

Chapter 6

Linear Recurrences

“Everything goes, everything comes back; eternally rolls the wheel of being.” (Friedrich Nietzsche)

This chapter is dedicated to linear recurrences, a special type of equations that defines a sequence, that is a series of terms of the form

$$a_0, a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$$

recursively, that is such that each term a_n is defined as a function of the preceding terms. A recursive linear recurrence must be accompanied by initial conditions, that is information about some of the first terms such as a_0 or a_0, a_1 .

Example 60. The Fibonacci sequence is defined by

$$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, \quad f_1 = 1.$$

To compute f_2 , we have

$$f_2 = f_1 + f_0 = 1.$$

To compute f_3 , we have

$$f_3 = f_2 + f_1 = 1 + 1 = 2.$$

We will see in this chapter two methods to solve linear recurrences involving one or two preceding terms.

Recurrence Relation

A *recurrence relation* is an equation that *recursively defines a sequence*, i.e., each term of the sequence is defined as a function of the preceding terms

A recursive formula must be accompanied by *initial conditions* (information about the beginning of the sequence).

Fibonacci Sequence

$$f_n = f_{n-1} + f_{n-2} \text{ with } f_0 = 0, f_1 = 1$$

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...



Leonardo
Pisano Bigolio
(c. 1170-c.
1250)

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The first method is called *backtracking*, and consists of taking a linear recurrence defining a_n , and replace the terms a_{n-1}, a_{n-2}, \dots with the relation that defines a_n , but where n is replaced by $n-1, n-2$, etc. This is best illustrated on an example.

Example 61. Consider the linear recurrence

$$a_n = a_{n-1} + 3, \quad a_1 = 2.$$

Then

$$\begin{aligned} a_{n-1} &= a_{n-2} + 3 \\ a_{n-2} &= a_{n-3} + 3 \\ a_{n-3} &= a_{n-4} + 3 \end{aligned}$$

and so on and so forth. Therefore

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3 \\ &= (a_{n-3} + 3) + 6 = a_{n-3} + 3 \cdot 3 \\ &= \dots \\ &= a_1 + 3(n-1). \end{aligned}$$

The last equality follows because a generic term is of the form $a_{n-i} + 3i$, therefore when $n-i = 1$, $i = n-1$. By plugging the initial condition, we conclude

$$a_n = 2 + 3(n-1).$$

Once the solution has been found, you may wonder how to check whether this is the right answer. One way to do it is by proving it by induction!

Example 62. Let us provide a proof by induction for the above example. Define $P(n) = "a_n = 2 + 3(n-1)"$. Then $P(1) = "a_1 = 2"$, which is the initial condition, is true. Suppose $P(k) = "a_k = 2 + 3(k-1)"$ is true. We want to prove $P(k+1)$.

$$\begin{aligned} a_{k+1} &= a_k + 3 \\ &= 2 + 3(k-1) + 3 \\ &= 2 + 3k \end{aligned}$$

as desired.

Solving Recurrence Relation

- **Backtracking** is a technique for finding explicit formula for recurrence relation
 - E.g., say $a_n = a_{n-1} + 3$ and $a_1 = 2$
 - $a_n = a_{n-1} + 3 = (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3$
 $= (a_{n-3} + 3) + 2 \cdot 3 = a_{n-3} + 3 \cdot 3$
 $= (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3$
 \dots
 $= a_1 + (n-1) \cdot 3$
 $a_n = 2 + (n-1) \cdot 3$
-

Homogeneous Relation Of Degree d

A **linear homogeneous relation** of degree d is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}$$

Examples

- The Fibonacci sequence
- The relation: $a_n = 2a_{n-1}$ (degree 1)
- But **not** the relation: $a_n = 2a_{n-1} + 1$

The **characteristic equation** of the above relation is

$$x^d = c_1 x^{d-1} + c_2 x^{d-2} + \dots + c_d$$

Definition 37. A *linear homogeneous relation* of degree d is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}.$$

Its *characteristic equation* is

$$x^d = c_1 x_{d-1} + c_2 x_{d-2} + \dots + c_d.$$

The characteristic equation is obtained from $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}$ by replacing a_i by x^i :

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} - \dots - c_d x^{n-d} = 0,$$

then factor out x^{n-d} to get

$$x^{n-d}(x^d - c_1 x^{d-1} - c_2 x^{d-2} - \dots - c_d) = 0.$$

Example 63. The Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ is a homogeneous relation. Let us compute its characteristic equation:

$$x^n - x^{n-1} - x^{n-2} = 0 \iff x^{n-2}(x^2 - x - 1) = 0$$

therefore $x^2 - x - 1 = 0$ is the characteristic equation.

