Biostat 602 Winter 2017

Lecture Set 15

Asymptotic Evaluation of Estimators

Reading: CB 10.1

Consistency

Asymptotic Evaluation of Point Estimators

When the sample size n approaches infinity, the behaviors of an estimator are unknown as its *asymptotic* properties.

Definition - Consistency: Let $W_n = W_n(X_1, \cdots, X_n) = W_n(\mathbf{X})$ be a sequence of estimators for $\tau(\theta)$. We say W_n is consistent for estimating $\tau(\theta)$ if $W_n \xrightarrow{P} \tau(\theta)$ under P_θ for every $\theta \in \Omega$.

$W_n \xrightarrow{P} \tau(\theta)$ (converges in probability to $\tau(\theta)$) means that, given any $\epsilon > 0$.

$$\lim_{n\to\infty} \Pr(|W_n - \tau(\theta)| \geq \epsilon) = 0$$

$$\lim_{n\to\infty} \Pr(|W_n - \tau(\theta)| < \epsilon) = 1$$

Consistency implies that the probability of W_n being close to $\tau(\theta)$ approaches to 1 as n goes to ∞.

Tools for proving consistency

- Use definition (complicated)
- Chebychev's Inequality

$$\begin{aligned}
\Pr(|W_n - \tau(\theta)| \geq \epsilon) &= \Pr((W_n - \tau(\theta))^2 \geq \epsilon^2) \\
&\leq \frac{\mathrm{E}[W_n - \tau(\theta)]^2}{\epsilon^2} \\
&= \frac{\mathrm{MSE}(W_n)}{\epsilon^2} = \frac{\mathrm{Bias}^2(W_n) + \mathrm{Var}(W_n)}{\epsilon^2}
\end{aligned}$$

Need to show that both $\mathrm{Bias}(W_n)$ and $\mathrm{Var}(W_n)$ converge to zero

Theorem 10.1.3: If W_n is a sequence of estimators of $\tau(\theta)$ satisfying

- $\lim_{n->\infty} \text{Bias}(W_n) = 0$.
- $\lim_{n->\infty} \text{Var}(W_n) = 0$.

for all θ, then W_n is consistent for $\tau(\theta)$

Consistency of $\overline{X}$

Theorem 5.5.2 - Weak Law of Large Numbers: Let $X_1, \cdots, X_n$ be iid random variables with $\text{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2 < \infty$. Then $\overline{X}$ converges in probability to μ, i.e. $\overline{X} \xrightarrow{P} \mu$.

Consistent sequence of estimators

Theorem 10.1.5: Let W_n is a consistent sequence of estimators of $\tau(\theta)$. Let a_n, b_n be sequences of constants satisfying

1. $\lim_{n\to\infty} a_n = 1$
2. $\lim_{n\to\infty} b_n = 0$.

Then $U_n = a_n W_n + b_n$ is also a consistent sequence of estimators of $\tau(\theta)$.

Continuous Mapping Theorem - Theorem 5.5.4: If W_n is consistent for θ ($W_n \xrightarrow{P} \theta$) and g is a continuous function, then $g(W_n)$ is consistent for $g(\theta)$ ($g(W_n) \xrightarrow{P} g(\theta)$).

Example 1: $X_1, \cdots, X_n$ are iid samples from a distribution with mean μ and variance $\sigma^2 < \infty$.

1. Show that $\overline{X}$ is consistent for μ.
2. Show that $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is consistent for σ^2.
3. Show that $\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is consistent for σ^2.

Proof: By law of large numbers, $\overline{X}$ is consistent for μ.

Also

- $\mathrm{Bias}(\overline{X}) = \mathrm{E}(\overline{X}) - \mu = \mu - \mu = 0$.
- $\mathrm{Var}(\overline{X}) = \mathrm{Var}\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \sigma^2/n$.
- $\lim_{n\to\infty}\mathrm{Var}(\overline{X}) = \lim_{n\to\infty}\frac{\sigma^2}{n} = 0$.

By Theorem 10.1.3. $\overline{X}$ is consistent for μ.

