Topics in Macro 2

Week 9 - Second Part - Part II - Exercise II

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TSE

Tuesday (17:00-18:30)

Toulouse School of Economics

TD Second Part: Fiscal Multipliers (Weeks 6 to 10)

Part I

- Exercise I: Habit Persistence and The Keynesian Multiplier (Week 6)
- Exercise II: A Benchmark Model (Week 7)
- Exercise III: Consumption, Labor Supply and the Multiplier (Week 7)

Part II

- Exercise I: Taxes on the Labor Input and the Multiplier (Week 8)
- Exercise II: Public Spending in Utility Function and the Multiplier (Week 8)
- Exercise III: Labor Supply, Public Spending in Utility and the Multiplier (Week 9)

Part III

- **Exercise I: Endogenous Public Spending (Week 9)**
- Exercise II: Externality in Production and the Multiplier (Week 10)
- Exercise III: Externality in Labor Supply and the Multiplier (Week 10)

Exercise I: Endogenous Public Spending

The Economy

Utility:

$$ln(c_t) - \frac{\eta}{1+\nu} n_t^{1+\nu}$$

Budget constraint:

$$c_t \leq w_t n_t - T_t + \Pi_t$$

Technology:

$$y_t = a n_t^{\alpha}$$

Profits:

$$\Pi_t = y_t - w_t n_t$$

Government:

$$T_t = g_t$$

Stabilizing government spending rule:

$$g_t = \left(\frac{y_t}{y_{t-1}}\right)^{-\varphi_g} exp(u_t)$$

Question 1. Determine the optimality condition of the households and then deduce the Marginal Rate of Substitution (MRS)

$$\max_{c_t, n_t} \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu} \text{ subject to } c_t \leq w_t n_t - T_t + \pi_t$$

Lagrangean:

$$L = \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu} - \lambda_t [c_t - w_t n_t + T_t - \pi_t]$$

F.O.C.

$$n_t : \ \eta n_t^{\nu} = \lambda_t w_t$$
$$c_t : \ 1/c_t = \lambda_t$$

Then:

$$\frac{\eta n_t^{\nu}}{1/c_t} = w_t$$

Question 2. Determine the optimality condition of the firm.

$$\max_{n_t} n_t^{\alpha} - w_t n_t$$

F.O.C.

$$\alpha n_t^{\alpha-1} = w_t$$

Question 3. Determine the equilibrium output.

Definition of a **competitive equilibrium**: An equilibrium are quantities $c_t, n_t^s, n_t^d, y_t, T_t, g_t$ and prices w_t such that:

1. c_t, n_t^s solve the consumer's problem.
$$\frac{\eta n_t^{\nu}}{1/c_t} = w_t \tag{1}$$
2. n_t^d solve the firm's problem.
$$\alpha n_t^{\alpha-1} = w_t \tag{2}$$
3. Government budget balance: $T_t = g_t$.
$$T_t = g_t \tag{3}$$
4. Goods and labor market clearing: $n_t^d = n_t^s$ and $y_t = c_t + g_t$.
$$n_t^d = n_t^s \tag{4}$$
$$y_t = an_t^{\alpha} = c_t + g_t \tag{5}$$

Recall: We want to compute $\frac{dy_t}{dg_t}$. So let's express everything in terms of y_t and g_t.

- Equilibrium conditions (4) and (5), and optimality (1) and (2) imply:

$$\frac{\eta n_t^{\nu}}{1/(c_t)} = \alpha n_t^{\alpha-1}$$
$$\frac{\eta n_t^{\nu}}{1/(y_t - g_t)} = \alpha n_t^{\alpha-1}$$
$$\frac{\eta n_t^{\nu} n_t}{1/(y_t - g_t)} = \alpha n_t^{\alpha-1} n_t$$

Then:

$$\frac{\eta n_t^{1+\nu}}{1/(y_t - g_t)} = \alpha y_t$$

- From the production function $y_t = a n_t^{\alpha}$:

$$n_t = \left(\frac{y_t}{a}\right)^{1/\alpha}$$

We know:

$$\frac{\eta n_t^{1+\nu}}{1/(y_t - g_t)} = \alpha y_t$$

And

$$n_t = \left(\frac{y_t}{a}\right)^{1/\alpha}$$

Combining both expressions:

$$\eta \left(\frac{y_t}{a}\right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

Question 4. Compute the log-linearization of equilibrium output around the deterministic steady-state.

