Lecture 9. Properties of cond exp

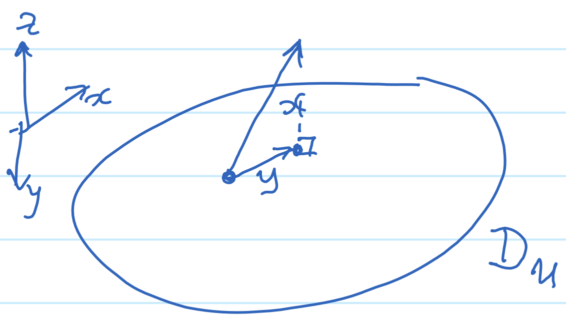
Wednesday, October 4, 2017 9:41 AM

$X \qquad (\Omega, \mathcal{F}, P)$

Y measurable on $\mathcal{U} \subset \mathcal{F}$

$$\mathbb{E}(Y;A) = \int_A Y\,dP = \mathbb{E}(X;A), \quad \forall A \in \mathcal{U}$$

$$Y = \mathbb{E}(X|\mathcal{U})$$



$x^T \cdot y$

$\|x\| = \sqrt{x^T \cdot x}$

$$x = a_1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + a_2 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + a_3 \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$$

basis vectors

$$y = \arg\min_{a \in D_\mathcal{U}} \underbrace{(x-a)^T \cdot (x-a)}_{\|x-a\|^2}$$

distance between x and a

Characteristic features of projections

$$x - y \perp a, \quad \forall a \in D_\mathcal{U}$$

Mapped the linear algebra understanding to functional spaces

Functional Analysis

Instead of vectors x we have functions $X(\omega)$
instead of dot-product we have
scalar product

$$\langle X, Y \rangle = \mathbb{E}(X \cdot Y) = \int X \cdot Y \, dP$$

A_k, $P(A_k) = \Delta P$

$$\int X \cdot Y \, dP \approx \underbrace{\left(\sum_k x_k \cdot y_k\right)}_{x^T \cdot y} \cdot \Delta P$$

$x = \begin{pmatrix} x_1 \\ \vdots \end{pmatrix}$

$y = \begin{pmatrix} y_1 \\ \vdots \end{pmatrix}$

Example $D_{\mathcal{U}}$ was the space of simple functions

Definition $\mathbb{E}(Y; A) = \mathbb{E}(X; A)$

is equivalent to

$$x - y \perp a, \quad \forall a \in D_{\mathcal{U}}$$

written for r.v. X, Y

Kantorovich, Akilov Functional Analysis

Pergamon Press 1982 goes back to 1959

Properties of conditional expectations

1) Linearity

$$\mathbb{E}(a + b \cdot X \mid \mathcal{U}) = a + b \cdot \mathbb{E}(X \mid \mathcal{U}) \quad (*)$$

Proof:

$$\exists \; \tilde{Y} = a + b \cdot \underbrace{\mathbb{E}(X \mid \mathcal{U})}_{Y}$$

By def. Y is $\mathcal{U}$-measurable

$\tilde{Y}^{-1}((c, d)) = \{\omega : c < \tilde{Y} < d\} =$

$$= \left\{ \omega : \frac{c-a}{b} < Y < \frac{d-a}{b} \right\} =$$

w/o loss of generality consider $b > 0$

$$= Y^{-1}\left(\left(\frac{c-a}{b}, \frac{d-a}{b}\right)\right) \in \mathcal{U} \quad b/c\ Y \text{ is } \mathcal{U}\text{-measurable}$$

$$\underline{\mathbb{E}(\tilde{Y}; A)} = \int_A \tilde{Y}\, dP \underset{\text{by lin of } \int}{=} a + b \cdot \int_A Y\, dP =$$

$$\underset{\text{by def of } \mathbb{E}(X|\mathcal{U})}{=} a + b \cdot \int_A X\, dP =$$

$$\underset{\text{lin of } \int}{=} \int_A (a + bX)\, dP = \underline{\mathbb{E}\{a + bX ; A\}}$$

$$\Downarrow$$

$$(*) \qquad \tilde{Y} = \mathbb{E}(a + bX \mid \mathcal{U}) \qquad \square$$

DF Multivariate random variable $(X_1, \dots, X_n)$ is function $\Omega \to \mathbb{R}^n$, where $X_1, \dots, X_n$ are r.v. defined on a common measurable space $(\Omega, \mathcal{F})$

Associated with $\mathbb{R}^n$ is a σ-algebra $\mathcal{B}^n$ of Borel subsets of $\mathbb{R}^n$

$X = (X_1, \dots, X_n)$ is measurable with

$P_X(B) = P(\{\omega : X(\omega) \in B\})$, $B \in \mathcal{B}^n$, joint probability measure

Will use sample space so that $X^{-1}(B) \in \mathcal{B}^n$

2) $\exists$ X_1, X_2 two r.v. on common measurable space $(\Omega, \mathcal{F})$

$$X = (X_1, X_2) : \Omega \xrightarrow{X} \mathbb{R}^2$$

$$P_X(B) = \mathbb{P}(X \in B) = \iint_B X \, dP, \quad B \in \mathcal{B}^2$$

$$\underbrace{\mathbb{E}(X_1 + X_2 \mid \mathcal{U})}_{\tilde{Y}} = \underbrace{\mathbb{E}\{X_1 \mid \mathcal{U}\}}_{Y_1} + \underbrace{\mathbb{E}\{X_2 \mid \mathcal{U}\}}_{Y_2}, \quad \mathcal{U} \subset \mathcal{B}^n$$

