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MODULE 2.1

RANDOM NUMBERS

What are random numbers?

- **True random numbers**
  - Truly non-deterministic numbers
  - Conceptually, easy to imagine.
  - Practically very hard to prove true randomness
  - See http://random.org and see what they do

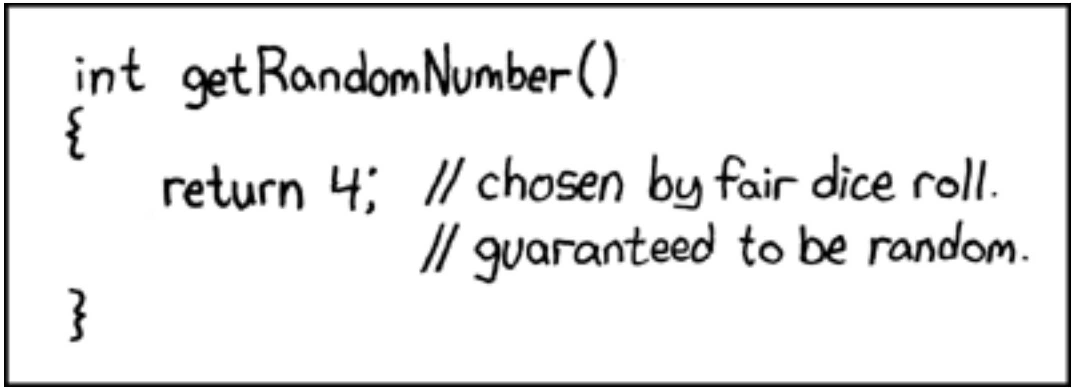
- **Pseudo-random numbers**
  - A deterministic sequence of apparent random numbers calculated from a seed
  - Good pseudo-random numbers should be very hard to guess the next numbers, just based on observations

Usage of random numbers

- **Statistical methods**
  - Resampling : Permutation & Bootstrapping
  - Simulation of data
  - Stochastic processes for estimation : MCMC, Gibbs sampling

- **Cryptography**
  - Generating "hard-to-predict" pseudo-random numbers given seed is closely related to encrypting the seed into a sequence of random bits
  - Creating a "hard-to-fabricate" digital signature or a certificate is also closely related to generating a good hash function, which are reproducible pseudo-random numbers given seeds

What are good random numbers?

