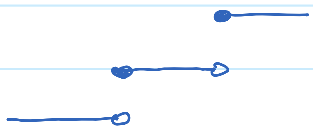
Lecture 6. Leb decomp, integr

Monday, September 25, 2017 9:57 AM

Defined CDFs $F(x) = P(X \le x) =$
$= P(\{\omega : X(\omega) \le x\})$



$F(x) \to 1, \quad x \to +\infty$
$\qquad\ \ 0, \quad x \to -\infty$

Example of a non-measurable function
Suppose a subset A is not measurable
Then $I_A(\omega)$ is not going to be measurable

$P(X = x) = 0 \iff F$ - continuous

Classes of distributions

1) Discrete $\{x_1, \dots, x_n, \dots\}$
Range of X has countably many elements
$p_k = P(X = x_k), \quad k = 1, 2 \dots, n, \dots$

$\sum_k p_k = 1 \qquad P(\{\omega : X(\omega) = x_k\})$

2) Absolutely continuous distributions
When a pdf exists
$\exists f : \quad P(X \in B) = \int f(x)\,dx$

$$\exists f: \quad \mathbb{P}(X \in B) = \int_B f(x)\,dx$$

3) Singular distributions (wrt Lebesgue measure)

a) There are no atoms $\Rightarrow$ CDF is continuous

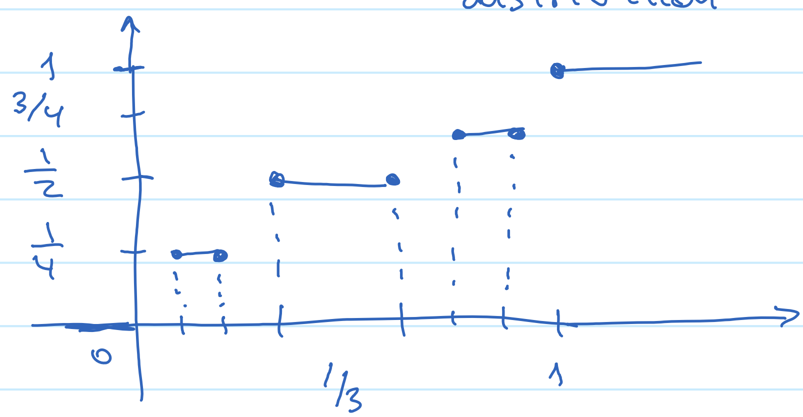
b) Its support has measure 0

$\boxed{\text{TH}}$ Lebesgue decomposition theorem

w/o proof

There are only 3 classes of distributions possible

Cantor function: Example of a singular distribution



$n=1$

$n=2$

$\frac{1}{3^n}$ size of the current interval where I define the function

"Jumps" $\Delta F = \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$

No jumps in the limit

$\Rightarrow F$ is continuous

What is the length of segment where F is Const.

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\dots=$$

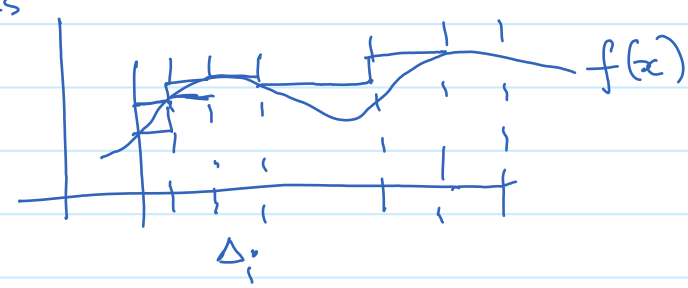
$$=\frac{1}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\dots\right)=$$

$$=\frac{1}{3}\ \frac{1}{1-\frac{2}{3}}=1$$

$\square$

Riemann integral $\int f(x)\,dx$

Darboux sums



$\lim\limits_{\max \Delta_i \to 0}$

$\sup\limits_{\Delta_i} f \to$ Upper Darboux sum

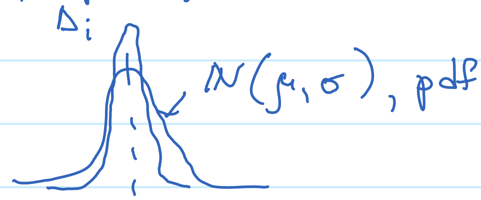
$\inf\limits_{\Delta_i} f \to$ Lower Darboux sums

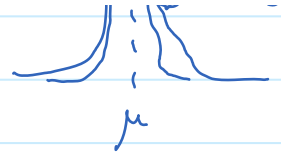
lim of Darboux sums

Stieltjes integral

$$\int f(x)\,dF(x) := \lim_{\max \Delta_i \to 0} \left(\sup_{\Delta_i} f\right)\cdot\left(F(x_i)-F(x_{i-1})\right)$$

Dirac's δ function





$$\delta(x-\mu) = \lim_{\sigma \to 0} N\text{pdf} \qquad \delta(x-\mu) = \begin{cases} \infty, & x = \mu \\ 0, & x \neq \mu \end{cases}$$

$$\int_{\mu-}^{\mu+} \delta(x-\mu)\,dx = 1$$

Consider

$$\varphi = f + \sum_i \delta(x - x_i) \cdot p_i \ , \quad p_i \geq 0$$

] f is continuous, $\int f(x)\,dx + \sum_i p_i = 1$

$\Rightarrow F(x) = \int_0^x \varphi(\xi)\,d\xi$ is a mixed discrete continuous CDF

Can write Stieltjes integral

$\int \psi(x)\,dF(x)$ in "Riemann" form

$$= \int \psi(x)\,\varphi(x)\,dx$$

Example of Riemann-non-integrable function

] Q be a set of rational numbers on $[0,1]$

$$I_Q(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

Q and $\bar{Q}$ are dense in $\mathbb{R}$

Upper Darboux for $\int I_Q\,dx$ will be $= 1$

Lower will be $= 0$

$\Rightarrow$ Not Riemann-integrable

Lebesgue measure of $\{\omega : I_Q = 1\}$

$= 0$

Lebesgue integral, Expectation

Lebesgue integral, Expectation

$$\mathbb{E}X := \int_{\Omega} X(\omega)\, dP(\omega)$$

$$\mathbb{E}(X; A) := \int_{A} X(\omega)\, dP(\omega) =$$

$$= \int_{\Omega} X(\omega) \cdot I_A(\omega)\, dP(\omega) =$$

$$= \mathbb{E}(X \cdot I_A)$$

DF Simple r.v.

r.v. $X(\omega)$ is called simple if the range of X is a finite set $\{x_1, \dots, x_n\}$, $n < \infty$

$$F_k = \{\omega : X(\omega) = x_k\} \in \mathcal{F}$$

$$(\Omega, \mathcal{F}, P)$$

measurable means $P(F_k)$ is defined

Then X can be represented as

$$X(\omega) = \sum_{k=1}^{n} x_k \cdot I_{F_k}(\omega)$$