1. Sets and Events
   (a) Algebra
   Closed under
      i. Complement
      ii. Finite union
   (b) σ-algebra
   Closed under
      i. Complement
      ii. Countable union
   (c) σ-algebra generated by $\mathcal{C}$
   The intersection of all the σ-algebras that contain $\mathcal{C}$.
   (d) Borel sets on the real line

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}), \quad \mathcal{C} = \{(a,b], -\infty \leq a \leq b < \infty\}$$

2. Probability Spaces: $(\Omega, \mathcal{B}, P)$
   (a) Probability measure P:
      i. $P(A) \geq 0$
      ii. $P(\sum_1^\infty A_n) = \sum_1^\infty P(A_n)$
      iii. $P(\Omega) = 1$
   (b) Proeprties of P
      i. $P(\lim A_n) = \lim P(A_n)$ for monotonic A_n.
      ii. $P(\liminf A_n) \leq \liminf P(A_n) \leq \limsup P(A_n) \leq P(\limsup A_n)$
3. Integration and Expectation
   (a) Simple functions: $X(\omega) = \sum_{i=1}^k a_i 1_{A_i}(\omega)$.
      i. Measurability Theorem: Suppose $X(\omega) \geq 0$ for all ω. Then X is measurable iff $0 \leq X_n \uparrow X$ for some simple functions X_n.
4. Convergence
   (a) Almost sure convergence: $P(\lim X_n = X) = 1$.
      i. If X_i is iid with distribution $F(x) < 1$ for all x, then $\bigvee_1^n X_i \uparrow \infty$ $a.s.$
   (b) Convergence in probability: $\lim P[|X_n - X| > \epsilon] = 0$
      i. Caucy criterion: $X_n \xrightarrow{P}$ iff $X_n - X_m \xrightarrow{P} 0$.

ii. Subsequence criterion: $X_n \overset{P}{\to} X$ iff each subsequence X_{n_k} contains a $X_{n_{k_i}} \overset{a.s.}{\to} X$.

(c) L_p convergence: $E(|X_n - X|^p) \to 0$.

i. Variation of Scheffe: $X_n \overset{L_1}{\to} X$ iff

$$\sup_{A \in \mathcal{B}} |\int_A X_n dP - \int_A X dP| \to 0.$$

ii. If $X_n \overset{L_p}{\to} X$, then $\|X_n\|_p \to \|X\|_p$.

iii. Let $p \geq 1$ and $X_n \in L_p$. The following are equivalent:

A. X_n is L_p convergent.

B. X_n is L_p Cauchy ($\|X_n - X_m\|_p \to 0$).

C. $|X_n|^p$ is ui and X_n converges ip.

(d) Properties

i. Connections

A. If $X_n \overset{a.s.}{\to} X$, then $X_n \overset{P}{\to} X$.

B. If $X_n \overset{L_p}{\to} X$, then $X_n \overset{P}{\to} X$.

ii. Continuous transformation: Let g be continuous.

A. If $X_n \overset{a.s.}{\to} X$, then $g(X_n) \overset{a.s.}{\to} g(X)$.

B. If $X_n \overset{P}{\to} X$, then $g(X_n) \overset{P}{\to} g(X)$.

iii. Lebesgue Dominated Convergece: If $X_n \overset{P}{\to} X$ and $|X_n| \leq \xi$ for some $\xi \in L_1$, then $E(X_n) \to E(X)$.

(e) Uniformly integrable: $\sup\limits_{t \in T} \int_{[|X_t|>a]} |X_t| dP \to 0$ as $a \to \infty$.

i. X_t is ui if

A. $T = \{1\}$.

B. $|X_t| \leq Y$ for some $Y \in L_1$.

C. $T = \{1, \ldots, n\}$.

D. $|X_t| \leq |Y_t|$ for some ui $\{Y_t\}$.

ii. $\{|X_n|^p\}$ is ui if $\sup_n E(|X_n|^{p+\delta}) < \infty$ for some $\delta > 0$.

iii. Let $X_t \in L_1$. Then $\{X_t\}$ is ui iff

A. For all $\epsilon > 0$, there exists ξ such that $\sup\limits_{t \in T} \int_A |X_t| dP < \epsilon$ for all $P(A) \leq \xi$.