Solution - consistency for σ^2

$$
\begin{aligned}
\frac{\sum(X_i - \overline{X})^2}{n} &= \frac{\sum(X_i^2 + \overline{X}^2 - 2X_i\overline{X})}{n} \\
&= \frac{\sum X_i^2 + n\overline{X}^2 - 2\overline{X}\sum_{i=1}^{n} X_i}{n} = \frac{\sum X_i^2}{n} - \overline{X}^2
\end{aligned}
$$

By law of large numbers,

$$\frac{1}{n}\sum X_i^2 \xrightarrow{P} \mathrm{E}X^2 = \mu^2 + \sigma^2$$

Note that $\overline{X}^2$ is a function of $\overline{X}$. Define $g(x) = x^2$, which is a continuous function. Then $\overline{X}^2 = g(\overline{X})$ is consistent for μ^2. Therefore,

$$\frac{\sum(X_i - \overline{X})^2}{n} = \frac{\sum X_i^2}{n} - \overline{X}^2 \xrightarrow{P} (\mu^2 + \sigma^2) - \mu^2 = \sigma^2$$

So, $\sum(X_i - \overline{X})^2/n$ is consistent for σ^2.

Define $S_n^2 = \frac{1}{n-1}\sum(X_i - \overline{X})^2$, and $(S_n^*)^2 = \frac{1}{n}\sum(X_i - \overline{X})^2$.

$$S_n^2 = \frac{1}{n-1}\sum(X_i - \overline{X})^2 = (S_n^*)^2 \cdot \frac{n}{n-1}$$

Because $(S_n^*)^2$ was shown to be consistent for σ^2 previously, and $a_n = \frac{n}{n-1} \to 1$ as $n \to \infty$, by Theorem 10.1.5, S_n^2 is also consistent for σ^2.

Example 2 - Exponential Suppose $X_1, \cdots, X_n$ *iid* Exponential(β).

1. Propose a consistent estimator of the population median.
2. Propose a consistent estimator of $\Pr(X \leq c)$ where c is constant.

Consistent estimator for the median

First, we need to express the median in terms of the parameter β.

$$\begin{aligned}
\int_0^m \frac{1}{\beta} e^{-x/\beta} dx &= \frac{1}{2} \\
-e^{-x/\beta}\Big|_0^m &= \frac{1}{2} \\
1 - e^{-m/\beta} &= \frac{1}{2} \\
\text{median} &= m = \beta \log 2
\end{aligned}$$

By law of large numbers, $\overline{X}$ is consistent for $\mathrm{E}(X) = \beta$. Applying continuous mapping Theorem to $g(x) = x \log 2$, $g(\overline{X}) = \overline{X} \log 2$ is consistent for $g(\beta) = \beta \log 2$ (median).

Consistent estimator of $\Pr(X \leq c)$

$$\begin{aligned}\Pr(X \leq c) &= \int_0^c \frac{1}{\beta} e^{-x/\beta} dx \\ &= 1 - e^{-c/\beta}\end{aligned}$$

As $\overline{X}$ is consistent for β, $1 - e^{-c/\beta}$ is continuous function of β. By continuous mapping Theorem, $g(\overline{X}) = 1 - e^{-c/\overline{X}}$ is consistent for $\Pr(X \leq c) = 1 - e^{-c/\beta} = g(\beta)$

Consistent estimator of $\Pr(X \leq c)$ **- Alternative Method**

Define $Y_i = I(X_i \leq c)$. Then Y_i *iid* Bernoulli(p) where $p = \Pr(X \leq c)$.

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq c)$$

is consistent for p by the Weak Law of Large Numbers.

Theorem 10.1.6 - Consistency of MLEs

Suppose X_i *iid* $f(x|\theta)$. Let $\hat{\theta}$ be the MLE of θ, and $\tau(\theta)$ be a continuous function of θ. Then under "regularity conditions" on $f(x|\theta)$, the MLE of $\tau(\theta)$ (i.e. $\tau(\hat{\theta})$) is consistent for $\tau(\theta)$. The regularity conditions, described in 10.6.2, include iid, identifiability, differentiability, parameter space containing open set.

