Proofs of Neyman-Pearson Lemma, Corollary 1, Karlin-Rubin Theorem, Theorem 2 of SPRT and Wald's indentity

February 13, 2018

Please let me know of any typos.

1 Neyman-Pearson Lemma

First, look at the trivial case with $\alpha = 0$ or 1. Let

$$\phi(x) = \begin{cases} 1, & \frac{p_1(x)}{p_0(x)} > \infty \text{ (or 0)} \\ 0, & \frac{p_1(x)}{p_0(x)} < \infty \text{ (or 0)} \end{cases}$$

Then

$$E_0\phi(x) = \int_{\mathscr{X}} p_0(x)\phi(x)d\mu(x) = \int_{\{x:\phi(x)=1\}} p_0(x)d\mu(x) = 0 \text{ (or 1)}$$

The size of test $\phi(x)$ is 0 (or 1).

Moreover, the power is:

$$\beta_\phi(\theta_1) = E_1\phi(x) = \int_{\mathscr{X}} p_0(x)\phi(x)d\mu(x) = \int_{\{x:\phi(x)=1\}} p_0(x)d\mu(x) = 0 \text{ (or 1)}$$

The power of test $\phi(x)$ is 0 (or 1).

Consider an arbitrary test $\phi'(x)$ with size 0 (or 1), then this test would always accept (or reject) the null hypothesis. Then the type 2 error would always be 1 (or 0), and the power would always be 0 (or 1). Therefore, for $\alpha = 0$ or 1, we have

$$\beta_{\phi}(\theta_1) \geq \beta_{\phi'}(\theta_1)$$

Based on definition, $\phi(x)$ is a UMP level 0 (or 1) test.

Second, in the followings we only consider the case where $0 < \alpha < 1$.

Part (i):

Define

$$h(c) = P_0(\frac{p_1(x)}{p_0(x)} \leq c), \ \text{c} \geq 0$$

Since $h(c)$ is nondecreasing, right-continuous with $\lim_{c\to\infty} h(c) = 1$, then $h(c)$ as a CDF of random variable $\frac{p_1(x)}{p_0(x)}$.

(a) If $h(0) < 1-\alpha$, then there exists a constant k such that $h(k^-) \leq 1-\alpha$ and $h(k) \geq 1-\alpha$ due to the monotonicity.

(a.1) If $h(k) = 1-\alpha$ then

$$h(k) = P_0(\frac{p_1(x)}{p_0(x)} \leq k) = 1-\alpha$$

$$P_0(\phi(x) = 0) = 1-\alpha$$

$$P_0(\phi(x) = 1) = \alpha$$

(a.2) If $h(k) > 1-\alpha$ then define

$$\phi(x) = \begin{cases} 1, & \frac{p_1(x)}{p_0(x)} > k \\ \frac{\alpha-(1-h(k))}{h(k)-h(k^-)} & \frac{p_1(x)}{p_0(x)} = k \\ 0, & \frac{p_1(x)}{p_0(x)} < k \end{cases}$$

So,

$$E_0\phi(x) = \int_{\mathscr{X}} p_0(x)\phi(x)d\mu(x)$$

$$= \int_{<k} 0 + \int_{=k} \frac{\alpha-(1-h(k))}{h(k)-h(k^-)} + \int_{>k} 1 \ p_0(x)d\mu(x)$$

$$= P_0(\frac{p_1(x)}{p_0(x)} = k)\frac{\alpha - (1-h(k))}{h(k)-h(k^-)} + P_0(\frac{p_1(x)}{p_0(x)} > k)$$
$$= 1 - h(k) + (h(k) - h(k^-))\frac{\alpha - (1-h(k))}{h(k)-h(k^-)} = \alpha$$

(b) If $h(0) = 1-\alpha$, then take $k = 0$ in the proof of (a.1).

(c) If $h(0) > 1-\alpha$, then take $k = 0$ in the proof of (a.2).

Part(ii).

