Biostat 602 **Assignment 9** **Due: Tuesday, April 4**

Homework Problems

1. Suppose that $X_1, \cdots, X_n$ be *i.i.d.* observations from a Uniform$(0, \theta)$ distribution.

   (a) Show that $W_n = 2\overline{X}$ is a consistent estimator for θ.

   (b) Show that $V_n = X_{(n)}$ is a consistent estimator for θ.

   (c) Show that $\sqrt{n}(W_n - \theta)$ converges in distribution. Identify the limiting distribution along with its parameters.

   (d) Show that $n(\theta - V_n)$ converges in distribution. Identify the limiting distribution along with its parameters.

   (e) Compare the approximate variances of W_n and V_n. Which estimator becomes more efficient (has less variance) as n grows large?

2. Suppose that $X_1, \cdots, X_n$ be *i.i.d.* observations from a Bernoulli(p) distribution. Let $\hat{p} = \overline{X}$ be the MLE of p. We are interested in estimating $\tau(p) = p(1-p)$.

   (a) When $p \neq \frac{1}{2}$, show that the MLE $\tau(\hat{p}) = \hat{p}(1-\hat{p})$ is asymptotically efficient using Definition 10.1.11, instead of using Theorem 10.1.12. In other words, show directly that the asymptotic variance of $\tau(\hat{p})$ (using Delta Method) equals the Cramer Rao Lower Bound for variance of unbiased estimators of $\tau(p)$.

   (b) When $p = \frac{1}{2}$, use the second-order delta method in Theorem 5.5.26 to find the limiting distribution of $\tau(\hat{p})$.

3. The lognormal distribution is a popular choice for describing the distribution of the concentration X of an air pollutant. Suppose that the random variable

   $$Y = \ln(X) \sim N(\mu, \sigma^2)$$

   so X has a *lognormal*(μ, σ^2) distribution where $-\infty < \mu < \infty, 0 < \sigma^2 < \infty$ are unknown. Suppose that $X_1, \cdots, X_n$ constitute a random sample from the lognormal distribution for X.

   (a) Let $G_n = \prod_{i=1}^{n} X_i^{1/n}$ define the sample *Geometric Mean*. Show that G_n is a consistent estimator of e^{μ}.

(b) Show that for some suitable $\tau(\mu, \sigma^2),\ \nu(\mu, \sigma^2)$

$$\sqrt{n}\left(G_n - \tau(\mu, \sigma^2)\right) \xrightarrow{d} \mathcal{N}\left(0, \nu(\mu, \sigma^2)\right).$$

Identify both $\tau(\mu, \sigma^2),$ and $\nu(\mu, \sigma^2)$.

Suppose that it is of interest to consider the maximum likelihood estimator $\hat{\xi}$ of $\xi = E(X)$, the true mean concentration of the pollutant X in the original scale.

(c) Find the MLE $\hat{\xi}$ of the parameter ξ.

(d) Find explicit expressions for $\mathrm{E}(\hat{\xi})$ and $\mathrm{Var}(\hat{\xi})$. Use the following hints if needed

  (i) The sample mean and sample variance of Y follow known distributions.
  (ii) The moment-generating function of $Z \sim N(\mu, \sigma^2)$ is $\mathrm{E}[e^{tZ}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.
  (iii) The moment-generating function of $Z \sim \chi^2_k$ is $\mathrm{E}[e^{tZ}] = (1 - 2t)^{-\frac{k}{2}}$.

(e) Find the limiting expectation and variance $\lim_{n\to\infty} \mathrm{E}(\hat{\xi})$ and $\lim_{n\to\infty} \mathrm{Var}(\hat{\xi})$, and determine whether $\hat{\xi}$ is consistent or not. If needed, use the fact below as a hint.

$$\lim_{n\to\infty}\left(1 - \frac{c}{n}\right)^{n/c} = \frac{1}{e}, \qquad (c > 0)$$

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Practice Problems

(a) C&B Exercise 10.1

(b) C&B Exercise 10.3

(c) C&B Exercise 10.9

(d) C&B Exercise 10.11

(e) C&B Exercise 10.23