Asymptotic Normality

Definition: Asymptotic Normality A statistic (or an estimator) $W_n(\mathbf{X})$ is *asymptotically normal* if

$$\sqrt{n}(W_n - \tau(\theta)) \xrightarrow{d} \mathcal{N}(0, \nu(\theta))$$

for all θ
where $\xrightarrow{d}$ stands for "converge in distribution"

- $\tau(\theta)$: "asymptotic mean"
- $\nu(\theta)$: "asymptotic variance"

We denote $W_n \sim \mathcal{AN}\left(\tau(\theta), \frac{\nu(\theta)}{n}\right)$.

A general definition of asymptotic normality is defined as $k_n(T_n - \tau(\theta)) \xrightarrow{d} N(0, \nu(\theta))$. The definition above is when $k_n = \sqrt{n}$.

Given a statistic $W_n(\mathbf{X})$, for example $\overline{X}$, $s^2_{\mathbf{X}}$, $e^{-\overline{X}}$

$$\sqrt{n}(W_n - \tau(\theta)) \quad \xrightarrow{d} \quad \mathcal{N}(0, \nu(\theta)) \qquad \text{for all } \theta$$

$$\iff W_n \quad \sim \quad \mathcal{AN}\left(\tau(\theta), \frac{\nu(\theta)}{n}\right)$$

Tools to show asymptotic normality

1. Central Limit Theorem
2. Slutsky Theorem
3. Delta Method (Theorem 5.5.24)

Central Limit Theorem 5.5.14

Assume X_i *iid* $f(x|\theta)$ with finite mean $\mu(\theta)$ and variance $\sigma^2(\theta)$. Then

$$\begin{aligned}\overline{X} &\sim \mathcal{AN}\left(\mu(\theta), \frac{\sigma^2(\theta)}{n}\right) \\ \Leftrightarrow \sqrt{n}\left(\overline{X} - \mu(\theta)\right) &\xrightarrow{d} \mathcal{N}(0, \sigma^2(\theta))\end{aligned}$$

Theorem 5.5.17 - Slutsky's Theorem If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$, where a is a constant,

1. $Y_n \cdot X_n \xrightarrow{d} aX$
2. $X_n + Y_n \xrightarrow{d} X + a$

Theorem 5.5.24 - Delta Method Assume $W_n \sim \mathcal{AN}\left(\theta, \frac{\nu(\theta)}{n}\right)$. If a function g satisfies $g'(\theta) \neq 0$, then

$$g(W_n) \sim \mathcal{AN}\left(g(\theta), [g'(\theta)]^2 \frac{\nu(\theta)}{n}\right)$$

Example 3 - Estimator of $\Pr(X \leq c)$ Define $Y_i = I(X_i \leq c)$. Then Y_i *iid* Bernoulli(p) where $p = \Pr(X \leq c)$.

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq c)$$

is consistent for p. Therefore,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} I(X_i \leq c) &\sim \mathcal{AN}\left(\mathrm{E}(Y), \frac{\mathrm{Var}(Y)}{n}\right) \\ &= \mathcal{AN}\left(p, \frac{p(1-p)}{n}\right)\end{aligned}$$

Example 4: Let $X_1, \cdots, X_n$ be iid samples with finite mean μ and variance σ^2. Define

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

By Central Limit Theorem,

$$\begin{aligned}
\overline{X} \quad &\sim \quad \mathcal{AN}\left(\mu, \frac{\sigma^2}{n}\right) \\
\Leftrightarrow \sqrt{n}(\overline{X} - \mu) \quad &\xrightarrow{d} \quad \mathcal{N}(0, \sigma^2) \\
\Leftrightarrow \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \quad &\xrightarrow{d} \quad \mathcal{N}(0, 1)
\end{aligned}$$

$$\frac{\sqrt{n}(\overline{X} - \mu)}{S_n} \quad = \quad \frac{\sigma}{S_n} \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma}$$

We showed previously $S_n^2 \xrightarrow{P} \sigma^2 \Rightarrow S_n \xrightarrow{P} \sigma \Rightarrow \sigma/S_n \xrightarrow{P} 1$. Therefore, By Slutsky's Theorem $\frac{\sqrt{n}(\overline{X}-\mu)}{S_n} \xrightarrow{d} \mathcal{N}(0, 1)$.

