Problem 1 (a). It's known that $\sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2 \sim \chi^2_{n-1}$. It follows that $(n-1)\sigma_1^2/\sigma^2 \sim \chi^2_{n-1}$, and $(n+1)\sigma_2^2/\sigma^2 \sim \chi^2_{n-1}$. Thus, their risk functions are given by, respectively,

- $$R(\theta,\sigma_1^2) = E\left(\sigma_1^2\sigma^{-2}-1\right)^2 = \frac{1}{(n-1)^2}E\left\{\frac{(n-1)\sigma_1^2}{\sigma^2}-(n-1)\right\}^2 = \frac{1}{(n-1)^2}Var\left\{\chi^2_{n-1}\right\} =$$
$$= \frac{1}{(n-1)^2}\cdot 2(n-1) = \boxed{\frac{2}{n-1}}.$$

- $$R(\theta,\sigma_2^2) = \boxed{\frac{2}{n+1}}.$$

- Since $\frac{X_i}{\sigma}$ iid $N(\frac{\mu}{\sigma},1)$, $\sum_{i=1}^{n}X_i^2/\sigma^2$ follows a non-central χ^2 dist'n with DF $= n$ and non-centrality parameter $\lambda = \frac{n\mu^2}{\sigma^2}$. Then $E\left(\sum_{i=1}^{n}X_i^2/\sigma^2\right) = n+\lambda = n+\frac{n\mu^2}{\sigma^2}$ and $Var\left(\sum_{i=1}^{n}X_i^2/\sigma^2\right) = 2(n+2\lambda) = 2\left(n+2\frac{n\mu^2}{\sigma^2}\right)$. Thus.

$$R(\theta,\sigma_3^2) = E\left(\frac{1}{n+2}\sum_{i=1}^{n}X_i^2/\sigma^2-1\right)^2 = Var\left(\frac{1}{n+2}\chi^2_n(\lambda)\right)^2 + \left\{E\left(\frac{1}{n+2}\chi^2_n(\lambda)-1\right)\right\}^2$$
$$= \frac{2(n+2\lambda)}{(n+2)^2} + \left\{\frac{n+\lambda}{n+2}-1\right\}^2 = \frac{2(n+2\lambda)+(\lambda-2)^2}{(n+2)^2}$$
$$= \frac{2n+4+\lambda^2}{(n+2)^2} = \boxed{\frac{2}{n+2}+\frac{n^2}{(n+2)^2}\frac{\mu^4}{\sigma^4}}$$

(b) Let us show that a Bayes estimator δ_π with respect to a proper prior $\pi(\theta)$ is admissible. In fact, if δ_π is inadmissible, there must exist a decision rule δ that dominates δ_π; that is, $R(\theta,\delta)\le R(\theta,\delta_\pi)$, with at least θ^* at which $R(\theta^*,\delta) < R(\theta^*,\delta_\pi)$. Because of the continuity of the risk function $R(\theta,\cdot)$, $\in \mathcal{B}_\pi$ there exists a neighborhood around θ^*, say $N(\theta^*)$, over which $R(\theta,\delta) < R(\theta,\delta_\pi)$, $\theta\in N(\theta^*)$ and $\pi(N(\theta^*))>0$. Then their Bayes risk is

$$R_\pi(\delta) = \int_{\Theta} R(\theta,\delta)\,d\pi(\theta) = \int_{N(\theta^*)} + \int_{N^c(\theta^*)} R(\theta,\delta)\,d\pi(\theta) < \int R(\theta,\delta_\pi)\,d\pi(\theta)$$
$$= R_\pi(\delta_\pi).$$ This is impossible because δ_π is the Bayes rule.

Now we show that $\sigma_3^2 = \frac{1}{n+2}\sum_{i=1}^n X_i^2$ is admissible when $X_1, \cdots, X_n$ iid $N(0, \sigma^2)$, $\sigma > 0$. ②
This is the special case with $\mu = 0$. It is sufficient to show that σ_3^2 is a Bayes estimator with respect to a proper prior.

