Biostatistics 682: Applied Bayesian Inference
Lecture 12: Bayesian Linear Regression

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Bayesian linear regression

- Linear regression is the most common statistical model.
- The multiple linear regression model is

$$Y_i \sim \mathrm{N}\left(\beta_0 + \sum_{j=1}^{p} X_{i,j}\beta_j, \sigma^2\right),$$

for $i = 1, \ldots, n$. Y_i are independently across the n observations.

- Bayesian and classical linear regression are similar if $n >> p$ and the priors are uninformative

Review of least squares

- The least squares estimate of $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^{\mathrm{T}}$ is

$$\hat{\beta}_{\mathrm{OLS}} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} (Y_i - \mu_i)^2,$$

  where $\mu_i = \beta_0 + X_{i,1}\beta_1 +, \ldots, +X_{i,p}\beta_p$.
- $\hat{\beta}_{\mathrm{OLS}}$ is unbiased even if the errors are non-Gaussian.
- If the errors are Gaussian then the likelihood is proportional to

$$\prod_{i=1}^{n} \exp\left\{-\frac{(Y_i - \mu_i)^2}{2\sigma^2}\right\} = \exp\left\{-\frac{\sum_{i=1}^{n}(Y_i - \mu_i)^2}{2\sigma^2}\right\}.$$

- Therefore, if the errors are Gaussian $\hat{\beta}_{\mathrm{OLS}}$ is also the MLE.

Review of least squares

- Linear regression is often simpler to describe using linear algebra notation.
- Let $\mathbf{Y} = (Y_1, \ldots, Y_n)^{\mathrm{T}}$ be the response vector and $\mathbf{X}$ be the $n \times (p+1)$ matrix of covariates.
- Then the mean of $\mathbf{Y}$ is $\mathbf{X}\beta$ and the least squares solution is

$$\boldsymbol{\beta}_{\mathrm{OLS}} = \arg\min_{\boldsymbol{\beta}} (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^{\mathrm{T}}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{Y}.$$

- If the errors are Gaussian then the sampling distribution is

$$\hat{\boldsymbol{\beta}}_{\mathrm{OLS}} \sim \mathrm{N}\left[\boldsymbol{\beta}, \sigma^2(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\right].$$

- If the variance σ^2 is estimated using the mean squared residual error then the sampling distribution is multivariate t.

- The likelihood remains

$$Y_i \mid \boldsymbol{\beta}, \sigma^2 \sim \mathrm{N}\left(\beta_0 + X_{i,1}\beta_1 + \ldots + X_{i,p}\beta_p, \sigma^2\right)$$

  independent for $i = 1, \ldots, n$ observations.
- A Bayesian analysis also requires priors for $\boldsymbol{\beta}$ and σ.
- We will focus on prior specification since this piece is uniquely Bayeisan.

Priors

- For the purpose of setting priors, it is helpful to standardize both the response and each covariate to have mean zero and variance one.
- Many priors for β have been considered:
  - Improper priors
  - Gaussian priors
  - Double exponential priors
  - Many, many more ...

Improper priors

- The Jeffrey's prior is flat $\pi(\boldsymbol{\beta}) \propto 1$.
- This is improper, but the posterior is proper under the same conditions required by least squares.
- If σ is known then

$$\boldsymbol{\beta} \mid \mathbf{Y} \sim \mathrm{N}\left[\hat{\boldsymbol{\beta}}_{\mathrm{OLS}}, \sigma^2(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\right].$$

- How is this result different from the least squares?

Improper priors

- We rarely know σ^2 in practice.
- The Jeffreys prior for $(\boldsymbol{\beta}, \sigma^2)$ is

$$\pi(\boldsymbol{\beta}, \sigma^2) \propto 1/\sigma^2,$$

  which is the limit case of an inverse gamma distribution with shape and rate parameters approaching zero.
- Then the posterior of $\boldsymbol{\beta}$ follows a multivariate t centered on $\hat{\boldsymbol{\beta}}_{\mathrm{OLS}}$.

