Numerical Solution of Partial Differential Equations I
Homework #1

Due on March 31, 2022 at 10:00pm

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Problem 1

If a function $f(x)$ extremizes the functional

$$J(g) = \int_a^b L\left(x, g(x), g'(x)\right) dx$$

over all functions $g \in C^2[a,b]$ with $g(a) = g(b) = 0$, prove that $f(x)$ satisfies the differential equation

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$$

where L is assumed to be continuously differentiable.

Problem 2

Consider the energy functional

$$J(v) = \int_D \frac{1}{p}|\nabla v|^p - fv\,dx \quad (1 \le p < \infty)$$

defined over all functions $v \in W^{1,p}(D)$, prove that its minimizer u satisfies the p-Laplace equation

$$-\Delta_p u(x) = f(x) \quad \text{in } D,$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$. (hint: use Euler-Lagrange equation for a function of several variables)

Problem 3

Determine the Green's function for the two-point boundary value problem

$$\begin{gathered} -u''(x) = f(x) \quad \text{in } (0,1), \\ u(0) = u'(1) = 0, \end{gathered}$$

and then employ it to derive the exact solution for the right-hand-side $f(x) = x^2$.

Problem 4

For approximating the 2nd-order derivative of $u(x)$ on an equidistant mesh, calculate the truncation error of the finite difference scheme

$$\frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2}.$$

Problem 5

Consider the Poisson equation on a unit square

$$\begin{aligned} -\Delta u(x,y) &= 4\pi^2 \sin(2\pi x)(2\cos(2\pi y) - 1) && \text{in } D = (0,1)^2, \\ u(x,y) &= 0 && \text{on } \partial D, \end{aligned}$$

where the exact solution is given by $u(x,y) = \sin(2\pi x)(\cos(2\pi y) - 1)$, write a MATLAB program that uses the 5-point central difference scheme on an $n \times n$ uniform mesh, and verify that second order accuracy can be achieved by performing a grid refinement study.

Problem 6

Consider the following singularly perturbed differential equations with $\varepsilon = 10^{-2}$

$$
\begin{aligned}
-\varepsilon u''(x) + u'(x) &= 1 \quad \text{in } (0,1), \\
u(0) = u(1) &= 0,
\end{aligned}
$$

write a MATLAB program that employs the central difference, upwind and Il'in-Allen-Southwell difference schemes on an equidistant mesh.