Biostat 602 Winter 2017

Lecture Set 19

Interval Estimation

Reading: CB Chapter 9

Interval Estimation

In Chapter 7, we have focused on $\hat{\theta}(\mathbf{X})$, which is a *point estimator* of θ, i.e. a single value as a guess for the unknown parameter. Such an estimator does not incorporate any margin of error in the estimation. This motivates interval estimation which provides a set of values as possible values for the sample space and has the capability of incorporating the error in estimation.

Interval Estimator

Let $[L(\mathbf{X}), U(\mathbf{X})]$, where $L(\mathbf{X})$ and $U(\mathbf{X})$ are functions of sample $\mathbf{X}$ and $L(\mathbf{X}) \leq U(\mathbf{X})$. Based on the observed sample $\mathbf{x}$, we can make an inference that

$$\theta \in [L(\mathbf{X}), U(\mathbf{X})]$$

Then we call $[L(\mathbf{X}), U(\mathbf{X})]$ an interval estimator of θ.

Three types of intervals

- Two-sided interval $[L(\mathbf{X}), U(\mathbf{X})]$
- One-sided (with lower-bound) interval $[L(\mathbf{X}), \infty)$
- One-sided (with upper-bound) interval $(-\infty, U(\mathbf{X})]$

Example 1: Let $X_i \sim \mathcal{N}(\mu, 1)$. Define

1. A point estimator of μ : $\overline{X}$

$$\Pr(\overline{X} = \mu) = 0$$

2. An interval estimator of μ : $[\overline{X} - 1, \overline{X} + 1]$

$$\begin{aligned}
\Pr(\mu \in [\overline{X} - 1, \overline{X} + 1]) &= \Pr(\overline{X} - 1 \leq \mu \leq \overline{X} + 1) \\
&= \Pr(\mu - 1 \leq \overline{X} \leq \mu + 1) \\
&= \Pr(-\sqrt{n} \leq \sqrt{n}(\overline{X} - \mu) \leq \sqrt{n}) \\
&= \Pr(-\sqrt{n} \leq Z \leq \sqrt{n}) \longrightarrow 1
\end{aligned}$$

as $n \to \infty$, where $Z \sim \mathcal{N}(0, 1)$.

For specific values of n, there is a positive probability content. For example, with $n = 4$, the above probability equals

$$\Pr(-2 \leq Z \leq 2) = .9544.$$

Thus we have over a 95% chance of covering the unknown parameter μ with the interval estimator. In moving from a point to an interval estimator resulted in increased confidence in our estimation.

Some Definitions

Coverage Probability: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of θ, its *coverage probability* is defined as

$$\Pr(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

In other words, it is the probability that the random interval $[L(\mathbf{X}), U(\mathbf{X})]$ covers the parameter θ.

Confidence Coefficient: The *confidence coefficient* associated with an interval estimator is defined as

$$\inf_{\theta \in \Omega} \Pr(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

Confidence Interval: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of θ, if its confidence coefficient is $1 - \alpha$, we call it a $(1 - \alpha)$ *confidence interval*

Confidence Set: If a set of estimators has confidence coefficient is $1 - \alpha$, we call it a $(1 - \alpha)$ *confidence set*

Expected Length: Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of θ, its *expected length* is defined as

$$\mathrm{E}[U(\mathbf{X}) - L(\mathbf{X})]$$

where $\mathbf{X}$ are random samples from $f_{\mathbf{X}}(\mathbf{x}|\theta)$. In other words, it is the average length of the interval estimator.

How to construct confidence interval?

A confidence interval can be obtained by inverting the acceptance region of a test. There is a one-to-one correspondence between tests and confidence intervals (or confidence sets).

Example 2: $X_i \sim \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. Consider $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$. As previously shown, level α LRT test reject H_0 if and only if

$$\left| \frac{\overline{X} - \theta_0}{\sigma/\sqrt{n}} \right| > z_{\alpha/2}$$

Equivalently, we accept H_0 if $\left|\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}}\right| \leq z_{\alpha/2}$. Accepting $H_0 : \theta = \theta_0$ implies we believe our data "agrees with" the hypothesis $\theta = \theta_0$.

$$-z_{\alpha/2} \leq \frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} \leq z_{\alpha/2}$$

$$\theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{X} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$$

The *Acceptance region* is

$$\left\{ \mathbf{x} : \theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{x} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\}.$$

As this is size α test, the probability of accepting H_0 is $1 - \alpha$.

$$\begin{aligned} 1 - \alpha &= \Pr\left(\theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{X} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right) \\ &= \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \theta_0 \leq \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right) \end{aligned}$$

Since θ_0 is arbitrary,

$$1 - \alpha = \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \theta \leq \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right)$$

Therefore, $[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}]$ is $(1 - \alpha)$ confidence interval (CI).

