Topics in Macro 2

Week 6 - Second Part - TD 1 and TD 2

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TSE

Tuesday (17:00-18:30)

TD Second Part: Fiscal Multipliers (4 Weeks)

Part I

- Exercise I: Habit Persistence and The Keynesian Multiplier (Week 1)
- Exercise II: A Benchmark Model (Week 1)
- Exercise III: Consumption, Labor Supply and the Multiplier

Part II

- Exercise I: Taxes on the Labor Input and the Multiplier
- Exercise II: Public Spending in Utility Function and the Multiplier
- Exercise III: Labor Supply, Public Spending in Utility and the Multiplier

Part III

- Exercise I: Endogenous Public Spending
- Exercise II: Externality in Production and the Multiplier
- Exercise III: Externality in Labor Supply and the Multiplier

Objectives

1. What is a Fiscal multiplier?

   $\frac{dY}{dG}$ = Fiscal Mult.

2. What is the lag operator

   L such that $Lx_t = x_{t-1}$, $L^2 x_t = x_{t-2}$

3. What is log-linearization of a function

Exercise I: Habit Persistence and The Keynesian Multiplier

Model with habit persistence in consumption

Consumption:

Persistence P.

$$C_t = \lambda C_{t-1} + c(1-\lambda)(Y_t - T_t)$$

Aggregate resources:

$$Y_t = C_t + G_t$$

Lag operator:

$$LC_t = C_{t-1}$$

$$\frac{dY}{dG} = ?$$

Question 1. Assume first that $T_t = 0$. Using the lag operator L, determine the equilibrium output as a function of G_t. More precisely, determine the function $B(L)$, such that equilibrium output is expressed as:

$$Y_t = B(L)G_t$$

$$C_t = \lambda C_{t-1} + c(1-\lambda)Y_t$$

$$C_t - \lambda C_{t-1} = c(1-\lambda)Y_t$$

$$(1-\lambda L)C_t = c(1-\lambda)Y_t$$

$$C_t = c(1-\lambda)\underbrace{(1-\lambda L)^{-1}}_{\text{Series of operations}}Y_t$$

$$C_t + G_t = Y_t$$

$$c(1-\lambda)(1-\lambda L)^{-1}Y_t + G_t = Y_t$$

$$\left[1-c(1-\lambda)(1-\lambda L)^{-1}\right]Y_t = G_t$$

$$Y_t = \underbrace{\left[1-c(1-\lambda)(1-\lambda L)^{-1}\right]^{-1}}_{\text{Fiscal Mult.}}G_t$$

Answer: $Y_t = \frac{1-\lambda L}{1-c(1-\lambda)-\lambda L}G_t$

Question 2. Compute the short run multiplier. This multiplier is obtained by imposing $L = 0$ in the representation $Y_t = B(L)G_t$.

$L = 0$,

$$C_t = \lambda C_{t-1} + c(1-\lambda)Y_t$$
$$= \cancel{\lambda L C_t} + c(1-\lambda)Y_t$$
$$= c(1-\lambda)Y_t$$

$$Y_t = \underbrace{\boxed{\left[1 - c(1-\lambda)(1-\lambda L)^{-1}\right]^{-1}}} G_t = \underbrace{B(L)} G_t$$

$$B(0) = \left[1 - c(1-\lambda)(1-\cancel{\lambda(0)})^{-1}\right]^{-1} = \frac{1}{1 - c(1-\lambda)}$$

Answer: $B(0) = \frac{1}{1-c(1-\lambda)}$. Maximum for $\lambda = 0$.

Question 3. Compute the long run multiplier. This multiplier is obtained by imposing $L = 1$ in the representation $Y_t = B(L)G_t$.

$$C_t = C_{t-1} = L C_t$$

$$C_t = L C_t \iff L = 1$$

$$B(L) = \left[1 - c(1-\lambda)(1-\lambda L)^{-1}\right]^{-1}$$

$$B(1) = \left[1 - c\cancel{(1-\lambda)}\cancel{(1-\lambda)^{-1}}\right]^{-1} = \frac{1}{1-c}$$

Answer: $B(1) = \frac{1-\lambda}{1-c(1-\lambda)-\lambda} = \frac{1}{1-c}$

Question 4. Discuss the two obtained government spending multipliers.

$$B(0) = B(1) \text{ if } \lambda = 0 \text{ and}$$
$$B(0) < B(1) \text{ if } \lambda > 0$$

$$C_t = \lambda C_{t-1} + c(1-\lambda)Y_t$$

If $\lambda = 0 \Rightarrow C_t = cY_t$

Question 5. Assume now that $G_t = T_t$. Using the lag operator L, determine the equilibrium output as a function of G_t. More precisely, determine the function $B(L)$, such that equilibrium output is expressed as $Y_t = B(L)G_t$. Compute the short run and long-run multipliers. Discuss.