Example 5: Delta Method $X_1, \cdots, X_n$ *iid* Bernoulli(p) where $p \neq \frac{1}{2}$, we want to know the asymptotic distribution of $\overline{X}(1 - \overline{X})$. By central limit Theorem,

$$\begin{aligned}
\frac{\sqrt{n}(\overline{X} - p)}{\sqrt{p(1-p)}} \quad &\xrightarrow{d} \quad \mathcal{N}(0, 1) \\
\Leftrightarrow \overline{X} \quad &\sim \quad \mathcal{AN}\left(p, \frac{p(1-p)}{n}\right)
\end{aligned}$$

Define $g(y) = y(1 - y)$, then $\overline{X}(1 - \overline{X}) = g(\overline{X})$.

$$g'(y) = (y - y^2)' = 1 - 2y$$

By Delta Method,

$$
\begin{aligned}
g(\overline{X}) = \overline{X}(1-\overline{X}) &\sim \mathcal{AN}\left(g(p), [g'(p)]^2 \frac{p(1-p)}{n}\right) \\
&= \mathcal{AN}\left(p(1-p), (1-2p)^2 \frac{p(1-p)}{n}\right)
\end{aligned}
$$

Example 6 - Normal MLE Let $X_1, \cdots, X_n$ *iid* $\mathcal{N}(\mu, \sigma^2) \qquad \mu \neq 0$. Find the asymptotic distribution of MLE of μ^2.

Solution:

1. It can be easily shown that MLE of μ is $\overline{X}$.
2. By the invariance property of MLE, MLE of μ^2 is $\overline{X}^2$.
3. By central limit theorem, we know that

$$
\overline{X} \sim \mathcal{AN}\left(\mu, \frac{\sigma^2}{n}\right)
$$

4. Define $g(y) = y^2$, and apply Delta Method.

$$
\begin{aligned}
g'(y) &= 2y \\
\overline{X}^2 &\sim \mathcal{AN}\left(g(\mu), [g'(\mu)]^2 \frac{\sigma^2}{n}\right) \\
&\sim \mathcal{AN}\left(\mu^2, (2\mu)^2 \frac{\sigma^2}{n}\right)
\end{aligned}
$$

Asymptotic Efficiency

If both estimators are consistent and asymptotic normal, we can compare their asymptotic variance.

Definition 10.1.16 : Asymptotic Relative Efficiency If two estimators W_n and V_n satisfy

$$\sqrt{n}[W_n - \tau(\theta)] \xrightarrow{d} \mathcal{N}(0, \sigma_W^2)$$

$$\sqrt{n}[V_n - \tau(\theta)] \xrightarrow{d} \mathcal{N}(0, \sigma_V^2)$$

The asymptotic relative efficiency (ARE) of V_n with respect to W_n is

$$ARE(V_n, W_n) = \frac{\sigma_W^2}{\sigma_V^2}$$

If $ARE(V_n, W_n) \geq 1$ for every $\theta \in \Omega$, then V_n is asymptotically more efficient than W_n.

Example 7: Let X_i *iid Poisson*(λ). Consider estimating

$$\Pr(X = 0) = e^{-\lambda}$$

Our estimators are

$$\begin{aligned} W_n &= \frac{1}{n}\sum_{i=1}^{n} I(X_i = 0) \\ V_n &= e^{-\overline{X}} \end{aligned}$$

Determine which one is more asymptotically efficient estimator.

