Lecture 13. Convergence

Wednesday, October 25, 2017 10:12 AM

Dependence

Copulae

Sklar theorem → any dependence can be represented by a Copula

Frechét - Hoeffding bounds

Fubini Theorem $\iint \to$ repeated $\int$

Convolutions

Convergence of sequences of random variables

Convergence in probability $X_n \xrightarrow{P} X$

- $\forall \varepsilon > 0 \quad P(|X_n - X| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$

$\Updownarrow$

- $\varepsilon - \delta$ language

$$\forall \varepsilon > 0, \delta > 0 \quad \exists N: \quad \forall n > N$$

$$P(|X_n - X| > \varepsilon) < \delta$$

$\Updownarrow$

- $\sup\limits_{k \geq N} P(|X_k - X| > \varepsilon) \underset{N}{\longrightarrow} 0$

$\Updownarrow$

- $\forall \varepsilon > 0 \quad \limsup\limits_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$

$x_n \in \mathbb{R}$

$x_n \underset{n\to\infty}{\longrightarrow} x$

$\Leftrightarrow$ all subsequence of $\{x_n\}$ have the same limit

$\Leftrightarrow \sup\limits_{n > N} |x_n - x| \underset{N\to\infty}{\longrightarrow} 0$

$\Leftrightarrow$ limsup = liminf

What does it mean that $X_n \not\xrightarrow{P} X$?

$$\forall N \ \exists\, n_1 > N : \quad P(|X_{n_1} - X| > \varepsilon) \geq \delta > 0$$

for some $\varepsilon > 0$

Take $N = n_1 \Rightarrow \exists\, n_2$: $\nearrow$ is the case, ...

$\Rightarrow$ $\exists$ a subsequence

$$X_{n_k} : \quad P(|X_{n_k} - X| \geq \varepsilon) \geq \delta > 0$$

for all k

Almost sure convergence

$$X_n \xrightarrow{a.s.} X \quad \text{wrt } P$$

- $X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)$ for all $\omega \in \Omega$ except perhaps for sets of P-measure 0

- $P(X_n \to X) = 1$

- ε- language

  $\forall\, \omega$ except for sets of measure 0

  $\forall\, \varepsilon > 0 \ \exists\, N : \ |X_n(\omega) - X(\omega)| < \varepsilon, \ \forall\, n > N$

  $\Updownarrow$

  $$\sup_{k \geq n} |X_k(\omega) - X(\omega)| < \varepsilon$$

  $n > N$

$k \geq n$

$n > N$

Connection with limits of sets

$$\bar{A}_n = \{ \omega : |X_n(\omega) - X(\omega)| < \varepsilon \}$$

$$A_n = \{ \omega : |X_n(\omega) - X(\omega)| \geq \varepsilon \}$$

- $P(\liminf_{n\to\infty} \bar{A}_n) = 1$
- $P(\limsup_{n\to\infty} A_n) = 0$

$\liminf \bar{A}_n$ = not ($\bar{A}_n$ not happening at most finitely many times) =

= not (A_n happening at most fin. many times)

= A_n happening ∞ many times =

= $\limsup_{n\to\infty} A_n$

What does it mean that $X_n \overset{a.s.}{\not\to} X$?

- $P(X_n \to X) < 1$
- $\exists A : P(A) > 0$, $\underbrace{X_n(\omega) \not\to X(\omega)}$, $\forall \omega \in A$

$\exists$ subsequence X_{n_k} :

$|X_{n_k}(\omega) - X(\omega)| > \varepsilon$,

$\forall k$ for some $\varepsilon > 0$

$\Downarrow$ $\quad \forall k$ for some $\varepsilon > 0$

$$\sup_{n \geq N} |X_n - X| > \varepsilon \quad \forall N, \forall \omega \in A$$

TH $\quad X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$

Proof: Goes by contradiction

$\exists \; X_n \not\xrightarrow{P} X$, but $X_n \xrightarrow{a.s.} X$

$\Downarrow$

$\exists$ a non-converging subsequence X_{n_k}:

$$P(\underbrace{|X_{n_k} - X| \geq \varepsilon}_{A_k}) \geq \delta > 0, \quad \forall k$$