$$C_t = \lambda C_{t-1} + c(1-\lambda)(Y_t - G_t)$$

$$C_t = (1-\lambda L)^{-1} c(1-\lambda)(Y_t - G_t)$$

$$C_t + G_t = Y_t$$

$$(1-\lambda L)^{-1} c(1-\lambda)(Y_t - G_t) + G_t = Y_t$$

$$(1-\lambda L)^{-1} c(1-\lambda) Y_t - (1-\lambda L)^{-1} c(1-\lambda) G_t + G_t = Y_t$$

$$\cancel{[1-(1-\lambda L)^{-1} c(1-\lambda)]} G_t = \cancel{[1-(1-\lambda L)^{-1} c(1-\lambda)]} Y_t$$

$$Y_t = G_t$$

$$\frac{dY_t}{dG_t} = 1$$

$$B(L) = 1$$

$$\forall L$$

Answer: $Y_t = G_t$. Fiscal multiplier $B(L) = 1 = B(0) = B(1)$.

Exercise II: A Benchmark Model

$$ax + by = c$$
$$dx + fy = g$$
$$x = \log(z)$$
$$y = \log(v)$$

$$\frac{dY}{dG} = ?$$

$$Y = f(G) \Rightarrow \frac{dY}{dG} = f'(G)$$

$$\begin{aligned} Y &= g(Y, G, x(Y)) \\ G &= h(z, w(Y)) \end{aligned} \Longrightarrow \frac{dY}{dG} = ?$$

Log-linearization in 3 steps

1. Take logs

$$y_t = F(x_t)$$

$$log(y_t) = log(F(x_t))$$

2. 1st order Taylor expansion around the steady state

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m = \frac{y - y_2}{x - x_2}, \quad y = f(x)$$

$$\overset{y}{log(y_t)} \approx \overset{y_2}{log(\bar{y})} + \overset{m}{\frac{1}{\bar{y}}} * \overset{x \;\; x_2}{(y_t - \bar{y})}$$

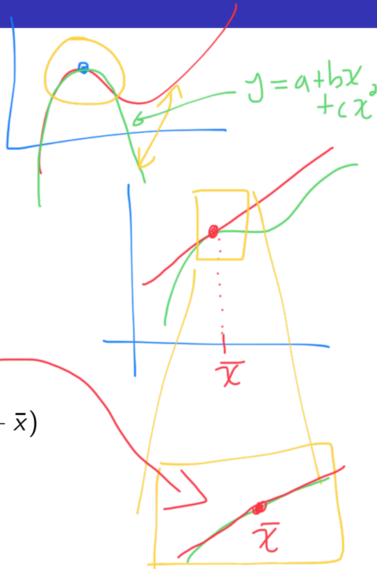
$$x \rightarrow \bar{x}$$

$$log(F(x_t)) \approx log(F(\bar{x})) + \frac{F'(\bar{x})}{F(\bar{x})} * (x_t - \bar{x})$$

3. Express everything in deviations

$$\Delta\% x = \frac{x_2 - x_1}{x_1}$$

$$\boxed{\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}}}$$



Example 1

$F(x) \approx F(\bar{x}) + F'(\bar{x})(x-\bar{x})$

Production function $y_t = Ak_t^{\alpha} n_t^{1-\alpha}$.

1) $\log(y_t) = \log(A) + \alpha \log(k_t) + (1-\alpha)\log(n_t)$

$\log(\bar{y}) = \log(A) + \alpha \log(\bar{k}) + (1-\alpha)\log(\bar{n})$

2) F.O.T.A. $\log(y_t) \approx \log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y})$

$$\cancel{\log(\bar{y})} + \underbrace{\frac{1}{\bar{y}}(y_t - \bar{y})}_{\hat{y}_t} = \cancel{\log(A)} + \alpha\left[\cancel{\log(\bar{k})} + \frac{1}{\bar{k}}(k_t - \bar{k})\right] + (1-\alpha)\left[\cancel{\log(\bar{n})} + \frac{1}{\bar{n}}(n_t - \bar{n})\right]$$

3) Dev: $\boxed{\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{n}_t}$

Example 2

$F(x,y) \simeq F(\bar{x},\bar{y}) + F_x(\bar{x},\bar{y})(x-\bar{x}) + F_y(\bar{x},\bar{y})(y-\bar{y})$

Resource constraint: $y_t = c_t + g_t$.

1) Logs $\log(y_t) = \log(c_t + g_t) \Rightarrow \log(\bar{y}) = \log(\bar{c} + \bar{g})$

2) 1st Taylor $\cancel{\log(\bar{y})} + \frac{1}{\bar{y}}(y_t - \bar{y}) = \cancel{\log(\bar{c}+\bar{g})} + \frac{1}{\bar{c}+\bar{g}}\boxed{(c_t - \bar{c}) \cdot \frac{\bar{c}}{\bar{c}}}$

$+ \frac{1}{\bar{c}+\bar{g}}(g_t - \bar{g}) \cdot \frac{\bar{g}}{\bar{g}}$

3) Deviat.

$$\hat{y}_t = \frac{\bar{c}}{\bar{c}+\bar{g}}\hat{c}_t + \frac{\bar{g}}{\bar{c}+\bar{g}}\hat{g}_t$$

Benchmark model

Representative household:

$$U_t = \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu}$$

dL

Budget constraint:

$$c_t \leq w_t n_t - T_t + \pi_t$$

Technology:

$$y_t = a n_t^{\alpha}$$

$$\frac{dY_t}{dg_t}$$

Profits:

$$\pi_t = y_t - w_t n_t$$

Taxes:

$$T_t = g_t$$

Market clearing condition:

$$y_t = c_t + g_t$$

Question 1. Determine the optimality condition of the households and then deduce the Marginal Rate of Substitution (MRS).

Answer: $\frac{\eta n_t^{\nu}}{1/c_t} = w_t.$

Question 2. Determine the optimality condition of the firm.

Answer: $a\alpha n_t^{\alpha-1} = w_t$.

Question 3. Determine the equilibrium output.

Answer: Solution to $\eta \left(\frac{y_t}{a}\right)^{\frac{\nu+1}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$.

Question 4. Compute the log-linearization of equilibrium output around the steady-state.

Answer: $\left(\nu + 1 + \frac{\alpha s_g}{1-s_g}\right) \widehat{y}_t = \frac{\alpha s_g}{1-s_g} \widehat{g}_t.$

Question 5. Compute the output multiplier and discuss the value of this multiplier with respect to ν and α.

Answer: $\frac{dy_t}{dg_t} = \frac{\alpha s_g}{(1-s_g)(\nu+1)+\alpha s_g}$.

Question 6. Compute the consumption multiplier and discuss the value of this multiplier with respect to ν and α.

Answer: $\frac{d\hat{c}_t}{d\hat{g}_t}$.

Appendix

Taylor Approximation (Wikipedia)

Definition

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

where $n!$ denotes the factorial of n. In the more compact sigma notation, this can be written as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. (The derivative of order zero of f is defined to be f itself and $(x-a)^0$ and $0!$ are both defined to be 1.)

For example, for a function $f(x,y)$ that depends on two variables, x and y, the Taylor series to second order about the point (a, b) is

$$f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big)$$

Lag polynomial

$\boxed{y_t = \rho y_{t-1} + x_t}$ (AR(1) process) with $\rho \in (0,1)$

$$\Rightarrow y_t - \rho y_{t-1} = x_t \Rightarrow \boxed{(1-\rho L) y_t = x_t} \Rightarrow \boxed{y_t = (1-\rho L)^{-1} x_t}$$

But what is $(1-\rho L)^{-1}$?

$$y_t = \boxed{(1-\rho L)^{-1}} x_t$$

$$
\begin{aligned}
y_t &= \rho y_{t-1} + x_t \\
&= \rho[\rho y_{t-2} + x_{t-1}] + x_t \\
&= \rho^2 y_{t-2} + \rho x_{t-1} + x_t \\
&= \rho^2[\rho y_{t-3} + x_{t-2}] + \rho x_{t-1} + x_t \\
&= \rho^3 y_{t-3} + \rho^2 x_{t-2} + \rho x_{t-1} + x_t \\
&\ \ \vdots \\
&= \rho^k y_{t-k} + \rho^{k-1} x_{t-k-1} + \dots + x_t \\
&= \rho^k y_{t-k} + \sum_{\tau=0}^{k-1} \rho^\tau x_{t-\tau}
\end{aligned}
$$

If $k \to \infty$ $\rho^k \to 0$

$$\Rightarrow \boxed{y_t = \sum_{\tau=0}^{\infty} \rho^\tau x_{t-\tau}}$$

And $\sum_{\tau=0}^{\infty} \rho^\tau x_{t-\tau} = \sum_{\tau=0}^{\infty} \rho^\tau L^\tau x_t$

So we have:

$$\boxed{(1-\rho L)^{-1} = \sum_{\tau=0}^{\infty} \rho^\tau L^\tau}$$