BIOSTAT 802 HW#2 Solution

Winter, 2018

Solution of Problem 1

(a) Consider

$$
\begin{aligned}
f(\boldsymbol{x} \mid \xi, \eta) &= \prod_{i=1}^{n} f(x_i \mid \xi, \eta) \\
&= \prod_{i=1}^{\xi} f(x_i \mid \xi, \eta) \prod_{i=\xi+1}^{n} f(x_i \mid \xi, \eta) \\
&= \prod_{i=1}^{\xi} \eta e^{-\eta x_i} \prod_{i=\xi+1}^{n} c\eta e^{-c\eta x_i} \\
&= c^{n-\xi}\eta^{n} \exp\left[-\eta\left(\sum_{i=1}^{\xi} x_i + c\sum_{i=\xi+1}^{n} x_i\right)\right]
\end{aligned}
$$

The joint posterior is

$$\pi(\xi, \eta \mid \boldsymbol{x}) \propto f(\boldsymbol{x} \mid \xi, \eta)\pi(\eta)\pi(\xi)$$

Integrating out η, we find

$$
\begin{aligned}
\pi(\xi \mid \boldsymbol{x}) &= \int_0^{\infty} f(\boldsymbol{x} \mid \xi, \eta)\pi(\xi) d\eta \\
&= \pi(\xi) \int_0^{\infty} c^{n-\xi}\eta^{n} \exp\left[-\eta\left(\sum_{i=1}^{\xi} x_i + c\sum_{i=\xi+1}^{n} x_i\right)\right] d\eta
\end{aligned}
$$

Applying integration by part (or take the hint), we find

$$
\begin{aligned}
\pi(\xi \mid \boldsymbol{x}) &\propto \pi(\xi) c^{n-\xi} n! x_1^{-(n+1)} \left(\sum_{i=1}^{\xi} z_i + c\sum_{i=\xi+1}^{n} z_i\right)^{-(n+1)} \\
&\propto \pi(\xi) c^{n-\xi} \left(\sum_{i=1}^{\xi} z_i + c\sum_{i=\xi+1}^{n} z_i\right)^{-(n+1)}
\end{aligned}
$$

(b) The exponential distribution belongs to scale family, thus $X_1, \ldots, X_n$ can be equivalently presented by $\eta Y_1, \eta Y_2, \ldots, c\eta Y_n$, where $Y_1, \ldots, Y_n$ are i.i.d $\exp(1)$ random variables. Then it is clear $P(Z_2, .., Z_n) = P(Y_2/Y_1, \ldots, cY_n/Y_1)$ is not a function of η.

(c) Apply the change of variable formula, and note the absolute value of determinant of Jacobian

$$\left|\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x_1, z_2, \ldots, z_n)}\right| = x_1^{n-1},$$

we find that

$$f(x_1, z \mid \xi, \eta) = c^{n-\xi} \eta^n x_1^{n-1} \exp\left[-\eta \left(\sum_{i=1}^{\xi} z_i + c \sum_{i=\xi+1}^{n} z_i\right) x_1\right]$$

Therefore, by applying integration by part

$$\begin{aligned} f(z \mid \xi, \eta) &= \int_0^\infty f(x_1, z \mid \xi, \eta) dx_1 \\ &\propto c^{n-\xi} \left(\sum_{i=1}^{\xi} z_i + c \sum_{i=\xi+1}^{n} z_i\right)^{-n} \end{aligned}$$

Accordingly, apply the Bayes rule

$$\begin{aligned} \pi(\xi \mid z) &\propto \pi(\xi) f(z \mid \xi) \\ &= \pi(\xi) c^{n-\xi} \left(\sum_{i=1}^{\xi} z_i + c \sum_{i=\xi+1}^{n} z_i\right)^{-n} \end{aligned}$$

(d.) Under the new prior, re-do the integration in (1), we find

$$\pi(\xi \mid x) \propto \pi(\xi) c^{n-\xi} \left(\sum_{i=1}^{\xi} z_i + c \sum_{i=\xi+1}^{n} z_i\right)^{-n},$$

and the paradox no longer occurs.

(c) When $\beta \leq \max_i (x_i) = X_{(n)}$, the Bayes estimator becomes $\delta(\underset{\sim}{x}) = \frac{n+\alpha}{n+\alpha-1} X_{(n)}$.