Define these events A_k

Consider $\quad B_m = \bigcup_{k=m}^{\infty} A_k$, $\quad \downarrow$ sets in m

$$B = \limsup A_k = \bigcap_{m=1}^{\infty} B_m = \{\text{inf \# of } A_k\text{s are happening}\}$$

by def of limsup

Note: fin or inf # of A_ks are not happening

$\forall \omega \in B \quad \forall N \; \exists n > N$:

$|X_n(\omega) - X(\omega)| > \varepsilon$

A_n is happening

$\Downarrow$

$X_n(\omega) \not\to X(\omega)$, $\forall \omega \in B$

Let's show that $P(B) > 0$

B_m are $\downarrow$ sets $\quad P(\underset{\bigcup_{k=m}^{\infty} A_k}{B_m}) \geq P(A_m) \geq \delta > 0$

By continuity of $P \Rightarrow$

$$P(B_m) \to P(B) \geq \delta > 0$$

$\qquad \qquad ||$

$\limsup A_n$ $\Leftarrow$

$$P(\limsup_{n\to\infty} A_n) > 0$$

(!?) Contradiction with $X_n \xrightarrow{a.s.} X$

$\square$

a.s. convergence is stronger than convergence in pr

$\xrightarrow{P} \not\Rightarrow \xrightarrow{a.s.}$ From conv. in pr it does not follow conv. a.s.

Example: Moving impulse example

$\Omega = [0,1]$

$\mathcal{F} = \mathcal{B}([0,1])$

$A_{in} = \left[\frac{i-1}{n}, \frac{i}{n}\right], \ i = 1, \dots, n$

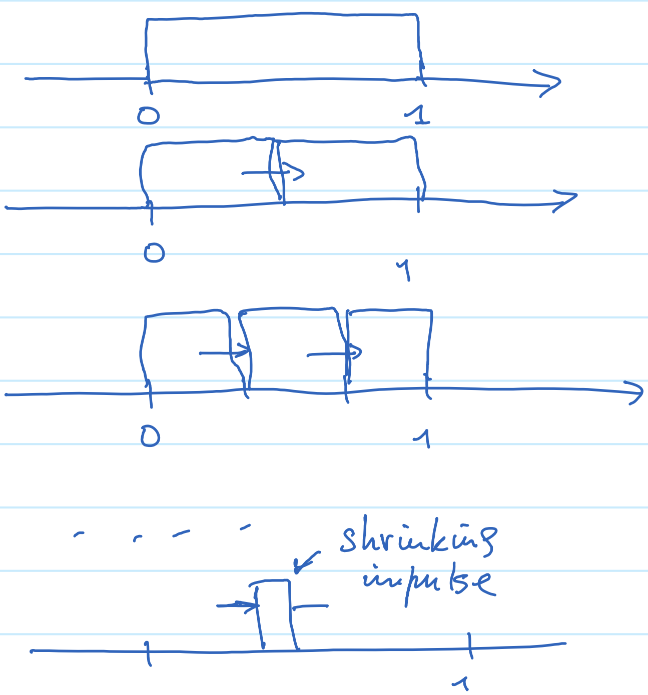
$X_{in} = I_{A_{in}}$

$n=1$ X_{11}

$n=2$ X_{12}, X_{22}

$n=3$ X_{13}, X_{23}, X_{33}

$\vdots$



$P(|X_{in}| > \varepsilon) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ We have $\xrightarrow{P}$

However, $P(\{\omega : X_{in} \to 0\}) = 0 \Rightarrow$ we do not have $\xrightarrow{a.s.}$

Note:

$$\mathbb{E}(|X_{in}|^r) \xrightarrow[n\to\infty]{} 0 \Rightarrow \xrightarrow{L^r} \not\Rightarrow \xrightarrow{a.s.}$$

more about L^r convergence later