Biostat 602 Winter 2017

Lecture Set 17

Hypothesis Testing
Likelihood Ratio Test

Reading: CB 8.2

Likelihood Ratio Tests (LRT)

Definition Let $L(\theta|\mathbf{x})$ be the likelihood function of θ. The likelihood ratio test statistic for testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_0^c$ is

$$\lambda(\mathbf{x}) \;=\; \frac{\sup_{\theta\in\Omega_0} L(\theta|\mathbf{x})}{\sup_{\theta\in\Omega} L(\theta|\mathbf{x})} = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}$$

where $\hat{\theta}$ is the MLE of θ over $\theta \in \Omega$, and $\hat{\theta}_0$ is the MLE of θ over $\theta \in \Omega_0$ (restricted MLE).

The *likelihood ratio test* is a test that rejects H_0 if and only if $\lambda(\mathbf{x}) \leq c$ where $0 \leq c \leq 1$.

c is obtained from the size condition of the test, namely

$$\sup_{\theta\in\Omega_0} \beta(\theta) = \alpha$$

where $\beta(\theta) = \Pr(\mathbf{X} \in R|\theta) = \Pr(\text{reject } H_0|\theta)$ is the power function of the test.

LRT based on sufficient statistics

Theorem 8.2.4: If $T(\mathbf{X})$ is a sufficient statistic for θ, $\lambda^*(t)$ is the LRT statistic based on T, and $\lambda(\mathbf{x})$ is the LRT statistic based on $\mathbf{x}$ then

$$\lambda^*[T(\mathbf{x})] = \lambda(\mathbf{x})$$

for every $\mathbf{x}$ in the sample space.

Proof: By Factorization Theorem, the joint pdf of $\mathbf{x}$ can be written as

$$f(\mathbf{x}|\theta) = g(T(\mathbf{x})|\theta)h(\mathbf{x})$$

and we can choose $g(t|\theta)$ to be the pdf or pmf of $T(\mathbf{x})$. Then, the LRT statistic based on $T(\mathbf{X})$ is defined as

$$\lambda^*(t) = \frac{\sup_{\theta\in\Omega_0} L(\theta|T(\mathbf{x}) = t)}{\sup_{\theta\in\Omega} L(\theta|T(\mathbf{x}) = t)} = \frac{\sup_{\theta\in\Omega_0} g(t|\theta)}{\sup_{\theta\in\Omega} g(t|\theta)}$$

LRT statistic based on $\mathbf{X}$ is

$$\begin{aligned}
\lambda(\mathbf{x}) &= \frac{\sup_{\theta\in\Omega_0} L(\theta|\mathbf{x})}{\sup_{\theta\in\Omega} L(\theta|\mathbf{x})} \\
&= \frac{\sup_{\theta\in\Omega_0} f(\mathbf{x}|\theta)}{\sup_{\theta\in\Omega} f(\mathbf{x}|\theta)} \\
&= \frac{\sup_{\theta\in\Omega_0} g(T(\mathbf{x})|\theta)h(\mathbf{x})}{\sup_{\theta\in\Omega} g(T(\mathbf{x})|\theta)h(\mathbf{x})} \\
&= \frac{\sup_{\theta\in\Omega_0} g(T(\mathbf{x})|\theta)}{\sup_{\theta\in\Omega} g(T(\mathbf{x})|\theta)} = \lambda^*(T(\mathbf{x}))
\end{aligned}$$

The simplified expression of $\lambda(\mathbf{x})$ should depend on $\mathbf{x}$ only through $T(\mathbf{x})$, where $T(\mathbf{x})$ is a sufficient statistic for θ.

Example 1: Consider $X_1, \cdots, X_n$ i.i.d. $\mathcal{N}(\theta, \sigma^2)$ where σ^2 is known.

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned}$$

Find a size α LRT.

