1.4 Irreducibility

The first concept of "stochastic stability" that we study is that of irreducibility. It is a first measure of the sensitivity of the Markov chain to the initial distribution μ or initial state x_0. It is crucial in Markov chain Monte Carlo as it guarantees convergence no matter what the initial conditions. Thus, we avoid the need of a detailed study of the probability transition kernel in order to determine acceptable initial conditions (to guarantee convergence).

Recall that if $\mathcal{X}$ is countable, then for any two distinct states x, $y \in \mathcal{X}$ we say that x leads to y (written $x \to y$) if $L(x, y) > 0$ and that they communicate (written $x \leftrightarrow y$) if $L(x, y) > 0$ and $L(y, x) > 0$. By convention, $x \to x$.

Proposition 4 *The communication relation "$\leftrightarrow$" is an equivalence relation. The equivalence classes $C(x) = \{y : x \leftrightarrow y\}$ form a partition of the state space $\mathcal{X}$.*

Definition 11 (Irreducible Spaces and Absorbing Sets) *Let $\mathcal{X}$ be a countable state space for a Markov chain $\mathbf{\Phi}$.*

(i) If $\exists\, x \ni C(x) = \mathcal{X}$, then we say that $\mathcal{X}$, or the chain $\mathbf{\Phi}$, is irreducible.

(ii) If $P(y, C(x)) = 1 \,\forall\, y \in C(x)$, then $C(x)$ is called an absorbing set.

The problem with the communication relation "$\leftrightarrow$" is that it does not hold in general state spaces. Consider the case when $\mathcal{X} = \mathbb{R}$ and $P(x, A) = \Phi(A; 0, 1)$. Then it is obvious that $L(x, y) = 0$ for all singletons, y. We could consider a Borel measurable set A and determine whether or not $L(x, A) > 0$. If yes, then $x \to A$, but $L(A, x) = 0$, so A does not lead to x. In general state spaces, therefore, we cannot say in general whether we return to single states x.

Therefore, we define an analog to irreducibility. The analog that forms the basis for modern general state space analysis is φ-irreducibility.

1.4.1 φ- and ψ-irreducibility

Definition 12 (φ-irreducibilty) *A Markov chain $\mathbf{\Phi}$ is φ-irreducible if there exists a measure φ on $\mathcal{B}(\mathcal{X})$ such that, for all $x \in \mathcal{X}$ and for all $A \in \mathcal{B}(\mathcal{X})$, $L(x, A) > 0$ whenever $\varphi(A) > 0$.*

There are several other equivalent definitions of φ-irreducibility that prove useful.

Proposition 5 *The following are equivalent definitions of φ-irreducibility: A Markov chain $\mathbf{\Phi}$ is φ-irreducible if there exists a measure φ on $\mathcal{B}(\mathcal{X})$ such that, for all $x \in \mathcal{X}$ and for all $A \in \mathcal{B}(\mathcal{X})$ with $\varphi(A) > 0$*

(i) $L(x, A) > 0 \quad \Longleftrightarrow$

(ii) $P^n(x, A) > 0$, *for some* $n > 0$, *possibly depending on* x *and* $A \quad \Longleftrightarrow$

(iii) $U(x, A) > 0 \quad \Longleftrightarrow$

(iv) $K_{1/2}(x, A) > 0$.

Proof:

The assumption of φ-irreducibility precludes many obvious forms of reducible behavior. The definition guarantees that sets of positive φ-measure are reached with positive probability, regardless of the starting value. Therefore the chain cannot break up into separate pieces.

However, what we would also like is the reverse implication, that sets B such that $\varphi(B) = 0$ are avoided with probability one from most starting points. Instead of trying to restrict an irreducibility measure to sets of non-zero measure, it has been more fruitful to extend φ-irreducibility to a "maximal" irreducibility measure ψ. ψ is maximal in the sense that if ϕ satisfies Definition 12, then $\phi \prec \psi$. That is ψ dominates ϕ or ϕ is absolutely continuous with respect to ψ: if $\psi(A) = 0$, then $\phi(A) = 0$ (alternatively if $\phi(A) > 0$, then $\psi(A) > 0$).

Proposition 6 *If $\mathbf{\Phi}$ is φ-irreducible for some measure φ, then there exists a probability measure ψ on $\mathcal{B}(\mathcal{X})$ such that*

(i) *$\mathbf{\Phi}$ is ψ-irreducible;*

(ii) *for any other measure ξ, the chain $\mathbf{\Phi}$ is ξ-irreducible if and only if $\xi \prec \psi$.*

(iii) *if $\psi(A) = 0$, then $\psi\{y : L(y, A) > 0\} = 0$.*

(iv) *the probability measure ψ is equivalent to*

$$\psi'(A) := \int_{\mathcal{X}} K_{1/2}(y, A)\phi'(dy)$$

for any finite irreducibility measure ϕ'. Equivalent means $\psi' \prec \psi$ and $\psi \prec \psi'$.

Proof:

Proposition 6(iii), now guarantees that we avoid "negligible" sets B with $\psi(B) = 0$, with probability one irrespective of the initial state. And by Proposition 6(i) for sets B such that $\psi(B) > 0$, we have that $L(y, B) > 0$ so we have positive probability of reaching any non-negligible set B from any initial state.

Definition 13 (ψ-irreducible) *A Markov chain is called ψ-irreducible if it is φ-irreducible for some measure φ and the measure ψ is a maximal irreducible measure satisfying the conditions of Proposition 6. We write*

$$\mathcal{B}^{+}(\mathcal{X}) := \{A \in \mathcal{B}(\mathcal{X}) : \psi(A) > 0\}$$

for the sets of positive ψ-measure. By the equivalence of maximal irreducibility measures, $\mathcal{B}^{+}(\mathcal{X})$ is unique defined.

Definition 14 (Full and Absorbing Sets) *A set $A \in \mathcal{B}(\mathcal{X})$ is said to be*

(i) full if $\psi(A^c) = 0$.

(ii) absorbing if $P(x, A) = 1$ for $x \in A$.

Definition 15 (Accessibility) *We say that a set $B \in \mathcal{B}(\mathcal{X})$ is accessible from another set $A \in \mathcal{B}(\mathcal{X})$ if $L(x, B) > 0$ for all $x \in A$.*

A set $B \in \mathcal{B}(\mathcal{X})$ is uniformly accessible from another set $A \in \mathcal{B}(\mathcal{X})$ if there exists a $\delta > 0$ such that

$$\inf_{x \in A} L(x, B) \geq \delta. \tag{1}$$

When (1) holds, we write $A \rightsquigarrow B$.

1.5 Pseudo-atoms and Small Sets

1.5.1 Psuedo-atoms

Most of Markov chain theory on a general state space can be develop analogously to the countable state space case when $\mathcal{X}$ contains an atom for the chain $\mathbf{\Phi}$.

Definition 16 *A set $\boldsymbol{\alpha} \in \mathcal{B}(\mathcal{X})$ is called an atom for $\mathbf{\Phi}$ if there exists a measure ν on $\mathcal{B}(\mathcal{X})$ such that*

$$P(x, A) = \nu(A), \quad \forall\, x \in \boldsymbol{\alpha},\ \forall\, A \in \mathcal{B}(\mathcal{X}).$$

If $\mathbf{\Phi}$ is ψ-irreducible and $\psi(\boldsymbol{\alpha}) > 0$, then $\boldsymbol{\alpha}$ is called an accessible atom.