1(a) Data is collected as a form of the following table.

Subject	item 1	2	...	K
1				
2				
⋮				
n				

$Y_{ij} \in \{1, 0\} \overset{iid}{\sim} \text{Bernoulli}(p_{ij})$, $p_{ij} = \dfrac{e^{\alpha_j \theta_i - \beta_j}}{1 + e^{\alpha_j \theta_i - \beta_j}}$ $\uparrow$ $\beta_j \downarrow$, $\alpha_j \uparrow$

correct answer: 1; incorrect answer: 0; i: subject i; j: item j

$\eta_j = (\alpha_j, \beta_j)$, $\eta = (\eta_1, \cdots, \eta_n)$

$0 \le \theta_i \sim \Phi(\theta)$ unobserved ability

- The probability space consists of the following three elements:

$\mathcal{X} = \{0,1\}^{nK}$, a collection of all $n \times K$ matrices of 0 or 1.

$\mathcal{B}_x$ = a collect of all subsets of matrices of dimensions $n \times K$ or less.

$$\mathcal{P} = \left\{ P_\eta : \prod_{i=1}^{n} \int_{R_+} \prod_{j=1}^{K} p_{ij}^{Y_{ij}} (1 - p_{ij})^{1 - Y_{ij}} \phi(\theta_i)\, d\theta_i, \quad \eta = (\alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K) \in R_+^{2K} \right\}$$

where $p_{ij} = P(Y_{ij} = 1 \mid \eta_j, \theta_i)$, $j = 1, \cdots, K$, $i = 1, \cdots, n$.

(b) The action space is $(A, \mathcal{B}_A)$

$A = R_+^{2K}$ or R^{2K}, including impossible values

$\mathcal{B}_A = \{$all possible subsets of $A\}$.

(c) The observed likelihood is

$$L(\eta, x) = \prod_{i=1}^{n} \int_{R_+} \prod_{j=1}^{K} p_{ij}^{Y_{ij}}(\eta_j, \theta_i) \{1 - p_{ij}(\eta_j, \theta_i)\}^{1 - Y_{ij}} \phi(\theta_i)\, d\theta_i, \quad x \in \mathcal{X}$$

and the log observed likelihood is

$$\ell(\eta, x) = \sum_{i=1}^{n} \int_{R_+} \sum_{j=1}^{K} p_{ij}^{Y_{ij}}(\eta_j, \theta_i) \{1 - p_{ij}(\eta_j, \theta_i)\}^{1 - Y_{ij}} \phi(\theta_i)\, d\theta_i, \quad x \in \mathcal{X}$$

The decision function $\delta : (\mathcal{X}, \mathcal{B}_x) \to (A, \mathcal{B}_A)$ is given by the MLE:

$$\delta(x) = \underset{\eta}{\operatorname{argmax}}\, \ell(\eta, x), \quad x \in \mathcal{X}, \quad \text{if } A = R^{2K}$$

$$\text{or} = \max\{0, \underset{\eta}{\operatorname{argmax}}\, \ell(\eta, x)\}, \quad \text{if } A = R_+^{2K}.$$

(d) The squared loss function is

$$L(\eta, a) = \sum_{j=1}^{K} \{(a_{j1} - \alpha_j)^2 + (a_{j2} - \beta_j)^2\} = \sum_{j=1}^{K} (a_j - \eta_j)^T (a_j - \eta_j)$$

Risk function is $E_{P_\eta} L(\eta, \delta(x)) = \int_{\mathcal{X}} \sum_{j=1}^{K} (\delta_j(x) - \eta_j)^T (\delta_j(x) - \eta_j)\, dP_\eta(dx) = \sum_{j=1}^{K} \{\text{bias}_j^2 + \text{var}_j\}$

(10) **Problem 3** Consider a probability space $(\mathcal{X}, \mathcal{B}_x, \mathcal{P} = \{P_\theta, \theta \in \Theta\})$. Show the following statements.

5 (a) If $\delta_0(x)$ is the unique minimax decision rule with respect to the loss function $L(\theta, a)$, then δ_0 is admissible.

5 (b) If $\delta_0(x)$ is an admissible decision rule and has a constant risk function on Θ, then δ_0 is the minimax decision rule.

