Lecture 15. Cauchy, cont, Cantelli

Monday, November 6, 2017 10:04 AM

Convergence of sequences of r.v. X_n

$$\text{a.s.} \underset{\text{add. cond.}}{\overset{}{\rightleftarrows}} \xrightarrow{P} \underset{\text{add. cond.}}{\Leftarrow} L^r$$

$$\xrightarrow{P} \ \overset{1/}{\Downarrow} \ \rightsquigarrow \Leftrightarrow \xrightarrow{d}$$

$$\sum_{n=1}^{\infty} P(|X_n - X| \geq \varepsilon) < \infty, \ \forall \varepsilon \implies X_n \xrightarrow[a.s.]{} X$$

$$\implies X_n \xrightarrow[p]{} X$$

L^r defined as $\mathbb{E}(|X_n - X|^r) \to 0 \implies X_n \xrightarrow[P]{} X$

Lemma Borel-Cantelli: Zero-One Laws

$\exists$ $\{A_n\}$ sequence of sets

$$A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k$$

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

Then $P(A) = 0$

$$\underbrace{B_n = \bigcup_{k\geq n} A_k}_{\text{decreasing sets}} \qquad P(A) = \lim_{n\to\infty} P(B_n)$$

Limits of sets

$$P(A) = \lim_{n\to\infty} P\Big(\bigcup_{k\geq n} A_k\Big) \leq \lim_{n\to\infty} \underbrace{\sum_{k\geq n} P(A_k)}=0$$

$n\to\infty \quad k\geq n \qquad\qquad n\to\infty \quad \underbrace{k\geq n}$

$\xrightarrow[n\to\infty]{} 0$ b/c the series converges

$\Rightarrow \quad P(A)=0 \quad \square$

Note from calculus of sequences of #s: $x_n \to x \iff x_n$ being a Cauchy sequence (Fundamental) $|x_n - x_m| \xrightarrow[m,n\to\infty]{} 0$

DF $\{X_n\}$ is Cauchy when

a) In probability

$\forall \varepsilon>0,\ P(|X_n - X_m| > \varepsilon) \xrightarrow[m,n\to\infty]{} 0$

b) a.s.

$P\left(\sup_{n\geq m} |X_n - X_m| > \varepsilon\right) \xrightarrow[m\to\infty]{} 0$

c) L^r

$\mathbb{E}\left(|X_n - X_m|^r\right) \xrightarrow[m,n\to\infty]{} 0$

Note on norms and scalar/inner products

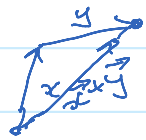
DF $\|\cdot\|$ is a functional $X \xrightarrow{\|\cdot\|} \#$ such that

$\|\cdot\| \geq 0$

$\|a\cdot X\| = |a|\,\|X\|$ homogeneity

$\|X+Y\| \leq \|X\| + \|Y\|$ triangle inequality

$\|X\| = 0 \iff X \equiv 0$



#	1	2	3
X	x_1	x_2	x_3

Examples of norms

Euclidean

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots x_k^2} = \sqrt{x^T\cdot x} \qquad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

$$\|f(\cdot)\|_{L^r} = \left(\int f^r \, d\mu\right)^{1/r}$$ wrt to some measure μ

$$\|f(\cdot)\|_{L^2} = \sqrt{\int f^2(x)\, d\mu(x)}$$

DF Inner product (Scalar product)

$\langle X, X \rangle \geq 0$

$\langle X, Y \rangle = \langle Y, X \rangle$

$\langle X, a \cdot Y \rangle = a \langle X, Y \rangle$

$\langle X, Y+Z \rangle = \langle X, Y \rangle + \langle X, Z \rangle$

Example of $\|\cdot\|$ $\quad \|X\| = \sqrt{\langle X, X \rangle}$

other useful inequalities

Cauchy-Schwarz

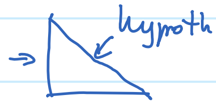
$$|\langle X, Y \rangle| \leq \|X\| \cdot \|Y\|$$

Hölder $\quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 0$

$$\|X \cdot Y\|_1 \leq \|X\|_p \cdot \|Y_p\|$$

Reverse triangle

$$\|X - Y\| \geq \|X\| - \|Y\|$$



Pithagorean Th. $\quad \exists \langle X, Y \rangle = 0$

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$$

TH $\quad X_n \xrightarrow[\substack{P \\ a.s. \\ L^r}]{} X \quad$ iff X_n is Cauchy P, a.s., L^r, respectively

Sufficiency take w/o proof.

