Lecture 16. Unif int

Wednesday, November 8, 2017 10:11 AM

Cauchy sequences $\underset{\substack{a.s.\\ P\\ L^r}}{\Longleftrightarrow}$ Conv.

Continuity Th $\exists$ $X_n \to X$, f cont.

$\Rightarrow$ $f(X_n) \to f(X)$

Proved for a.s., P

DF $\{X_n\}$ is uniformly integrable (u.i.) when

$$\sup_n \mathbb{E}(|X_n|; |X_n| > A) \xrightarrow[A\to\infty]{} 0$$

$A > 0$

or $$\sup_n \int_{|X_n|>A} |X_n| \, dP \xrightarrow[A\to\infty]{} 0$$

$X_n > A \cup X_n < -A$

Th Uniformly integrable sequences have uniformly finite expectations

$\exists$ X_n is u.i. $\Rightarrow$ $\sup_n \mathbb{E}|X_n| \le c < \infty$

c does not depend on n

Proof: $\mathbb{E}|X_n| = \underbrace{\mathbb{E}(|X_n|; |X_n| > A)}_{\to 0} + \underbrace{\mathbb{E}(|X_n|; |X_n| \le A)}_{\le A}$

b/c X_n is u.i.

$\sup \mathbb{E}|X_n| \le \tilde{C}(A) + A$

n for all A large enough $\square$

TH $\quad \exists\ X_n \xrightarrow[P]{} X$ and $\{X_n\}$ is u.i. $\Rightarrow$

$\mathbb{E}|X| < \infty$

$\mathbb{E}|X_n - X| \xrightarrow[n\to\infty]{} 0$, $\quad X_n \xrightarrow{L^1} X$

if additionally $\{|X_n|^r\}$ is u.i. $\Rightarrow$

$\Rightarrow \mathbb{E}(|X_n|^r) < \infty$ and $X_n \xrightarrow{L^r} X$

if $r \geq 1$, $X_n \xrightarrow{L^r} X$, $\mathbb{E}(|X|^r) < \infty \Rightarrow |X_n|^r$ is u.i.

w/o proof.

TH $\quad$ Continuity

$\exists\ X_n \xrightarrow[P]{} X$, f is continuous,

$f(X_n)$ is a u.i. sequence

$\Rightarrow \mathbb{E}|f(X_n) - f(X)| \to 0 \qquad (1)$

$\mathbb{E}f(X_n) \to \mathbb{E}f(X) \qquad (2)$

Proof:

$$\left.\begin{array}{l} X_n \xrightarrow[P]{} X \Rightarrow f(X_n) \xrightarrow[P]{} f(X) \\ \text{by Cont. Th for } \xrightarrow[P]{} \\ f(X_n) \text{ is u.i.} \end{array}\right\} \Rightarrow f(X_n) \xrightarrow{L^1} f(X)$$

i.e. $\mathbb{E}|f(X_n) - f(X)| \xrightarrow[n\to\infty]{} 0 \Rightarrow (1)$

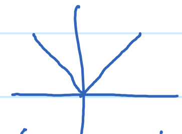
$|\mathbb{E}f(X_n) - \mathbb{E}f(X)| \leq \mathbb{E}|f(X_n) - f(X)| \xrightarrow[n\to\infty]{} 0$

$$|\mathbb{E}f(X_n) - \mathbb{E}f(X)| \leq \mathbb{E}|f(X_n)-f(X)| \xrightarrow[n\to\infty]{} 0$$

Jensen inequality

φ – convex

$\varphi(\mathbb{E}(\cdot)) \leq \mathbb{E}(\varphi(\cdot))$

$|\cdot|$:  convex $\Rightarrow$

$\Rightarrow \mathbb{E}(|\cdot|) \geq |\mathbb{E}(\cdot)|$

$\square$

Corollary: X_n is u.i. $\Rightarrow \mathbb{E}X_n \to \mathbb{E}X$
$X_n \xrightarrow[p]{} X$

Lemma $\exists\ |X_n| < Y \Rightarrow X_n$ is u.i.
$\mathbb{E}(Y) < \infty$, $Y \geq 0$

Proof:

$$\mathbb{E}(|X_n|; |X_n| > A) \leq \mathbb{E}(Y; Y > A) \xrightarrow[A\to\infty]{} 0 \Rightarrow$$

$$A < |X_n| < Y \Rightarrow Y > A$$

$$\Rightarrow \sup_n \mathbb{E}(|X_n|; |X_n| > A) \xrightarrow[A\to\infty]{} 0 \quad \text{u.i.}$$

$\square$

TH Dominated convergence

$\exists\ X_n \xrightarrow[p]{} X$, $|X_n| < Y$, $\mathbb{E}Y < \infty \Rightarrow$
$Y \geq 0$

$$\Rightarrow \mathbb{E}(X) < \infty \text{ and } \mathbb{E}X_n \to \mathbb{E}X,\ \mathbb{E}|X_n - X| \xrightarrow[n\to\infty]{} 0$$

Proof: Lemma $\Rightarrow X_n$ is u.i.
$+\ X_n \xrightarrow[p]{} X$
$\Rightarrow \mathbb{E}X_n \to \mathbb{E}X$

$\square$

Lemma $\exists\ \mathbb{E}|X_n|^{1+\alpha} < C < \infty$, $\alpha > 0 \Rightarrow$
$\Rightarrow |X_n|$ is u.i.