Multivariate normal prior

- Another common prior for is Zellner's g-prior

$$\boldsymbol{\beta} \mid \sigma^2 \sim \mathrm{N}\left[0, \frac{\sigma^2}{g}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\right].$$

- This prior is proper assuming $\mathbf{X}$ is full rank.
- Then

$$\boldsymbol{\beta} \mid \sigma^2, \mathbf{Y}, \mathbf{X} \sim \mathrm{N}\left[\frac{1}{1+g}\hat{\boldsymbol{\beta}}_{\mathrm{OLS}}, \frac{1}{1+g}\sigma^2(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\right].$$

- This shrinks the least squares estimate towards zero.
- g controls the amount of shrinkage.
- $g = 1/n$ is common, and called the unit information prior.

Univariate Gaussian priors

- If there are many covariates or the covariates are collinear, then $\hat{\boldsymbol{\beta}}_{\text{OLS}}$ is unstable.
- Independent priors can counteract collinearity

$$\beta_j \sim \mathrm{N}(0, \sigma^2/g)$$

  independent over j.
- The posterior mode is

$$\arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} (Y_i - \mu_i)^2 + g \sum_{j=1}^{p} \beta_j^2.$$

- In classical statistics, this is known as the ridge regression solution and is used to stabilize the least squares solution.

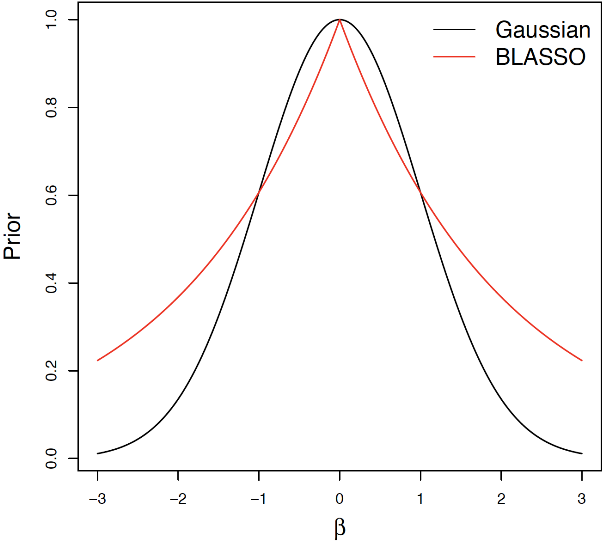
BLASSO

- An increasingly-popular prior is the double exponential or Bayesian LASSO prior
- The prior is $\beta_j \sim \mathrm{DE}(\tau^2)$ which has the probability density function

$$\pi(\beta_j) \propto \exp\left(-\frac{|\beta|}{\tau^2}\right).$$

- The square in the Gaussian prior is replaced with an absolute value
- The shape of the PDF is thus more peaked at zero
- The BLASSO prior favors settings where there are many β_j near zero and a few large β_j.
- That is, p is large but most of the covariates are noise.

BLASSO



BLASSO

- The posterior model is

$$\arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} (Y_i - \mu_i)^2 + \tau^{-2} \sum_{j=1}^{p} |\beta_j|.$$

- In classical statistics, this is known as the LASSO solution
- It is popular because it adds stability by shrinking estimates towards zero, and also sets some coefficient to zero
- Covariates with coefficients set to zero and can be excluded from the model.
- LASSO performs variable selection and estimation simultaneously.