Let ϕ be a test satisfying (7) and (8), $\tilde{\phi}$ be an arbitrary level α test, i.e., $E_0\tilde{\phi}(x) \leq \alpha$.

Define

$$S^+ = \{x : \phi(x) > \tilde{\phi}(x)\} \text{ and } S^- = \{x : \phi(x) < \tilde{\phi}(x)\}$$

Because of the definition (7) of test ϕ, we know that on S^+, $\phi(x) = 1(>0)$, namely $\frac{p_1(x)}{p_0(x)} \geq k$, $k \geq 0$, and on S^-, $\phi(x) = 0(<1)$, namely $\frac{p_1(x)}{p_0(x)} \leq k$, $k \geq 0$.

Thus, on $S^+ \cup S^-$, we always have

$$\{\phi(x) - \tilde{\phi}(x)\}\{p_1(x) - kp_0(x)\} \geq 0$$

If follows that

$$\int_{\mathscr{X}} \{\phi(x) - \tilde{\phi}(x)\}\{p_1(x) - kp_0(x)\}d\mu(x) \geq 0$$

$$E_1\phi(x) - E_1\tilde{\phi}(x) = \int_{\mathscr{X}} \{\phi(x) - \tilde{\phi}(x)\}p_1(x)d\mu(x) \geq k\int_{\mathscr{X}} \{\phi(x) - \tilde{\phi}(x)\}p_0(x)d\mu(x)$$
$$= k\{E_0\phi(x) - E_0\tilde{\phi}(x)\} \geq 0$$

Therefore, $\phi(x)$ is a UMP level α test.

Part(iii)

First, we prove the first statement. Let $\tilde{\phi}$ be an arbitrary level α UMP test. According to part(i), there exists a test ϕ that satisfies (7) and (8). Using the same notations in part(ii) for S^+ and S^-, we have the union $S = S^+ \cup S^-$. Also, denote $S_1 = S \cap \{x : \frac{p_1(x)}{p_0(x)} \neq k\}$.

Suppose that $\mu(S_1) > 0$, we have on S_1

$$\{\phi(x) - \tilde{\phi}(x)\}\{p_1(x) - kp_0(x)\} > 0$$

From the arguments of part(ii), we would have $E_1\phi(x) > E_1\tilde{\phi}(x)$. This contradicts to the fact that $\tilde{\phi}(x)$ is a level α UMP test. This implies that $\mu(S_1) = 0$. In other words, we must have $\phi(x) = \tilde{\phi}(x)$ on the set $\{x : \phi(x) = 0\} \cup \{x : \phi(x) = 1\}$. That is, $\tilde{\phi}(x)$ takes the form of (7) a.e. μ.

Second, we prove the second statement by contradiction. Suppose $\tilde{\phi}(x)$ is a level α UMP test with $E_1\tilde{\phi}(x) < 1$ but $E_0\tilde{\phi}(x) < \alpha$.
Set

$$\phi(x) = \min\{1, \tilde{\phi}(x) + \alpha - E_0\tilde{\phi}(x)\}$$

Then $E_0\phi(x) \leq E_0\{\tilde{\phi}(x) + \alpha - E_0\tilde{\phi}(x)\} = \alpha$. This implies that ϕ is a level α test. Note that for all $x \in \mathscr{X}$, $\phi(x) \geq \tilde{\phi}(x)$.

Consider the power of test ϕ

$$E_1\phi(x) = \int_{\{x:\tilde{\phi}(x)=1\}} 1 + \int_{\{x:1-(\alpha-E_0\tilde{\phi}(x))\leq\tilde{\phi}(x)<1\}} 1 + \int_{\{x:\tilde{\phi}(x)<1-(\alpha-E_0\tilde{\phi}(x))\}} (\tilde{\phi}(x)+\alpha-E_0\tilde{\phi}(x))p_1(x)d\mu(x)$$

$$\geq \int_{\{x:\tilde{\phi}(x)=1\}} 1 + \int_{\{x:1-(\alpha-E_0\tilde{\phi}(x))\leq\tilde{\phi}(x)<1\}} \tilde{\phi}(x) + \int_{\{x:\tilde{\phi}(x)<1-(\alpha-E_0\tilde{\phi}(x))\}} \tilde{\phi}(x)p_1(x)d\mu(x) = E_1\tilde{\phi}(x)$$

