

Proof: (i) Let $A \in \mathcal{B}(\mathcal{X})$. Then

$$\begin{aligned}
\int_{\mathcal{X}} \mu^o_{\boldsymbol{\alpha}}(dy)P(y, A) &= \int_{\boldsymbol{\alpha}} \mu^o_{\boldsymbol{\alpha}}(dy)P(y, A) + \int_{\boldsymbol{\alpha}^c} \mu^o_{\boldsymbol{\alpha}}(dy)P(y, A) \\
&= \mu^o_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})P(\boldsymbol{\alpha}, A) + \int_{\boldsymbol{\alpha}^c} \sum_{n=1}^{\infty} {}_{\boldsymbol{\alpha}}P^n(\boldsymbol{\alpha}, dy)P(y, A) \\
&\leq P(\boldsymbol{\alpha}, A) + \sum_{n=1}^{\infty} \int_{\boldsymbol{\alpha}^c} {}_{\boldsymbol{\alpha}}P^n(\boldsymbol{\alpha}, dy)P(y, A) \\
&= {}_{\boldsymbol{\alpha}}P(\boldsymbol{\alpha}, A) + \sum_{n=2}^{\infty} {}_{\boldsymbol{\alpha}}P^n(\boldsymbol{\alpha}, A) \\
&= \mu^o_{\boldsymbol{\alpha}}(A).
\end{aligned}$$

Hence, $\mu^o_{\boldsymbol{\alpha}}$ is a subinvariant measure.

Now, $\mu^o_{\boldsymbol{\alpha}}$ is invariant if and only if $\mu^o_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = 1$. But by definition, $\mu^o_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = U_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = L(\boldsymbol{\alpha}, \boldsymbol{\alpha})$. But if $L(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 1$, then $\mathbf{\Phi}$ is recurrent by Lemma 9.

(ii) Let μ be any subinvariant measure with $\mu(\boldsymbol{\alpha}) = 1$. We have

$$\begin{aligned}
\mu(A) &\geq \int_{\mathcal{X}} \mu(dy)P(y, A) \\
&\geq \int_{\boldsymbol{\alpha}} \mu(dy)P(y, A) \\
&= \mu(\boldsymbol{\alpha})P(\boldsymbol{\alpha}, A) = P(\boldsymbol{\alpha}, A).
\end{aligned}$$

Now assume that $\mu(A) \geq \sum_{m=1}^{n} {}_{\boldsymbol{\alpha}}P^m(\boldsymbol{\alpha}, A)$ for all A. Then by subinvariance

$$\begin{aligned}
\mu(A) &\geq \mu(\boldsymbol{\alpha})P(\boldsymbol{\alpha}, A) + \int_{\boldsymbol{\alpha}^c} \mu(dx)P(x, A) \\
&\geq P(\boldsymbol{\alpha}, A) + \int_{\boldsymbol{\alpha}^c} \left(\sum_{m=1}^{n} {}_{\boldsymbol{\alpha}}P^m(\boldsymbol{\alpha}, A) \right) P(x, A) \\
&= \sum_{m=1}^{n+1} {}_{\boldsymbol{\alpha}}P^m(\boldsymbol{\alpha}, A).
\end{aligned}$$

Taking the limit as $n \uparrow \infty$ shows that $\mu(A) \geq \mu^o_{\boldsymbol{\alpha}}(A), \quad \forall\, A \in \mathcal{B}(\mathcal{X})$.

Next, suppose $\boldsymbol{\Phi}$ is recurrent so that $\mu^o_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = 1$. If μ and $\mu^o_{\boldsymbol{\alpha}}$ differ, then $\mu(A) > \mu^o_{\boldsymbol{\alpha}}(A)$ for some $A \in \mathcal{B}(\mathcal{X})$. By ψ-irreducibility there exists an n such that $P^n(x, \boldsymbol{\alpha}) > 0$ for all $x \in \mathcal{X}$, since $\psi(\boldsymbol{\alpha}) > 0$ ($\boldsymbol{\alpha}$ is an accessible atom). Then

$$\begin{aligned}
1 = \mu(\boldsymbol{\alpha}) &\geq \int_{\mathcal{X}} \mu(dx)P^n(x, \boldsymbol{\alpha}) \\
&= \int_A \mu(dx)P^n(x, \boldsymbol{\alpha}) + \int_{A^c} \mu(dx)P^n(x, \boldsymbol{\alpha}) \\
&> \int_A \mu^o_{\boldsymbol{\alpha}}(dx)P^n(x, \boldsymbol{\alpha}) + \int_{A^c} \mu^o_{\boldsymbol{\alpha}}(dx)P^n(x, \boldsymbol{\alpha}) \\
&= \int_{\mathcal{X}} \mu^o_{\boldsymbol{\alpha}}(dx)P^n(x, \boldsymbol{\alpha}) = \mu^o_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = 1.
\end{aligned}$$

This leads to a contradiction so that $\mu(A) = \mu^o_{\boldsymbol{\alpha}}(A)$. Therefore, $\mu = \mu^o_{\boldsymbol{\alpha}}$ and $\mu^o_{\boldsymbol{\alpha}}$ is the unique (sub)invariant measure.

(*iii*) If $\mu^o_{\boldsymbol{\alpha}}$ is finite, then it is invariant (by Proposition 19(*iv*) on page 45). Also by the construction of $\mu^o_{\boldsymbol{\alpha}}$ we have

$$\mu^o_{\boldsymbol{\alpha}}(\mathcal{X}) = \sum_{n=1}^{\infty} {}_{\boldsymbol{\alpha}}P^n(\boldsymbol{\alpha}, \mathcal{X}) = \sum_{n=1}^{\infty} P_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}} \geq n) = \mathbb{E}_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}}).$$

Therefore an invariant probability measure exists if and only if the mean return time to $\boldsymbol{\alpha}$ is finite. In the above the first equality holds by definition. The second equality holds because

$${}_{\boldsymbol{\alpha}}P^n(\boldsymbol{\alpha}, \mathcal{X}) := P_{\boldsymbol{\alpha}}(\Phi_n \in \mathcal{X}; \tau_{\boldsymbol{\alpha}} \geq n) = P_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}} \geq n).$$

The third equality holds because

$$\sum_{n=1}^{\infty} P_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}} \geq n) = \sum_{n=1}^{\infty} \mathbb{E}_{\boldsymbol{\alpha}}[\mathbb{I}(\tau_\alpha \geq n)] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\sum_{n=1}^{\infty} \mathbb{I}(\tau_\alpha \geq n)\right] = \mathbb{E}_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}}).$$