1D wave equation with finite elements
INF5620

Fredrik E Pettersen

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The purpose of this project is to derive and analyze a finite element method for the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Leftrightarrow D_t D_t u = c^2 D_x D_x u$$
$$x \in [0, L], t \in (0, T]$$

with initial and boundary conditions

$$u(0,t) = U_0(t), u_x(L) = 0, u(x,0) = I(0), u_t(x,0) = V(x)$$

a)

If we now use a finite difference method on this equation we can formulate a series of spatial problems as follows

$$D_t D_t u = u_{tt} = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}$$
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 u_{xx}^n$$
$$u_j^{n+1} = \Delta t^2 c^2 u_{xx}^n + 2u_j^n - u_j^{n-1}$$

where $n = 0, 1, ..., N_x$

b)

Using the Galerkin method we can now transform the series of spatial problems to variational form. I have used the testfunction $v(x)$ which is part of the function space of P1 elements. Assume (for later) that

$$v(x) = \phi_i, (i = 0, 1, ..., N) \text{ and } u^n \simeq \sum_{j=0}^{N_x} \phi_j c_j^n \tag{1}$$

We want the inner product $(R, v) = 0$ where $R = u_j^{n+1} - \Delta t^2 c^2 u_{xx}^n - 2u_j^n + u_j^{n-1} = 0$. This means that

$$(R, v) = \left(u_j^{n+1} - \Delta t^2 c^2 u_{xx}^n - 2u_j^n + u_j^{n-1}, v\right)$$
$$= \int_0^L \left(u_j^{n+1} - \Delta t^2 c^2 u_{xx}^n - 2u_j^n + u_j^{n-1}\right) \cdot v dx$$
$$= \int_0^L u_j^{n+1} v dx - 2\int_0^L u_j^n v dx + \int_0^L u_j^{n-1} v dx - \underbrace{\Delta t^2 c^2 \int_0^L u_{xx}^n v dx}_{=\Delta t^2 c^2 \left([u_x v]_0^L - \int_0^L u_x^n v_x dx \right)}$$
$$= \int_0^L u_j^{n+1} v dx - 2\int_0^L u_j^n v dx + \int_0^L u_j^{n-1} v dx + \int_0^L u_x^n v_x dx = 0$$

which leads us to the final variational form

$$\left(u^{n+1}, v\right) - 2\left(u^n, v\right) + \left(u^{n-1}, v\right) - C^2\left(u_x^n, v_x\right) = 0, \quad \forall\, v \in V$$

c)

Using the approximation of u in (1) we can set up the variational for as a linear system

$$\int_0^L u_j^{n+1} v dx - 2\int_0^L u_j^n v dx + \int_0^L u_j^{n-1} v dx + \int_0^L u_x^n v_x dx =$$
$$\sum_j^{N_x} \int_0^L (\phi_j \phi_i) c_j^{n+1} dx - 2\sum_j^{N_x} \int_0^L (\phi_j \phi_i) c_j^n dx + \sum_j^{N_x} \int_0^L (\phi_j \phi_i) c_j^{n-1} dx - C^2 \sum_j^{N_x} \int_0^L (\phi_j' \phi_i') c_j^n dx$$

If we now set $M_{ij} = \int_0^L (\phi_j \phi_i) dx$ and $K_{ij} = \int_0^L (\phi_j' \phi_i') dx$ we get

$$M_{ij} = \int_0^L (\phi_j \phi_i) dx \implies M_{00} = \int_{-1}^1 (\phi_0 \phi_0) dx$$