Biostat 602 **Assignment 10** **Due: Thursday, April 13**

Homework Problems

1. Let X_1, X_2 be $i.i.d$ observations from Uniform$(\theta, \theta+1)$, $\theta \geq 0$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\begin{array}{lcl} \phi_1(X_1) & : & \text{Reject } H_0 \text{ if } X_1 > 0.95 \\ \phi_2(X_1, X_2) & : & \text{Reject } H_0 \text{ if } X_1 + X_2 > C \end{array}$$

   (a) Find the value of C so that ϕ_2 has the same size as ϕ.

   (b) Calculate the power function of ϕ_1 and ϕ_2 obtained from part (a).

   (c) Show that ϕ_2 from part (a) is not uniformly more powerful test than ϕ_1.

   (d) Find a test that is always equally or more powerful than ϕ_2 and ϕ_1 with the same size. Justify your answer.

   { **Hint:** Consider the test

$$\text{Reject } H_0 \text{ if } X_1 + X_2 > C \text{ or } X_1 \geq 1 \text{ or } X_2 \geq 1$$

   where C is as in part (a). }

   (e) **(Optional)** Consider now a random sample of size n, $n \geq 2$ from the same population for testing the same hypothesis. Demonstrate that the test

$$\text{Reject } H_0 \text{ if } X_{(n)} \geq 1 \text{ or } X_{(1)} \geq k,$$

   where k is a constant, is UMP of its size. Determine k such that the test has size α, $0 < \alpha < 1$.

2. Let $X_1, \ldots, X_n$ be an $i.i.d$ random sample of size n with pdf $f(x)$, the functional form of which is unknown. We are interested in testing the null hypothesis

$$H_0 : f(x) = f_0(x)$$

   against the alternative hypothesis

$$H_1 : f(x) = f_1(x)$$

   where the functional forms of f_0 and f_1 are known. They have no unknown parameters and they have the same support.

   (a) Rewrite H_0 and H_1 parametrically by considering the following pdf:

$$f(x|\theta) = (1-\theta) f_0(x) + \theta f_1(x),\ 0 \leq \theta \leq 1.$$

(b) Show that if

$$
\begin{aligned}
f_0(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad x \in R \\
f_1(x) &= \frac{1}{2} \exp\left(-|x|\right), \quad x \in R,
\end{aligned}
$$

then the rejection region for the Uniformly Most Powerful (UMP) level α test of H_0 against H_1 is given by

$$\sum_{i=1}^{n} (|x_i| - 1)^2 > k$$

for some k that depends on α. (Note that it is not necessary to find a closed form of k in terms of α. Describing mathematical relationship between k and α should be sufficient).

(c) Calculate both probability of type I error and probability of type II error for the test in part (b) when $n = 1$ and (i) $k = 1$ or (ii) $k = \frac{1}{4}$. Present the results to three digits beyond the decimal point. (You may use calculator, R, or table of standard normal distribution if needed).

(d) Determine whether the two tests in (c) are unbiased or not.

3. Let $X_1, \ldots, X_n$ be an *i.i.d.* random sample from the following pdf

$$f(x|\theta) = \frac{1}{2\theta^3} x^2 e^{-\frac{x}{\theta}}, \qquad 0 < \theta < \infty, 0 < x < \infty$$

(a) Find the UMP level α test of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ with $\theta_1 > \theta_0$.

(b) Using the fact $\frac{2}{\theta}\sum_{i=1}^{n} X_i \sim \chi^2_{(6n)}$, construct the level α UMP test of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ when $n = 4, \theta_0 = 1, \theta_1 = 3$, and $\alpha = 0.05$. Calculate the power of this UMP test. Present the results to three digits beyond the decimal point. (You may use calculator, R, or table of chi-squared distribution if needed).

(c) Find the level α UMP test for $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ (for arbitrary n, θ_0, α).

(d) Find a size α LRT for $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ based on sufficient statistics. Justify whether the LRT is the level α UMP test or not.

4. Suppose that the time X to death of non-smoking heart transplant patients follows the distribution with pdf

$$f_X(x|\beta) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \; 0 < x < \infty, \beta > 0$$

Further, suppose that the time Y to death of smoking heart transplant patients follows the distribution with pdf

$$f_Y(y|\gamma) = \frac{1}{\gamma} e^{-\frac{y}{\gamma}}, \; 0 < y < \infty, \gamma > 0$$

Let $X_1, \ldots, X_n$ be an *i.i.d.* random sample from $f_X(x|\beta)$ and $Y_1, \ldots, Y_n$ be an *i.i.d.* random sample from $f_Y(y|\gamma)$.

It is of interest to make inference about the parameter $\theta = \beta - \gamma$.

   (a) Find a constant k such that the estimator $k(\overline{X} - \overline{Y})$ is an unbiased estimator of θ.
   (b) Based on the estimator from (b), construct an asymptotic level α Wald test for $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$.
   (c) Construct an asymptotic level α LRT for $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$.

5. Suppose $X_1, \ldots, X_n$ is an *i.i.d.* random sample with pdf

$$f_X(x|\theta) = \theta \exp(-\theta x), \qquad x \geq 0, \theta > 0$$

   (a) Derive asymptotic size α LRT for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.
   (b) Derive asymptotic size α score test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.
   (c) Derive asymptotic size α Wald for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

Practice Problems

1. C&B Exercise 8.17
2. C&B Exercise 8.20
3. C&B Exercise 8.28
4. C&B Exercise 8.29

5. C&B Exercise 8.31
6. C&B Exercise 8.39
7. C&B Exercise 10.31
8. C&B Exercise 10.34
9. C&B Exercise 10.35
10. C&B Exercise 10.36
11. C&B Exercise 10.37