Solution: Asymptotic Distribution of V_n

$V_n(\mathbf{X}) = e^{-\overline{X}}$. By CLT

$$\overline{X} \sim \mathcal{AN}(\mathrm{E}X, \mathrm{Var}X/n) \sim \mathcal{AN}(\lambda, \lambda/n)$$

Define $g(y) = e^{-y}$, then $V_n = g(\overline{X})$ and $g'(y) = -e^{-y}$. By Delta Method

$$\begin{aligned} V_n = e^{-\overline{X}} &\sim \mathcal{AN}\left(g(\lambda), [g'(\lambda)]^2 \frac{\lambda}{n}\right) \\ &\sim \mathcal{AN}\left(e^{-\lambda}, e^{-2\lambda}\frac{\lambda}{n}\right) \end{aligned}$$

Asymptotic Distribution of W_n

Define $Z_i = I(X_i = 0)$

$$\begin{aligned} W_n &= \frac{1}{n}\sum_{i=1}^{n} I(X_i = 0) = \overline{Z} \\ Z_i &\sim \mathit{Bernoulli}(\mathrm{E}(Z)) \\ \mathrm{E}(Z) &= \Pr(X = 0) = e^{-\lambda} \\ \mathrm{Var}(Z) &= e^{-\lambda}(1 - e^{-\lambda}) \end{aligned}$$

By CLT,

$$\begin{aligned} W_n = \overline{Z} &\sim \mathcal{AN}\left(\mathrm{E}(Z), \mathrm{Var}(Z)/n\right) \\ &\sim \mathcal{AN}\left(e^{-\lambda}, \frac{e^{-\lambda}(1 - e^{-\lambda})}{n}\right) \end{aligned}$$

Calculating ARE

$$\begin{aligned}
ARE(W_n, V_n) &= \frac{e^{-2\lambda}\lambda/n}{e^{-\lambda}(1-e^{-\lambda})/n} \\
&= \frac{\lambda}{e^{\lambda}(1-e^{-\lambda})} \\
&= \frac{\lambda}{e^{\lambda}-1} \\
&= \frac{\lambda}{\left(1+\lambda+\frac{\lambda^2}{2}+\frac{\lambda^3}{3!}+\cdots\right)-1} \\
&\leq 1 \qquad (\forall \lambda \geq 0)
\end{aligned}$$

Therefore $W_n = \frac{1}{n}\sum I(X_i = 0)$ is less efficient than V_n (MLE), and ARE attains maximum at $\lambda = 0$.

Asymptotic Efficiency

Asymptotic Efficiency for iid samples

A sequence of estimators W_n is asymptotically efficient for $\tau(\theta)$ if for all $\theta \in \Omega$,

$$\begin{aligned}
\sqrt{n}(W_n - \tau(\theta)) &\xrightarrow{d} \mathcal{N}\left(0, \frac{[\tau'(\theta)]^2}{I(\theta)}\right) \\
\Longleftrightarrow W_n &\sim \mathcal{AN}\left(\tau(\theta), \frac{[\tau'(\theta)]^2}{nI(\theta)}\right) \\
I(\theta) &= \mathrm{E}\left[\left\{\frac{\partial}{\partial\theta}\log f(X|\theta)\right\}^2\right] \\
&= -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right] \text{ (with Lemma 7.3.11)}
\end{aligned}$$

Note: $\frac{[\tau'(\theta)]^2}{nI(\theta)}$ is the C-R bound for unbiased estimators of $\tau(\theta)$.

Theorem 10.1.12 Let $X_1, \cdots, X_n$ be iid samples from $f(x|\theta)$. Let $\hat{\theta}$ denote the MLE of θ. Under regularity conditions, $\hat{\theta}$ is consistent and asymptotically normal for θ, i.e.

$$\sqrt{n}(\hat{\theta} - \theta) \ \xrightarrow{d} \ \mathcal{N}\left(0, \frac{1}{I(\theta)}\right) \text{ for every } \theta \in \Omega$$

And if $\tau(\theta)$ is continuous and differentiable in θ, then

$$\sqrt{n}(\tau(\hat{\theta}) - \tau(\theta)) \ \xrightarrow{d} \ \mathcal{N}\left(0, \frac{[\tau'(\theta)]^2}{I(\theta)}\right)$$

- The regularity condition includes the ones in 10.1.6, plus finite three-times differentiabilble log-likelihood functions (See 10.6.2)
- Note that the asymptotic variance of $\tau(\hat{\theta})$ is Cramer-Rao lower bound for unbiased estimators of $\tau(\theta)$.

