SICP
Solution to Exercise 1.13

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Exercise 1.13: Prove that $\mathrm{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers (see 1.2.2) to prove that $\mathrm{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$.

Proof. We use strong induction. Let $n \in \mathbb{N}$ be arbitrary, and suppose that for all $k < n$,

$$\mathrm{Fib}(k) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$$

If $n = 0$, then

$$\begin{aligned}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0}{\sqrt{5}} \\ &= \frac{1-1}{\sqrt{5}} = 0 = \mathrm{Fib}(0).\end{aligned}$$

If $n = 1$, then

$$\begin{aligned}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} \\ &= \frac{\sqrt{5}}{\sqrt{5}} = 1 = \mathrm{Fib}(1).\end{aligned}$$

For $n \geq 2$, applying the inductive hypothesis to $n-2$ and $n-1$, we have

$$\begin{aligned}\mathrm{Fib}(n) &= \mathrm{Fib}(n-2) + \mathrm{Fib}(n-1) \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}\end{aligned}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(1 + \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(1 + \frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}}$$

Since $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$
and similarly $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2}$.

Substituting into the formula for $\text{Fib}(n)$ yields

$$\text{Fib}(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}$$

Therefore, $\text{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$ as required.

$\square$

Now, we will prove that $\text{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$.

Proof. Since $\text{Fib}(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$, it can be rewritten as

$$\frac{\varphi^n}{\sqrt{5}} = \text{Fib}(n) + \frac{\psi^n}{\sqrt{5}}$$

Note that, since $\left|\frac{\psi}{\sqrt{5}}\right| < \frac{1}{2}$, it follows that $\left|\frac{\psi^n}{\sqrt{5}}\right| < \left|\frac{\psi}{\sqrt{5}}\right| < \frac{1}{2}$, and therefore $-\frac{1}{2} < \frac{\psi^n}{\sqrt{5}} < \frac{1}{2}$.

Now adding $\text{Fib}(n)$ to the inequality $-\frac{1}{2} < \frac{\psi^n}{\sqrt{5}} < \frac{1}{2}$ gives

$$\text{Fib}(n) - \frac{1}{2} < \text{Fib}(n) + \frac{\psi^n}{\sqrt{5}} < \text{Fib}(n) + \frac{1}{2}$$

.

Since $\frac{\varphi^n}{\sqrt{5}} = \text{Fib}(n) + \frac{\psi^n}{\sqrt{5}}$, then

$$\text{Fib}(n) - \frac{1}{2} < \frac{\varphi^n}{\sqrt{5}} < \text{Fib}(n) + \frac{1}{2}$$

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Which shows that $\text{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$.

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