Example: Maximum Likelihood Estimation

A more flexible alternative to the exponential distribution is the *Weibull* model, with density

$$f(y) \quad = \quad (\lambda\gamma)y^{\gamma-1}\exp\{-\lambda y^{\gamma}\}$$

- We analyze the data set used in the previous example.

- Since it is required that $\lambda > 0$ and $\gamma > 0$, we reparameterize: $\lambda = e^{\beta}$, $\gamma = e^{\alpha}$, then set $\boldsymbol{\theta} = (\beta, \alpha)^T$

- We will derive $L_i(\boldsymbol{\theta})$, $\ell_i(\boldsymbol{\theta})$, $U_i(\boldsymbol{\theta})$ and $J_i(\boldsymbol{\theta})$, then program Newton-Raphson using IML, then
  - solve for $\widehat{\boldsymbol{\theta}}$
  - estimate $SE(\widehat{\theta}_j)$ for $j = 1, 2$

Likelihood and Related Functions: Weibull Case

- Under the reparameterized model,

$$\begin{aligned} f_i &= e^{\beta+\alpha} Y_i^{\exp\{\alpha\}-1} \exp\left\{-e^{\beta} Y_i^{\exp\{\alpha\}}\right\} \\ &= L_i \end{aligned}$$

- We then obtain

$$\ell_i = \beta + \alpha + (e^{\alpha} - 1)\log Y_i - e^{\beta} Y_i^{\exp\{\alpha\}}$$

- Setting up the score equation,

$$\begin{aligned} \frac{\partial \ell_i}{\partial \beta} &= 1 - e^{\beta} Y_i^{\exp\{\alpha\}} \\ \frac{\partial \ell_i}{\partial \alpha} &= 1 + e^{\alpha} \log Y_i - e^{\beta+\alpha} Y_i^{\exp\{\alpha\}} \log Y_i \end{aligned}$$

- Information matrix,

$$
\begin{aligned}
\frac{\partial^2 \ell_i}{\partial \beta^2} &= -e^{\beta} Y_i^{\exp\{\alpha\}} \\
\frac{\partial^2 \ell_i}{\partial \alpha^2} &= e^{\alpha} \log Y_i \\
&\quad -e^{\beta+\alpha} \log Y_i Y_i^{\exp\{\alpha\}} \{1 + e^{\alpha} \log Y_i\} \\
\frac{\partial^2 \ell_i}{\partial \alpha \partial \beta} &= -Y_i^{\exp\{\alpha\}} e^{\beta+\alpha} \log Y_i
\end{aligned}
$$

- Computing MLE through Newton-Raphson: Refer to SAS code ...