The above equality would achieve if both $\{x : 1 - (\alpha - E_0\tilde{\phi}(x)) \leq \tilde{\phi}(x) < 1\}$ and $\{x : \tilde{\phi}(x) < 1 - (\alpha - E_0\tilde{\phi}(x))\}$ are empty sets. However, in this case, $\tilde{\phi}(x)$ would be 1 a.e. μ, and $E_1\tilde{\phi}(x)$ would be 1 (contradiction). Therefore, we would have $E_1\phi(x) > E_1\tilde{\phi}(x)$. This contradicts to the fact that $\tilde{\phi}(x)$ is a level α UMP test. Then $E_0\tilde{\phi}(x)$ has to be α.

2 Corollary 1

First note that $\phi_1(x) \equiv \alpha$ is a size α test with the power $E_1\phi_1(x) = \alpha$. Let ϕ be a level α UMP test, and we would have $E_1\phi(x) \geq E_1\phi_1(x) = \alpha$.

Second, if $E_1\phi(x) = \alpha$, then $E_1\phi(x) = E_1\phi_1(x) = \alpha$. Since ϕ is a level α UMP test, then ϕ_1 is also a level α UMP test. Also N-P (iii) implies that ϕ_1 must have the form in equation (7). However, since $0 < \alpha < 1$, i.e., $\phi_1(x) \neq 0$, $\phi_1(x) \neq 1$, then

$$\frac{p_1(x)}{p_0(x)} = k, \text{ a.e.}\mu$$

$$\Leftrightarrow \int_{\mathscr{X}} p_1(x)d\mu(x) = k \int_{\mathscr{X}} p_0(x)d\mu(x) \Leftrightarrow k = 1 \text{ a.e.}\mu \Leftrightarrow p_1(x) = p_0(x) \text{ a.e.}\mu$$

It contradicts to the fact that $P_0 \neq P_1$. Therefore, in this case, the power $\beta > \alpha$.

3 Karlin-Rubin theorem

Part(i)

The existence of $\phi(x)$ in (9) satisfying (10) can be proved by following the arguments given in the proof of part(i) in Neyman-Peason Lemma, where $\frac{p_1(x)}{p_0(x)}$ is replaced by $T(x)$ and constant c is chosen by $P_0(T(x) > c) \leq \alpha \leq P_0(T(x) \geq c)$.

Now we show that a test satisfying (9) and (10) is a level α UMP test. Taking an arbitrary $\theta_1 \in \Theta_k \subset \Theta$, $\theta_1 > \theta_0$. Consider the following simple hypothesis:

$$H' : \theta = \theta_0 \text{ against } K' : \theta = \theta_1$$

Denote

$$k \equiv \frac{f(x;\theta_1)}{f(x,\theta_0)}|_{T(x)=c}$$

Then, the MLR ensures that a test in (9) can be rewritten as

$$\phi(x) = \begin{cases} 1, & \frac{f(x;\theta_1)}{f(x;\theta_0)} > k \\ 0, & \frac{f(x;\theta_1)}{f(x;\theta_0)} < k \end{cases}$$

Because of the existence proved above, we know that ϕ is a level α test. By Neyman-Pearson lemma part(ii), the above test ϕ must be a level α UMP test for H' against K' (For testing H against K, ϕ should be constructed independent of θ_1).

