Solution to Assignment 5

5.8: **(a)** Let $K_n = \sup_{\omega\in\Omega} |X(\omega) - X_n(\omega)|$. Note that X is a bounded variable as the X_n are uniformly bounded and converge uniformly to X. So all of the variables are integrable and by monotonicity and the modular inequality we have

$$|EX_n - EX| = \big|E[X_n - X]\big| \le E|X_n - X| \le K_n \to 0.$$

5.9: By Fubini's theorem,

$$\begin{aligned}\int \big(F(x+a) - F(x)\big)\,dx &= \iint 1_{(x,x+a]}(u)\,F(du)\,dx \\ &= \iint 1_{(x,x+a]}(u)\,dx\,F(du) = \int a\,F(du) = a.\end{aligned}$$

5.13: Since $X_n(\omega) = 0$ for $n > 1/\omega$, we have $X_n \to 0$. Also, $EX_n = 1/\log n \to 0$. But

$$\sup_n X_n(\omega) = \frac{n}{\log n}, \qquad \text{achieved when} \qquad n \in [1/\omega - 1, 1/\omega),$$

and so

$$E\big[\sup_n X_n\big] = \sum_{n\ge 1} \frac{1}{(n+1)\log n} = \infty.$$

(Note that $\int_e^M (x\log x)^{-1}\,dx = \log\log M \to \infty$ as $M \to \infty$.)

5.15: **(a)** As $n \to \infty$,

$$\frac{n}{X} 1_{[X>n]} \to 0,$$

so the results follow by dominated convergence because

$$\left|\frac{n}{X} 1_{[X>n]}\right| \le 1.$$

5.19: **(a)** Since $\int f(x,y)\,\mu_1(dx) = 0$ and $\int f(x,y)\,\mu_2(dy) = 1$, we have

$$\iint f(x,y)\,\mu_2(dy)\,\mu_1(dx) = 1 \quad \text{and} \quad \iint f(x,y)\,\mu_1(dx)\,\mu_2(dy) = 0.$$

(b) No, μ_2 is not σ-finite. If $\mu_2(B_n) = \#B_n < \infty$, then $\bigcup_n B_n$ will be countable and cannot be the whole interval $(0,1)$.

5.32: **(a)** Since $EX = 0$ and $EXY = EX^3 = 0$, $\text{Cov}(X,Y) = EXY - (EX)(EY) = 0$. **(b)** $EX = EY = 0$, and so

$$\begin{aligned}\text{Cov}(X,Y) &= EXY = E(U^2 - V^2)\\ &= EU^2 - EV^2 = \text{Var}(U) - \text{Var}(V) = 0.\end{aligned}$$

(c) By part (b), X and Y are uncorrelated. But they are not independent. For instance,

$$P(X = 12, Y = 0) = \frac{1}{36} \neq \frac{1}{216} = P(X = 12)P(Y = 0).$$