Solution to Assignment 2

2.18: If $D \stackrel{\text{def}}{=} (A \cup B \cup C)^c = \emptyset$, then

$$\sigma(A,B,C) = \{\emptyset, \Omega, A, B, C, A\cup B, A \cup C, B \cup C\},$$

and the probabilities for the events in this σ-field (other than A, B, and C) are

$$P(\emptyset) = 0,\ P(\Omega) = 1,\ P(A\cup B) = 0.9,\ P(A\cup C) = 0.7,\ P(B\cup C) = 0.4.$$

(If $D \neq \emptyset$, then the σ-algebra also contains D and unions of D with sets in the class listed except $\emptyset$. As D is a null set its presence or absense will not effect the probability of the event. For instance $P(A \cup D) = P(A) = 0.6$.)

2.22: Fix $\epsilon > 0$. Since $F(x) \to 0$ as $x \to -\infty$, there exists $a < 0$ so that $F(a) < \epsilon/2$, and since $F(x) \to 1$ as $x \to \infty$, there exists $b > 0$ such that $F(b) > 1-\epsilon/2$. Since continuous functions on a compact domain must be uniformly continuous, the restriction of F to $[a, b]$ is uniformly continuous. Hence there must exist $\delta > 0$ so that

$$|F(x) - F(y)| < \epsilon/2, \qquad \forall x, y \in [a,b],\ |x-y| < \delta.$$

It then follows that

$$|F(x) - F(y)| < \epsilon, \qquad \forall |x-y| < \delta.$$

For instance, if $x \le b < y$ with $|x - y| < \delta$

$$|F(x) - F(y)| \le |F(a) - F(x)| + |F(y) - F(a)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

2.23: **(a)** With $k = 2$, this asserts that $(a, b]$ is

$$(-\infty, b]\backslash\Big((-\infty,(a_1,b_2)] \cup (-\infty,(b_1,a_2)]\Big).$$

A point x lies in this set iff $x \le b$ and $x_1 > a_1$ and $x_2 > a_2$. **(b)** The sets in the generating class are Borel, and using part (a) the σ-field contains all rectangles. **(c)** This follows because

$$(-\infty,a]\bigcap(-\infty,b] = (-\infty,(\min\{a_1,b_1\},\min\{a_2,b_2\})].$$

(d) This follows by Corollary 2.2.1 with $\mathcal{P}$ the π-system in part (c). **(e)** (1) If $x_1 \to \infty$ and $x_2 \to \infty$, then $(-\infty, x] \to \mathbb{R}^2$ and $F(x) = P\big((-\infty,x]\big) \to P(\mathbb{R}^2) = 1$. (2) If $x_1 \to -\infty$ or $x_2 \to -\infty$, then $(-\infty, x] \to \emptyset$ and $F(x) = P\big((-\infty,x]\big) \to$

$P(\emptyset) = 0$. (3) By the addition law (inclusion-exclusion with two events), for $a < b$,

$$P\left((-\infty, (a_1, b_2)] \bigcup (-\infty, (b_1, a_2)]\right) = F(a_1, b_2) + F(b_1, a_2) - F(a),$$

and since $(-\infty, b]$ is the disjoint union of

$$(-\infty, (a_1, b_2)] \bigcup (-\infty, (b_1, a_2)] \qquad \text{and} \qquad (a, b],$$

we have

$$P\big((a, b]\big) = F(b) - F(a_1, b_2) - F(b_1, a_2) + F(a).$$

(f) If $x \downarrow a$, then $(-\infty x] \to (-\infty, a]$, and so

$$F(x) = P\big((-\infty, x]\big) \to P\big((-\infty, a]\big) = F(a).$$

(g) Take $\mathcal{S} = \{(a, b] : a, b \in \mathbb{R}^2\}$, let

$$H(a, b) = F(b) - F(a_1, b_2) - F(b_1, a_2) + F(a),$$

and define P on $\mathcal{S}$ by $P(a, b] = H(a, b)$. By the extension theorem, if P is σ-additive on $\mathcal{S}$ is can be extended uniquely to a probability measure on the Borel sets. To establish this we can proceed as in class. L1: If $I_1, I_2, \ldots$ are disjoint in $\mathcal{S}$, and $\bigcup I_n \subset I \in \mathcal{S}$, then $P(I) \geq \sum_n P(I_n)$. Proof. Chopping I into disjoint rectangles it is not hard to see that $\sum_{j=1}^n P(I_j) \leq P(I)$, and L1 follows letting $n \to \infty$. L2: If $[a, b] \subset \bigcup_{i=1}^n (a_i, b_i)$, then $H(a, b) \leq H(a_i, b_i)$. Proof. This also follows by chopping the region into disjoint rectangles and using the fact that H is non-negative. L3: If $(a, b] \subset (a_n, b_n]$, then $P(a, b] \leq \sum_n P(a_n, b_n]$. Proof. Fix $\epsilon > 0$, let $\mathbf{1} = \binom{1}{1}$, and choose δ_k, $k \geq 1$ so that

$$H(a_k, b_k + \delta_k \mathbf{1}) - H(a_k, b_k) \leq \epsilon / 2^k, \qquad k = 1, 2, \ldots.$$

Then

$$[a + \epsilon, b] \subset \bigcup_n (a_k, b_k + \delta_k).$$

By compactness, for some N,

$$[a + \epsilon, b] \subset \bigcup_{n=1}^N (a_k, b_k + \delta_k).$$

So by L2,

$$H(a + \epsilon, b) \leq \sum_{n=1}^N \left(H(a_n, b_n) + \frac{\epsilon}{2^n}\right) \leq \epsilon + \sum_{n=1}^\infty H(a_n, b_n).$$

Since ϵ is arbitrary and F is continuous from above,

$$H(a,b) \le \sum_n H(a_n, b_n).$$

Combining L1 and L3, P is σ-additive.