Example 8: Suppose $X_1, \cdots, X_n$ *iid* Exponential(β) and let $\tau(\beta) = \Pr(X \leq c)$ for a known constant c. Consider the two consistent estimators

1. Is $W(\mathbf{X}) = 1 - e^{-\frac{c}{\overline{X}}}$ asymptotically efficient for $\tau(\beta)$?
2. Is $U(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^n I(X_i \leq c)$ asymptotically efficient for $\tau(\beta)$?

Solution 1: By invariance property, $W(\mathbf{X}) = \tau(\hat{\beta})$ is MLE. So it is asymptotically efficient.

Solution 2: Let $Y_i = I(X_i \leq c) \sim Bernoulli(1 - e^{-\frac{c}{\beta}})$.

$$
\begin{aligned}
\mathrm{E}Y_i &= 1 - e^{-\frac{c}{\beta}} \\
\mathrm{Var}Y_i &= e^{-\frac{c}{\beta}}(1 - e^{-\frac{c}{\beta}}) \\
U(\mathbf{X}) &= \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq c) = \overline{Y} \sim \mathcal{AN}\left(1 - e^{-\frac{c}{\beta}}, \frac{e^{-\frac{c}{\beta}}(1 - e^{-\frac{c}{\beta}})}{n}\right) \\
I(\beta) &= \frac{1}{\beta^2} \\
\frac{[\tau'(\beta)]^2}{nI(\beta)} &= \frac{c^2 e^{-\frac{2c}{\beta}}}{n\beta^2} \leq \frac{e^{-\frac{c}{\beta}}(1 - e^{-\frac{c}{\beta}})}{n}
\end{aligned}
$$

So $U(\mathbf{X})$ is not asymptotically efficient.

Summary - Consistency and Efficiency

Consistency

- $W_n(\mathbf{X}) \longrightarrow \mathrm{P}\tau(\theta)$.
- Use W.L.L.N., Theorem 10.1.3, Continuous Mapping Theorem.