Denote $\beta_\phi(\theta)$ the power function of ϕ. If we can show part(ii), the proof of this part is done, i.e., ϕ is level α UMP test for H against K. This is because

$$\beta_\phi(\theta) \leq \beta_\phi(\theta_0) = \alpha, \theta \in \Theta_H \text{ } (\phi \text{ is a size } \alpha \text{ test})$$

$$\beta_\phi(\theta) \geq \beta_{\tilde{\phi}}(\theta), \theta \in \Theta_K \text{ (because } \theta_1 \text{ is arbitrary in the above proof)}$$

Part(ii)

To show the monotonicity of $\beta_\phi(\theta)$, we need to show $\beta_\phi(\theta_1) \leq \beta_\phi(\theta_2)$ for any $\theta_1 < \theta_2$, $\theta_1, \theta_2 \in \Theta$. Consider a hypothesis test problem:

$$H'' : \theta = \theta_1 \text{ against } K'' : \theta = \theta_2$$

By Neyman-Pearson Lemma part(ii), ϕ is a level $\beta_\phi(\theta_1)$ UMP test. Moreover, Corollary 1 implies that $\beta_\phi(\theta_1) \leq \beta_\phi(\theta_2)$. This completes the proof of $\beta_\phi(\theta)$ being a nondecreasing function.

Furthermore, when $\theta_1 \neq \theta_2, P_{\theta_1} \neq P_{\theta_2}$, as long as on the set $\{\theta_1 : 0 < \beta_\phi(\theta_1) < 1\}$, we must have $\beta(\theta_1) < \beta(\theta_2)$. This completes the proof of part(ii) as well as part(i).

Part(iii)

Let $\tilde{\phi}$ be any test satisfying $E_{\theta_0}\tilde{\phi}(x) = \alpha$, and ϕ be the test defined in (9). We want to prove that $\beta_\phi(\theta) \leq \beta_{\tilde{\phi}}(\theta), \forall \theta < \theta_0$.

For an arbitrary $\theta_1 < \theta_0$, consider a hypothesis problem:

$$H''' : \theta = \theta_1 \text{ against } K''' : \theta = \theta_0$$

Then the test ϕ defined in (9) is a level $E_{\theta_1}\phi(x)$ UMP test, and its power is $\beta_\phi(\theta_0)$. Note that $E_{\theta_0}\tilde{\phi}(x) = \alpha = \beta_\phi(\theta_0)$. We claim that $E_{\theta_1}\tilde{\phi}(x) \geq E_{\theta_1}\phi(x)$. If not, $E_{\theta_1}\tilde{\phi}(x) < E_{\theta_1}\phi(x)$, then $\tilde{\phi}$ is a level $E_{\theta_1}\phi(x)$ UMP test, because $E_{\theta_0}\tilde{\phi}(x) = \alpha = \beta_\phi(\theta_0)$ with ϕ being a level a level $E_{\theta_1}\phi(x)$ UMP test. Due to the fact that $\beta_{\tilde{\phi}}(\theta_0) = \alpha < 1$, Neyman-Pearson Lemma (iii) implies that

$$E_{\theta_1}\tilde{\phi}(x) = \beta_\phi(\theta_1) \text{ contradiction}$$

In other words, there does not exist a test $\tilde{\phi}$ that satisfies $E_{\theta_0}\tilde{\phi}(x) = \alpha$ and $\beta_{\tilde{\phi}}(\theta_1) < \beta_\phi(\theta_1)$. Therefore, among all tests satisfying (10), the test defined in (9) minimizes $\beta(\theta)$ for any $\theta < \theta_0$.

4 Theorem 2 of SPRT

Let $Z = log(\frac{f_{\theta_2}(\mathbf{x})}{f_{\theta_1}(\mathbf{x})})$, and $Z_i = log(\frac{f_{\theta_2}(x_i)}{f_{\theta_1}(x_i)})$, $i = 1, 2, ..., n$.

Let $a = logA$, $b = logB$, then $a < 0 < b$.

Let $c = b - a$.

