Multiplier Analysis

Fernando Ramos

March 29, 2018

Introduction

This document shows how to rewrite an Auto Regressive Distributed Lag Model (ARDL) into an Infinite Distributed Lag Model (IDL) to see the total effect that an independent variable has on the dependent variable.

Models

Auto Regressive Distributed Lag Model

This model is ARDL(1,1) such that it has 1 lag of the dependent variable and 1 lag of the independent variable as explanatory variables.

$$Y_t = \delta + \theta_1 Y_{t-1} + \delta_0 X_t + \delta_1 X_{t-1} + V_t$$

Infinite Distributed Lag Model

This model has infinite lags of the dependent variable X and the effect of X is distributed throughout an infinite number of lagged periods.

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \beta_3 X_{t-3} + \beta_4 X_{t-4} + \ldots$$

Alternatively, this model can be rewritten as

$$Y_t = \alpha + \sum_{s=0}^{\infty} \beta_s X_{t-s} + e_t$$

Rewriting the Models

Now I rewrite both of the models using the lag operator L which imposes lags on a given variable. For example, $LX_t = X_{t-1}$ and $L^2 X_t = X_{t-2}$, etc. For the ARDL Model, I move all the Y terms on the left side and then transform the left side to be Y_t. Also, note that (1- $\theta_1 L)^{-1}(1 - \theta_1 L) = 1$.

ARDL(1,1) Using Lag Notation

$$
\begin{gathered}
Y_t = \delta + \theta_1 L Y_t + \delta_0 X_t + \delta_1 L X_t + V_t \\
Y_t - \theta_1 L Y_t = \delta + \delta_0 X_t + \delta_1 L X_t + V_t \\
(1 - \theta_1 L) Y_t = \delta + \delta_0 X_t + \delta_1 L X_t + V_t \\
(1 - \theta_1 L)^{-1} (1 - \theta_1 L) Y_t = (1 - \theta_1 L)^{-1} (\delta + \delta_0 X_t + \delta_1 L X_t + V_t) \\
Y_t = (1 - \theta_1 L)^{-1} \delta + (1 - \theta_1 L)^{-1} \delta_0 X_t + (1 - \theta_1 L)^{-1} \delta_1 L X_t + (1 - \theta_1 L)^{-1} V_t \\
Y_t = \underline{(1 - \theta_1 L)^{-1} \delta} + \underline{(1 - \theta_1 L)^{-1} (\delta_0 + \delta_1 L) X_t} + \underline{(1 - \theta_1 L)^{-1} V_t}
\end{gathered}
$$

IDL Using Lag Notation

$$
Y_t = \underline{\alpha} + \underline{(\beta_0 + \beta_1 L + \beta_2 L^2 + \beta_3 L^3 + \beta_4 L^4 + \cdots)} X_t + \underline{e_t}
$$

Equate The Colored, Underlined Terms

The IDL terms are on the left side while the ARDL terms are on the right side.

Equate the Intercepts

$$
\begin{gathered}
\alpha = (1 - \theta_1 L)^{-1} \delta \\
(1 - \theta_1 L) \alpha = (1 - \theta_1 L)(1 - \theta_1 L)^{-1} \delta \\
(1 - \theta_1 L) \alpha = \delta \\
L\alpha = \alpha \ so, \ \alpha - \theta_1 \alpha = \delta \\
(1 - \theta_1) \alpha = \delta \\
\alpha = \frac{\delta}{1 - \theta_1}
\end{gathered}
$$

Equate the Slope Coefficients of $\mathbf{X}_t$

$$
\begin{gathered}
\beta_0 + \beta_1 L + \beta_2 L^2 + \beta_3 L^3 + \beta_4 L^4 + \ldots = (1 - \theta_1 L)^{-1} (\delta_0 + \delta_1 L) \\
(1 - \theta_1 L)(\beta_0 + \beta_1 L + \beta_2 L^2 + \beta_3 L^3 + \beta_4 L^4 + \ldots) = (1 - \theta_1 L)(1 - \theta_1 L)^{-1} (\delta_0 + \delta_1 L) \\
(1 - \theta_1 L)(\beta_0 + \beta_1 L + \beta_2 L^2 + \beta_3 L^3 + \beta_4 L^4 + \ldots) = \delta_0 + \delta_1 L \\
\beta_0 - \beta_0 \theta_1 L + \beta_1 L - \beta_1 \theta_1 L^2 + \beta_2 L^2 - \beta_2 \theta_1 L^3 + \beta_3 L^3 - \beta_3 \theta_1 L^4 + \beta_4 L^4 - \beta_4 \theta_1 L^5 + \ldots = \delta_0 + \delta_1 L \\
\beta_0 + (\beta_1 - \beta_0 \theta_1) L + (\beta_2 - \beta_1 \theta_1) L^2 + (\beta_3 - \beta_2 \theta_1) L^3 + (\beta_4 - \beta_3 \theta_1) L^4 + \ldots = \delta_0 + \delta_1 L
\end{gathered}
$$

Set the terms equal to each other according to their L term. If a L^k for $k \in (0, \infty)$ term on the left side does not have an equivalent L^k term on the right side, then set equal to 0.

$$\beta_0 = \delta_0$$
$$\beta_1 - \beta_0\theta_1 = \delta_1 \Rightarrow \beta_1 = \delta_1 + \beta_0\theta_1$$
$$\beta_2 - \beta_1\theta_1 = 0 \Rightarrow \beta_2 = \beta_1\theta_1$$
$$\beta_3 - \beta_2\theta_1 = 0 \Rightarrow \beta_3 = \beta_2\theta_1$$
$$\beta_4 - \beta_3\theta_1 = 0 \Rightarrow \beta_4 = \beta_3\theta_1$$
$$\vdots$$

Total Effect

$$\sum_{s=0}^{\infty} \beta_s = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 \ldots$$
$$= \delta + \delta_1 + \beta_0\theta_1 + \beta_1\theta_1 + \beta_2\theta_1 + \beta_3\theta_1 + \beta_4\theta_1 \ldots$$
$$= \delta + (\delta_1 + \beta_0)(1 + \theta_1 + \theta_1^2 + \theta_1^3 + \theta_1^4 + \ldots)$$

Applying Geometric Series, $\left(1 + r + r^2 + r^3 + r^4 \ldots = \frac{1}{1-r}\right)$ **where** $r = \theta_1$ **and** $\beta_0 = \delta_0$**,**

$$\sum_{s=0}^{\infty} \beta_s = \delta + \frac{\delta_0 + \delta_1}{1 - \theta_1}$$