BIOSTAT 802 Homework #2

Due on February 12, 2018 in class

Problem 1 (An Old Qual Exam Problem): Consider n independent exponentially distributed random variables $X_1, ..., X_n$, such that the first ξ of these variables have mean $1/\eta$, the remaining $n-\xi$ variables have mean $1/(c\eta)$, where ξ is a random variable taking a value in $\{1, 2, ..., (n-1)\}$. Assume c is known, $c \neq 1$, and η is a positive random variable. We are interested in making Bayesian inference with respect to ξ based on a sample of $\mathbf{x} = (X_1 = x_1, \ldots, X_n = x_n)$.

(a) The first Bayesian statistician chooses independent priors for η and ξ: the prior for ξ is denoted by $\pi(\xi)$ and is proper, i.e., $\pi(\xi = 1) + \cdots + \pi(\xi = n-1) = 1$; the prior for η is a flat improper prior, i.e., $\pi(\eta) \propto 1$. Define $Z_i = X_i/X_1$. Derive the posterior distribution $\pi(\xi|\mathbf{x})$, and show it depends on the observed data only through $\mathbf{z} = (Z_2 = z_2, \ldots, Z_n = z_n)$, i.e., $\pi(\xi|\mathbf{x}) = \pi(\xi|\mathbf{z})$.
Hint: it may be useful to know $\int_0^\infty y^n e^{-py} dy = n!/p^{n+1}$.

(b) The second Bayesian statistician notices the result by the first Bayesian statistician, and takes the same prior distribution but directly works on the transformed data $\mathbf{z}$. Without extensive calculation, show that $(Z_2, \ldots, Z_n)$ is an ancillary statistic of η.

(c) Show that the likelihood function

$$f(\mathbf{z}|\xi) \propto \left(\sum_{i=1}^{\xi} z_i + c \sum_{i=\xi+1}^{n} z_i \right)^{-n} c^{n-\xi},$$

where $z_1 = 1$, and derive the corresponding posterior distribution for $\pi(\xi|\mathbf{z})$ using the above likelihood.

(d) Explain that there is no choice of prior distribution $\pi(\xi)$ that can reconcile the results obtained by the two Bayesian statisticians. Therefore it becomes a paradox. Show that if the flat improper prior for η is changed to $\pi(\eta) \propto 1/\eta$, the paradox no longer occurs. (In this case, $\pi(\eta)$ is still improper, but it is can be represented as a limit of some prior distribution.)

Problem 2 (An Old Qual Exam Problem): Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed (*i.i.d.*) sample drawn from a uniform distribution on the interval $(0, \theta)$, i.e. $X_i|\theta \overset{iid}{\sim} UNIF(0, \theta)$, $\theta > 0$. Consider a *Pareto* prior distribution, $\theta \sim PA(\alpha, \beta)$, whose density is given by

$$p(\theta) = \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}}, \quad \theta \geq \beta; \alpha, \beta > 0.$$

(a) Show that the family of Pareto distributions is the conjugate for the uniform distribution, that is, show that the posterior distribution of θ is also a Pareto distribution.

(b) Derive the posterior mean (i.e. Bayes Estimator under the quadratic loss function) and the posterior variance.

(c) Compare the Bayes estimator given in part (b) with the maximum likelihood estimator of θ in terms of mean squared error (MSE) when $\beta \in (0, \max_i(x_i))$. In this comparison θ is regarded as a non-random value.

Problem 3 Let $X_1, \ldots, X_n$ be *iid* sample from a uniform $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Consider the absolute error loss $L(\theta, a) = |\theta - a|$, $\theta \in (-\infty, \infty)$ and $a \in (-\infty, \infty)$.

(a) For a prior distribution $\Pi_\alpha(\theta)$ being $U(-\alpha, \alpha), \alpha > 0$, find the Bayes solution for parameter θ.

(b) Denote the Bayes estimator given in part (a) by $\delta_{\Pi_\alpha}(x_1, \ldots, x_n)$. Show that $\delta_{\Pi_\alpha}(x_1, \ldots, x_n) \to \delta(x_1, \ldots, x_n) \stackrel{def}{=} \frac{1}{2}(\min x_i + \max x_i)$ as $\alpha \to \infty$.

(c) Show that the limiting decision rule $\delta(x_1, \ldots, x_n) = \frac{1}{2}(\min x_i + \max x_i)$ given in part (b) is the minimax estimator of θ.