First let us show that there exists an integer r such that when r is large enough,

$$P_\theta(|\sum_{i=1}^{r} Z_i| \le c) = P_{\theta,r} < 1$$

Case(i)

If $E_\theta Z^2 = \infty$, then $r = 1$ is an obvious choice. This is because

$$P_\theta(|Z_1| \le c) = P_\theta(Z_1^2 \le c^2) = p < 1$$

Otherwise, if $P_\theta(Z_1^2 \le c^2) = 1$ then $E_\theta Z^2 \le c^2 < \infty$, contradiction.

Case(ii) If $E_\theta Z^2 = 0$, then $Z = 0$ a.e. μ. However, this contradicts to condition (ii).

Case(iii) If $E_\theta Z^2 < \infty$, then by Cauchy-Schwarz inequality we would get $E_\theta|Z| < \infty$. Denote $\sigma^2 \equiv var(Z)$ and $\mu \equiv E_\theta(Z)$. Then by CLT:

$$\lim_{r\to\infty} P_\theta(|\sum_{i=1}^{r} Z_i| \le c) = \lim_{r\to\infty} P_\theta(-\frac{c}{\sqrt{r}} \le \frac{1}{\sqrt{r}}\sum_{i=1}^{r} Z_i \le \frac{c}{\sqrt{r}})$$

$$= \lim_{r\to\infty} P_\theta(\frac{1}{\sqrt{r}}\sum_{i=1}^{r} Z_i \le \frac{c}{\sqrt{r}}) - \lim_{r\to\infty} P_\theta(\frac{1}{\sqrt{r}}\sum_{i=1}^{r} Z_i < -\frac{c}{\sqrt{r}})$$

$$= \lim_{r\to\infty} \Phi(\frac{c}{\sqrt{r}\sigma} - \frac{\sqrt{r}d}{\sigma}) - \lim_{r\to\infty} \Phi(-\frac{c}{\sqrt{r}\sigma} - \frac{\sqrt{r}d}{\sigma}) = 0$$

This proves the above claim.

Now, for an integer $m > 0$,

$$\{N > mr\} = \{\text{the number of draws to stop sampling} > mr\}$$

$$= \{Z_1 \in (a,b), Z_1 + Z_2 \in (a,b), ..., \sum_{i=1}^{mr} Z_i \in (a,b)\}$$

$$\subset \{|\sum_{i=1}^{r} Z_i| \le c, |\sum_{i=r+1}^{2r} Z_i| \le c, ..., |\sum_{i=(m-1)r+1}^{mr} Z_i| \le c\}$$

Thus,

$$P_\theta(N > mr) \le \prod_{j=1}^{m} P_\theta(|\sum_{i=(j-1)r+1}^{jr} Z_i| \le c) = P_{\theta,r}^m \to 0, \text{ as } m \to \infty$$

This implies that $P_\theta(N < \infty) = 1$.

5 Wald's identity

$$E(\sum_{i=1}^{N} X_i) = \sum_{n=1}^{\infty} P(N = n) E(\sum_{i=1}^{n} X_i | N = n)$$

$$= \sum_{n=1}^{\infty} P(N = n) \sum_{i=1}^{n} E(X_i | N = n)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(N = n) E(X_i | N = n)$$

$$= \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} P(N = n) E(X_i | N = n)$$

$$= \sum_{i=1}^{\infty} P(N \ge i) E(X_i | N \ge i)$$

Note that $\{N \ge i\} \equiv \{N < i\}^c$, where $\{N < i\} \in \mathscr{B}(X_1, ..., X_{i-1})$, so $\{N \ge i\} \in \mathscr{B}(X_1, ..., X_{i-1})$. Since $X_1, X_2, ...$ are iid random variables, X_i is independent of $\{N \ge i\}$. Therefore,

$$E(\sum_{i=1}^{N} X_i) = \sum_{i=1}^{\infty} P(N \ge i) E(X_i) = E(X) \sum_{i=1}^{\infty} P(N \ge i)$$

$$= E(X) \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} P(N = j) = E(X) \sum_{j=1}^{\infty} \sum_{i=1}^{j} P(N = j) = E(X) \sum_{j=1}^{\infty} j P(N = j) = E(X)E(N)$$