Diffusion equation with Finite elements
INF5620

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About the problem

The goal of this project is to solve the diffusion equation in 2 spatial dimensions by the Finite Element Method, using the FEniCS software package. Preferably with some non-linear terms which are to be specified later. The equation is then on the form

$$\rho C(u)\frac{\partial u}{\partial t} = \nabla \cdot \alpha(u)\nabla u + f(x, y, t) \tag{1}$$

First simplified discretization

To simply get started we begin by simplifying equation (1) quite a lot. If we set

$$\rho = C(u) = \alpha(u) = 1$$

we are left with the simple two-dimensional diffusion equation with a source term.

$$\frac{\partial u}{\partial t} = \nabla^2 u + f \tag{2}$$

Using Backward Euler discretization in time we get

$$u^n = \Delta t \nabla^2 u + \Delta t f + u^{n-1}$$

and set

$$R = u^n - \Delta t \nabla^2 u - \Delta t f - u^{n-1} = 0$$

We define a functionspace V and demand

$$\begin{aligned}
&(R, v) = 0 \;\; \forall \;\; v \in V \\
\implies &\int_\Omega u^n v dx - \Delta t \int_\Omega \nabla^2 u^n v dx - \Delta t \int_\Omega f v dx - \int_\Omega u^{n-1} v dx = 0 \\
&\int_\Omega u^n v dx - \Delta t \big([\nabla u^n v]_0^L - \int_\Omega \nabla u^n \nabla v dx - \Delta t \int_\Omega f v dx \big) - \int_\Omega u^{n-1} v dx = 0
\end{aligned}$$

Defining $a = \int_\Omega u^n v dx$ and $\mathcal{L} = \Delta t \int_\Omega \nabla u^n \nabla v dx - \Delta t \int_\Omega f v dx + \int_\Omega u^{n-1} v dx$ we get the so called weak form which is what we plug into the FEniCS software. Of course we will need to specify the function $f(x, y, t)$ and what basisfunctions should span the functionspace and so on. So let us do this now, and focus on the details later. Choosing Lagrange polynomials of degree 1 we know that a polynomial of degree 2 in space and degree 1 in time. A suitable solution to equation 2 is

$$\begin{aligned}
u(x, y, t) &= 1 + x^2 + \alpha y^2 + \beta t \\
\frac{\partial u}{\partial t} &= \beta. \quad \nabla^2 u = 2 + 2\alpha \\
\implies f(x, y, t) &= \beta - 2 - 2\alpha
\end{aligned}$$

This solution should be reproduced to (almost) machine precision.

```
Max error, t=0.20: 0.00000000000003997
Max error, t=0.30: 0.00000000000004796
...
Max error, t=1.90: 0.00000000000009948
Max error, t=2.00: 0.00000000000010347
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We can also compare the solution from the FEM to a finite difference scheme. The equation in question is in this case an even simpler one

$$\frac{\partial u}{\partial t} = \nabla^2 u \tag{3}$$

on the domain $x, y \in [0, 1]$ with Dirichlet boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$

Slightly more complex