4a) <u>VOR</u>: $a, b \in \mathbb{N} \setminus \{1\}$, $p \in \mathbb{P}$, $n_p, m_p \in \mathbb{N}$

<u>BEH</u>: $\text{lcm}(a,b) = a \cdot b / \gcd(a,b)$

<u>BEW</u>: <u>z</u>: (1) $\gcd(a,b) = \prod p^{\min(n_p, m_p)}$

(2) $\text{lcm}(a,b) = \prod p^{\max(m_p, m_p)}$

(3) Behauptung

Sei $a \in \mathbb{N}$. Die Menge $V(a) = \{n \in \mathbb{N} \mid a \mid n\}$

Die Menge $T(a) = \{n \in \mathbb{N} \mid n \mid a\}$

(1) z: (1.1) $g = \prod p^{\min(m_p, n_p)} \in T(a) \cap T(b)$

(1.2) g ist $\gcd(a,b)$

(1.1) $\min(m_p, n_p) \leq m_p \Rightarrow g \mid a$ Somit ist $g \in T(a) \cap T(b)$
$\min(m_p, n_p \leq n_p \Rightarrow g \mid b$

(1.2) Sei $t \in T(a) \cap T(b)$ bel. mit $t = \prod p^{t_p}$

$t \mid a \Rightarrow t_p \leq m_p$
$t \mid b \Rightarrow t_p \leq n_p$ $\Rightarrow t_p \leq \min(m_p, n_p) \Rightarrow t \mid g$

$\Rightarrow t \leq g \overset{\text{Vor}}{\Rightarrow} g = \gcd(a,b)$ #

(2) z: (1.1) $l = \prod p^{\max(m_p, n_p)} \in V(a) \cap V(b)$

$\max(m_p, n_p) \geq m_p \Rightarrow a \mid l$
$\max(m_p, n_p) \geq n_p \Rightarrow b \mid l$ $\Rightarrow l \in V(a) \cap V(b)$

(1.2) Sei $v \in V(a) \cap V(b)$ bel. mit $v = \prod p^{v_p}$

$a \mid v \Rightarrow m_p \leq v_p$
$b \mid v \Rightarrow n_p \leq v_p$ $\Rightarrow v_p \geq \max(m_p, n_p) \Rightarrow l \mid v \Rightarrow l \leq v$

$\Rightarrow l = \text{lcm}(a,b)$

#

(3) $\gcd(a,b) \cdot \text{lcm}(a,b)$

$= \prod p^{\min(m_p, n_p)} \cdot \prod p^{\max(m_p, n_p)}$

$= \prod p^{\min(m_p, n_p) + \max(m_p, n_p)}$

$= \prod p^{m_p + n_p}$

$= \prod p^{m_p} \cdot \prod p^{n_p}$

$= a \cdot b$

$\Rightarrow \text{lcm}(a,b) = a \cdot b / \gcd(a,b)$ $\square$