B. $\sup\limits_{t \in T} E(|X_t|) < \infty$.

(f) Inequalities

i. Schwartz

$$\|XY\|_1 \leq \|X\|_2 \|Y\|_2.$$

ii. Holder: If $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $E(|X|^p) < \infty$, $E(|Y|^q) < \infty$, then
$$\|XY\|_1 \leq \|X\|_p \|Y\|_q.$$

iii. Minkowski: Let $1 \leq p < \infty$ and $X, Y \in L_p$. Then
$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p.$$

iv. Jensen: If $u : \mathbb{R} \to \mathbb{R}$ is convex, $E|X| < \infty$ and $E|u(X)| < \infty$, then
$$E(u(X)) \geq u(E(X)).$$

(g) Quantile estimation: Fill in the blank.

1 Laws of large numbers

1. Tail equivalence: $\sum_n P[X_n \neq X'_n] < \infty$.
   (a) $\sum_n (X_n - X'n)$ converges a.s.
   (b) $\sum_n X_n$ converges a.s. iff $\sum_n X'_n$ converges a.s.
   (c) If $\frac{1}{a_n} \sum_{j=1}^n X_j \overset{a.s.}{\to} X$ for $a_n \uparrow \infty$, then $\frac{1}{a_n} \sum_{j=1}^n X'_j \overset{a.s.}{\to} X$.
2. General weak law of large numbers: If $\{X_n\}$ are independent,
$$\sum_{j=1}^n P[|X_j| > n] \to 0,$$
and
$$\frac{1}{n^2} \sum_{j=1}^n E X_j^2 1_{[|X_j| \leq n]} \to 0,$$
then
$$\frac{S_n - a_n}{n} \overset{P}{\to} 0,$$
where
$$a_n = \sum_{j=1}^n E(X_j 1_{[|X_j| \leq n]}).$$
   (a) If $\{X_n\}$ are iid with $E(X_n) = \mu$ and $E(X_n^2) < \infty$, then $\frac{1}{n} S_n \overset{P}{\to} \mu$.
   (b) If $\{X_n\}$ are iid with $E(X_n) = \mu$ and $E(|X_1|) < \infty$, then $\frac{1}{n} S_n \overset{P}{\to} \mu$.
   (c) If $\{X_n\}$ are iid with
$$\lim_{x \to \infty} x P[|X_1| > x] = 0$$
then
$$\frac{S_n}{n} - E(X_1 1_{[|X_1| \leq n]}) \overset{P}{\to} 0$$

3. Sum of independent r.v.

   (a) If $\{X_n\}$ is independent and $\alpha > 0$, then

   $$P[\sup_{j \leq N} |S_j| > 2\alpha] \leq \frac{1}{1-c} P[|S_N| > \alpha]$$

   where

   $$c = \sum_{j \leq N} P[|S_N - S_j| > \alpha]$$

   (b) If $\{X_n\}$ is independent, then TFAE:

      i. S_n converges i.p.
      ii. S_n is Cauchy i.p.
      iii. S_n converges a.s.
      iv. S_n is Cauchy a.s.

   (c) If $\{X_n\}$ is independent and $\sum_n Var(X_n) < \infty$, then $\sum_n (X_n - \mu_n)$ converges a.s. and L_2.