Consider a prior $\sigma^2 \sim IG(\alpha, \beta)$. Then the posterior $\sigma^2 | \underset{\sim}{x} \sim IG(\frac{n}{2}, \frac{\sum X_i^2}{2})$. Under the squared error loss function $L(\sigma^2, a) = (a\sigma^{-2} - 1)^2 = \frac{1}{\sigma^4}(a - \sigma^2)^2$, the Bayes estimator is the posterior mean

$$\delta_\pi(\underset{\sim}{x}) = \frac{\int \frac{1}{\sigma^4}\sigma^2 \pi(\sigma^2|x)\, d\sigma^2}{\int \frac{1}{\sigma^4}\pi(\sigma^2|x)\, d\sigma^2} = \frac{E_{\sigma^2|\underset{\sim}{x}}\left(\frac{1}{\sigma^2}\right)}{E_{\sigma^2|\underset{\sim}{x}}\left(\frac{1}{\sigma^4}\right)} = \frac{\sum X_i^2}{n+2} \hat{=} \sigma_3^2.$$

According to the fact proved above, σ_3^2 is admissible when $X_1, \cdots, X_n$ iid $N(0, \sigma^2)$.

Next, we show that σ_3^2 is also admissible when $X_1, \cdots, X_n$ iid $N(\mu, \sigma^2)$, $\mu \in R$, $\sigma > 0$.
If σ_3^2 is not admissible, there exists a decision rule $\delta^*(x)$, such that

$$R(\sigma^2, \delta^*) \le R(\sigma^2, \sigma_3^2) \quad \text{for all } \mu \in R \text{ and } \sigma > 0,$$

That is

$$E_{\underset{\sim}{x}|\mu,\sigma^2}(\delta^* - \sigma^2)^2 \le E_{\underset{\sim}{x}|\mu,\sigma^2}(\sigma_3^2 - \sigma^2), \quad \text{for all } \mu \in R \text{ and } \sigma > 0.$$

Now consider a decision rule $\delta^{**}(\underset{\sim}{x}) = \frac{\delta^* + \sigma_3^2}{2}$. Because the quadratic loss function is strictly convex, we have

$$E_{\underset{\sim}{x}|\mu,\sigma^2}(\delta^{**} - \sigma^2)^2 \le E_{\underset{\sim}{x}|\mu,\sigma^2}\left\{(\delta^* - \sigma^2)^2 + (\sigma_3^2 - \sigma^2)^2\right\}/2$$

$$\le E_{\underset{\sim}{x}|\mu,\sigma^2}(\sigma_3^2 - \sigma^2)^2, \quad \text{for all } \mu \in R \text{ and } \sigma > 0,$$

Note that the above inequality will be strictly less for $\delta^* \neq \sigma_3^2$. In particular, the above inequality holds with $\mu = 0$ and $\sigma > 0$, namely

$$E_{\underset{\sim}{x}|\mu=0,\sigma^2}(\delta^{**} - \sigma^2)^2 \le E_{\underset{\sim}{x}|\mu=0,\sigma^2}(\sigma_3^2 - \sigma^2)^2, \quad \text{for all } \sigma^2 > 0;$$

or, $R(\sigma^2; \delta^{**}) \le R(\sigma^2, \sigma_3^2)$, for all $\sigma^2 > 0$.
On the other hand, there must exist a $\sigma_0^2 > 0$ at $R(\sigma_0^2, \sigma_3^2) < R(\sigma_0^2, \delta^*)$, so at this point $(0, \sigma_0^2)$, $P(\delta^* \neq \sigma_3^2 \,|\, \mu = 0, \sigma_0^2) > 0$. Otherwise, $\delta^* = \sigma_3^2$ a.s. $(\mu = 0, \sigma_0^2)$, $R(\sigma^2, \sigma_3^2) \equiv R(\sigma^2, \delta^*)$ for all $\sigma^2 > 0$.

At this point $(\mu=0, \sigma_0^2)$, we have

$$R(\sigma_0^2, \delta^{**}) < R(\sigma_0^2, \sigma_3^2),$$

which implies that σ_3^2 is not adminible for $X_1, \cdots, X_n$ iid $N(0, \sigma^2)$. Contradiction!

(c) Because $R(\theta, \sigma_2^2) < R(\theta, \sigma_1^2)$, for all θ, σ_1^2 is not adminble.

Because at $\mu=0$, $R(\sigma^2, \sigma_3^2) = \frac{2}{n+2} < R(\sigma^2, \sigma_2^2)$, so σ_2^2 is inadminible. It is true for any known $\mu=\mu_0$, i.e. when $X_1, \cdots, X_n$ iid $N(\mu_0, \sigma^2)$, σ_2^2 is inadminible. In general, it is easy to see that

$$R(\sigma^2, \sigma_3^2) < R(\sigma^2, \sigma_2^2), \quad \text{when } \frac{\mu^4}{\sigma^4} < \frac{2(n+2)}{n^2(n+1)}$$

$$R(\sigma^2, \sigma_3^2) \geq R(\sigma^2, \sigma_2^2), \quad \text{when } \frac{\mu^4}{\sigma^4} \geq \frac{2(n+2)}{n^2(n+1)}$$

Thus, it is not clear if σ_2^2 is inadminible or not.

