Project 4, FYS4150

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November 16, 2012

About the problem

The algorithm

Analytic solution

As a comparison we can find the analytic solution to this problem as follows.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = u(d,t) = 0 \tag{1}$$

$$x \in [0,d], \quad D = d = u(0,t) = 1 \tag{2}$$

We see right away that the boundary $x = 0$ could give us some problems, so we start off with a small trick

$$u(x,t) = v(x,t) = u(x,t) - u_s(x)$$

where the $u_s = 1 - x$ term is the steady-state solution to equation 1. This trick leaves us with new boundary conditions un $u(x,t)$

$$v(0,t) = u(0,t) - u_s(0,t) = 1, \;\; u_s(0,t) = 1 \implies u(0,t) = 0$$

which makes the whole procedure much simpler. We now assume that $u(x,t)$ can be separated into factors

$$u(x,t) = F(x)G(t) \implies \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \to F(x)\frac{\partial G}{\partial t} = G(t)\frac{\partial^2 F}{\partial x^2}$$

$$\frac{1}{G(t)}\frac{\partial G}{\partial t} = -k^2 = \frac{1}{F(x)}\frac{\partial^2 F}{\partial x^2}$$

We start with the simplest of the equations which is the time dependence

$$\frac{1}{G(t)}\frac{\partial G}{\partial t} = -k^2$$

$$G(t) = Ce^{-k^2 t}$$

and leave it like this for now. The x-dependent equation is somewhat more complicated

$$\frac{1}{F(x)}\frac{\partial^2 F}{\partial x^2} = -k^2$$

$$F(x) = A\sin(kx) + B\cos(kx)$$

$$F(0) = A\sin(0) + B\cos(0) = 0 \implies B = 0$$

$$F(d) = A\sin(kd) = 0 \implies k = \frac{m\pi}{d} = \pi m, \;\; A = A_m$$

we now combine all the equations to determine $v(x,t)$

$$v(x,t) = 1 - x + \sum_{m=1}^{\infty} B_m e^{-(m\pi)^2 t}\sin(m\pi x)$$

$$v(x,0) = 1 - x + \sum_{m=1}^{\infty} B_m \sin(m\pi x)$$

$$\implies \int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \delta_{mn} = \int_0^1 (x-1)\sin(m\pi x)dx = -\frac{2}{m\pi} = B_m$$

This gives us the full analytical solution

$$v(x,t) = 1 - x - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}e^{-(m\pi)^2 t}\sin(m\pi x) \tag{3}$$

which satisfies all initial and boundary conditions.

Results

Stability and precision

Final comments