1D wave equation with finite elements
INF5620

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November 15, 2012

The purpose of this project is to derive and analyze a finite element method for the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Leftrightarrow D_t D_t u = c^2 D_x D_x u$$
$$x \in [0, L], t \in (0, T]$$

with initial and boundary conditions

$$u(0,t) = U_0(t), u_x(L) = 0, u(x,0) = I(0), u_t(x,0) = V(x)$$

a)

If we now use a finite difference method on this equation we can formulate a series of spatial problems as follows

$$D_t D_t u = u_{tt} = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}$$
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 u_{xx}^n$$
$$u_j^{n+1} = \Delta t^2 c^2 u_{xx}^n + 2u_j^n - u_j^{n-1}$$

where $n = 0, 1, ..., N_x$

b)

Using the Galerkin method we can now transform the series of spatial problems to variational form. I have used the testfunction $v(x)$ which is part of the function space of P1 elements. Assume (for later) that

$$v(x) = \phi_i, (i = 0, 1, ..., N) \text{ and } u^n \simeq \sum_{j=0}^{N_x} \phi_j c_j^n \tag{1}$$

We want the inner product $(R, v) = 0$ where $R = u_j^{n+1} - \Delta t^2 c^2 u_{xx}^n - 2u_j^n + u_j^{n-1} = 0$. This means that

$$(R, v) = \left(u_j^{n+1} - \Delta t^2 c^2 u_{xx}^n - 2u_j^n + u_j^{n-1}, v\right)$$
$$= \int_0^L \left(u_j^{n+1} - \Delta t^2 c^2 u_{xx}^n - 2u_j^n + u_j^{n-1}\right) \cdot v dx$$
$$= \int_0^L u_j^{n+1} v dx - 2\int_0^L u_j^n v dx + \int_0^L u_j^{n-1} v dx - \underbrace{\Delta t^2 c^2 \int_0^L u_{xx}^n v dx}_{=\Delta t^2 c^2 \left([u_x v]_0^L - \int_0^L u_x^n v_x dx\right)}$$
$$= \int_0^L u_j^{n+1} v dx - 2\int_0^L u_j^n v dx + \int_0^L u_j^{n-1} v dx - C^2 \int_0^L u_x^n v_x dx = 0$$

which leads us to the final variational form

$$\left(u^{n+1}, v\right) - 2\left(u^{n}, v\right) + \left(u^{n-1}, v\right) - C^2\left(u_x^n, v_x\right) = 0, \quad \forall\, v \in V$$

where

$$C^2 = -\Delta t^2 c^2 \tag{2}$$

c)

Using the approximation of u in (1) we can set up the variational for as a linear system

$$\int_0^L u_j^{n+1} v dx - 2\int_0^L u_j^n v dx + \int_0^L u_j^{n-1} v dx + \int_0^L u_x^n v_x dx =$$

$$\sum_j^{N_x}\int_0^L(\phi_j\phi_i)c_j^{n+1}dx - 2\sum_j^{N_x}\int_0^L(\phi_j\phi_i)c_j^{n}dx + \sum_j^{N_x}\int_0^L(\phi_j\phi_i)c_j^{n-1}dx - C^2\sum_j^{N_x}\int_0^L(\phi_j'\phi_i')c_j^{n}dx$$

If we now set $M_{ij} = \int_0^L(\phi_j\phi_i)dx$ and $K_{ij} = \int_0^L(\phi_j'\phi_i')dx$ we get

$$M_{ii} = \int_{-1}^{1}(\phi_i\phi_i)dx, \quad M_{ij} = M_{ji} = \int_{-1}^{1}(\phi_j\phi_i)dx$$

$$K_{ii} = \int_{-1}^{1}(\phi_i'\phi_i')dx, \quad K_{ij} = K_{ji} = \int_{-1}^{1}(\phi_j'\phi_i')dx$$