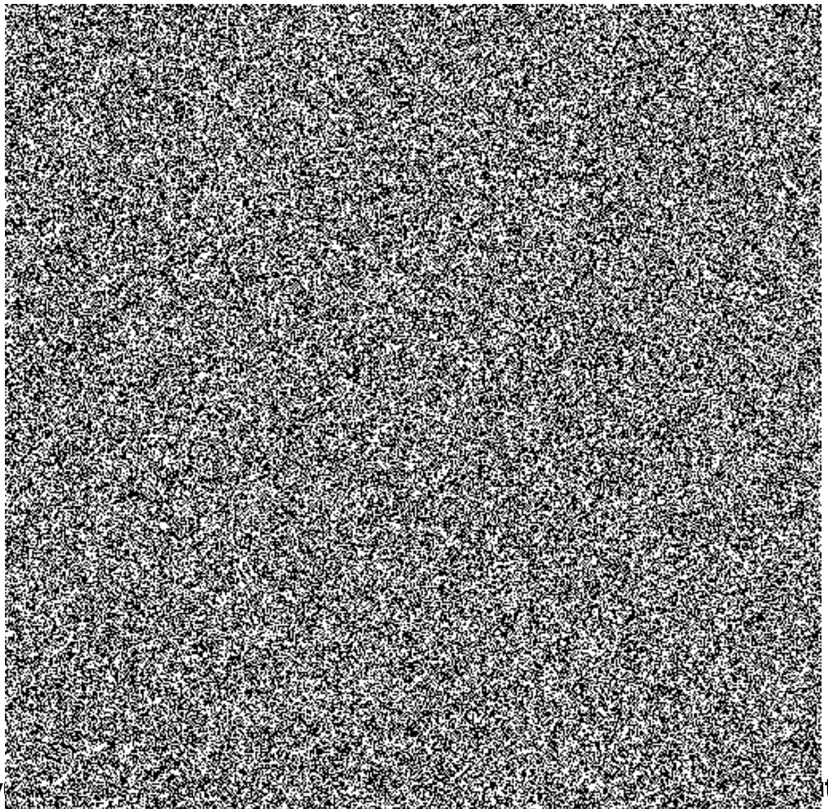


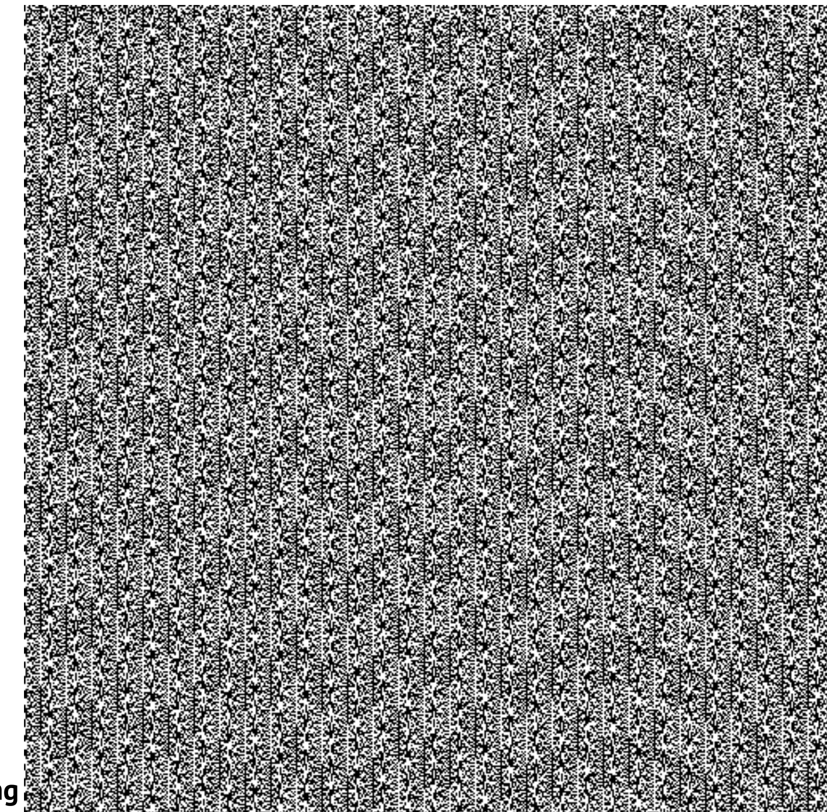
https://xkcd.com/221/

http://ardiri.com/blog/secure_number_generator_for_the_arduino

Good vs. bad pseudo-random numbers

- Which image generated from pseudo-random numbers looks random?





Playing a rock-paper-scissors game

- Suppose that you are playing 100 rock-paper-scissors games against a world champion of rock-paper-scissors, betting $1 per game.
- Your opponent is known to be excellent in predicting what your next move will be.
- What is the best strategy to minimize your loss?

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- Your opponent is known to be excellent in predicting what your next move will be.
- What is the best strategy to minimize your loss?

Answer: Use random.org

C++ rand() – An easy but poor choice.

```
#include <cstdlib>
#include <ctime>
#include <cstdio>
int main(int argc, char** argv) {
  srand( argc < 2 ? std::time(NULL) : atoi(argv[1]) );
  for(int i=0; i < 10; ++i)
    printf("%ld ", rand() % 2 );
  printf("\n");
  for(int i=0; i < 10; ++i)
    printf("%.5lf ", (rand() + 0.5) / (RAND_MAX+1.0));
  printf("\n");
  return 0;
}
```

In most case, it works fine, but who knows?

```
$ ./rand1
1 1 1 0 0 0 1 1 1 1
0.97964 0.73368 0.99384 0.51616 0.12086 0.36270 0.89395
0.57954 0.35506 0.53798
$ ./rand1
1 1 1 1 1 0 1 0 1 1
0.74664 0.77251 0.65577 0.58531 0.22779 0.42210 0.23625
0.59494 0.12189 0.67166
$ ./rand1 12345
1 0 0 0 1 1 0 0 0 1
0.33315 0.19511 0.26720 0.79270 0.98388 0.14889 0.33915
0.03411 0.26621 0.16742
$ ./rand1 12345
1 0 0 0 1 1 0 0 0 1
0.33315 0.19511 0.26720 0.79270 0.98388 0.14889 0.33915
0.03411 0.26621 0.16742
```