Confidence intervals and level α test

Theorem 9.2.2

1. For each $\theta_0 \in \Omega$, let $A(\theta_0)$ be the acceptance region of a level α test of $H_0 : \theta = \theta_0$ Define a set $C(\mathbf{x}) = \{\theta : \mathbf{x} \in A(\theta)\}$, then the random set $C(\mathbf{x})$ is a $1 - \alpha$ confidence set.

2. Conversely, if $C(\mathbf{x})$ is a $(1 - \alpha)$ confidence set for θ, for any θ_0, define the acceptance region of a test for the hypothesis $H_0 : \theta = \theta_0$ by $A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}$. Then the test has level α.

In other words, if we invert the acceptance region of the test statistic, we can obtain confidence interval, and vice versa.

Example 3: For $X_i \sim \mathcal{N}(\theta, \sigma^2)$, the acceptance region $A(\theta_0)$ is a subset of the sample space

$$A(\theta_0) = \left\{ \mathbf{x} : \theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{x} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\}$$

The confidence set $C(\mathbf{x})$ is a subset of the parameter space

$$\begin{aligned} C(\mathbf{x}) &= \left\{ \theta : \theta - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{x} \leq \theta + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\} \\ &= \left\{ \theta : \overline{x} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \theta \leq \overline{x} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\} \end{aligned}$$

There is no guarantee that the confidence set obtained from Theorem 9.2.2 is an interval, but it is so quite often

1. To obtain $(1-\alpha)$ two-sided CI $[L(\mathbf{X}), U(\mathbf{X})]$, we invert the acceptance region of a level α test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$
2. To obtain a lower-bounded CI $[L(\mathbf{X}), \infty)$, then we invert the acceptance region of a test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$, where $\Omega = \{\theta : \theta \geq \theta_0\}$.
3. To obtain a upper-bounded CI $(-\infty, U(\mathbf{X})]$, then we invert the acceptance region of a test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta < \theta_0$, where $\Omega = \{\theta : \theta \leq \theta_0\}$.

Example 4: Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$ where both parameters are unknown.

1. Find $1-\alpha$ two-sided CI for μ
2. Find $1-\alpha$ upper bound for μ

Solution - Two-sided CI

The testing problem is $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$. The LRT test rejects if and only if

$$\left|\frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}}\right| > t_{n-1,\alpha/2}$$

The acceptance region is

$$A(\mu_0) = \left\{\mathbf{x} : \left|\frac{\overline{x} - \mu_0}{s_{\mathbf{x}}/\sqrt{n}}\right| \leq t_{n-1,\alpha/2}\right\}$$

The confidence set is

$$\begin{aligned} C(\mathbf{x}) &= \left\{\mu : \left|\frac{\overline{x}-\mu}{s_{\mathbf{x}}/\sqrt{n}}\right| \le t_{n-1,\alpha/2}\right\} \\ &= \left\{\mu : -t_{n-1,\alpha/2} \le \frac{\overline{x}-\mu}{s_{\mathbf{x}}/\sqrt{n}} \le t_{n-1,\alpha/2}\right\} \\ &= \left\{\mu : \overline{x} - \frac{s_{\mathbf{x}}}{\sqrt{n}} t_{n-1,\alpha/2} \le \mu \le \overline{x} + \frac{s_{\mathbf{x}}}{\sqrt{n}} t_{n-1,\alpha/2}\right\} \end{aligned}$$

Solution - upper-bounded CI

The CI is $(-\infty, U(\mathbf{X})]$. We need to invert a testing procedure for $H_0 : \mu = \mu_0$ vs $H_1 : \mu < \mu_0$.

$$\Omega_0 = \{(\mu, \sigma^2) : \mu = \mu_0, \sigma^2 > 0\}$$

$$\Omega = \{(\mu, \sigma^2) : \mu \le \mu_0, \sigma^2 > 0\}$$

LRT statistic is

$$\lambda(\mathbf{x}) = \frac{L(\hat{\mu}_0, \hat{\sigma}_0^2 | \mathbf{x})}{L(\hat{\mu}, \hat{\sigma}^2 | \mathbf{x})}$$

where $(\hat{\mu}_0, \hat{\sigma}_0^2)$ is the MLE restricted to Ω_0, and$(\hat{\mu}, \hat{\sigma}^2)$ is the MLE restricted to Ω, and

within Ω_0, $\hat{\mu}_0 = \mu_0$, and $\hat{\sigma}_0^2 = \frac{\sum_{i=1}^n (X_i - \mu_0)^2}{n}$

