Solution to Assignment 1

1.8: We have

$$\liminf A_n = \bigcup_n \bigcap_{k\ge n} A_k = \bigcup_n BC = BC$$

and

$$\limsup A_n = \bigcap_n \bigcup_{k\ge n} A_k = \bigcap_n (B\cup C) = B\cup C.$$

1.18: Since $\Omega \in \mathcal{F}$, if $A\in\mathcal{F}$ then $A^c = \Omega\setminus A \in \mathcal{F}$. Thus $\mathcal{F}$ is closed under complements. If A and B lie in $\mathcal{F}$, the by deMorgan's law

$$A\cup B = \big(A^cB^c\big)^c = \big(A^c\setminus B\big)^c,$$

which lies in $\mathcal{F}$ as $\mathcal{F}$ is closed under complements and differences. Thus $\mathcal{F}$ is closed under unions.

1.24: Clearly $\Omega \in \mathcal{A}$, and $\mathcal{A}$ is closed under complements by the symmetry of the definition. Suppose A and B lie in $\mathcal{A}$. If $\#A<\infty$ and $\#B<\infty$, then $\$(A\cup B) < \#A + \#B < \infty$ and so $A\cup B\in\mathcal{A}$. If $\#A^c<\infty$, then $\#(A\cup B)^c = \#(A^cB^c)\le \#A^c<\infty$ and so $A\cup B\in\mathcal{A}$. Similarly, if $\#B^c<\infty$ then $A\cup B\in\mathcal{A}$. As one of these cases must hold, $\mathcal{A}$ is closed under finite unions and must be a field. But $\mathcal{A}$ is not closed under countable unions for if $A_n=\{2n\}\in\mathcal{A}$, $n\ge 1$, then

$$\bigcup_n A_n = \{\text{even natural numbers}\}\notin\mathcal{A}.$$

So $\mathcal{A}$ is not a σ-field.

1.34: Let $\mathcal{C}$ be the class of all sets of the given form. Since σ-fields are closed under unions, intersections, and complements, if $B, B'\in\mathcal{B}$ then

$$AB\cup A^cB' \in \sigma\big(\mathcal{B}\cup\{A\}\big),$$

and hence $\mathcal{C}\subset\sigma\big(\mathcal{B}\cup\{A\}\big)$. So the desired result will follow if we show that $\mathcal{C}$ is a σ-field. Since $\Omega = A\Omega\cup A^c\Omega$, we have $\Omega\in\mathcal{C}$. Next, if $B,B'\in\mathcal{B}$, then

$$(AB\cup A^cB')^c = AB^c\cup A^cB'^c\in\mathcal{C}.$$

Thus $\mathcal{C}$ is closed under complements. Finally, if $AB_n \cup A^cB'_n$, $n \geq 1$, are sets in $\mathcal{C}$, with B_n, B'_n, $n \geq 1$, in $\mathcal{B}$, then

$$\bigcup_n \bigl(AB_n \cup A^cB'_n\bigr) = \bigl(A \cup_n B_n\bigr) \cup \bigl(A^c \cup_n B'_n\bigr),$$

which lies in $\mathcal{C}$ as $\cup_n B_n \in \mathcal{B}$ and $\cup_n B'_n \in \mathcal{B}$. Thus $\mathcal{C}$ is a σ-field.

2.4: By monotonicity, the value P_1 assigns to A must satisfy

$$P_1(A) \leq P_1(B) = P(B), \quad \forall B \in \mathcal{B} \text{ with } B \supset A.$$

So the largest possible value for $P_1(A)$ will be

$$\pi^* = \inf_{B \in \mathcal{B}, B \supset A} P(B),$$

the outer measure of A. We will seek an extension with $P_1(A) = \pi^*$. Let A_n, $n \geq 1$, be a sequence of sets in $\mathcal{B}$ with $A_n \supset A$ and $P(A_n) \to \pi^*$, and define

$$\tilde{A} = \bigcap_n A_n.$$

Since $\tilde{A} \supset A$ and $\tilde{A} \in \mathcal{B}$, we have $P(\tilde{A}) \geq \pi^*$. By continuity, since $\cap_{k=1}^n A_k \uparrow \tilde{A}$,

$$P(A_n) \leq P\left(\bigcap_{k=1}^n A_k\right) \to P(\tilde{A}),$$

and so

$$P(\tilde{A}) = \liminf_{n\to\infty} P\left(\bigcap_{k=1}^n A_k\right) \leq \liminf_{n\to\infty} P(A_n) = \pi^*.$$

Thus $P(\tilde{A}) = \pi^*$. So it seems we can make $P_1(A)$ as large as possible by making A and $\tilde{A}$ equivalent. Specifically, if $B, B' \in \mathcal{B}$, the extension will be given by

$$P_1(AB \cup A^cB') = P(\tilde{A}B) + P(\tilde{A}^cB').$$

To show that P_1 is a probability measure, we first note that if A and $B \in \mathcal{B}$ are disjoint, then $P(\tilde{A}B) = 0$. To see this, note that since $A \subset \tilde{A}B^c$, we have $P(\tilde{A}B^c) \geq \pi^*$, and so

$$\pi^* = P(\tilde{A}) = P(\tilde{A}B) + P(\tilde{A}B^c) \geq P(\tilde{A}B) + \pi^*.$$