Using std::time for seed could be dangerous

```
$ ./rand1;./rand1
0 1 0 0 1 0 0 0 0 0
0.70750 0.02907 0.60001 0.29918 0.35208 0.43513 0.18139
0.65265 0.02598 0.58701
0 1 0 0 1 0 0 0 0 0
0.70750 0.02907 0.60001 0.29918 0.35208 0.43513 0.18139
0.65265 0.02598 0.58701
```

C++ random() – A lower-risk choice.

```
#include <cstdlib>
#include <ctime>
#include <cstdio>
int main(int argc, char** argv) {
  srand( argc < 2 ? std::time(NULL) : atoi(argv[1]) );
  for(int i=0; i < 10; ++i)
    printf("%ld ", random() % 2 );
  printf("\n");
  for(int i=0; i < 10; ++i)
    printf("%.5lf ", (random() + 0.5) / (RAND_MAX+1.0));
  printf("\n");
  return 0;
}
```

C++11 <random> – A more principled choice

```
#include <random>
#include <cstdio>
int main(int argc, char** argv) {
  std::default_random_engine prg;
  if ( argc > 1 ) prg.seed(atoi(argv[1]));
  else { std::random_device r; prg.seed(r());}
  std::binomial_distribution<int> bdist(1, 0.5);
  for(int i=0; i < 10; ++i)
    printf("%d ", bdist(prg));
  printf("\n");
  std::uniform_real_distribution<double> udist(0.0, 1.0);
  for(int i=0; i < 10; ++i)
    printf("%.5lf ", udist(prg));
  printf("\n");
  return 0;
}
```

<random> is supported in C++11 standard. If it does not compile, you may need to add **--std=c++11** flag

Separates these three:
(1) random_device (to pick a random seed)
(2) PRG (pseudo-number generater) engine
(3) distribution to convert random bits to numbers

Running examples

```
$ ./rand3
1 1 1 0 0 1 0 1 0 1
0.80351 0.73279 0.53399 0.13210 0.53778 0.53161 0.98968
0.40718 0.81156 0.32475
$ ./rand3
0 1 1 1 1 0 0 1 0 1
0.49421 0.22141 0.97281 0.35042 0.26444 0.74354 0.11054
0.03712 0.25954 0.45112
$ ./rand3 12345
0 0 0 1 0 1 1 1 1 0
0.59747 0.26928 0.12623 0.85370 0.75281 0.78732 0.35292
0.89005 0.04774 0.86126
$ ./rand3 12345
0 0 0 1 0 1 1 1 1 0
0.59747 0.26928 0.12623 0.85370 0.75281 0.78732 0.35292
0.89005 0.04774 0.86126
```

No collision at the same time

```
$ ./rand3; ./rand3
0 0 1 1 1 0 1 0 1 1
0.80122 0.83184 0.33650 0.20144 0.35302 0.50180 0.76823
0.98229 0.02290 0.04000
0 1 0 1 1 1 0 0 0 0
0.49672 0.36860 0.77740 0.69564 0.14145 0.34864 0.20053
0.87646 0.32895 0.90694
```

In UNIX or Mac OS X system, `std::random_device` looks up `/dev/urandom` to draw random numbers
(You may try `$ xxd /dev/urandom | head`)

Pseudo-random numbers in python

- Randomly sampling a single value

```
import random
print(random.uniform(0,1))
print(random.randint(0,1))
```

```
0.963765991622
0
```

Pseudo-random numbers in python

- **Randomly sampling multiple values**

```
import numpy as np
print(np.random.random(10))
print(np.random.randint(0,2,10))
```

```
[ 0.34386851  0.92403653  0.76340613  0.98810838  0.60215624  0.77208185
  0.3582923   0.28345326  0.41182332  0.20186387]
[1 1 1 1 0 1 1 0 1 1]
```

Pseudo-random numbers in python

- Setting seeds

```
import random
random.seed(12345)
print(random.uniform(0,1))
print(random.randint(0,1))
random.seed(12345)
print(random.uniform(0,1))
print(random.randint(0,1))
```

```
0.416619872545
0
0.416619872545
0
```

```
import numpy as np
np.random.seed(12345)
print(np.random.random(10))
np.random.seed(12345)
print(np.random.random(10))
```

```
[ 0.92961609  0.31637555  0.18391881  0.20456028  0.56772503  0.5955447
  0.96451452  0.6531771   0.74890664  0.65356987]
[ 0.92961609  0.31637555  0.18391881  0.20456028  0.56772503  0.5955447
  0.96451452  0.6531771   0.74890664  0.65356987]
```

H

Pseudo-random numbers in python

- A cryptographically secure way

```
import os
print(int(os.urandom(1).encode('hex'),16))
print(int(os.urandom(4).encode('hex'),16))
```

```
44
2164963379
```

Pseudo-number generator in R

Random {base} R Documentation

Random Number Generation

Description

`.Random.seed` is an integer vector, containing the random number generator (RNG) **state** for random number generation in R. It can be saved and restored, but should not be altered by the user.

`RNGkind` is a more friendly interface to query or set the kind of RNG in use.

`RNGversion` can be used to set the random generators as they were in an earlier R version (for reproducibility).

`set.seed` is the recommended way to specify seeds.

Usage

```
.Random.seed <- c(rng.kind, n1, n2, \dots)

RNGkind(kind = NULL, normal.kind = NULL)
RNGversion(vstr)
set.seed(seed, kind = NULL, normal.kind = NULL)
```

Default is “Mersenne-Twister”

Arguments

`kind`

character or `NULL`. If `kind` is a character string, set R's RNG to the kind desired. Use `"default"` to return to the R default

Pseudo-random number generation in R

```
runif(10)
```

```
 [1] 0.7950476 0.3546202 0.1492300 0.8132865 0.6745914 0.8969490 0.8494247
 [8] 0.1997723 0.5724314 0.3892715
```

```
rbinom(10,1,0.5)
```

```
 [1] 1 1 0 1 0 1 1 0 0 1
```

Using seed in R

```
set.seed(12345)
runif(10)
```

```
 [1] 0.7209039 0.8757732 0.7609823 0.8861246 0.4564810 0.1663718 0.3250954
 [8] 0.5092243 0.7277053 0.9897369
```

```
set.seed(12345)
runif(10)
```

```
 [1] 0.7209039 0.8757732 0.7609823 0.8861246 0.4564810 0.1663718 0.3250954
 [8] 0.5092243 0.7277053 0.9897369
```

Pseudo-random numbers in Rcpp

```
#include <Rcpp.h>
#include <cstdio>
using namespace Rcpp;
// [[Rcpp::export]]
void foo() {
  // create a vector of random numbers and print
  NumericVector x = runif(10);
  for(int i=0; i < 10; ++i) printf("%.5lf ",x[i]);
  printf("\n");
  NumericVector y = rbinom(10,1,0.5);
  for(int i=0; i < 10; ++i) printf("%d ",(int)y[i]);
  printf("\n");
  // create a single random number and print
  printf("%.5lf\n", R::runif(0,1));
  printf("%d\n",(int)R::rbinom(1,0.5));
}
```

```
foo()
```

```
0.45373 0.32675 0.96542 0.70748 0.64454 0.38983 0.69854 0.54406 0.22647 0.48456
1 0 0 1 0 0 1 1 1 0
0.78219
0
```

```
set.seed(12345)
foo()
```

```
0.72090 0.87577 0.76098 0.88612 0.45648 0.16637 0.32510 0.50922 0.72771 0.98974
0 0 1 0 0 0 0 0 0 1
0.45373
0
```

```
set.seed(12345)
foo()
```

```
0.72090 0.87577 0.76098 0.88612 0.45648 0.16637 0.32510 0.50922 0.72771 0.98974
0 0 1 0 0 0 0 0 0 1
0.45373
0
```

Randomly sampling from a 'non-uniform' distribution

- Supposed that you only have a uniformly distributed random number generator `runif()`

- You need to implement your own `rnorm(mu, beta)`

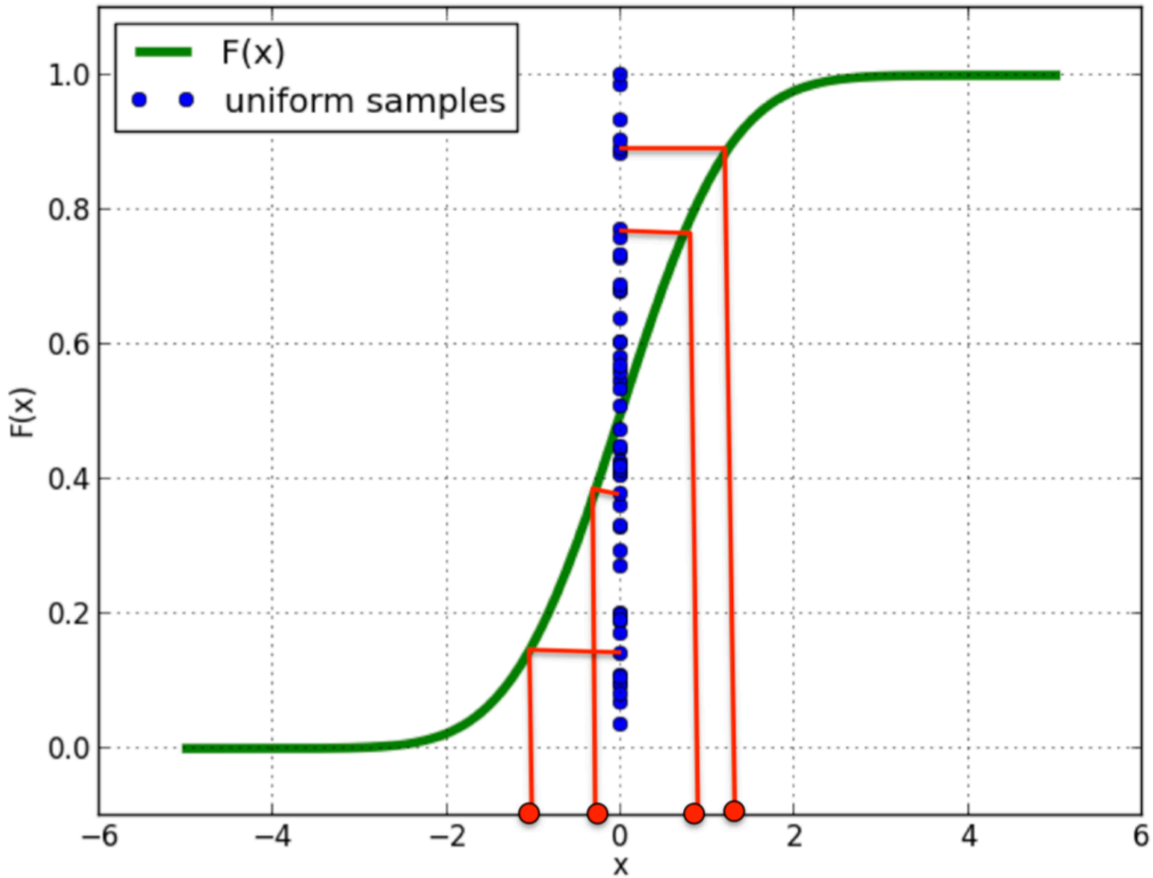
- How?

Inverse transform sampling

- **Assumption**
  - You have CDF ***F(·)*** and inverse CDF ***F⁻¹(·)*** of a known distribution.
- **Method**
  1. Sample a random variable U from Uniform(0,1)
  2. Define $X = F^{-1}(U)$
  3. Then X follows the distribution that defines ***F(·)***
- **Example : exponential distribution**
  - Density: $f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$
  - CDF : $F(x) = 1 - \frac{1}{\beta} e^{-\frac{x}{\beta}}$
  - Transformation: $X = -\beta \log(1 - U)$

DIY : Try to prove why

Illustration of inverse transform sampling



Credit: Kenny Chowdhary

Sampling from a Gaussian distribution

- **Using inverse transform sampling**
  - Inverse CDF must be known.
  - But normal distribution does not have a close form.
  - Hard to achieve without specialized library to compute inverse CDF

- **Using central limit theorem**
  - In principle, averaging A LOT of uniformly random samples should provide a normally distributed variables
  - But this is slow and imprecise.