Clearly, the MLE of θ is $X_{(n)}$. Then,

$$MSE(\delta) = E_\theta \left\{ \delta(\underset{\sim}{X}) - \theta \right\}^2 = E_\theta \left\{ \frac{n+\alpha}{n+\alpha-1} X_{(n)} - \theta \right\}^2$$

$$= E_\theta \left\{ \frac{n+\alpha}{n+\alpha-1} X_{(n)} - \frac{n+\alpha}{n+\alpha-1}\theta + \underbrace{\frac{n+\alpha}{n+\alpha-1}\theta - \theta}_{\frac{\theta}{n+\alpha-1}} \right\}^2$$

$$= E_\theta \left\{ \left(\frac{n+\alpha}{n+\alpha-1}\right)\left(X_{(n)} - \theta\right) + \frac{\theta}{n+\alpha-1} \right\}^2$$

$$= E_\theta \left(\frac{n+\alpha}{n+\alpha-1}\right)^2 \left(X_{(n)} - \theta\right) + 2\left(\frac{n+\alpha}{n+\alpha-1}\right)\left(\frac{\theta}{n+\alpha-1}\right) \underbrace{E_\theta\left(X_{(n)} - \theta\right)}_{\text{bias of MLE}} + \left(\frac{\theta}{n+\alpha-1}\right)^2$$

$$= \left(\frac{n+\alpha}{n+\alpha-1}\right)^2 MSE(\hat{\theta}_{MLE}) + 2\left(\frac{n+\alpha}{n+\alpha-1}\right)\frac{\theta^2}{(n+\alpha-1)(n+1)} + \frac{\theta^2}{(n+\alpha-1)^2} \quad = \frac{\theta}{n+1}$$

$$= \underbrace{\left(\frac{n+\alpha}{n+\alpha-1}\right)^2}_{>1} MSE(\hat{\theta}_{MLE}) + \underbrace{\frac{3n+2\alpha+1}{(n+\alpha-1)^2(n+1)}\theta^2}_{>0}$$

$$> MSE(\hat{\theta}_{MLE}).$$

Solution of Problem 2

(a) Because $f(x_i|\theta) = \frac{1}{\theta} 1[0 \le x_i \le \theta]$ and $f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \frac{1}{\theta^n} 1[X_{(n)} \le \theta]$,

the posterior density is

$$\pi(\theta|\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}|\theta)\pi(\theta)}{f(\underset{\sim}{x})} \propto \frac{1}{\theta^n} \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}} 1[X_{(n)} \le \theta] 1[\beta \le \theta]$$

$$= \frac{\alpha\beta^\alpha}{\theta^{n+\alpha+1}} 1[\max(X_{(n)}, \beta) \le \theta]$$

$$\propto \frac{1}{\theta^{n+\alpha+1}} 1[\max(X_{(n)}, \beta) \le \theta]$$

This is the density kernel of a pareto distribution Pareto $(n+\alpha, \max(X_{(n)}, \beta))$, which implies that Pareto is the conjugate for the uniform dist'n.

(b) We can show that if $X \sim PA(\alpha, \beta)$, then $E(X) = \frac{\alpha\beta}{\alpha-1}$ $(\alpha > 1)$ and $var(X) = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}$.

Using the above fact, we obtain the Bayes estimator of θ, namely its posterior mean,

$$E(\theta|\underset{\sim}{x}) = \frac{(n+\alpha)\max(X_{(n)}, \beta)}{n+\alpha-1},$$

$$Var(\theta|\underset{\sim}{x}) = \frac{(n+\alpha)\{\max(X_{(n)}, \beta)\}^2}{(n+\alpha-1)^2(n+\alpha-2)}, \quad (n \ge 2)$$

(c) Under the absolute loss function, the Bayes estimator is the median of the posterior dist'n. Let $\beta(\underset{\sim}{x}) = \max(X_{(n)}, \beta)$. Then, the median $\delta^\pi(\underset{\sim}{x})$ satisfies

This part has been removed from the homework

$$\int_{\beta(\underset{\sim}{x})}^{\delta^\pi} \pi(\theta|x)\,d\theta = \int_{\delta^\pi}^{\infty} \pi(\theta|x)\,d\theta = \frac{1}{2}$$

or

$$\int_{\beta(\underset{\sim}{x})}^{\delta^\pi} \theta^{-(n+\alpha+1)}\,d\theta = \int_{\delta^\pi}^{\infty} \theta^{-(n+\alpha-1)}$$

or

$$(\delta^\pi)^{-(n+\alpha)} - (\beta(\underset{\sim}{x}))^{-(n+\alpha)} = -(\delta^\pi)^{-(n+\alpha)}$$

or

$$\delta^\pi(\underset{\sim}{x}) = \beta(\underset{\sim}{x})\, 2^{\frac{1}{n+\alpha}}.$$

3(a). We will derive the Bayes solution by the posterior median under the absolute error loss $L(\theta,a)=|\theta-a|$. First, the posterior is

$$\pi(\theta|\underset{\sim}{x}) \propto \pi(\underset{\sim}{x}|\theta)\,\pi_\alpha(\theta) = \prod_{i=1}^{n} 1\left[\theta-\tfrac{1}{2} \le x_i \le \theta+\tfrac{1}{2}\right] \frac{1}{2\alpha} 1[-\alpha<\theta<\alpha]$$