(a) If δ_0 is not admissible, then there exists a $\tilde{\delta} \in \mathcal{D}$ such that

$$R(\theta, \tilde{\delta}) \leq R(\theta, \delta_0), \text{ for all } \theta \in \Theta,$$

and there is at least one $\theta \in \Theta$ at which $R(\theta, \tilde{\delta}) < R(\theta, \delta_0)$.

Then, $\sup_{\theta \in \Theta} R(\theta, \tilde{\delta}) \leq \sup_{\theta \in \Theta} (\theta, \delta_0) \overset{\checkmark \text{ minimax rule}}{\leq} \sup_{\theta \in \Theta} (\theta, \delta), \delta \in \mathcal{D}$

It implies that $\tilde{\delta}$ is a minimax decision rule because δ_0 is a minimax rule. Due to the uniqueness, $\tilde{\delta} \equiv \delta_0$, which is contradictive to the fact that their risk functions are not the exactly same.

Thus, δ_0 must be admissible.

(b) If δ_0 is not the minimax decision rule, there exists a $\tilde{\delta} \in \mathcal{D}$, such that

$$\sup_{\theta} R(\theta, \tilde{\delta}) < \sup_{\theta} R(\theta, \delta_0) = c.$$

Then, $R(\theta, \tilde{\delta}) < R(\theta, \delta_0) = c$, all $\theta \in \Theta$.

That is, δ_0 is not admissible. Contradiction!

Therefore, δ_0 is a minimax decision rule.

2(a) The loss function is

$$L(\theta, \delta_c(\underset{\sim}{x}, \underset{\sim}{y})) = \theta\, 1(\theta > 0)\, 1(\bar{x} \leq \bar{y} + c) - \theta\, 1(\theta \leq 0)\, 1(\bar{x} > \bar{y} + c)$$

2(b) $\bar{X} \sim N(\mu_1, \frac{1}{m})$ and $\bar{Y} \sim N(\mu_2, \frac{1}{n})$, and they are independent. Then,

$$\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \tfrac{1}{m} + \tfrac{1}{n}) = N(\theta, \tfrac{1}{m} + \tfrac{1}{n}).$$

Thus the risk function is

$$\begin{aligned}
R(\theta, \delta_c) &= \theta\, 1(\theta \geq 0)\, P(\bar{X} - \bar{Y} \leq c) - \theta\, 1(\theta < 0)\, P(\bar{X} - \bar{Y} > c) \\
&= \theta\, 1(\theta \geq 0)\, \Phi\left(\frac{c - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) - \theta\, 1(\theta < 0) \left\{1 - \Phi\left(\frac{c - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right)\right\} \\
&= \theta \left\{1(\theta \geq 0) + 1(\theta < 0)\right\} \Phi\left(\frac{c - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) - \theta\, 1(\theta < 0) \\
&= \theta\, \Phi\left(\frac{c - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) - \theta\, 1(\theta < 0)
\end{aligned}$$

2(c) For any $c \in (-\infty, \infty)$, because $\Phi\left(\frac{c - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) \uparrow$ $c \uparrow$ there always exist $c^* < c$, $c^{**} > c$ (e.g. $c^* = c - 1$ and $c^{**} = c + 1$).

$$\begin{aligned}
R(\theta, \delta_c) &= \theta\, \Phi\left(\frac{c - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) - \theta\, 1(\theta < 0) \\
&> \underbrace{\theta\, \Phi\left(\frac{c^* - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) 1(\theta > 0) + \theta\, \Phi\left(\frac{c^{**} - \theta}{\sqrt{\frac{1}{m} + \frac{1}{n}}}\right) 1(\theta < 0) - \theta\, 1(\theta < 0)} \\
&= R(\theta, \delta^*)
\end{aligned}$$

where

$$\delta^* = \begin{cases} a_0, & \text{if } \bar{X} < \bar{Y} + c^*, \\ a_1, & \text{if } \bar{X} > \bar{Y} + c^{**}, \\ \phi, & \text{o/w.} \end{cases}$$

So, δ_c is not admissible

Note that no decision made incurs no loss nor risk.