Lecture. 19. Char F and weak conv

Monday, November 20, 2017 9:26 AM

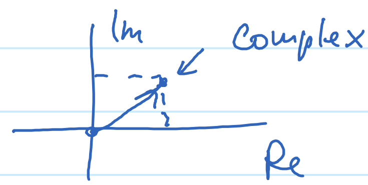
Characteristic functions

$X \qquad \varphi_X(t) = \mathbb{E}\{e^{itX}\}$

$i = \sqrt{-1}$

Euler $e^{itx} = \cos x + i \sin x$

Complex plain



$|e^{itX}| = 1$

φ_X always exists, because $|\varphi_X| \le 1$

Looking forward to results like

$\varphi_n \to \varphi \Rightarrow X_n \rightsquigarrow X$

Statistics: Consistency: $X_n \xrightarrow[p]{} c$ ↗ LLN

Weak convergence $\hat{X}_n \rightsquigarrow X$ ↗ CLT

↓ usually $\mathcal{N}(\cdot,\cdot)$

Convolution

$X \perp Y$

$\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)$

$\exists \; \{X_k\}_{k=1}^n$ i.i.d. r.v.

$\varphi_{\sum_{k=1}^n X_k}(t) = \varphi_{X_1}^n(t)$

Moments

X r.v.

$\varphi_X(t)$ – char. funct.

$$\varphi_X^{(k)}(0) = i^k \underbrace{\mathbb{E}(X^k)}_{M_k}$$

Taylor expansion

$$\varphi_X(t) = \sum_{k=1}^{n} \frac{(it)^k}{k!} M_k + o(|t|^n)$$

Examples

(1) $\exists$ X is degenerate : $P(X=c)=1$
non-random

$$\varphi_X(t) = \mathbb{E}\{e^{itX}\} = e^{itc}$$

Consequence : if $\varphi_n \to e^{itc}$
then $X_n \xrightarrow[P]{} c$

(2)

$\mathbb{N}$ distribution ; $X \sim \mathbb{N}(0,1)$

$$f_X(x) = e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}$$

Characteristic function

$$\varphi_X(t) = \mathbb{E}\{e^{itX}\} =$$

$$= \int_{-\infty}^{\infty} e^{itx - \frac{x^2}{2}} dx \cdot \frac{1}{\sqrt{2\pi}}$$

Write
φ for φ_X for brevity

$$\int^{\infty} \quad itx - \frac{x^2}{2} \ldots$$

$$\varphi'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ix\, e^{itx - \frac{x^2}{2}}\, dx =$$

Integrating by parts

$$= -\frac{1}{\sqrt{2\pi}} \int e^{itx} \cdot i \cdot \underbrace{d e^{-\frac{x^2}{2}}}_{-e^{-\frac{x^2}{2}} \cdot x} =$$

$$= -\frac{1}{\sqrt{2\pi}}\, i e^{itx} \cdot e^{-\frac{x^2}{2}} \Big|_{x=-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t \cdot e^{itx - \frac{x^2}{2}}\, dx =$$

$$i\, d e^{itx} = \underbrace{i^2}_{-1} t \cdot e^{itx} dx$$

$$= -t\,\varphi(t)$$

$$\varphi'(t) = -t\varphi(t) \qquad \varphi'(t) = \frac{d\varphi}{dt}$$

$$\frac{d\varphi}{\varphi} = -t\, dt = -d\frac{t^2}{2}$$

$$d \log\varphi = d\left(-\frac{t^2}{2}\right)$$

$$\varphi = C_1 \cdot e^{-\frac{t^2}{2}}$$

$$\varphi(0) = 1 = \mathbb{E}\{e^{itX}\}\Big|_{t=0} \Rightarrow C_1 = 1$$

$$\boxed{\varphi_{\mathbb{N}(0,1)}(t) = e^{-t^2/2}}$$

$$\boxed{X \sim \mathbb{N}(\mu, \sigma^2) \qquad \varphi_X(t) = e^{it\mu - \frac{t^2\sigma^2}{2}}}$$

Multivariate Characteristic functions

Multivariate Characteristic functions

$\exists$ X is a r. vector

DF $$\varphi_X(t) = \mathbb{E}\{e^{it^T \cdot X}\}$$

t — a vector

Example:

$X \sim \mathbb{N}(\mu, \Sigma)$ $\quad \varphi_X(t) = e^{it^T \mu - \frac{1}{2} t^T \cdot \Sigma \cdot t}$

μ — vector, Σ — matrix

TH $\quad \varphi$ uniquely determines the distribution

(1) $\varphi_X \equiv \varphi_Y := \varphi \iff X$ and Y have same distributions

$\equiv$ — uniformly over the argument

(2) Inversion formula for densities

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t)\, dt$$

(3) $$F_X(y) - F_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-ity}}{it} \varphi(t)\, dt$$