Problem 2: (From the 2016 Qualifying Exam) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution with the density function,

$$f(x|\theta) = 2x/\theta^2, \;\; 0 \le x \le \theta,$$

and 0, otherwise, where parameter $\theta > 0$.

(a) First compute $E(X_1|\theta)$ and then, using the sample $X_1, \ldots, X_n$, derive an unbiased estimator of θ and its variance.
Solution: $E(X|\theta) = 2\int_0^\theta x^2 dx/\theta^2 = 2\theta/3$. So $E(3\bar{X}/2) = \theta$. Now $var(X|\theta) = 2\int_0^\theta x^3 dx/\theta^2 - 4\theta^2/9 = \theta^2/18$. Therefore, $var(3\bar{X}/2) = 9\theta^2/(4 \times 18n) = \theta^2/(8n)$.

(b) Obtain the maximum likelihood estimator of θ. Is it unbiased? What is the mean square error of the maximum likelihood estimator of θ? Justify your answer.
Solution: Let $Y = max_{1 \le i \le n} X_i$ be the largest order statistic. The likelihood function is

$$L(\theta|x_1, x_2, \ldots, x_n) \propto \theta^{-2n}$$

for $\theta \ge y$ and 0 otherwise. The likelihood is maximized when at $\widehat{\theta}_{ML} = Y$.

Note that, $Pr(X \le x|\theta) = (x/\theta)^2$ and, hence, $Pr(Y \le y) = (y/\theta)^{2n}$ and $f(y|\theta) = 2ny^{2n-1}/\theta^{2n}$ for $0 \le y \le \theta$ and 0 otherwise. It follows that $E(Y|\theta) = 2n\theta/(2n+1)$. The maximum likelihood estimate is biased and the bias is $-\theta/(2n+1)$.

Note that $var(Y|\theta) = 2n\theta^2/(2n+2) - 4n^2\theta^2/(2n+1)^2 = n\theta^2/\{(n+1)(2n+1)^2\}$. Thus, the mean square error is

$$\frac{n\theta^2}{(n+1)(2n+1)^2} + \frac{\theta^2}{(2n+1)^2} = \frac{\theta^2}{(n+1)(2n+1)}$$

(c) Assuming a prior $\pi(\theta) \propto \theta^{-2}$, find the posterior density of θ and its posterior mean.
Solution: The posterior density, therefore, is,

$$\pi(\theta|y) = (2n+1)y^{2n+1}\theta^{-2n-2}, \;\; \theta \ge y$$

and 0 otherwise. The posterior mean is

$$\widehat{\theta}_B = E(\theta|y) = (2n+1)y^{2n+1}\int_y^\infty \theta^{-2n-1} d\theta = (2n+1)y/(2n).$$

Note that $E(\theta^2|y) = (2n+1)y^{2n+1}\int_y^\infty \theta^{-2n} d\theta = (2n+1)y^2/(2n-1)$. The posterior variance is

$$\begin{aligned} &(2n+1)y^2/(2n-1) - (2n+1)^2y^2/(4n^2) \\ &= (2n+1)y^2/(4n^2(2n-1)) \end{aligned}$$

(d) As a frequentist, compute the sampling variance of the posterior mean in (c). Compare the properties of the estimators of θ derived in (a), (b) and (c). Which estimator will you choose and why?

Solution: The posterior mean is an unbiased estimate of θ because $E(\widehat{\theta}_B|\theta) = (2n + 1)E(Y)/2n = \theta$.

Thus, the sampling variance of $\widehat{\theta}_B = \theta^2/\{4n(n+1)\}$. The Relative efficiency of the posterior mean relative to the sample mean is

$$RE(c|a) = \frac{4n(n+1)}{8n} = (n+1)/2.$$

The relative efficiency of the posterior mean relative to the maximum likelihood estimate is

$$RE(c|b) = 4n/(2n+1).$$

The posterior mean is asymptotically infinitely better than the estimate in (a). Asymptotically, the relative efficiency of the posterior mean compared to the maximum likelihood estimate is 2. The posterior mean is also the uniformly minimum variance unbiased estimator.