Singletons are always atoms. When $\mathcal{X}$ is countable and the chain is ψ-irreducible then every point is an accessible atom. However, on general state spaces, accessible atoms may not exist.

For the random walk on $\mathbb{R}$ defined by

$$\Phi_k = \Phi_{k-1} + W_k, \quad k > 0,$$

where $W_k \sim N(0,1)$, $\forall k > 0$ and Φ_0 is chosen from an arbitrary distribution, accessible atoms do not exist.

So, if this simple example does not contain accessible atoms, then what is the point in developing a theory for general state spaces that contain atoms? It seems that the class of general state spaces that contain accessible atoms is too small to be of practical importance. However, there exists "artificial atoms" for φ-irreducible chains. These atoms are found for "strongly aperiodic" chains by construction. We will construct a "split chain" $\check{\mathbf{\Phi}}$ evolving on a split state space $\check{\mathcal{X}} = \mathcal{X}_0 \cup \mathcal{X}_1$, where $\mathcal{X}_i$ is a copy of $\mathcal{X}$ in such a way that

(i) the chain $\mathbf{\Phi}$ is the marginal chain $\check{\mathbf{\Phi}}$, in the sense that $P(\Phi_k \in A) = P(\check{\Phi}_k \in A_0 \cup A_1)$ for appropriate initial distributions, and

(ii) the "bottom level" $\mathcal{X}_1$ is an accessible atom for $\check{\mathbf{\Phi}}$.

Before we get into this, though there are two consequences of the existence of atoms that will prove useful later on.

Proposition 7 *Suppose there is an atom $\boldsymbol{\alpha}$ in $\mathcal{B}(\mathcal{X})$ such that $\sum_{n=1}^{\infty} P^n(x, \boldsymbol{\alpha}) > 0$ for all $x \in \mathcal{X}$. Then $\boldsymbol{\alpha}$ is an accessible atom and $\mathbf{\Phi}$ is ν-irreducible with $\nu(A) = P(x, A)$ for all $x \in \boldsymbol{\alpha}$ and $A \in \mathcal{B}(\mathcal{X})$.*

Proof:

Proposition 8 *If $L(x, A) > 0$ for some $x \in \boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is an atom, then $\boldsymbol{\alpha} \rightsquigarrow A$.*

Proof:

In order to construct the split chain, we have to consider sets that satisfy the following

Definition 17 (Minorization Condition) *For some $\delta > 0$, some $C \in \mathcal{B}(\mathcal{X})$ and some probability measure ν such that $\nu(C) = 1$ and $\nu(C^c) = 0$*

$$P(x, A) \geq \delta \nu(A) \mathbb{I}_C(x), \quad \forall x \in \mathcal{X}, \forall A \in \mathcal{B}(\mathcal{X}).$$

This definition ensures that the chain has probabilities uniformly bounded below by multiples of ν for every $x \in C$.

Now we will construct the split chain. We first split the space $\mathcal{X}$ by writing $\check{\mathcal{X}} = \mathcal{X} \times \{0, 1\}$, where $\mathcal{X}_i := \mathcal{X} \times \{i\}$, $i = 0, 1$. These are thought of as copies of $\mathcal{X}$ equipped with copies $\mathcal{B}(\mathcal{X}_i)$ of the σ-algebra $\mathcal{B}(\mathcal{X})$.

Let $\mathcal{B}(\check{\mathcal{X}})$ denote the σ-algebra of subsets of $\check{\mathcal{X}}$ generated by $\mathcal{B}(\mathcal{X}_0)$ and $\mathcal{B}(\mathcal{X}_1)$: that is $\mathcal{B}(\check{\mathcal{X}})$ is the smallest σ-algebra containing sets of the form $A_0 := A \times \{0\}$, $A_1 := A \times \{1\}$, $A \in \mathcal{B}(\mathcal{X})$. We denote the elements of $\check{\mathcal{X}}$ by x_0 and x_1 where x_0 denotes members of the upper level $\mathcal{X}_0$ and x_1 denotes members of the lower level $\mathcal{X}_1$.

Suppose λ is some measure on $\mathcal{B}(\mathcal{X})$. We split λ into two measures, one on $\mathcal{X}_0$ and one on $\mathcal{X}_1$ by defining the measure λ^* on $\mathcal{B}(\check{\mathcal{X}})$:

$$\left.\begin{array}{lcl} \lambda^*(A_0) & = & \lambda(A \cap C)(1-\delta) + \lambda(A \cap C^c) \\ \lambda^*(A_1) & = & \lambda(A \cap C)\delta \end{array}\right\} \tag{2}$$

where δ and C are the constant and the set in the minorization condition. Note that λ is the marginal measure induced by λ^*: for any $A \in \mathcal{B}(\mathcal{X})$ we have

$$\lambda^*(A_0 \cup A_1) = \lambda^*(A_0) + \lambda^*(A_1) = \lambda(A).$$

When $A \subset C^c$, $\lambda^*(A_0) = \lambda(A)$—only subsets of C are effectively split by this construction.

