Practice Final Exam—Statistics 621

The final will be a closed book exam, but you are allowed two formula sheet. Show your work for full or partial credit.

(1) Let $X_1, \dots, X_k$ be i.i.d. from the (discrete) uniform distribution on $\{1, \dots, n\}$. Call a pair of indices $\{i, j\}$ a *match* if $X_i = X_j$.

(a) Find the expected number of matches.
(b) Find the variance of the number of matches.
(c) Find the probability of *exactly* one match.
(d) Find the limit of the probability in part (c) as $n \to \infty$ and $k \to \infty$ with $k/\sqrt{n} \to c > 0$. Hint: In this limit,

$$\frac{n!}{n^k (n-k)!} \to e^{-c^2/2}.$$

(2) Let X_i, $i \geq 1$, be i.i.d. with $P(X_i \leq x) = x^2$, $x \in [0, 1]$. Define $M_n = \min\{X_1, \dots, X_n\}$. Find constants c_n, $n \geq 1$, so that $c_n M_n \Rightarrow M$, with M nondegenerate, and give the cumulative distribution function for M.

(3) Let X_k, $k \geq 1$ be independent random variables with

$$P(X_k = \pm k) = \frac{1}{2k^\alpha} \qquad \text{and} \qquad P(X_k = 0) = 1 - 1/k^\alpha, \quad k \geq 1,$$

with $\alpha > 0$ a fixed constant. Take $S_n = X_1 + \cdots + X_n$ and $\overline{X}_n = S_n/n$.

(a) For which α will S_n have an almost sure limit as $n \to \infty$?
(b) For which α will $\overline{X}_n \to 0$ in L_2?
(c) For which α will $\overline{X}_n \to 0$ almost surely?

(4) Suppose $\mu = EX > 0$ and $\sigma^2 = \mathrm{Var}(X) < \infty$. Then by Chebyshev's inequality, $P(X \leq 0) \leq \sigma^2/\mu^2$.

(a) Derive a better lower bound for $P(X \leq 0)$ by contrasting in indicator of interest with a multiple of $(X - 1/c)^2$.
(b) Show that the lower bound you derived in part (a) is sharp, finding a distribution where the probability of interest equals its lower bound.

(5) Let X_k, $k \geq 1$, be i.i.d., all uniformly distributed on $(0, e)$, and define

$$Y_n = \sqrt[n]{\prod_{i=1}^{n^2} X_i}.$$

Show that $Y_n \Rightarrow Y$ as $n \to \infty$, giving the cumulative distribution function for Y.

(6) Suppose $P^* \ll P$ are probability measures and that

$$Y \stackrel{\text{def}}{=} \frac{dP^*}{dP} > 0.$$

Let E and E^* denote expectation under P and P^*. Show that if $X \geq 0$ and $XY \in L_1(P)$, then

$$E^*(X|\mathcal{G}) = \frac{E(XY|\mathcal{G})}{E(Y|\mathcal{G})}.$$

(7) Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ be increasing sigma-fields, let X be an integrable random variable, and define $X_n = E[X|\mathcal{B}_n]$. Then by smoothing X_n, $n \geq 1$, is a martingale, and $EX_n = EX$.

(a) Show that $E|X_n| \leq E|X|$.

(b) If $B \in \mathcal{B}_n$, show that

$$E\big[|X_n|1_B\big] = E\big[|X|1_B\big],$$

and that

$$E\big[|X|1_B\big] \leq E\big[|X|1_{[|X|>t]}\big] + tP(B), \quad \forall\, t > 0.$$

(c) Show that the variables X_n, $n \geq 1$, are uniformly integrable.