Let us focus on quadratic characteristic equations, that is of the form

$$x^2 - c_1 x - c_2 = 0$$

which corresponds to linear recurrences of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}.$$

Suppose that $x^2 - c_1 x - c_2 = 0$ has two distinct real roots s_1, s_2 , then

$$s_1^2 - c_1 s_1 - c_2 = 0, \quad s_2^2 - c_1 s_2 - c_2 = 0.$$

Therefore

$$s_1^n - c_1 s_1^{n-1} - c_2 s_1^{n-2} = 0, \quad s_2^n - c_1 s_2^{n-1} - c_2 s_2^{n-2} = 0$$

and we have that if s is a solution of $x^2 - c_1 x - c_2 = 0$ then s^n is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. This tells us that solutions of a_n are composed of s_1^n, s_2^n . Note the term "composed" is used, because if a sequence a'_n also satisfies the recurrence of a_n , then $a_n + a'_n$ satisfies the recurrence of a_n as well, as does multiples of a_n (see Exercise 53).

Theorem

If the *characteristic equation* $x^2 - c_1x - c_2 = 0$ (of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$) has

- two distinct roots s_1, s_2 , then the explicit formula for the sequence a_n is

$$u \cdot s_1^n + v \cdot s_2^n$$

- a single root s , then the explicit formula for a_n is

$$u \cdot s^n + v \cdot n \cdot s^n$$

where u & v are determined by initial conditions.

Example

Determine the number of bit strings (i.e., comprising 0/1s) of length n that contains *no adjacent 0s*.

- C_n = this number of bit strings
 - A binary string with no adjacent 0s is constructed by
 - Adding “1” to any string w of length $n-1$ satisfying the condition, or
 - Adding “10” to any string v of length $n-2$ satisfying the condition
 - Thus $C_n = C_{n-1} + C_{n-2}$ where $C_1=2$ (0,1), $C_2=3$ (01, 10, 11)
-

This means that the final solution is really a composition of s_1^n, s_2^n , namely a solution for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is given by

$$a_n = us_1^n + vs_2^n,$$

where u, v depend on the initial conditions (that is on a_0, a_1).

Suppose now that $x^2 - c_1x - c_2 = 0$ has one double real root s , that is $x^2 - c_1x - c_2 = (x - s)^2$. Then

$$s^2 - c_1s - c_2 = 0,$$

and s^n is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ as for the case of two distinct roots. We obtained the characteristic equation from

$$x^n - c_1x^{n-1} - c_2x^{n-2} = x^{n-2}(x^2 - c_1x - c_2) = 0.$$

If s is a root of this equation, then s is a root of its derivative:

$$nx^{n-1} - c_1(n-1)x^{n-2} - c_2(n-2)x^{n-3} = 0.$$

Therefore s satisfies both

$$s^n = c_1s^{n-1} + c_2s^{n-2}, \quad ns^n = c_1(n-1)s^{n-1} + c_2(n-2)s^{n-2}. \quad (6.1)$$

If we combine s^n and ns^n , as we did for s_1^n and s_2^n , we get

$$a_n = us^n + vns^n,$$

and $a_{n+1} = us^{n+1} + v(n+1)s^{n+1}$. We are left to check

$$\begin{aligned} c_1a_{n+1} + c_2a_{n+2} &= c_1(us^{n+1} + v(n+1)s^{n+1}) + c_2(us^{n+2} + v(n+2)s^{n+2}) \\ &= u[c_1s^{n+1} + c_2s^{n+2}] + v[c_1(n+1)s^{n+1} + c_2(n+2)s^{n+2}] \\ &= us^{n+2} + v(n+2)s^{n+2} \\ &= a_{n+2} \end{aligned}$$

using (6.1), which is consistent with our recurrence relation.

Example 64. Suppose we want to determine the number of bit strings of length n that contains no adjacent zeroes. We denote this number by C_n . We first observe that there are two ways of obtaining such sequences from a smaller sequence. One may take any string:

- of length $n - 1$ satisfying the condition and add a 1 (one cannot add a 0, since the sequence may finish by 0),
- of length $n - 2$ satisfying the condition and add 10 (one cannot add 00, or 01 since the string could finish by 0, and 11 is included above).

