IE 526: Stochastic Calculus for Finance

Project: Valuation of Auto-Callable Contingent Interest Notes

Shixue Feng, Yingshan He, Yuli Tang

Part 1 Introduction

In this project, we are going to develop and implement a finite difference method to value Auto Callable Contingent Interest Notes Linked to the Class B Shares of United Parcel Service which is issued by JP Morgan Chase. The estimated value of the notes, when the terms of the notes were set, was $935.50 per $1,000 principal amount note. Overall, our model is based on a similar fashion as a barrier option's valuation with underlying asset of the UPS stock, so we decide to use a Black Scholes PDE to evaluate the notes, and then impose boundary conditions on the PDE, finally solve the result by the Crank-Nicolson Method. To be specific, we set two possible situations in our model: triggered and not-triggered with different boundaries and evaluate the value of both. Then we set not-triggered as our base model, replace its low boundary with another one's low boundary's value in order to simulate a barrier condition.

As for why we choose Crank-Nicolson Method, we discover that Explicit Finite Difference Method exists many limitations in practice. For example, it is difficult to converge, unstable and the maximum stock steps in the finite difference grid is limited by the large volatility. At the same time, Crank-Nicolson Method is implicit in time and has less error in the time derivative since it is of $(\Delta t)^2$ rather than Δt. So compared with Explicit Finite Difference Method, Crank-Nicolson Method is more precise, more stable and more unconstraint.

Part 2 Model

2.1 Key Terms Introduction

Important Timings:

- Underlying Asset: The UPS Stock.
- Pricing Date (t_0): March 23, 2020. This is the day we seem as the Initial Date of evaluation.
- Original Issue Date: March 26, 2020 ($t_0 + 3$), which is the day paying the principal to the Issuer.

- Review Dates: June 23, 2020, September 23, 2020, December 23, 2020, March 23, 2021 and June 23, 2021.
  - ✧ Coupon Date (t_1): June 23, 2020
  - ✧ Call Dates: September 23, 2020 (t_2), 2020, December 23 (t_3), March 23, 2021 (t_4)
  - ✧ Final Review Date (T): June 23, 2021. This is the day we seem as the Maturity Date of evaluation.
- Interest Payment Dates: June 26, 2020 ($t_1 + 3$), September 28, 2020 ($t_2 + 5$), December 29, 2020 ($t_3 + 6$), March 26, 2021 ($t_4 + 3$) and June 28, 2021 (T + 5), which are the days you may gain the interest coupon payment.
- Monitoring Period: From but excluding March 23, 2020 (Pricing Date) to and including June 23, 2021 (Final Review Date).

Important Values:

- Principal: $1000
- Initial Value: $91.90, which is the closing price of one share of the Reference Stock on the Pricing Date.
- Contingent Interest Payments (coupon): $32.50, which is 32.5% of principal. You will receive it whenever the closing price on any Review date is greater than the Interest Barrier.
- Automatic Call: Whenever the closing price on any Call Date is greater than the Initial Value, you will receive the principal and a coupon payment at the next nearest Interest Payment Date, and no more payment you will receive in the future.
- Interest Barrier/Trigger Value: $64.33, which is 70% of the Initial Value.
- Trigger Event: If any day during the Monitoring Period has the closing price less than the Trigger Value, then we assume the note is triggered.

2.2 Mechanism Analysis

Since we set two different situations in our model which are triggered and not-triggered, this part will analyze the payment at each timing under the two situations. We will set the closing price of one share stock as S_0 (at t_0), …, S4 (at t_4), S_T (at T).

1. ***Triggered***

   (a) At Coupon Date (t_1)

   If $S_1 \geq 70\% * S_0$, a payment of coupon ($32.50) will be paid at $t_1 + 3$, so the current value of it will be $\$\mathbf{32.50} * e^{-r*\frac{3}{365}}$, then move on;

   If $S_1 < 70\% * S_0$, there will be **no payment** this time and move on.