Solution - Using sufficient statistics: Note that in this case, $T(\mathbf{X}) = \overline{X}$ is a sufficient statistic for θ.

$$\begin{aligned} T &\sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right) \\ \lambda(t) &= \frac{\sup_{\theta\in\Omega_0} L(\theta|t)}{\sup_{\theta\in\Omega} L(\theta|t)} = \frac{\sqrt{\frac{1}{2\pi\sigma^2/n}}\ \exp\left[-\frac{(t-\theta_0)^2}{2\sigma^2/n}\right]}{\sup_{\theta\in\Omega}\sqrt{\frac{1}{2\pi\sigma^2/n}}\ \exp\left[-\frac{(t-\theta)^2}{2\sigma^2/n}\right]} \end{aligned}$$

The numerator is fixed, and MLE in the denominator is $\hat{\theta} = t$. Therefore the LRT statistic is

$$\lambda(t) = \exp\left[-\frac{n(t-\theta_0)^2}{2\sigma^2}\right]$$

LRT rejects H_0 if and only if

$$\begin{aligned} \lambda(t) = \exp\left[-\frac{n(t-\theta_0)^2}{2\sigma^2}\right] &\leq c \\ \Longrightarrow \left|\frac{t-\theta_0}{\sigma/\sqrt{n}}\right| &\geq \sqrt{-2\log c} = c^* \end{aligned}$$

Note that

$$\begin{aligned} T = \overline{X} &\sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right) \\ \frac{T-\theta_0}{\sigma/\sqrt{n}} &\sim \mathcal{N}(0,1) \end{aligned}$$

A size α test satisfies

$$\begin{aligned} \sup_{\theta \in \Omega_0} \Pr\left(\left|\frac{T-\theta}{\sigma/\sqrt{n}}\right| \geq c^*\right) &= \alpha \\ \Pr\left(\left|\frac{T-\theta_0}{\sigma/\sqrt{n}}\right| \geq c^*\right) &= \alpha \\ \Pr\left(|Z| \geq c^*\right) &= \alpha \\ \Pr(Z \geq c^*) + \Pr(Z \leq -c^*) &= \alpha \end{aligned}$$

$$|Z| = \left|\frac{T-\theta}{\sigma/\sqrt{n}}\right| \geq z_{\alpha/2}$$

LRT with nuisance parameters

Example 2: Let $X_1, \cdots, X_n$ be i.i.d $\mathcal{N}(\theta, \sigma^2)$ where both θ and σ^2 are unknown. Obtain a LRT for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

1. Specify Ω and Ω_0
2. Find size α LRT.

Solution - Ω and Ω_0

$$\begin{aligned} \Omega &= \{(\theta, \sigma^2) : \theta \in \mathbb{R}, \sigma^2 > 0\} \\ \Omega_0 &= \{(\theta, \sigma^2) : \theta \leq \theta_0, \sigma^2 > 0\} \end{aligned}$$

Size α LRT

$$\lambda(\mathbf{x}) = \frac{\sup_{\{(\theta,\sigma^2):\theta\leq\theta_0,\sigma^2>0\}} L(\theta, \sigma^2|\mathbf{x})}{\sup_{\{(\theta,\sigma^2):\theta\in\mathbb{R},\sigma^2>0\}} L(\theta, \sigma^2|\mathbf{x})}$$

For the denominator, the MLE of θ and σ^2 are

$$\begin{cases} \hat{\theta} = \overline{X} \\ \hat{\sigma}^2 = \frac{\sum(X_i - \overline{X})^2}{n} = \frac{n-1}{n} s_{\mathbf{X}}^2 \end{cases}$$

For numerator, we need to maximize $L(\theta, \sigma^2|\mathbf{x})$ over the region $\theta \leq \theta_0$ and $\sigma^2 > 0$.

$$L(\theta, \sigma^2|\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[-\frac{\sum_{i=1}^n (x_i - \theta)^2}{2\sigma^2}\right]$$

Maximizing Numerator

Step 1: fix σ^2, likelihood is maximized when $\sum_{i=1}^{n}(x_i - \theta)^2$ is minimized over $\theta \leq \theta_0$.

$$\hat{\theta}_0 = \begin{cases} \overline{x} & \text{if } \overline{x} \leq \theta_0 \\ \theta_0 & \text{if } \overline{x} > \theta_0 \end{cases}$$

Step 2: Now, we need to maximize likelihood (or log-likelihood) with respect to σ^2 and we substitute $\hat{\theta}_0$ for θ.

$$\begin{aligned} l(\hat{\theta}, \sigma^2|\mathbf{x}) &= -\frac{n}{2}\left(\log 2\pi + \log \sigma^2\right) - \frac{\sum(x_i - \hat{\theta}_0)^2}{2\sigma^2} \\ \frac{\partial \log l}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{\sum(x_i - \hat{\theta}_0)^2}{2(\sigma^2)^2} = 0 \\ \hat{\sigma}_0^2 &= \frac{\sum_{i=1}^{n}(x_i - \hat{\theta}_0)^2}{n} \end{aligned}$$