Proof:

$\mathbb{E}(|X_n|; |X_n| > A) < \mathbb{E}(|X_n|^{1+\alpha}; |X_n| > A) < \frac{C}{\ }$

Proof:

$$\mathbb{E}(|X_n|; |X_n|>A) \le \mathbb{E}\left(\frac{|X_n|^{1+\alpha}}{A^\alpha}; |X_n|>A\right) \le \frac{C}{A^\alpha}$$

$$|X_n|>A \Rightarrow |X_n|^\alpha > A^\alpha \Rightarrow |X_n|^{1+\alpha} > A^\alpha \cdot |X_n| \Rightarrow$$

$$\frac{|X_n|^{1+\alpha}}{A^\alpha} > |X_n|$$

$$\frac{C}{A^\alpha} \xrightarrow[A\to\infty]{} 0, \quad \alpha>0$$

$\square$

Corollary: Replace $|X_n|$ by $|X_n|^r$ in the above Lemma:

$$\mathbb{E}|X_n|^{r+\alpha} < C < \infty \Rightarrow |X_n|^r \text{ is u.i.}$$

Corollary to Lemma and the Dominated Conv. Th

$\exists\ X_n \xrightarrow[P]{} X$ and $\mathbb{E}|X_n|^{r+\alpha} < C < \infty$, $\alpha>0$

$$\Rightarrow X_n \xrightarrow{L^r} X$$

Proof:

$\left.\begin{array}{l}\Rightarrow \text{u.i.}\\ \xrightarrow[P]{}\end{array}\right\} \Rightarrow L^r$ conv.

$\square$

Corollary: $\exists\ X_n \xrightarrow[P]{} X$ and f is continuous and bounded

$$\Rightarrow \mathbb{E}|f(X_n) - f(X)| \to 0$$

$\left.\begin{array}{l} f_n \text{ - bounded} \Rightarrow f_n \text{ is u.i}\\ \text{\quad}\!\!\!\!\!\!\!\!\!\!\! \quad\quad\quad + \text{ Cont.}\\ \quad\quad\quad + \xrightarrow[P]{}\end{array}\right\} \Rightarrow \mathbb{E}|f_n - f| \to 0$ Cont. Th

$f_n \overset{\cdot}{=} f(X_n)$

$\square$

Convergence in distribution

Convergence in distribution

(L) w/o proof

$X_n \rightsquigarrow X$ $\Rightarrow \exists \tilde{X}_n, \tilde{X}$: they have same CDFs

$F_n \rightsquigarrow F$ ← CDF

$\tilde{X}_n \sim F_n$

$X \sim F$

such that

$\tilde{X}_n \xrightarrow[a.s.]{} \tilde{X}$

(L) $\exists X_n \rightsquigarrow X$, $Y_n \xrightarrow[P]{} 0 \Rightarrow$

$X_n + Y_n \rightsquigarrow X$

Proof: Take $\delta > 0$, $x \pm \delta$ are continuity points of $F(x)$

$$\limsup_{n\to\infty} P(X_n + Y_n < x) =$$

$$= \limsup_n P(\underbrace{X_n + Y_n < x \cap Y_n > -\delta}_{\Downarrow\; X_n < x - Y_n < x + \delta}) \le$$

b/c $P(Y_n > -\delta) \to 1$

$$\le \limsup_n P(X_n < x + \delta) = F(x + \delta)$$

by cont of P

Similar argument gives

$$\liminf_{n\to\infty} P(X_n + Y_n < x) \geq F(x - \delta)$$

b/c $\delta > 0$ is arbitrary, $x \pm \delta$ is a continuity point

$$\Rightarrow \limsup F_{X_n+Y_n} = \liminf F_{X_n+Y_n} = F(x)$$

$$\Rightarrow \lim F_{X_n+Y_n} = F(x) \quad \square$$

(L) $X_n \rightsquigarrow X$, $Y_n \xrightarrow[P]{} 1 \Rightarrow X_n \cdot Y_n \xrightarrow[d]{} X$

Similar proof.