Step 1: Take Logs.

$$\eta\left(\frac{y_t}{a}\right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

$$log(\eta) + \frac{1+\nu}{\alpha} log(y_t) - \frac{1+\nu}{\alpha} log(a) = log(\alpha) + log(y_t) - log(y_t - g_t)$$

Step 2: Taylor approximation.

$$log(\eta) + \frac{1+\nu}{\alpha}[log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y})] - \frac{1+\nu}{\alpha} log(a) =$$

$$log(\alpha) + log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) - [log(\bar{y} - \bar{g}) + \frac{1}{\bar{y}-\bar{g}}(y_t - \bar{y}) - \frac{1}{\bar{y}-\bar{g}}(g_t - \bar{g})]$$

In the steady state:

$$log(\eta) + \frac{1+\nu}{\alpha} log(\bar{y}) - \frac{1+\nu}{\alpha} log(a) = log(\alpha) + log(\bar{y}) - log(\bar{y} - \bar{g})$$

Then:

$$\frac{1+\nu}{\alpha}[\frac{1}{\bar{y}}(y_t - \bar{y})] = \frac{1}{\bar{y}}(y_t - \bar{y}) - [\frac{1}{\bar{y}-\bar{g}}(y_t - \bar{y}) - \frac{1}{\bar{y}-\bar{g}}(g_t - \bar{g})]$$

Step 3: Express everything in deviations from the steady state.

$$\frac{1+\nu}{\alpha}[\frac{1}{\bar{y}}(y_t - \bar{y})] = \frac{1}{\bar{y}}(y_t - \bar{y}) - [\frac{1}{\bar{y}-\bar{g}}(y_t - \bar{y}) - \frac{1}{\bar{y}-\bar{g}}(g_t - \bar{g})]$$

$$\frac{1+\nu}{\alpha}\hat{y}_t = \hat{y}_t - [\frac{1}{\bar{y}-\bar{g}}(y_t - \bar{y})\frac{\bar{y}}{\bar{y}} - \frac{1}{\bar{y}-\bar{g}}(g_t - \bar{g})\frac{\bar{g}}{\bar{g}}]$$

$$\frac{1+\nu}{\alpha}\hat{y}_t = \hat{y}_t - [\frac{\bar{y}}{\bar{y}-\bar{g}}\hat{y}_t - \frac{\bar{g}}{\bar{y}-\bar{g}}\hat{g}_t]$$

Rearranging:

$$\left(\frac{1+\nu}{\alpha} - 1 + \frac{\bar{y}}{\bar{y}-\bar{g}}\right)\hat{y}_t = \frac{\bar{g}}{\bar{y}-\bar{g}}\hat{g}_t$$

Define:

$$s_g = \frac{\bar{g}}{\bar{y}}$$

Then:

$$\frac{\bar{g}}{\bar{y} - \bar{g}} = \frac{\bar{g}}{\bar{y} - \bar{g}} * \frac{1/\bar{y}}{1/\bar{y}} = \frac{s_g}{1 - s_g}$$

And:

$$\frac{\bar{y}}{\bar{y} - \bar{g}} = \frac{\bar{y}}{\bar{y} - \bar{g}} * \frac{1/\bar{y}}{1/\bar{y}} = \frac{1}{1 - s_g}$$

Then:

$$\left(\frac{1 + \nu}{\alpha} + \frac{1}{1 - s_g} - 1 \right) \hat{y}_t = \frac{s_g}{1 - s_g} \hat{g}_t$$

Rearranging:

$$\left(\frac{1+\nu}{\alpha}+\frac{s_g}{1-s_g}\right)\hat{y}_t=\frac{s_g}{1-s_g}\hat{g}_t$$

Multiply both sides by α.

$$\left(1+\nu+\frac{\alpha s_g}{1-s_g}\right)\hat{y}_t=\frac{\alpha s_g}{1-s_g}\hat{g}_t$$

So we have that:

$$y_t=\frac{\alpha s_g}{(\nu+1)(1-s_g)+\alpha s_g}\hat{g}_t$$

Question 5. Discuss the dynamic effects on output and consumption of an increase in the discretionary part u_t of the government policy.