Combining the results together

$$\lambda(\mathbf{x}) = \begin{cases} 1 & \text{if } \overline{x} \leq \theta_0 \\ \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} & \text{if } \overline{x} > \theta_0 \end{cases}$$

Constructing LRT

LRT test rejects H_0 if and only if $\overline{x} > \theta_0$ and

$$\begin{aligned}
\left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} &\leq c \\
\left(\frac{\sum(x_i - \overline{x})^2/n}{\sum(x_i - \theta_0)^2/n}\right)^{n/2} &\leq c \\
\frac{\sum(x_i - \overline{x})^2}{\sum(x_i - \theta_0)^2} &\leq c^* \\
\frac{\sum(x_i - \overline{x})^2}{\sum(x_i - \overline{x})^2 + n(\overline{x} - \theta_0)^2} &\leq c^* \\
\frac{1}{1 + \frac{n(\overline{x} - \theta_0)^2}{\sum(x_i - \overline{x})^2}} &\leq c^*
\end{aligned}$$

$$\begin{aligned}
\frac{n(\overline{x} - \theta_0)^2}{\sum(x_i - \overline{x})^2} &\geq c^{**} \\
\frac{\overline{x} - \theta_0}{s_{\mathbf{X}}/\sqrt{n}} &\geq c^{***}
\end{aligned}$$

LRT test rejects H_0 if

$$\frac{\overline{x} - \theta_0}{s_{\mathbf{X}}/\sqrt{n}} \geq c^{***}$$

The next step is to specify c^{***} to get size α test (can you figure out?).

Unbiased Test

Definition: If a test always satisfies

$$\Pr(\text{reject } H_0 \text{ when } H_0 \text{ is false }) \geq \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is true })$$

Then the test is said to be unbiased.

Alternative Definition: Recall that $\beta(\theta) = \Pr(\text{reject } H_0)$. A test is unbiased if

$$\beta(\theta') \geq \beta(\theta)$$

for every $\theta' \in \Omega_0^c$ and $\theta \in \Omega_0$.

Example 3: Let $X_1, \cdots, X_n$ be i.i.d. $\mathcal{N}(\theta, \sigma^2)$ where σ^2 is known, testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$. LRT test rejects H_0 if

$$\frac{\overline{x} - \theta_0}{\sigma/\sqrt{n}} > c.$$

$$\begin{aligned}
\beta(\theta) &= \Pr\left(\frac{\overline{X} - \theta_0}{\sigma/\sqrt{n}} > c\right) \\
&= \Pr\left(\frac{\overline{X} - \theta + \theta - \theta_0}{\sigma/\sqrt{n}} > c\right) \\
&= \Pr\left(\frac{\overline{X} - \theta}{\sigma/\sqrt{n}} + \frac{\theta - \theta_0}{\sigma/\sqrt{n}} > c\right) \\
&= \Pr\left(\frac{\overline{X} - \theta}{\sigma/\sqrt{n}} > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)
\end{aligned}$$

Note that $X_i \sim \mathcal{N}(\theta, \sigma^2)$, $\overline{X} \sim \mathcal{N}(\theta, \sigma^2/n)$, and $\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$. Therefore, for $Z \sim \mathcal{N}(0,1)$

$$\beta(\theta) \;=\; \Pr\left(Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

Because the power function is increasing function in θ,

$$\beta(\theta') \geq \beta(\theta)$$

always holds when $\theta \leq \theta_0 < \theta'$. Therefore the LRTs are unbiased.

Question: Can the same test be biased when hypotheses change?

Example 4: Same framework as before.

- New hypotheses : $H_0 : \theta = \theta_0$, $H_1 : \theta \neq \theta_0$.
- Same test : $R = \left\{\frac{\overline{x}-\theta_0}{\sigma/\sqrt{n}} > c\right\}$.

Testing unbiasedness

The power function $\beta(\theta)$ is still an increasing function. Therefore, if $\theta_+ > \theta_0 > \theta_-$, then

$$\beta(\theta_+) > \beta(\theta_0) > \beta(\theta_-)$$

where both $\beta(\theta_+)$ and $\beta(\theta_-)$ are power but $\beta(\theta_0)$ is Type I error.

Hence, power can be smaller than the Type I error when $\theta < \theta_0$, so the test is biased.

