Biostat 801 Homework 8

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1 Problem

Let

$$Y_{n,i}(x) = 1_{[X_i \in (x-a_n, x+a_n]]} \overset{iid}{\sim} \text{Bern}(p_n(x))$$

where

$$p_n(x) = F((x-a_n, x+a_n]).$$

Then

$$p_n(x) \to 0 \quad \text{and} \quad \frac{p_n(x)}{2a_n} \to f(x)$$

as $a_n \to 0$. Moreover,

$$\begin{aligned} E[\hat{f}_n(x)] =& E\left[\frac{\hat{F}_n(x+a_n) - \hat{F}_n(x-a_n)}{2a_n}\right] \\ =& E\left[\frac{\sum_{i=1}^n Y_{n,i}(x)}{2na_n}\right] \\ =& \frac{np_n(x)}{2na_n} \\ =& \frac{p_n(x)}{2a_n} \to f(x) \quad \text{as} \quad a_n \to 0 \end{aligned}$$

2 Problem

From the previous problem, we have

$$\begin{aligned} Var[\hat{f}_n(x)] =& \frac{Var \sum_{i=1}^n Y_{n,i}(x)}{4n^2 a_n^2} \\ =& \frac{np_n(x)(1-p_n(x))}{4n^2 a_n^2} \\ =& \frac{1}{2na_n} \frac{p_n(x)}{2a_n}(1-p_n(x)) \\ \to& 0 \cdot f(x) \cdot (1-0) = 0 \end{aligned}$$

as $a_n \to 0$ and $na_n \to \infty$.

3 Problem

We have

$$E\{(\hat{f}_n(x) - E[\hat{f}_n(x)])^2\} = Var[\hat{f}_n(x)] \to 0$$

as $na_n \to \infty$, so

$$\hat{f}_n(x) \xrightarrow{L_2} E[\hat{f}_n(x)].$$

Moreover, since $E[\hat{f}_n(x)] \to f(x)$,

$$E[\hat{f}_n(x)] \xrightarrow{L_2} f(x).$$

Then by the triangular inequality,

$$\hat{f}_n(x) \xrightarrow{L_2} f(x),$$

which implies

$$\hat{f}_n(x) \xrightarrow{p} f(x).$$

Thus $\hat{f}_n$ is a consistent estimator for f.

4 Problem

Let

$$W_{n,i}(x) = \frac{Y_{n,i}(x)}{2a_n}.$$

Then

$$\hat{f}_n(x) = \frac{1}{n}\sum_{i=1}^{n} W_{n,i}(x) = \bar{W}_n(x).$$

$$E[\hat{f}_n(x)] = \frac{1}{n}\sum_{i=1}^{n} E[W_{n,i}(x)] = E[W_{n,1}(x)] = \frac{p_n(x)}{2a_n}$$

and

$$Var[W_{n,1}(x)] = \frac{p_n(x)}{2a_n}\frac{(1-p_n(x))}{2a_n} \to \frac{f(x)}{2a_n}$$

as $a_n \to 0$. Then as $n \to \infty$ and $a_n \to 0$,

$$\frac{\bar{W}_n(x) - E[W_{n,1}(x)]}{\sqrt{Var[W_{n,1}(x)]/n}} \to \sqrt{2na_n}\frac{\hat{f}_n(x) - E\hat{f}_n(x)}{\sqrt{f(x)}}$$

and by CLT,

$$\frac{\bar{W}_n(x) - E[W_{n,1}(x)]}{\sqrt{Var[W_{n,1}(x)]/n}} \xrightarrow{d} N(0,1).$$

Then by Slutsky,

$$\sqrt{2na_n}\frac{\hat{f}_n(x) - E\hat{f}_n(x)}{\sqrt{f(x)}} \xrightarrow{d} N(0,1).$$