# Biostat 801 Homework 6

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## 1 Problem

Let $m \geq 1$. Then

$$\begin{aligned}
P[|X-Y| > 1/m] &\leq P[|X-X_n| + |Y-X_n| > \frac{1}{m}] \\
&\leq P([|X-X_n| > \frac{1}{2m}] \cup [|Y-X_n| > \frac{1}{2m}]) \\
&\leq P[|X-X_n| > \frac{1}{2m}] + P[|Y-X_n| > \frac{1}{2m}] \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $P[|X-Y| > \frac{1}{m}] = 0$. Then

$$\begin{aligned}
P[X \neq Y] = P[|X-Y| > 0] &= P(\bigcup_{m=1}^{\infty} [|X-Y| > \frac{1}{m}]) \\
&= \lim_{m\to\infty} P[|X-Y| > \frac{1}{m}] = \lim_{m\to\infty} 0 = 0
\end{aligned}$$

## 2 Problem

### 2.1

We have

$$P[||X| - |X_n|| < \epsilon] \leq P[|X - X_n| < \epsilon] \to 0.$$

Thus $|X_n| \xrightarrow{P} |X_n|$.

### 2.2

1. To show: $X_n Y \xrightarrow{P} XY$.
Proof: We have

$$\begin{aligned}
&P[|X_n Y - XY| > \epsilon] \\
=&P[|X_n - X||Y| > \epsilon] \\
=&P([|X_n - X||Y| > \epsilon] \cap [|Y| > M]) \cup ([|X_n - X||Y| > \epsilon] \cap [|Y| \leq M]) \\
\leq&P[|Y| > M] + P[|X_n - X|M > \epsilon] \\
\rightarrow&P[|Y| > M] \text{ as } n \to \infty
\end{aligned}$$

Since $M \geq 1$ is arbitrary and $P[|Y| > M] \to 0$ as $M \to \infty$, we conclude that $\lim_{n\to\infty} P[|X_n Y - XY| > \epsilon] = 0$.

2. To show: $X_n Y_n \xrightarrow{P} XY_n$.
Proof: Following the previous arguement, we have

$$P[|X_n Y - XY| > \epsilon] \leq P[|Y_n| > M] + P[|X_n - X|M > \epsilon]$$

Since

$$Y_n \xrightarrow{P} Y \Rightarrow |Y_n| \xrightarrow{P} |Y| \Rightarrow |Y_n| \xrightarrow{d} |Y|,$$

we have

$$\lim_{n\to\infty} P[|Y_n| > M] = P[|Y| > M].$$

Then

$$\lim_{n\to\infty} P[|X_n Y - XY| = P[|Y| > M] \to 0$$

as we let $M \to \infty$.

3. To show: $X_n Y_n \xrightarrow{P} XY$.
Proof: We have

$$\begin{aligned}
&P[|X_n Y_n - XY| > \epsilon] \\
\leq&P[|X_n Y_n - X_n Y| + |X_n Y - XY| > \epsilon] \\
\leq&P[|X_n Y_n - X_n Y| > \frac{\epsilon}{2}] + P[|X_n Y - XY| > \frac{\epsilon}{2}] \\
\rightarrow&0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

## 3 Problem

### 3.1

Since $X_n \xrightarrow{a.s.} X$,

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall \omega \in A$$

for some $A \subset \Omega$ with $P(A^C) = 0$. Then

$$\lim_{n\to\infty} |X_n(\omega)| = |X(\omega)| \quad \forall \omega \in A.$$

Thus $|X_n| \xrightarrow{a.s.} |X|$.

### 3.2

We have

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall \omega \in A$$

and

$$\lim_{n\to\infty} Y_n(\omega) = Y(\omega) \quad \forall \omega \in B$$

for some $A, B \subset \Omega$ with $P(A^C) = P(B^C) = 0$. Then

$$P((AB)^C) = P(A^C \cup B^C) \leq P(A^C) + P(B^C) = 0$$

and

$$\lim_{n\to\infty} X_n(\omega)Y_n(\omega) = X(\omega)Y(\omega) \quad \forall \omega \in AB.$$

Thus $X_nY_n \overset{a.s.}{\to} XY$.