Uniformly Most Powerful Test (UMP)

Definition: Let $\mathcal{C}$ be a class of tests between $H_0 : \theta \in \Omega$ vs $H_1 : \theta \in \Omega_0^c$. A test in $\mathcal{C}$, with power function $\beta(\theta)$ is *uniformly most powerful (UMP) test* in class $\mathcal{C}$ if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Omega_0^c$ and every $\beta'(\theta)$, which is a power function of another test in $\mathcal{C}$.

UMP level α test

Consider $\mathcal{C}$ to be the set of all the level α test. The UMP test in this class is called a UMP level α test.

UMP level α test has the smallest type II error probability for any $\theta \in \Omega_0^c$ in this class.

- A UMP test is "uniform" in the sense that it is most powerful for every $\theta \in \Omega_0^c$.
- For simple hypothesis such as $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$, UMP level α test always exists.

Neyman-Pearson Lemma

Theorem 8.3.12 - Neyman-Pearson Lemma: Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding to θ_i is $f(\mathbf{x}|\theta_i)$, $i = 0, 1$, using a test with rejection region R that satisfies

$$\begin{aligned} \mathbf{x} \in R \qquad & \text{if } f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0) \\ \mathbf{x} \in R^c \qquad & \text{if } f(\mathbf{x}|\theta_1) < kf(\mathbf{x}|\theta_0) \end{aligned} \tag{8.3.1}$$

for some $k \geq 0$ and

$$\alpha = \Pr(\mathbf{X} \in R|\theta_0). \tag{8.3.2}$$

Then,

- (Sufficiency) Any test that satisfies 8.3.1 and 8.3.2 is a UMP level α test
- (Necessity) If there exist a test satisfying 8.3.1 and 8.3.2 with $k > 0$, then every UMP level α test is a size α test (satisfies 8.3.2), and every UMP level α test satisfies 8.3.1 except perhaps on a set A satisfying $\Pr(\mathbf{X} \in A|\theta_0) = \Pr(\mathbf{X} \in A|\theta_1) = 0$.

Example 5: Let $X \sim Binomial(2, \theta)$, and consider testing $H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$.

Calculating the ratios of the pmfs given,

$$\frac{f(0|\theta_1)}{f(0|\theta_0)} = \frac{1}{4}, \qquad \frac{f(1|\theta_1)}{f(1|\theta_0)} = \frac{3}{4}, \qquad \frac{f(2|\theta_1)}{f(2|\theta_0)} = \frac{9}{4}$$

- Suppose that $k < 1/4$, then the rejection region $R = \{0, 1, 2\}$, and UMP level α test always rejects H_0. Therefore

$$\alpha = \Pr(\text{reject } H_0|\theta = \theta_0 = 1/2) = 1.$$

- Suppose that $1/4 < k < 3/4$, then $R = \{1, 2\}$, and UMP level α test rejects H_0 if $x = 1$ or $x = 2$.

$$\alpha = \Pr(\text{reject } H_0|\theta = \frac{1}{2}) = \Pr(x \neq 0|\theta = 1/2) = \frac{3}{4}$$

- Suppose that $3/4 < k < 9/4$, then UMP level α test rejects H_0 if $x = 2$

$$\alpha = \Pr(\text{reject} H_0|\theta = 1/2) = \Pr(x = 2|\theta = 1/2) = \frac{1}{4}$$

- If $k > 9/4$ the UMP level α test will always not reject H_0, and $\alpha = 0$

Example 6 – Normal Distribution: $X_i \sim \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.

$$\begin{aligned}
f(\mathbf{x}|\theta) &= \prod_{i=1}^{n} \left[\frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{(x_i - \theta)^2}{2\sigma^2} \right\} \right] \\
\frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)} &= \frac{\exp\left\{ -\frac{\sum_{i=1}^{n}(x_i-\theta_1)^2}{2\sigma^2} \right\}}{\exp\left\{ -\frac{\sum_{i=1}^{n}(x_i-\theta_0)^2}{2\sigma^2} \right\}} \\
&= \exp\left[-\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right] \\
&= \exp\left[\frac{\sum_{i=1}^{n}(x_i - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right] \\
&= \exp\left[\frac{n(\theta_0^2 - \theta_1^2) + 2\sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} \right]
\end{aligned}$$

