Solution to Assignment 4

4.17: **(c)** Clearly $P(X \geq n) \geq P(X_n = n) = \lambda^n e^{-\lambda}/n!$. For the other bound, since $\binom{n+k}{k} \geq 1$, we have $(k+n)! \geq k!n!$. Using this,

$$P(X_n \geq n) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+n}}{(n+k)!} \leq e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+n}}{n!k!} = \frac{\lambda^n}{n!}.$$

By Stirling's approximation, as $n \to \infty$

$$\log n! = n \log n - n + O(\log n),$$

and so

$$\log P(X_n > x) = x \log \lambda - x \log x + x + O(\log x)$$

as $x \to \infty$. Taking $Y_n = \log(\log n) X_n / \log n$, as $n \to \infty$,

$$\log P(Y_n > c) = -c \log n + O\left(\frac{\log n}{\log(\log n)}\right).$$

If $\epsilon > 0$ and $c = 1 + 2\epsilon$, then for n large enough we have

$$\log P(Y_n > c) \leq -(1+\epsilon) \log n \quad \text{or} \quad P(Y_n > 1 + 2\epsilon) \leq \frac{1}{n^{1+\epsilon}}.$$

Thus

$$\sum_n P(Y_n > 1 + 2\epsilon) < \infty.$$

But if $c = 1 - 2\epsilon$, for n large enough we have

$$\log P(Y_n > 1 - 2\epsilon) \geq -(1-\epsilon) \log n \quad \text{or} \quad P(Y_n > 1 + 2\epsilon) \leq \frac{1}{n^{1-\epsilon}}.$$

Thus

$$\sum_n P(Y_n > 1 - 2\epsilon) = \infty.$$

By the Borel zero-one law,

$$P(Y_n > 1 + 2\epsilon, \text{i.o.}) = 0 \qquad \text{and} \qquad P(Y_n > 1 - 2\epsilon, \text{i.o.}) = 1.$$

Since $\epsilon > 0$ can be arbitrarily small, these imply (with the reasoning detailed in Example 4.5.2) that $\limsup Y_n = 1$ with probability one.

4.18: If $B \in \mathcal{P}$, then by additivity

$$\begin{aligned} P(A^c B) &= P(B) - P(AB) \\ &= P(B) - P(A)P(B) \\ &= \big(1 - P(A)\big) P(B) \\ &= P(A^c) P(B). \end{aligned}$$

Thus A^c is independent of $\mathcal{P}$. Noting that $\emptyset$ and Ω are also independent of $\mathcal{P}$, we have $\sigma(A)$ independent of $\mathcal{P}$. But $\mathcal{P}$ is a π-system, so by Theorem 4.1.1, $\sigma(A)$ is independent of $\sigma(\mathcal{P})$. As $A \in \sigma(\mathcal{P})$, A is independent of itself, and we must have $P(A) = 0$ or 1.

4.21: **(a)** Take $n_k \to \infty$ so that $P(A_{n_k}) \geq 1 - 1/3^k$. Then by Boole's inequality,

$$P(\cap_{k\geq 1} A_{n_k}) = 1 - P(\cup_{k\geq 1} A_{n_k}^c) \geq 1 - \sum_{k\geq 1} P(A_{n_k}^c) \geq 1 - \sum_{k\geq 1} 3^{-k} = 1/2.$$

4.24: This holds because

$$\begin{aligned} P(A_n, \text{i.o.}) &= P\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} A_k\right) \\ &= \lim_{n\to\infty} P\left(\bigcup_{k\geq n} A_k\right) \\ &= \liminf_{n\to\infty} P\left(\bigcup_{k\geq n} A_k\right) \\ &\geq \liminf_{n\to\infty} P(A_n) \\ &\geq \epsilon. \end{aligned}$$

5.3: Noting that for $n \geq 1$,

$$P(L_1 > n) = P(R_2 \neq 1, \ldots, R_n \neq 1) = \frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{n-1}{n} = \frac{1}{n},$$

we have

$$P(L_1 = n) = P(L_1 > n-1) - P(L_1 > n) = \frac{1}{n(n-1)}, \qquad n \geq 2.$$

Since $L_1 1_{[L_1 \leq n]}$ are simple functions increasing to L_1,

$$EL_1 = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k-1} = \infty.$$

5.6: **(a)** Since $X1_{[|X|>n]} \to 0$ (pointwise) as $n \to \infty$, this follows by dominated convergence as

$$|X1_{[|X|>n]}| \leq |X| \in L_1.$$

(b) Using the decomposition in the hint,

$$\int_{A_n} |X|\,dP \le MP(A_n) + \int_{[|X|>M]} |X|\,dP,$$

and so, since $P(A_n) \to 0$,

$$\limsup \int_{A_n} |X|\,dP \le \int_{[|X|>M]} |X|\,dP.$$

The desired result follows because this inequality holds for all M and the upper bound tends to zero as $M \to \infty$ by part (a).

(c) First note that if $Y = \sum_1^k c_i 1_{B_i}$ is a nonnegative simple function with $c_i > 0$ for all i, and if $P(N) = 0$, then

$$\int_N Y\,dP = \int 1_N Y\,dP = \sum_1^k c_i P(NB_i) = 0.$$

Also, if

$$EY = \sum_i c_i P(B_i) = 0,$$

then $P(B_i) = 0$, $i = 1, \ldots, k$, and so

$$P(Y > 0) = P\left(\bigcup B_i\right) = 0.$$

Next, consider the assertion in the problem with $A = \Omega$. Let Y_n, $n \ge 1$, be nonnegative simple functions increasing to $|X|$. Suppose $E|X| = 0$. Since $0 \le Y_n \le |X|$, we have $EY_n = 0$, and thus $P(Y_n > 0) = 0$. Since $[Y_n > 0] \uparrow [|X| > 0]$, by continuity we then have $P(|X| > 0) = \lim P(Y_n > 0) = 0$. So $E|X| = 0$ implies $P(|X| > 0) = 0$. For the converse, suppose $P(|X| > 0) = 0$. Since $Y_n \le |X|$, event $[Y_n > 0] \subset N \stackrel{\text{def}}{=} [|X| > 0]$ and so $Y_n = 1_N Y_n$. But then, since Y_n is simple and $P(N) = 0$,

$$EY_n = E1_N Y_n = \int_N Y_n\,dP = 0$$

and $E|X| = \lim EY_n = 0$. Thus $E|X| = 0$ if and only if $P(|X| > 0) = 0$. If we change X to $1_A X$ in this assertion, noting that $[|1_A X| > 0] = A \bigcap [|X| > 0]$, we have $\int_A |X|\,dP = 0$ if and only if $P(A \bigcap [|X| > 0]) = 0$, as desired.