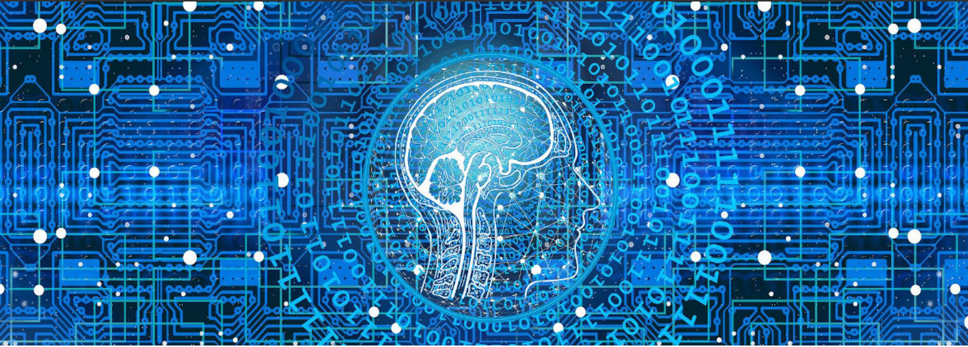

Chapter 8: Satisfiability and Model Construction

Source A: Foundations of Artificial Intelligence by J. Boedecker, W. Burgard, F. Hutter, B. Nebel (Uni. Freiburg)

Motivation

SAT solving is the best available technology for practical solutions to many NP-hard problems Formal verification

- Verification of software
  - Ruling out unintended states (null-pointer exceptions, etc.)
  - Proving that the program computes the right solution
- Verification of hardware (Pentium bug, etc)

Practical approach: encode into SAT & exploit the rapid progress in SAT solving

- Solving CSP instances in practice
- Solving graph coloring problems in practice

Contents

Logical deduction vs. satisfiability

Propositional Logic - typical algorithmic questions:

- Logical deduction
  - Given: A logical theory (set of propositions)
  - Question: Does a proposition logically follow from this theory?
  - Reduction to unsatisfiability, which is coNP-complete (complementary to NP problems)
- Satisfiability of a formula (SAT)
  - Given: A logical theory
  - Wanted: Model of the theory
  - Example: Configurations that fulfill the constraints given in the theory
  - Can be "easier" because it is enough to find one model
  - Possible Application: Find software bugs, i.e. find a state that is not desired

The Satisfiability Problem (SAT)

- Given
  - Propositional formula ϕ in CNF
- Wanted
  - Model of ϕ.
  - or proof, that no such model exists.

SAT can be formulated as a Constraint-Satisfaction-Problem ($\rightarrow$ search):

- CSP-Variables = Symbols of the alphabet
- Domain of values = $\{$T,F$\}$
- Constraints given by clauses

The DPLL algorithm

The DPLL algorithm (Davis, Putnam, Logemann, Loveland, 1962) corresponds to backtracking with inference in CSPs:

- Recursive call DPLL (Δ, l) with
  - Δ: set of clauses
  - l: variable assignment
- Result: satisfying assignment that extends l or "unsatisfiable" if no such assignment exists.
  - First call by DPLL(Δ, $\emptyset$)
- Inference in DPLL:
  - Simplify: if variable v is assigned a value d, then all clauses containing v are simplified immediately (corresponds to forward checking)
  - Variables in unit clauses ($=$ clauses with only one variable) are immediately assigned (corresponds to minimum remaining values ordering in CSPs)

The DPLL Procedure

DPLL Function

Given a set of clauses Δ defined over a set of variables Σ, return "satisfiable" if Δ is satisfiable. Otherwise return "unsatisfiable".

- 1. If $\Delta = \emptyset$ return "satisfiable"
- 2. If $\in \Delta$ return "unsatisfiable"
- 3. Unit-propagation Rule: If Δ contains a unit-clause C, assign a truth-value to the variable in C that satisfies C, simplify Δ to Δ' and return DPLL(Δ').
- 4. Splitting Rule: Select from Σ a variable v which has not been assigned a truth-value. Assign one truth value t to it, simplify Δ to Δ' and call DPLL(Δ')
  - a. If the call returns "satisfiable", then return "satisfiable".
  - b. Otherwise assign the other truth-value to v in Δ, simplify to Δ" and return DPLL(Δ").