(b) At Call Dates (t_2, t_3, t_4)

If S_2 / S_3 / S_4 ≥ S_0, the notes will be automatically called for a cash payment of principal ($1000) + coupon ($32.50) at $t_2 + 5$ / $t_3 + 6$ / $t_4 + 3$, so the current value of it will be $(\$1000 + \$32.50) * e^{-r*\frac{5/6/3}{365}}$, then the note will finish and no more payments will be paid in the future;

If $S_0 > S_2$ / S_3 / S_4 ≥ 70%*S_0, a payment of coupon ($32.50) will be paid at $t_2 + 5$ / $t_3 + 6$ / $t_4 + 3$, so the current value of it will be $\$32.50 * e^{-r*\frac{5/6/3}{365}}$, then move on;

If S_2 / S_3 / S_4 < 70%*S_0, there will be **no payment** this time and move on.

(c) At Final Review Date (T)

If S_T ≥ S_0, a payment of principal ($1000) + coupon ($32.50) will be paid at T + 5, so the current value of it will be $(\$1000 + \$32.50) * e^{-r*\frac{5}{365}}$**,** then the note will finish;

If $S_0 > S_T$ ≥ 70%*S_0, a payment of principal & stock return ($1,000*$S_T$ / S_0) + coupon ($32.50) will be paid at T + 5, so the current value of it will be $(\$1,000 * \frac{S_T}{S_0} + \$32.50) * e^{-r*\frac{5}{365}}$, then the note will finish;

If S_T < 70%*S_0, there will be only a payment of principal & stock return ($1,000*$S_T$ / S_0) at T + 5, and the current value of it will be $(\$1,000 * \frac{S_T}{S_0}) * e^{-r*\frac{5}{365}}$, then the note will finish.

2. ***Not Triggered*** *(S_t ⩾ 70%*S_0 always happen)*

(a) At Coupon Date (t_1)

A payment of coupon ($32.50) will be paid at $t_1 + 3$, and the current value of it will be $\$32.50 * e^{-r*\frac{3}{365}}$, then move on.

(b) At Call Dates (t_2, t_3, t_4)

If S_2 / S_3 / S_4 ≥ S_0, the notes will be automatically called for a cash payment of principal ($1000) + coupon ($32.50) at $t_2 + 5$ / $t_3 + 6$ / $t_4 + 3$, so the current value of it will be $(\$\mathbf{1000} + \$\mathbf{32.50}) * \boldsymbol{e}^{-\boldsymbol{r}*\frac{5/6/3}{365}}$, then the note will finish and no more payments will be paid in the future;

If S_2 / S_3 / S_4 < S_0, a payment of coupon ($32.50) will be paid at at $t_2 + 5$ / $t_3 + 6$ / $t_4 + 3$, so the current value of it will be $\$\mathbf{32.50} * \boldsymbol{e}^{-\boldsymbol{r}*\frac{5/6/3}{365}}$, then move on.

(c) At Final Review Date (T)

A payment of principal ($1000) + coupon ($32.50) will be paid at T + 5, so the current value of it will be $(\$\mathbf{1000} + \$\mathbf{32.50}) * \boldsymbol{e}^{-\boldsymbol{r}*\frac{5}{365}}$, then the note will finish;

2.3 Parameter Selection

Now let us introduce details in parameters. Since we choose a model of option, what we primarily need are the interest rate r, the dividend rate δ and the implied volatility σ. All of them are chosen or computed from the information in Bloomberg.

- **The Risk Free Rate $\boldsymbol{r}$**:

2021/6/16	0.991958467	FUTURE
2021/6/23	INTerpolate	-->
2021/9/15	0.990731684	FUTURE

We choose to use future compounded interest rate as our risk free rate. First we use the interpolation method to compute the future discount rate ρ on March 23, 2020. The two nearest dates are June 16, 2021 and Sept 15, 2021, and each future discount rate is 0.991958467 and 0.990731684 as the picture above. So the interpolation future discount rate is:

$$\rho = 0.991958467 + (0.990731684 - 0.991958467) * \frac{7}{91} = 0.9918641$$

Then through the formula (From March 23, 2020 to June 23, 2021 is 457 days):

$$\rho(0, T) = e^{-r(0,T)*T}, \qquad T = \frac{457}{365}$$

Finally we can get a $r = 0.65246\%$.