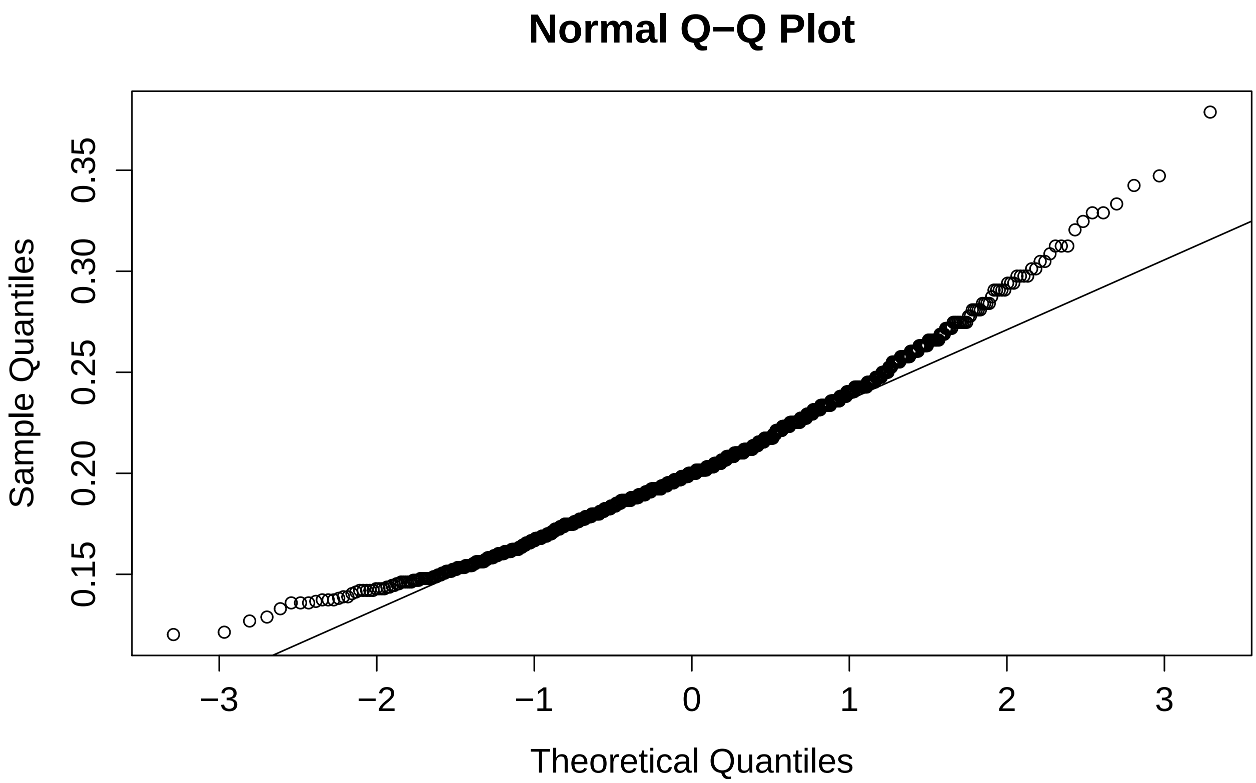
Asymptotic Normality and Efficiency

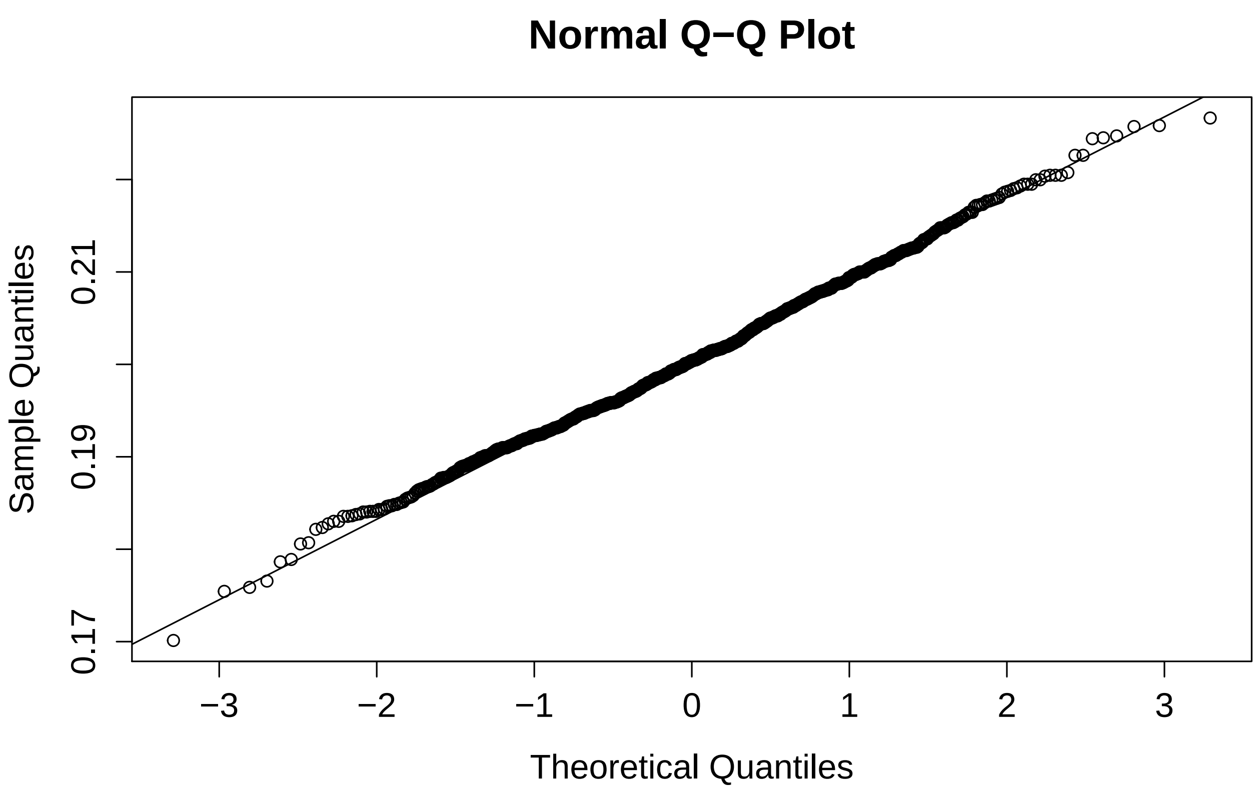
- Asymptotic behavior of mean (consistency) and variance (efficiency).
- Useful tools are C.L.T, Slutsky's Theorem, and Delta Method.
- Asymptotic Relative Efficiency (ARE) allows to compare the efficiency between two consistent and asymptotically normal estimators.
- Asymptotically efficient : asymptotic variance approaches CR-bound.
- MLE is always asymptotically efficient (under mild condition).

