Lecture 12. Fubini, Bounds, Convergence

Wednesday, October 18, 2017 9:50 AM

HW4

min and max confusion

$\mathbb{E}(\max X_i) < \infty$ not true

Plan for the nearest future

HW5 due on Friday

No new HW until next Wednesday

30th Wednesday → Midterm

Topics all the way until convergence

Next Monday No In-class lecture

I will post a practice Mid-term

Posted Panopto lecture capture on

Solutions for the practice exam

I will make more office hours

No office hours next Monday

New Office hours on Friday 20th 1-2 pm

Tuesday 24th 5-6 pm

Previous lecture

Independence

probabilities of intersections any # of events factorize

r.v.s and sets

CDFs will also factorize

X is $\sigma(Y)$ - measurable then X and Y

are functionally related

Copulae

$C(u,v)$ $\quad\quad\quad (u,v) \in [0,1] \times [0,1]$

$\downarrow$ joint uniform CDF $\quad\quad\quad \downarrow$ Cartesian Product of sets

Grounded

$C(0,v) = C(u,0) = 0$

With margins

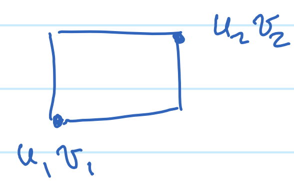
$C(1,v) = v$

$C(u,1) = u$

DF $\quad C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0$

$u_1 \leq u_2$

$v_1 \leq v_2$



Th $\quad$ Sklar

Any joint distribution of r.v. (X,Y) can be represented as

$$F_{(X,Y)} = C(F_X, F_Y)$$

$\downarrow$ joint CDF $\quad\quad \downarrow\ \swarrow$ marginal CDFs

w/o proof.

$X \perp Y$

$C(u,v) = u \cdot v$

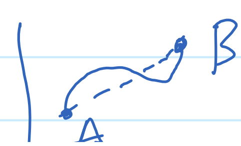
Note: Copulae are not the only way to model dependence

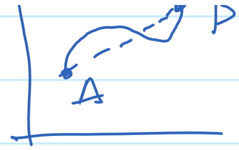
Conditional specifications:

$[X \mid Y]$, $[Y \mid X]$ } define conditional models simultaneously

TH $\quad$ Fréchet - Hoeffding bounds

$\swarrow$ any Copula satisfies the bounds





any Copula satisfies the bounds

$$\max(0, u+v-1) \le C(u,v) \le \min(u,v)$$

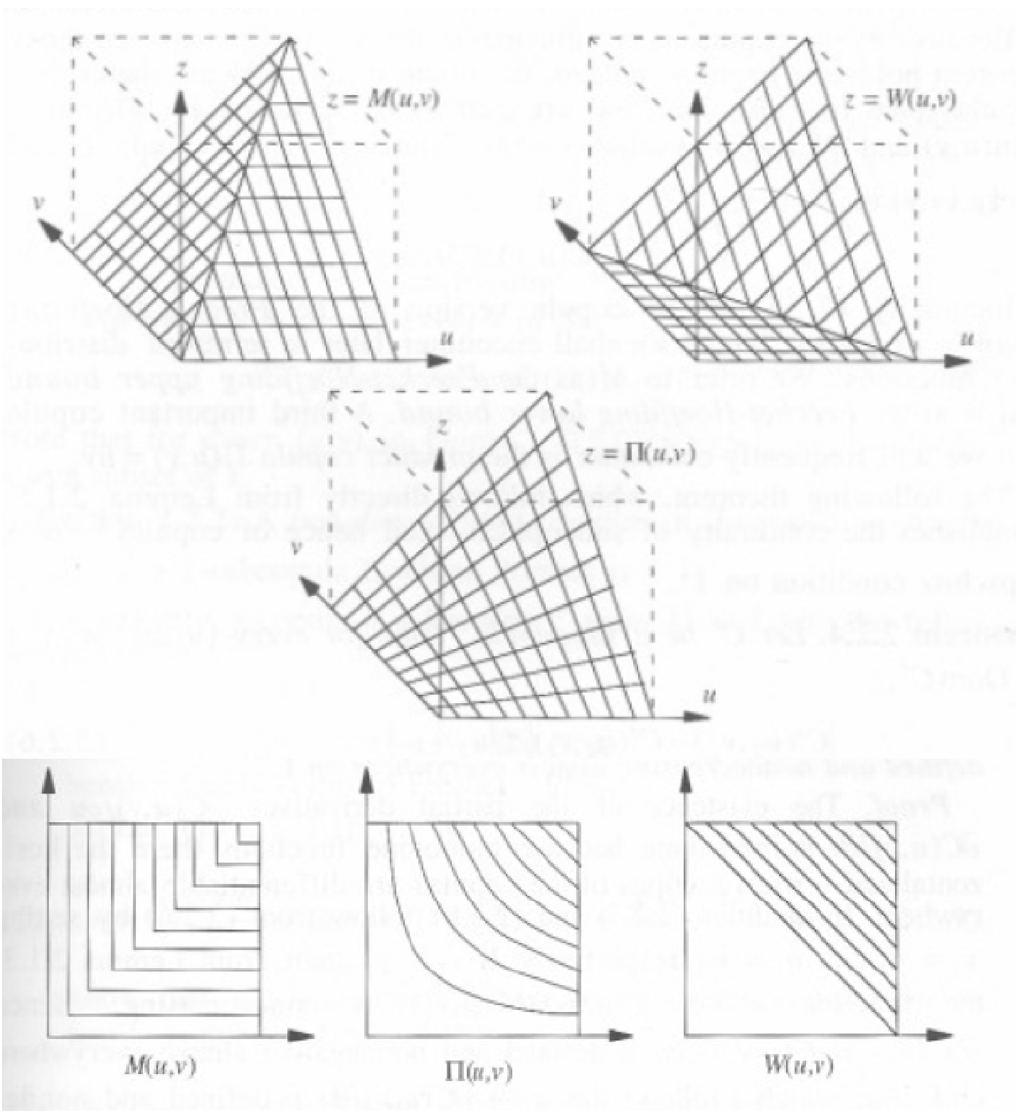
$$\le u\cdot v \le$$

extreme neg. dep. Copula

extreme positive dep.

$x = \varphi(t)$

$y = \psi(t)$



Iterated Expectations

Expectations of functions of independent random variables

r.v.

$$X(\omega) = (X_1(\omega), \dots, X_n(\omega))^T$$

defined on a sample space

$$(\mathbb{R}^n, \mathcal{B}^n, P_X)$$

Lebesgue Integral

$$\int_{A \in \mathbb{R}^n} g(x)\, dP_X(x) := \mathbb{E}(g(X); A)$$

some function

$$A = \{(x_1, \dots, x_n)^T, x_1 \in A_1, \dots, x_n \in A_n\} = A_1 \times A_2 \times \dots \times A_n$$

If $X_1, \dots, X_n$ are mutually independent

$$P_X(\underset{\substack{\shortparallel \\ (x_1, \dots, x_n)^T}}{x}) = \prod_i P_{X_i}(x_i)$$

TH Fubini $\exists\, X_1 \perp X_2$, r.v.s $X = (X_1, X_2)^T$

If $\int g(x)\, dP_X(x)$ exists then

$$\int g(x)\, dP_X(x) = \int \Big[\underbrace{\int g(x_1, x_2)\, dP_{X_2}(x_2)}_{\text{over a line}} \Big] dP_{X_1}(x_1)$$

over the plane

order of $1 \rightleftarrows 2$ can be interchanged

w/o proof. You can find the proof usually in Appendices to Probability or Measure Theory textbooks

Example: $X_1 \perp X_2$

$X_1 + X_2 \sim$? Convolution

$$F_{X_1+X_2} = P_X\left(X_1+X_2 \le x\right) = \int_{x_1+x_2\le x} dP_X(x_1,x_2) =$$

$$= \int_{x_1\in\mathbb{R}} \left[\int_{x_2 < x - x_1} dP_{X_2}(x_2)\right] dP_{X_1}(x_1) =$$

Fubini

$$= \int_{-\infty}^{\infty} F_{X_2}(x-x_1)\, dF_{X_1}(x_1) \rightarrow \text{Convolution}$$

Convergence of random sequences

DF Convergence in probability (Consistency)

$X_n \xrightarrow[n\to\infty]{P} X$ if

$$\forall \varepsilon > 0 \quad P\left(|X_n - X| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$

Any ε-deviation of X_n from X becomes increasingly unlikely

In $\varepsilon-\delta$ language

$\forall \varepsilon > 0,\ \forall \delta > 0 \quad \exists N:$

$$P(|X_n - X| > \varepsilon) < \delta, \quad \forall n > N$$

$$\sup_{k>N} P(|X_k - X| > \varepsilon) \rightarrow 0$$

DF Almost sure convergence a.s. (Strong Consistency)
everywhere a.e.

$X_n \xrightarrow[a.s.]{} X$ wrt measure P

when $X_n(\omega) \to X(\omega)$ for all ω, except perhaps a set of measure 0.

Will connect both definitions to limits of sets
Will show that conv. a.s. implies conv in prob.