- **The Dividend Yield $\boldsymbol{\delta}$:**

Dividend Yield	4.7901

We directly choose the fixed dividend yield as 4.7901% from the excel since the change in the dividend rate will not have huge impact in the valuation.

- **The Implied Volatility σ:**

Since our Monitoring Period is about 15 months, the Strike price will be chosen as 100 or 70, so our implied volatility is between 45.116% and 55.591% from the form above. Since different implied volatility will influence the valuation importantly, then we will try several values in this range.

2.4 Model Building

Grid Setup:

We set the minimum stock price as $S_L=0$, the maximum stock price is $S_U=3*S_0$, the time period is $T=457/365$, the maximum S-step is $j_{max}=300$, the maximum t-step is $i_{max}=457$, so $\Delta S=0.01*S_0$, $\Delta t=1/365$.

Boundaries:

As we mentioned before, we set two situations and set each with a grid to make a valuation. Therefore, they have different boundaries as follows:

(a) Triggered:

- Up Boundary Condition (not include the coupon at Review Dates except for the final review date):

$$V_{j_{max}}^{i} = (\$1000 + \$32.50) * e^{-r*(T_{NAC} - i\Delta t)}$$

T_{NAC}: Sept 28, Dec 29, Mar 26, June 28 t_1: June 26, 2020

- Low Boundary Condition:

$$V_{j_{min}}^{i} = 0$$

Since when triggered S→0, stock return = -1

- Terminal Boundary Condition:

$$V_{T_j}^{i_{max}} = \begin{cases} (\$1000 + \$32.5) * e^{-r*\frac{5}{365}}, j \geq 100 \\ \left(\$1{,}000 * \frac{j\Delta S}{S0} + \$32.50\right) * e^{-r*\frac{5}{365}}, 70 \leq j < 100 \\ \left(\$1{,}000 * \frac{j\Delta S}{S0}\right) * e^{-r*\frac{5}{365}}, j < 70 \end{cases}$$

(b) Not Triggered:

- Up Boundary Condition (not include the coupon at Review Dates except for the final review date):

$$V_{j_{max}}^{i} = (\$1000 + \$32.50) * e^{-r*(T_{NAC} - i\Delta t)}$$

T_{NAC}: Sept 28, Dec 29, Mar 26, June 28 t_1: June 26, 2020

- Low Boundary Condition:

$$V_{j_{min}}^{i} = triggered\ value\ in\ the\ same\ position$$

- Terminal Boundary Condition:

$$V_{T_j}^{i_{max}} = (\$1000 + \$32.5) * e^{-r*\frac{5}{365}}$$

Crank-Nicolson Method with LU Decomposition:

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1}}{\Delta t}$$

$$\frac{\partial V}{\partial S} = \frac{1}{4\Delta S}(V_{j+1}^{i} - V_{j-1}^{i} + V_{j+1}^{i+1} - V_{j-1}^{i+1})$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{2\Delta S^2}(V_{j+1}^{i} - 2V_j^{i} + V_{j-1}^{i} + V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1})$$

Recall the BSM equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-d)S\frac{\partial V}{\partial S} - rV = 0$$

Because we have already known V^{i+1}, we can rearrange the equations

$$a_j V_{j-1}^i + b_j V_j^i + c_j V_{j+1}^i = d_j^i$$

where:

$$a_j = \frac{1}{4}(\sigma^2 j^2 - (r-d)j)$$
$$b_j = -\frac{\sigma^2 j^2}{2} - \frac{r}{2} - \frac{1}{\Delta t}$$
$$c_j = \frac{1}{4}(\sigma^2 j^2 + (r-d)j)$$
$$d_j^i = -\frac{1}{4}(\sigma^2 j^2 - (r-d)j)V_{j+1}^{i+1} - \left(-\frac{\sigma^2 j^2}{2} - \frac{r}{2} - \frac{1}{\Delta t}\right)V_j^{i+1} - \frac{1}{4}(\sigma^2 j^2 + (r-d)j)V_{j+1}^{i+1}$$

