1 Convergence

1. Convergence a.s. implies convergence i.p.
2. (a) Convergence i.p. iff Cauchy i.p.
   (b) $X_n \stackrel{P}{\to} X$ iff each subsequence X_{n_k} contains a further subsequence $X_{n_{k(i)}} \stackrel{a.s.}{\to} X$.
3. Let $EX_n < \infty$. Then X_n is ui iff
   (a) $\sup_n \int_A |X_n| dP \to 0$ for $P(A) \to 0$ with $A \in \mathcal{B}$.
   (b) $\sup_n E|X_n| < \infty$
4. $\{|X_n|^p\}$ is ui if $\sup_n E(|X_n|^{p+\delta}) < \infty$ for some $\delta > 0$.
5. Let $p \geq 1$ and $X_n \in L_p$. The following are equivalent:
   (a) X_n is L_p convergent.
   (b) X_n is L_p Cauchy ($\|X_n - X_m\|_p \to 0$).
   (c) $|X_n|^p$ is ui and X_n converges ip.

2 Law of Large Numbers

1. General weak law of large numbers: If $\{X_n\}$ are independent,

$$\sum_{j=1}^{n} P[|X_j| > n] \to 0,$$

and

$$\frac{1}{n^2} \sum_{j=1}^{n} EX_j^2 1_{[|X_j| \leq n]} \to 0,$$

then

$$\frac{S_n - a_n}{n} \stackrel{P}{\to} 0,$$

where

$$a_n = \sum_{j=1}^{n} E(X_j 1_{[|X_j| \leq n]}).$$

2. If $\{X_n\}$ is independent, then TFAE:
   (a) S_n converges i.p.
   (b) S_n is Cauchy i.p.
   (c) S_n converges a.s.
   (d) S_n is Cauchy a.s.

3. If $\{X_n\}$ is independent and $\sum_n Var(X_n) < \infty$, then $\sum_n (X_n - \mu_n)$ converges a.s. and L_2.

4. If X_n is iid, then

$$E|X_1| < \infty \Rightarrow \bar{X}_n \overset{a.s.}{\to} \mu$$
$$EX_1^2 < \infty \Rightarrow S_n^2 \overset{a.s.}{\to} \sigma^2$$

5. Let X_n be independent. Then $\sum_n X_n$ converges a.s. if and only if there exists $c > 0$ such that

   (a) $\sum_n P[|X_n| > c] < \infty$
   (b) $\sum_n Var(X_n 1_{[|X_n| \leq c]}) < \infty$
   (c) $\sum_n E(X_n 1_{[|X_n| \leq c]})$ converges

3 Convergence in Distribution

1. Thm: If F is proper, then all four convergence are equivalent. (Write $F_n \Rightarrow F$.)

2. Thm (Scheffe):

$$\sup_{B \in \mathcal{B}(\mathbb{R})} |F_n(B) - F(B)| = \frac{1}{2} \int |f_n(x) - f(x)| dx$$

3. Thm (Baby Shorohod): If $X_n \Rightarrow X_0$, then there exist $X_n^{\#}$ on $([0,1], \mathcal{B}([0,1]), \lambda)$ such that $X_n \overset{d}{=} X_n^{\#}$ and $X_n^{\#} \overset{a.s.}{\to} X_0^{\#}$.

4. Thm (Continuous mapping): If $X_n \Rightarrow X_0$ and $h : \mathbb{R} \to \mathbb{R}$ satisfies $P[X_0 \in (\mathcal{C}(h))^c] = 0$, then $h(X_n) \Rightarrow h(X_0)$. Moreover, if h is bounded, then $Eh(X_n) \to Eh(X_0)$.

5. Thm (Delta method):

$$\sqrt{n} \left(\frac{g(\bar{X}) - g(\mu)}{\sigma g'(\mu)} \right) \Rightarrow N(0,1)$$

6. Thm (Portmanteau): Let F_n be proper. TFAE:

   (a) $F_n \Rightarrow F_0$
   (b) $Eg(X_n) \to Eg(X_0)$ for any bounded continuous $g : \mathbb{R} \to \mathbb{R}$.
   (c) $F_n(A) \to F(A)$ for any $A \in \mathcal{B}(\mathbb{R})$ with $F_0(\partial(A)) = 0$ (boundary has zero probability).

7. Thm (Slutsky): If $X_n \Rightarrow X$ and $X_n - Y_n \overset{P}{\to} 0$, then

8. Thm (Convergence to type):

   (a) If

   $$\frac{X_n - b_n}{a_n} \Rightarrow U, \quad \frac{X_n - \beta_n}{\alpha_n} \Rightarrow V, \tag{1}$$

   then

   $$\frac{\alpha_n}{a_n} \to A > 0, \quad \frac{\beta_n - b_n}{a_n} \to B \in \mathbb{R} \tag{2}$$

   and

   $$V \overset{d}{=} \frac{U - B}{A}. \tag{3}$$

   (b) If eq. (2) holds, then either of eq. (1) implies the other and eq. (3) holds.

4 Central Limit Theorem