to be able to sample from the posterior distribution, which is stationary. It is quite obvious that if we construct a Markov chain that does not possess a stationary distribution, we will not be sampling from the posterior. More importantly, we need to also show that the n-step transition kernels converge to this stationary distribution, no matter what the initial distribution, and will do so in the next section. Thus once $P^n(\mu, A) = \pi(A)$ we are guaranteed that $P^{n+1}(\mu, A) = \pi(A)$ as well (once we start sampling from the posterior distribution, we will always be sampling from the posterior).

Definition 29 (Invariant measure) *If a σ-finite measure π on $\mathcal{B}(\mathcal{X})$ has the property*

$$\pi(A) = \int_{\mathcal{X}} \pi(dx)P(x, A), \quad A \in \mathcal{B}(\mathcal{X}),$$

then we call π invariant.

If an invariant measure is finite, then we can renormalize it to be a probability measure. Obviously, this is the situation in which we are most interested.

Definition 30 (Positive and null chains) *If $\mathbf{\Phi}$ is ψ-irreducible and admits an invariant probability measure π, then we call $\mathbf{\Phi}$ a positive chain. If $\mathbf{\Phi}$ does not admit such a measure, we call it null.*

Recall that a process is stationary if for any k, the marginal distribution of $\{\Phi_n, \ldots, \Phi_{n+k}\}$ does not change as n varies. In general, Markov chains are not stationary (e.g. consider the a chain with initial distribution δ_x). However, with an appropriate choice for the initial distribution for Φ_0 we can produce a stationary process $\{\Phi_n, n \in \mathbb{N}_+\}$.

For Markov chains we only need to consider first step stationarity to generate an entire stationary process. Suppose π is the initial invariant measure (initial distribution of the chain) with

$$\pi(A) = \int_{\mathcal{X}} \pi(dw)P(w, A).$$

Now, by iterating and the Chapman-Kolmogorov equations we have all n and $A \in \mathcal{B}(\mathcal{X})$

$$\begin{aligned}
\pi(A) &= \int_{\mathcal{X}} \pi(dw)P(w,A) \\
&= \int_{\mathcal{X}} \left[\int_{\mathcal{X}} \pi(dx)P(x,dw)\right] P(w,A) \\
&= \int_{\mathcal{X}} \pi(dx) \int_{\mathcal{X}} P(x,dw)P(w,A) \\
&= \int_{\mathcal{X}} \pi(dx)P^2(x,A) \\
&= \vdots \\
&= \int_{\mathcal{X}} \pi(dx)P^n(x,A) = P_\pi(\Phi_n \in A).
\end{aligned}$$

From the Markov property $\mathbf{\Phi}$ is stationary if and only if the distribution of Φ_n does not depend on n (time).

Lemma 8 *Let $\mathbf{\Phi}$ be a Markov chain and if $A \in \mathcal{B}(\mathcal{X})$ is uniformly transient with $U(x,A) \leq M$ for $x \in A$, then $U(x,A) \leq 1 + M$ for all $x \in \mathcal{X}$.*

Proof:

Proposition 18 *If the chain $\mathbf{\Phi}$ is positive, then it is recurrent.*

Proof:

Positive chains are often referred to as positive recurrent to reinforce the fact that they are recurrent.

Definition 31 (Positive Harris chains) *If $\mathbf{\Phi}$ is Harris recurrent and positive, then $\mathbf{\Phi}$ is called a positive Harris (recurrent) chain.*

Now we set out to show that an invariant probability measure exists and that it is unique, up to a multiplicative constant, for certain chains. We will begin by showing that chains that admit atoms are positive, and then extend to strongly aperiodic chains and then to recurrent chains.

Definition 32 (Subinvariant measures) *if μ is σ-finite and satisfies*

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dw)P(w,A), \quad A \in \mathcal{B}(\mathcal{X}) \tag{11}$$

then μ is called subinvariant.

Iterating we get

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dx)P^n(x,A). \tag{12}$$

Multiplying by $a(n)$, where a is a sampling distribution on $\mathbb{N}_+$, and then summing we get

$$\mu(A) \geq \int_{\mathcal{X}} \mu(dx)K_a(x,A). \tag{13}$$

Equations (12) and (13) tell us, respectively, that if μ is a subinvariant measure for $\mathbf{\Phi}$ is it also a subinvariant measure for any m-skeleton and for any sampled chain.

Proposition 19 *If $\mathbf{\Phi}$ is ψ-irreducible and if μ is any measure satisfying (11) with $\mu(A) < \infty$ for some $A \in \mathcal{B}^+(\mathcal{X})$, then*

(i) *μ is σ-finite and thus μ is a subinvariant measure;*

(ii) *$\psi \prec \mu$;*

(iii) *if C is ν_a-petite, then $\mu(C) < \infty$;*

(iv) *if $\mu(\mathcal{X}) < \infty$, then μ is invariant.*

Proof: