Solution to the Practice Midterm Exam

(1) By subadditivity,

$$P(M \le 0) = P\left(\bigcup_{i=1}^{n}[X_i \le 0]\right) \le \sum_{i=1}^{n} P(X_i \le 0)$$
$$= n - \sum_{i=1}^{n} P(X_i > 0) < 1.$$

(2) $\mathcal{C}$ is a field: it contains $\emptyset$ and Ω and is clearly closed under complements. If either A or B in $\mathcal{C}$ are finite, then AB will be finite and lie in $\mathcal{C}$. If A and B are both infinite, then $(AB)^c = A^c \bigcup B^c$ will be finite, and again AB will lie in $\mathcal{C}$. So $\mathcal{C}$ is closed under intersections. But $\mathcal{C}$ is not closed under countable unions. For instance, the even numbers $E = \{2, 4, \ldots\}$ is not in $\mathcal{C}$ but is the countable union $\bigcup_{k\ge1}\{2k\}$ of sets in $\mathcal{C}$. Hence, $\mathcal{C}$ is not a σ-field.

(3) Yes, $\mathcal{C}$ is a σ-field. It contains $\emptyset$ and Ω because $P(\emptyset) = 0$ and $P(\Omega) = 1$, and it is closed under complements because $P(B^c) = 1 - P(B)$. Finally, suppose that B_n, $n \ge 1$, lie in $\mathcal{C}$. If $P(B_n) = 0$ for all n, then by subadditivity

$$0 \le P\left(\bigcup_n B_n\right) \le \sum_n P(B_n) = 0,$$

so $P\big(\bigcup_n B_n\big) = 0$ and $\bigcup_n B_n \in \mathcal{C}$. If instead $P(B_k) = 1$ for some k, then by monotonicity

$$1 \ge P\left(\bigcup_n B_n\right) \ge P(B_k) = 1,$$

so $P\big(\bigcup_n B_n\big) = 1$ and again $\bigcup_n B_n \in \mathcal{C}$. Thus $\mathcal{C}$ is closed under countable unions.

(4) **(a)** Sets $S_2 = X^{-1}\big([\pi^2, \infty)\big)$ and $S_4 = X^{-1}\big([0, 49]\big)$ lie in $\sigma(X)$, but S_1 and S_3 are not in $\sigma(X)$. **(b)** $F\big(\{0\}\big) = P\big((-\infty, 0]\big) = 1/2$ and $F\big([1, \infty)\big) = P\big([1, \infty)\big) = 1/(2e)$.

(5) Since

$$P(Y_n \ge c) = P(X_n \ge c \log n) = \frac{1}{n^c},$$

$$\sum_n P(Y_n \geq c) = \infty, \qquad c < 1;$$

and

$$\sum_n P(Y_n \geq c) < \infty, \qquad c > 1.$$

Hence $P(Y_n \geq c, \text{i.o.}) = 0$ for $c > 1$, and $P(Y_n \geq c, \text{i.o.}) = 1$ for $c < 1$. It follows that $\limsup Y_n = 1$ almost surely.

(6) **(a)** For $\epsilon \in (0,1)$,

$$P\big(\inf_{n\geq 1} nX_n \geq \epsilon\big) = \prod_{n\geq 1} P(X_n \geq \epsilon/n) = \prod_{n\geq 1}\Big(1 - \frac{\epsilon}{n}\Big) = 0.$$

So by continuity, $P(\inf_{n\geq 1} nX_n > 0) = 0$. **(b)** By continuity,

$$P\big(\inf_{n\geq 1} n^2X_n > 0\big) = \lim_{\epsilon\downarrow 0} P\big(\inf_{n\geq 1} n^2X_n \geq \epsilon\big) = \lim_{\epsilon\downarrow 0}\prod_{n\geq 1}\Big(1 - \frac{\epsilon}{n^2}\Big) = 1.$$

(c) For $c \in (0,1)$,

$$\sum_{n\geq 1} P(n^2X_n < c) = \sum_{n\geq 1}\frac{c}{n^2} < \infty,$$

and so $P(n^2X_n < c, \text{i.o.}) = 0$. Hence $\liminf n^2X_n \geq c$ almost surely, and then by continuity, $\liminf n^2X_n = \infty$ almost surely.

(7) By dominated convergence,

$$\begin{aligned} nEh\big(1/(n^2Z^2)\big) &= \int nh\left(\frac{1}{n^2z^2}\right)\phi(z)\,dz \\ &= \int h\left(\frac{1}{x^2}\right)\phi(x/n)\,dx \to \frac{1}{\sqrt{2\pi}}\int h\left(\frac{1}{x^2}\right)\,dx. \end{aligned}$$

(8) Since $F^2(t) = P(X \leq t, Y \leq t) = E\big[1_{[X\vee Y\leq t]}\big]$, by Fubini's theorem

$$\int F^2(t)e^{-t}\,dt = E\int 1_{[X\vee Y\leq t]}e^{-t}\,dt = E\big[e^{-X\vee Y}\big].$$

(9) Since $P(X < y) = E\big[1_{[X<y]}\big]$, by Fubini's theorem

$$\int_0^\infty P(X<y)\,dy = E\int_0^\infty 1_{[X<y]}e^{-y}\,dy = E\big[e^{-X^+}\big].$$

(10) If $X_n \xrightarrow{p} 0$, then $|X_n|/(1+|X_n|) \xrightarrow{p} 0$ as the function $x \rightsquigarrow |x|/(1+|x|)$ is continuous. Then $E\big[|X_n|/(1+|X_n|)\big] \to 0$ by

Lebesgue dominated convergence. For the converse, for any $\epsilon > 0$ we have

$$\frac{|X_n|}{1+|X_n|} \geq \frac{\epsilon}{1+\epsilon} 1_{|X_n|>\epsilon}.$$

Hence, if $E\big[|X_n|/(1+|X_n|)\big] \to 0$,

$$P\big(|X_n| > \epsilon\big) \leq \frac{1+\epsilon}{\epsilon} E\left[\frac{|X_n|}{1+|X_n|}\right] \to 0.$$

(11) For any $t > 0$,

$$\sup_n E|c_n X_n| 1_{[c_n X_n > t]} = \sup\{p_n c_n : c_n > t\}.$$

This tends to zero as $t \to \infty$ if and only if $p_n c_n \to 0$.