Definition 18 (Small sets) A set $C \in \mathcal{B}(\mathcal{X})$ is called a small set if there exists an m > 0 and a non-trivial measure ν_m on $\mathcal{B}(\mathcal{X})$ such that

$$P^{m}(x, B) \ge \nu_{m}(B), \quad \forall x \in C, \forall B \in \mathcal{B}(\mathcal{X}).$$

When this holds, we say that C is ν_m -small.

Recall that the probability transition kernel $P^n(x,\cdot)$ is a probability measure on $(\mathcal{X},\mathcal{B}(\mathcal{X}))$. Therefore there exists a Lebesgue decomposition into its absolutely continuous and singular parts with respect to some σ -finite measure ϕ on $\mathcal{B}(\mathcal{X})$: for any fixed x and $B \in \mathcal{B}(\mathcal{X})$

$$P^{n}(x,B) = \int_{B} p^{n}(x,y)\phi(dy) + P_{\perp}(x,B),$$
 (6)

where $p^n(x,y)$ is the density of $P^n(x,\cdot)$ with respect to ϕ and P_{\perp} and ϕ are mutually singular. (Recall that two measures ν and μ are mutually singular, $\mu \perp \nu$, if there exists disjoint sets A and B such that $A \cup B = \mathcal{X}$ and $\nu(A) = \mu(B) = 0$. Thus, the two measures "live" on different subspaces of \mathcal{X} .)

Theorem 3 Suppose ϕ is a σ -finite measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Suppose $A \in \mathcal{B}(\mathcal{X})$ with $\phi(A) > 0$ such that

$$B \subseteq A, \ \phi(B) > 0 \implies \sum_{k=1}^{\infty} P^k(x, B) > 0, \quad x \in A.$$

Then, for every n, the density p^n can be chosen to be a measurable function on \mathcal{X}^2 and there exists $C \subseteq A$, an m > 1, and a $\delta > 0$ such that $\phi(C) > 0$ and

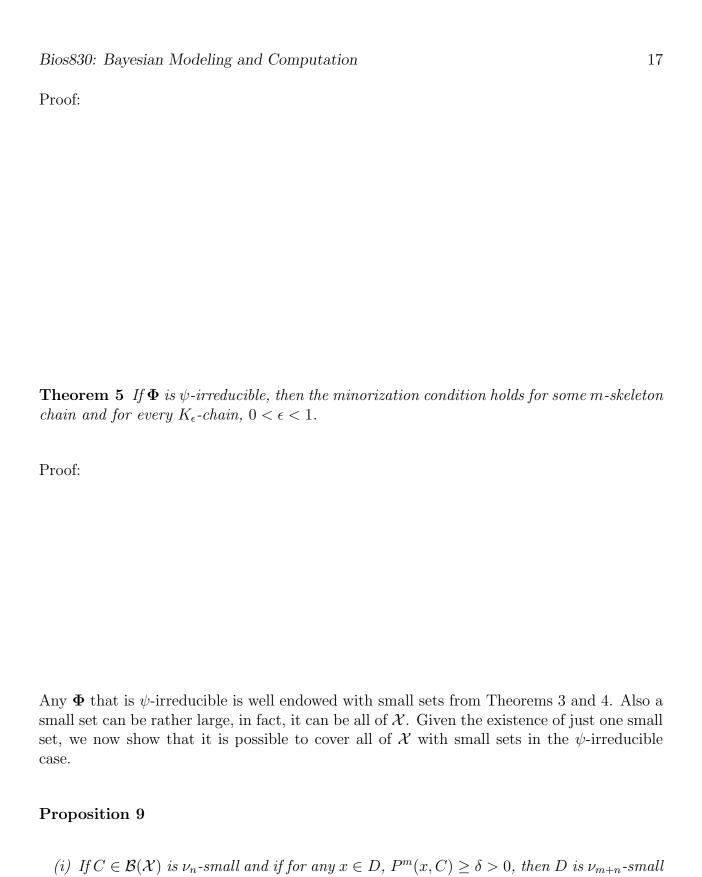
$$p^m(x,y) > \delta, \quad x,y \in C.$$
 (7)

Proof: (Meyn & Tweedie, pp. 103–105).

The key point in this theorem is that we can define a version of the densities of the probability transition kernel such that (7) holds uniformly over $x \in C$.

Now the main theorem of this section:

Theorem 4 If Φ is ψ -irreducible, then for every $A \in \mathcal{B}^+(\mathcal{X})$ there exists $m \geq 1$ and a ν_m -small set $C \subseteq A$ such that $C \in \mathcal{B}^+(\mathcal{X})$ and $\nu_m(C) > 0$.



where ν_{m+n} is a multiple of ν_n .

(ii) If Φ is ψ -irreducible, then there exists a countable collection C_i of small sets in $\mathcal{B}(\mathcal{X})$ such that

$$\mathcal{X} = \bigcup_{i=1}^{\infty} C_i.$$

(iii) If Φ is ψ -irreducible and $C \in \mathcal{B}^+(\mathcal{X})$ is ν_n -small, then there exists an $M \geq 1$ and a measure ν_M such that C is ν_M -small and $\nu_M(C) > 0$.

Proof:

The covering in Proposition 9 will be used in the following section.