Solution of BIOSTAT 802 HW3

Solution of Problem 1:
Let $\Theta_H = \{P_0\}$ and $\Theta_K = \{P_1, P_2\}$. The hypothesis of interest is

$$H : \Theta_H \quad \text{versus} \quad K : \Theta_K.$$

To begin we calculate likelihood ratios of $P_i(x)/P_0(x), i = 1, 2$, that are listed in the following table:

	1	2	3	4	5	6
$P_1(x)/P_0(x)$	2	2.5	4	2	∞	78/98
$P_2(x)/P_0(x)$	3	2.5	6	0	∞	72/98

From this table, it is clear that $x = 5$ leads to the action of rejection as the corresponding LRs is ∞.

(i) Consider the case of $\alpha = 0.01$. According to the **Neyman-Pearson Lemma**, we can construct an MP test $\phi_1(x)$ for $H' : P_0$ versus $K' : P_1$. That is, there exists a constant k_1 such that

$$\phi_1(x) = \begin{cases} 1, & P_1(x)/P_0(x) > k_1; \\ \gamma_1(x), & P_1(x)/P_0(x) = k_1; \\ 0, & P_1(x)/P_0(x) < k_1, \end{cases}$$

where the cutoff k_1 is so chosen that its size $E_0\phi_1(X) = 0.01$. Here we choose $k_1 = 4$, and must take $\gamma_1(x) = 1/2$ (at $x = 3$), so $x = 3$ leads to the 50% chance of rejection! In this case, the size (or the probability of type I error) is, under P_0, 0.00 + (1/2) (0.02) = 0.01, as desired. In summary, the critical function of the MP test is given by

$$\phi_1(x) = \phi_2(x) = 1(x = 5) + (1/2)1(x = 3),$$

where 1(.) is the indicator function.

Following the above steps, similarly we can construct an MP test for $H'' : P_0$ against $K'' : P_2$. The N-P Lemma ensures that there exists a constant k_2 such that the test $\phi_2(x)$ takes the following form:

$$\phi_2(x) = \begin{cases} 1, & P_2(x)/P_0(x) > k_2; \\ \gamma_2(x), & P_2(x)/P_0(x) = k_2; \\ 0, & P_2(x)/P_0(x) < k_2, \end{cases}$$

where the cutoff k_2 is so chosen that its size $E_0\phi_2(X) = 0.01$. Here we choose $k_2 = 6$, and must take $\gamma_2(x) = 1/2$ (at $x = 3$), so $x = 3$ leads to the 50% chance of rejection! In this case, the size (or the probability of type I error) is 0.00 + (1/2) (0.02) = 0.01, as desired. In summary, the critical function of the MP test $\phi_2(x)$ is given by

$$\phi_2(x) = 1(x = 5) + (1/2)1(x = 3).$$

Although k_1 and k_2 are different, the rejection rule is the same $\phi_1(x) = \phi_2(x)$ on any data x. In other words, the rejection rule is uniform over the sample space regardless of alternative P_1 or P_2. Thus, the UMP test of P_0 against P_1 and P_2 is of the form

$$\phi(x) = 1(x = 5) + (1/2)1(x = 3).$$

(ii) Similarly, for $\alpha = 0.05$, the most powerful tests of P_0 against P_1 and P_2 take the following forms, respectively,

$$\phi_1(x) = 1(x \in \{2, 3, 5\}) + \gamma_1(x)1(x \in \{1, 4\}),$$
$$\phi_2(x) = 1(x \in \{2, 3, 5\}),$$

where $\gamma_1(x) \neq 0$ in order to satisfy the size condition $E_0\phi_1(X) = 0.05$. Because $\phi_1(x) \neq \phi_2(x)$ on the sample space, these two MP tests cannot be unified, and thus there is no UMP test for the hypothesis H versus K.

(iii) For $\alpha = 0.07$, the most powerful tests of P_0 against P_1 and P_2 take the following forms, respectively,

$$\phi_1(x) = 1(x \in \{2, 3, 5\}) + \gamma(x)1(x \in \{1, 4\}),$$
$$\phi_2(x) = 1(x \in \{1, 2, 3, 5\}),$$

where $\gamma(x) = 1(x = 1)$. Although $\phi_1(x)$ and $\phi_2(x)$ have different analytic forms, they are effectively the same critical function defining the same rejection region. Thus, there is a UMP test for the hypothesis H versus K, that is, $\phi(x) = 1(x \in \{1, 2, 3, 5\})$.