4. Strong laws of large numbers

   (a) If $0 < a_n \uparrow \infty$ and $\sum_n \frac{x_n}{a_n}$ converges, then

   $$\lim_{n \to \infty} a_n^{-1} \sum_{k=1}^{n} x_k = 0$$

   (b) If is $\{X_n\}$ independent with $E(X_n^2) < \infty$ and

   $$\sum_n Var(\frac{X_n}{b_n}) < \infty$$

   for $b_n \uparrow \infty$, then

   $$\frac{S_n - E(S_n)}{b_n} \overset{a.s.}{\to} 0$$

5. Strong law of large numbers for iid

   (a) If X_n is iid, then TFAE:

      i. $E|X_1| < \infty$
      ii. $|\frac{X_n}{n}| \overset{a.s.}{\to} 0$
      iii. $\sum_n P[|X_1| \geq \epsilon n] < \infty$ for all $\epsilon > 0$.

   (b) If X_n is iid, then $\bar{X}_n \overset{a.s.}{\to} c < \infty$ iff $\mu < \infty$, in which case $c = \mu$.

   (c) If X_n is iid, then

   $$E|X_1| < \infty \Rightarrow \bar{X}_n \overset{a.s.}{\to} \mu$$
   $$EX_1^2 < \infty \Rightarrow S_n^2 \overset{a.s.}{\to} \sigma^2$$

(d) Let $X_n \stackrel{\text{iid}}{\sim} F(x)$ and define

$$\hat{F}_n(x,\omega) = \frac{1}{n}\sum^{n} 1_{[X_j \le x]}(\omega)$$

Then

$$\sum_x |\hat{F}_n(x) - F(x)| \stackrel{a.s.}{\to} 0$$

6. Let X_n be independent. Then $\sum_n X_n$ converges a.s. if and only if there exists $c > 0$ such that

   (a) $\sum_n P[|X_n| > c] < \infty$

   (b) $\sum_n Var(X_n 1_{[|X_n| \le c]}) < \infty$

   (c) $\sum_n E(X_n 1_{[|X_n| \le c]})$ converges

2 Convergence in distribution

1. Distribution function:

   (a) $0 \le F(X) \le 1 \forall x \in \mathbb{R}$

   (b) F is nondecreasing

   (c) F is right continuous

   (d) $F(-\infty) = 0$, $F(\infty) = 1$ (To be proper/nondefective/prob. df)

   (e) $\mathcal{C}(F) := \{x : F \text{ is continuous at } x\}$

      i. $(\mathcal{C}(F))^c$ is at most countable

      ii. Interval of continuity: endpoints are finite and are in $\mathcal{C}(F)$.

2. Lem: If F is a pdf on a dense set, then its extension to $\mathbb{R}$ is a pdf and is unique.

3. Notions of convergence

   (a) Vague: Convergence for any $(a, b]$ with $a, b \in \mathcal{C} \cap \mathbb{R}$

   (b) Proper: Vague convergence and the limit is proper

   (c) Weak: Convergence for any $(-\infty, a]$ with $a \in \mathcal{C}$.

   (d) Compete: Weak convergence and the limit is proper.

4. Thm: If F is proper, then all four convergence are equivalent. (Write $F_n \Rightarrow F$.)

5. Rmk: The Weak limit is unique.

6. Def: Convergence in total variation: $\sup_{A \in \mathcal{B}(\mathbb{R})} |F_n(A) - F(A)| \to 0$.

7. Thm (Scheffe):

$$\sup_{B\in\mathcal{B}(\mathbb{R})} |F_n(B) - F(B)| = \frac{1}{2}\int |f_n(x) - f(x)|dx$$

8. Prop: Suppose $k \to \infty$ and $n - k \to \infty$. Let $\sigma_n^2 = \frac{k}{n}(1 - \frac{k}{n})$. Then

$$X_n := \frac{n^{1/2}(U_{(k,n)} - k/n)}{\sigma_n} \to N(0,1) \text{ in TV}$$

9. If $X_n \xrightarrow{P} X$, then $X_n \Rightarrow X$.

10. Thm (Baby Shorohod): If $X_n \Rightarrow X_0$, then there exist $X_n^\#$ on $([0,1], \mathcal{B}([0,1]), \lambda)$ such that $X_n \stackrel{d}{=} X_n^\#$ and $X_n^\# \xrightarrow{a.s.} X_0^\#$.

