Lecture 21 Lyapunov CLT, Z, M intro

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WLLN

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow[p]{} \mathbb{E}X \iff \varphi'(0) \text{ exists and } = i\cdot\mu$$

$\{X_i\}$ are i.i.d.

CLT $\mathbb{E}X_1 = \mu$ i.i.d $\{X_i\}_{i=1}^{n}$

$\mathrm{Var}(X_1) = \sigma^2$

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) \rightsquigarrow \mathcal{N}(0, \sigma^2)$$

multivariate CLT

Cramer - Wold

$$\varphi_X(t) = \mathbb{E}\{e^{it^TX}\}$$

X, t ↳ vector

$\varphi_{\underbrace{t^TX}_{\text{a scalar}}}$ - univariate

$X_n \rightsquigarrow X \iff$

$t^TX_n \rightsquigarrow t^TX$, $\forall t$

Extensions of CLTs

X_i are not identically distributed

still independent

Lindeberg - Feller CLT

$X_{n1}, \dots, X_{nk_n}$ $\perp$ r.v.

with finite covariances

Lindeberg condition

$$\forall \varepsilon > 0 \quad \sum_{i=1}^{k_n} \mathbb{E}\left(\|X_{ni}\|^2; \|X_{ni}\| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$

k_n

$$\forall \varepsilon > 0 \quad \sum_{i=1}^{k_n} \mathbb{E}\left(\|X_{ni}\|^2 ; \|X_{ni}\| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$

$$\sum_{i=1}^{k_n} \mathrm{Cov}(X_{ni}) \to \Sigma \quad \text{fixed cov. matrix}$$

$$\Rightarrow \sum_{i=1}^{K_n} \left(X_{ni} - \mathbb{E}(X_{ni})\right) \rightsquigarrow \mathcal{N}(0, \Sigma)$$

Lyapunov condition

$$L_s = \sum_{k=1}^{k_n} \mathbb{E}\|X_{nk}\|^s \quad , \quad s > 2$$

Note:
In practice
try $s=3$

Lindeberg was replaced by Lyapunov

$$L_s \xrightarrow[n\to\infty]{} 0$$

(L) Lyapunov $\Rightarrow$ Lindeberg

Proof: $\sum_k \mathbb{E}\left(\|X_{nk}\|^2 ; \|X_{nk}\| > \varepsilon\right) \le$

$$\|X_{nk}\| > \varepsilon \Rightarrow \frac{\|X_{nk}\|}{\varepsilon} > 1 \Rightarrow \left(\frac{\|X_{nk}\|}{\varepsilon}\right)^{s-2} > 1$$

$$\|X_{nk}\|^2 < \|X_{nk}\|^2 \cdot \left(\frac{\|X_{nk}\|}{\varepsilon}\right)^{s-2} = \frac{\|X_{nk}\|^s}{\varepsilon^{s-2}}$$

$$\frac{1}{\varepsilon^{s-2}} \sum_k \mathbb{E}\left(\|X_{nk}\|^s ; \|X_{nk}\| > \varepsilon\right) \le$$

$$\le \frac{1}{\varepsilon^{s-2}} \sum_k \mathbb{E}\left(\|X_{nk}\|^s\right) = \frac{L_s}{\varepsilon^{s-2}} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{Lindeberg}$$

$\square$

A version of Lyapunov CLT

- Expose normalization
- Scalar version

$\{X_i\}_{i=1}^n$ indep. r.v.s

$$\mathbb{E}(X_i) = \mu_i \qquad \mathrm{Var}(X_i) = \sigma_i^2 < \infty$$

$\exists\ Y_i = X_i - \mu_i \qquad \mathbb{E}\, Y_i = 0 \qquad$ centered X_i

$$T_n = \sum_{i=1}^{n} Y_i$$

$$S_n^2 = Var(T_n) = \sum_{i=1}^{n} \sigma_i^2$$

Standardize T_n

$\frac{T_n}{S_n}$ will have mean $= 0$, var $= 1$

If $\exists\ s > 2$: $\qquad \frac{\sum_{i=1}^{n} \mathbb{E}(|Y_i|^s)}{S_n^s} \xrightarrow[n\to\infty]{} 0$

Lyapunov condition

Then

$$\frac{T_n}{S_n} \rightsquigarrow \mathbb{N}(0,1)$$

Lyapunov inequality

$[\mathbb{E}(|X|^\alpha)]^{1/\alpha}$ is an increasing function of $\alpha \geq 0$

Example:

$$\mathbb{E}(|X|) \leq \left(\mathbb{E}(X^2)\right)^{1/2} \leq \mathbb{E}(|X|^s)^{1/s}, \quad s > 2$$

Proof:

$|\cdot|^r$ is a convex function when $r \geq 1$

Jensen $\quad \mathbb{E}(|\cdot|^r) \geq (\mathbb{E}|\cdot|)^r$

$$\left.\begin{array}{l} \text{Take} \quad 0 < \alpha < \beta \\ |\cdot| = |X|^\alpha \\ r = \frac{\beta}{\alpha} \end{array}\right\} \Rightarrow$$

$$\mathbb{E}\left(|X|^{\alpha\cdot\frac{\beta}{\alpha}}\right) \geq \left(\mathbb{E}(|X|^\alpha)\right)^{\beta/\alpha} \Rightarrow$$

Take 1/beta of both parts

$$\Rightarrow \left[\mathbb{E}(|X|^\beta)\right]^{1/\beta} \geq \left(\mathbb{E}(|X|^\alpha)\right)^{1/\alpha}$$

Lyapunov condition

$$\frac{\left(\sum_i \mathbb{E}|Y_i|^s\right)^{1/s}}{S_n} \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow$$

$\forall \varepsilon > 0 \ \exists N: \ n > N$

$$\left(\sum_{i=1}^n \sigma_i^2\right)^{1/2} \leq \left(\sum_i \mathbb{E}|Y_i|^s\right)^{1/s} < S_n \cdot \varepsilon$$

Lyapunov inequality

Let's see if Lindeberg and Lyapunov are generalization of the classical CLT

When $\sigma_i^2 = \sigma^2$ b/c in X_i are i.i.d.

$$S_n^2 = n \cdot \sigma^2, \quad S_n = \sqrt{n} \cdot \sigma$$

$s > 2 \qquad \delta > 0, \ 2 + \delta = s$

Lyapunov condition

$$\frac{\sum_i \mathbb{E}\left(|Y_i|^{2+\delta}\right)}{\left(\sqrt{n}\cdot\sigma\right)^{2+\delta}} = \frac{n \cdot \mathbb{E}\left(|Y_1|^{2+\delta}\right)}{n \cdot n^{\delta} \cdot \sigma^{2+\sigma}} =$$

$$= \frac{C}{n^{\delta}} \xrightarrow[n\to\infty]{} 0 \quad \text{i.e.}$$

When $\mathbb{E}|Y_1|^{2+\delta} < \infty$, classical CLT satisfies Lyapunov $\Rightarrow$

$\Rightarrow$ satisfies Lindeberg

M and Z estimators

$X_1, \dots, X_n$ sample

Model parameterized by $\theta \in \Theta$

space of θs

M-estimators are defined as maximizing some random functions $M_n(\theta)$

M-estimates: $\hat{\theta}_n = \underset{\theta \in \Theta}{\arg\max}\ M_n(\theta)$

Z-estimation, Estimating equations

Random functions $\Psi_n(\theta)$

$\hat{\theta}_n$ is a root of $\Psi_n(\theta) = 0$

$\downarrow$

Z-estimator

In both cases we want $\hat{\theta}_n \xrightarrow[p]{} \theta^*$ (Consistency)

θ^* $\to$ true parameter value

$M_n(\theta) \xrightarrow[p]{} M(\theta)$, pointwise

$\underset{\theta \in \Theta}{\arg\max}\ M(\theta) = \theta^*$

$\Psi_n(\theta) \to \Psi(\theta)$, pointwise

root of $\Psi(\theta) = 0$ is θ^*

LLN

if M_n is defined as $\frac{1}{n}\sum_i m_\theta(X_i) = \hat{\mathbb{E}}(m_\theta(X))$

then

pointwise convergence will follow from LLN

We want

conditions invoking the whole functions M_n, M

$M_n \xrightarrow[p]{} M$ stronger than pointwise

$\Rightarrow \arg\max M_n \xrightarrow[p]{} \arg\max M$

$\hat{\theta}_n \qquad\qquad \theta^*$

$$\hat{\theta}_n \qquad \xrightarrow{p} \qquad \theta^*$$

Uniform convergence ; ULLN
uniform

Operator notation

$$\mathbb{E} f = \int f \, dP := P f$$

$f = f(X)$
X is r.v. $\sim P$

$$\hat{\mathbb{E}} f = \frac{1}{n} \sum_i f(X_i) := \underbrace{\mathbb{P}_n}_{\text{empirical distribution operator}} f$$

Define

$$M_n(\theta) = \mathbb{P}_n m_\theta$$

$$\Psi_n(\theta) = \mathbb{P}_n \psi_\theta$$

MLE $\quad m_\theta(x) = \underbrace{\log [p_\theta(x) \, dx]}$
X is continuous

$$\ell = \mathbb{P}_n m_\theta \qquad = M_n \quad \longrightarrow M$$

Score equs

$$\dot{\ell} = \mathbb{P}_n \dot{m}_\theta \qquad = \Psi_n \quad \longrightarrow \Psi$$