Example (1)

$$\Delta = \{\{a, b, \neg c\}, \{\neg a, \neg b\}, \{c\}, \{a, \neg b\}\}$$

1. Unit-propagation rule: $c \mapsto T$
   $\{\{a, b\}, \{\neg a, \neg b\}, \{a, \neg b\}\}$
2. Splitting rule:

2a. $a \mapsto F$
$\{\{b\}, \{\neg b\}\}$

3a. Unit-propagation rule:
$b \mapsto T$
$\{\square\}$

Example (1)

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   $\{\{b\}, \{\neg b\}\}$

3a. Unit-propagation rule:
   $b \mapsto T$
   $\{\square\}$

2b. $a \mapsto T$
   $\{\{\neg b\}\}$

3b. Unit-propagation rule: $b \mapsto F$
   $\{\}$

Example (2)

$$\Delta = \{\{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\}\}$$

- 1. Unit-propagation rule: $d \rightarrow T$
  - $\{\{a, \neg b, \neg c\}, \{b\}, \{c\}\}$
- 2. Unit-propagation rule: $b \rightarrow T$
  - $\{\{a, \neg c\}, \{c\}\}$
- 3. Unit-propagation rule: $c \rightarrow T$
  - $\{\{a\}\}$
- 4. Unit-propagation rule: $a \rightarrow T$
  - $\{\}$

Properties of DPLL

- DPLL is complete, correct, and guaranteed to terminate.
- DPLL constructs a model, if one exists.
- In general, DPLL requires exponential time (splitting rule!)
  - $\rightarrow$ Heuristics are needed to determine which variable should be instantiated next and which value should be used.
- DPLL is polynomial on Horn clauses (see next slides).
- In current SAT competitions, DPLL-based procedures have shown the best performance.

DPLL on Horn Clauses (0)

- Horn Clauses constitute an important special case, since they require only polynomial runtime of DPLL.
- Definition: A Horn clause is a clause with maximally one positive literal E.g., $\neg A_1 \vee ... \vee \neg A_n \vee B$ or $\neg A_1 \vee ... \vee \neg A_n$ (n = 0 is permitted).
- Equivalent representation: $\neg A_1 \vee ... \vee \neg A_n \vee B \Leftrightarrow \bigwedge_i A_i \Rightarrow B$
    - $\rightarrow$ Basis of logic programming (e.g., PROLOG)

DPLL on Horn Clauses (1)

Note

- 1. The simplifications in DPLL on Horn clauses always generate Horn clauses
- 2. If the first sequence of applications of the unit propagation rule in DPLL does not lead to termination, a set of Horn clauses without unit clauses is generated
- 3. A set of Horn clauses without unit clauses and without the empty clause is satisfiable, since
  - All clauses have at least one negative literal (since all non-unit clauses have at least two literals, where at most one can be positive (Def. Horn))
  - Assigning $false$ to all variables satisfies formula

DPLL on Horn Clauses (2)

... still note

- 4. It follows from 3.:
  - a. every time the splitting rule is applied, the current formula is satisfiable
  - b. every time, when the wrong decision (= assignment in the splitting rule) is made, this will be immediately detected (e.g., only through unit propagation steps and the derivation of the empty clause).
- 5. Therefore, the search trees for n variables can only contain a maximum of n nodes, in which the splitting rule is applied (and the tree branches).
- 6. Therefore, the size of the search tree is only polynomial in n and therefore the running time is also polynomial.

"Average" Complexity of the Satisfiability Problem

"Average" Complexity of the Satisfiability Problem

How Good is DPLL in the Average Case?

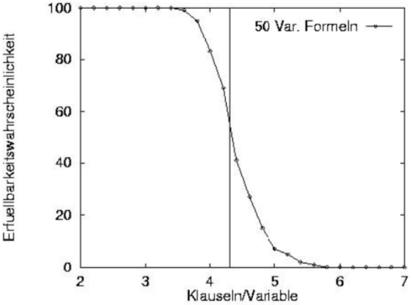
- We know that SAT is NP-complete, i.e., in the worst case, it takes exponential time.
- This is clearly also true for the DPLL-procedure. $\rightarrow$ Couldn't we do better in the average case?
- For CNF-formulae, in which the probability for a positive appearance, negative appearance and non-appearance in a clause is $1/3$, DPLL needs on average quadratic time (Goldberg 79)!
  - $\rightarrow$ The probability that these formulae are satisfiable is, however, very high.

Phase Transitions ...

- Conversely, we can, of course, try to identify hard to solve problem instances.
- Cheeseman et al. came up with the following plausible conjecture:
  - All NP-complete problems have at least one order parameter and the hard to solve problems are around a critical value of this order parameter. This critical value (a $phasetransition$) separates one region from another, such as over-constrained and under-constrained regions of the problem space.
  - over-constrained: you have more and more clauses and hence it becomes more and more difficult to find a solution
  - under-constrained: you have much more variables than constrains and hence a solution is easy to find
- Confirmation for graph coloring and Hamiltonian path ..., later also for other NP-complete problems.

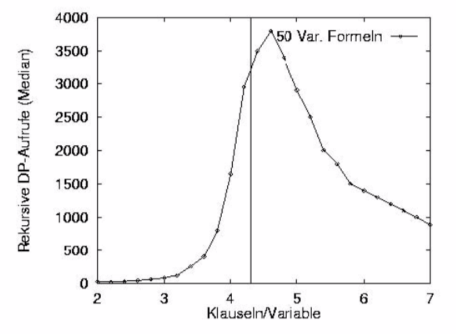
Phase Transitions with 3-SAT

- Constant clause length model (Mitchell et al.): Clause length k is given. Choose variables for every clause k and use the complement with probability 0.5 for each variable.
- Phase transition for 3-SAT with a clause/variable ratio of approx. 4.3:



Empirical Difficulty

- The Davis-Putnam (DPLL) Procedure shows extreme runtime peaks at the phase transition:
- Note: Hard instances can exist even in the regions of the more easily satisfiable/unsatisfiable instances!



