Lecture 2 generating sigma algebras

Sunday, September 10, 2017 10:36 PM

Last lecture

$\mathcal{A}$ algebra $\quad A \subset \Omega, \; A \in \mathcal{A}$

Ω set of elementary events $\quad \omega \in \Omega$

$\omega \in \Omega$

$\in$ "is an element of"

$\subset$ "is a subset of"

σ-algebra $\mathcal{F}$, closed over Countable $\cup$, $\cap$, $\overline{(\cdot)}$

$(\Omega, \mathcal{F})$ – measurable space

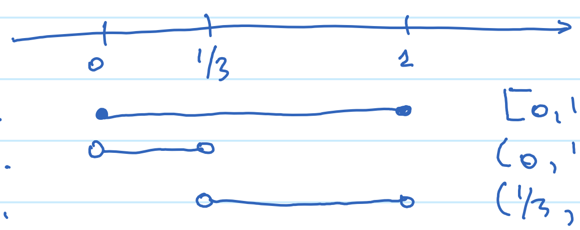
For a given Ω does at least one $\mathcal{F}$ exist ?
$\uparrow$ σ-algebra

Yes:
$\mathcal{F}$ containing all possible subsets of Ω is a σ-algebra

Examples: $\quad \Omega = [0,1]$

Take sets $\quad \{[0,1], (0, 1/3), (1/3, 1)\}$

$\mathcal{F}$ or $\mathcal{A}$ based on $\uparrow$



1. $[0,1]$
2. $(0, 1/3)$
3. $(1/3, 1)$
4. $\emptyset$
5. $\{0, [\frac{1}{3}, 1]\}$
6. $\{[0, 1/3], 1\}$

7.  $\{(0,\frac{1}{3}), (\frac{1}{3},1)\}$

8. $\{0, 1/3, 1\}$

DF For any class $\mathcal{A}$ of subsets of Ω, the smallest σ-algebra that contains $\mathcal{A}$ will be called the σ-algebra generated by $\mathcal{A}$

Notation $\sigma(\mathcal{A})$

DF Consider $\Omega = \mathbb{R}$

The σ-algebra $\mathcal{B}$ generated by open intervals (a,b) ($\pm\infty$ allowed for a,b) is called Borel σ-algebra

DF Sets in $\mathcal{B}$ are called Borel sets

$$\{a\} = \bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, a+\frac{1}{n}\right)$$

Single points as elements of σ-algebra are called singletons

$$[a,b) = \bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, b\right)$$

$[a,b]$ also a Borel set

Example: What is an $\mathcal{F}$ generated by $\Omega = [0,1]$, intervals $(a,1]$, ?
$0 < a < 1$

This notation means a set of intervals

Listing sets

$(a,1]$ $\qquad$ $[0,a]$

$[b,1] = \bigcap_{n=1}^{\infty} (b-\frac{1}{n},1]$ $\qquad$ $[0,b)$

$(a,b) = [0,b) \cap (a,1]$, $a<b$

$\{a\} = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, a+\frac{1}{n})$

$[b,1] = \bigcap_{n=1}^{\infty} (b-\frac{1}{n}, 1]$ $\qquad [0,b)$ $\qquad \{a\} = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, a+\frac{1}{n})$

$[0,1)$ $\qquad [a,b] = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$

$[0,1]$ $\qquad \emptyset$

Complements

Got the whole $\mathcal{B}$ on $[0,1]$

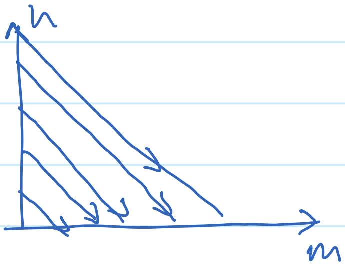
Note: Set of intervals $(a,1]$ is different from, say, $[a,1]$

σ (all possible singletons of Ω) ?

DF Countable sets are the ones whose elements we can enumerate, i.e. map to integers.

Example: Rational numbers are $\frac{m}{n}$ ← integers are countable

Cantor



Properties of countable sets (C)

Any subset of a countable set is countable

Any countable $\cap$ or $\cup$ of countable sets is countable

DF Cocountable set is a set whose complement is countable

Complement

TH $\sigma\{\text{all singletons}\} = \{\text{either countable or cocountable}\}$

Note: Based on a uniform distribution, probabilities of such sets will be either 0 or 1, which is not very useful, b/c

$P\{\text{Countable set}\} = 0$

$P\{\text{Cocountable set}\} = P\{\text{Compl. of countable}\} = 1 - 0 = 1$

for P defined using for example length-based measures (Lebesgue)

Proof.

cocountable

based measures (Lebesgue)

Proof.

$\mathcal{S} = \sigma\{\{a\}\}$

↳ singletons

Countable   cocountable

$\mathcal{F} = \{A \subset \Omega : \text{ either } A \in C \text{ or } \bar{A} \in C\}$

$A \in CC$

at this point we do not know if this is a σ-algebra

Need to show $\mathcal{S} = \mathcal{F}$

all countable sets are of the form $\cup_i \{a_i\} \in \mathcal{S}$

all cocountable complements of $" \in \mathcal{S}$

$\Rightarrow \mathcal{F} \subset \mathcal{S}$

Now, let's prove that $\mathcal{F}$ is a σ-algebra

We need to show that $\cup, \cap, \overline{(\cdot)}$ of sets in $\mathcal{F}$ are again in $\mathcal{F}$

when $A \in C, B \in C \Rightarrow A \cap B, A \cup B \in C$ by property of countable sets

taking complement of the previous line we have the same statement for CC

$A \in C, \ B \in CC \Rightarrow A \cap B \in C$ because $A \cap B \subset A \in C$

$\Rightarrow A \cap B \in \mathcal{F}$

$\overline{A \cup B} = \underset{\in C}{\bar{A}} \cap \underset{\in CC}{\bar{B}} \in C \Rightarrow A \cup B \in CC \Rightarrow A \cup B \in \mathcal{F}$

$A \in C \Rightarrow \bar{A} \in CC \Rightarrow \bar{A} \in \mathcal{F}$

really only need
to show that
collection is closed
wrt $\cap$ (or $\cup$)
given that it is
closed over the
complement operation

$\mathfrak{F}$ contains singletons b/c they are countable sets

So $\mathfrak{F}$ is a σ-algebra containing singletons

Now b/c $\mathcal{S}$ is the smallest σ-algebra containing singletons, and $\mathfrak{F} \subset \mathcal{S}$, it must be

$$\mathfrak{F} \equiv \mathcal{S}$$

↳ "for all elements" □