Within Ω, the MLE is

$$\begin{cases} \hat{\mu} = \overline{X} \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n} & \text{if } \overline{X} \le \mu_0 \\ \hat{\mu} = \mu_0 \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \mu_0)^2}{n} & \text{if } \overline{X} > \mu_0 \end{cases}$$

$$\lambda(\mathbf{x}) = \begin{cases} 1 & \text{if } \overline{X} > \mu_0 \\ \dfrac{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}_0^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^n (X_i-\mu_0)^2}{2\hat{\sigma}_0^2}\right\}}{\left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^n (X_i-\overline{X})^2}{2\hat{\sigma}^2}\right\}} & \text{if } \overline{X} \leq \mu_0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } \overline{X} > \mu_0 \\ \left(\frac{\frac{n-1}{n}s_{\mathbf{X}}^2}{\frac{n-1}{n}s_{\mathbf{X}}^2 + (\overline{X}-\mu_0)^2}\right)^{\frac{n}{2}} & \text{if } \overline{X} \leq \mu_0 \end{cases}$$

For $0 < c < 1$, LRT test rejects H_0 if $\overline{X} \leq \mu_0$ and

$$\left(\frac{\frac{n-1}{n}s_{\mathbf{X}}^2}{\frac{n-1}{n}s_{\mathbf{X}}^2 + (\overline{X}-\mu_0)^2}\right)^{\frac{n}{2}} < c$$

$$\left(\frac{\frac{n-1}{n}}{\frac{n-1}{n} + \frac{(\overline{X}-\mu_0)^2}{s_{\mathbf{x}}^2}}\right)^{\frac{n}{2}} < c$$

$$\frac{(\overline{X}-\mu_0)^2}{s_{\mathbf{X}}^2} > c^*$$

$$\frac{\mu_0 - \overline{X}}{s_{\mathbf{X}}/\sqrt{n}} > c^{**}$$

c^{**} is chosen to satisfy

$$\begin{aligned}
\alpha &= \Pr(\text{reject } H_0|\mu_0) \\
&= \Pr\left(\frac{\mu_0 - \overline{X}}{s_{\mathbf{X}}/\sqrt{n}} > c^{**}\right) \\
&= \Pr\left(\frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} < -c^{**}\right) \\
&= \Pr(T_{n-1} < -c^{**}) \\
1 - \alpha &= \Pr(T_{n-1} > -c^{**}) \\
c^{**} &= -t_{n-1,1-\alpha} = t_{n-1,\alpha}
\end{aligned}$$

Therefore, LRT level α test reject H_0 if

$$\frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} \;<\; -t_{n-1,\alpha}$$

Acceptance region is

$$A(\mu_0) = \left\{\mathbf{x} : \frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} \geq -t_{n-1,\alpha}\right\}$$

Inverting the above to get CI

$$
\begin{aligned}
C(\mathbf{X}) &= \{\mu : \mathbf{X} \in A(\mu)\} \\
&= \left\{\mu : \frac{\overline{X} - \mu}{s_{\mathbf{X}}/\sqrt{n}} \geq -t_{n-1,\alpha}\right\} \\
&= \left\{\mu : \overline{X} - \mu \geq -\frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha}\right\} \\
&= \left\{\mu : \mu \leq \overline{X} + \frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha}\right\} \\
&= \left(-\infty, \overline{X} + \frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha}\right]
\end{aligned}
$$

Solution - lower-bounded CI

LRT level α test reject H_0 if and only if

$$
\frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} \ > \ t_{n-1,\alpha}
$$

Acceptance region is

$$
A(\mu_0) = \left\{\mathbf{x} : \frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} \leq t_{n-1,\alpha}\right\}
$$

Confidence interval is

$$
\begin{aligned}
C(\mathbf{X}) &= \{\mu : \mathbf{X} \in A(\mu)\} = \left\{\mu : \frac{\mathbf{X} - \mu}{s_{\mathbf{X}}/\sqrt{n}} \leq t_{n-1,\alpha}\right\} \\
&= \left\{\mu : \mu \geq \overline{X} - \frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha}\right\} \\
&= \left[\overline{X} - \frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha}, \infty\right)
\end{aligned}
$$

Example 4: Let $X_1, \cdots, X_n$ be iid sample from exponential distribution with mean θ. What is a $1-\alpha$ confidence interval for the estimator of θ?

