Biostat 602 Winter 2017

Lecture Set 11

Best Unbiased Estimation

Unbiasedness

- If there are at least two unbiased estimators, there are infinitely many. If T_1, T_2 are unbiased estimators of $\tau(\theta)$, then so is

$$\omega T_1 + (1-\omega)T_2$$

  for any $0 \le \omega \le 1$.
- If there is a best unbiased estimator, then it is unique.

Strategies for finding best unbiased estimator

Cramer-Rao Lower Bound

1. Calculate joint score function

$$u_n(\theta|\mathbf{x}) = \frac{\partial}{\partial\theta} \log L(\theta|\mathbf{x}).$$

2. Express u_n if possible as

$$u_n(\theta|\mathbf{x}) = a(\theta)[W(\mathbf{x}) - \tau(\theta)].$$

   Then $W(\mathbf{x})$ is the best unbiased estimator for $\tau(\theta)$ and attains CRLB.

- If "regularity conditions" are satisfied, then we have a Cramer-Rao bound for unbiased estimators of $\tau(\theta)$.
  - It helps to confirm an estimator is the best unbiased estimator of $\tau(\theta)$ if it happens to attain the CR-bound.
  - If an unbiased estimator of $\tau(\theta)$ has variance greater than the CR-bound, it does NOT mean that it is not the best unbiased estimator.
- When "regularity conditions" are not satisfied, $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$ is no longer a valid lower bound.
  - There may be unbiased estimators of $\tau(\theta)$ that have variance smaller than $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$.

Lehmann-Scheffé

Use complete sufficient statistic to find the best unbiased estimator for $\tau(\theta)$.

1. Find complete sufficient statistic T for θ.
2. Obtain $\phi(T)$, an unbiased estimator of $\tau(\theta)$ using either of the following two ways
   - Guess a function $\phi(T)$ such that $\mathrm{E}[\phi(T)] = \tau(\theta)$.
   - Guess an unbiased estimator $h(\mathbf{X})$ of $\tau(\theta)$. Construct $\phi(T) = \mathrm{E}[h(\mathbf{X})|T]$, then $\mathrm{E}[\phi(T)] = \mathrm{E}[h(\mathbf{X})] = \tau(\theta)$.

   In either case, $\phi(T)$ is the best unbiased estimator of $\tau(\theta)$.

Example 1: Let $X_1, \ldots, X_n$ be *i.i.d.* observations from the distribution with pdf

$$f_X(x|\theta) = e^{-(x-\theta)} \exp(-e^{-(x-\theta)}), \qquad -\infty < \theta < \infty, \ -\infty < x < \infty$$

(a) Find a Cramer-Rao lower bound to the variance of unbiased estimators of θ.

(b) Find a function $\tau(\theta)$ for which there exists an unbiased estimator whose variance attains the Cramer-Rao bound.

(c) Find the best unbiased estimator for $\tau(\theta)$ found in (b).

Solution: (a) The distribution belongs to an exponential family with $c(\theta) = e^{\theta}$, $h(x) = e^{-x}$, $w(\theta) = -e^{\theta}$, $t(x) = e^{-x}$. The Fisher information per observation can be calculated as

$$
\begin{aligned}
\log L(\theta|x) &= -(x-\theta) - \exp(-x+\theta) \\
u(\theta|x) &= \frac{\partial}{\partial\theta}\log L(\theta|x) = 1 - \exp(-x+\theta) \\
I(\theta) &= -\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\log L(\theta|X)\right] \\
&= -\mathrm{E}\left[\frac{\partial}{\partial\theta}u(\theta|X)\right] = \mathrm{E}\left[\exp(-X+\theta)\right] = 1
\end{aligned}
$$

The Cramer-Rao lower bound of θ is $\frac{1}{nI(\theta)} = \frac{1}{n}$.

(b, c) The score function of joint likelihood is

$$
\begin{aligned}
\log L(\theta|\mathbf{x}) &= -\left(\sum x_i - n\theta\right) - \exp\left(-\sum x_i + n\theta\right) \\
u_n(\theta|\mathbf{x}) &= \frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) = n - n\exp\left(-\sum x_i + n\theta\right) \\
&= -n\exp(n\theta)\left[\exp\left(-\sum x_i\right) - \exp\left(-n\theta\right)\right] \\
&= a(\theta)[W(\mathbf{x}) - \tau(\theta)]
\end{aligned}
$$

so $\tau(\theta) = \exp\left(-n\theta\right)$ and $W(\mathbf{x}) = \exp\left(-\sum x_i\right)$ is its best unbiased estimator whose variance attains CRLB.

Example 2: Let $X_1, \cdots, X_n$ be i.i.d. Uniform$(0, \theta)$. Find the best unbiased estimator for (1) θ, (2) θ^2, (3) $1/\theta$.

Solution - UMVUE of θ: $T(\mathbf{X}) = X_{(n)}$ is a complete and sufficient statistic for θ.

- $f_T(t) = n\theta^{-n}t^{n-1}I(0 < t < \theta)$.
- $\mathrm{E}[T] = \mathrm{E}[X_{(n)}] = \int_0^\theta tn\theta^{-n}t^{n-1}dt = \frac{n}{n+1}\theta$ (biased)
- $\mathrm{E}[\phi(T)] = \mathrm{E}\left[\frac{n+1}{n}X_{(n)}\right] = \theta$.

$\frac{n+1}{n}X_{(n)}$ is the best unbiased estimator of θ.

Estimating θ^2

$$\begin{aligned} \mathrm{E}[X_{(n)}^2] &= \int_0^\theta t^2 n\theta^{-n}t^{n-1}dt \\ &= n\theta^{-n}\int_0^\theta t^{n+1}dt = n\theta^{-n} \times \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta^2 \end{aligned}$$

So $\phi(X_{(n)}) = \frac{n+2}{n}X_{(n)}^2$ is the best unbiased estimator for θ^2.

Estimating $1/\theta$

$$\begin{aligned} \mathrm{E}[X_{(n)}^{-1}] &= \int_0^\theta t^{-1} n\theta^{-n}t^{n-1}dt \\ &= n\theta^{-n}\int_0^\theta t^{n-2}dt = n\theta^{-n} \times \frac{\theta^{n-1}}{n-1} = \frac{n}{n-1}\theta^{-1} \end{aligned}$$

So $\phi(X_{(n)}) = \frac{n-1}{n}X_{(n)}^{-1}$ is the best unbiased estimator for θ^{-1}.

Example 3: Let $X_1, \cdots, X_n$ i.i.d. Binomial(k, θ). Find the best unbiased estimator of the probability of exactly one success from a Binomial(k, θ).

Solution: The quantity we need to estimate is

$$\tau(\theta) = \Pr(X = 1|\theta) = k\theta(1-\theta)^{k-1}$$

We know that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i \sim$ Binomial(kn, θ) and it is a complete sufficient statistic. So we need to find a $\phi(T)$ that satisfies $\mathrm{E}[\phi(T)] = \tau(\theta)$.

There is no immediately evident unbiased estimator of $\tau(\theta)$ as a function of T. Start with a simple-minded estimator

$$W(\mathbf{X}) = \begin{cases} 1 & X_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

The expectation of W is

$$\begin{aligned} \mathrm{E}[W] &= \sum_{x_1=0}^{k} W(x_1) \binom{k}{x_1} \theta^{x_1}(1-\theta)^{k-x_1} \\ &= k\theta(1-\theta)^{k-1} \end{aligned}$$

and hence it is an unbiased estimator of $\tau(\theta) = k\theta(1-\theta)^{k-1}$. The best unbiased estimator of $\tau(\theta)$ is

$$\phi(T) = \mathrm{E}[W|T] = \mathrm{E}\left[W(\mathbf{X}) \middle| T(\mathbf{X})\right]$$

$$
\begin{aligned}
\phi(t) &= \mathrm{E}\left[W(\mathbf{X})\middle|\sum_{i=1}^{n} X_i = t\right] = \Pr\left[X_1 = 1\middle|\sum_{i=1}^{n} X_i = t\right] \\
&= \frac{\Pr(X_1 = 1, \sum_{i=1}^{n} X_i = t)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{\Pr(X_1 = 1, \sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{\Pr(X_1 = 1)\Pr(\sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{[k\theta(1-\theta)^{k-1}]\left[\binom{k(n-1)}{t-1}\theta^{t-1}(1-\theta)^{k(n-1)-t-1}\right]}{\binom{kn}{n}\theta^t(1-\theta)^{kn-t}} = k\frac{\binom{k(n-1)}{t-1}}{\binom{kn}{t}}
\end{aligned}
$$

Therefore, the unbiased estimator of $k\theta(1-\theta)^{k-1}$ is

$$
\phi\left(\sum_{i=1}^{n} X_i\right) = k\frac{\binom{k(n-1)}{\sum X_i - 1}}{\binom{kn}{\sum X_i}}
$$

Example 4: Let X_1, X_2 be *i.i.d.* observations from the pdf

$$f_X(x|\lambda) = \lambda e^{-\lambda x}, \qquad x \geq 0,\ \lambda > 0.$$

1. Show that the distribution of X_1 conditional on $Z = z$ is Uniform$(0, z)$.
2. Prove the best unbiased estimators of $\Pr(X_1 > 1) = e^{-\lambda}$ is

$$T(X_1, X_2) = \begin{cases} 0, & \text{if } X_1 + X_2 \leq 1 \\ \\ \frac{X_1+X_2-1}{X_1+X_2}, & \text{if } X_1 + X_2 > 1 \end{cases}$$

Solution: Note that X_1, X_2 are i.i.d. $Exp(1/\lambda)$ and so the $Z = X_1 + X_2$ is distributed as a $Gamma(2, 1/\lambda)$ random variable with pdf

$$f_Z(z|\lambda) = \lambda^2 z e^{-\lambda z}, \qquad z > 0,\ \lambda > 0.$$

The conditional pdf of $X_1|Z = z$ is

$$f(x_1|z, \lambda) \ = \ \frac{f(x_1, z|\lambda)}{f_Z(z|\lambda)} = \frac{\lambda^2 e^{-\lambda z}}{\lambda^2 z e^{-\lambda z}} = \frac{1}{z}$$

when $0 < x_1 < z$. If $x_1 > z$, the pdf is zero. Therefore, the conditional pdf of X_1 given z is Uniform$(0, z)$.

1. A naive unbiased estimator of $\Pr(X_1 > 1)$ is $W = I(X_1 > 1)$.
2. We know that $Z = X_1 + X_2$ is a complete sufficient statistic.
3. By Theorem 7.3.23, the best unbiased estimator of $\Pr(X_1 > 1)$ can be obtained by conditional expectation $\mathrm{E}[W|Z]$.

Because $\Pr(X|Z)$ is uniformly distributed between 0 and Z,

$$\mathrm{E}[W|Z] = \Pr(X_1 > 1|Z) = \begin{cases} 0 & \text{if } Z \leq 1 \\ 1 - \frac{1}{Z} = \frac{Z-1}{Z} & \text{if } Z > 1 \end{cases}$$

Therefore $E[W|X_1 + X_2] = T(X_1, X_2)$ is the best unbiased estimator of $\Pr(X_1 > 1) = e^{-\lambda}$.