Definition 18 (Small sets) *A set $C \in \mathcal{B}(\mathcal{X})$ is called a small set if there exists an $m > 0$ and a non-trivial measure ν_m on $\mathcal{B}(\mathcal{X})$ such that*

$$P^m(x, B) \geq \nu_m(B), \quad \forall\, x \in C, \forall\, B \in \mathcal{B}(\mathcal{X}).$$

When this holds, we say that C is ν_m-small.

Recall that the probability transition kernel $P^n(x, \cdot)$ is a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Therefore there exists a Lebesgue decomposition into its absolutely continuous and singular parts with respect to some σ-finite measure ϕ on $\mathcal{B}(\mathcal{X})$: for any fixed x and $B \in \mathcal{B}(\mathcal{X})$

$$P^n(x, B) = \int_B p^n(x, y)\phi(dy) + P_{\perp}(x, B), \tag{6}$$

where $p^n(x, y)$ is the density of $P^n(x, \cdot)$ with respect to ϕ and $P_{\perp}$ and ϕ are mutually singular. (Recall that two measures ν and μ are mutually singular, $\mu \perp \nu$, if there exists disjoint sets A and B such that $A \cup B = \mathcal{X}$ and $\nu(A) = \mu(B) = 0$. Thus, the two measures "live" on different subspaces of $\mathcal{X}$.)

Theorem 3 *Suppose ϕ is a σ-finite measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Suppose $A \in \mathcal{B}(\mathcal{X})$ with $\phi(A) > 0$ such that*

$$B \subseteq A,\ \phi(B) > 0 \implies \sum_{k=1}^{\infty} P^k(x, B) > 0, \quad x \in A.$$

Then, for every n, the density p^n can be chosen to be a measurable function on $\mathcal{X}^2$ and there exists $C \subseteq A$, an $m > 1$, and a $\delta > 0$ such that $\phi(C) > 0$ and

$$p^m(x, y) > \delta, \quad x, y \in C. \tag{7}$$

Proof: (Meyn & Tweedie, pp. 103–105).

The key point in this theorem is that we can define a version of the densities of the probability transition kernel such that (7) holds uniformly over $x \in C$.

Now the main theorem of this section:

Theorem 4 *If $\mathbf{\Phi}$ is ψ-irreducible, then for every $A \in \mathcal{B}^+(\mathcal{X})$ there exists $m \geq 1$ and a ν_m-small set $C \subseteq A$ such that $C \in \mathcal{B}^+(\mathcal{X})$ and $\nu_m(C) > 0$.*

Proof:

Theorem 5 *If $\mathbf{\Phi}$ is ψ-irreducible, then the minorization condition holds for some m-skeleton chain and for every K_ϵ-chain, $0 < \epsilon < 1$.*

Proof:

Any $\mathbf{\Phi}$ that is ψ-irreducible is well endowed with small sets from Theorems 3 and 4. Also a small set can be rather large, in fact, it can be all of $\mathcal{X}$. Given the existence of just one small set, we now show that it is possible to cover all of $\mathcal{X}$ with small sets in the ψ-irreducible case.

Proposition 9

(i) If $C \in \mathcal{B}(\mathcal{X})$ is ν_n-small and if for any $x \in D$, $P^m(x, C) \geq \delta > 0$, then D is ν_{m+n}-small where ν_{m+n} is a multiple of ν_n.

(ii) *If* $\mathbf{\Phi}$ *is* ψ*-irreducible, then there exists a countable collection* C_i *of small sets in* $\mathcal{B}(\mathcal{X})$ *such that*

$$\mathcal{X} = \bigcup_{i=1}^{\infty} C_i.$$

(iii) *If* $\mathbf{\Phi}$ *is* ψ*-irreducible and* $C \in \mathcal{B}^{+}(\mathcal{X})$ *is* ν_n*-small, then there exists an* $M \geq 1$ *and a measure* ν_M *such that* C *is* ν_M*-small and* $\nu_M(C) > 0$.

Proof:

The covering in Proposition 9 will be used in the following section.