Solution: We can use LRT test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

The LRT statistic is given by

$$
\begin{aligned}
\lambda(\mathbf{x}) &= \frac{\frac{1}{\theta_0^n} e^{-\sum x_i/\theta_0}}{\sup_\theta \frac{1}{\theta^n} e^{-\sum x_i/\theta}} \\
&= \frac{\frac{1}{\theta_0^n} e^{-\sum x_i/\theta_0}}{\frac{1}{(\sum x_i/n)^n} e^{-n}} \\
&= \left(\frac{\sum x_i}{n\theta_0}\right)^n e^{n-\sum x_i/\theta_0}
\end{aligned}
$$

The acceptance region is given by

$$
A(\theta_0) = \left\{\mathbf{x} : \left(\frac{\sum x_i}{\theta_0}\right)^n e^{-\sum x_i/\theta_0} \geq k\right\}
$$

where k is chosen to be $\Pr(\mathbf{X} \in A(\theta_0)|\theta_0) = 1-\alpha$. Inverting this acceptance region gives the $1-\alpha$ confidence set

$$
\begin{aligned}
C(\mathbf{x}) &= \left\{\theta : \left(\frac{\sum x_i}{\theta}\right)^n e^{-\sum x_i/\theta} \geq k\right\} \\
&= \left\{\theta : L\left(\sum x_i\right) \leq \theta \leq U\left(\sum x_i\right)\right\}
\end{aligned}
$$

where L and U are functions satisfying

$$
\left(\frac{\sum x_i}{L(\sum x_i)}\right)^n e^{-\sum x_i/L(\sum x_i)} = \left(\frac{\sum x_i}{U(\sum x_i)}\right)^n e^{-\sum x_i/U(\sum x_i)} = k
$$

Finally,

$$\frac{\sum x_i}{L(\sum x_i)} = a \qquad\qquad \frac{\sum x_i}{U(\sum x_i)} = b \qquad (a > b)$$

where a, b satisisfy the following two conditions

$$a^n e^{-a} = b^n e^{-b} \tag{1}$$

$$\Pr\left(\tfrac{1}{a}\sum X_i \le \theta < \tfrac{1}{b}\sum X_i\right) = \Pr\left(b \le \frac{\sum X_i}{\theta} \le a\right) = 1 - \alpha \tag{2}$$

The fact that $\frac{2\sum X_i}{\theta} \sim \chi^2_{2n}$ can be used to select a, b.

Example of asymptotic confidence interval

Example 5: Let $X_1, \cdots, X_n$ be iid from a distribution with mean μ and finite variance σ^2. Construct asymptotic $(1-\alpha)$ two-sided interval for μ

Solution: Recall that $\overline{X}$ is the method of moment estimator for μ.
By law of large number, $\overline{X}$ is consistent for μ, and by central limit theorem,

$$\overline{X} \sim \mathcal{AN}\left(\mu, \frac{\sigma^2}{n}\right)$$

Consider testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$. The Wald statistic

$$Z_n = \frac{\overline{X} - \mu_0}{S_n}$$

where

$$S_n = \sqrt{\frac{\sum_{i=1}^n (X_i - \overline{X})^2}{(n-1)n}}$$

is chosen as a consistent estimator of $\sigma/\sqrt{n}$. From previous lectures, we know that

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \xrightarrow{P} \sigma^2$$

$$\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{(n-1)n}} \xrightarrow{P} \frac{\sigma}{\sqrt{n}}$$

The Wald level α test is

$$\left|\frac{(\overline{X} - \mu_0)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}}\right| > z_{\alpha/2}$$

The acceptance region is

$$A(\mu_0) = \left\{\mathbf{x} : \left|\frac{(\overline{x} - \mu_0)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}}\right| \le z_{\alpha/2}\right\}$$

and so the $(1-\alpha)$ CI is

$$\begin{aligned}
C(\mathbf{x}) &= \{\mu : \mathbf{x} \in A(\mu)\} \\
&= \left\{\mu : \left|\frac{(\overline{x} - \mu)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}}\right| \le z_{\alpha/2}\right\} \\
&= \left[\overline{x} - \tfrac{1}{\sqrt{n}}\sqrt{\tfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}\, z_{\alpha/2},\ \overline{x} + \tfrac{1}{\sqrt{n}}\sqrt{\tfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}\, z_{\alpha/2}\right]
\end{aligned}$$

