Definition 26 (Harris recurrence) *A set A is called Harris recurrent if*

$$Q(x, A) = P_x(\eta_A = \infty) = 1, \quad x \in A.$$

A chain $\mathbf{\Phi}$ is called Harris (recurrent) if it is ψ-irreducible and every set $A \in \mathcal{B}^+(\mathcal{X})$ is Harris recurrent.

A standard alternative definition of an Harris recurrence set is that $L(x, A) = 1$ for $x \in A$, however they are equivalent. Nevertheless, using $Q(x, A) = 1$ highlights the strengthening of recurrence to Harris recurrence from an expected infinite number of visits to a set to an almost surely infinite number of visits.

Proposition 17 *Suppose for some one set $A \in \mathcal{B}(\mathcal{X})$ we have $L(x, A) \equiv 1$ for $x \in A$. Then $Q(x, A) = L(x, A)$ for all $x \in \mathcal{X}$ and A is Harris recurrent.*

Proof:

The most difficult proof of this section is the following.

Theorem 11 *(i) Suppose that $D \rightsquigarrow A$ for any D, $A \in \mathcal{B}(\mathcal{X})$. Let μ be any initial distribution of the chain $\mathbf{\Phi}$. Then*

$$\{\mathbf{\Phi} \in D \, i.o.\} \subseteq \{\mathbf{\Phi} \in A \, i.o.\} \quad a.s. \; [P_\mu]$$

and $Q(y, D) \leq Q(y, A)$ for all $y \in \mathcal{X}$.

(ii) If $\mathcal{X} \rightsquigarrow A$, then A is Harris recurrent and $Q(x, A) \equiv 1$ for all $x \in \mathcal{X}$.

Proof: See Meyn & Tweedie, p. 202. The proof involves Martingales and would take too much time to give the necessary background to show/prove the Martingale Convergence Theorem.

This leads us to the following strengthening of Harris recurrence...

Theorem 12 *If $\mathbf{\Phi}$ is Harris recurrent, then $Q(x, B) = 1$ for every $x \in \mathcal{X}$ and every $B \in \mathcal{B}^+(\mathcal{X})$.*

Proof:

Let D be any Harris recurrent set. Let $D^\infty = \{y : L(y, D) = 1\}$. Then $D \subseteq D^\infty$ and D^∞ is absorbing. We call D a maximal absorbing set if $D = D^\infty$.

Definition 27 (Maximal Harris sets) *A set H is called maximal Harris if H is a maximal absorbing set such that $\mathbf{\Phi}$ restricted to H is Harris recurrent.*

Recall the following definitions

Definition 28 (Full and Absorbing Sets) *A set $A \in \mathcal{B}(\mathcal{X})$ is said to be*

(i) full if $\psi(A^c) = 0$.

(ii) absorbing if $P(x, A) = 1$ for $x \in A$.

In order to prove Theorem 13 we need the following three lemmas:

Lemma 5 *Suppose that $\mathbf{\Phi}$ is ψ-irreducible. Then*

(i) every absorbing set is full.

(ii) every full set contains a non-empty absorbing set.

Proof:

Lemma 6 *If* $\mathbf{\Phi}$ *is a Markov chain and if* $A \in \mathcal{B}(\mathcal{X})$ *satisfies* $L(x, A) \leq \epsilon < 1$ *for* $x \in A$, *then* $U(x, A) \leq (1 - \epsilon)^{-1}$ *for all* $x \in \mathcal{X}$.

Proof:

Lemma 7 *If* $\mathbf{\Phi}$ *is* ψ*-irreducible and* $A \in \mathcal{B}(\mathcal{X})$ *with* $\psi(A) = 0$, *then* A *is transient.*

Proof: See Meyn & Tweedie, p. 186.

Theorem 13 *If* $\mathbf{\Phi}$ *is recurrent, then*

$$\mathcal{X} = H \cup N$$

where H *is a non-empty maximal Harris set and* N *is transient. Furthermore,* $\psi(N) = 0$.

Proof:

This theorem states that a recurrent Markov chain and a Harris recurrent Markov chain differ only by the existence of a ψ-null set N on which recurrence does not hold.

Theorem 14 *Suppose that* $\mathbf{\Phi}$ *is* ψ*-irreducible and aperiodic. Then* $\mathbf{\Phi}$ *is Harris if and only if each skeleton is Harris.*

Proof:

The existence of a stationary distribution

Now we investigate the conditions necessary for a Markov chain to possess a stationary distribution. This is quite important for the development of MCMC theory—for we want