We can find the V_j^i values by solving them. To accelerate the computation, we can simplify the matrix inversion process by using LU Decomposition. It splits the original matrix into two matrixes 'L' & 'U', but in our practice we only consider for two extra variables derived from a_j, b_j and c_j. We set these two variables as $alpha$ and q.

$$alpha_j = b_j - \frac{a_j * c_{j-1}}{alpha_{j-1}}, \qquad alpha_1 = b_1$$
$$q_j{}^i = b_j{}^i - \frac{a_j}{alpha_{j-1}} * q_{j-1}{}^i, \qquad q_1{}^i = b_1{}^i$$

Finally we can easily compute the values by $V_j^i = \begin{cases} \frac{q_{j_{max}}{}^i}{d_{j_{max}}} \\ \frac{q_j{}^i - c_j * V_{j+1}^i}{d_j}, \quad j = 1, \ldots, j_{max} - 1 \end{cases}$.

Some Important Changes in the Valuation:

1. Add Coupon:

When the stock price is above $70\% * S_0$ at t_2, t_3 or t_4 ($j \geqslant 70$, $i = 92, 184, 275, 365$), we will add the original value with $\$32.50 * e^{-r*\frac{3/5/6/3}{365}}$ respectively, and then move on the computation to the previous Δt.

2. Early Exercise:

When the notes should be auto-called at t_1, t_2, t_3 or t_4 ($j \geqslant 100$, $i = 184, 275, 365$), we will directly replace those values with $(\$1000 + \$32.50) * e^{-r*\frac{5/6/3}{365}}$ respectively, and then move on the computation to the previous Δt.

Part 3

3.1 Result

Our default implied volatility in the model is 55.591%, and the result is $859.9359934075098, which seems like a number far way from the estimation $935.50 by JP Morgan. So we decide to do more sensitivity analysis with the implied volatility and grid steps setting.

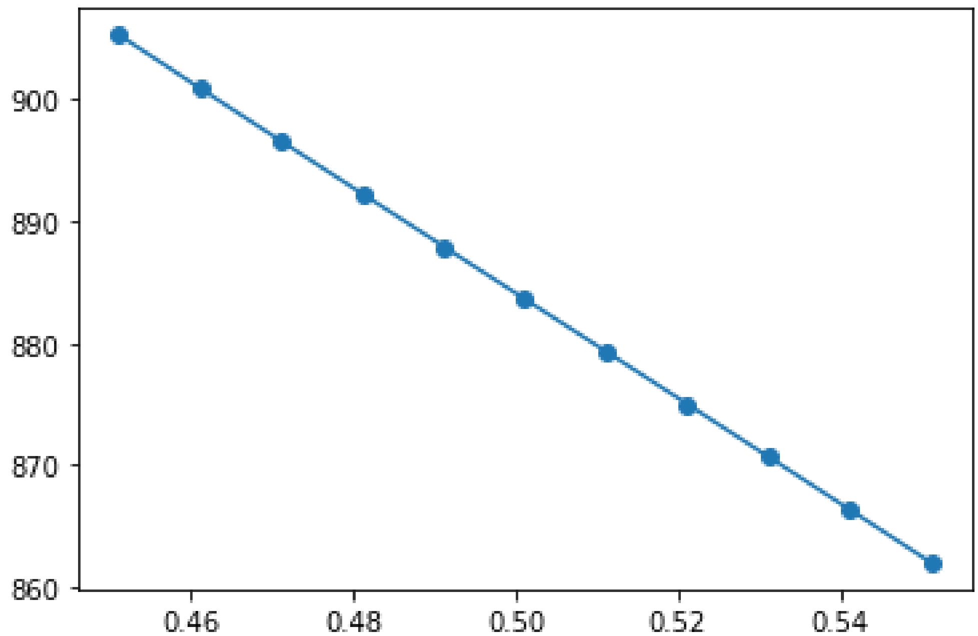
3.2 Sensitivity Analysis

- **Volatility Changes:**

  We first test different volatilities in the range (45.116%, 55.591%), the results are in a form and showed as a figure (i_{max}=457, j_{max}=300):