11. Lem: If $X_n \Rightarrow X_0$, then $F_n^{\leftarrow}(t) \to F_0^{\leftarrow}(t)$ for $t \in (0,1) \cap \mathcal{C}(F_0^{\leftarrow})$.

12. Thm (Continuous mapping): If $X_n \Rightarrow X_0$ and $h : \mathbb{R} \to \mathbb{R}$ satisfies $P[X_0 \in (\mathcal{C}(h))^c] = 0$, then $h(X_n) \Rightarrow h(X_0)$. Moreover, if h is bounded, then $Eh(X_n) \to Eh(X_0)$.

13. Thm (Delta method):

$$\sqrt{n}\left(\frac{g(\bar{X}) - g(\mu)}{\sigma g'(\mu)}\right) \Rightarrow N(0,1)$$

14. Thm (Portmanteau): Let F_n be proper. TFAE:
    (a) $F_n \Rightarrow F_0$
    (b) $Eg(X_n) \to Eg(X_0)$ for any bounded continuous $g : \mathbb{R} \to \mathbb{R}$.
    (c) $F_n(A) \to F(A)$ for any $A \in \mathcal{B}(\mathbb{R})$ with $F_0(\partial(A)) = 0$ (boundary has zero probability).

15. Prop: If $X_n \Rightarrow c \in \mathbb{R}$, then $X_n \xrightarrow{P} c$.

16. Thm (Slutsky): If $X_n \Rightarrow X$ and $X_n - Y_n \xrightarrow{P} 0$, then $Y_n \Rightarrow X$.

17. Thm (Second convergence together)

18. Thm (Hoeffding and Robbins)

19. Thm (Convergence to type):
    (a) If

$$\frac{X_n - b_n}{a_n} \Rightarrow U, \quad \frac{X_n - \beta_n}{\alpha_n} \Rightarrow V, \tag{1}$$

then

$$\frac{\alpha_n}{a_n} \to A > 0, \quad \frac{\beta_n - b_n}{a_n} \to B \in \mathbb{R} \tag{2}$$

and

$$V \stackrel{d}{=} \frac{U - B}{A}. \tag{3}$$

(b) If eq. (2) holds, then either of eq. (1) implies the other and eq. (3) holds.

3 Central Limit Theorem

3.1 Characteristic Functons

1. Def: The characteristic function of $X \sim F$ is

$$\phi(t) = E[\exp\{itX\}] = E[\cos(tX) + i\sin(tX)] = E\cos(tX) + iE\sin(tX)$$

Note

$$|\phi(t)| \leq E|\exp(itX)| = 1$$

2. Properties
   (a) $\phi(t)$ is uniformly continuous
   (b) $|\phi(t)| \leq 1$, $\phi(0) = 0$
   (c)
   $$E[\exp\{it(aX + b)\}] = \phi(at)\exp(itb)$$
   (d) $\bar{\phi}(t) = \phi(-t) = E\exp(-itX)$
   (e) $Re(\phi(t))$ is even, $IM(\phi(t))$ is odd
   (f) $X_1 \perp X_2$, then $\phi_{X_1+X_2}(t) = \phi_1(t)\phi_2(t)$

3. Taylor expansion

$$\exp(ix) = \sum_0^n \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!}\int_0^x (x-s)^n \exp(is)ds$$

by integration by part which implies

$$|\exp(ix) - \sum_0^n \frac{(ix)^k}{k!}| \leq \frac{|X|^{n+1}}{(n+1)!}$$

Final result:

$$|\exp(ix)$$

4. Prop: $\int \phi(x - \mu)dx = 1$ for all complex μ

5. Corollary: If $Z \sim N(0, 1)$, then $E[\exp(itZ)] = \exp(-t^2/2)$.

6.