If $\mathbf{\Phi}$ is recurrent, then we will use π to denote the unique invariant measure. That is $\pi = \mu^0_{\boldsymbol{\alpha}}$.

This leads us to the following criterion for positivity.

Theorem 16 (Kac's Theorem) *If $\mathbf{\Phi}$ is ψ-irreducible and admits an atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathcal{X})$ (an accessible atom), then $\mathbf{\Phi}$ is positive recurrent if and only if $\mathbb{E}_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}}) < \infty$. If π is the invariant probability measure for $\mathbf{\Phi}$, then*

$$\pi(\boldsymbol{\alpha}) = \mathbb{E}^{-1}_{\boldsymbol{\alpha}}(\tau_{\boldsymbol{\alpha}}).$$

Proof:

1.6.3 The existence of an invariant measure—recurrent chains

Proposition 20 *Let* $\mathbf{\Phi}$ *be a strongly aperiodic Markov chain and let* $\check{\mathbf{\Phi}}$ *denote the split chain. Then*

(i) if the measure $\check{\pi}$ *is invariant for* $\check{\mathbf{\Phi}}$ *then the measure* π *on* $\mathcal{B}(\mathcal{X})$ *defined by*

$$\pi(A) = \check{\pi}(A_0 \cup A_1), \quad A \in \mathcal{B}(\mathcal{X}),$$

is invariant for $\mathbf{\Phi}$ *and* $\check{\pi} = \pi^*$.

(ii) If μ *is any subinvariant measure for* $\mathbf{\Phi}$ *then* μ^* *is subinvariant for* $\check{\mathbf{\Phi}}$*, and if* μ *is invariant then so is* μ^*.

Proof:

So now we have that a strongly aperiodic Markov chain admits an invariant measure if the split chain does. This leads us to

Proposition 21 *If* $\mathbf{\Phi}$ *is recurrent and strongly aperiodic, then* $\mathbf{\Phi}$ *admits a unique (up to constant multiples) subinvariant measure that is invariant.*

Proof:

Now we use the fact that the resolvent chain is strongly aperiodic to lift this result to the whole chain, even when the chain is not strongly aperiodic.

Lemma 10 *For any $\epsilon \in (0,1)$, a measure π is invariant with respect to the resolvent K_ϵ if and only if it is invariant with respect to P (the one step transition probability kernel for the original chain).*

Proof:

Using this lemma we can then prove

Proposition 22 *If $\mathbf{\Phi}$ is recurrent then $\mathbf{\Phi}$ has a unique (up to constant multiples) subinvariant measure that is invariant.*

Proof:

And if $\mathbf{\Phi}$ is aperiodic, we can show that it admits an invariant measure by using its skeletons.

Theorem 17 *Suppose that $\mathbf{\Phi}$ is ψ-irreducible and aperiodic. Then, for each m, a measure π is invariant for the m-skeleton if and only if it is invariant for the $\mathbf{\Phi}$.*

Hence, under aperiodicity, the chain $\mathbf{\Phi}$ is positive if and only if each of the m-skeletons is positive.

Proof: