Lecture 20 CLTs

Wednesday, November 22, 2017 9:59 AM

Characteristic functions

degenerate c.f. $\quad e^{itc}$

normal $\quad N(0,1) \quad e^{-\frac{t^2}{2}}$

$N(\mu, \sigma^2) \quad e^{it\mu - \frac{t^2\sigma^2}{2}}$

$N(\mu, \Sigma) \quad e^{it^T\mu - \frac{1}{2}t^T\Sigma t}$

φ uniquely determines the distribution

Lévy Th

(1) $\quad X_n \rightsquigarrow X \iff \varphi_n(t) \to \varphi(t), \ \forall t$

(2)

If $\varphi_n(t) \to \varphi(t), \ \forall t$

φ is continuous at $t=0$

$\Rightarrow$ φ is a characteristic function of r.v. X that is the limit in distribution of X_n

$X_n \rightsquigarrow X \sim \varphi$

If $X \sim \varphi$ has a symmetric distribution around 0

then $\operatorname{Im} \varphi = 0$ and φ is a real function

$[X] = [-X]$

$\uparrow$ distribution

$$\lim_{a\to\infty} \int_{-a}^{a} \sin(tx)\, dP(x) =$$

$$\mathbb{E}\{e^{itX}\} = \mathbb{E}\cos tX + i\,\mathbb{E}\sin tX \quad \Big\} = \lim \int_0^{a} - \int_0^{-a} = 0$$

Weak Law of Large numbers

TH $\exists\, X_n$ - r.v.s i.i.d. $\sim \varphi$

$$\hat{\mathbb{E}}X = \frac{1}{n}\sum_{k=1}^{n} X_k$$

$$\hat{\mathbb{E}}X \xrightarrow[p]{} \mathbb{E}X \quad \text{iff. } \varphi'(0) \text{ exists}$$

and $\varphi'(0) = i\cdot\underset{\substack{\text{a constant} \\ \mathbb{E}X}}{\mu}$

Proof:

$$\varphi_{\hat{\mathbb{E}}_n X}(t) = \mathbb{E}\{e^{it\hat{\mathbb{E}}_n X}\} = \mathbb{E}\{e^{i\frac{t}{n}\sum_{k=1}^{n} X_k}\} \underset{X_k \text{ are i.i.d}}{=}$$

$$= \prod_{k=1}^{n} \underbrace{\mathbb{E}\{e^{i\frac{t}{n}X_k}\}}_{\varphi\left(\frac{t}{n}\right)} = \varphi^n\left(\frac{t}{n}\right) =$$

Taylor expansion

$$\varphi(t) = \varphi(0) + \varphi'(0)\cdot t + o(t)$$

$$= \Big(1 + \underbrace{\varphi'(0)}_{i\mu = i\mathbb{E}X}\cdot\frac{t}{n} + \underbrace{o\left(\frac{t}{n}\right)}_{\text{as } n\to\infty}\Big)^n \xrightarrow[n\to\infty]{} e^{it\mu}$$

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

$$\mathbb{E}X \sim e^{it\mu}$$

↙ degenerate r.v.

$\square$

Sufficient condition for differentiability of φ at $t=0$

If $\varphi'(0)$ exists $\Rightarrow$ $\varphi'(0) = i\,\mathbb{E}X$

$\mathbb{E}X$ exists if $\mathbb{E}|X| < \infty$

Under the same condition we can interchange $(\ )'_t$ and $\mathbb{E}$ over X

$$\varphi'(t) = \frac{d}{dt}\mathbb{E}e^{itX} = \mathbb{E}\left(\frac{d}{dt}e^{itX}\right) = \mathbb{E}\left(iXe^{itX}\right)$$

$$|\varphi'(t)| \le \mathbb{E}\left\{|iXe^{itX}|\right\} = \mathbb{E}|X| < \infty$$

In particular $\varphi'(0) = i\,\mathbb{E}X$

Classical C. L. T. (Central Limit Theorem)

↓

for i.i.d. r.vs

$\exists\ X_i$ i.i.d. r.v. $i = 1, 2, \ldots$

$\mathbb{E}X_1 = \mu$

$\mathrm{Var}(X_1) = \sigma^2 < \infty$

$$\Rightarrow \quad \sqrt{n}\left(\hat{\mathbb{E}}_n X - \mu\right) \underset{n\to\infty}{\rightsquigarrow} \mathcal{N}(0, \sigma^2)$$

Proof:

w/o loss of generality $\mu = 0$, $\sigma^2 = 1$

$\exists\ X_1 \sim \varphi$

$$\varphi_{\sqrt{n}\hat{\mathbb{E}}_n X}(t) = \prod_{i=1}^{n} \varphi_{\frac{X_1}{\sqrt{n}}} = \varphi^n\left(\frac{t}{\sqrt{n}}\right) =$$

$$\varphi_{\sqrt{n}\hat{\mathbb{E}}_n X}(t) = \underbrace{\prod_{i=1} \varphi_{\frac{X_1}{\sqrt{n}}} = \varphi^{n}\left(\frac{t}{\sqrt{n}}\right)}_{\text{like in WLLN}} =$$

$\varphi''(0) = -\mathrm{Var}(X)$ b/c $\mu = 0$

$\varphi'(0) = 0$ b/c $\mu = 0$

$\varphi(0) = 1$

$$= \left(1 + i\underbrace{\mathbb{E}X_1}_{=0} \cdot \frac{t}{\sqrt{n}} - \frac{1}{2}\mathrm{Var}(X_1)\left(\frac{t}{\sqrt{n}}\right)^2 + o\left(\left(\frac{t}{\sqrt{n}}\right)^2\right)\right)^n \to$$

$$\to e^{-\frac{t^2}{2}\sigma^2} \sim \mathbb{N}(0, \sigma^2)$$

$$\varphi_{\sqrt{n}\hat{\mathbb{E}}_n X}(t) \longrightarrow e^{-\frac{t^2}{2}\sigma^2} = \varphi_{\mathbb{N}(0,\sigma^2)}$$

$\square$

Cramer–Wold device

Working with univariate φ, X in a multivariate situation

Let X be a vector-valued r.v.

$$\varphi_X(t) = \mathbb{E}\{e^{it^T X}\}$$

t → vector

Consider a r.v. $Y_t = t^T \cdot X$

$$\varphi_{Y_t}(\tau) = \mathbb{E}\{e^{i\tau \cdot (t^T \cdot X)}\} = \varphi(\tau \cdot t) \quad \forall t.$$

$$\varphi_{Y_t}(\tau) = \mathbb{E}\left\{ e^{i\tau\cdot(t^T\cdot X)} \right\} = \varphi(\tau; t), \quad \forall t$$

$\hookrightarrow$ scalar

$$\Rightarrow \quad X_n \rightsquigarrow X \text{ iff } \varphi_{X_n} \to \varphi_X \text{ iff } \varphi_{Y_{nt}} \to \varphi_{Y_t}, \ \forall t, \tau$$

$$\text{iff} \quad t^T X_n \rightsquigarrow t^T X, \ \forall t$$

$$\boxed{X_n \rightsquigarrow X \quad \text{iff.} \quad t^T X_n \rightsquigarrow t^T X, \ \forall t}$$

vectors

Cramer - Wold

Multivariate CLT

X_i are random vectors, i.i.d.

$\mathbb{E}X_1 = \mu$

$Cov(X_1) = \mathbb{E}(X_1 - \mu)\cdot(X_1-\mu)^T$ — matrix

$$\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \rightsquigarrow \mathcal{N}(0, Cov(X_1))$$

Proof:

$$t^T \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underbrace{(\underbrace{t^T X_i}_{Y_{it}} - \underbrace{t^T \mu}_{\mathbb{E}Y_{it}})}_{\text{i.i.d. with mean } 0 \text{ and variance } Var(t\cdot X_1) = t^T Cov(X_1)\cdot t} \rightsquigarrow$$

$$\rightsquigarrow \quad \mathcal{N}(0, t^T Cov(X_1) t)$$

Univariate CLT

$\overset{\Rightarrow}{\text{Cramer - Wold}}$

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_i - \mu) \rightsquigarrow \mathcal{N}(0, Cov(X_1))$$

$\sqrt{n}$

$\square$

CLTs for i. non-i.d

$\underbrace{\text{non}}_{\text{not identically distributed}}$

Lindeberg - Feller CLT

$\exists$ $X_{n1}, \dots, X_{nk_n}$ are $\perp$ r. vectors with finite (generally different) covariances

Such that

Lindeberg condition

$\forall \varepsilon > 0$ $\quad \sum_{i=1}^{k_n} \mathbb{E}\left(\|X_{ni}\|^2 ; \|X_{ni}\| > \varepsilon \right) \xrightarrow[n\to\infty]{} 0$

and $\sum_{i=1}^{k_n} \operatorname{Cov}(X_{ni}) \to \Sigma$ (matrix)

Then $\sum_{i=1}^{k_n} \left(X_{ni} - \mathbb{E}(X_{ni}) \right) \rightsquigarrow \mathcal{N}(0, \Sigma)$

w/o proof

Lyapunov (Ляпунов) CLT

Lyapunov functions $L_s = \sum_{k=1}^{n} \mathbb{E}\|X_{nk}\|^s$, $s > 2$

Lyapunov condition $L_s \xrightarrow[n\to\infty]{} 0$

L Lyapunov condition $\Rightarrow$ Lindeberg condition