Remember that for every M_{ii} and K_{ii} entry there are two possible function combinations because within each element there are defined two functions say $\phi_0 = \frac{1}{2}(1-x)$ and $\phi_1 = \frac{1}{2}(1+x)$. This means that for every diagonal entry we need to calculate the sum of two integrals (or just multiply by 2 since they are equal). The integrals become

$$M_{ii} = \int_{-1}^{1}(\phi_i\phi_i)dX = 2\frac{h}{8}\int_{-1}^{1}(1-X)^2dX = 2\frac{h}{8}\cdot\frac{8}{3} = \frac{2h}{3}$$

$$M_{ij} = \int_{-1}^{1}(\phi_i\phi_j)dX = \frac{h}{8}\int_{-1}^{1}(1-X)(1+X)dX = \frac{h}{8}\cdot\frac{4}{3} = \frac{h}{6}$$

$$K_{ii} = 2\int_{-1}^{1}\phi_i'\phi_i'dX = 2\int_{-1}^{1}(\frac{2}{h}\phi_i'\frac{2}{h}\phi_i')\frac{h}{2}dX = 2\frac{2}{h}\int_{-1}^{1}\frac{-1}{2}\frac{-1}{2}dX = 2\frac{1}{h}$$

$$K_{ij} = \int_{-1}^{1}\phi_i'\phi_j'dX = \frac{2}{h}\int_{-1}^{1}\frac{-1}{2}\frac{1}{2}dX = \frac{-1}{h}$$

This makes our matrices

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & \dots & & 0 \\ 1 & 4 & 1 & 0 & \dots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \\ & & & & & \\ 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & & \dots & 0 & 1 & 4 \end{pmatrix}, \quad K = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & 0 & \dots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \\ & & & & & \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & & \dots & 0 & -1 & 2 \end{pmatrix}$$

And thus naming $\sum_{j=0}^{N_x} c_j^n = \mathbf{c}^n$ we arrive at a linear system to be solved for each timestep

$$M\mathbf{c}^{n+1} - 2M\mathbf{c}^n + M\mathbf{c}^{n-1} - C^2K\mathbf{c}^n = 0 \tag{3}$$

d)

Say we now (for some well defined reason) would like to compare the i'th equation in (3) with the nummerical scheme

$$\left[D_tD_t\left(u+\frac{1}{6}D_xD_x\Delta x^2u\right)=c^2D_xD_xu\right]_i^n \tag{4}$$

All we need to do is to find the i'th equation and do some algebra. The i'th equation is

$$\begin{aligned}
&\frac{h}{6}(c_{i+1}^{n+1}+4c_i^{n+1}+c_{i-1}^{n+1})-\frac{2h}{6}(c_{i+1}^{n}+4c_i^{n}+c_{i-1}^{n})+\frac{h}{6}(c_{i+1}^{n-1}+4c_i^{n-1}+c_{i-1}^{n-1})\\
&=\frac{-C^2}{h}(c_{i+1}^n-2c_i^n+c_{i-1}^n)\\
&\frac{h}{6}(c_{i+1}^{n+1}-2c_{i+1}^{n}+c_{i+1}^{n-1})+\frac{2h}{3}(c_i^{n+1}-2c_i^n+c_i^{n-1})+\frac{h}{6}(c_{i-1}^{n+1}-2c_{i-1}^{n}+c_{i-1}^{n-1})\\
&=\frac{-C^2}{h}(c_{i+1}^n-2c_i^n+c_{i-1}^n)
\end{aligned}$$

Having done the calculation of (4) quite a few times now, I can easily recognize the left-hand-side of the animal above as the left-hand-side of (4) times h. If we simply divide by h and recognize C^2 from (2) we can set $h=\Delta x$ and arrive at the very scheme we were comparing with. This means that the finite element approach will use more points in the calculation of the next timestep than a normal finite difference scheme.

e)