2.24: **(a)** If B is the boundary, then $B^c = (0,1)^2$ and for any $\epsilon > 0$,
$P(B) = 1 - P(B^c) \le 1 - P\big((\epsilon, 1-\epsilon]^2\big) = 1 - (1-2\epsilon)^2$.
Letting $\epsilon \downarrow 0$, $P(B) = 0$. **(b)** 1/9. **(c)** $2(x \wedge y) - (x \wedge y)^2$, $x \wedge y \in [0,1]$.

3.2: The σ-fields are

$$\sigma(X_1) = \big\{\emptyset, (0,1]\big\}, \qquad \sigma(X_2) = \big\{\emptyset, (0,1], \{1/2\}, (0,1] \setminus \{1/2\}\big\},$$

and

$$\sigma(X_3) = \big\{\emptyset, (0,1], (0,1] \cap \mathbb{Q}, (0,1] \setminus \mathbb{Q}\big\}.$$

3.6: Since the events $\big[|X| < K\big] \uparrow \big[|X| < \infty\big]$, by continuity

$$P\big[|X| < K\big] \uparrow P\big[|X| < \infty\big] = 1.$$

Hence given $\epsilon > 0$ there exists $K > 0$ with $P\big[|X| < K\big] > 1 - \epsilon$. If we define $Y = X1_{\left[|X|<K\right]}$, then $\big[|X| < K\big]$ implies $[X = Y]$ and so

$$P[X = Y] \ge P\big[|X| < K\big] > 1 - \epsilon,$$

and thus $P[X \ne Y] < \epsilon$.

3.8: If $B \in \mathcal{B}$, then $[X = B] \in \mathcal{B}$, $[Y \in B] \in \mathcal{B}$, and

$$[Z \in B] = A[X \in B] \bigcup A^c[Y \in B]$$

lies in $\mathcal{B}$ as $A \in \mathcal{B}$ and $\mathcal{B}$ is closed under intersections and unions.

3.13: **(a)** Let $\mathcal{C} = \{(a,b]; a, b \in \mathbb{Q}\}$, a countable class. For any $c < d$,

$$[c,d] \bigcap_{a,b \in \mathbb{Q}, a<c<d<b} (a,b],$$

and so $[c,d] \in \sigma(\mathcal{C})$. Then, since

$$(c,d] = \bigcap_{n \ge 1} [c + 1/n, d],$$

we have $(c,d] \in \sigma(\mathcal{C})$. So

$$\sigma(\mathcal{C}) \supset \sigma\big((c,d] : c \le d\big) = \mathcal{B}(\mathbb{R}).$$

But all of the sets in $\mathcal{C}$ are Borel sets, so $\sigma(\mathcal{C}) \subset \mathcal{B}(\mathbb{R})$, and hence $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$. By Proposition 3.1.2, if X is a random variable

$$\sigma(X) = X^{-1}\big(\mathcal{B}(\mathbb{R})\big) = X^{-1}\big(\sigma(\mathcal{C})\big) = \sigma\big(X^{-1}(\mathcal{C})\big).$$

Thus $\sigma(X)$ is countably generated because $X^{-1}(\mathcal{C})$ is countable.
(b) If $\mathcal{B} = \sigma(B_n, n \geq 1)$ we can define

$$X = \sum_{n\geq 1} 3^{-n} 1_{B_n}.$$

Them X is measurable by Proposition 3.2.6 because

$$X = \limsup_{n\to\infty} \sum_{k=1}^{n} 3^{-k} 1_{B_k},$$

and the partial sums are measurable functions. For any $j \geq 1$,

$$0 \leq \sum_{k=1}^{\infty} 3^{-k} 1_{B_{k+j}} \leq \sum_{k=1}^{\infty} 3^{-k} = 2/3 < 1,$$

and we have

$$\lfloor 3X \rfloor = \left\lfloor 1_{B_1} + \sum_{k=1}^{\infty} 3^{-k} 1_{B_{k+1}} \right\rfloor = 1_{B_1}.$$

So $B_1 = \big[3X \in [1,2)\big] = \big[X \in [1/3, 2/3)\big]$. Thus $B_1 \in \sigma(X)$. Also, $B_1^c = \big[X \in [0, 1/3)\big]$. Next note that on B_1,

$$9X = 3 + 1_{B_2} + \sum_{k=1}^{\infty} 3^{-k} 1_{B_{k+2}}$$

and $1_{B_2} = \lfloor 9X - 3 \rfloor$. Similarly, on B_1^c we have $1_{B_2} = \lfloor 9X \rfloor$. So

$$B_2 = B_1 B_2 \cup B_1^c B_2 = \big[X \in [1/9, 2/9)\big] \cup \big[X \in [4/9, 5/9)\big].$$

Thus $B_2 \in \sigma(X)$. Continuing in this fashion $B_n \in \sigma(X)$ for all $n \geq 1$. Thus $\mathcal{B} = \sigma(X)$, as desired.