

# Topics in Macro 2

## Week 7 - Second Part - Part I - Exercise II

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# TD Second Part: Fiscal Multipliers (4 Weeks)

## Part I

- Exercise I: Habit Persistence and The Keynesian Multiplier (Week 6)
- Exercise II: A Benchmark Model (Week 7)
- Exercise III: Consumption, Labor Supply and the Multiplier (Week 7)

## Part II

- Exercise I: Taxes on the Labor Input and the Multiplier (Week 8)
- Exercise II: Public Spending in Utility Function and the Multiplier (Week 8)
- Exercise III: Labor Supply, Public Spending in Utility and the Multiplier (Week 9)

## Part III

- Exercise I: Endogenous Public Spending (Week 9)
- Exercise II: Externality in Production and the Multiplier (Week 10)
- Exercise III: Externality in Labor Supply and the Multiplier (Week 10)

## Exercise II: A Benchmark Model

Goal of the exercise:

- Compute the **Fiscal Multiplier** for a general equilibrium model.

$$FM = \frac{dy_t}{dg_t}$$

Challenge:

- Sometimes it is not possible to explicitly compute  $\frac{dy_t}{dg_t}$ .

Alternative:

- Log-linearization: compute  $\frac{dy_t}{dg_t}$  around the steady state.

# Log-linearization in 3 steps

1. Take logs

$$y_t = F(x_t)$$

$$\log(y_t) = \log(F(x_t))$$

2. 1st order Taylor expansion around the steady state

$$\log(y_t) \approx \log(\bar{y}) + \frac{1}{\bar{y}} * (y_t - \bar{y})$$

$$\log(F(x_t)) \approx \log(F(\bar{x})) + \frac{F'(\bar{x})}{F(\bar{x})} * (x_t - \bar{x})$$

3. Express everything in deviations

$$\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}}$$

## Example 1

Production function  $y_t = Ak_t^\alpha n_t^{1-\alpha}$ .

- **Step 1.** Take logs.

$$\log(y_t) = \log(A) + \alpha \log(k_t) + (1 - \alpha) \log(n_t)$$

- **Step 2.** First Order Taylor Approximation.

$$\log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) = \log(A) + \alpha[\log(\bar{k}) + \frac{1}{\bar{k}}(k_t - \bar{k})] + (1 - \alpha)[\log(\bar{n}) + \frac{1}{\bar{n}}(n_t - \bar{n})]$$

In the steady state:

$$\log(\bar{y}) = \log(A) + \alpha \log(\bar{k}) + (1 - \alpha) \log(\bar{n})$$

Then:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \alpha \frac{1}{\bar{k}}(k_t - \bar{k}) + (1 - \alpha) \frac{1}{\bar{n}}(n_t - \bar{n})$$

## Example 1

We know:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \alpha \frac{1}{\bar{k}}(k_t - \bar{k}) + (1 - \alpha) \frac{1}{\bar{n}}(n_t - \bar{n})$$

- **Step 3.** Express everything in deviations from the steady state. Recall:

$$\hat{x}_t = \frac{x_t - \bar{x}}{\bar{x}}$$

Then:

$$\hat{y}_t = \alpha \hat{k}_t + (1 - \alpha) \hat{n}_t$$

In words: if capital increases 1%, production will increase  $\alpha\%$  close to the steady state.

## Example 2

Resource constraint:  $y_t = c_t + g_t$ .

- **Step 1.** Take logs.

$$\log(y_t) = \log(c_t + g_t)$$

- **Step 2.** First Order Taylor Approximation.

$$\log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) = \log(\bar{c} + \bar{g}) + \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g})$$

In the steady state:

$$\log(\bar{y}) = \log(\bar{c} + \bar{g})$$

Then:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g})$$



## Example 2

We know:

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g})$$

Then (notice the trick):

$$\frac{1}{\bar{y}}(y_t - \bar{y}) = \frac{1}{\bar{c} + \bar{g}}(c_t - \bar{c}) * \frac{\bar{c}}{\bar{c}} + \frac{1}{\bar{c} + \bar{g}}(g_t - \bar{g}) * \frac{\bar{g}}{\bar{g}}$$

- **Step 3.** Express everything in deviations from the steady state.

$$\hat{y}_t = \frac{\bar{c}}{\bar{c} + \bar{g}} \hat{c}_t + \frac{\bar{g}}{\bar{c} + \bar{g}} \hat{g}_t$$

In words: if consumption increases 1%, production will increase  $\frac{\bar{c}}{\bar{c} + \bar{g}}$  % close to the steady state.

## Benchmark model

Representative household:

$$U_t = \ln c_t - \frac{\eta}{1 + \nu} n_t^{1+\nu}$$

Budget constraint:

$$c_t \leq w_t n_t - T_t + \pi_t$$

Technology:

$$y_t = a n_t^\alpha$$

Profits:

$$\pi_t = y_t - w_t n_t$$

Taxes:

$$T_t = g_t$$

Market clearing condition:

$$y_t = c_t + g_t$$

**Question 1.** Determine the **optimality** condition of the **households** and then deduce the Marginal Rate of Substitution (MRS). Problem:

$$\max_{c_t, n_t} \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu} \text{ subject to } c_t \leq w_t n_t - T_t + \pi_t$$

Lagrangean:

$$L = \ln c_t - \frac{\eta}{1+\nu} n_t^{1+\nu} - \lambda_t [c_t - w_t n_t + T_t - \pi_t]$$

F.O.C.

$$n_t : \eta n_t^\nu = \lambda_t w_t$$

$$c_t : 1/c_t = \lambda_t$$

Then:

$$\frac{\eta n_t^\nu}{1/c_t} = w_t$$

**Question 2.** Determine the optimality condition of the firm. Problem:

$$\max_{n_t} n_t^\alpha - w_t n_t$$

F.O.C.

$$\alpha n_t^{\alpha-1} = w_t$$

**Question 3.** Determine the **equilibrium output**.

Definition of a **competitive equilibrium**: An equilibrium are quantities  $c_t, n_t^s, n_t^d, y_t, T_t, g_t$  and prices  $w_t$  such that:

1.  $c_t, n_t^s$  solve the consumer's problem.

$$\frac{\eta n_t^\nu}{1/c_t} = w_t \quad (1)$$

2.  $n_t^d$  solve the firm's problem.

$$\alpha n_t^{\alpha-1} = w_t \quad (2)$$

3. Government budget balance:  $T_t = g_t$ .

$$T_t = g_t \quad (3)$$

4. Goods and labor market clearing:  $n_t^d = n_t^s$  and  $y_t = c_t + g_t$ .

$$n_t^d = n_t^s \quad (4)$$

$$y_t = \alpha n_t^\alpha = c_t + g_t \quad (5)$$

Recall: We want to compute  $\frac{dy_t}{dg_t}$ . So let's express everything in terms of  $y_t$  and  $g_t$ .

- Equilibrium conditions (4) and (5), and optimality (1) and (2) imply:

$$\begin{aligned}\frac{\eta n_t^\nu}{1/(c_t)} &= \alpha n_t^{\alpha-1} \\ \frac{\eta n_t^\nu}{1/(y_t - g_t)} &= \alpha n_t^{\alpha-1} \\ \frac{\eta n_t^\nu n_t}{1/(y_t - g_t)} &= \alpha n_t^{\alpha-1} n_t\end{aligned}$$

Then:

$$\frac{\eta n_t^{1+\nu}}{1/(y_t - g_t)} = \alpha y_t$$

- From the production function  $y_t = a n_t^\alpha$ :

$$n_t = \left(\frac{y_t}{a}\right)^{1/\alpha}$$

We know:

$$\frac{\eta n_t^{1+\nu}}{1/(y_t - g_t)} = \alpha y_t$$

And

$$n_t = \left(\frac{y_t}{a}\right)^{1/\alpha}$$

Combining both expressions:

$$\eta \left(\frac{y_t}{a}\right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

We want to compute  $\frac{dy_t}{dg_t}$  from

$$\eta \left( \frac{y_t}{a} \right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$

But this is hard, so an alternative to do it is to log-linearize and compute it *locally*.



