Now we can prove the main result of this section:

Theorem 9 *If $\mathbf{\Phi}$ is ψ-irreducible, then $\mathbf{\Phi}$ is either recurrent or transient.*

Proof:

Theorem 10 *Suppose that $\mathbf{\Phi}$ is ψ-irreducible and aperiodic.*

(i) If there is an m-skeleton chain $\mathbf{\Phi}^m$ that is transient, then $\mathbf{\Phi}$ is transient. If $\mathbf{\Phi}$ is transient, then every m-skeleton chain is transient.

(ii) If there is an m-skeleton chain $\mathbf{\Phi}^m$ that is recurrent, then $\mathbf{\Phi}$ is recurrent. If $\mathbf{\Phi}$ is recurrent, then every m-skeleton chain is recurrent.

Proof:

1.6.1 Harris recurrence

In this section we will consider a stronger concept of recurrence, called Harris recurrence. To develop this concept, we will not only consider the first hitting time τ_A of a set $A \in \mathcal{B}(\mathcal{X})$, or the expected value $U(\cdot, A)$ of η_A, but also the event that $\mathbf{\Phi} \in A$ infinitely often (i.o.), or $\eta_A = \infty$. This event (set) is defined by

$$\{\mathbf{\Phi} \in A \,\text{i.o.}\} := \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{\Phi_k \in A\}.$$

For $x \in \mathcal{X}$ and $A \in \mathcal{B}(\mathcal{X})$ let

$$Q(x, A) := P_x(\mathbf{\Phi} \in A \,\text{i.o.}).$$

Note that

$$Q(x, A) \leq L(x, A), \quad x \in \mathcal{X},\, A \in \mathcal{B}(\mathcal{X})$$

and by the strong Markov property

$$Q(x, A) = E_x[P_{\Phi_{\tau_A}}(\mathbf{\Phi} \in A \,\text{i.o.})\mathbb{I}(\tau_A < \infty)] = \int_A U_A(x, dy) Q(y, A).$$