Lecture 22 M, Z est

Wednesday, November 29, 2017 10:09 AM

Lyapunov CLT

Lyapunov condition $\Rightarrow$ Lindeberg condition

M and Z estimators

M-estimation

$M_n(\theta)$

$\downarrow$

random functions

$M_n(\theta) \xrightarrow[p]{} M(\theta)$

Estimator is defined as

a maximizer or

near-maximizer (up to $o_p(1)$)

$\max_\theta M_n(\theta) \Rightarrow \hat{\theta}_n$

True model

θ^* true parameter value

Define θ^* as a solution to

some maximization problem

$\max_\theta M(\theta) \Rightarrow \theta^*$

Consistency : $\hat{\theta}_n \xrightarrow[p]{} \theta^*$ LLN +

Normality : $\sqrt{n}(\hat{\theta}_n - \theta^*) \rightsquigarrow \mathcal{N}(0, \Sigma)$ CLT

Z- estimation

$\Psi_n(\cdot)$ random function

based on the sample

and approximating Ψ

$\Psi(\theta)$

θ^* is a solution to

$\Psi(\theta) = 0$

$\Psi_n(\cdot) \xrightarrow[P]{} \psi(\cdot)$

$\Psi_n(\theta) = 0 \Rightarrow \hat{\theta}_n$

deterministic functions

$M_n(\theta) = \frac{1}{n}\sum_i m_\theta(X_i)$

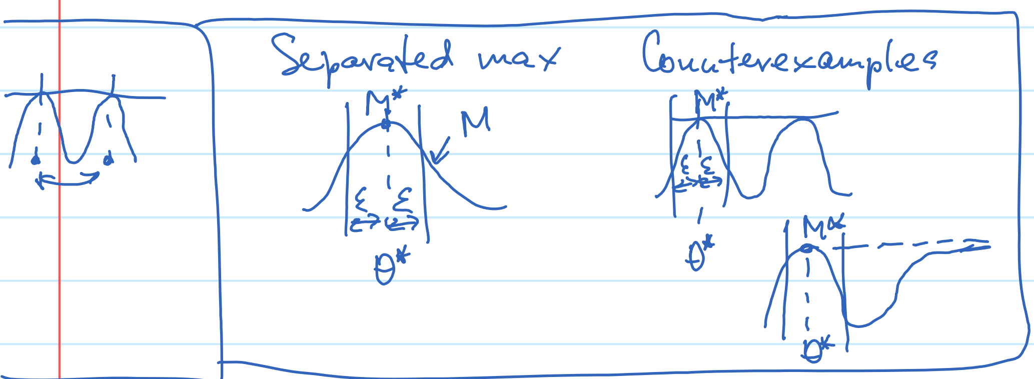
$\Psi_n(\theta) = \frac{1}{n}\sum_i \psi_\theta(X_i)$

Notation $\quad \mathbb{E}f(X) := Pf$

$\frac{1}{n}\sum_i f(X_i) := \mathbb{P}_n f$

$\downarrow$ empirical distribution

TH $\quad \left.\begin{array}{l}\text{Uniform convergence of } M_n \to M \\ \text{Separated true maximum}\end{array}\right\} \Rightarrow$ Consistency



] We have a sequence of estimators $\hat{\theta}_n$ such that

$M_n(\hat{\theta}_n) \geq M_n(\theta^*) - o_p(1)$ "approx maximizer"

$\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \xrightarrow[P]{} 0$ "Uniform convergence"

$\forall \varepsilon > 0, \quad \sup_{\theta : \|\theta - \theta^*\| \geq \varepsilon} M(\theta) < M(\theta^*)$ "separated true max"

Then

$\hat{\theta}_n \xrightarrow[P]{} \theta^*$

$$\hat{\theta}_n \xrightarrow[P]{} \theta^*$$

Proof: By the unif. conv. cond. $M_n(\theta^*) \xrightarrow[P]{} M(\theta^*)$

approx. maximizer cond $\Rightarrow$ $M(\theta^*) \le M_n(\hat{\theta}_n) + o_p(1)$

$$M(\theta^*) - M(\hat{\theta}_n) \le M_n(\hat{\theta}_n) - M(\hat{\theta}_n) + o_p(1) \le$$

$$\le \sup_\theta |M_n(\theta) - M(\theta)| + o_p(1) \xrightarrow[P]{} 0$$ by unif. conv.

$$M(\hat{\theta}_n) \xrightarrow[P]{} M(\theta^*)$$

b/c θ^* is a separated max

$\forall \varepsilon > 0 \ \exists \eta > 0$:

$M(\theta) < M(\theta^*) - \eta$, $\forall \theta : \|\theta - \theta^*\| \ge \varepsilon$

$\|\hat{\theta}_n - \theta^*\| \ge \varepsilon \Rightarrow M(\hat{\theta}_n) < M(\theta^*) - \eta$

$$P(\|\hat{\theta}_n - \theta^*\| \ge \varepsilon) \le P\{M(\theta^*) - M(\hat{\theta}_n) > \eta\} \xrightarrow[n \to \infty]{} 0$$

$\Rightarrow \hat{\theta}_n \xrightarrow[P]{} \theta^*$ $\square$

$M_n(\hat{\theta}_n)$

TH Z estimator consistency

Unif. conv. $\sup_\theta \|\Psi_n(\theta) - \Psi(\theta)\| \xrightarrow[P]{} 0$

Separ. of the root $\inf_{\theta : \|\theta - \theta^*\| \ge \varepsilon} \|\Psi(\theta)\| > 0 = \|\Psi(\theta^*)\|$

$\Rightarrow \forall \hat{\theta}_n : \Psi_n(\hat{\theta}_n) = o_p(1)$ approx root cond

we will have

$$\theta_n \xrightarrow[p]{} \theta$$

Proof:

Take $M_n = -\|\Psi_n\|$

$M = -\|\Psi\|$

reverse triangle inequality

$$\sup|M_n - M| = \sup\big|\,\|\Psi\| - \|\Psi_n\|\,\big| \le$$

$$\le \sup\|\Psi - \Psi_n\| \xrightarrow[p]{} 0$$

unif conv. cond

$$\sup_{\theta:\|\theta-\theta^*\|\geq\varepsilon} M(\theta) = \sup(-\|\Psi\|) = -\inf(\|\Psi\|) < 0 = M(\theta^*)$$

By the previous Th $\Rightarrow$ $\hat{\theta}_n \xrightarrow[p]{} \theta^*$ $\square$

Note: if M has a unique sup and M is continuous $\Big\}$ Separation cond is satisfied

Functions M_n, Ψ_n that satisfy ULLN are studied in Epirical processes

TH Consistency w/o uniform convergence

] $\theta \in \mathbb{R}$

Ψ_n random functions Ψ fixed function:

We have pointwise convergence

$$\Psi_n(\theta) \xrightarrow[p]{} \Psi(\theta), \quad \forall\theta$$

(a) Let Ψ_n be a continuous function with unique zero $\hat{\theta}_n$

or

(b) Ψ_n is a non-decreasing function with $\hat{\theta}_n$ being its asymptotic zero $\Psi_n(\hat{\theta}_n) = o_p(1)$

Let θ^* be such that $0 \in [\psi(\theta^* \pm \varepsilon)]$, $\forall\ \varepsilon > 0$

interval $[\psi(\ldots - \varepsilon), \psi(\ldots + \varepsilon)]$

Proof: