Lecture 11. Indep dep

Wednesday, October 11, 2017 10:10 AM

Monotone Conv. Th. for $\mathbb{E}(\cdot|\mathcal{U})$

$$0 \le X_n \uparrow X \implies \mathbb{E}(X_n|\mathcal{U}) \to \mathbb{E}(X|\mathcal{U})$$

Formulae of total probability

$$\mathbb{E}(X) = \mathbb{E}\{\mathbb{E}\{X|\mathcal{U}\}\}$$

$$\mathcal{U} \subset \sigma(X)$$

$$\mathrm{Var}(X) = \mathbb{E}\{\mathrm{Var}(X|\mathcal{U})\} + \mathrm{Var}\{\mathbb{E}(X|\mathcal{U})\}$$

$\mathcal{U}$-measurable functions Y can be taken out of $\mathbb{E}(\cdot|\mathcal{U})$

$$\mathbb{E}(X \cdot Y|\mathcal{U}) = Y \cdot \mathbb{E}\{X|\mathcal{U}\}$$

Y is $\mathcal{U}$-measurable

Corollary

$$\mathbb{E}(X|\mathcal{F}) = X$$

Random Processes

$X(t, \omega)$

$\mathcal{F}_t$ nested within $\mathcal{F}$ sequence of $\uparrow$ σ-algebras are filtrations

Adapted the process to filtration $\mathcal{F}_t = \sigma(\bar{X}_t)$

$$\mathbb{E}\{dX_t | \mathcal{F}_{t-}\} = 0 \qquad \text{Martingales}$$

$$\mathbb{E}\{dX_t \cdot dX_\tau\} = 0 \qquad \text{Uncorrelated increments}$$

Example. Martingales

Martingale are "noise" processes

$$\mathbb{E}\{X_t\} = 0$$

$$X_t = \int_0^t dX_\tau$$

$$\mathbb{E}\{X_t\} = \int_0^t \mathbb{E}\Big[\underbrace{\mathbb{E}\{dX_\tau \mid \mathcal{F}_{\tau-}\}}_{=0 \text{ by def.}}\Big] = 0$$

Example. Optimal Non-linear Estimation.

$\exists$ X - r.v. of interest, unobserved
$\exists$ Y - r.v., observed (example: a sample)

Want to find a function g:

$$\hat{X} = g(Y) \quad \text{estimate of } X$$

g - unknown function

$$\mathbb{E}\, g^2(Y) < \infty$$

Quadratic Loss/Risk function

$$[X - g(Y)]^2$$

The problem

$$\min_g \underbrace{\mathbb{E}[X - g(Y)]^2}_{\text{Average Loss/Risk}} \Rightarrow \hat{g}$$

MSE

By the properties of conditional expectations

$$\hat{X} = \hat{g}(Y) = \mathbb{E}(X \mid Y)$$

Independence and Dependence

Independence and Dependence

(DF) $X_1, \dots, X_n$ are independent if

$$P\left(\bigcap_{i=1}^{n} \{X_i \in B_i\}\right) = \prod_{i=1}^{n} P(\{X_i \in B_i\})$$

$B_i \in \mathcal{B}$

(DF) Collections of events $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent if

$\forall A_1 \in \mathcal{A}_1, \quad \forall A_2 \in \mathcal{A}_2$

$$\Rightarrow P(A_1 \cdot A_2) = P(A_1) \cdot P(A_2)$$

(TH) $\exists\, \mathcal{A}_1 \perp \mathcal{A}_2 \Rightarrow \sigma(\mathcal{A}_1) \perp \sigma(\mathcal{A}_2)$

Proof is based on set approximations and extention to the limit.

(TH) $X_1, \dots, X_n$ are independent iff

CDF
$$\underset{\text{joint}}{F(x_1, \dots, x_n)} = \prod_{i=1}^{n} \underset{\text{marginals}}{F(x_i)}$$

w/o proof.

(TH) $\exists\, X, Y$ r.v. and X is $\sigma(Y)$ - measurable

$\Rightarrow \exists$ a function f :

deterministic

$$X = f(Y)$$

Extreme form of dependence between X and Y

Proof.

Use the usual logic of simple functions approximations and then proceeding to the limit.

$\Downarrow$

We need to show the result for an indicator function.

$\rceil \quad X = I_A, \quad A \in \sigma(Y)$

$A \in \sigma(Y) \Rightarrow$ A must be an inverse image of some set $B \in \mathcal{B}$, the range of Y

$\exists B: \quad A = \{\omega : Y(\omega) \in B\}$

$$A = Y^{-1}(B)$$

Introduce an indicator $\chi_B(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}$

Consider

$$\chi_B(Y(\omega)) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} = I_A(\omega) = X$$

$$X = \chi_B(Y)$$

$\swarrow$ deterministic function

$\square$

Dependence modeling

Copulae

X, Y are r.v.

joint CDF $F(x,y)$

marginal CDFs $F_X(x)$, $F_Y(y)$

$\rceil$ X, Y are continuous

$U = F_X(X) \sim \text{Uniform}[0,1]$

$V = F_Y(Y) \sim \text{Uniform}[0,1]$

U and V are dep. r.v., generally
$\exists$ $C(u,v)$ joint CDF of U and V

$$F(x,y) = C(F_X(x), F_Y(y))$$
$\downarrow$ Copula

for continuous variable $\downarrow$ $\frac{\partial^2 C}{\partial u \partial v} \geq 0$ joint density

Properties: $C(0,v) = C(u,0) = 0$ grounded

$\left.\begin{array}{l} C(1,v) = v \\ C(u,1) = u \end{array}\right\}$ with margins

C is $\uparrow$ in u and v

Approximation of derivatives

$$\left.\frac{d f(x)}{d(x)}\right|_{x_1 \leq x \leq x_2} \approx \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta_{x_1}^{x_2} f}{x_2 - x_1}$$

$$\frac{\partial f(x,y)}{\partial x \partial y} \approx \frac{\Delta_{x_1}^{x_2} \Delta_{y_1}^{y_2} f}{(x_2 - x_1)\cdot(y_2 - y_1)}$$

$\Delta_{x_1}^{x_2} \Delta_{y_1}^{y_2} f = f(x_2,y_2) + f(x_1,y_1) - f(x_1,y_2) - f(x_2,y_1)$

DF Copulas are functions on $[0,1] \times [0,1]$ Satisfying

$$C(0,v) = C(u,0) = 0$$
$$C(1,v) = v; \quad C(u,1) = u$$

$$C(u_2,v_2) + C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) \geq 0$$
$$\forall 0 \leq u_1 \leq u_2 \leq 1$$

$\forall\ 0 \leq v_1 \leq v_2 \leq 1$

TH Sklar