Proof:

$$\overset{\iint_A Y_1+Y_2 \, dP}{\overset{\parallel}{\mathbb{E}(Y_1 + Y_2 ; A)}} \underset{\text{lin of } \iint}{=} \underbrace{\overset{\iint_A Y_1 dP}{\overset{\parallel}{\mathbb{E}(Y_1 ; A)}}}_{\mathbb{E}(X_1 ; A)} + \underbrace{\overset{\iint_A Y_2 dP}{\overset{\parallel}{\mathbb{E}(Y_2 ; A)}}}_{\mathbb{E}(X_2 ; A)} =$$

$$= \mathbb{E}(X_1 + X_2 ; A) \quad \square$$

3) $\exists$ $X_1 \leq X_2$ a.s. $\Rightarrow$ $\underbrace{\mathbb{E}(X_1 \mid \mathcal{U})}_{Y_1} \leq \underbrace{\mathbb{E}(X_2 ; \mathcal{U})}_{Y_2}$ a.s.

Proof:

$$\mathbb{E}(Y_1 ; A) = \iint_A Y_1 \, dP = \mathbb{E}(X_1 ; A) \leq \mathbb{E}(X_2 ; A) =$$

$$= \mathbb{E}(Y_2 ; A)$$

$$\Rightarrow \mathbb{E}(Y_2 ; A) - \mathbb{E}(Y_1 ; A) \geq 0$$

$$\mathbb{E}(Y_2 - Y_1 ; A) \geq 0$$

$$\iint_A \underbrace{(Y_2 - Y_1)}_{\mathcal{U}\text{-measurable}} dP \geq 0, \quad \forall A \in \mathcal{U}$$

if $\exists$ $B : P(B) > 0$ & $Y_2 - Y_1 < 0$ on B

then we would have $\iint_B (Y_2 - Y_1) dP < 0$

contradicts

then we would have $\iint_B (Y_2 - Y_1) dP < 0$ contradicts

$\Rightarrow Y_2 \geq Y_1$ a.s. $\square$

DF Conditional probability

$$P(A|\mathcal{U}) := \mathbb{E}\{I_A|\mathcal{U}\}$$

Also, $\mathbb{P}(A) = \mathbb{E}\{I_A\}$ when $\mathcal{U} = \mathcal{F}$

4) Chebyshev's inequality

$$\exists \; \boxed{X \geq 0, \quad x > 0 \Rightarrow P(X \geq x|\mathcal{U}) \leq \frac{\mathbb{E}(X|\mathcal{U})}{x}}$$

Proof:

$$\mathbb{P}(X \geq x|\mathcal{U}) := \mathbb{E}(I_{X \geq x}|\mathcal{U}) \leq \mathbb{E}\left(\frac{X}{x}|\mathcal{U}\right) = \frac{\mathbb{E}(X|\mathcal{U})}{x}$$

$$I_{X \geq x} = \begin{cases} 1, & X \geq x \\ 0, & X < x \end{cases} \leq \begin{cases} \frac{X}{x}, & X \geq x \\ \frac{X}{x}, & X < x \end{cases} = \frac{X}{x}$$

$\square$

$$\boxed{\exists f \uparrow, \; f \geq 0 \Rightarrow P(X \geq x|\mathcal{U}) \leq \frac{\mathbb{E}(f(X)|\mathcal{U})}{f(x)}}$$

Proof

$$\mathbb{P}(X \geq x|\mathcal{U}) = \mathbb{P}(\underbrace{f(X)}_{\text{new}X} \geq \underbrace{f(x)}_{\text{new}x}|\mathcal{U}) \leq \frac{\mathbb{E}(f(X)|\mathcal{U})}{f(x)}$$

$\square$

$\exists\, f(\cdot) = [(\cdot) - Y]^2$, $\quad Y = \mathbb{E}(X|\mathcal{U})$

then

$$P\left(|X - \mathbb{E}(X|\mathcal{U})| \geq x \,|\, \mathcal{U}\right) \leq \frac{\operatorname{Var}(X|\mathcal{U})}{x^2}$$

DF $\quad \operatorname{Var}(X|\mathcal{U}) = \mathbb{E}(X^2|\mathcal{U}) - \mathbb{E}^2(X|\mathcal{U})$

Proof:

$$P\left(|X - \mathbb{E}(X|\mathcal{U})| \geq x \,|\, \mathcal{U}\right) =$$

$$= P\Big(\underbrace{(X - \mathbb{E}(X|\mathcal{U}))^2}_{\text{new } X} \geq \underbrace{x^2}_{\text{new } x} \,\Big|\, \mathcal{U}\Big) \leq \frac{\mathbb{E}\left((X - \mathbb{E}(X|\mathcal{U}))^2 \,|\, \mathcal{U}\right)}{x^2} =$$

$$=$$

$$\mathbb{E}\left((X - \mathbb{E}(X|\mathcal{U}))^2 \,|\, \mathcal{U}\right) = \mathbb{E}(X^2|\mathcal{U}) - 2\cdot \overbrace{\mathbb{E}\left(X \cdot \mathbb{E}(X|\mathcal{U}) \,|\, \mathcal{U}\right)}^{\mathbb{E}^2(X|\mathcal{U})} +$$

$$+ \underbrace{\mathbb{E}\left(\mathbb{E}^2(X|\mathcal{U}) \,|\, \mathcal{U}\right)}_{\mathbb{E}^2(X|\mathcal{U})} = \mathbb{E}(X^2|\mathcal{U}) - \mathbb{E}^2(X|\mathcal{U}) = \operatorname{Var}(X|\mathcal{U})$$

Will prove later that $\mathcal{U}$-measurable functions inside $\mathbb{E}(\cdot|\mathcal{U})$ behave like constants.

$\square$