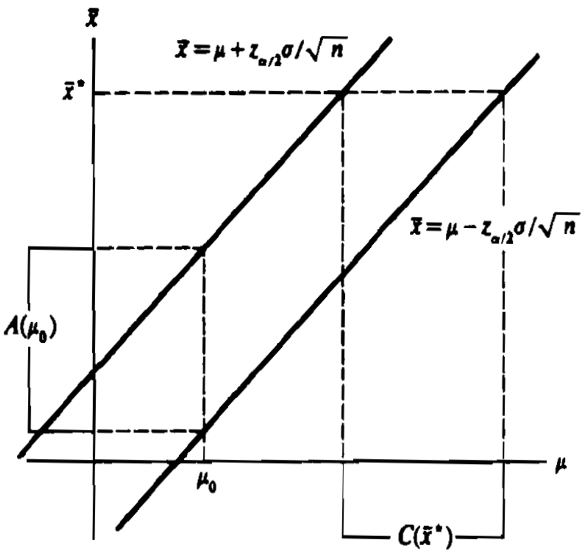

Figure 9.2.1. *Relationship between confidence intervals and acceptance regions for tests. The upper line is* $\bar{x} = \mu + z_{\alpha/2}\sigma/\sqrt{n}$ *and the lower line is* $\bar{x} = \mu - z_{\alpha/2}\sigma/\sqrt{n}$.

Discrete Distributions

Typically for discrete distributions, it is quite hard to get an explicit interval.

Example 6: Let $X_1, \cdots, X_n$ be iid $Bernoulli(p)$ an consider testing

$$H_0 : p = p_0 \; vs \; H_1 : p > p_0.$$

In this problem, $T = \sum_{i=1}^n X_i$ is a sufficient statistic. Since

$$\begin{aligned} f(\mathbf{x}|p) &= \prod_{i=1}^n p^{x_i}(1-p)^{x_i} \\ &= \left(\frac{p}{1-p}\right)^{\sum_{i=1}^n x_i} (1-p)^n \\ &= (1-p)^n \exp\left[\log\left(\frac{p}{1-p}\right) \sum_{i=1}^n x_i\right] \end{aligned}$$

conforms to an exponential family with $\omega(p) = \log\left(\frac{p}{1-p}\right)$ an increasing function of p, the family of pmf's has MLR in $T = \sum_{i=1}^n X_i$. So by Karlin-Rubin Theorem, the test that

$$\text{rejects } H_0 \text{ if } T > k(p_0)$$

is the UMP test of its size. We cannot get the size of the test to be exactly α, except for certain values of p_0, because of the discreteness of T.

The cut-off $k(p_0)$ is the integer between 0 and n that satisfies

$$\sum_{y=0}^{k(p_0)} \binom{n}{y} p_0^y (1-p_0)^{n-y} \;\geq\; 1-\alpha, \qquad \sum_{y=0}^{k(p_0)-1} \binom{n}{y} p_0^y (1-p_0)^{n-y} \;<\; 1-\alpha.$$

For each p_0, the acceptance region is given by

$$A(p_0) = \{t : t \le k(p_0)\}.$$

Correspondingly, for each value of t, the confidence set is

$$C(t) = \{p_0 : t \le k(p_0)\}.$$

While this is formally correct, this is not explicit. The $(1-\alpha)$ lower confidence bound can be shown to be given by

$$C(t) = \left\{ p_0 : p_0 > \sup_p \left\{ \sum_{y=0}^{t-1} \binom{n}{y} p^y (1-p)^{n-y} \ \ge \ 1-\alpha \right\} \right\}.$$

Pivotal Quantities

Pivotal quantities are quite useful in constructing confidence intervals.

Definition 9.2.6: A random variable $Q(\mathbf{X};\theta) = Q(X_1,\ldots,X_n;\theta)$ is a pivotal quantity if the distribution of $Q(\mathbf{X},\theta)$ is free of all parameters.

$Q(\mathbf{X};\theta)$ contains both parameters and statistics, but its distribution is free of θ. Note that a pivotal quantity is different from an ancillary statistic.

Examples

1. Consider $X_1,\cdots,X_n \sim \mathcal{N}(\mu,\sigma^2)$; σ^2 known.

$$Q(\mathbf{X};\mu) = \overline{X} - \mu$$

2. Consider $X_1,\cdots,X_n \sim \mathcal{N}(\mu,\sigma^2)$; both parameters unknown.

   - $Q_1(\mathbf{X};\mu,\sigma^2) = \frac{S_{\mathbf{X}}^2}{\sigma^2}$.
   - $Q_2(\mathbf{X};\mu,\sigma^2) = \frac{\overline{X}-\mu}{S_{\mathbf{X}}}$.

