Topics in Macro 2

Week 7 - Second Part - Exercise II

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TSE

Tuesday (17:00-18:30)

TD Second Part: Fiscal Multipliers (4 Weeks)

Part I

- Exercise I: Habit Persistence and The Keynesian Multiplier (Week 6)
- Exercise II: A Benchmark Model (Week 7)
- Exercise III: Consumption, Labor Supply and the Multiplier (Week 7)

Part II

- Exercise I: Taxes on the Labor Input and the Multiplier (Week 8)
- Exercise II: Public Spending in Utility Function and the Multiplier (Week 8)
- Exercise III: Labor Supply, Public Spending in Utility and the Multiplier (Week 9)

Part III

- Exercise I: Endogenous Public Spending (Week 9)
- Exercise II: Externality in Production and the Multiplier (Week 10)
- Exercise III: Externality in Labor Supply and the Multiplier (Week 10)

Exercise II: A Benchmark Model

Goal of the exercise:

- Compute the Fiscal Multiplier for a general equilibrium model.

$$FM = \frac{dy_t}{dg_t}$$

Challenge:

- Sometimes it is not possible to explicitly compute $\frac{dy_t}{dg_t}$.

Alternative:

- Log-linearization: compute $\frac{dy_t}{dg_t}$ around the steady state.

Log-linearization in 3 steps

1. Take logs

$$y_t = F(x_t)$$

$$log(y_t) = log(F(x_t))$$

2. 1st order Taylor expansion around the steady state

$$log(y_t) \approx log(\bar{y}) + \frac{1}{\bar{y}} * (y_t - \bar{y})$$

$$log(F(x_t)) \approx log(F(\bar{x})) + \frac{F'(\bar{x})}{F(\bar{x})} * (x_t - \bar{x})$$

3. Express everything in deviations

$$\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}}$$

Example 1

Production function $y_t = Ak_t^{\alpha} n_t^{1-\alpha}$.

- **Step 1**. Take logs.

$$log(y_t) = log(A) + \alpha log(k_t) + (1-\alpha) log(n_t)$$

- **Step 2**. First Order Taylor Approximation.

$$log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) = log(A) + \alpha[log(\bar{k}) + \frac{1}{\bar{k}}(k_t - \bar{k})] + (1-\alpha)[log(\bar{n}) + \frac{1}{\bar{n}}(n_t - \bar{n})]$$

In the steady state:

$$log(\bar{y}) = log(A) + \alpha log(\bar{k}) + (1-\alpha) log(\bar{n})$$

Then:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \alpha \frac{1}{\bar{k}}(k_t - \bar{k}) + (1-\alpha)\frac{1}{\bar{n}}(n_t - \bar{n})$$

Example 1

We know:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \alpha \frac{1}{\bar{k}}(k_t - \bar{k}) + (1-\alpha)\frac{1}{\bar{n}}(n_t - \bar{n})$$

- **Step 3**. Express everything in deviations from the steady state. Recall:

$$\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}}$$

Then:

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{n}_t$$

In words: if capital increases 1%, production will increase α% close to the steady state.

Example 2

Resource constraint: $y_t = c_t + g_t$.

- **Step 1**. Take logs.

$$log(y_t) = log(c_t + g_t)$$

- **Step 2**. First Order Taylor Approximation.

$$log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) = log(\bar{c} + \bar{g}) + \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g})$$

In the steady state:

$$log(\bar{y}) = log(\bar{c} + \bar{g})$$

Then:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g})$$

Example 2

We know:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g})$$

Then (notice the trick):

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) * \frac{\bar{c}}{\bar{c}} + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g}) * \frac{\bar{g}}{\bar{g}}$$

- **Step 3**. Express everything in deviations from the steady state.

$$\hat{y}_t = \frac{\bar{c}}{\bar{c} + \bar{g}}\hat{c}_t + \frac{\bar{g}}{\bar{c} + \bar{g}}\hat{g}_t$$

In words: if consumption increases 1%, production will increase $\frac{\bar{c}}{\bar{c}+\bar{g}}$% close to the steady state.