UMP level α test rejects H_0 if

$$
\begin{aligned}
& \exp\left[\frac{n(\theta_0^2-\theta_1^2)+2\sum_{i=1}^n x_i(\theta_1-\theta_0)}{2\sigma^2}\right] > k \\
\Longleftrightarrow \quad & \frac{n(\theta_0^2-\theta_1^2)+2\sum_{i=1}^n x_i(\theta_1-\theta_0)}{2\sigma^2} > \log k \\
\Longleftrightarrow \quad & \sum_{i=1}^n x_i > k^*
\end{aligned}
$$

$$
\alpha = \Pr\left(\sum_{i=1}^n X_i > k^* | \theta_0\right)
$$

Under H_0,

$$
\begin{aligned}
X_i &\sim \mathcal{N}(\theta_0, \sigma^2) \\
\overline{X} &\sim \mathcal{N}(\theta_0, \sigma^2/n) \\
\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} &\sim \mathcal{N}(0,1)
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \Pr\left(\sum_{i=1}^n X_i > k^* | \theta_0\right) \\
&= \Pr\left(Z > \frac{k^*/n-\theta_0}{\sigma/\sqrt{n}}\right)
\end{aligned}
$$

where $Z \sim \mathcal{N}(0,1)$.

$$
\begin{aligned}
\frac{k^*/n-\theta_0}{\sigma/\sqrt{n}} &= z_\alpha \\
k^* &= n\left(\theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}\right)
\end{aligned}
$$

Thus, the UMP level α test rejects H_0 if $\sum X_i > k^*$, or equivalently, rejects H_0 if $\overline{X} > k^*/n = \theta_0 + z_\alpha \sigma/\sqrt{n}$

Neyman-Pearson Lemma on Sufficient Statistics

Corollary 8.3.13: Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(\mathbf{X})$ is a sufficient statistic for θ and $g(t|\theta_i)$ is the pdf or pmf of T. Corresponding $\theta_i, i \in \{0, 1\}$. Then any test based on T with rejection region S is a UMP level α test if it satisfies

$$
\begin{aligned}
t \in S \qquad & \text{if } g(t|\theta_1) > k \cdot g(t|\theta_0) \text{ and} \\
t \in S^c \qquad & \text{if } g(t|\theta_1) < k \cdot g(t|\theta_0)
\end{aligned}
$$

for some $k > 0$ and $\alpha = \Pr(T \in S|\theta_0)$

Proof: The rejection region in the sample space is

$$
\begin{aligned}
R &= \{\mathbf{x} : T(\mathbf{x}) = t \in S\} \\
&= \{\mathbf{x} : g(T(\mathbf{x})|\theta_1) > kg(T(\mathbf{x})|\theta_0)\}
\end{aligned}
$$

By Factorization Theorem:

$$
\begin{aligned}
f(\mathbf{x}|\theta_i) &= h(\mathbf{x})g(T(\mathbf{x})|\theta_i) \\
R &= \{\mathbf{x} : g(T(\mathbf{x})|\theta_1)h(x) > kg(T(\mathbf{x})|\theta_0)h(x)\} \\
&= \{\mathbf{x} : f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0)\}
\end{aligned}
$$

By Neyman-Pearson Lemma, this test is the UMP level α test, and

$$\alpha \;=\; \Pr(\mathbf{X} \in R) = \Pr(T(\mathbf{X}) \in S|\theta_0)$$

Revisiting the Example of Normal Distribution

$X_i \sim \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. Consider testing

$$H_0 : \theta = \theta_0 \;\; vs. \;\; H_1 : \theta = \theta_1, \quad where \;\; \theta_1 > \theta_0.$$

It is known that $T = \overline{X}$ is a sufficient statistic for θ, where $T \sim \mathcal{N}(\theta, \sigma^2/n)$.

$$\begin{aligned}
g(t|\theta_i) &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{(t-\theta_i)^2}{2\sigma^2/n}\right\} \\
\frac{g(t|\theta_1)}{g(t|\theta_0)} &= \frac{\exp\left\{-\frac{(t-\theta_1)^2}{2\sigma^2/n}\right\}}{\exp\left\{-\frac{(t-\theta_0)^2}{2\sigma^2/n}\right\}} \\
&= \exp\left\{-\frac{1}{2\sigma^2/n}\left[(t-\theta_1)^2 - (t-\theta_0)^2\right]\right\} \\
&= \exp\left\{-\frac{1}{2\sigma^2/n}\left[\theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0)\right]\right\}
\end{aligned}$$