Example

- Now solve $C_n = C_{n-1} + C_{n-2}$ where $C_1=2, C_2=3$
- Characteristic equation: $x^2 - x - 1 = 0$
- Its roots are $(1 + \sqrt{5})/2$
 $(1 - \sqrt{5})/2$

Recall roots of quadratic eqn.

$$a.x^2 + b.x + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

- Thus

$$C_n = u.\left(\frac{1+\sqrt{5}}{2}\right)^n + v.\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example

Initial conditions give us:

$$C_1 = u.\left(\frac{1+\sqrt{5}}{2}\right) + v.\left(\frac{1-\sqrt{5}}{2}\right) = 2$$

$$\text{i.e., } \frac{u+v}{2} + \frac{(u-v)\sqrt{5}}{2} = 2$$

$$C_2 = u.\left(\frac{1+\sqrt{5}}{2}\right)^2 + v.\left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$$

$$\text{i.e., } \frac{3(u+v)}{2} + \frac{(u-v)\sqrt{5}}{2} = 3$$

Solving, we get

$$u = \frac{\sqrt{5}+3}{2\sqrt{5}}$$

$$v = \frac{\sqrt{5}-3}{2\sqrt{5}}$$

Therefore the linear recurrence involved is

$$C_n = C_{n-1} + C_{n-2}.$$

The characteristic equation is obtained from

$$x^n - x^{n-1} - x^{n-2} = x^{n-2}(x^2 - x - 1) = 0,$$

it is $x^2 - x - 1 = 0$. To find its roots, we compute

$$\frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

The solution is then

$$C_n = u\left(\frac{1 + \sqrt{5}}{2}\right)^n + v\left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

We are left to find u, v based on the initial conditions. They are $C_1 = 2$ (the strings are 0 and 1), while $C_2 = 3$ (the strings are 11, 01, 10). This gives us two equations for two unknowns:

$$\begin{aligned} u\left(\frac{1 + \sqrt{5}}{2}\right) + v\left(\frac{1 - \sqrt{5}}{2}\right) &= 2 \\ u\left(\frac{1 + \sqrt{5}}{2}\right)^2 + v\left(\frac{1 - \sqrt{5}}{2}\right)^2 &= 3. \end{aligned}$$

which can be simplified to

$$\begin{aligned} \frac{u + v}{2} + \sqrt{5}\frac{u - v}{2} &= 2 \\ 3\frac{u + v}{2} + \sqrt{5}\frac{u - v}{2} &= 3. \end{aligned}$$

Set $a = (u + v)/2$, $b = \sqrt{5}(u - v)/2$. We need to solve

$$a + b = 2, \quad 3a + b = 3 \Rightarrow b = \frac{3}{2}, \quad a = \frac{1}{2}.$$

Thus

$$v = \frac{\sqrt{5} - 3}{2\sqrt{5}}, \quad u = 1 - \frac{\sqrt{5} - 3}{2\sqrt{5}} = \frac{\sqrt{5} + 3}{2\sqrt{5}}.$$

Exercises for Chapter 6

Exercise 51. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

Exercise 52. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Exercise 53. Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Examples for Chapter 6

Linear Recurrence relations are often useful to analyze algorithms, such as divide-and-conquer algorithms. We will illustrate this using the game of Hanoi tower.

In this game, the goal is to move n disks ranked from the largest at the bottom to the smallest on top from one post to another. The only permitted action is to remove the top disk from a post and drop it onto another post. The rule is that a larger disk can never lie above a smaller disk on any post. When $n = 3$ disks, the Hanoi tower game can be solved in 7 steps. But say one would like to know how many steps it would take to solve it for $n = 50$ disks, how could this be figured out?

The method is to derive a linear recurrence relation, and then to solve it. To find a linear recurrence relation, notice that to solve the Hanoi tower game for $n = 3$ disks, the following steps are done:

1. Solve a Hanoi tower game for $n = 2$ disks,
2. Move the largest disk,
3. Solve another Hanoi tower game for $n = 2$ disks.

In fact, this is true in general, which yields the linear recurrence

$$T_n = 2T_{n-1} + 1,$$

where T_n denotes the number of steps for n disks.

We solve this linear recurrence using backtracking. Note that

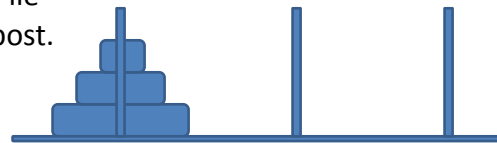
$$T_{n-1} = 2T_{n-2} + 1, \quad T_{n-2} = 2T_{n-3} + 1, \quad T_{n-3} = 2T_{n-4} + 1, \dots$$