Proof:

Informally, $$\int_{-\infty}^{y} e^{-itx}\, dx = \frac{-1}{it} \cdot e^{-itx} \Big|_{x=-\infty}^{y} = \frac{e^{\text{“}i\infty\text{”}} - e^{-ity}}{it}$$

$$\int_{-\infty}^{y} \ldots dx - \int_{-\infty}^{x} \ldots dx = \frac{e^{-itx} - e^{-ity}}{it}$$

(2) => (3)

Proof of (i): $\exists\ Z \sim N(0,1)\ ,\ Z \perp X$

Consider $\tilde{X} = X + \sigma \cdot Z$

$$\underbrace{f_{\tilde{X}}(x)}_{\text{convolution}} = \int_{-\infty}^{\infty} f_{\sigma Z}(x-y) \cdot \underbrace{f_X(y)\, dy}_{d F_X(y)} =$$

Fubini

$$= \mathbb{E}\left\{ f_{\sigma Z}(x - X) \right\} \underset{\sigma Z \sim N(0,\sigma^2)}{=} \frac{1}{\sqrt{2\pi}\,\sigma} \mathbb{E}\left\{ e^{-\frac{(x-X)^2}{2\sigma^2}} \right\}$$

Consider

$$\int e^{-itx} \varphi_{\tilde{X}}(t)\, dt = \mathbb{E}\left\{ \int e^{-it(X-x) - \frac{t^2\sigma^2}{2}} dt \right\} =$$

$$\varphi_{\tilde{X}}(t) = \varphi_X(t) \cdot \varphi_{\sigma Z}(t) = \mathbb{E}\left\{ e^{itX} \right\} \cdot \underbrace{e^{-\frac{t^2\sigma^2}{2}}}_{\varphi_{N(0,\sigma^2)}}$$

$$= \mathbb{E}\left\{ \int e^{it(X-x)} \cdot \underbrace{e^{-\frac{t^2}{2(1/\sigma)^2}} \frac{1}{\sqrt{2\pi}\ 1/\sigma}}_{\tilde{Z} \sim \text{pdf of } N(0, 1/\sigma^2)} \right\} \sqrt{2\pi}\ 1/\sigma =$$

$$= \mathbb{E}\left\{ e^{i(X-x)\tilde{Z}} \right\} \sqrt{2\pi} \frac{1}{\sigma} =$$

$$\varphi_{\tilde{Z}}(X-x) = e^{i(X-x)\cdot 0 - \frac{(X-x)^2\left(\frac{1}{\sigma}\right)^2}{2}} = e^{-\frac{(X-x)^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma} \sqrt{2\pi}\ \mathbb{E}\left\{ e^{-\frac{(X-x)^2}{2\sigma^2}} \right\}$$

$$\int e^{-itx} \varphi_{\tilde{X}}(t)\, dt = \frac{\sqrt{2\pi}}{\sigma} \mathbb{E}\left\{ e^{-\frac{(X-x)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\,\sigma} \right\} \cdot \sqrt{2\pi}\,\sigma =$$

$$\int e^{-itx}\varphi_{\tilde{X}}(t)\,dt = \frac{\sqrt{2\pi}}{\sigma}\,\mathbb{E}\left\{ e^{-\frac{(X-x)^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}\,\sigma}\right\}\cdot\sqrt{2\pi}\,\sigma =$$

$$= 2\pi\cdot f_{\tilde{X}}(x)$$

based on the form of $f_{\tilde{X}}$ obtained as a convolution pdf

i.e.

$$\forall \text{ r.v. } X \perp \sigma Z \sim \mathbb{N}(0,\sigma^2),\quad \tilde{X} = X+\sigma Z$$

$$f_{\tilde{X}}(x) = \frac{1}{2\pi}\int e^{-itx}\varphi_{\tilde{X}}(t)\,dt$$

By Slutsky

$$\tilde{X} = X+\sigma Z \rightsquigarrow X,\quad \sigma\to 0$$

and $\varphi_{\tilde{X}} \longrightarrow \varphi_X, \quad \sigma\to 0$

If X, Y have same char. funct. $\Rightarrow$

$\Rightarrow$ $X+\sigma Z$, $Y+\sigma Z$ have same lim in distribution

$$X \overset{d}{=} Y \quad \square$$

TH Lévy

(1) $X_n \rightsquigarrow X \iff \varphi_n(t)\to\varphi(t),\ \forall t$

(2) If $\varphi_n(t)\to\varphi(t),\ \forall t$ and φ is continuous at $t=0$

$\Rightarrow$ φ is a characteristic function of some r.v. X, and $X_n \rightsquigarrow X$

w/o proof