3. Consider $X_1, \cdots, X_n \sim Exp(\theta)$.

$$Q(\mathbf{X};\theta) = \frac{\sum_{i=1}^n X_i}{\theta}$$

4. Consider $X_1, \cdots, X_n \sim Uniform(\theta, \theta+1)$.

$$Q(\mathbf{X};\theta) = X_{(n)} - \theta$$

Pivotal quantity and location-scale family

Let $X_1, \cdots, X_n$ be a random sample from $f(x|\theta)$.

Location Family

$$f(x|\theta) \sim f_0(x-\theta) \quad \text{where} \quad f_0 \text{ is parameter free.}$$

Then

$$Q(\mathbf{X};\theta) = (\hat{\theta}_{MLE} - \theta) \quad \text{is a pivotal.}$$

Scale Family

$$f(x|\theta) \sim \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right) \quad \text{where} \quad f_0 \text{ is parameter free.}$$

Then

$$Q(\mathbf{X};\theta) = \frac{\hat{\theta}_{MLE}}{\theta} \quad \text{is a pivotal.}$$

Location-Scale Family

$$f(x|\mu,\sigma) \sim \frac{1}{\sigma} f_0\left(\frac{x-\mu}{\sigma}\right) \quad \text{where} \quad f_0 \text{ is parameter free.}$$

Then

$$Q(\mathbf{X};\mu,\sigma) = \frac{\hat{\mu}_{MLE} - \mu}{\hat{\sigma}_{MLE}} \quad \text{is a pivotal.}$$

Once we have a pivotal quantity, then for any specified α, we can find numbers a and b, which do not depend on θ, and satisfy

$$\Pr_{\theta}\left[a \le Q(\mathbf{X};\theta) \le b\right] \;\ge\; 1-\alpha.$$

So a $1-\alpha$ confidence set for θ is given by

$$C(\mathbf{x}) = \{\theta_0 : a \le Q(\mathbf{X};\theta_{\mathbf{0}}) \le b.\}$$

If θ is a real-valued parameter, and if for each $\mathbf{x} \in \mathcal{X}$, the pivotal $Q(\mathbf{X};\theta)$ is a monotone function of θ, then $C(\mathbf{x})$ will be an interval.

Example 7: Let $X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ where both parameters are unknown. Since $\mathcal{N}(\mu, \sigma^2)$ is a location-scale family, and

$$\hat{\mu}_{MLE} = \overline{X}, \quad \hat{\sigma}^2_{MLE} = \frac{(n-1)S^2_{\mathbf{X}}}{n},$$

$$Q(\mathbf{X};\mu,\sigma^2) = \frac{\overline{x}-\mu}{\sqrt{(n-1)S^2_{\mathbf{X}}/n}} \quad \text{is a pivot.}$$

Note that $T = \sqrt{n-1}\; Q = \frac{\overline{X}-\mu}{S_{\mathbf{X}}/\sqrt{n}} \;\sim\; t_{(n-1)}$ and hence

$$\Pr[a \le T \le b] = 1-\alpha$$

for specific percentiles of $t_{(n-1)}$. Making an equal tailed choice

$$a = -t_{(n-1),\alpha/2},\;\; b = t_{(n-1),\alpha/2}$$

and so a $1-\alpha$ confidence interval for μ is the familiar one

$$C(\mathbf{x}) = \left\{\mu : \overline{x} - t_{(n-1),\alpha/2}\,\frac{s_{\mathbf{x}}}{\sqrt{n}} \;\le\; \mu \;\le\; \overline{x} + t_{(n-1),\alpha/2}\,\frac{s_{\mathbf{x}}}{\sqrt{n}}\right\}.$$

Example 8: Let $X_1, \cdots, X_n \sim Exp(\theta)$. Find a pivotal and construct a equal-tailed $1-\alpha$ confidence interval for θ based on the pivotal.

Example 9: Let $X_1, \cdots, X_n \sim Exp(\mu, 1)$ with pdf

$$f(x|\mu) = e^{-(x-\mu)}\ I(x > \mu), \quad -\infty < \mu < \infty.$$

Find a pivotal and construct a equal-tailed $1-\alpha$ confidence interval for μ based on the pivotal.