Lecture 17 Char func

Monday, November 13, 2017 10:13 AM

Uniform integrability

X_n is u.i. $\Rightarrow$ $\sup_n \mathbb{E}|X_n| \le c < \infty$

$X_n \xrightarrow[p]{} X$, f cont., $f(X_n)$ u.i. $\Rightarrow$

$\mathbb{E}|f(X_n) - f(X)| \to 0$

$\mathbb{E} f(X_n) \to \mathbb{E} f(X)$

Weak convergence $\mathbb{E} f(X_n) \to \mathbb{E} f(X)$

$\forall$ cont. and bounded f

DCT theorem dom. convergence

$X_n \xrightarrow[p]{} X$, $|X_n| < Y \geq 0$ $\Rightarrow$

$\mathbb{E} Y < \infty$

$\Rightarrow$ $\mathbb{E} X < \infty$, $\mathbb{E} X_n \to \mathbb{E} X$

$\mathbb{E}|X_n - X| \xrightarrow[n\to\infty]{} 0$

Slutsky (Слуцкий), Cramer

] $X_n \rightsquigarrow X$ $\quad$ $Y_n \xrightarrow[p]{} 0$ $\Rightarrow$

$X_n + Y_n \rightsquigarrow X$

Proof:] F_n is CDF of X_n

F is CDF of X

$$P(X_n + Y_n \le x) = P(X_n \le x - Y_n) = \mathbb{E}\{P(X_n \le x - Y_n \mid Y_n)\} =$$

$$= \mathbb{E}\{F_n(x - Y_n)\}$$

$$\mathbb{E}\{F_n(x-|Y_n|)\} \le P(X_n+Y_n \le x) \le \mathbb{E}\{F_n(x+|Y_n|)\}$$

$$\mathbb{E}\{F_n(x \pm |Y_n|)\} = \mathbb{E}\{F_n(x \pm |Y_n|); |Y_n| \le \delta\} +$$

$$+ \underbrace{\mathbb{E}\{F_n(x \pm |Y_n|); |Y_n| > \delta\}}_{\le P(|Y_n|>\delta) \to 0 \text{ b/c } Y_n \xrightarrow{p} 0}$$

$$\mathbb{E}\{F_n(x-\delta); \dots\} \le \mathbb{E}\{F_n(x \pm |Y_n|); |Y_n| \le \delta\} \le \mathbb{E}\{F_n(x+\delta); \dots\}$$

$$\mathbb{E}\{F_n(x-\delta); \dots\} = F_n(x-\delta) \cdot P(|Y_n| \le \delta) \to F(x-\delta)\cdot 1$$

$$\mathbb{E}\{F_n(x+\delta); \dots\} = F_n(x+\delta) \cdot P(|Y_n| \le \delta) \to F(x+\delta)\cdot 1 \quad \text{b/c } X_n \rightsquigarrow X$$

δ is arbitrary $\Rightarrow$

$$\mathbb{E}(F_n(x \pm |Y_n|) \to F(x) \Rightarrow$$

$$\Rightarrow F_{X_n+Y_n} \to F \quad \square$$

Note: $X_n \rightsquigarrow X$, $Y_n \xrightarrow{p} 0 \Rightarrow X_n Y_n \xrightarrow{p} 0$
(Can be proved as an exercise)

TH $\quad X_n \rightsquigarrow c \Rightarrow X_n \xrightarrow{p} c$

Proof: $\quad P(|X_n - c| \le \varepsilon) =$

$$= P(c-\varepsilon \le X_n \le c+\varepsilon) =$$

$$= F_n(c+\varepsilon) - F_n(c-\varepsilon) \to \quad \text{b/c } X_n \rightsquigarrow c$$

$$= F_n(c+\varepsilon) - F_n(c-\varepsilon) \to$$

$$F(x) = I(x-c) = \begin{cases} 1, & x \geq c \\ 0, & x < c \end{cases}$$ is a CDF of $X : P(X=c)=1$

$$\to \underbrace{I(\varepsilon)}_{=1} - \underbrace{I(-\varepsilon)}_{=0} = 1 \quad \text{b/c } I \text{ right cont.}$$

$$\Rightarrow X_n \xrightarrow[p]{} c$$

$\square$

w/o proof.

TH Continuity

$X_n \rightsquigarrow X$, f - continuous $\Rightarrow f(X_n) \rightsquigarrow f(X)$

TH $X_n \rightsquigarrow X$, X_n u.i. $\Rightarrow \mathbb{E}X_n \to \mathbb{E}X$

TH Helly (from functional analysis)

The class of all CDFs including improper $Pr(X = \pm\infty) > 0$

is compact wrt weak convergence, i.e.

$\forall$ sequence F_n $\exists$ a converging subsequence

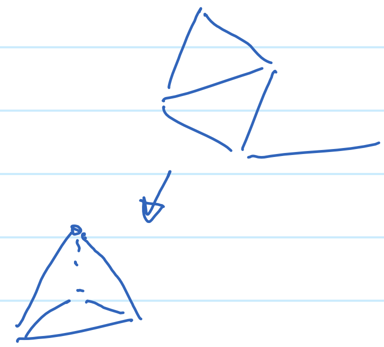
$F_{n_k}, F : \quad F_{n_k} \rightsquigarrow F$

are in the same class

If each/any subsequence $\rightsquigarrow F$, then $F_n \rightsquigarrow F$

Characteristic Functions

Complex Numbers



Laplace transform $\mathcal{L}(s) = \mathbb{E}\{e^{-s \cdot X}\}$

$X \geq 0$

$\sqrt{-1} = i$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Analytic function ⇒ all derivatives exist
⇒ can be expanded into a Power series

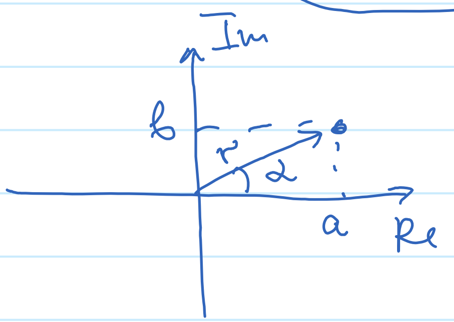
$x \to ix \Rightarrow$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

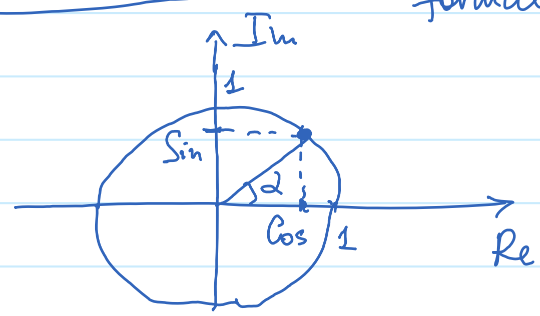
$i \times \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

times

$$e^{ix} = \cos x + i \cdot \sin(x)$$

Leonard Euler formula





$$z = a + b \cdot i = r \cdot (\cos\alpha + i \sin\alpha)$$

$$z = a + b \cdot i = r \cdot (\cos\alpha + i\sin\alpha)$$

$$r = \sqrt{a^2 + b^2} = \sqrt{z \cdot z^*}$$

$a = \mathrm{Re}(z)$ $\qquad$ $b = \mathrm{Im}(z)$

$$z^* := a - ib$$

$$(a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2$$

DF $\qquad$ Characteristic function

$$\varphi_X(t) = \mathbb{E}\{e^{itX}\}$$

$$|\varphi_X(t)| \le \mathbb{E}|e^{itX}| = 1$$

φ always exists!