**Question 4.** Compute the **log-linearization** of equilibrium output around the steady-state. Define  $s_g = \frac{\bar{g}}{\bar{y}}$  (government expenditures as a percentage of GDP in the steady state).

**Step 1:** Take Logs.

$$\eta \left( \frac{y_t}{a} \right)^{\frac{1+\nu}{\alpha}} = \frac{\alpha y_t}{y_t - g_t}$$
$$\log(\eta) + \frac{1+\nu}{\alpha} \log(y_t) - \frac{1+\nu}{\alpha} \log(a) = \log(\alpha) + \log(y_t) - \log(y_t - g_t)$$

**Step 2:** Taylor approximation.

$$\log(\eta) + \frac{1+\nu}{\alpha} \left[ \log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) \right] - \frac{1+\nu}{\alpha} \log(a) =$$
$$\log(\alpha) + \log(\bar{y}) + \frac{1}{\bar{y}}(y_t - \bar{y}) - [\log(\bar{y} - \bar{g}) + \frac{1}{\bar{y} - \bar{g}}(y_t - \bar{y}) - \frac{1}{\bar{y} - \bar{g}}(g_t - \bar{g})]$$

In the steady state:

$$\log(\eta) + \frac{1+\nu}{\alpha} \log(\bar{y}) - \frac{1+\nu}{\alpha} \log(a) = \log(\alpha) + \log(\bar{y}) - \log(\bar{y} - \bar{g})$$

Then:

$$\frac{1+\nu}{\alpha} \left[ \frac{1}{\bar{y}} (y_t - \bar{y}) \right] = \frac{1}{\bar{y}} (y_t - \bar{y}) - \left[ \frac{1}{\bar{y} - \bar{g}} (y_t - \bar{y}) - \frac{1}{\bar{y} - \bar{g}} (g_t - \bar{g}) \right]$$

**Step 3:** Express everything in deviations from the steady state.

$$\begin{aligned}\frac{1+\nu}{\alpha} \left[ \frac{1}{\bar{y}} (y_t - \bar{y}) \right] &= \frac{1}{\bar{y}} (y_t - \bar{y}) - \left[ \frac{1}{\bar{y} - \bar{g}} (y_t - \bar{y}) - \frac{1}{\bar{y} - \bar{g}} (g_t - \bar{g}) \right] \\ \frac{1+\nu}{\alpha} \hat{y}_t &= \hat{y}_t - \left[ \frac{1}{\bar{y} - \bar{g}} (y_t - \bar{y}) \frac{\bar{y}}{\bar{y}} - \frac{1}{\bar{y} - \bar{g}} (g_t - \bar{g}) \frac{\bar{g}}{\bar{g}} \right] \\ \frac{1+\nu}{\alpha} \hat{y}_t &= \hat{y}_t - \left[ \frac{\bar{y}}{\bar{y} - \bar{g}} \hat{y}_t - \frac{\bar{g}}{\bar{y} - \bar{g}} \hat{g}_t \right]\end{aligned}$$

Rearranging:

$$\left( \frac{1+\nu}{\alpha} - 1 + \frac{\bar{y}}{\bar{y} - \bar{g}} \right) \hat{y}_t = \frac{\bar{g}}{\bar{y} - \bar{g}} \hat{g}_t$$

Define:

$$s_g = \frac{\bar{g}}{\bar{y}}$$

Then:

$$\frac{\bar{g}}{\bar{y} - \bar{g}} = \frac{\bar{g}}{\bar{y} - \bar{g}} * \frac{1/\bar{y}}{1/\bar{y}} = \frac{s_g}{1 - s_g}$$