Notes on the Phase Transition

- When the probability of a solution is close to 1 (under-constrained), there are many solutions, and the first search path of a backtracking search is usually successful.
- If the probability of a solution is close to 0 (over-constrained), this fact can usually be determined early in the search.
- In the phase transition stage, there are many near successes ("close, but no cigar")
  - $\rightarrow$ (limited) possibility of predicting the difficulty of finding a solution based on the parameters
  - $\rightarrow$ (search intensive) benchmark problems are located in the phase region (but they have a special structure)

Local Search Procedures

Local Search Procedures

Local Search Methods for Solving Logical Problems

- In many cases, we are interested in finding a satisfying assignment of variables (example CSP), and we can sacrifice completeness if we can "solve" much larger instances this way.
- Standard process for optimization problems: Local Search
  - Based on a (random) configuration
  - Through local modifications, we hope to produce better configurations
  - $\rightarrow$ Main problem: local maxima

Dealing with Local Maxima

- As a measure of the value of a configuration in a logical problem, we could use the number of satisfied constraints/clauses.
- At first glance, local search seems inappropriate, considering that we want to find a global maximum (all constraints/clauses satisfied).
- However:
  - By restarting and/or injecting noise, we can often escape local maxima.
  - Local search can perform very well for SAT solving

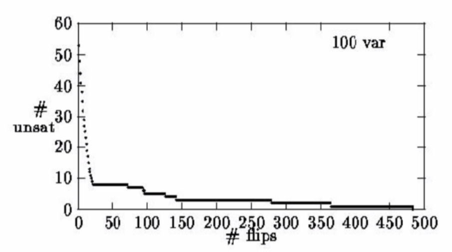
A pioneering local search method for SAT: GSAT (1993)

Procedure GSAT

```
INPUT: a set of clauses α, Max − Flips, and Max − Tries
OUTPUT: a satisfying truth assignment of α, if found
begin
        for i := 1 to Max − Tries
                T := a randomly-generated truth assignment
                for j := 1 to Max − Flips
                        if T satisfies α then return T
                        v := a propositional variable such that a change in its truth ...
                        ...assignment gives the largest increase in
the number of clauses of α that are satisfied by T
                        T := T with the truth assignment of v reversed
                end for
        end for
        return "no satisfying assignment found"
end
```

The Search Behavior of GSAT

- In contrast to many other local search methods, we must also allow sideways movements!
- Most time is spent searching on plateaus.



State of the Art

State of the Art

Practical Improvements of DPLL Algorithms

Clause Learning

- Consider an exemplary SAT problem
  - 26 variables $A, ..., Z$
  - Amongst many other clauses, we have $\{(\neg A, Y, Z)\}, \{(\neg A, \neg Y, Z)\}, \{(\neg A, Y, \neg Z)\}, \{(\neg A, \neg Y, \neg Z)\}$
  - We'll branch on variables in lexicographic order and try true first
- What will happen?
  - There is no satisfying assignment to the clauses above when $A = T$
  - For each assignment to variables $B, ..., X$, we'll have to rediscover this fact
  - Rather: reason about the variables that led to a conflict: $A, Y and Z$
  - We can 'Learn' (here: logically infer) a new clause: $\neg A$
  - Leads to conflict-directed clause learning (CDCL)
- Intelligent Backjumping
  - Closely related to clause learning
  - Jump back to the branching decision responsible for a conflict

Practical Improvements of SAT Algorithms

Both for DPLL/CDCL algorithms and local search algorithms

- Efficient data structures, indexing, etc
- Engineering ingenious heuristics
- Meta-algorithmic advances
  - Automated parameter tuning and algorithm configuration
  - Selection of the best-fitting algorithm based on instance characteristics
  - Selection of the best-fitting parameters based on instance characteristics
  - Use of machine learning to pinpoint what factors most affects performance

The Current State of the Art

- SAT competitions since beginning of the 90s
- Current SAT competitions (http://www.satcompetition.org/):
  - Largest "industrial" instances: $> 10,000,000$ variables
- Complete solvers dominate handcrafted and industrial tracks
- Incomplete local search solvers best on random satisfiable instances
- Best solvers use meta-algorithmic methods, such as algorithm configuration, selection, etc.

Concluding Remarks

- SAT solving very prominently uses resolution
- DPLL: combines resolution and backtracking
  - Very efficient implementation techniques
  - Good branching heuristics
  - Clause learning
- Incomplete randomized SAT-solvers
  - Perform best on random satisfiable problem instances
- State of the art
  - Typically obtained by automatic algorithm configuration & selection