We know:

$$\hat{y}_t = \frac{\alpha s_g}{(\nu+1)(1-s_g)+\alpha s_g}\hat{g}_t \tag{1}$$

And:

$$g_t = \left(\frac{y_t}{y_{t-1}}\right)^{-\varphi_g} exp(u_t)$$

But we need:

$$\frac{dy_t}{du_t} =?$$

To do this:

- Log-linearize g_t
- Plug it in (1)
- Go from $\frac{d\hat{y}_t}{du_t}$ to $\frac{dy_t}{du_t}$.

Step 1. Take logs

$$g_t = \left(\frac{y_t}{y_{t-1}}\right)^{-\varphi_g} exp(u_t)$$
$$log(g_t) = -\varphi_g(log(y_t) - log(y_{t-1})) + u_t$$

Step 2. Taylor expansion

$$log(\bar{g}) + \frac{1}{\bar{g}}(g_t - \bar{g}) = -\varphi_g \left(log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) - log(\bar{y}) - \frac{1}{\bar{y}}(y_{t-1} - \bar{y})\right) + u_t$$

Step 3. Deviations from steady state

$$log(\bar{g}) + \hat{g}_t = -\varphi_g\left(\hat{y}_t - \hat{y}_{t-1}\right) + u_t$$

Notice that:

$$\bar{g} = \left(\frac{\bar{y}}{\bar{y}}\right)^{-\varphi_g} exp(\bar{u})$$

$$log(\bar{g}) = \bar{u}$$

If $u_t \sim (0, \sigma)$, then

$$log(\bar{g}) = \bar{u} = E(u_t) = 0$$

$$\bar{g} = 1$$

So

$$\hat{g}_t = -\varphi_g \left(\hat{y}_t - \hat{y}_{t-1}\right) + u_t$$

We now know 2 things:

$$\hat{y}_t = \frac{\alpha s_g}{(\nu+1)(1-s_g)+\alpha s_g}\hat{g}_t = \kappa \hat{g}_t$$

And

$$\hat{g}_t = -\varphi_g\left(\hat{y}_t - \hat{y}_{t-1}\right) + u_t$$

So

$$\begin{aligned}
\hat{y}_t &= \kappa(-\varphi_g\left(\hat{y}_t - \hat{y}_{t-1}\right) + u_t) \\
\hat{y}_t(1+\kappa\varphi_g) &= \kappa\varphi_g \hat{y}_{t-1} + \kappa u_t \\
\hat{y}_t &= \frac{\kappa\varphi_g}{(1+\kappa\varphi_g)}\hat{y}_{t-1} + \frac{\kappa}{(1+\kappa\varphi_g)}u_t
\end{aligned}$$

Which has the form:

$$\hat{y}_t = \rho \hat{y}_{t-1} + b u_t$$

We know:

$$\hat{y}_t = \rho \hat{y}_{t-1} + b u_t$$

So

$$\begin{aligned}
\hat{y}_t &= \rho[\rho \hat{y}_{t-2} + b u_{t-1}] + b u_t \\
&= \rho^2 \hat{y}_{t-2} + \rho b u_{t-1} + b u_t \\
&= \rho^2 [\rho \hat{y}_{t-3} + b u_{t-2}] + b[\rho u_{t-1} + u_t] \\
&= \rho^3 \hat{y}_{t-3} + b[\rho^2 u_{t-2} + \rho u_{t-1} + u_t] \\
\dots \\
&= \rho^K y_{t-K} + b \sum_{i=0}^{K-1} \rho^i u_{t-i}
\end{aligned}$$

If $K \to \infty$, then

$$\hat{y}_t = b \sum_{i=0}^{\infty} \rho^i u_{t-i}$$

So, finally,

$$\frac{d\hat{y}_t}{du_{t-i}} = b\rho^i$$

Or equivalently

$$\frac{d\hat{y}_{t+i}}{du_t} = b\rho^i$$

Where

$$b = \frac{\kappa}{(1+\kappa\varphi_g)}$$
$$\rho = \frac{\kappa\varphi_g}{(1+\kappa\varphi_g)}$$
$$\kappa = \frac{\alpha s_g}{(\nu+1)(1-s_g)+\alpha s_g}$$