Implied Volatility σ	Valuation ($)
45.116%	905.2895389236793
46.116%	900.920291461191
47.116%	896.5755989314956
48.116%	892.2491027219
49.116%	887.9347700338178
50.116%	883.6268989948496
51.116%	879.3201246786533
51.116%	875.0094250857505
53.116%	870.690126286062
54.116%	866.3579060964573
55.116%	862.0087958456247
55.591%	859.9359934075096

Valuation of different implied volatilities σ



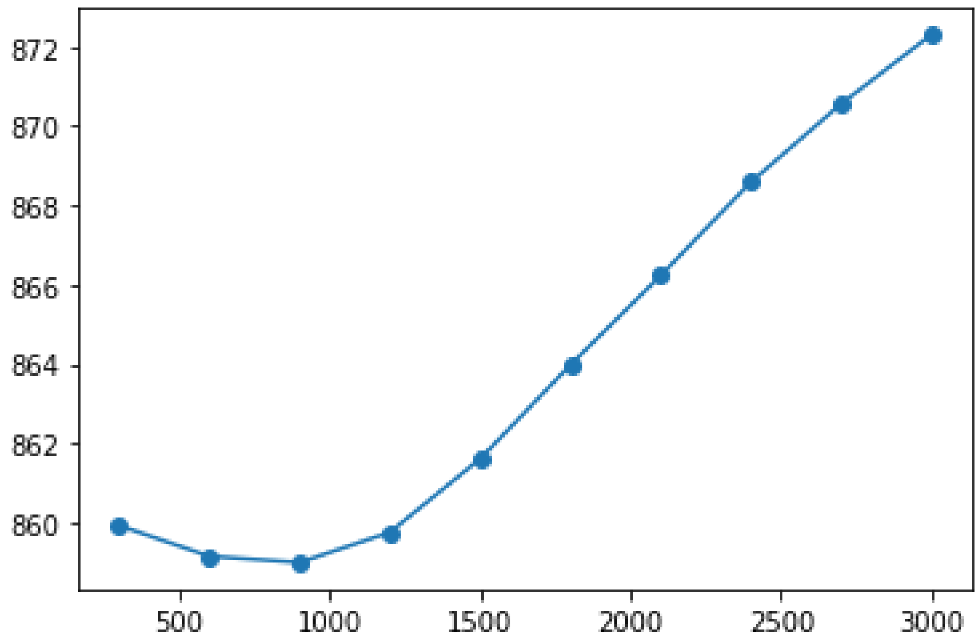
Based on the result, we can simply know that volatility importantly impact the valuation result. More specifically, when the volatility increases, the value will decrease.