- **Using a Box-Muller transformation**

Box-Muller Transformation

1. Sample two random variables U_1, U_2 from *Uniform(0,1)*

2. Define to following values
   - $R = \sqrt{-2\log U_1}$
   - $\Theta = 2\pi U_2$
   - $Z_0 = R\cos\Theta$
   - $Z_1 = R\sin\Theta$

3. Then Z_0, Z_1 are i.i.d. normally distributed random variables following *N(0,1)*

DIY : Can you prove it?

Sampling from a complex distribution

- What if CDF is not easy to obtain?

- How can we simulate from a mixture or normal?

$$f(x;\mu_1,\sigma_1^2,\mu_2,\sigma_2^2,\alpha) = \alpha f_{\mathcal{N}}(x;\mu_1,\sigma_1^2) + (1-\alpha) f_{\mathcal{N}}(x;\mu_2,\sigma_2^2)$$

Sampling from a complex distribution

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- How can we simulate from a mixture or normal?

$$f(x;\mu_1,\sigma_1^2,\mu_2,\sigma_2^2,\alpha) = \alpha f_{\mathcal{N}}(x;\mu_1,\sigma_1^2) + (1-\alpha) f_{\mathcal{N}}(x;\mu_2,\sigma_2^2)$$

```
w <- rbinom(1,1,alpha)
y <- rnorm(1,mu1,sigma1)
z <- rnorm(1,mu2,sigma2)
x <- w*y + (1-w)*z
```

An example:
Not necessarily the best

Sampling from a bivariate normal

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$$

- **Is this a correct implementation?**

```
x <- rnorm(1,mu.x,sigma.x)
y <- rnorm(1,mu.y,sigma.y)
```

Use conditional distribution unless independent

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{matrix} \mu_x \\ \mu_y \end{matrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$$

$$y|x \sim \mathcal{N}\left(\mu_y + \frac{\sigma_{xy}}{\sigma_x^2}(x - \mu_x), \sigma_y^2 \left(1 - \frac{\sigma_{xy}^2}{\sigma_x^2 \sigma_y^2} \right) \right)$$

```
x <- rnorm(1, mu.x, sigma.x)
y <- rnorm(1, mu.y + sigma.xy/sigma.x^2*(x-mu.x),
           sigma.y^2 - sigma.xy^2/sigma.x^2)
```

Sampling from a multivariate normal distribution

- **Problem setting**
  - Given mean vector and positive-definite covariance matrix $\boldsymbol{\mu}, V$
  - The goal is to randomly sample from $N(\boldsymbol{\mu}, V)$

- **One possible solution is..**
  1. Sample $\boldsymbol{x_1}$ from the marginal distribution $N(\mu_1, V_{11})$.
  2. Sample $\boldsymbol{x_2}$ from the conditional distribution $N(\mu_2, V_{12}V_{22}^{-1}(x_1 - \mu_1), V_{22} - V_{21}V_{11}^{-1}V_{12})$.
  3. … Sample $\boldsymbol{x_i}$ from the subsequent conditional distribution

- **Is this computationally efficient? Why or why not?**

Leveraging matrix decomposition

- **Cholesky decomposition**
  - For a positive-definite matrix V, there exists a upper-triangular matrix such at
$$V = U^T U$$
- **Let $z \sim N(0, I_n)$ a *i.i.d.* random variables, then**
$$\boldsymbol{x} = U^T \boldsymbol{z} + \boldsymbol{m} \sim N(\boldsymbol{m}, U^T U) = N(\boldsymbol{m}, V)$$
- **An example R code**

```
z <- rnorm(length(m))
U <- chol(V)
x <- m + t(U) %*% z
```

What's the time complexity of this method, compared to the previous one?

Algorithm to generate a random permutation

- Suppose that you have an array with n elements.

- The goal is to randomly shuffle the elements with equal probability for every possible permutation.

- How can we achieve this, assuming that you have a good uniform pseudo-random number generator between 0 and 1?

Summary : Random numbers

- True vs. Pseudo- random numbers
- Different implementations of pseudo-random number generators across different languages
- Sampling a random number from a complex distribution
- Sampling correlating random numbers.