Proof of necessity:

$$|X_n - X_m| = |X_n - X + X - X_m| \leq |X_n - X| + |X_m - X|$$

$$\Rightarrow \sup_{n \geq m} |X_n - X_m| \leq 2 \cdot \sup |X_n - X|$$

$$\Rightarrow \sup_{n \geq m} |X_n - X_m| \leq 2 \cdot \sup_{n \geq m} |X_n - X|$$

a.s. necessity:

$\forall \omega$ except perhaps for sets of measure 0

$\sup_{n \geq m} |X_n - X| \to 0$ b/c $X_n \xrightarrow[a.s.]{} X$

$\Rightarrow \sup_{n \geq m} |X_n - X_m| \to 0$ a.s.

$\Downarrow$

X_n is Cauchy a.s.

necessity in p:

$$P(|X_n - X_m| \geq \varepsilon) \leq \underbrace{P(|X_n - X| \geq \tfrac{\varepsilon}{2})}_{\to 0 \text{ b/c of } \xrightarrow{P}} + \underbrace{P(|X_m - X| \geq \tfrac{\varepsilon}{2})}_{\to 0}$$

$$\text{not}\left(|X_n - X| < \tfrac{\varepsilon}{2} \cap |X_m - X| < \tfrac{\varepsilon}{2}\right) =$$

$$= |X_n - X| \geq \frac{\varepsilon}{2} \cup |X_m - X| \geq \frac{\varepsilon}{2}$$

$$\Rightarrow P(|X_n - X_m| \geq \varepsilon) \xrightarrow[m,n \to \infty]{} 0, \quad \forall \varepsilon > 0 \Rightarrow$$

$\{X_n\}$ is Cauchy in p

necessity in $\mathcal{L}^r$

$$\|X\| = [\mathbb{E} X^r]^{1/r}$$

By triangle inequality for $\|\cdot\|$

$$\mathbb{E}(|X_n - X_m|^r) \leq \left[\underbrace{\mathbb{E}(|X_m - X|^r)^{1/r}}_{\xrightarrow[m \to \infty]{} 0} + \underbrace{\mathbb{E}(|X_n - X|^r)^{1/r}}_{\xrightarrow[n \to \infty]{} 0}\right]^r \to 0$$

b/c $X_n \xrightarrow{\mathcal{L}^r} X$

$$\xrightarrow[m,n \to \infty]{} 0 \Leftarrow$$

$\Downarrow$ X_n is Cauchy in $\mathcal{L}^r$

$\square$

TH Continuity

$\exists X_n \xrightarrow[n \to \infty]{} X$ and $f(\cdot)$ is a continuous function

(4) Continuity

$\exists\ X_n \xrightarrow[\substack{a.s.\\ P}]{} X$ and $f(\cdot)$ is a continuous function a.s.,

Then $f(X_n) \xrightarrow[\substack{a.s.\\ P}]{} f(X)$, respectively

Proof:

a.s. $P(\underbrace{X_n \to X}_{A}) = P(\underbrace{f \text{ is continuous}}_{B}) = 1$

$$P(AB) = \underset{1}{P(A)} + \underset{1}{P(B)} - \underset{1}{P(A+B)} = 1$$

$\Rightarrow$ Smth proved on $A \cap B$ will be valid a.s.

f is cont $\Rightarrow$

$\forall\ \underset{\text{any sequence of \#s}}{x_n} \to x \Rightarrow f(x_n) \to f(x)$

On AB $x_n = X_n(\omega)$ $x = X(\omega)$

$X_n \to X \Rightarrow f(X_n) \to f(X)$

$\Downarrow$

$P(f(X_n) \to f(X)) = 1$

$\square$ a.s.

P $X_n \xrightarrow[P]{} X$ By contradiction $\exists\ f(X_n) \underset{P}{\not\to} f(X)$

$\Rightarrow \exists$ subsequence X_{n_k}: $P(|f(X_{n_k}) - f(X)| \geq \varepsilon) \geq \delta > 0$ $\forall k$

$X_n \xrightarrow[P]{} X \Rightarrow X_{n_k} \xrightarrow[P]{} X \Rightarrow \exists\ X_{n_{k_m}} \xrightarrow[a.s.]{} X$

by the previous a.s. cont. Th $\Rightarrow$

$f(X_{n_{k_m}}) \xrightarrow[a.s.]{} f(X) \Rightarrow$

$\Rightarrow f(X_{n_{k_m}}) \xrightarrow[P]{} f(X)$

$$\Rightarrow f(x_{n_{k_m}}) \xrightarrow[P]{} f(x)$$

Contradiction with

$\square$