Benchmark model

Representative household:

$$U_t = \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu}$$

Budget constraint:

$$c_t \leq w_t n_t - T_t + \pi_t$$

Technology:

$$y_t = a n_t^{\alpha}$$

Profits:

$$\pi_t = y_t - w_t n_t$$

Taxes:

$$T_t = g_t$$

Market clearing condition:

$$y_t = c_t + g_t$$

Question 1. Determine the optimality condition of the households and then deduce the Marginal Rate of Substitution (MRS). Problem:

$$\max_{c_t, n_t} \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu} \text{ subject to } c_t \leq w_t n_t - T_t + \pi_t$$

Lagrangean:

$$L = \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu} - \lambda_t [c_t - w_t n_t + T_t - \pi_t]$$

F.O.C.

$$n_t : \ \eta n_t^{\nu} = \lambda_t w_t$$
$$c_t : \ 1/c_t = \lambda_t$$

Then:

$$\frac{\eta n_t^{\nu}}{1/c_t} = w_t$$

Question 2. Determine the optimality condition of the firm. Problem:

$$\max_{n_t} n_t^{\alpha} - w_t n_t$$

F.O.C.

$$\alpha n_t^{\alpha-1} = w_t$$

Question 3. Determine the equilibrium output.

Definition of a **competitive equilibrium**: An equilibrium are quantities $c_t, n_t^s, n_t^d, y_t, T_t, g_t$ and prices w_t such that:

1. c_t, n_t^s solve the consummer's problem.

$$\frac{\eta n_t^\nu}{1/c_t} = w_t \tag{1}$$

2. n_t^d solve the firm's problem.

$$\alpha n_t^{\alpha-1} = w_t \tag{2}$$

3. Government budget balance: $T_t = g_t$.

$$T_t = g_t \tag{3}$$

4. Goods and labor market clearing: $n_t^d = n_t^s$ and $y_t = c_t + g_t$.

$$n_t^d = n_t^s \tag{4}$$

$$y_t = an_t^\alpha = c_t + g_t \tag{5}$$

Recall: We want to compute $\frac{dy_t}{dg_t}$. So let's express everything in terms of y_t and g_t.

- Equilibrium conditions (4) and (5), and optimality (1) and (2) imply:

$$\frac{\eta n_t^{\nu}}{1/(c_t)} = \alpha n_t^{\alpha-1}$$

$$\frac{\eta n_t^{\nu}}{1/(y_t - g_t)} = \alpha n_t^{\alpha-1}$$

$$\frac{\eta n_t^{\nu} n_t}{1/(y_t - g_t)} = \alpha n_t^{\alpha-1} n_t$$

Then:

$$\frac{\eta n_t^{1+\nu}}{1/(y_t - g_t)} = \alpha y_t$$

- From the production function $y_t = a n_t^{\alpha}$:

$$n_t = \left(\frac{y_t}{a}\right)^{1/\alpha}$$

We know:

$$\frac{\eta n_t^{1+\nu}}{1/(y_t - g_t)} = \alpha y_t$$

And

$$n_t = \left(\frac{y_t}{a}\right)^{1/\alpha}$$

Combining both expressions:

$$\eta \left(\frac{y_t}{a}\right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

We want to compute $\frac{dy_t}{dg_t}$ from

$$\eta \left(\frac{y_t}{a}\right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

But this is hard, so an alternative to do it is to log-linearize and compute it *locally*.

Question 4. Compute the log-linearization of equilibrium output around the steady-state. Define $s_g = \frac{\bar{g}}{\bar{y}}$ (government expenditures as a percentage of GDP in the steady state).

Step 1: Take Logs.

$$\eta\left(\frac{y_t}{a}\right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

$$log(\eta) + \frac{1+\nu}{\alpha} log(y_t) - \frac{1+\nu}{\alpha} log(a) = log(\alpha) + log(y_t) - log(y_t - g_t)$$

Step 2: Taylor approximation.

$$log(\eta) + \frac{1+\nu}{\alpha}\left[log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y})\right] - \frac{1+\nu}{\alpha} log(a) =$$

$$log(\alpha) + log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) - \left[log(\bar{y} - \bar{g}) + \frac{1}{\bar{y}-\bar{g}}(y_t - \bar{y}) - \frac{1}{\bar{y}-\bar{g}}(g_t - \bar{g})\right]$$

In the steady state:

$$log(\eta) + \frac{1+\nu}{\alpha} log(\bar{y}) - \frac{1+\nu}{\alpha} log(a) = log(\alpha) + log(\bar{y}) - log(\bar{y} - \bar{g})$$

Then:

$$\frac{1+\nu}{\alpha}[\frac{1}{\bar{y}}(y_t - \bar{y})] = \frac{1}{\bar{y}}(y_t - \bar{y}) - [\frac{1}{\bar{y} - \bar{g}}(y_t - \bar{y}) - \frac{1}{\bar{y} - \bar{g}}(g_t - \bar{g})]$$

Step 3: Express everything in deviations from the steady state.