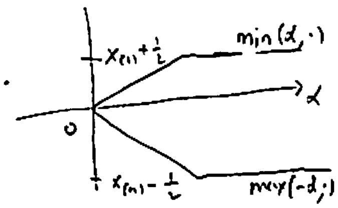
$$\propto 1\Big[\theta-\tfrac{1}{2} \le \underbrace{\min x_i}_{X_{(1)}} \le \underbrace{\max x_i}_{X_{(n)}} \le \theta+\tfrac{1}{2}\Big] 1[-\alpha<\theta<\alpha]$$

$$= 1\left[X_{(n)}-\tfrac{1}{2} \le \theta \le X_{(1)}+\tfrac{1}{2}\right] 1[-\alpha<\theta<\alpha]$$

$$= 1\left[\max\{-\alpha, X_{(n)}-\tfrac{1}{2}\} \le \theta \le \min\{\alpha, X_{(1)}+\tfrac{1}{2}\}\right]$$

The median of the posterior is

$$\delta_{\pi_\alpha}(\underset{\sim}{x}) = \frac{1}{2}\left[\max\{-\alpha, X_{(n)}-\tfrac{1}{2}\} + \min\{\alpha, X_{(1)}+\tfrac{1}{2}\}\right].$$



3(b)

$$\delta_{\pi_\alpha}(\underset{\sim}{x}) \to \delta(x) \hat{=} \frac{1}{2}\{X_{(n)}+X_{(1)}\}, \quad \text{as } \alpha\to\infty.$$

We know that $\underbrace{X_i-\theta+\frac{1}{2}}_{U_i} \overset{iid}{\sim} U(0,1)$.

Note that $\delta(x)$ is not necessarily a Bayes estimator with a proper prior.

3(c) $U_{(1)} = X_{(1)}-\theta+\frac{1}{2} \sim \text{Beta}(1,n)$, and $U_{(n)} = X_{(n)}-\theta+\frac{1}{2} \sim \text{Beta}(n,1)$.

Then $\delta(x) = \frac{1}{2}\left\{(X_{(n)}-\theta+\frac{1}{2}) + (X_{(1)}-\theta+\frac{1}{2})\right\} + (\theta-1)$ and

$$L(\theta,a) = |\theta-\delta(x)| = \left|\frac{1}{2}\left\{(X_{(n)}-\theta+\tfrac{1}{2})+(X_{(1)}-\theta+\tfrac{1}{2})\right\}-1\right|$$

$$= \left|\frac{1}{2}\left\{(U_{(n)}-\tfrac{n}{n+1}) + (U_{(1)}-\tfrac{1}{n+1})\right\}\right|$$ ← whose dist'n does not depend on θ.

$$R(\theta,\delta) = E_{x|\theta}|\theta-\delta(x)| = E\left|\frac{1}{2}\left\{(U_{(n)}-\tfrac{n}{n+1}) + (U_{(1)}-\tfrac{1}{n+1})\right\}\right| = \text{constant}.$$

Then δ is a minimax estimator. See the proof in the next page.

Suppose Bayes estimators $\delta_{\pi_\alpha}(\underset{\sim}{x}) \to \delta(\underset{\sim}{x})$, $\alpha \to \infty$ and the risk functions

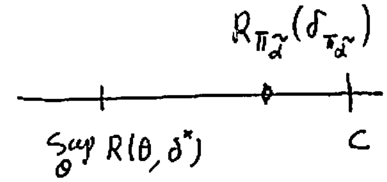
$$R(\theta, \delta_{\pi_\alpha}) \to R(\theta, \delta) = \text{constant over } \theta \in \Theta$$

If $\delta(x)$ is not a minimax estimator, then exists a $\delta^*(\underset{\sim}{x})$ such that

$$\sup_\theta R(\theta, \delta^*) < \sup_\theta R(\theta, \delta) = C$$

Because $R_{\pi_\alpha}(\delta_{\pi_\alpha}) = \int_\Theta R(\theta, \delta_{\pi_\alpha})\, d\pi_\alpha(\theta) \to R(\theta, \delta) = C$, as $\alpha \to \infty$

there exists an $\tilde{\alpha}$ such that



$$\sup_\theta R(\theta, \delta^*) < R_{\pi_{\tilde\alpha}}(\delta_{\pi_{\tilde\alpha}})$$

Then

$$R_{\pi_{\tilde\alpha}}(\delta^*) = \int_\Theta R(\theta, \delta^*)\, \pi_{\tilde\alpha}(\theta) \le \sup_{\theta \in \Theta} R(\theta, \delta^*) < R_{\pi_{\tilde\alpha}}(\delta_{\pi_{\tilde\alpha}})$$

This implies that $\delta_{\pi_{\tilde\alpha}}$ is not a Bayes estimator. Contradiction.