Now we need to split the chain $\mathbf{\Phi}$ to form a chain $\check{\mathbf{\Phi}}$ which lives on $(\check{\mathcal{X}}, \mathcal{B}(\check{\mathcal{X}}))$. For $x_i \in \check{\mathcal{X}}$ and $\check{A} \in \mathcal{B}(\check{\mathcal{X}})$, define the split kernel $\check{P}(x_i, \check{A})$ by

$$\check{P}(x_0, \check{A}) = P^*(x, \check{A}), \qquad x_0 \in \mathcal{X}_0 \setminus C_0; \tag{3}$$

$$\check{P}(x_0, \check{A}) = (1-\delta)^{-1}(P^*(x, \check{A}) - \delta\nu^*(\check{A})), \qquad x_0 \in C_0; \tag{4}$$

$$\check{P}(x_1, \check{A}) = \nu^*(\check{A}), \qquad x_1 \in \mathcal{X}_1, \tag{5}$$

where ν, C and δ are the measure, set and constant in the minorization condition.

Outside C_0, $\check{\mathbf{\Phi}}$ behaves as $\mathbf{\Phi}$ (Equation 3) moving on the top half $\mathcal{X}_0$ of the split space. When $\mathbf{\Phi}$ enters C, we split $\check{\mathbf{\Phi}}$. $\check{\mathbf{\Phi}}$ moves to C_1 with probability δ and stays in C_0 with probability $1-\delta$. If $\check{\mathbf{\Phi}}$ stays in C_0, we have the modified law (4). The bottom half $\mathcal{X}_1$ is an atom (5), by construction. By Equation 2, $\check{P}(x_i, \mathcal{X}_1 \setminus C_1) = 0$ for all $x_i \in \check{\mathcal{X}}$. Hence, by the Chapman-Kolmogorov equations $\check{P}^n(x_i, \mathcal{X}_1 \setminus C_1) = 0$ for all $x_i \in \check{\mathcal{X}}$ and all $n > 0$. Therefore, the atom $C_1 \subset \mathcal{X}_1$ is the only part of the bottom level that is reached with positive probability. Since C_1 is an atom, we will use the notation $\check{\boldsymbol{\alpha}} := C_1$, to emphasize this fact.

Note that the minorization condition is only used in (4). However, without it, $\check{P}$ would not be a probability law on $\mathcal{B}(\check{\mathcal{X}})$.

Theorem 2 *The following hold for the split and original chain:*

(i) The chain $\mathbf{\Phi}$ is the marginal chain of $\check{\mathbf{\Phi}}$: i.e., for any initial distribution λ on $\mathcal{B}(\mathcal{X})$ and any $A \in \mathcal{B}(\mathcal{X})$,

$$\int_{\mathcal{X}} P^n(x, A)\lambda(dx) = \int_{\check{\mathcal{X}}} \check{P}^n(x_i, \check{A})\lambda^*(dx_i).$$

(ii) $\check{\mathbf{\Phi}}$ *is* φ^**-irreducible if and only if* $\mathbf{\Phi}$ *is* φ*-irreducible.*

(iii) If $\mathbf{\Phi}$ *is* φ*-irreducible and* $\varphi(C) > 0$*, then* $\check{\mathbf{\Phi}}$ *is* ν^**-irreducible and* $\check{\boldsymbol{\alpha}}$ *is an accessible atom for* $\check{\mathbf{\Phi}}$*.*

Proof:

1.5.2 Small sets

A small set is a set for which the minorization condition holds, at least for an m-skeleton chain. For the splitting construction of the previous section, then, the existence of small sets is of considerable importance as they ensure the splitting method is not vacuous. Small sets act in many ways like atoms, and thus are sometimes referred to as "pseudo-atoms". The central results regarding small sets is that for a ψ-irreducible chain every set $A \in \mathcal{B}^+(\mathcal{X})$ contains a small set: C is small with $C \subseteq A$ and $C \in \mathcal{B}^+(\mathcal{X})$. Therefore, every ψ-irreducible chain has an m-skeleton that can be split, and for which the atomic structure of the split chain can be exploited.