$$\frac{1+\nu}{\alpha}[\frac{1}{\bar{y}}(y_t-\bar{y})] = \frac{1}{\bar{y}}(y_t-\bar{y}) - [\frac{1}{\bar{y}-\bar{g}}(y_t-\bar{y}) - \frac{1}{\bar{y}-\bar{g}}(g_t-\bar{g})]$$
$$\frac{1+\nu}{\alpha}\hat{y}_t = \hat{y}_t - [\frac{1}{\bar{y}-\bar{g}}(y_t-\bar{y})\frac{\bar{y}}{\bar{y}} - \frac{1}{\bar{y}-\bar{g}}(g_t-\bar{g})\frac{\bar{g}}{\bar{g}}]$$
$$\frac{1+\nu}{\alpha}\hat{y}_t = \hat{y}_t - [\frac{\bar{y}}{\bar{y}-\bar{g}}\hat{y}_t - \frac{\bar{g}}{\bar{y}-\bar{g}}\hat{g}_t]$$

Rearranging:

$$\left(\frac{1+\nu}{\alpha} - 1 + \frac{\bar{y}}{\bar{y}-\bar{g}}\right)\hat{y}_t = \frac{\bar{g}}{\bar{y}-\bar{g}}\hat{g}_t$$

Define:

$$s_g = \frac{\bar{g}}{\bar{y}}$$

Then:

$$\frac{\bar{g}}{\bar{y} - \bar{g}} = \frac{\bar{g}}{\bar{y} - \bar{g}} * \frac{1/\bar{y}}{1/\bar{y}} = \frac{s_g}{1 - s_g}$$

And:

$$\frac{\bar{y}}{\bar{y} - \bar{g}} = \frac{\bar{y}}{\bar{y} - \bar{g}} * \frac{1/\bar{y}}{1/\bar{y}} = \frac{1}{1 - s_g}$$

Then:

$$\left(\frac{1 + \nu}{\alpha} + \frac{1}{1 - s_g} - 1\right) \hat{y}_t = \frac{s_g}{1 - s_g} \hat{g}_t$$

Rearranging:

$$\left(\frac{1+\nu}{\alpha} + \frac{s_g}{1-s_g}\right)\hat{y}_t = \frac{s_g}{1-s_g}\hat{g}_t$$

Multiply both sides by α.

$$\left(1+\nu + \frac{\alpha s_g}{1-s_g}\right)\hat{y}_t = \frac{\alpha s_g}{1-s_g}\hat{g}_t$$

Question 5. Compute the output multiplier and discuss the value of this multiplier with respect to ν and α.

We know:

$$\left(1+\nu+\frac{\alpha s_g}{1-s_g}\right)\hat{y}_t = \frac{\alpha s_g}{1-s_g}\hat{g}_t$$

Then:

$$\frac{(1+\nu)(1-s_g)+\alpha s_g}{1-s_g}\hat{y}_t = \frac{\alpha s_g}{1-s_g}\hat{g}_t$$

Then:

$$\hat{y}_t = \frac{\alpha s_g}{(1+\nu)(1-s_g)+\alpha s_g}\hat{g}_t$$

So we have that:

$$\frac{\hat{y}_t}{\hat{g}_t} = \frac{\alpha s_g}{(\nu+1)(1-s_g)+\alpha s_g}$$

Remember that we want $\frac{dy_t}{dg_t}$. Notice that:

$$\frac{x_t - \bar{x}}{\bar{x}} = \frac{\Delta x}{\bar{x}} = \frac{dx}{\bar{x}} = \hat{x}_t$$

Then:

$$dx = \hat{x}_t \bar{x}$$

So,

$$\frac{dy_t}{dg_t} = \frac{\hat{y}_t \bar{y}}{\hat{g}_t \bar{g}} * \frac{\bar{y}}{\bar{g}} = \frac{\alpha s_g}{(\nu + 1)(1 - s_g) + \alpha s_g} * \frac{\bar{y}}{\bar{g}}$$

Finally,

$$\frac{dy_t}{dg_t} = \frac{\alpha}{(\nu + 1)(1 - s_g) + \alpha s_g}$$

Discussion.

We know:

$$\frac{dy_t}{dg_t} = \frac{\alpha}{(1 - s_g)(\nu + 1) + \alpha s_g}$$

- ν

- α

Question 6. Compute the consumption multiplier and discuss the value of this multiplier with respect to ν and α.

Answer: $\frac{d\hat{c}_t}{d\hat{g}_t}$.

Appendix

Taylor Approximation (Wikipedia)

Definition

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

where $n!$ denotes the factorial of n. In the more compact sigma notation, this can be written as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. (The derivative of order zero of f is defined to be f itself and $(x-a)^0$ and $0!$ are both defined to be 1.)

For example, for a function $f(x, y)$ that depends on two variables, x and y, the Taylor series to second order about the point (a, b) is

$$f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big)$$