UMP level α test rejects H_0 if

$$\begin{aligned}
& \exp\left\{-\frac{1}{2\sigma^2/n}\left[\theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0)\right]\right\} > k \\
\Longleftrightarrow\;\; & \frac{1}{2\sigma^2/n}\left[-(\theta_1^2 - \theta_0^2) + 2t(\theta_1 - \theta_0)\right] > \log k \\
\Longleftrightarrow\;\; & \overline{X} = t > k^*
\end{aligned}$$

Under H_0, $\overline{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$. Now,

$$\begin{aligned}
\Pr(\text{reject } H_0|\theta_0) &= \alpha \\
\alpha &= \Pr(\overline{X} > k^*|\theta_0) \\
&= \Pr\left(\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} > \frac{k^*-\theta_0}{\sigma/\sqrt{n}}\right) \\
&= \Pr\left(Z > \frac{k^*-\theta_0}{\sigma/\sqrt{n}}\right) \\
\frac{k^*-\theta_0}{\sigma/\sqrt{n}} &= z_\alpha \\
k^* &= \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}
\end{aligned}$$

Monotone Likelihood Ratio (Karlin-Rubin)

Definition: A family of pdfs or pmfs $\{g(t|\theta) : \theta \in \Omega\}$ for a univariate random variable T with real-valued parameter θ have a monotone likelihood ratio if $\frac{g(t|\theta_2)}{g(t|\theta_1)}$ is an increasing (or non-decreasing) function of t for every $\theta_2 > \theta_1$ on $\{t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0\}$.

Note: we may define MLR using decreasing function of t. But all following theorems are stated according to the definition.

Examples of Monotone Likelihood Ratio

- Normal, Poisson, Binomial have the MLR Property (Exercise 8.25)
- If T is from an exponential family with the pdf or pmf

$$g(t|\theta) = h(t)c(\theta)\exp[w(\theta) \cdot t]$$

  Then T has an MLR if $w(\theta)$ is a non-decreasing function of θ.

Proof: Suppose that $\theta_2 > \theta_1$.

$$\begin{aligned}\frac{g(t|\theta_2)}{g(t|\theta_1)} &= \frac{h(t)c(\theta_2)\exp[w(\theta_2)t]}{h(t)c(\theta_1)\exp[w(\theta_1)t]} \\ &= \frac{c(\theta_2)}{c(\theta_1)}\exp[\{w(\theta_2) - w(\theta_1)\}t]\end{aligned}$$

If $w(\theta)$ is a non-decreasing function of θ, then

1. $w(\theta_2) - w(\theta_1) \geq 0$ and
2. $\exp[\{w(\theta_2) - w(\theta_1)\}t]$ is an increasing function of t.

Therefore, $\frac{g(t|\theta_2)}{g(t|\theta_1)}$ is a non-decreasing function of t, and T has MLR if $w(\theta)$ is a non-decreasing function of θ.

Karlin-Rubin Theorem

Theorem 8.3.17: Suppose $T(\mathbf{X})$ is a sufficient statistic for θ and the family $\{g(t|\theta) : \theta \in \Omega\}$ is an MLR family. Then

1. For testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$, the UMP level α test is given by rejecting H_0 is and only if $T > t_0$ where $\alpha = \Pr(T > t_0|\theta_0)$.
2. For testing $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$, the UMP level α test is given by rejecting H_0 if and only if $T < t_0$ where $\alpha = \Pr(T < t_0|\theta_0)$.

Example 7: Let $X_i \sim \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known, Find the UMP level α test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.

Solution: Here $T(\mathbf{X}) = \overline{X}$ is a sufficient statistic for θ, and $T \sim \mathcal{N}(\theta, \sigma^2/n)$. Therefore

$$
\begin{aligned}
g(t|\theta) &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{(t-\theta)^2}{2\sigma^2/n}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{t^2+\theta^2-2t\theta}{2\sigma^2/n}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{t^2}{2\sigma^2/n}\right\} \exp\left\{-\frac{\theta^2}{2\sigma^2/n}\right\} \exp\left\{\frac{t\theta}{\sigma^2/n}\right\} \\
&= h(t)c(\theta)\exp[w(\theta)t]
\end{aligned}
$$

where $w(\theta) = \frac{\theta}{\sigma^2/n}$ is an increasing function in θ. Therefore T has an MLR property.