Solution of Problem 2

(a) $\text{Prob(error)} = P(\text{accept } H_0 \mid H_1) P(H_1) + P(\text{reject } H_0 \mid H_0) P(H_0)$

$$= \pi E_0 \psi(X) + (1-\pi) E_1 [1 - \psi(X)].$$

(b) Let $\pi_0 = \pi$ and $\pi_1 = 1-\pi$. The above prob. of error may be expressed as follows.

$$\text{Prob(error)} = \pi_1 \int_{\mathcal{X}} (1-\psi(x))\, dP_1(x) + \pi_0 \int_{\mathcal{X}} \psi(x)\, dP_0(x)$$

$$= \pi_1 - \pi_1 \int \psi(x) p_1(x)\, dx + \pi_0 \int \psi(x) p_0(x)\, dx$$

$$= \pi_1 + \int_{\mathcal{X}} \psi(x) \{ \pi_0 p_0(x) - \pi_1 p_1(x) \}\, dx,$$

where π_1 is a constant independent data x and decision rule $\psi(\cdot)$.

Set $\mathcal{X}_1 = \{x : \pi_0 p_0(x) - \pi_1 p_1(x) > 0\} = \{x : \frac{\pi_0}{\pi_1} p_0(x) > p_1(x)\}$

$\mathcal{X}_2 = \{x : \pi_0 p_0(x) - \pi_1 p_1(x) = 0\} = \{x : p_1 = \frac{\pi_0}{\pi_1} p_0(x)\}$

$\mathcal{X}_3 = \{x : \pi_0 p_0(x) - \pi_1 p_1(x) < 0\} = \{x : p_1 > \frac{\pi_0}{\pi_1} p_0(x)\}$,

and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3$, and they are disjoint sets. Since $0 \le \psi(x) \le 1$,

$$\psi(x) = \arg\min_{0 \le \psi \le 1} \int \psi(x) \{ \pi_0 p_0(x) - \pi_1 p_1(x) \}\, dx = \begin{cases} 1, & x \in \mathcal{X}_3; \\ 0, & x \in \mathcal{X}_1, \end{cases}$$

namely,

$$\psi(x) = \begin{cases} 1, & \text{if } p_1(x) > \frac{\pi_0}{\pi_1} p_0(x) = k\, p_0(x), \\ 0, & \text{if } p_1(x) < \frac{\pi_0}{\pi_1} p_0(x) = k\, p_0(x), \end{cases} \qquad k = \frac{\pi_0}{\pi_1}$$

(c) $P(H_0 \mid X = x) = \dfrac{P(X=x \mid H_0) P(H_0)}{P(X=x)} = \dfrac{p_0(x)\pi_0}{\pi_0 p_0(x) + \pi_1 p_1(x)}$, by the Bayes formula.

Likewise $P(H_1 \mid X = x) = \dfrac{p_1(x)\pi_1}{\pi_0 p_0(x) + \pi_1 p_1(x)}$.

Given $X = x$, reject H_0 if $p_1(x) > \frac{\pi_0}{\pi_1} p_0(x) \iff P(H_1 \mid X=x) > P(H_0 \mid X=x)$

while accept H_0 if $p_1(x) < \frac{\pi_0}{\pi_1} p_0(x) \iff P(H_0 \mid X=x) > P(H_1 \mid X=x)$.

This means that this Bayes test makes decision according to larger ~~posterior~~ posterior probability.

Solution of Problem 3 :

Since $T(x)$ is a sufficient statistic, the conditional distribution of $X|T$ is independent of parameter $\theta \in \Theta$. That is, $\phi(t) = E[\psi(X)|t]$ is parameter θ free. Also, $0 \leq \psi(X) \leq 1$, so,

$$0 \leq \phi(t) \leq 1.$$

Therefore, $\phi(t)$ defines a critical function, or a test. Now, look at the power function of $\phi(t)$, because of the property of conditional expectation

$$E_\theta \phi(T) = E_\theta\{E[\psi(X)|T]\} = E_\theta \psi(X),$$

i.e. the two tests have the same power function.