Biostat 602 **Assignment 8** **Due: Thursday, March 23**

Homework Problems

1. Let $X_1, \cdots, X_n$ be an *i.i.d.* random sample from Exponential(θ) distribution. Assuming $n \geq 2$, let θ have a prior distribution of InverseGamma(α, β) distribution.

   (a) Show that the family of inverse gamma distribution is the conjugate for the exponential distribution.

   (b) Calculate the posterior mean (Bayes estimator under squared error loss) and the posterior variance at hyperparameter $\alpha = 1$ and $\beta > 0$.

   (c) Compare the Bayes estimator in part (b) with the MLE of θ with respect to the Frequentist's criteria of bias and MSE (where θ is non-random and the expectation is taken under the sampling distribution $f(\mathbf{x}|\theta)$).

   The pdf of Exponential(θ) distribution is

$$f(x|\theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right)$$

   The pdf of InverseGamma(α, β) distribution is

$$f(x|\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right)$$

2. Consider the weighted squared error loss function

$$L(\theta, a) = \omega(\theta)(\theta - a)^2,$$

   where $\omega(\theta)$ is a nonnegative weight function, and a is the estimator.

   (a) Show that the Bayes estimator directly minimizing the posterior expected loss function with respect to the estimate a is given by

$$a^* = \frac{\mathrm{E}\left[\theta\omega(\theta)|\mathbf{X}\right]}{\mathrm{E}\left[\omega(\theta)|\mathbf{X}\right]}.$$

(b) Let $X_1, \cdots, X_n$ be a random sample from a Bernoulli distribution with probability of success $p \in (0,1)$. Consider the $Uniform(0,1)$ prior on p. Find the Bayes estimator of p the with respect to this prior and the loss function

$$L(p,a) = \frac{(p-a)^2}{p(1-p)}$$

(c) Calculate the Risk function for the Bayesian estimator in part (b).

3. The LINEX (LINear-EXponential) loss function introduced by Zellner (1986) is given by

$$L(\theta,a) = e^{c(a-\theta)} - c(a-\theta) - 1,$$

where $c > 0$ is a constant. As the constant c varies, the loss function varies from very assymmetric to an almost symmetric one.

(a) Verify that $L(\theta,a)$ is a nonnegative function of the difference $a-\theta$. For $c = .2, .5, 1$ plot $L(\theta,a)$ as a function of $a-\theta$.

(b) Show that the Bayes estimator of θ with respect to a prior π is given by

$$\hat{\theta}_B = -\frac{1}{c}\log \mathrm{E}(e^{-c\theta}|\mathbf{X}).$$

(c) Let $X_1, \cdots, X_n$ be a random sample from a $\mathcal{N}(\theta,\sigma^2)$ where σ^2 is known. Suppose that θ has a noninformative prior given by

$$\pi(\theta) = 1, \quad \theta \in \mathcal{R}.$$

Show that the Bayes estimator of θ with respect to this prior is given by

$$\hat{\theta}_B(\overline{X}) = \overline{X} - \left(c\sigma^2/(2n)\right).$$

(d) Calculate the posterior expected loss for $\hat{\theta}_B(\overline{X})$ and $\overline{X}$ using LINEX loss.

(e) Calculate the posterior expected loss for $\hat{\theta}_B(\overline{X})$ and $\overline{X}$ using (unweighted) squared-error loss.

4. Consider independent random variables $X_1, X_2, \ldots, X_n$ such that $X_i \sim Poisson(\lambda_i)$. Further assume that λ_i are i.i.d. from $Exponential(\beta)$.

(a) Under the assumption that β is known, find Bayes estimator of λ_i under squared error loss, i.e. obtain $\mathrm{E}(\lambda_i|X_i)$.

(b) (**Empirical Bayes**) Instead of assuming known β in part (a), let us estimate β through the following steps.

- First obtain the marginal distribution $m(\mathbf{X})$ whose distribution involves β.
- Next obtain the maximum likelihood estimator of β using $m(\mathbf{X})$. Call it $\hat{\beta}$.
- Plug in $\hat{\beta}$ in the Bayes estimator in part (a) in place of (known) β. The resulting estimator is called the **empirical Bayes** estimator of λ_i.

5. Let $X_1, X_2, \ldots, X_n$ be a i.i.d. random sample from a $Poisson(\lambda)$ distribution. Assume the following hierarchy

$$X_i|\lambda \sim Poisson(\lambda), \quad \lambda|\beta \sim Exp(\beta), \quad \pi(\beta) = \frac{1}{\beta} \text{ (noninformative).}$$

Then find Bayes estimator of λ under squared error loss.