Lecture 14. conv continued

Wednesday, November 1, 2017 10:13 AM

Convergence of sequences of r.v.

Convergence in probability $\xrightarrow[P]{}$

a.s. $\xrightarrow[a.s.]{}$ / $\xrightarrow[a.e.]{}$

in L^r $\xrightarrow{L^r}$ $\|X_n - X\|_{L^r} \to 0$

in distribution $\xrightarrow{d}$

weak convergence $\rightsquigarrow$

with additional conditions

$$\text{a.s.} \Rightarrow \xrightarrow[P]{} \Leftarrow L^r$$

$$\Downarrow$$

$$\xrightarrow[d]{}$$

$$\rightsquigarrow$$

$\boxed{\text{TH}}$ $\xrightarrow[P]{}$ + monotonicity $\Rightarrow$ $\overrightarrow{a.s}$

If 1) $X_n \downarrow$ or $\uparrow$

a.s., uniformly over ω

wrt to n for fixed ω

happens $\forall \omega$ except perhaps a set of measure 0

2) $X_n \xrightarrow[P]{} X$

Then $X_n \xrightarrow[a.s.]{} X$

Proof:

w/o loss of generality $X = 0$

$X_n \geq 0$

$X_n \downarrow$

By contradiction: $\exists$ $X_n \not\xrightarrow[a.s.]{} X \Rightarrow$

$\exists A : P(A) > 0$ and

$$\sup_{n \geq N} |X_n - X| > \varepsilon \quad \forall N, \forall \omega \in A$$

this means

$$A \subset \left\{ \sup_{n \geq N} |X_n - X| > \varepsilon \right\}$$

$$P(A) \leq P\left(\sup_{n \geq N} |X_n - X| > \varepsilon \right)$$

$$0 < P(A) \leq P\left(\underbrace{\sup_{n \geq N} X_n}_{= X_N} > \varepsilon \right)$$

$$P(X_n > \varepsilon) \geq P(A) > \delta > 0, \quad \forall n$$

Contradicts $X_n \xrightarrow[p]{} X$

$\square$

Corollary:

$$X_n \xrightarrow[a.s.]{} X \quad \text{iff} \quad \sup_{k \geq n} |X_k - X| \xrightarrow[p]{} 0$$

Proof:

$$X_n \xrightarrow[a.s.]{} X \quad \text{iff} \quad \sup_{k \geq n} |X_k - X| \xrightarrow[a.s.]{} 0$$

by properties of sequences of numbers

$x_n = X_n(\omega)$

$x_n \to x$

Sufficiency:

Conv. in p $\quad \sup_{k \geq n} |X_k - X| \xrightarrow[p]{} 0$

monotonicity $\quad \sup_{k \geq n} |X_k - X|$

$\Rightarrow \sup_{k \geq n} |X_k - X| \xrightarrow[a.s.]{} 0$

$\Downarrow$

$X_n \xrightarrow[a.s.]{} X$

Necessity

$\xrightarrow[a.s.]{}$ implies $\xrightarrow[p]{}$

$X_n \xrightarrow[a.s.]{} X \Rightarrow \sup_{k \geq n} |X_k - X| \xrightarrow[a.s.]{} 0 \Rightarrow$

$\Rightarrow \sup_{k \geq n} |X_k - X| \xrightarrow[p]{} 0$

$\square$

Yet another additional condition

TH

$\exists \sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty, \ \forall \varepsilon$

Then

$X_n \xrightarrow[a.s.]{} X$

series converges i.e. $P(|X_n - X| > \varepsilon)$ goes to 0 faster than they otherwise could in $\xrightarrow[p]{}$

Note:

$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty \Rightarrow X_n \xrightarrow[p]{} X$

Proof:

$$P\left(\bigcup_{k \geq n} \{|X_k - X| > \varepsilon\}\right) \leq \underbrace{\sum_{k \geq n} P(\{|X_k - X| > \varepsilon)}_{}$$

$\to 0$ by properties of sequences of numbers b/c series $\sum_{n=1}^{\infty} P(\dots) < \infty$

$$n=1$$

By continuity of $\mathbb{P}$ $\Rightarrow$

$$\lim_{n\to\infty} P\left(\bigcup_{k\geq n}\{\ldots\}\right) = P\Big(\underbrace{\bigcap_n \bigcup_{k\geq n}}_{\limsup_{n\to\infty}}\{\ldots\}\Big) =$$

$$= P\left(\limsup_{n\to\infty}\{\ldots\}\right) = 0$$

$\Updownarrow$ By a.s. equiv. statement in Lecture 13

$$X_n \xrightarrow[a.s.]{} X$$

$\square$

Corollary

$\exists$ $X_n \xrightarrow[p]{} X$, any such sequence

Then we can construct a subsequence that converges a.s.

Proof:

$$X_n \xrightarrow[p]{} X \Rightarrow P(|X_n - X| > \varepsilon) \to 0 \Rightarrow$$

$$\Rightarrow \forall \varepsilon > 0 \ \exists N(k) : P(|X_n - X| > \varepsilon) < \frac{1}{k^2} \ \forall n \geq N(k)$$

Take a k large enough

Series $\frac{1}{k^{1+\text{smth}>0}}$ will converge (calculus)

$$\Rightarrow \sum_k P(|X_{N(k)} - X| > \varepsilon) \leq \sum_k \frac{1}{k^2} < \infty$$

$$\Rightarrow X_{N(k)} \xrightarrow[a.s.]{} X$$

by the previous Th

$\square$

L^r - convergence $\overset{\text{def}}{\Rightarrow}$ $\mathbb{E}(|X_n - X|^r) \to 0$

TH $\quad X_n \xrightarrow{L^r} X \implies X_n \xrightarrow[p]{} X$

Proof:

usually $r \geq 1$

Chebyshev's inequality

$$P\left(|X_n - X|^r > \varepsilon\right) \leq \frac{\mathbb{E}\left(|X_n - X|^r\right)}{\varepsilon} \xrightarrow[n\to\infty]{} 0$$

$$\shortparallel$$

$$P\left(|X_n - X| > \underbrace{\varepsilon^{1/r}}_{\text{new}, \forall \varepsilon}\right) \xrightarrow[n\to\infty]{} 0 \quad , \forall \varepsilon > 0$$

$$\implies \quad X_n \xrightarrow[p]{} X$$

$\square$

Note:

$$X_n \xrightarrow[a.s.]{} \not\Rightarrow X_n \xrightarrow{L^r} X$$

$$X_n \xrightarrow{L^r} X \not\Rightarrow X_n \xrightarrow[a.s.]{} X \quad \text{impulse example}$$

DF $\quad$ Convergence in distribution

$X_n \xrightarrow{d} X$ when

CDF $F_{X_n} \to F_X$ at each point of continuity of F_X

Note: Monotonic and bounded functions can have at most a countable # of discontinuities of the bounded jump type

DF $\quad$ Weak convergence

CDF $F_n \rightsquigarrow F$ if $\forall$ continuous and bounded function f

$$\mathbb{E} f(X_n) \to \mathbb{E} f(X) = \int f(x)\, dF(x)$$

$$'' \int f(x)\, dF_n(x)$$

TH $\xrightarrow{d} \iff \rightsquigarrow$ w/o proof