Problem 3

$$\pi(\lambda \mid y, \alpha, \beta) \propto \pi(\lambda \mid \alpha, \beta)\, \pi(y \mid \lambda, \alpha, \beta)$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{\frac{-\lambda}{\beta}}\right)\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right) \propto \lambda^{\alpha+y-1} e^{-\left(1+\frac{1}{\beta}\right)\lambda}$$

$$\sim \text{Gamma}\left(\alpha+y-1, \frac{1}{1+\frac{1}{\beta}}\right)$$

$$\pi(\alpha \mid \lambda, y, \beta) \propto \pi(\alpha \mid \beta)\, \pi(\lambda \mid \alpha, \beta)\, \pi(y \mid \lambda, \alpha, \beta)$$

$$= \left(\frac{1}{\Gamma(a) b^{a}} \alpha^{a-1} e^{\frac{-\alpha}{b}}\right)\left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{\frac{-\lambda}{\beta}}\right)\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right)$$

$$\propto \alpha^{a-1} e^{\frac{-\alpha}{b}} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1}$$

$$\pi(\beta \mid \lambda, y, \alpha) \propto \pi(\beta \mid \alpha)\, \pi(\lambda \mid \beta, \alpha)\, \pi(y \mid \lambda, \beta, \alpha)$$

$$= \left(\frac{1}{\Gamma(c) d^{c}} \beta^{c-1} e^{\frac{-\beta}{d}}\right)\left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \lambda^{\alpha-1} e^{\frac{-\lambda}{\beta}}\right)\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right)$$

$$\propto \beta^{c-1} e^{\frac{-\beta}{d}} \frac{1}{\beta^{\alpha}} e^{\frac{-\lambda}{\beta}}$$