Finding a UMP level α test

By Karlin-Rubin Theorem, UMP level α test rejects H_0 iff $T > t_0$

$$
\begin{aligned}
\alpha &= \Pr(T > t_0|\theta_0) \\
&= \Pr\left(\frac{T-\theta_0}{\sigma/\sqrt{n}} > \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\middle|\,\theta_0\right) \\
&= \Pr\left(Z > \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\right)
\end{aligned}
$$

where $Z \sim \mathcal{N}(0,1)$.

$$
\begin{aligned}
\frac{t_0-\theta_0}{\sigma/\sqrt{n}} &= z_\alpha \\
\Rightarrow t_0 &= \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha
\end{aligned}
$$

UMP level α test rejects H_0 if $T = \overline{X} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$.

Testing $H_0 : \theta \geq \theta_0$ **vs.** $H_1 : \theta < \theta_0$

UMP level α test rejects H_0 if $T < t_0$ where

$$
\begin{aligned}
\alpha &= \Pr(T < t_0|\theta_0) = \Pr\left(\frac{T-\theta_0}{\sigma/\sqrt{n}} < \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\middle|\,\theta_0\right) \\
&= \Pr\left(Z < \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\right) \\
1-\alpha &= \Pr\left(Z \geq \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\right) \\
\frac{t_0-\theta_0}{\sigma/\sqrt{n}} &= z_{1-\alpha} \\
t_0 &= \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} = \theta_0 - \frac{\sigma}{\sqrt{n}} z_\alpha
\end{aligned}
$$

Therefore, the test rejects H_0 if $T < t_0 = \theta_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$

Example 8: Normal Example with Known Mean Let $X_i \sim \mathcal{N}(\mu_0, \sigma^2)$ where σ^2 is unknown and μ_0 is known. Find the UMP level α test for testing $H_0 : \sigma^2 \leq \sigma_0^2$ vs. $H_1 : \sigma^2 > \sigma_0^2$. Let $T = \sum_{i=1}^n (X_i - \mu_0)^2$ is sufficient for σ^2.

To check whether T has MLR property, we need to find $g(t|\sigma^2)$.

$$
\begin{aligned}
\frac{X_i - \mu_0}{\sigma} &\sim \mathcal{N}(0,1) \\
\left(\frac{X_i - \mu_0}{\sigma}\right)^2 &\sim \chi_1^2 \\
Y &= T/\sigma^2 = \sum_{i=1}^n \left(\frac{X_i - \mu_0}{\sigma}\right)^2 \sim \chi_n^2 \\
f_Y(y) &= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} y^{\frac{n}{2}-1} e^{-\frac{y}{2}}
\end{aligned}
$$

$$
\begin{aligned}
f_T(t) &= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \left(\frac{t}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}} \left|\frac{dy}{dt}\right| \\
&= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \left(\frac{t}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}} \frac{1}{\sigma^2} \\
&= \frac{t^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{t}{2\sigma^2}} \\
&= h(t) c(\sigma^2) \exp[w(\sigma^2) t]
\end{aligned}
$$

where $w(\sigma^2) = -\frac{1}{2\sigma^2}$ is an increasing function in σ^2. Therefore, $T = \sum_{i=1}^n (X_i - \mu_0)^2$ has the MLR property.

By Karlin-Rubin Theorem, UMP level α rejects s H_0 if and only if $T > t_0$ where t_0 is chosen such that $\alpha = \Pr(T > t_0 | \sigma_0^2)$. Note that $\frac{T}{\sigma^2} \sim \chi_n^2$. Hence

$$\begin{aligned}
\Pr(T > t_0|\sigma_0^2) &= \Pr\left(\frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2}\bigg|\, \sigma_0^2\right) \\
\frac{T}{\sigma_0^2} &\sim \chi_n^2 \\
\Pr\left(\chi_n^2 > \frac{t_0}{\sigma_0^2}\right) &= \alpha \\
\frac{t_0}{\sigma_0^2} &= \chi_{n,\alpha}^2 \\
t_0 &= \sigma_0^2 \chi_{n,\alpha}^2
\end{aligned}$$

where $\chi_{n,\alpha}^2$ satisfies $\int_{\chi_{n,\alpha}^2}^{\infty} f_{\chi_n^2}(x)dx = \alpha$.

Example 9: Let $X_1, \cdots, X_n \sim \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known.
Consider testing $H_0 : \theta = \theta_0$ versus an alternative hypothesis.