Spike and Slab Priors

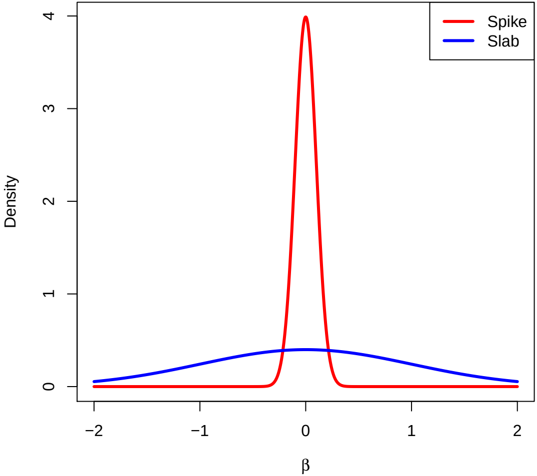
- Mixture Prior

$$\beta_j \mid \gamma_j \sim (1-\gamma_j)\mathrm{N}(0, c_0^2) + \gamma_j \mathrm{N}(0, c_1^2).$$

$$\gamma_j \sim \mathrm{Bernoulli}(\pi).$$

- The constant c_0^2 is small, so that if $\gamma_j = 0$, "β_j could be safely estimated by 0".
- The constant c_1^2 is large, so that if $\gamma_j = 1$, "a non-zero estimate of β_j should probably be included in the final model".
- It works well for computing the marginal inclusion probability of each covariate and for model averaging
- This model is computationally convenient and extremely flexible

Spike and Slab Priors



Posterior computation for Bayesian linear model

- With flat or Gaussian (with fixed prior variance) priors the posterior is available in closed-form and Monte Carlo sampling is not needed
- With Gaussian priors all full conditionals are Gaussian or inverse gamma, and so Gibbs sampling is simple and fast
- With the BLASSO prior the full conditionals are more complicated
    - There is a trick to make all full conditional conjugate so that Gibbs sampling can be used
    - Metropolis sampling works fine too
- With the Spike Slab prior the full conditionals are available
- JAGS can handle all of them

Summarizing the results

- The standard summary is a table with marginal means and 95% intervals for each β_j.
- This becomes unwieldy for large p
- Picking a subset of covariates is a crucial step in a linear regression analysis
- Common methods include cross-validation and information criteria.

Predictions

- Say we have a new covariate vector $\mathbf{X}_{\text{new}}$ and we would like to predict the corresponding response Y_{new}.
- A plug-in approach would fix $\boldsymbol{\beta}$ and σ at their posterior means $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}$ to make predictions
$$Y_{\text{new}} \mid \boldsymbol{\beta}, \hat{\sigma}^2 \sim \mathrm{N}(\mathbf{X}_{\text{new}}\hat{\boldsymbol{\beta}}, \hat{\sigma}^2).$$
- However, this plug-in approach suppresses uncertainty about $\boldsymbol{\beta}$ and σ^2.
- Therefore these prediction intervals will be slightly too narrow leading to under coverage.

Posterior predictive distribution (PPD)

- We should really account for all uncertainty when making predictions, including our uncertainty about $\boldsymbol{\beta}$ and σ^2.
- We really want to PPD

$$
\begin{aligned}
\pi(Y_{\text{new}} \mid \mathbf{Y}) &= \int \pi(Y_{\text{new}}, \boldsymbol{\beta}, \sigma^2 \mid \mathbf{Y}) d\boldsymbol{\beta} d\sigma^2 \\
&= \int \pi(Y_{\text{new}} \mid \boldsymbol{\beta}, \sigma^2) \pi(\boldsymbol{\beta}, \sigma^2 \mid \mathbf{Y}) d\boldsymbol{\beta} d\sigma^2
\end{aligned}
$$

- Marginalizing over the model parameters accounts for their uncertainty

- MCMC naturally gives draws from Y_{new}'s PPD
  - For MCMC iteration t we have $\boldsymbol{\beta}^{(t)}$ and $\sigma^{2(t)}$.
  - For MCMC iteration t we sample

$$Y_{\text{new}}^{(t)} \sim \mathrm{N}\left(\mathbf{X}\boldsymbol{\beta}^{(t)}, \sigma^{2(t)}\right).$$

  - $Y_{\text{new}}^{(1)}, \ldots, Y_{\text{new}}^{(S)}$ are samples from the PPD.

- Thus, "Bayesian methods" naturally quantify uncertainty.
- JAGS can handle it.