Solutions to Biostat 682 Homework 3

Self-grading

1. Let $k \sim G(\nu/2, \nu/2)$ and $\theta \mid \mu, \sigma^2, k \sim \mathrm{N}(\mu, \sigma^2/k)$. Show that $\theta \mid\mid \mu, \sigma^2 \sim t_\nu(\mu, \sigma^2)$.

2. Find Jeffrey's priors for the unknown parameters in the following models:

   (a) $Y \sim \text{Multinomial}(\theta_1, \ldots, \theta_k, n)$ with n known.

   (b) $Y \sim \mathrm{N}(\mu, \sigma^2)$ with μ known and consider σ^{-2} as the parameters that we need to specify the prior.

   (c) $Y \sim \mathrm{N}(\mu, \sigma^2)$ and consider (μ, σ^{-2}) as the parameters that we need to specify the prior.

3. Suppose data $(y_1, \ldots, y_J)$ follow a multinomial distribution with parameters $(\theta_1, \ldots, \theta_J)$. Also, suppose that $\theta = (\theta_1, \ldots, \theta_J)$ has a Dirichlet prior distribution. Let $\alpha = \theta_1/(\theta_1 + \theta_2)$.

   (a) Write the marginal posterior distribution for α.

   (b) Show that this distribution is identical to the posterior distribution for α obtained by treating y_1 as an observation from the binomial distribution with probability α and sample size $y_1 + y_2$, ignoring the data $y_3, \ldots, y_n$.

4. Consider multivariate model $y_1, \ldots, y_n \sim \mathrm{MVN}(\mu, \Sigma)$. We assume that non-informative prior

$$\pi(\mu, \Sigma) \propto |\Sigma|^{-(d+1)/2}.$$

Show that the marginal posterior distribution of μ is a multivariate t distribution with degrees of freedom $n - d$, location parameter $\bar{y}$ and scale matrix $S/\{n(n-d)\}$.

$$\mu \mid y_1, \ldots, y_n \sim t_{n-d}[\bar{y}, S/\{n(n-d)\}].$$

where $\bar{y} = n^{-1} \sum_{i=1}^n y_i$ and $S = \sum_{i=1}^n (y_i - \bar{y})(y_i - \bar{y})^{\mathrm{T}}$. Note: if x follows a d-dimensional multivariate t distribution with degrees of freedom ν, location parameter θ and scale matrix S. Its density function is given by

$$\frac{\Gamma\{(\nu+d)/2\}}{\Gamma(\nu/2)\nu^{d/2}\pi^{d/2}|S|^{1/2}} \left[1 + \frac{1}{\nu}(x-\theta)^{\mathrm{T}} S^{-1} (x - \theta)\right]^{-(\nu+d)/2}.$$