Example 9: Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from Negative Binomial (r, p). The following R code shows how close the MLEs resemble their large-sample properties. For this example, we take $r = 5, p = 0.2$. We generate a random sample of size n from this random sample and calculate the MLE of p. We repeat this process k times.

```
n=5
r=5
p=0.2
k=1000
mle=NULL
for(i in 1:k){
  x=rnbinom(n,size=r,p)
  mle[i] = (n*r)/((n*r) + sum(x))
}
mean(mle) # mle is consistent for p = 0.2

[1] 0.2072064

n*(var(mle)) # CRLB = (1-p)*p^2/r = 0.0064

[1] 0.007368217
qqnorm(mle)
qqline(mle)
```

We then repeat the process with n = 100. mean(MLE) = .2006, asymptotic variance = .00731.


Normal Q−Q Plot
Sample Quantiles
0.15
0.20
0.25
0.30
0.35
−3
−2
−1
0
1
2
3
Theoretical Quantiles


Normal Q−Q Plot
Sample Quantiles
0.17
0.19
0.21
−3
−2
−1
0
1
2
3
Theoretical Quantiles

Central Limit Theorem for Sample Median

Result: Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from a pdf/pmf f that is differentiable. Define μ to be the *median* of the population, i.e. $P(X_i \le \mu) = 1/2$. Let M_n be the sample median. Then, as $n \longrightarrow \infty$,

$$\sqrt{n}(M_n - \mu) \xrightarrow{d} \mathcal{N}(0, 1/\left[2f(\mu)\right]^2).$$

Example 10: Suppose f denotes the pdf of Cauchy with median θ, i.e.

$$f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}, \qquad -\infty < x < \infty.$$

Thus, $f(\theta) = 1/\pi$. With M_n denoting the sample median based on a random sample from Cauchy,

$$\sqrt{n}(M_n - \mu) \xrightarrow{d} \mathcal{N}(0, \pi^2/4).$$

Example 11: Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from $\mathcal{N}(\mu, \sigma^2)$. Consider the sample mean $\overline{X}$ and the sample median M_n to be competing estimators for μ which is both the mean and the median. Note that $f(\mu) = 1/(\sqrt{2\pi}\sigma)$ and so

$$\sqrt{n}(M_n - \mu) \xrightarrow{d} \mathcal{N}(0, \pi\sigma^2/2).$$

Since $\sqrt{n}(\overline{X} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, hence

$$ARE(\overline{X}, M_n) = 2/\pi = 0.64.$$

Example 12: Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from $Gamma(3, \beta)$. Consider

$$T_1(\mathbf{X}) = \overline{X}, \quad \text{and} \quad T_2(\mathbf{X}) = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}}$$

to be the sample arithmetic mean and sample harmonic mean respectively.

(a) Show that $W = T_1/3$ and $V = T_2/2$ are both consistent estimators of β.

(b) Prove the asymptotic normality results for both W and V.

(c) Find the ARE of V with respect to W.

Note: If $X \sim Gamma(\alpha, \beta)$ then $Y = 1/X \sim Inverse\ Gamma(\alpha, \beta^{-1})$ with

$$E(Y) = \frac{1}{\beta(\alpha - 1)}, \quad \alpha > 1, \quad Var(Y) = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}, \quad \alpha > 2.$$