1. When the alternative hypothesis is $H_1 : \theta_1 < \theta_0$, does UMP level α test exist? If yes, what is it?
2. When the alternative hypothesis is $H_1 : \theta_1 > \theta_0$, does UMP level α test exist? If yes, what is it?
3. When the alternative hypothesis is $H_1 : \theta_1 \neq \theta_0$, does UMP level α test exist? If yes, what is it?
4. Are the tests above unbiased?

$H_1 : \theta < \theta_0$

A level α test should satisfy $\Pr(\mathbf{X} \in R|\theta_0) \leq \alpha$.

As $\overline{X}$ is sufficient and its distribution has an MLR as shown in the previous example, by Karlin-Rubin Theorem, the rejection region of UMP level α test is

$$\overline{X} < -\frac{\sigma z_\alpha}{\sqrt{n}} + \theta_0$$

$H_1 : \theta > \theta_0$

As $\overline{X}$ is sufficient and its distribution has an MLR as shown in the previous example, by Karlin-Rubin Theorem, the rejection region of UMP level α test is

$$\overline{X} > \frac{\sigma z_\alpha}{\sqrt{n}} + \theta_0$$

$H_1 : \theta \neq \theta_0$

1. When $\theta < \theta_0$, $\beta_1(\theta) = \Pr(\mathbf{X} \in R_1) = \Pr\left(\overline{X} < -\frac{\sigma z_\alpha}{\sqrt{n}} + \theta_0\right)$ is the largest among level α tests.
2. If UMP level α test exists, the rejection region must be R_1 by the necessity condition of Neyman-Pearson Lemma.
3. When $\theta > \theta_0$, $\beta_2(\theta) = \Pr(\mathbf{X} \in R_2) = \Pr\left(\overline{X} > \frac{\sigma z_\alpha}{\sqrt{n}} + \theta_0\right)$ is the largest among level α tests.
4. Accordingly, $\beta_1(\theta)$ is not the power function of a UMP level α test.
5. Therefore, UMP level α test does not exist.

Are these tests unbiased?

Test based on $\overline{X} < -\frac{\sigma z_\alpha}{\sqrt{n}} + \theta_0$

1. When $\theta < \theta_0$, $\beta_1(\theta) > \beta_1(\theta_0)$.
2. When $\theta > \theta_0$, $\beta_1(\theta) < \beta_1(\theta_0)$.
3. Therefore, the test is not unbiased.

Test based on $\overline{X} > \frac{\sigma z_\alpha}{\sqrt{n}} + \theta_0$

1. When $\theta > \theta_0$, $\beta_2(\theta) > \beta_2(\theta_0)$.
2. When $\theta < \theta_0$, $\beta_2(\theta) < \beta_2(\theta_0)$.
3. Therefore, the test is not unbiased.

UMPU test

What is the optimal test for the two-sided test?

Consider a class of unbiased tests. Define a rejection region

$$|\overline{X} - \theta_0| > \frac{\sigma z_{\alpha/2}}{\sqrt{n}}$$

1. The test is unbiased. $\beta_3(\theta) > \beta_3(\theta_0)$ for all $\theta \neq \theta_0$.
2. The test is indeed the UMP test in the class of unbiased level α test.
3. This test is called a UMPU level α test.
4. Proving that the test is UMPU level α test is a little more complicated than UMP.

Example 8: Let $X_1, X_2, \ldots, X_n \sim Uniform(0, \theta)$. Consider testing

$$H_0 : \theta \leq \theta_0 \ vs.\ H_1 : \theta > \theta_0.$$

(a) Show that the family of $Uniform(0, \theta)$ has MLR in $X_{(n)}$.

(b) Find a size α UMP test for the above testing problem.

Example 9: Suppose that $X_1, \cdots, X_n$ are *i.i.d.* observations from Exponential(θ), and $Y_1, \cdots, Y_m$ are *i.i.d.* observations from Exponential(μ). Assume that $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ are independent between them.

(a) Find the LRT statistic of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$

(b) Show that the LRT from part (a) can be represented as a function of the following statistic T.

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i}$$

(Note that it is possible to construct a size α LRT using the fact that T follows a beta distribution under the null hypothesis.)

```
x=c(seq(0,1,by=0.0001))
z=(x^5)*((1-x)^15)
plot(x,z)
abline(h=.000002)
```