Then

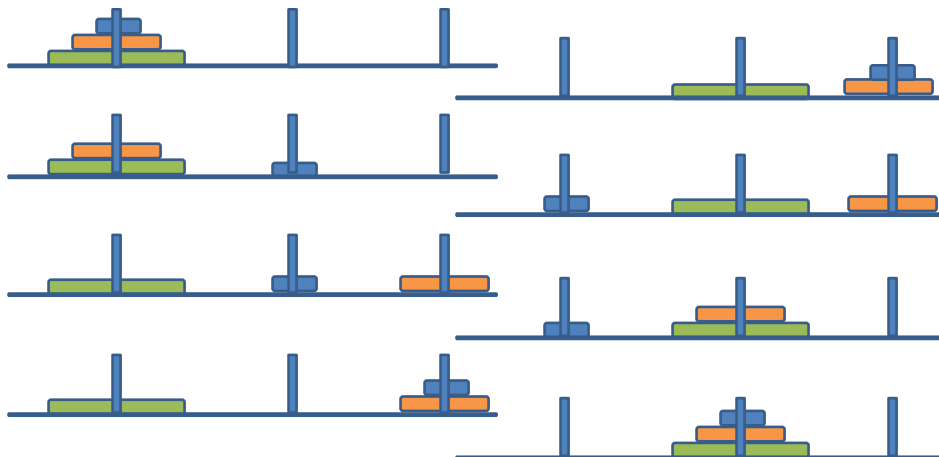
$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2T_{n-2} + 1) = 4T_{n-2} + 3 \\ &= 4(2T_{n-3} + 1) + 3 = 8T_{n-3} + 7 \\ &= 8(2T_{n-4} + 1) + 7 = 16T_{n-4} + 15 \\ &= \dots \end{aligned}$$

Hanoi Tower

- **Goal:** move all n disks in the same order, but on a different post.
- **Only permitted action:** remove the top disk from a post and drop it onto another post.
- **Rule:** a larger disk can never lie above a smaller disk on any post.

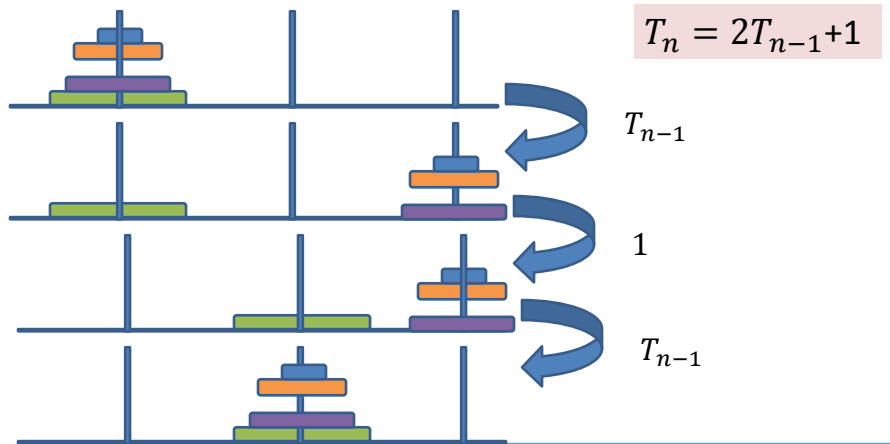


Hanoi Tower ($n=3$)



Find a Recurrence

- T_n = minimum number of steps needed to move an n -disk tower from one post to another



Backtracking

$$T_1 = 1 \quad T_n = 2T_{n-1} + 1$$

Backtracking

$$T_1 = 1 \quad T_n = 2T_{n-1} + 1$$

$$\begin{aligned} T_n &= 2T_{n-1} + 1 = 2(2T_{n-2} + 1) + 1 \quad \leftarrow 3 \\ &= 4T_{n-2} + 3 = 4(2T_{n-3} + 1) + 3 \quad \leftarrow 7 \\ &= 8T_{n-3} + 7 \quad \leftarrow 15 \end{aligned}$$

$$T_n = 2^n - 1$$

Induction

- $P(n) = "T_n = 2^n - 1"$
 - Basis step: $P(1) = "T_1 = 1"$
 - Inductive step: suppose $P(n)$ is true.
 - To show, $P(n+1)$.
 - $T_{n+1} = 2T_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$
-

We notice that a general term is of the form

$$2^i T_{n-i} + (2^i - 1),$$

therefore when $n - i = 1$, $i = n - 1$, and we get

$$T_n = 2^{n-1} T_1 + (2^{n-1} - 1)$$

with $T_1 = 1$. Thus finally

$$T_n = 2^n - 1.$$

Once a solution has been found by backtracking, it is advised to confirm that the solution is sound, by performing a proof by induction. Here $P(n) = "T_n = 2^n - 1"$. The basis step is $P(1) = "T_1 = 2^1 - 1 = 1"$ which is true. The induction step is to assume that $P(k) = "T_k = 2^k - 1"$ is true. We need to prove $P(k + 1)$. But

$$T_{k+1} = 2T_k + 1 = 2 \cdot 2^k + 1 = 2^{k+1}.$$

as desired!