- **S-step Number Changes:**

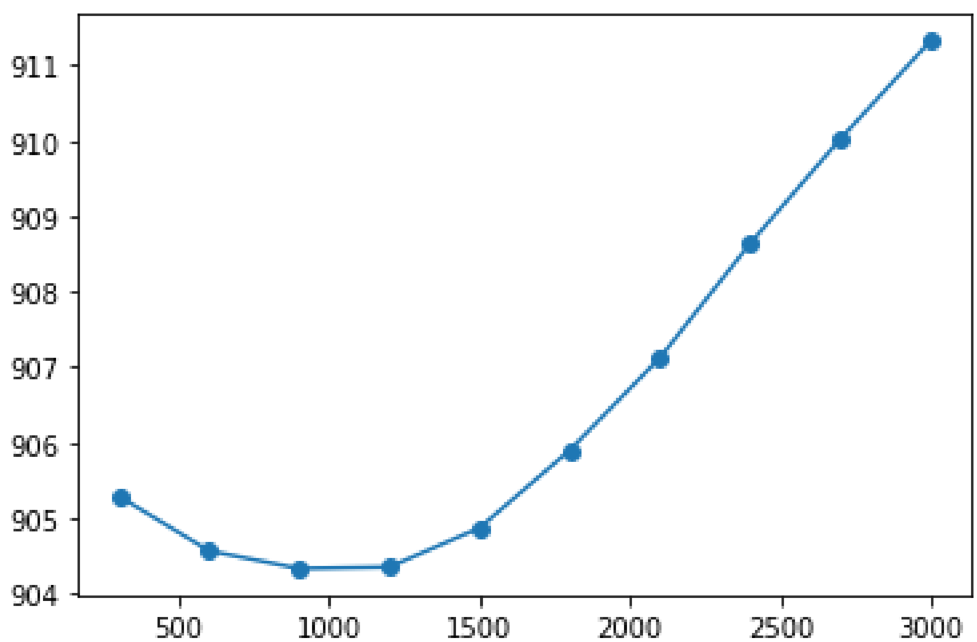
Then we test different step numbers of S in the range (300, 3000), the results are in a form and showed as a figure (i_{max}=457):

j_{max}	Valuation ($) ($\sigma$=55.591%)	Valuation ($) ($\sigma$=45.116%)
300	859.9359934075096	905.2895389236793
600	859.1625479855596	904.5653736061456
900	859.0063986256266	904.3301058599694
1200	859.7669783783052	904.3512668649419
1500	861.61962592848	904.8704659813073
1800	863.9818217019579	905.9011672450903
2100	866.2350181549999	907.1272349108801
2400	868.5860315917766	908.641032049663
2700	870.5595709215969	910.0271597661228
3000	872.2985781829526	911.3276155282804

Valuation of different S-step Numbers (σ=55.591%)



Valuation of different S-step Numbers (σ=45.116%)



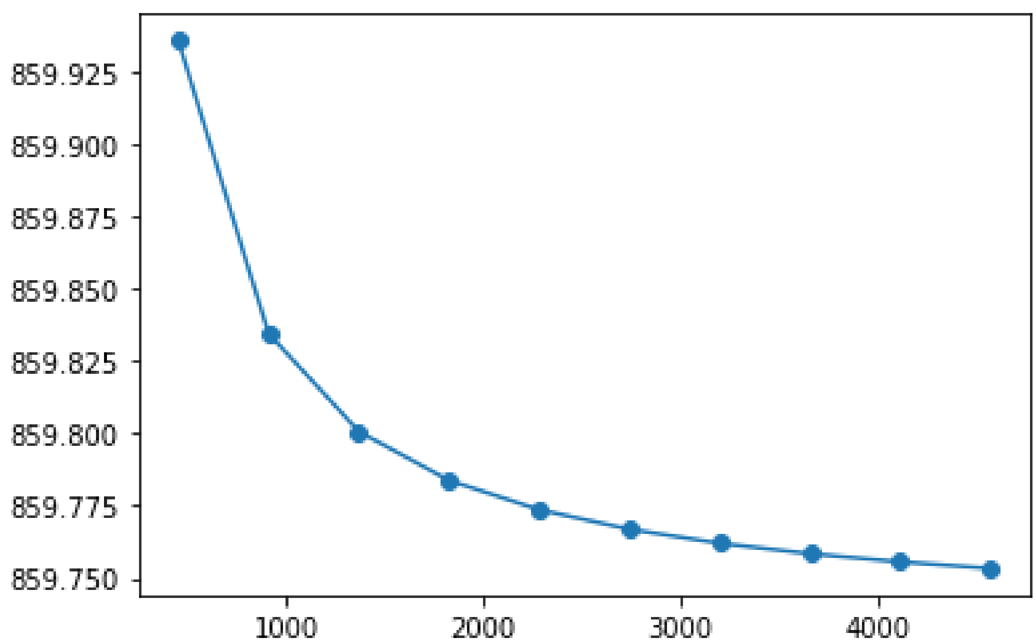
Based on the result, we can simply know that increasing S-step Number importantly impact the valuation result. More specifically, when the number increases, the value will decrease a little bit and then increase a lot.

