Biostat 801 Homework 9

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1 Problem 1

From the previous homework, we know that

$$2na_n\hat{f}_n(x) = \sum Y_{ni}(x)$$

where

$$B_{n,i}(x) = 1_{[X_i \in (x-a_n, x+a_n]]} \overset{iid}{\sim} \text{Bern}(p_n(x))$$

and

$$p_n(x) = F((x-a_n, x+a_n]).$$

Moreover, since $a_n \to 0$ and $na_n \to \infty$ as $n \to \infty$, we have

$$p_n(x) \to 0, \quad \frac{p_n(x)}{2a_n} \to f(x), \quad np_n(x) \to \infty.$$

Furthermore,

$$\text{Var}[2na_n\hat{f}_n(x)] = np_n(x)[1-p_n(x)]$$

and

$$\begin{aligned}\text{Cov}[2na_n\hat{f}_n(x), 2na_n\hat{f}_n(y)] &= \text{Cov}\left[\sum B_{ni}(x), \sum B_{ni}(y)\right] \\ &= \sum_{i,j} \text{Cov}[B_{ni}(x), B_{nj}(y)] \\ &= \sum_i \text{Cov}[B_{ni}(x), B_{ni}(y)] \to 0\end{aligned}$$

since

$$(x-a_n, x+a_n] \cap (y-a_n, y+a_n] = \emptyset, \quad \forall n > N$$

Let

$$Y_n = \sqrt{2na_n}\begin{bmatrix}\hat{f}_n(x) - E\hat{f}_n(x) \\ \hat{f}_n(y) - E\hat{f}_n(y)\end{bmatrix}$$

The problem will be the same as last homework if $x = y$, so here we assume $x \neq y$. Then

$$\begin{aligned}\operatorname{Var}[\alpha^T Y_n] =& \frac{1}{2na_n}\{\alpha_1^2 \operatorname{Var}[2na_n \hat{f}_n(x)] \\ &+2\alpha_1\alpha_2 \operatorname{Cov}[2na_n\hat{f}_n(x), 2na_n\hat{f}_n(y)] + \alpha_2^2 \operatorname{Var}[2na_n\hat{f}_n(y)]\} \\ \rightarrow& \frac{1}{2na_n}\{\alpha_1^2 np_n(x)[1-p_n(x)] + 0 + \alpha_2^2 np_n(x)[1-p_n(x)] \\ \rightarrow& \alpha_1^2 f(x) + \alpha_2^2 f(y)\end{aligned}$$

2 Problem 2

Let $r \geq 1$ and

$$h(x) := |x|^r$$

Then h is a convex function, so

$$h(\frac{1}{2}[x+y]) \leq \frac{1}{2}[h(x) + h(y)],$$

that is,

$$|\frac{1}{2}[x+y]|^r \leq \frac{1}{2}(|x|^r + |y|^r) \Rightarrow |x+y|^r \leq 2^{r-1}(|x|^r + |y|^r)$$

so

$$E|x+y|^r \leq 2^{r-1}(E|x|^r + E|y|^r)$$

3 Problem 3

We show that $Y_n \xrightarrow{d} N(0, D)$ where

$$D = \begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix}$$

By Cramer-Wold, we just need to show

$$\alpha^T Y_n \xrightarrow{d} N(0, C)$$

where

$$C = \alpha^T D \alpha = \begin{bmatrix} \alpha_1^2 f(x) & 0 \\ 0 & \alpha_2^2 f(y) \end{bmatrix}$$

for arbitrary $\alpha \in \mathbb{R}^2$. We check the Lyapunov condition. Let

$$A_{ni}(x,y) = \alpha_1 B_{ni}(x) + \alpha_2 B_{ni}(y)$$

We have

$$
\begin{aligned}
&\sum E|A_{ni}(x,y) - E[A_{ni}(x,y)]|^3 \\
\le &\sum \{\alpha_1[1+p_n(x)] + \alpha_2[1+p_n(y)]\} E|A_{ni}(x,y) - E[A_{ni}(x,y)]|^2 \\
= &\sum \{\alpha_1[1+p_n(x)] + \alpha_2[1+p_n(y)]\} \operatorname{Var}[A_{ni}(x,y)] \\
= &n\{\alpha_1[1+p_n(x)] + \alpha_2[1+p_n(y)]\} \operatorname{Var}[A_{n1}(x,y)]
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\sum E|A_{ni}(x,y) - E[A_{ni}(x,y)]|^3}{\{\sum \operatorname{Var}[A_{n1}(x,y)]\}^{\frac{3}{2}}} \\
= &\frac{n\{\alpha_1[1+p_n(x)] + \alpha_2[1+p_n(y)]\} \operatorname{Var}[A_{n1}(x,y)]}{\{n \operatorname{Var}[A_{n1}(x,y)]\}^{\frac{3}{2}}} \\
= &\frac{\alpha_1[1+p_n(x)] + \alpha_2[1+p_n(y)]}{\sqrt{n \operatorname{Var}[A_{n1}(x,y)]}} \to 0
\end{aligned}
$$

since

$$
n \operatorname{Var}[A_{n1}(x,y)] \to \alpha_1 n p_n(x) + 2\alpha_1\alpha_2 p_n(x) p_n(y) + \alpha_2^2 p_n(y) \to \infty
$$

Thus the Lyapunov condition holds, so

$$
\frac{2na_n(A_{ni}(x,y) - E[A_{ni}(x,y)])}{\sqrt{\alpha_1 n p_n(x)[1-p_n(x)] + \alpha_2 n p_n(y)[1-p_n(y)]}} \xrightarrow{d} N(0,1) = (*)
$$

By Slutsky,

$$
\begin{aligned}
Y_n = &(*) \frac{\sqrt{\alpha_1 n p_n(x)[1-p_n(x)] + \alpha_2 n p_n(y)[1-p_n(y)]}}{2na_n} \\
= &N(0,1)\sqrt{\alpha_1 f(x) + \alpha_2 f(y)}
\end{aligned}
$$

This completes the proof.

4 Problem 4

This follows from the same argument as in the previous homework. We have

$$
\begin{aligned}
&\sqrt{2na_n}(\alpha_1(E\hat{f}_n(x) - f(x)) + \alpha_2(E\hat{f}_n(y) - f(y))) \\
= &\sqrt{2na_n^3}[\alpha_1 f'(x) + \alpha_2 f'(y) + O(1)]
\end{aligned}
$$

so we need

$$
na_n^3 \to 0
$$