Solution to Assignment 3

3.16: Let $\mathcal{T}$ denote the topology of open sets in $\mathbb{R}$. Suppose $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$. By the definition of continuity, if $X$ is open, $X^{-1}(\mathcal{T}) \subset \mathcal{T}$. Since $\sigma(\mathcal{T}) = \mathcal{B}(\mathbb{R})$, by Proposition 3.1.2,

$$X^{-1}\big(\mathcal{B}(\mathbb{R})\big) = X^{-1}\big(\sigma(\mathcal{T})\big) \subset \sigma\big(X^{-1}(\mathcal{T})\big) \subset \sigma(\mathcal{T}) = \mathcal{B}(\mathbb{R}) \subset \mathcal{B}.$$

Thus $X$ is measurable. Conversely, if all continuous functions are measurable, then the identity function $Y(x) = x$, $x \in \mathbb{R}$ is measurable. But then

$$\mathcal{B}(\mathbb{R}) = Y^{-1}\big(\mathcal{B}(\mathbb{R})\big) \subset \mathcal{B}.$$

3.18: By finite additivity and monotonicity,

$$\begin{aligned} P(X \in A) &= P(X \in A, X = Y) + P(X \in A, X \neq Y) \\ &\leq P(Y \in A) + P(X \neq Y). \end{aligned}$$

Reversing $X$ and $Y$,

$$P(Y \in A) \leq P(X \in A) + P(X \neq Y).$$

Together these give

$$|P(X \in A) - P(Y \in A)| \leq P(X \neq Y),$$

proving the stated bound.

3.23: First note that if $X_1, \ldots, X_n$ are random variables, then $X = (X_1, \ldots, X_n)$ is a random vector. To see this, let $x$ be an arbitrary vector in $\mathbb{R}^n$. Then $[X_n \leq x_n] \in \mathcal{B}$, and so

$$[X \leq x] = \bigcap_n [X_n \leq x_n] \in \mathcal{B}$$

as $\mathcal{B}$ is closed under intersections. Thus if $\mathcal{C} \stackrel{\text{def}}{=} \big\{(-\infty, x] : x \in \mathbb{R}^n\big\}$, we have $X^{-1}(\mathcal{C}) \subset \mathcal{B}$, and $X$ is measurable by Proposition 3.2.1 because $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^k)$. By Proposition 3.2.2, since subtraction $(x, y \rightsquigarrow x - y)$ is a continuous function from $\mathbb{R}^2 \to \mathbb{R}$, the differences $X_i - X_j$ are random variables. So

$$[X_i = X_j] = (X_i - X_j)^{-1}\big(\{0\}\big) \in \mathcal{B}$$

and then

$$T \stackrel{\text{def}}{=} \bigcup_{i<j} [X_i = X_j] \in \mathcal{B}.$$

Similarly, $[X_i \geq X_n]$ and $[X_i \geq X_n]T^c$ are events in $\mathcal{B}$. It then follows that $171_T$ and $1_{[X_i \geq X_n]T^c}$, $1 \leq i \leq n$, are random variables. Together these random variables form a random vector, and their sum $R_n$ is then a random variable by Proposition 3.2.1 because $x \rightsquigarrow \sum_k x_k$ is a continuous function from $\mathbb{R}^{n+1}$ to $\mathbb{R}$.

4.4: **(b)** Let $Z = 1_{[0,1/2)}$. By Corollary 4.2.1, it is sufficient to show that

$$P[Y \leq y, Z \leq z] = P[Y \leq y]P[Z \leq z], \qquad \forall y, z \in \mathbb{R}. \tag{1}$$

Suppose $y \in (0, 1/4)$ and $z \in [0, 1)$. Then

$$[Z \leq z] = [0, 1/2), \qquad [Y \leq y, Z \leq z] = \left[0, \frac{1 - \sqrt{1-4y}}{2}\right],$$

and

$$[Y \leq y] = \left[0, \frac{1 - \sqrt{1-4y}}{2}\right] \bigcup \left[\frac{1 + \sqrt{1-4y}}{2}, 1\right].$$

In this case, (1) holds because

$$P[Y \leq y, Z \leq z] = \frac{1 - \sqrt{1-4y}}{2}, \qquad P[Y \leq y] = 1 - \sqrt{1-4y},$$

and $P[Z \leq z] = 1/2$. In all other cases (1) is clear.

4.7: If $A, B, C$ are independent, then by the addition law

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

and

$$\begin{aligned} P\big((A \cup B)C\big) &= P(AB \cup BC) \\ &= P(AC) + P(BC) - P(ABC) \\ &= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C) \\ &= \big[P(A) + P(B) - P(AB)\big]P(C) \\ &= P(A \cup B)P(C). \end{aligned}$$

By additivity,

$$P(A) = P(AB) + P(AB^c)$$

and so

$$P(A \setminus B) = P(AB^c) = P(A) - P(AB) = P(A) - P(A)P(B).$$

Using additivity in the same way,

$$\begin{aligned} P\big((A \setminus B)C\big) &= P\big((AC) \setminus B\big) = P(AC) - P(ABC) \\ &= P(A)P(C) - P(A)P(B)P(C) = P(A \setminus B)P(C). \end{aligned}$$

4.9: An algebraic proof using the Borel zero-one law is possible, but a more probabilistic approach seems nicer. If $B$ and $C$ are independent events with $P(B \cup C) = 1$ and $P(B) < 1$, then by the addition law

$$P(B \cup C) = P(B) + \big(1 - P(B)\big)P(C) = 1,$$

and this can only hold if $P(C) = 1$. Suppose the probability of the union is one. If we define

$$B_m = \bigcup_{n=1}^{m} A_n \qquad \text{and} \qquad C_m = \bigcup_{n=m+1}^{\infty} A_n,$$

then

$$\begin{aligned} P(B_m) = 1 - P(B_m^c) &= 1 - P\left(\bigcap_{n=1}^{m} B_n^c\right) \\ &= 1 - \prod_{n=1}^{m}\big(1 - P(A_n)\big) < 1, \end{aligned}$$

$B_m$ and $C_m$ are independent, and

$$P(B_m \cup C_m) = P\left(\bigcup_{n\geq 1} A_n\right) = 1.$$

Thus $P(B_m) = 1$ for every $n \geq 1$, and

$$P(A_n, \text{i.o.}) = \lim_{m\to\infty} P(B_m) = 1.$$

The converse is immediate. Finally, $A_1 = \Omega$ and $A_2 = A_3 = \cdots = \emptyset$ gives an example showing why the condition $P(A_n) < 1$ is necessary.

4.11: By continuity, $P[|X_n| > b] \to 0$ as $b \to \infty$, and we can define $b_n < \infty$ by

$$b_n = \inf\big\{b > 1 : P[|X_n| > b] < 1/2^n\big\}.$$

Since

$$\sum_{n\geq 1} P[|X_n| > b_n] \leq \sum_{n\geq 1} 1/2^n = 1 < \infty,$$

we have $P[|X_n|/b_n > 1, \text{i.o.}]$ by the Borel-Cantelli Lemma. On $[|X_n|/b_n > 1, \text{i.o.}]^c$, we have $X_n/(nb_n) \to 0$, so with $c_n \stackrel{\text{def}}{=} nb_n$, we have

$$P[X_n/c_n \to 0] = 1.$$