And:

$$\frac{\bar{y}}{\bar{y} - \bar{g}} = \frac{\bar{y}}{\bar{y} - \bar{g}} * \frac{1/\bar{y}}{1/\bar{y}} = \frac{1}{1 - s_g}$$

Then:

$$\left( \frac{1 + \nu}{\alpha} + \frac{1}{1 - s_g} - 1 \right) \hat{y}_t = \frac{s_g}{1 - s_g} \hat{g}_t$$

Rearranging:

$$\left( \frac{1 + \nu}{\alpha} + \frac{s_g}{1 - s_g} \right) \hat{y}_t = \frac{s_g}{1 - s_g} \hat{g}_t$$

Multiply both sides by  $\alpha$ .

$$\left( 1 + \nu + \frac{\alpha s_g}{1 - s_g} \right) \hat{y}_t = \frac{\alpha s_g}{1 - s_g} \hat{g}_t$$

**Question 5.** Compute the **output multiplier** and **discuss** the value of this multiplier with respect to  $\nu$  and  $\alpha$ .

We know:

$$\left(1 + \nu + \frac{\alpha s_g}{1 - s_g}\right) \hat{y}_t = \frac{\alpha s_g}{1 - s_g} \hat{g}_t$$

Then:

$$\frac{(1 + \nu)(1 - s_g) + \alpha s_g}{1 - s_g} \hat{y}_t = \frac{\alpha s_g}{1 - s_g} \hat{g}_t$$

Then:

$$\hat{y}_t = \frac{\alpha s_g}{(1 + \nu)(1 - s_g) + \alpha s_g} \hat{g}_t$$

So we have that:

$$\frac{\hat{y}_t}{\hat{g}_t} = \frac{\alpha s_g}{(\nu + 1)(1 - s_g) + \alpha s_g}$$

Remember that we want  $\frac{dy_t}{dg_t}$ . Notice that:

$$\frac{x_t - \bar{x}}{\bar{x}} = \frac{\Delta x}{\bar{x}} = \frac{dx}{\bar{x}} = \hat{x}_t$$

Then:

$$dx = \hat{x}_t \bar{x}$$

So,

$$\frac{dy_t}{dg_t} = \frac{\hat{y}_t \bar{y}}{\hat{g}_t \bar{g}} * \frac{\bar{y}}{\bar{g}} = \frac{\alpha s_g}{(\nu + 1)(1 - s_g) + \alpha s_g} * \frac{\bar{y}}{\bar{g}}$$

Finally,

$$\frac{dy_t}{dg_t} = \frac{\alpha}{(\nu + 1)(1 - s_g) + \alpha s_g}$$

## Discussion.

We know:

$$\frac{dy_t}{dg_t} = \frac{\alpha}{(1 - s_g)(\nu + 1) + \alpha s_g}$$

- $\nu$

- $\alpha$



**Question 6.** Compute the **consumption multiplier** and discuss the value of this multiplier with respect to  $\nu$  and  $\alpha$ .

Answer:  $\frac{d\hat{c}_t}{d\hat{g}_t}$ .

# Appendix

## Taylor Approximation (Wikipedia)

### Definition

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The Taylor series of a [real](#) or [complex-valued function](#)  $f(x)$  that is [infinitely differentiable](#) at a [real](#) or [complex number](#)  $a$  is the [power series](#)

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

where  $n!$  denotes the [factorial](#) of  $n$ . In the more compact [sigma notation](#), this can be written as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n,$$

where  $f^{(n)}(a)$  denotes the  $n$ th [derivative](#) of  $f$  evaluated at the point  $a$ . (The derivative of order zero of  $f$  is defined to be  $f$  itself and  $(x-a)^0$  and  $0!$  [are both defined to be 1](#).)

For example, for a function  $f(x, y)$  that depends on two variables,  $x$  and  $y$ , the Taylor series to second order about the point  $(a, b)$  is

$$f(a, b) + (x-a)f_x(a, b) + (y-b)f_y(a, b) + \frac{1}{2!} \left( (x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right)$$