- **T-step Number Changes:**

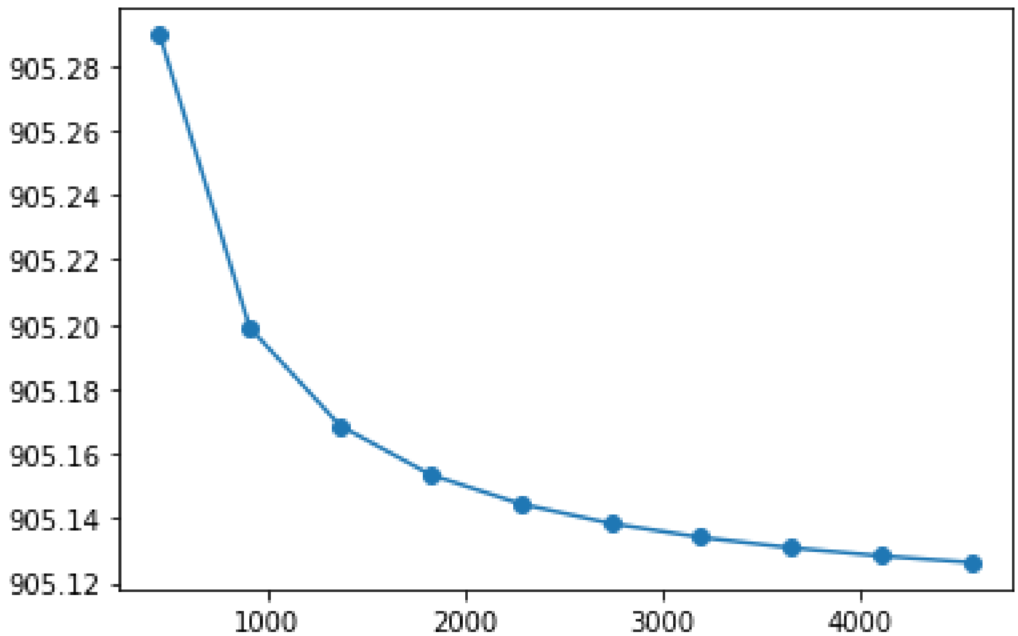
Finally we test different step numbers of T in the range (457, 4570), the results are in a form and showed as a figure (j_{max}=300):

i_{max}	Valuation ($) ($\sigma$=55.591%)	Valuation ($) ($\sigma$=45.116%)
457	859.9359934075096	905.2895389236793
914	859.8345824145885	905.198914334468
1371	859.8007425736462	905.1686889222153
1828	859.7838158349932	905.1535729479509
2285	859.7736576054682	905.1445023124616
2742	859.7668845404291	905.1384547830312
3199	859.7620461898157	905.1341349036956
3656	859.7584171822078	905.1308948762592
4113	859.7555944757705	905.128374784946
4570	859.7533362192243	905.1263586678199

Valuation of different T-step Numbers (σ=55.591%)



Valuation of different T-step Numbers (σ=45.116%)



Based on the result, we can simply know that increasing T-step Number don't impact the valuation result very much, especially compared with the change brought by S-step number's increase. When the number increases, the value will decrease a little bit.

3.3 Conclusion

From the results of our tests, we can know that increasing the implied volatility will decrease the value of the notes. This is easy to understand, since when the volatility is high, the stock price will more possibly touch the barrier and lose money at maturity. We also discover that when the T-steps increase in our grid, the value will decrease and converge to a value; but there is no converge when increasing the S-steps, and the value changes importantly when the S-step changes. Therefore, the Crank-Nicolson Method is not as stable as we assume, and since our value result is still far away from the real estimation from JP Morgan, there must be a room for improvement.