Let us do some analysis on the finite difference scheme (4) we are actually using for each timestep. Say that the solution u is some sine-cosine combination represented by the complex exponential $u(x,t)=e^{i(kp\Delta x-\tilde{\omega}n\Delta t}$. We start off by doing all the derivatives

$$\begin{aligned}
D_tD_tu&=\frac{e^{ikp\Delta x}}{\Delta t^2}\left(e^{-i\tilde{\omega}(n+1)\Delta t}-2e^{-i\tilde{\omega}n\Delta t}+e^{-i\tilde{\omega}(n-1)\Delta t}\right)\\
&=\frac{e^{ikp\Delta x}}{\Delta t^2}\big(-2e^{-i\tilde{\omega}n\Delta t}+e^{-i\tilde{\omega}n\Delta t}\big(\underbrace{e^{i\tilde{\omega}\Delta t}+e^{-i\tilde{\omega}\Delta t}}_{2-4\sin^2(\frac{\tilde{\omega}\Delta t}{2})}\big)\big)\\
&=\frac{-4}{\Delta t^2}e^{i(kp\Delta x-\tilde{\omega}n\Delta t)}\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)
\end{aligned}$$

$$\begin{aligned}
D_xD_xu&=\frac{e^{-in\tilde{\omega}\Delta t}}{\Delta x^2}\left(e^{ik(p+1)\Delta x}-2e^{ikp\Delta x}+e^{ik(p-1)\Delta x}\right)\\
&=\frac{e^{-in\tilde{\omega}\Delta t}}{\Delta x^2}\big(-2e^{ikp\Delta x}+e^{ikp\Delta x}\big(\underbrace{e^{ik\Delta x}+e^{-ik\Delta x}}_{2-4\sin^2(\frac{k\Delta x}{2})}\big)\big)\\
&=\frac{-4}{\Delta x^2}e^{i(kp\Delta x-\tilde{\omega}n\Delta t)}\sin^2\left(\frac{k\Delta x}{2}\right)
\end{aligned}$$

and finally for the "cross" term we have allready done the nececary calculations for the time derivative and we are left with

$$\begin{aligned}
D_tD_t(\frac{\Delta x^2}{6}D_xD_xu)&=D_tD_t\left(\frac{-4}{6}e^{i(kp\Delta x-\tilde{\omega}n\Delta t)}\sin^2\left(\frac{k\Delta x}{2}\right)\right)\\
&=\frac{-4}{6}e^{ikp\Delta x}\sin^2\left(\frac{k\Delta x}{2}\right)D_tD_te^{-i\tilde{\omega}n\Delta t}\\
&=\frac{16}{6\Delta t^2}e^{i(kp\Delta x-\tilde{\omega}n\Delta t)}\sin^2\left(\frac{k\Delta x}{2}\right)\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)
\end{aligned}$$

so if we insert all the derivatives in the scheme (4) we get the following animal

$$\begin{aligned}
&\frac{-4}{\Delta t^2} e^{i(kp\Delta x - \tilde{\omega} n \Delta t)} \sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) + \frac{16}{6\Delta t^2} e^{i(kp\Delta x - \tilde{\omega} n \Delta t)} \sin^2\left(\frac{k\Delta x}{2}\right) \sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) \\
&= c^2 \frac{-4}{\Delta x^2} e^{i(kp\Delta x - \tilde{\omega} n \Delta t)} \sin^2\left(\frac{k\Delta x}{2}\right) \\
&\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) - \frac{2}{3}\sin^2\left(\frac{k\Delta x}{2}\right)\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right) = C^2 \sin^2\left(\frac{k\Delta x}{2}\right) \\
&\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)\left(1 - \sin^2\left(\frac{k\Delta x}{2}\right) + \frac{1}{3}\sin^2\left(\frac{k\Delta x}{2}\right)\right) = C^2 \sin^2\left(\frac{k\Delta x}{2}\right) \\
C^2 &= \underbrace{\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)\cot\left(\frac{k\Delta x}{2}\right)}_{\simeq 0} + \frac{1}{3}\sin^2\left(\frac{k\Delta x}{2}\right)
\end{aligned}$$

The maximum value the sine term can be is 1 therefore

$$C^2 \leq \frac{1}{3} \implies C \leq \frac{1}{\sqrt{3}}$$