Lecture 5. CDFs

Wednesday, September 20, 2017 10:08 AM

Properties of probabilities

Continuity B_n a sequence of decreasing sets

$$P(B_n) \to P(\underbrace{\cap_n B_n}_{B = \lim B_n})$$

if B_n is increasing sequence of sets with limit $B = \bigcup_n B_n$ then

$$P(B_n) \to P(B)$$

Measures P defined on a collection of sets $\mathcal{A}$ can be uniquely extended to $\sigma(\mathcal{A})$

Carathéodory theorem

Random variable is a measurable function of ω

Random variables

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space

Consider $X(\omega)$ a function of ω:

$$\Omega \xrightarrow{X} \mathbb{R}$$

σ-algebras: $\mathcal{F}$ $\mathcal{B}$ Borel

DF if for any $B \in \mathcal{B}$ the set

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$$

then X is called measurable

We are going towards notation $P(X \in [a,b])$

$P(X \le x)$

DF Measurable function $X(\omega)$ is called a random variable

Mapped probability spaces

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, P_X)$$

$$\mathcal{F} \xrightarrow{P} [0,1] \qquad \mathcal{B} \xrightarrow{P_X} [0,1]$$

DF $$P_X(B) = P(\underbrace{\{\omega : X(\omega) \in B\}}_{A})$$

$$A = X^{-1}(B)$$

Will use P(smth that invokes X) notation meaning this is P_X

DF CDF
function
istribution
umulative

Take $(-\infty, x]$ intervals

$$F_X(x) = P_X((-\infty, x]) =$$

$$= P(X \le x) = P(\{\omega : X(\omega) \le x\})$$

Example:

$\Omega = [a, b]$

Experiment : choosing $\omega \in [a, b]$

Lebesgue measure = length = μ

$$P([a, x]) = \frac{\mu([a, x])}{\mu([a, b])}$$

$X(\omega) = \omega$

$\omega \in [a, b]$ $\qquad$ $\mathcal{B}$ built as $\sigma([a, x])$

$$F(x) = P(X \le x) = \frac{x-a}{b-a}, \qquad X, x \in [a,b]$$

$$X \sim \text{Uniform}([a,b]) \qquad F(x) = \underset{X}{P}([a,x])$$

$$\Omega = \mathbb{R}$$

$$F(x) = \begin{cases} 0, & x < a \\ (x-a)/(b-a), & a \le x \le b \\ 1, & x > b \end{cases}$$

$$F(x) = P_X((-\infty, x])$$

$$P_X(B) = \frac{\mu(B \cap [a,b])}{b-a}, \qquad B \in \mathcal{B} \text{ on } \mathbb{R}$$

Properties of CDFs

(1) Monotonicity

if $x_1, x_2 \in \mathbb{R}$ $\quad x_1 < x_2 \Rightarrow F(x_1) \le F(x_2)$

$$x_1 < x_2 \Rightarrow (X \le x_1 \Rightarrow X \le x_2)$$

$$\{\omega : X \le x_1\} \subset \{\omega : X(\omega) \le x_2\}$$

$$P\{\quad\downarrow\quad\} \le P\{\quad\downarrow\quad\}$$

$$F(x_1) \le F(x_2) \quad \square$$

(2) Limits of CDFs

$$\lim_{x \to -\infty} F(x) = 0$$

$$\lim_{x \to +\infty} F(x) = 1$$

Take $x_n \downarrow -\infty$ monotonically

$A_n = \{X \le x_n\}$ decreasing with the limit $\bigcap_n A_n = \emptyset$

By continuity of Probabilities

$$P_X(A_n) \to P_X(\emptyset) = 0$$

$$\overset{\shortparallel}{F(x_n)} \longrightarrow 0$$

F is also monotonic and bounded $\Rightarrow \lim F(x)$ exists $\Rightarrow \lim$ is the same for all sequences $x_n \to -\infty$ $\Rightarrow F(x) \underset{x \to -\infty}{\longrightarrow} 0$

Now $x \to +\infty$:

$x_n \uparrow +\infty$ monotonically

$B_n : \{X \le x_n\}$ increasing sets with the limit

$$\bigcup_n B_n = \mathbb{R} = \Omega$$

By continuity $F(x_n) \underset{n \to \infty}{\longrightarrow} P_X(\mathbb{R}) = 1$

F is bounded and monotonic $\Rightarrow \lim_{x \to \infty} F(x)$ exists

$\Rightarrow$ all sequences $F(\tilde{x}_n)$, $\tilde{x}_n \to \infty$ have same limit

$\Rightarrow \lim_{x \to \infty} F(x) = 1$

What about continuity?

3 $F(x) = P(X \le x)$ is right-continuous

Take: $x = x_0$

$A = \{X \le x_0\}$

$A_n = \{X \le x_n\}$ $x_n \downarrow x_0$, arbitrary

decreasing sets

$$\bigcap_n A_n = A$$

By continuity of $\mathbb{P}$: $\underset{X}{P}(A_n) \to \underset{X}{P}(A)$

$$\overset{\parallel}{F(x_n)} \longrightarrow \overset{\parallel}{F(x_0)}$$

$\lim_{x \downarrow x_0} F(x)$ exists b/c $F(x)$ is monotonic and bounded $\Rightarrow F(x_n)$ have same lim for any sequence $x_n \to x_0$

$$\Downarrow$$

$$F(x) \underset{x \to x_0}{\to} F(x_0)$$

$\square$

Note: Calculus: Monotonic functions can only have at most a countable # of discont. only of the jump type

DF Set is called countable if its elements can be mapped to $\mathbb{N}$

What about left-continuity?

Consider $\qquad x \le y$

$$F(y) - F(x-0) =$$

$$= \lim_{n\to\infty} \left(F(y) - F\left(x - \tfrac{1}{n}\right) \right) =$$

$$= \lim_{n\to\infty} \Big[\underset{X}{P}(\overset{B}{(-\infty, y]}) - \underset{X}{P}(\overset{A}{(-\infty, x - \tfrac{1}{n}]}) \Big] =$$

$A \subset B$

$P(B) = P(A) + P(\bar{A} \cdot B)$

$\Rightarrow P(B) - P(A) = P(\bar{A} \cdot B)$

$$= \lim_{n\to\infty} P_X\left(\left(x - \tfrac{1}{n}, y\right] \right) =$$

decr. sets

$$= \underset{X}{P}\left(\bigcap_{n=1}^{\infty} \left(x - \tfrac{1}{n}, y\right] \right) =$$

Continuity

$= P_X([x,y])$

Just proved that

$$P_X([x,y]) = P(x \le X \le y) = F(y) - F(x-0)$$

Corollary:

Take $x=y \Rightarrow$

$$P(X=x) = F(x) - F(x-0)$$

F is also left continuous iff $P(X=x)=0, \forall x$

X has no atoms

F has no jumps

For continuous r.v. X, F is continuous

TH CDF uniquely specifies the probability space

Suppose F is a function: ①, ②, ③ are satisfied

Take $\mathcal{F} = \sigma((\cdot,\cdot]) = \mathcal{B}$

Any $(\cdot,\cdot] = \bigcup_i (a_i, b_i]$ for some a_i, b_i

↓ disjoint

$P((a,b]) := F(b) - F(a)$

Caratheodory theorem, I can extend to the whole $\mathcal{B}$

Take $X(\omega) = \omega$

All properies of P work with this construction, except continuity

↓

goes w/o proof.

DF The probability space with $X(\omega)=\omega$ is called sample space