Using this, if AB and AB' are disjoint with $B, B' \in \mathcal{B}$, then A and BB' are disjoint, and by the addition law

$$\begin{aligned} P(\tilde{A}B \cup \tilde{A}B') &= P(\tilde{A}B) + P(\tilde{A}B') - P(\tilde{A}BB') \\ &= P(\tilde{A}B) + P(\tilde{A}B'). \end{aligned}$$

Iterating this, if AB_k, $1 \le k \le n$, are disjoint, then

$$P\left(\bigcup_{k=1}^{n} \tilde{A}B_k\right) = \sum_{k=1}^{n} P(\tilde{A}B_k).$$

Using this, we next show that P_1 is σ-additive. Let D_n, $n \ge 1$, be disjoint sets in $\mathcal{B}_1$, and write

$$D_n = AB_n \cup A^c B_n', \text{ with } B_n, B_n' \in \mathcal{B}.$$

Then

$$D = \bigcup_n D_n = (A \cup_n B_n) \cup (A^c \cup_n B_n'),$$

and so

$$P_1(D) = P(\cup_n \tilde{A}B_n) + P(\cup_n \tilde{A}^c B_n').$$

Since $A^c B_n'$, $n \ge 1$, are disjoint and $\tilde{A}^c B_n' \subset A^c B_n'$, the sets $\tilde{A}^c B_n'$, $n \ge 1$ are disjoint, and so

$$P(\cup_n \tilde{A}^c B_n') = \sum_n P(\tilde{A}^c B_n').$$

By continuity and the identity above,

$$P(\cup_n \tilde{A}B_n) = \lim_{n\to\infty} P(\cup_{k=1}^n \tilde{A}B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} P(\tilde{A}B_k) = \sum_n P(\tilde{A}B_n).$$

So

$$P_1(D) = \sum_n \big[P(\tilde{A}B_n) + P(\tilde{A}^c B_n')\big] = \sum_n P_1(D_n).$$

For the other axioms, from the definition $P_1(B) \ge 0$ for all $B \in \mathcal{B}_1$, and

$$P_1(\Omega) = P(\tilde{A}) + P(\tilde{A}^c) = 1.$$

Thus P_1 is a probability measure on $(\Omega, \mathcal{B}_1)$.

2.8: In the example given in the problem,

$$\begin{aligned} P_1\big(\{a,b\}\big) &= P_1\big(\{d,c\}\big) = P_1\big(\{a,c\}\big) = P_1\big(\{b,d\}\big) \\ &= P_2\big(\{a,b\}\big) = P_2\big(\{d,c\}\big) = P_2\big(\{a,c\}\big) = P_2\big(\{b,d\}\big) = 1/2, \end{aligned}$$

and so P_1 and P_2 agree on $\mathcal{C}$. But $\sigma(\mathcal{C}) = 2^{\{a,b,c,d\}}$, and P_1 and P_2 do not agree on $\sigma(\mathcal{C})$, because, for instance, $P_1\big(\{a\}\big) \neq P_2\big(\{a\}\big)$. Note that $\mathcal{C}$ is not a π-system as $\{a,b\}\{a,c\} = \{a\} \notin \mathcal{C}$.

2.11: Using Boole's and D'Morgan's laws,

$$P\big((\cap_n B_n)^c\big) = P(\cup_n B_n^c) \leq \sum_n P(B_n^2) = 0,$$

and so $P(\cap_n B_n) = 1$.

2.12: If $B \in \mathcal{C}$, then $\mathcal{C}_B = \{B\} \subset \mathcal{C}$ is a countable class with $B \in \sigma(\mathcal{C}_B)$, and so $\mathcal{C} \subset \mathcal{G}$. And $\Omega \in \mathcal{G}$ as $\Omega \in \mathcal{C}_B$ for any $\mathcal{C}_B$. If $B \in \mathcal{G}$ with $B \in \mathcal{C}_B$ for some countable $\mathcal{C}_B$, then $B^c \in \mathcal{G}$ because $B^c \in \mathcal{C}_B$. So $\mathcal{G}$ is closed under complements. Finally, let B_n, $n \geq 1$, be sets in $\mathcal{G}$. Then there exist countable classes $\mathcal{C}_{B_n} \subset C$, $n \geq 1$, with $B_n \in \sigma(\mathcal{C}_{B_n})$. But then $\mathcal{C}_B \stackrel{\text{def}}{=} \cup_n \mathcal{C}_{B_n} \subset \mathcal{C}$ is a countable class and $B \stackrel{\text{def}}{=} \cup_n B_n \in \sigma(\mathcal{C}_B)$, and so $B \in \mathcal{G}$. Thus $\mathcal{G}$ is a σ-field and must contain $\sigma(\mathcal{C})$. (In fact, $\mathcal{G} = \sigma(\mathcal{C})$ as all of the sets of $\mathcal{G}$ lie in $\sigma(\mathcal{C})$.) The desired result follows.